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THE 7-CONNECTED COBORDISM RING AT $p = 3$

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ABSTRACT. In this paper, we study the cobordism spectrum $MO\langle 8\rangle$ at the prime 3. This spectrum is important because it is conjectured to play the role for elliptic cohomology that Spin cobordism plays for real K-theory. We show that the torsion is all killed by 3, and that the Adams-Novikov spectral sequence collapses after only 2 differentials. Many of our methods apply more generally.

INTRODUCTION

In algebraic topology, the complex cobordism spectrum MU is a sort of universal example of a well-behaved cohomology theory. Virtually every commonly studied theory admits an orientation from MU. The most significant exception is real K-theory, KO. Now elliptic cohomology is supposed to be a higher analog of KO. So one would not expect it to admit an orientation from MU either. In Witten's interpretation [Wit] of the level 1 elliptic genus as the index of the equivariant Dirac operator on the free loop space LM of a manifold M, one needs LM to be Spin. The easiest way to guarantee this is to take M to be a manifold such that the classifying map of its tangent bundle, $M \longrightarrow BO$, lifts to $BO\langle 8\rangle$, the 7-connected cover of BO. This indicates that whatever the final version of elliptic cohomology is, it should admit an orientation of $MO\langle 8\rangle$, which is the Thom spectrum built from $BO\langle 8\rangle$.

Now $MO\langle 8\rangle$ is not very well understood. It has torsion in its homotopy at the primes 2 and 3, and if we localize it by inverting 2 and 3, it splits into a wedge of suspensions of the Brown-Peterson spectrum BP. This means that the homotopy groups modulo torsion are easily computed, but the multiplicative structure is unknown. It is certainly not polynomial. There have been low-dimensional calculations of $MO\langle 8\rangle$ at the prime 2, the most recent of which is due to Gorbunov and Mahowald [GM]. However, at the prime 3 virtually nothing is known. We attempt to remedy that in this paper.

There is actually a candidate for level 1 elliptic cohomology at each of the primes 2 and 3. In each case, the spectrum involved is called EO_2, and they are special cases of a more general construction due to Hopkins and Miller [HM]. Their homotopy groups are completely known, and much of the algebra structure is known as well. But at the moment, there is no solid evidence relating them to elliptic cohomology. It would strengthen the case considerably if there

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was an orientation $MO\langle 8\rangle \longrightarrow EO_2$. This is another reason to try to understand $MO\langle 8\rangle$. In fact, we show in this paper that $MO\langle 8\rangle$ behaves very similarly to EO_2. Hopkins and Mahowald (personal communication) have constructed a connective, uncompleted version of EO_2, called $eo(2)$, at $p = 3$. The conjecture is that $MO\langle 8\rangle$ at $p = 3$ should be an amalgam of BP and $eo(2)$ just as MSU at $p = 2$ is an amalgam of BP and ko [Pen].

There is a standard strategy to try to compute the homotopy of a Thom spectrum such as $MO\langle 8\rangle$. First one computes the cohomology of the base, in our case $BO\langle 8\rangle$, as a module over the Steenrod algebra $\mathscr{A}$. Here we would take mod 3 cohomology, so use the mod 3 Steenrod algebra. Then one computes the operations on the Thom class, or equivalently, the homomorphism $\mathscr{A} \longrightarrow H^*MO\langle 8\rangle$. These two things together give the structure of $H^*MO\langle 8\rangle$ as a module over $\mathscr{A}$, and one then applies the Adams spectral sequence to get at the homotopy.

The main problem with this strategy for $MO\langle 8\rangle$ at $p = 3$ is in calculating the $\mathscr{A}$-module structure on $H^*BO\langle 8\rangle$. One certainly knows the structure of H^*BSO, and there is a fibration

$$K(\mathbf{Z}, 3) \longrightarrow BO\langle 8\rangle \longrightarrow BSO$$

but the Serre spectral sequence only gives limited information about the $\mathscr{A}$-action. There are many $\mathscr{A}$-extensions that are hard to compute. So we need a different method. The method we use is based on Hopf rings. It turns out that $BO\langle 8\rangle$ localized at 3 is homotopy equivalent to $\mathbf{BP}\langle \mathbf{1}\rangle_8$, the 8th space in the Ω-spectrum for $BP\langle 1\rangle$. One can then use the results in [Wil, RW] to calculate the $\mathscr{A}$-action.

This approach also sheds light on the homomorphism $\mathscr{A} \longrightarrow H^*MO\langle 8\rangle$ defined by the Thom class. First, notice that this homomorphism is the map induced on cohomology by

$$MO\langle 8\rangle \longrightarrow MSO \longrightarrow H\mathbf{F}_3.$$

At $p = 3$, this map factors through BP, so it will certainly kill the 2-sided ideal generated by the Bockstein. Denote by $\mathscr{P}$ the sub-Hopf algebra of $\mathscr{A}$ generated by the reduced powers. Note that P^1 in dimension 4 must go to 0 in $H^*MO\langle 8\rangle$ for dimensional reasons, so we have a map

$$\mathscr{P}/\mathscr{P}(P^1) \longrightarrow H^*MO\langle 8\rangle.$$

We show in the first section that this map is in fact injective. It turns out to be not very much harder to calculate the kernel of the corresponding map for all $MO\langle k\rangle$ at odd primes, and all $MU\langle k\rangle$, so we do so. This calculation is originally due to Rosen, who used a different method in his unpublished thesis [Ros].

Now it follows from the general theory of $\mathscr{A}$-module coalgebras that Rosen's result puts some fairly tight restrictions on the $\mathscr{A}$-module structure of $H^*MO\langle 8\rangle$. We also know, in the particular case of $MO\langle 8\rangle$ at $p = 3$, that the cohomology is evenly graded. Let X denote the 8-skeleton of BP, which is a 3-cell complex which is P^1-free. Then these considerations lead to a proof that $MO\langle 8\rangle \wedge X$ is a wedge of suspensions of BP. Essentially we just have to put back in the P^1 that $H^*MO\langle 8\rangle$ does not see.

This result has a number of nice corollaries about the global structure of the homotopy of $MO\langle 8\rangle$. For example, we show that the 3-torsion is all killed by 3 itself. There are also elements in $\pi_* MO\langle 8\rangle$ analogous to v_1^k, for each $k > 1$, as constructed in [Hov]. We show that the 3-torsion coincides with the v_1-torsion, and that the v_1-torsion is all killed by any of the v_1-elements above. This is all contained in the second section.

Since $MO\langle 8\rangle \wedge X$ is a wedge of suspensions of BP, any X-resolution of the sphere becomes an Adams-Novikov resolution of $MO\langle 8\rangle$ upon smashing with $MO\langle 8\rangle$. But there is a very simple X-resolution of the sphere, used by the second author in [Rav, Chapter 7]. The resulting Adams-Novikov resolution of $MO\langle 8\rangle$ puts severe restrictions on its Adams-Novikov spectral sequence. We find in the third section that the spectral sequence must collapse at E_{10} after at most d_5 and d_9. This is precisely what happens in EO_2, and the pattern of differentials appears to be the same.

At this point, we must do some calculation to learn more. In the fourth section we calculate far enough to see that the Adams-Novikov spectral sequence does not collapse at E_6, so one really does need two differentials. Here we use the Adams spectral sequence to compute with. This is certainly not the best method for computing in $MO\langle 8\rangle$. We have a better spectral sequence for doing so which we will describe in a future paper. Here we calculate through dimension 32. This calculation leads to several conjectures about the behavior further out. The main thing stopping us from proving these conjectures at this time is a more thorough understanding of the homology of $MO\langle 8\rangle$.

Our methods do not apply to $MO\langle k\rangle$ for k larger than 8, for two closely related reasons. Firstly, the homology of $\mathbf{BP}\langle \mathbf{1}\rangle_n$ is, so far as we know, not known as a comodule over the dual Steenrod algebra when n is larger than $2p+2$. Secondly, one expects torsion in the homology of $MO\langle k\rangle$ for $k > 8$ and in the homology of $MU\langle k\rangle$ for $k > 6$. One might ask if there is a Thom spectrum mapping to $MO\langle k\rangle$ for which these problems disappear. In fact, there is. One can build p-local Thom spectra over the spaces of the $BP\langle r\rangle$ spectrum, and if r is large enough relative to the connectivity k, these problems do not arise. We then get bounded torsion results for these Thom spectra. This is explained in the last section.

The authors would like to thank several people for their help with this paper. This paper grew out of unpublished notes of David Pengelley and the second author, so we thank David for letting us use them. We thank Haynes Miller and Mike Hopkins for teaching us about the EO_n, and for several helpful conversations. In particular, the argument for proving the bounded torsion theorem is due to Mike. We thank Neil Strickland for teaching the first author the correct way to calculate in Hopf rings. We thank Chuck Giffen and Nick Kuhn for help in defining p-local Thom spaces. We thank Bob Bruner for sharing his program for computing Ext. And we thank Mark Mahowald for pointing out the torsion in the homology of $MO\langle k\rangle$ for $k > 8$.

Let us fix notation we will use throughout the paper. We will mostly be working in the p-local category, whether we are dealing with spaces or spectra. When we need a notation for the p-localization of X, we will use $X_{(p)}$. If E is a ring spectrum, $\mu : E \wedge E \longrightarrow E$ will denote its multiplication, and $\eta : S^0 \longrightarrow E$ will denote its unit. We reserve the letter H for the mod p homology spectrum, so that H_*H is the dual Steenrod algebra. $T: X \wedge Y \longrightarrow Y \wedge X$ will denote the twist map.

1. Rosen's Theorem

Our goal in this section is to determine which Steenrod reduced powers act trivially on the Thom class in $H^0 MO\langle k\rangle$. First let us recall some well-known facts and notation.

The mod p Steenrod algebra will be denoted by $\mathscr{A}$, and its dual by $\mathscr{A}_*$. For $p = 2$, $\mathscr{A}$ is generated by the Steenrod squares Sq^{2^i}, and $\mathscr{A}_*$ is a polynomial algebra

$$\mathscr{A}_* = P(\xi_1, \xi_2, \dots) \text{ with } |\xi_n| = 2^n - 1.$$

The diagonal is defined by

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i.$$

There is a canonical anti-automorphism c, and we let ζ_n denote $c\xi_n$, so that we have

$$\Delta(\zeta_n) = \sum_{i=0}^{n} \zeta_i \otimes \zeta_{n-i}^{2^i}.$$

As usual, $A(n)$ denotes the sub-Hopf algebra of $\mathscr{A}$ generated by Sq^{2^i} for $i \leq n$, and it is convenient to let $A(-1) = 0$.

We denote by $\mathscr{P}$ the quotient of $\mathscr{A}$ by the 2-sided ideal generated by Sq^1. $\mathscr{P}$ is also a Hopf algebra, and its dual $\mathscr{P}_* = P(\xi_1^2, \xi_2^2, \dots)$. We denote by $P(n)$ the sub-Hopf algebra of $\mathscr{P}$ generated by Sq^{2^i} for $1 \leq i \leq n+1$ and let $P(-1) = 0$.

For p odd, let us denote by $\mathscr{P}$ the sub-Hopf algebra of the mod p Steenrod algebra generated by the reduced powers. Recall that $\mathscr{P}^*$ is generated by the P^{p^n}, which have degree $2p^n(p-1)$. The dual $\mathscr{P}_*$ is a polynomial algebra $P(\xi_1, \xi_2, \dots)$ with $|\xi_n| = 2(p^n - 1)$ and

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i.$$

There is a canonical anti-automorphism c as at $p = 2$ and we denote $c\xi_n$ by ζ_n. Then we have

$$\Delta\zeta_n = \sum_{i=0}^{n} \zeta_i \otimes \zeta_{n-i}^{p^i}.$$

Denote by $P(n)$ the sub-Hopf algebra of $\mathscr{P}$ generated by P^{p^i} for $i \leq n$, and let $P(-1) = 0$. Notice that, for any prime p, $H^*BP = \mathscr{P}$.

Let us recall the result of Bahri-Mahowald [BM]. Given an integer r, let $\phi(r)$ denote the dimension of the rth nonzero homotopy group of BSO. An explicit formula for $\phi(r)$ is as follows: write $r = 4a + b$, where $0 \leq b \leq 3$, and let $\phi(r) = 8a + 2^b$. Let f denote the map of ring spectra

$$f : MO\langle\phi(r)\rangle \longrightarrow MO \longrightarrow H\mathbf{F}_2.$$

Then Bahri and Mahowald show that the kernel of $H^*(f)$ is the left ideal generated by the augmentation ideal of $A(r-1)$.

The goal of this section is to investigate the analogous question when p is odd, and also for $MU\langle k\rangle$. If $k \geq 2$ and p is odd, let f denote the map of (p-local) ring spectra

$$f : MO\langle k\rangle \longrightarrow MSO \longrightarrow BP.$$

Similarly, at any prime, let f denote the map of ring spectra

$$f : MU\langle k\rangle \longrightarrow MU \longrightarrow BP.$$

Theorem 1.1 (Rosen [Ros]). (1) *Suppose p is odd and $tq + 4 \leq k \leq (t+1)q$. Then the kernel of $H^*f : \mathscr{P} \longrightarrow H^*MO\langle k\rangle$ is the left ideal generated by the augmentation ideal of $P(t-1)$.*

(2) *Suppose p is an arbitrary prime, and $tq + 2 \leq k \leq (t+1)q$. Then the kernel of $H^*f : \mathscr{P} \longrightarrow H^*MU\langle k\rangle$ is the left ideal generated by the augmentation ideal of $P(t-1)$.*

To prove this theorem, we will also need the equivalent dual statement: under the hypotheses above the image of $H_*f : H_*MO\langle k\rangle \longrightarrow \mathscr{P}_*$ is

$$P(\zeta_1^{p^t}, \zeta_2^{p^{t-1}}, \dots, \zeta_t^p, \zeta_{t+1}, \dots).$$

Rosen's proof of this theorem is similar to the proof of Bahri-Mahowald [BM]. Because Rosen's proof is unpublished, and because we need the formalism of our proof later, we present our proof here.

We will first prove the easy half, that the image is contained within the polynomial algebra above.

Lemma 1.2. (1) *Suppose p is odd, and $k \geq tq + 2$. Then $P^{p^{t-1}}U = 0$, where U is the Thom class in either $H^0MO\langle k\rangle$ or in $H^0MU\langle k\rangle$.*

(2) *Suppose $p = 2$, and $k \geq 2t + 2$. Then $\mathrm{Sq}^{2^t}U = 0$, where U is the Thom class in $H^0MU\langle k\rangle$.*

Proof. First assume p is odd. It suffices to prove that $P^{p^{t-1}}U = 0$ in $H^*MO\langle tq+4\rangle$, since there is a map $MU\langle tq+2\rangle \longrightarrow MO\langle tq+4\rangle$ compatible with the Thom class. For this we use Giambalvo's calculation of $H^*BO\langle k\rangle$ [Giam]. Given an integer n, let $\alpha(n)$ denote the sum of the digits in the p-adic expansion of n. He shows that the image of H^*BO in $H^*BO\langle tq+4\rangle$ is a polynomial algebra on classes Θ_i in degree $4i$, where $\alpha(2i-1) \geq (p-1)t + 1$. In particular, the image is 0 in positive degrees less than $4p^t$. By the Thom isomorphism theorem, the image of H^*MSO in $H^*MO\langle tq+4\rangle$ is also 0 in positive degree less than $4p^t$. Since $P^{p^{t-1}}U$ is in degree $2p^t(p-1) < 4p^t$, it must be 0. A similar argument works when $p = 2$ using the results of Stong [St]. □

To prove the other half of Rosen's theorem, we use a completely different method. We will outline it here. We first point out that the space $BO\langle k\rangle$ is the kth space in the Ω-spectrum for connective real K-theory, ko. When localized at p, ko splits as a wedge of $(p-1)/2$ shifted copies of $BP\langle 1\rangle$. There is a corresponding decomposition of the p-localization of $BO\langle k\rangle$ into a product of spaces in the Ω-spectrum of $BP\langle 1\rangle$. The homology of these spaces form a Hopf ring which is not completely understood, but there is a map from the homology Hopf ring of BP, which is completely understood. We then use the Hopf ring distributive law to deduce the rest of Rosen's theorem.

We first recall the nth space functor from spectra to spaces. We work in a good category of spectra, as for example the one used in [LMM]. There a spectrum E is a sequence of spaces, and we have the nth space functor $\mathbf{E}_n$. If X is a space, then $E^n(X)$ is homotopy classes of maps from X to $\mathbf{E}_n$. The nth space functor is right adjoint to the functor that takes the space X to the nth desuspension of its suspension spectrum. The main properties of this functor are summarized in the following proposition.

Proposition 1.3. (1) *If E is a CW spectrum, then $\mathbf{E}_n$ has the homotopy type of a CW complex.*

(2) *The nth space functor takes cofiber sequences to fiber sequences, and locally finite wedges to products.*

(3) $\Sigma\mathbf{E}_n \simeq \mathbf{E}_{n+1}$.

(4) *If E is connective, then $\mathbf{E}_n$ is $(n-1)$-connected.*

(5) *The nth space functor commutes with p-localization.*

The proof of this proposition will be left to the reader, except for the first part, which can be found in [LMM, p. 52]. The second part is a consequence of the general facts that right adjoints preserve limits, and that many colimits in spectra are also limits. We will also comment on the last part. There is no reason that $\mathbf{E}_n$ should be connected, but if we write $\mathbf{E}_n = \pi_{-n}E \times E'$, then E' is nilpotent since its fundamental group is abelian. We can therefore define the p-localization of $\mathbf{E}_n$ to be $\pi_{-n}(E_{(p)}) \times E'_{(p)}$. But we will only need this for connected spaces.

Recall that $\mathbf{Z} \times BU = \mathbf{ku}_0$ and $\mathbf{Z} \times BO = \mathbf{ko}_0$ are the 0th spaces of the connective K-theory spectra. We need to identify some of the other spaces in these Ω-spectra.

Lemma 1.4. *We have the following homotopy equivalences of H-spaces.*

(1) $BU\langle 2i\rangle \simeq \mathbf{ku}_{2i}$.

(2) $BO\langle 8i\rangle \simeq \mathbf{ko}_{8i}$.

(3) $BO\langle 4i\rangle_{(p)} \simeq (\mathbf{ko}_{(p)})_{4i}$. *(Recall p is odd here.)*

Proof. Using the cofibre sequence

$$\Sigma^2 ku \xrightarrow{\times v} ku \longrightarrow H\mathbf{Z}$$

we get a fibration of infinite loop spaces

$$\mathbf{ku}_{n+2} \longrightarrow \mathbf{ku}_n \longrightarrow K(\mathbf{Z}, n).$$

By induction, we have an H-space equivalence $BU\langle 2n\rangle \simeq \mathbf{ku}_{2n}$. Obstruction theory shows the composite

$$BU\langle 2n+2\rangle \longrightarrow BU\langle 2n\rangle \longrightarrow \mathbf{ku}_{2n}$$

lifts to an H-map

$$BU\langle 2n+2\rangle \longrightarrow \mathbf{ku}_{2n+2}$$

which is an isomorphism on homotopy groups. The first part follows. The second and third parts are similar. Multiplication by $v \in ku_2$ is replaced by multiplication by $v \in ko_8$ and $w \in ko_4$ respectively. The base of the fibration is no longer an Eilenberg-Mac Lane space, but the obstruction theory still works. □

Now we must recall the well-known p-local splittings of ku and ko. Recall the p-local spectrum $BP\langle 1\rangle$, whose homotopy is $BP\langle 1\rangle_* = \mathbf{Z}_{(p)}[v_1]$, where $|v_1| = 2(p-1)$. There is an obvious ring homomorphism $\pi_*BP\langle 1\rangle \longrightarrow \pi_*ku_{(p)}$ that takes v_1 to v^{p-1}, and we then have $\pi_*ku_{(p)} \simeq \pi_*BP\langle 1\rangle[v]/(v^{p-1} - v_1)$. We have a corresponding multiplicative splitting of spectra

$$ku_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i}BP\langle 1\rangle.$$

The multiplication on the right-hand side is defined as follows. If $i+j < p-1$, we have

$$\Sigma^{2i}BP\langle 1\rangle \wedge \Sigma^{2j}BP\langle 1\rangle \xrightarrow{\mu} \Sigma^{2(i+j)}BP\langle 1\rangle$$

and if $i+j \geq p-1$ we have

$$\Sigma^{2i}BP\langle 1\rangle \wedge \Sigma^{2j}BP\langle 1\rangle \xrightarrow{\mu} \Sigma^{2(i+j)}BP\langle 1\rangle \xrightarrow{\times v_1} \Sigma^{2(i+j-p+1)}BP\langle 1\rangle.$$

In particular, multiplication by $v \in \pi_2 ku$ corresponds in the splitting to the identity map on the summands $\Sigma^{2i}BP\langle 1\rangle$ where $0 < i < p-1$ and takes the summand $\Sigma^{2(p-1)}BP\langle 1\rangle$ to $BP\langle 1\rangle$ by multiplication by v_1. Similar remarks hold for ko at odd primes p, except there are $(p-1)/2$ summands.

By the preceding lemma, we then get a product decomposition of $BU\langle k\rangle_{(p)}$ and, if p is odd, of $BO\langle k\rangle_{(p)}$. We summarize in the following corollary, where we use the notation $[v_1]$ for the map $\mathbf{BP\langle 1\rangle}_{i+q} \longrightarrow \mathbf{BP\langle 1\rangle}_i$ corresponding to multiplication by v_1.

Corollary 1.5. (1) *If k is even, there is a p-local decomposition of H-spaces*

$$BU\langle k\rangle_{(p)} \simeq \prod_{i=0}^{p-2} \mathbf{BP\langle 1\rangle}_{k+2i}.$$

The map $BU\langle k+2\rangle \longrightarrow BU\langle k\rangle$ corresponds to the identity map on the factors $\mathbf{BP\langle 1\rangle}_{k+2i}$ when $0 < i < p-1$ and to $[v_1] : \mathbf{BP\langle 1\rangle}_{k+q} \longrightarrow \mathbf{BP\langle 1\rangle}_k$ on the remaining factor.

(2) *If k is divisible by 4 and p is odd, there is a p-local decomposition of H-spaces*

$$BO\langle k\rangle \simeq \prod_{i=0}^{(p-3)/2} \mathbf{BP\langle 1\rangle}_{k+4i}.$$

The map $BO\langle k+4\rangle \longrightarrow BO\langle k\rangle$ corresponds to the identity map on the factors $\mathbf{BP\langle 1\rangle}_{k+4i}$ when $0 < i < (p-1)/2$ and to $[v_1] : \mathbf{BP\langle 1\rangle}_{k+q} \longrightarrow \mathbf{BP\langle 1\rangle}_k$ on the remaining factor.

Recall that we are trying to determine the image of the map

$$H_*MO\langle k\rangle \longrightarrow H_*MSO \longrightarrow H_*BP = \mathscr{P}_*.$$

By the Thom isomorphism theorem, it suffices to determine the image of the algebra map

$$H_*\,BO\langle k\rangle \longrightarrow H_*BSO \longrightarrow H_*MSO \longrightarrow H_*BP.$$

However, by the above corollary, we have

$$H_*BSO = H_*\,BO\langle 4\rangle = H_*\,\mathbf{BP}\langle \mathbf{1}\rangle_4 \otimes \cdots \otimes H_*\,\mathbf{BP}\langle \mathbf{1}\rangle_q\,.$$

There is a natural map $H_*\,\mathbf{BP}_i \longrightarrow H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_i$, which, by [Wil], is a surjection for $i \le 2q$.

The Hopf algebra $H_*\,\mathbf{BP}_k$ is computed by Ravenel-Wilson in [RW]. To describe it, we need to introduce some notation. First, given a class $z \in \pi_i BP = BP^{-i}(*)$, there is a corresponding map $* \longrightarrow \mathbf{BP}_{-i}$. If we choose a generator for $\mathbf{F}_p = H_0(*)$, we get a class $[z] \in H_0\,\mathbf{BP}_{-i}$. From the canonical orientation in BP^2CP^∞ we get a map $CP^\infty \longrightarrow \mathbf{BP}_2$. Let $\beta_j \in H_{2j}CP^\infty$ be dual to $x^j \in H^*CP^\infty = Z[x]$. Define $b_j \in H_{2j}\,\mathbf{BP}_2$ to be its image. These elements together generate $H_*\,\mathbf{BP}_k$ in the following sense. There is a circle product

$$\mathbf{BP}_i \times \mathbf{BP}_j \xrightarrow{\circ} \mathbf{BP}_{i+j}$$

coming from the ring spectrum structure of BP. On the other hand, there is the usual product

$$\mathbf{BP}_k \times \mathbf{BP}_k \xrightarrow{*} \mathbf{BP}_k$$

coming from the infinite loop space structure on $\mathbf{BP}_k$ that makes $H_*\,\mathbf{BP}_k$ into a ring. Together, these structures make $H_*\,\mathbf{BP}_*$ or $H_*\,\mathbf{E}_*$ for any ring spectrum E, into a Hopf ring [RW].

One of the results of Ravenel-Wilson is that $H_*\,\mathbf{BP}_k$ is a polynomial algebra on generators of the form $[v^I]\circ b^J$. Here I and J are finite sequences of nonnegative integers, $[v^I] = [v_1^{i_1}v_2^{i_2}\cdots]$, and $b^J = b_{p^0}^{\circ j_0}\circ b_{p^1}^{\circ j_1}\circ\cdots$. There are conditions on I and J which we will not state yet, as they become much simpler in $H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_k$

Under the map $H_*\,\mathbf{BP}_k \longrightarrow H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_k$, the elements $[v_i]$ go to 0 for $i > 1$. Therefore, for $k \le 2q$, $H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_k$ is generated multiplicatively by elements $[v_1^i]\circ b^J$, where the conditions on i and J are that if $i > 0$ then all elements j_n of J must be less than p, and that

$$2\sum_n j_n - qi = k.$$

Now let us consider again the map

$$H_*BSO \longrightarrow H_*MSO \longrightarrow H_*BP.$$

The generator $[v_1^i]\circ b^J \in H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_k$ occurs in dimension $\sum 2j_np^n \cong \sum j_n$ (mod q). Using the identity $2\sum j_n - qi = k$, we find that the only generators occurring in dimensions divisible by q in H_*BSO come from the factor $H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_q$. Therefore, all the other factors must go to 0 for dimensional reasons.

Putting the last few paragraphs together, we have proved the following lemma.

Lemma 1.6. (1) *If p is odd, and $tq+4 \leq k \leq (t+1)q$, the image of $H_*MO\langle k\rangle$ in $\mathscr{P}_*$ is the same as the image of the composite*

$$H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_{tq} \xrightarrow{[v_1^{t-1}]} H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_q \longrightarrow H_*BSO \longrightarrow H_*MSO \longrightarrow H_*BP.$$

(2) *If p is arbitrary and $tq+2 \leq k \leq (t+1)q$, the image of $H_*MU\langle k\rangle$ in $\mathscr{P}_*$ is the same as the image of the composite*

$$H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_{tq} \xrightarrow{[v_1^{t-1}]} H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_q \longrightarrow H_*BU \longrightarrow H_*MU \longrightarrow H_*BP.$$

We know from Lemma 1.2 that the image in question is contained in

$$P(\zeta_1^{p^{t-1}}, \zeta_2^{p^{t-2}}, \ldots, \zeta_{t-1}^p, \zeta_t, \ldots),$$

and we must now show that the image is that large.

Now the map $H_*BSO \longrightarrow H_*MSO \longrightarrow H_*BP$ is onto, and since all the other factors go to 0, the map $H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_q \longrightarrow H_*BP$ must be onto. There is only one generator in $H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_q$ in dimension $2(p^n-1)$, namely $x_n = [v_1^{n-1}] \circ b_1^{\circ p-1} \circ \cdots \circ b_{p^{n-1}}^{\circ p-1}$. Thus, the image of x_n must be congruent modulo decomposables to $\zeta_n \in \mathscr{P}_*$.

It is evident that x_n is in the image of $[v_1^{t-1}]$ when $n \geq t$. But we can use properties of Hopf rings to find more elements in the image. Suppose we have a Hopf ring $B(*)$ defined over a ring R. Then each $B(n)$ is a Hopf algebra over R, equipped with a counit $\epsilon : B(n) \longrightarrow R$, and a unit for the $*$-product, which we denote $[0_n]$. Note that, if our Hopf ring is of the form $E_*\,\mathbf{G}_*$ for two ring spectra E and G, and if $z \in \pi_*G$, then $\epsilon([z]) = 1$. For $x \in B(m)$, we have

$$[0_n] \circ x = \epsilon(x)[0_{n+m}].$$

Denote by $I(*)$ the kernel of ϵ. Then $I(*)$ is a Hopf ideal, which we call the augmentation ideal and typically denote by simply I. We will need to consider the Hopf ideal I^{*k} as well, which is the ideal consisting of sums of terms of the form $x_1 * x_2 * \cdots * x_k$, where each x_i is in I. To see that this is really a Hopf ideal, one needs the Hopf ring distributive law, which we will recall here. Given $a \in B(n)$, write

$$\Delta a = \sum a' \otimes a''.$$

Then

$$a \circ (b * c) = \sum \pm (a' \circ b) * (a'' \circ c).$$

Since we will only be considering elements in even degree in each $B(n)$ and only for even n, we can ignore the signs. There is also a right distributive law of the same form. Finally, given a ring homomorphism $R \longrightarrow S$, one can extend scalars to get a Hopf ring defined over S.

Now, form a power series $b(s) = \sum b_i s^i$ Here $b_0 = [0_2]$, and we think of $b(s) \in H_*\,\mathbf{BP}\langle\mathbf{1}\rangle_2\,[[s]]$ as an element in a Hopf ring defined over $Z[[s]]$ using the extension of scalars above. That is to say, the circle and star products on $Z[[s]]$ are just the usual product. Now $b(s)$ is a group-like element in the Hopf algebra $H_* \mathbf{BP}\langle\mathbf{1}\rangle_2 [[s]]$, so that

$$\psi(b(s)) = b(s) \otimes b(s).$$

(This is an easy computation in $H_*\mathbf{CP}^\infty$.) In this circumstance, the Hopf ring distributive law reduces to the ordinary distributive law. So if $x, y \in H_* \mathbf{BP}\langle\mathbf{1}\rangle_*$, we have

$$(x * y) \circ b(s) = (x \circ b(s)) * (y \circ b(s)).$$

In particular,

$$y^{*p^k} \circ b(s) = (y \circ b(s))^{*p^k}.$$

If we take y to be constant and look at the coefficient of s^{p^i}, we find that

$$y^{*p^k} \circ b_{p^i} = (y \circ b_{p^{i-k}})^{*p^k}.$$

Here, if $i < k$, $y^{*p^k} \circ b_{p^i} = 0$.

The other ingredient we need is the main relation of Ravenel-Wilson [RW]. Recall this says that if E and G are complex oriented ring spectra, then in $E_* \mathbf{G}_* [[s]]$, we have

$$b([p]_E(s)) = [p]_{[G]}(b(s)).$$

Here the series on the left is just $b(ps) = [0_2]$ if E is mod p homology. But the p-series on the right is not the usual one—ordinary addition is replaced by the star product, and multiplication is replaced by the circle product.

Lemma 1.7. *We have*

$$[0_2] \equiv b(s)^{*p} * ([v_1] \circ b(s)^{\circ p}) \pmod{I^{*p^2}}.$$

Thus,

$$b_i^{*p} \equiv -[v_1] \circ b_i^{\circ p} \pmod{I^{*p+1}}.$$

Proof. Recall that $[p](x) = px +_F v_1 x^p$ in the formal group law associated to $BP\langle 1\rangle$. We therefore have

$$[p]_{BP\langle 1\rangle}(b(s)) = b(s)^{*p} * ([v_1] \circ b(s)^{\circ p}) * \prod([a_{kl}] \circ (b(s)^{*p})^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl}).$$

Note that each term in this $*$-product is congruent to $[0_2]$ modulo I^{*p}, except $[v_1] \circ b(s)^{\circ p}$. Thus, using the main relation above, we have

$$[v_1] \circ b(s)^{\circ p} \equiv [0_2] \pmod{I^{*p}}.$$

Now, by the distributive law, $(b(s)^{*p})^{\circ k} = (b(s)^{\circ k})^{*p^k}$. Since each $\circ$-factor $[a_{kl}]$, $[v_1^l]$, and $b(s)$ is group-like, the distributive law gives

$$[a_{kl}] \circ (b(s)^{\circ k})^{*p^k} \circ [v_1^l] \circ b(s)^{\circ pl} = ([a_{kl}] \circ b(s)^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl})^{*p^k}.$$

But, since $[v_1] \circ b(s)^{\circ p} \equiv [0_2] \pmod{I^{*p}}$, the factor inside the parentheses is congruent to $[0_2] \pmod{I^{*p}}$ as well. Since we are working in characteristic p, raising to either the pth $*$-power or the pth $\circ$-power is additive, and we find that

$$[a_{kl}] \circ (b(s)^{*p})^{\circ k} \circ [v_1^l] \circ b(s)^{\circ pl} \equiv [0_2] \pmod{I^{*p^{k+1}}}.$$

This completes the first part of the lemma.

The second part of the lemma is obtained by expanding the series in the first part. We have

$$(\sum b_i^{*p} s^{pi}) * (\sum [v_1] \circ b_i^{\circ p} s^{pi}) \equiv [0_2] \pmod{I^{*p^2}}.$$

If we work mod I^{*p+1} instead, all the cross terms in this $*$-product disappear, and we are left with

$$b_i^{*p} \equiv -[v_1] \circ b_i^{\circ p} \pmod{I^{*p+1}}. \quad \square$$

Note that, in dimension $2p$, there is no room for $p+1$-fold $*$-decomposables in $H_* \mathbf{BP}\langle \mathbf{1} \rangle_2$. Thus we have

$$b_1^{*p} = -[v_1] \circ b_1^{\circ p}.$$

Using this, we can now complete the proof of Rosen's theorem.

Lemma 1.8. *In the Hopf ring* $H_* \mathbf{BP}\langle \mathbf{1} \rangle_*$ *we have the following relations.*

(1) $$b_1^{*p^n} = (-1)^n [v_1^n] \circ b_1^{\circ p} \circ b_p^{\circ p-1} \circ \cdots \circ b_{p^{n-1}}^{\circ p-1}.$$

(2) $$x_k^{*p^n} = [v_1^{n+k-1}] \circ b_1^{\circ p} \circ b_p^{\circ p-1} \circ \cdots \circ b_{p^{n-1}}^{\circ p-1} \circ b_{p^n}^{\circ p-2} \circ b_{p^{n+1}}^{\circ p-1} \circ \cdots \circ b_{p^{n+k-1}}^{\circ p-1}.$$

Proof. We proceed by induction. We have

$$b_1^{*p^n} = (b_1^{*p^{n-1}})^{*p} = ((-1)^{n-1}[v_1^{n-1}] \circ b_1^{\circ p} \circ b_p^{\circ p-1} \circ \cdots \circ b_{p^{n-2}}^{\circ p-1})^{*p}.$$

Using the distributive law one element at a time starting from the right, we get

$$b_1^{*p^n} = (-1)^{n-1} b_1^{*p} \circ b_p^{\circ p-1} \circ \cdots \circ b_{p^{n-1}}^{\circ p-1}.$$

Then applying the relation $b_1^{*p} = -[v_1] \circ b_1^{\circ p}$ completes the proof of the first part.

For the second part, recall that

$$x_k = [v_1^{k-1}] \circ b_1^{\circ p-1} \circ \cdots \circ b_{p^{k-1}}^{\circ p-1}.$$

Because $[v_1]$ is primitive, we find that

$$x_k^{*p^n} = [v_1^{k-1}] \circ (b_1^{\circ p-1} \circ \cdots \circ b_{p^{k-1}}^{\circ p-1})^{*p^n}.$$

Using the Hopf ring distributive law to remove as many factors from the right, one at a time as possible, we get

$$x_k^{*p^n} = [v_1^{k-1}] \circ (b_1^{*p^n}) \circ b_{p^n}^{\circ p-2} \circ b_{p^{n+1}}^{\circ p-1} \circ \cdots \circ b_{p^{n+k-1}}^{\circ p-1}.$$

Now the first part completes the proof. $\square$

We saw in Lemma 1.2 and Lemma 1.6 that the image of $H_* \mathbf{BP}\langle \mathbf{1} \rangle_{tq}$ in $\mathscr{P}_*$ is contained in the subring $P(\zeta_1^{p^{t-1}}, \zeta_2^{p^{t-2}}, \ldots, \zeta_{t-1}^p, \zeta_t, \ldots)$. The preceding lemma tells us that this image contains classes y_i for $i < t$ which are congruent to $\zeta_i^{p^{t-i}}$ modulo $p^{t-i}+1$-fold decomposables of $\mathscr{P}_*$, and elements y_i for $i \geq t$ which are congruent to ζ_i modulo decomposables. It follows, by either comparing ranks or induction, that the image is in fact all of

$P(\zeta_1^{p^{t-1}}, \zeta_2^{p^{t-2}}, \dots, \zeta_{t-1}^p, \zeta_t, \dots)$. This completes the proof of Rosen's theorem.

2. Consequences of Rosen's Theorem

In this section, we will see that Rosen's theorem implies that one can smash $MO\langle 8\rangle$ with a small finite spectrum X and get a wedge of suspensions of BP. This in turn gives bounds on the torsion and the v_1-torsion in the homotopy of $MO\langle 8\rangle$. We begin with a more general theorem.

Theorem 2.1. (1) *Let R denote a p-local finite type connective commutative ring spectrum equipped with a map $f: R \longrightarrow H\mathbf{F}_p$ of ring spectra such that the kernel of $H^*f: \mathscr{A} \longrightarrow H^*R$ is $\mathscr{A}(IA(n))$. Let X denote a finite spectrum such that H^*X is a free $A(n)$-module. Then $R \wedge X$ is a wedge of suspensions of $H\mathbf{F}_p$.*

(2) *Let R denote a p-local finite type connective commutative ring spectrum equipped with a map $f: R \longrightarrow BP$ of ring spectra such that the kernel of $H^*f: \mathscr{P} \longrightarrow H^*R$ is $\mathscr{P}(IP(n))$. Suppose also that H^*R is evenly graded. Let X denote a finite spectrum such that H^*X is evenly graded and a free $P(n)$-module. Then $R \wedge X$ is a wedge of suspensions of BP.*

Note that finite spectra X as in the above theorem do exist, by the results of Mitchell [Mit] for the $A(n)$ case and J. Smith [Sm] for the $P(n)$ case.

Proof. We will begin with the first part of the theorem, and we will show that $H^*(R \wedge X) = H^*R \otimes H^*X$ is a free $\mathscr{A}$-module. To do this we use a characterization of free $\mathscr{A}$-modules, due to Adams-Margolis [AM] at $p = 2$, and Moore-Peterson [MP] at odd primes, but proved most cleanly by Miller-Wilkerson in [MW]. We begin at $p = 2$. Let $P_t^s \in \mathscr{A}$ denote the dual of $\xi_t^{2^s}$. Then if $s < t$, $(P_t^s)^2 = 0$, so we can take the P_t^s-homology group of an $\mathscr{A}$-module. At p odd, we have two kinds of differentials: P_t^s and Q_t, where P_t^s is as above and Q_t is the dual of τ_t. We again have $Q_t^2 = 0$, but if $s < t$, we now have $(P_t^s)^p = 0$. We then define the P_t^s homology groups of an $\mathscr{A}$-module by taking the kernel of P_t^s modulo the image of $(P_t^s)^{p-1}$.

Miller and Wilkerson show that, if B is a sub-Hopf algebra of $\mathscr{A}$, and M is a bounded below B-module, then M is free if and only if $H(M, x) = 0$ for all differentials $x \in B$. In particular, for X as in the first part of the theorem, we have $H(H^*X, x) = 0$ for all $x \in A(n)$. On the other hand, Margolis shows in [Mar, pp.356-358] that, under the hypotheses on R as in the first part of the theorem, $H(H^*R, x) = 0$ for $x \notin A(n)$. Margolis restricts himself to the prime 2, but in fact his argument works for an arbitrary prime. The crucial step is Theorem 19.21 of [Mar]. Now in general there is no Kunneth theorem for x-homology, but there is a spectral sequence, so that $H(H^*R \otimes H^*X, x) = 0$ for all $x \in \mathscr{A}$, so $H^*(R \wedge X)$ is a free $\mathscr{A}$-module. Choosing generators gives a map to a wedge of suspensions of $H\mathbf{F}_p$, which is an isomorphism on mod p homology. Since R is finite type, we then get a homotopy equivalence as required.

We use a similar method to prove the second part of the theorem, where we replace $\mathscr{A}$ by $\mathscr{P}$. The theorem of Miller and Wilkerson does not apply directly to $\mathscr{P}$ at $p = 2$, since we defined $\mathscr{P}$ as a quotient Hopf algebra rather than a sub-Hopf algebra. But one can use the doubling isomorphism between the category of $\mathscr{A}$-modules and the category of evenly graded $\mathscr{P}$-modules. We get that a $\mathscr{P}$-module M is free over a sub-Hopf algebra B of $\mathscr{P}$ if and only if $H(M, x) = 0$ for all differentials $x \in B$. Note that the differentials are the doubles of the old ones: that is, they are the P_t^{s+1} for $s < t$. Doubling the theorems of Margolis, we find that, if the kernel of $H^*f : \mathscr{P} \longrightarrow H^*R$ is $\mathscr{P}(IP(n))$, then $H(H^*R, x) = 0$ for all differentials $x \notin P(n)$. Smashing with X then gives us a free $\mathscr{P}$-module, though not a free $\mathscr{A}$-module. At odd primes, of course, we do not need the doubling. In any case, choosing a generator gives a map to (a suspension of) $H\mathbf{F}_p$, which will lift to BP since $R \wedge X$ is evenly graded. Then we get a mod p homology equivalence from $R \wedge X$ to a wedge of suspensions of BP which is then a homotopy equivalence. □

We can apply the first part of this theorem directly to $MO\langle k\rangle$ at $p = 2$, using the result of Bahri-Mahowald mentioned in the previous section [BM]. But to apply the second part of this theorem to $MO\langle k\rangle$ at an odd prime or to $MU\langle k\rangle$, we have to know the homology is evenly graded. Now $H_* \mathbf{BP}_n$ is always evenly graded, and the map

$$H_* \mathbf{BP}_n \longrightarrow H_* \mathbf{BP}\langle \mathbf{1}\rangle_n$$

is onto for $n \leq 2p+2$ by [Wil]. Thus $H_*MO\langle 8\rangle$ is evenly graded for p odd, and $H_*MU\langle 6\rangle$ is evenly graded for arbitrary p.

In these small cases, we can find explicit models for the finite spectra X used in Theorem 2.1 . Indeed, for $p > 3$, we can take $X = S^0$ and we recover the fact that $MO\langle 8\rangle$ and $MU\langle 6\rangle$ split into a wedge of suspensions of BP when localized at such a prime. For $p = 3$, we have to find a finite spectrum which is free over $P(0)$, which is the subalgebra generated by P^1. Let Y denote the 8-skeleton of BP, a 3-cell complex where the 4-cell is attached to the 0-cell by α_1 and the 8-cell to the 4-cell by α_1. In fact, the 8-cell is also attached to the 0-cell, but we will see that this attachment is irrelevant to us. Y is then obviously free over $P(0)$. At $p = 2$, we need to find a finite spectrum that is free over $P(1)$, which is the double of $A(1)$. Here one can double the construction of $A(1)$ in [DM] to get a complex Z with 8 cells in dimensions 0 through 12. We have then proved the following corollary.

Corollary 2.2. *Let Y and Z denote the finite spectra above. Then*:

(1) $MO\langle 8\rangle_{(3)} \wedge Y$ *and* $MU\langle 6\rangle_{(3)} \wedge Y$ *are wedges of suspensions of* BP.
(2) $MU\langle 6\rangle_{(2)} \wedge Z$ *is a wedge of suspensions of* BP.
(3) $MU\langle 6\rangle_{(p)}$ *and* $MO\langle 8\rangle_{(p)}$ *are wedges of suspensions of* BP *when* $p > 3$.

In particular, for any p, the Bousfield class of $MU\langle 6\rangle_{(p)}$ and $MO\langle 8\rangle_{(p)}$ is the same as that of BP.

The Bousfield class part of this corollary was previously conjectured by both of the authors [Hov]. It will be extended to all of the $MU\langle k\rangle$ and the $MO\langle k\rangle$ at odd primes in the last section of the paper.

Theorem 2.3. *The 3-torsion in $\pi_*MO\langle 8\rangle$ and in $\pi_*MU\langle 6\rangle$ is all killed by 3 itself. The 2-torsion in $\pi_*MU\langle 6\rangle$ is all killed by 16.*

Proof. The strategy in both cases is the same, and is due to Mike Hopkins. Let us first start with any ring spectrum R satisfying the hypotheses of part two of

Theorem 2.1. We will show the torsion in π_*R is bounded. We can assume the bottom cell of the finite spectrum X in that theorem is in dimension 0. The resulting map $R \longrightarrow R \wedge X$ arising from inclusion of the bottom cell must kill all the torsion in π_*R, since $R \wedge X$ is a wedge of suspensions of BP. On the other hand, let $\overline{X} \xrightarrow{j} S^0$ denote the fiber of the inclusion of the bottom cell $S^0 \longrightarrow X$. Then $\overline{X}$ has cells in only odd degrees, and therefore $[\overline{X}, S^0]$ is finite. Thus, there is some N such that $p^N j$ is null. We then get a map back $R \wedge X \longrightarrow R$ such that the composite

$$R \longrightarrow R \wedge X \longrightarrow R$$

is multiplication by p^N. It follows that p^N kills all the torsion in π_*R.

To determine specific bounds, we must look at the individual spectrum X. For the 3-cell complex Y, $\overline{Y} \simeq \Sigma^3 C(\alpha_1)$. It is not too hard to see that

$$[\Sigma^3 C(\alpha_1), S^0] = \mathbf{Z}/9$$

generated by a class x which is α_1 on the bottom cell. This shows that 9 kills the 3-torsion in $\pi_*MO\langle 8\rangle$ and in $\pi_*MU\langle 6\rangle$. To see that 3 actually kills the torsion, note that $3x$ is the composite

$$\Sigma^3 C(\alpha_1) \longrightarrow S^7 \xrightarrow{\alpha_2} S^0.$$

But we have shown in [Hov] that α_2, and indeed everything in the image of J above dimension 3, goes to 0 in both $\pi_*MU\langle 6\rangle$ and in $\pi_*MO\langle 8\rangle$. Hence $3x$ will be 0 upon smashing with either $MU\langle 6\rangle$ or $MO\langle 8\rangle$, so 3 will kill the 3-torsion.

Now consider the 2-local 8-cell spectrum Z. We must find the smallest k such that

$$\overline{Z} \longrightarrow S^0 \xrightarrow{\eta} MU\langle 6\rangle \xrightarrow{\times 2^k} MU\langle 6\rangle$$

is null. By Spanier-Whitehead duality, $[\overline{Z}, MU\langle 6\rangle] \cong \pi_{11}(MU\langle 6\rangle \wedge W)$ where H^*W is $P(1)$ minus the top class ($Sq^4Sq^4Sq^4$) as a $P(1)$-module.

The structure of $H^*MU\langle 6\rangle$ through dimension 12 can be computed using the Serre spectral sequences relating BSU to $BU\langle 6\rangle$ and $BU\langle 6\rangle$ to $BU\langle 8\rangle$, or by the Hopf ring methods of the preceding section. We find 6 generators over $\mathbf{F}_2$: 1 in degree 0, x in degree 6, c_4 in degree 8, y in degree 10, z and c_6 in degree 12. Here $Sq^8 1 = c_4$, $Sq^2 x = c_4$, $Sq^4 x = y$, $Sq^4 c_4 = c_6$, and $Sq^2 y = z$. All other Steenrod operations follow from these. One can then use Bruner's program [Br] for calculating Ext, or calculate by hand, to determine the E_2-term of the Adams spectral sequence for $MU\langle 6\rangle \wedge W$ through dimension 12. The hand calculation is not terribly difficult: the high point is that the Sq^8 in $MU\langle 6\rangle$ means that in $MU\langle 6\rangle \wedge W$ one gets all of $\mathscr{P}$ through dimension 12 except $Sq^4Sq^4Sq^4$. This means one must calculate the Ext groups of the extension

$$0 \longrightarrow \Sigma^{12}\mathbf{F}_2 \longrightarrow \mathscr{P} \longrightarrow N \longrightarrow 0$$

where the class in dimension 12 goes to $Sq^4Sq^4Sq^4$. One finds a $Z/16$ in dimension 11 in $\mathrm{Ext}(N)$ starting in Adams filtration 1 on a class b.

This proves the theorem, but one might hope that an Adams differential from the 12-stem could lower the bound to 8. (There are no differentials from the 11-stem to the 10-stem because the 10-stem is all infinite h_0-towers.) But in fact, there are no differentials from the 12-stem to the 11-stem. Indeed, consider the cofibration

$$MU\langle 6\rangle \wedge W \longrightarrow MU\langle 6\rangle \wedge X \longrightarrow \Sigma^{12}MU\langle 6\rangle$$

arising from putting in the missing top cell of $P(1)$. $MU\langle 6\rangle \wedge X$ is a wedge of suspensions of BP, and it is easy to see what its Adams spectral sequence looks like in dimension 12. It looks precisely the same as that of $MU\langle 6\rangle \wedge W$, except it has a class corresponding to $v_1^3 v_2$ that is not in $MU\langle 6\rangle \wedge Z$, and there is no class bh_1. The class bh_1 is a permanent cycle since b is. If any other element supported a differential, we would necessarily have a class in $\pi_*(MU\langle 6\rangle \wedge X)$ that is not hit, yet a multiple of it is hit. This would lead to torsion in $\pi_{12}\Sigma^{12}MU\langle 6\rangle = \mathbf{Z}$, so there are no differentials. $\square$

We conjecture that the correct exponent for the 2-torsion in $MU\langle 6\rangle$ is in fact 8, though this method can only give 16.

We now give some similar results for v_1-torsion. Recall that an element $x \in \pi_* X$ for X a p-local spectrum is said to be v_1-torsion if x maps to 0 under the natural map

$$X \longrightarrow L_{K(1)}X.$$

(The failure of the telescope conjecture makes this definition problematic for higher v_n—see [MS].) Recall from [Hov] that an element $v \in \pi_* R$ for a ring spectrum R is called a v_1-element if

$$L_{K(1)}R = (v^{-1}R)_p.$$

Here the subscript denotes p-completion. Another way of saying this is to say that the $K(1)$-Hurewicz image of v is a (possibly fractional) power of v_1, and that the $K(n)$-Hurewicz image of v is nilpotent for $n > 1$. Given such a v_1-element v, we say that R has bounded v_1-torsion with respect to v if there is an N such that $v^N x = 0$ for all v_1-torsion elements x.

Recall from [Hov] that there are v_1-elements in every $MU\langle k\rangle$ and $MO\langle k\rangle$. There are v_1-elements in $\pi_{4k}MO\langle 8\rangle_{(3)}$ and $\pi_{4k}MU\langle 6\rangle_{(3)}$ for all $k > 1$. These elements induce the kth power of the Adams map on $MO\langle 8\rangle \wedge M(3)$ and on $MU\langle 6\rangle \wedge M(3)$, where $M(3)$ denotes the mod 3 Moore spectrum. There are also v_1-elements in $\pi_{8k}MU\langle 6\rangle_{(2)}$ for $k > 0$.

Theorem 2.4. *Localize all spectra at $p = 3$.*

1. *The 3-torsion in $\pi_* MO\langle 8\rangle$ coincides with the v_1-torsion in $\pi_* MO\langle 8\rangle$. Similarly for $MU\langle 6\rangle$.*
2. *Let v denote a v_1-element in $\pi_* MO\langle 8\rangle$ that induces a power of the Adams map on $MO\langle 8\rangle \wedge M(3)$, such as those mentioned above. Then $x \in \pi_* MO\langle 8\rangle$ is v_1-torsion if and only if $vx = 0$. Similarly for $MU\langle 6\rangle$.*

Proof. We again use the 3-cell complex Y, and we will just do the $MO\langle 8\rangle$ case. Since BP has no v_1-torsion, any v_1-torsion element in $\pi_* MO\langle 8\rangle$ must map to 0 under the map $MO\langle 8\rangle \longrightarrow MO\langle 8\rangle \wedge Y$, and therefore must be 3-torsion. Conversely, since $L_{K(1)}MO\langle 8\rangle = L_{K(1)}MSO$ [Hov] is torsion-free, any 3-torsion element is also v_1-torsion. This proves the first part of the theorem.

For the second part, suppose $x \in \pi_* MO\langle 8\rangle$ is v_1-torsion. Then x is 3-torsion. Note that x, and all the 3-torsion, will survive the map $MO\langle 8\rangle \xrightarrow{i}$

$MO\langle 8\rangle \wedge M(3)$ since the 3-torsion is killed by 3. Let $v \in \pi_{4k} MO\langle 8\rangle$ denote a v_1-element which indvuces $1 \wedge A^k$ on $MO\langle 8\rangle \wedge M(3)$, where A is the Adams map and k is necessarily at least 2. It will suffice to prove that $i(vx) = 0$, or equivalently, $(1 \wedge A^k)i(x) = 0$.

We will show that the composite

$$\Sigma^3 C(\alpha_1) \longrightarrow S^0 \longrightarrow MO\langle 8\rangle \wedge M(3) \xrightarrow{1\wedge A} MO\langle 8\rangle \wedge \Sigma^{-4} M(3)$$
$$\xrightarrow{1\wedge A} MO\langle 8\rangle \wedge \Sigma^{-8} M(3)$$

is null. We will then get a map

$$MO\langle 8\rangle \wedge M(3) \wedge Y \xrightarrow{r} MO\langle 8\rangle \wedge \Sigma^{-8} M(3)$$

such that the composite

$$MO\langle 8\rangle \wedge M(3) \longrightarrow MO\langle 8\rangle \wedge M(3) \wedge Y \xrightarrow{r} MO\langle 8\rangle \wedge \Sigma^{-8} M(3)$$

is $1 \wedge A^2$. It then follows that $(1 \wedge A^2)i(x) = 0$.

Now

$$[\Sigma^3 C(\alpha_1), \Sigma^{-4} M(3)] = \mathbf{Z}/3 \oplus \mathbf{Z}/3.$$

The generators are a map which is α_2 on the bottom cell of $\Sigma^3 C(\alpha_1)$, and the composite

$$\Sigma^3 C(\alpha_1) \longrightarrow S^7 \xrightarrow{\hat{\beta}_1} \Sigma^{-4} M(3),$$

where $\hat{\beta}_1$ is β_1 on the top cell of the Moore spectrum. Both these generators map to the same class upon smashing with $MO\langle 8\rangle$, since α_2 goes to 0. The composite of $\hat{\beta}_1$ with the Adams map is an element of $\pi_{15} M(3)$, which is generated by α_4. It will then go to 0 in $\pi_{15} MO\langle 8\rangle \wedge M(3)$, and we are done. □

We think that the v_1-torsion in $\pi_* MU\langle 6\rangle_{(2)}$ is also bounded, but this method cannot prove that. The 2-torsion and the v_1-torsion do not coincide in this case, though every v_1-torsion element is a 2-torsion element.

3. Adams-Novikov resolutions

In this section, we will use the results of the previous section to construct an economical Adams-Novikov resolution for $MO\langle 8\rangle_{(3)}$. We will see that this resolution forces the Adams-Novikov spectral sequence to collapse after at most two differentials. This section is essentially a combination of the Russian approach to MSU, pioneered by Novikov [Nov] and extended by Botvinnik and Vershinin [Bot, Ver], with the last chapter of [Rav].

Let Y denote the 3-cell spectrum considered in the previous section. There is a very simple Y-resolution of the sphere, considered in [Rav, Section 7.4]:

$$\begin{array}{ccccccccc} S^0 & \longleftarrow & \Sigma^3 C(\alpha_1) & \longleftarrow & S^{10} & \longleftarrow & \Sigma^{13} C(\alpha_1) & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ Y & & \Sigma^3 Y & & \Sigma^{10} Y & & \Sigma^{13} Y & & \end{array}$$

Since $MO\langle 8\rangle \wedge Y$ is a wedge of suspensions of BP, when we smash this resolution with $MO\langle 8\rangle$ we get an Adams-Novikov resolution of $MO\langle 8\rangle$. We then get the following proposition.

Proposition 3.1. *There is a spectral sequence converging to $\pi_* MO\langle 8\rangle_{(3)}$ which agrees with the Adams-Novikov spectral sequence from E_2 onwards and whose E_1-term is*

$$\pi_*(MO\langle 8\rangle \wedge Y) \otimes E(\alpha) \otimes P(\beta).$$

Here $\pi_*(MO\langle 8\rangle \wedge Y)$ is in filtration 0, α is in bidegree $(1, 4)$, and β is in bidegree $(2, 10)$. The tensor product is taken over $\mathbf{Z}_{(3)}$, and α and β have infinite additive order in E_1.

Note that in E_2, α corresponds to the usual α_1 and β corresponds to β_1. (Both α and β are permanent cycles for dimensional reasons, and easy low dimensional computation shows that they survive to E_∞.) In particular, in E_2, they both have order 3. This is not true yet in E_1. This means there must be a d_1 starting on the 1-line that kills 3β. In particular, d_1 cannot be a derivation. One expects a multiplicative formula for d_1 similar to a Bockstein spectral sequence, but we do not yet know if such a formula exists. Note that $MO\langle 8\rangle \wedge Y$ is a ring spectrum, as it splits off of $MO\langle 8\rangle \wedge T(1)$, where $T(1)$ is the ring spectrum used in the last two chapters of [Rav].

Note that we have the usual sparseness phenomenon in this Adams-Novikov spectral sequence, so that d_i is 0 unless $i = 4k + 1$ for some k.

The proposition above has the following corollary.

Corollary 3.2. *In the Adams-Novikov spectral sequence for $\pi_* MO\langle 8\rangle_{(3)}$, $\times\beta$: $E_2^s \longrightarrow E_2^{s+2}$ is an isomorphism if $s > 0$ and is surjective if $s = 0$.*

Proof. Inspect the chain complex whose homology is E_2. That chain complex is

$$\pi_*(MO\langle 8\rangle \wedge Y) \xrightarrow{f} \pi_*(MO\langle 8\rangle \wedge \Sigma^4 Y) \xrightarrow{g} \pi_*(MO\langle 8\rangle \wedge \Sigma^{12} Y)$$
$$\xrightarrow{f} \pi_*(MO\langle 8\rangle \wedge \Sigma^{16} Y) \xrightarrow{g} \cdots . \quad \square$$

This property continues to be true, in a weaker sense, as we get farther along in the spectral sequence.

Lemma 3.3. *In the Adams-Novikov spectral sequence for $\pi_* MO\langle 8\rangle_{(3)}$, the map*

$$\times\beta : E_k^s \longrightarrow E_k^{s+2}$$

is surjective for all s and is injective if $s \geq k - 1$.

Proof. We proceed by induction. The cases $k = 1, 2$ have already been discussed. So suppose $k > 2$. We will first prove multiplication by β is surjective. Suppose $\bar{x} \in E_k^{s+2}$, where $x \in E_{k-1}^{s+2}$ has $d_{k-1}x = 0$. By induction, there is a $y \in E_{k-1}^s$ such that $x = y\beta$. Thus

$$0 = d_{k-1}(y\beta) = d_{k-1}(y)\beta$$

since d_{k-1} is a derivation. But, again by induction, $\times\beta$ is surjective on E_{k-1} in filtrations at least $k - 2$. The class $d_{k-1}(y)$ has filtration higher than this, so in fact $d_{k-1}(y)=0$. Thus y survives to a class $\bar{y}\in E_k^s$ with $\bar{y}\beta=\bar{x}$. This shows that multiplication by β is surjective.

We will now show multiplication by β on E_k is injective in filtrations at least $k-1$. So suppose $\bar{x}\beta=0$, where $\bar{x}\in E_k^s$ and $s\geq k-1$. Then there is a $y\in E_{k-1}^{s+3-k}$ such that $d_{k-1}y=x\beta$. Since $s\geq k-1$, we have $s+3-k\geq 2$, so, by surjectivity, there is a $z\in E_{k-1}^{s+1-k}$ such that $y=z\beta$. Then

$$d_{k-1}(z)\beta=d_{k-1}(y)=x\beta.$$

By induction, multiplication by β is injective here, so in fact $d_{k-1}(z)=x$. This means $\bar{x}=0$, as required. □

Note that this proof fails, and in fact the lemma is false, for the spectral sequence used in [Rav, Section 7.4], because in that case none of the differentials are derivations. It is the fact that this spectral sequence is really the Adams-Novikov spectral sequence, so that the differentials are derivations, that makes the lemma work.

Now the homotopy class corresponding to β must be nilpotent. Since in this case β is just β_1, there are in fact specific bounds, but in general a class of positive Novikov filtration in a ring spectrum is nilpotent by the nilpotence theorem of [DHS]. So some power of β must be killed by a differential. Choose the least k such that β^k does not survive the spectral sequence.

Lemma 3.4. *There is an $x\in E_{2k-1}^1$ such that $d_{2k-1}(x)=\beta^k$.*

Proof. There is some i and some class $x\in E_i^{2k-i}$ such that $d_i(x)=\beta^k$. If $2k-i>1$, write $x=y\beta$, for some $y\in E_i^{2k-i-2}$. Then $d_i(y)\beta=\beta^k$, and $d_i(y)$ has filtration $\geq i-1$. Thus $d_i(y)=\beta^{k-1}$, which is impossible. □

Theorem 3.5. *The Adams-Novikov spectral sequence for $\pi_*MO\langle 8\rangle_{(3)}$ collapses at E_{2k}, and $E_\infty^s=E_{2k}^s=0$ for $s>2k-2$.*

Proof. Suppose we have a class $z\in E_{2k-1}^s$, where $s\geq 2k$ and $d_{2k-1}(z)=0$. Then we can write $z=y\beta^k$ for some $y\in E_{2k-1}^{s-2k}$. Thus $d_{2k-1}(y)\beta^k=0$. But $d_{2k-1}(y)$ is in high enough filtration for this to mean that $d_{2k-1}(y)=0$.

Now from the preceding lemma, there is a class $x\in E_{2k-1}^1$ such that $d_{2k-1}(x)=\beta^k$. Thus $d_{2k-1}(xy)=y\beta^k=z$. Thus $E_{2k}^s=0$ is $s\geq 2k$. The spectral sequence therefore collapses at E_{2k}. To see that $E_{2k}^{2k-1}=0$, note that $\times\beta: E_{2k}^{2k-1}\longrightarrow E_{2k}^{2k+1}$ is injective. □

Note that the class $\alpha\beta^3\in E_2^{7,40}$ cannot survive the spectral sequence. Indeed, it is in the image of the E_2 term of the sphere, and does not survive that spectral sequence, as is well-known. One can consult the charts in [Rav] to see how it dies in the sphere. In $MO\langle 8\rangle$, $\alpha\beta^3$ is a permanent cycle, and can only be hit by a d_5. But then Lemma 3.3 shows that there must be a class $w\in E_2^{0,24}$ with $d_5w=\alpha\beta^2$.

This in turn implies, just as in the Hopkins-Miller calculation of EO_2, that $d_9(w^2\alpha)=\beta^5$. We will sketch their argument briefly, but first we present a simpler argument that shows that β^5 must be 0 in $\pi_*MO\langle 8\rangle$. First note that the class $w\alpha$ is a d_5-cycle, and since there is nothing in filtration greater than 5 in dimension 26 (even in the E_1 term above), it is a permanent cycle. Choose a homotopy class ϵ detected by $w\alpha$. This class ϵ is very similar to the class in $\pi_{37}S^0$ commonly denoted ϵ or ϵ'. In fact, the argument below shows that $\beta\epsilon$ is the image of ϵ' under the unit map $S^0 \longrightarrow MO\langle 8\rangle$. The class ϵ is a representative for the Toda bracket $\langle\alpha, \alpha, \beta^2\rangle$. We will see in the next section that there is no indeterminancy. Now by Toda bracket manipulation, up to a unit multiple we have

$$\alpha\epsilon = \langle\alpha, \alpha, \alpha\beta^2\rangle = \langle\alpha, \alpha, \alpha\rangle\beta^2 = \beta^3.$$

Again there is no indeterminancy, as we will see later. Hence we have $\beta^5 = \beta^2\alpha\epsilon = 0$.

The argument of Hopkins and Miller relies on Steenrod operations in the Novikov spectral sequence and the Kudo transgression theorem relating differentials to these operations. The main references are [May] and Bruner's part of [BMMS]. Given a cocommutative Hopf algebroid A over an $\mathbf{F}_p$-algebra R and a comodule algebra M over A, there are Steenrod operations in $\mathrm{Ext}_A(R, M)$. These were originially constructed by May in the Hopf algebra case, and Bruner gives the generalization to Hopf algebroids. Therefore there are Steenrod operations in the E_2-term of the Adams spectral sequence based on $BP \wedge M(p)$ of any ring spectrum E. However, we need some additional structure on E to be sure the differentials behave well with respect to these operations. In fact, we need E to be an H_∞ ring spectrum. Thom spectra such as $MO\langle 8\rangle$ are always H_∞ ring spectra, so that is much simpler for us than the analogous fact for EO_2. Bruner shows how the H_∞ ring structure allows one to relate differentials and operations. Unfortunately, he does not prove the Kudo transgression theorem (Theorem 3.4 of [May]) in this situation. His methods do apply though.

The Kudo transgression formula tells us that $d_5(w) = \alpha\beta^2$ implies that

$$d_9(w^2\alpha\beta^2) = \beta P^2(\alpha\beta^2).$$

Here we have used β for both the Bockstein and the homotopy class. The context should make clear which is meant. The Cartan formula applies, and using the fact that $\beta P^0\alpha = \beta$ we find that $\beta P^2(\alpha\beta^2) = \beta^7$. Since multiplication by β is injective in this range, we get $d_9(w^2\alpha) = \beta^5$ as required. This is true in the spectral sequence based on $BP \wedge M(3)$, but since everything in the Novikov spectral sequence in positive filtration is killed by 3, it must also hold in the Novikov spectral sequence.

This differential may not really occur: it could be that β^5 is killed by a d_5. But then Lemma 3.3 shows β^3 must also be hit by a d_5. We will see later that this does not happen. In any case, we have proved the following theorem.

Theorem 3.6. *The Adams-Novikov spectral sequence for $\pi_*MO\langle 8\rangle_{(3)}$ has $E_{10} = E_\infty$ and $E_\infty^s = 0$ for $s > 8$.*

Note that the same theorem is true for EO_2 at the prime 3 [HM]. We also point out that this method may be applicable to $MU\langle 6\rangle$ at $p = 2$ as well. There one would need a Z-resolution of S^0, where Z is one of the models for the double of $A(1)$. We do not know if a compact such resolution exists, but if so, one would get an E_1-term for the Adams-Novikov spectral sequence for

$MU\langle 6\rangle$ that would look something like

$$\pi_*(MU\langle 6\rangle \wedge Z) \otimes P(\eta, \nu, w, \kappa)/(\nu^3, \eta\nu, w\nu, w^2 - \eta^2\kappa).$$

We have given these classes the names they normally have in $\pi_* S^0$: η is the degree 1 Hopf map, ν the degree 3 Hopf map, and κ a class in bidegree $(4, 24)$. The class w in bidegree $(3, 14)$ is not in the homotopy of the sphere, and does not seem to get involved in $MU\langle 6\rangle$ either. The ring $P(\eta, \nu, w, \kappa)/\sim$ is basically $\mathrm{Ext}_{P(1)}(\mathbf{F}_2, \mathbf{F}_2)$, except that all of the classes in those Ext-groups have order 2, and here they have infinite order. Unfortunately, one does not get such a nice periodicity result here: there are two classes with infinite multiplicative order, η and κ. But η^4 should be killed by a d_3, so what should happen is that $\times\kappa : E_4^s \longrightarrow E_4^{s+4}$ is surjective, and, if $s > 0$, is an isomorphism. If that were so, the whole spectral sequence would collapse with a flat vanishing line as soon as a power of κ is killed. This should happen by a d_{23} to kill κ^6, as it does in EO_2 at the prime 2 [HM]. So we expect, in the Adams-Novikov spectral sequence for $MU\langle 6\rangle$ at $p = 2$, that $E_{24} = E_\infty$ and that $E_\infty^s = 0$ for $s > 20$.

4. Calculations

In this section, we calculate enough of the Adams spectral sequence for $MO\langle 8\rangle$ at $p = 3$ to see that β_1^3 is nonzero, so that the Adams-Novikov spectral sequence really collapses at E_{10} and not at E_6. The calculations suggest some conjectures about this Adams spectral sequence and about the homology of $MO\langle 8\rangle$.

We first calculate $H_*(BO\langle 8\rangle ; \mathbf{F}_3) = H_*(\mathbf{BP}\langle \mathbf{1}\rangle_8 ; \mathbf{F}_3)$ through dimension 32. Recall we have the elements $b_i \in H_{2i}\,\mathbf{BP}\langle \mathbf{1}\rangle_2$ and $[v_1] \in H_0\,\mathbf{BP}\langle \mathbf{1}\rangle_{-4}$. The elements b_i are the image of the corresponding elements β_i in $H_{2i}CP^\infty$ dual to the powers of the generator, under the complex orientation $CP^\infty \longrightarrow \mathbf{BP}\langle \mathbf{1}\rangle_2$. Through this range, the homology is generated multiplicatively by the classes in the following table, together with the class 1 in dimension 0.

Dim	*Gen*	*Dim*	*Gen*
8	$x_8 = b_1^{\circ 4}$	24	$x_{24} = b_1^{\circ 3} \circ b_9$
12	$x_{12} = b_1^{\circ 3} \circ b_3$	24	$y_{24} = b_3^{\circ 4}$
16	$x_{16} = b_1^{\circ 2} \circ b_3^{\circ 2}$	28	$x_{28} = b_1^{\circ 2} \circ b_3 \circ b_9$
20	$x_{20} = b_1 \circ b_3^{\circ 3}$	32	$x_{32} = b_1 \circ b_3^{\circ 2} \circ b_9$

In general, to find the generators in dimension $4k$, one takes the 3-adic expansion of $2k$:

$$2k = \sum a_i 3^i.$$

If the sum of the a_i is greater than 2, one has a single generator x_{4k} which is the circle product of the $b_{3^i}^{\circ a_i}$ and an appropriate power of $[v_1]$. If $2k = 3^i + 3^j$ with $0 < i < j$ both positive and distinct, there are 2 generators, namely $x_{4k} = b_{3^i-1}^{\circ 3} \circ b_{3^j}$ and $y_{4k} = b_{3^i} \circ b_{3^j-1}^{\circ 3}$. Otherwise, there is just one generator $x_{4k} = b_{3^i} \circ b_{3^j-1}^{\circ 3}$.

Now $H_*BO\langle 8\rangle$ is not a polynomial ring. In fact, we know from [Sin] that it has some generators which are truncated at height 3, one in each dimension of the form $3^i + 3^j$ where $i < j$. In particular, x_8, as the only element in dimension 8, must have cube 0. There are no other multiplicative relations in our range, as can be easily checked by counting dimensions. But there are multiplicative relations in higher dimensions. For example, there has to be a class in dimension 20 whose cube is 0. Hopf ring relations tell us that x_{20}^3 is congruent to 0 modulo 4-fold star-decomposables. But it is probably not 0 on the nose, and will have to be modified by adding a multiple of x_8x_{12} to produce a class whose cube is actually 0.

We now must compute the Steenrod algebra coaction through our range. This is completely mechanical: the b_i come from the homology of CP^∞ where the coaction is completely known. Indeed, we have

$$\begin{aligned}\psi(b_1) &= 1 \otimes b_1.\\ \psi(b_3) &= 1 \otimes b_3 + 2\zeta_1 \otimes b_1.\\ \psi(b_9) &= 1 \otimes b_9 + 2\zeta_1 \otimes b_7 + \zeta_1^2 \otimes b_5 + 2\zeta_1^3 \otimes b_3.\end{aligned}$$

Here we have used ζ_i instead of ξ_i as is frequently convenient in dealing with cobordism theories. The elements b_7 and b_5 are $*$-decomposable in H_*CP^∞, but of course the map $CP^\infty \longrightarrow \mathbf{BP}\langle\mathbf{1}\rangle_2$ is not an H-space map. In fact, it is part of the standard map

$$CP^\infty \longrightarrow BU \simeq_{(3)} \mathbf{BP}\langle\mathbf{1}\rangle_2 \times \mathbf{BP}\langle\mathbf{1}\rangle_4 .$$

This map is well-known to take the classes β_j to multiplicative generators. In particular, we must have

$$b_5 \equiv \pm[v_1] \circ b_1^{\circ 2} \circ b_3, \qquad b_7 \equiv \pm[v_1] \circ b_1 \circ b_3^{\circ 2}$$

modulo decomposables. By use of the Hopf ring relations developed in the first section, and careful calculation, we find the following formulae for the coaction in $H_* BO\langle 8\rangle$ through dimension 32.

$$\begin{aligned}
\psi x_8 &= 1 \otimes x_8.\\
\psi x_{12} &= 1 \otimes x_{12} + 2\zeta_1 \otimes x_8.\\
\psi x_{16} &= 1 \otimes x_{16} + \zeta_1 \otimes x_{12} + \zeta_1^2 \otimes x_8.\\
\psi x_{20} &= 1 \otimes x_{20} + 2\zeta_1^3 \otimes x_8.\\
\psi x_{24} &= 1 \otimes x_{24} + 2\zeta_1^3 \otimes x_{12} + (2\zeta_2 + \zeta_1^4) \otimes x_8.\\
\psi y_{24} &= 1 \otimes y_{24} + 2\zeta_1 \otimes x_{20} + 2\zeta_1^3 \otimes x_{12} + \zeta_1^4 \otimes x_8.\\
\psi x_{28} &= 1 \otimes x_{28} + 2\zeta_1 \otimes x_{24} + 2\zeta_1^3 \otimes x_{16} + (2\zeta_2 + 2\zeta_1^4) \otimes x_{12} + (\zeta_1\zeta_2 + 2\zeta_1^5) \otimes x_8.\\
\psi x_{32} &= 1 \otimes x_{32} + \zeta_1 \otimes x_{28} + \zeta_1^2 \otimes x_{24} + 2\zeta_1^3 \otimes x_{20} +\\
&\qquad 2\zeta_2 \otimes x_{16} + 2\zeta_1\zeta_2 \otimes x_{12} + (2\zeta_1^2\zeta_2 + \zeta_1^6) \otimes x_8.
\end{aligned}$$

To get the coaction on the homology of $MO\langle 8\rangle$, we apply the Thom isomorphism. The ring structure in homology is preserved by the Thom isomorphism, but the coaction is the composite below, dual to the usual description of the action of $\mathscr{A}$ on a Thom spectrum.

$$H_*BO\langle 8\rangle \xrightarrow{\Delta} H_*BO\langle 8\rangle \otimes H_*BO\langle 8\rangle \xrightarrow{f\otimes\psi} \mathscr{P}_* \otimes \mathscr{P}_* \otimes H_*BO\langle 8\rangle$$
$$\xrightarrow{\mu\otimes 1} \mathscr{P}_* \otimes H_*BO\langle 8\rangle.$$

Here we have identified $H_*MO\langle 8\rangle$ and $H_*BO\langle 8\rangle$, and f is the map considered in the first section. So we must explicitly compute both the diagonal and f. The diagonal in H_*CP^∞ is easy. In particular, we have

$$\Delta b_1 = [0_2]\otimes b_1 + b_1 \otimes [0_2].$$

Since $[0_2]\circ x = 0$ for x in the augmentation ideal, this means that for all the generators except $b_3^{\circ 4} = y_{24}$ in our range, we have

$$\Delta x = [0_8]\otimes x + x\otimes [0_8].$$

To compute $\Delta b_3^{\circ 4}$, we have

$$\Delta b_3 = [0_2]\otimes b_3 + b_1\otimes b_2 + b_2\otimes b_1 + b_3\otimes [0_2].$$

It is unclear just what b_2 is in $H_*\mathbf{BP}\langle \mathbf{1}\rangle_2$, but it must be ab_1^{*2} for some $a\in \mathbf{F}_3$. Recall the Hopf ring relations

$$b_1\circ (x*y) = 0.$$
$$b_1^{*2}\circ (x*y) = 2(b_1\circ x)*(b_1\circ y).$$

for x, y in the augmentation ideal. These give

$$\Delta y_{24} = 1\otimes y_{24} + 2a^2 x_8\otimes x_8^2 + 2a^2x_8^2\otimes x_8 + y_{24}\otimes 1.$$

Recall from the second section that the image of f is $P(\zeta_1^3, \zeta_2, \zeta_3, \ldots)$. In particular $f(x_8) = 0$, so since f is an algebra map, $f(x_8^2) = 0$ as well. Thus, if we denote the coaction in $H_*MO\langle 8\rangle$ by ψ and the coaction in $H_*BO\langle 8\rangle$ by ψ', we have, in our range,

$$\psi(x) = \psi'(x) + f(x)\otimes 1.$$

To get explicit formulas for f, one can work one's way up dimension by dimension, using coassociativity to determine f at each stage. Of course $f(x_8) = f(x_{20}) = 0$, and one has two possibilities for $f(x_{12})$, namely ζ_1^3 and $2\zeta_1^3$. By changing the generator $x\in H^2CP^\infty$, one can assume $f(x_{12}) = \zeta_1^3$. The resulting formulas for f are given below.

$$\begin{array}{rclrcl}
f(x_8) &=& 0. & f(x_{12}) &=& \zeta_1^3.\\
f(x_{16}) &=& \zeta_2 & f(x_{20}) &=& 0.\\
f(x_{24}) &=& 2\zeta_1^6. & f(y_{24}) &=& 2\zeta_1^6.\\
f(x_{28}) &=& \zeta_1^3\zeta_2. & f(x_{32}) &=& 2\zeta_2^2.
\end{array}$$

It is then straightforward, though tedious, to analyze the comodule structure of $H_*MO\langle 8\rangle$ in this range. We get a splitting

$$H_*MO\langle 8\rangle \cong M\oplus \Sigma^{16}\mathscr{P}_* \oplus \Sigma^{24}\mathscr{P}_* \oplus \Sigma^{28}\mathscr{P}_* \oplus \Sigma^{32}\mathscr{P}_*.$$

Here M is an extension

$$0\longrightarrow \Sigma^8 N\longrightarrow M\longrightarrow N\longrightarrow 0$$

where

$$N = (\mathscr{P}//P(0))_* = P(\zeta_1^3, \zeta_2, \zeta_3, \ldots).$$

It is easy to verify that such an extension is given by $P^1\zeta_1^3$, where P^1 is thought of as acting downward, and in this case it is $\Sigma^8 1$.

We would then like to compute $\mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_3, H_*MO\langle 8\rangle)$, the E_2-term of the Adams spectral sequence. Of course,

$$\mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_3, \mathscr{P}_*) = P(a_0, a_1, \ldots)$$

where the a_i are in bidegree $(1, 2\cdot 3^i - 1)$. To compute $\mathrm{Ext}(N)$, we use the isomorphisms

$$\mathrm{Ext}_{\mathscr{A}_*}(N) = \mathrm{Ext}_{\mathscr{P}_*}(N \otimes P(a_0, a_1, \ldots)) = \mathrm{Ext}_{P(0)_*}(P(a_0, a_1, \ldots)).$$

The first isomorphism can be found for example in [Rav, Theorem 4.4.3], where one sees that the $P(0)_*$-comodule structure on $P(a_0, a_1, \ldots)$ is determined by

$$\psi a_1 = 1 \otimes a_1 + \xi_1 \otimes a_0$$

with the other a_i being primitive.

It is then easy to calculate

$$\mathrm{Ext}(N) = P(a_1^3, a_2, \ldots) \otimes P(a_0, \alpha_1, \alpha_2, \beta_1)/R,$$

where $\alpha_1 \in \mathrm{Ext}^{1,4}$, $\alpha_2 \in \mathrm{Ext}^{2,9}$ and $\beta_1 \in \mathrm{Ext}^{2,12}$. The relations R are generated by

$$\alpha_1^2, \alpha_2^2, a_0\alpha_1, a_0\alpha_2, \alpha_1\alpha_2 - a_0\beta_1.$$

Of course $a_0, \alpha_1, \alpha_2, \beta_1$ are in the image of the map $\mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_3) \longrightarrow \mathrm{Ext}_{\mathscr{A}_*}(N)$.

To calculate $\mathrm{Ext}(M)$, we must then calculate the coboundary map

$$\delta : \mathrm{Ext}^{s,t}(N) \longrightarrow \mathrm{Ext}^{s+1,t}(\Sigma^8 N).$$

The coboundary map preserves multiplication by elements in $\mathrm{Ext}_{\mathscr{A}_*}(\mathbf{F}_3)$. It is 0 on a_1^3 and a_1^6 for dimensional reasons. We still must determine it on the classes a_2, $a_1^3 a_2$, and a_2^2. A cobar representative for a_2 is

$$[\overline{\tau}_2]\zeta_2 + [\overline{\tau}_1]\zeta_1^3 + [\overline{\tau}_2]1.$$

Here $\overline{\tau}_i$ denotes the conjugate of τ_i. It follows that $\delta a_2 = \Sigma^8 \alpha_2$. If one knew that δ were a derivation it would follow easily that

$$\delta(a_1^3 a_2) = \Sigma^8 a_1^3 \alpha_2 \delta(a_2^2) = 2\Sigma^8 a_2 \alpha_2.$$

We do not know how to see that δ is a derivation, so one must check these by hand. But that can be done. The resulting Adams E_2-term is displayed in Figure 1, without the BP summands.

There are some extensions and multiplicative behavior in the E_2 term that we need to determine. The extensions are $a_0 A = z\alpha_1$, shown on the chart by a

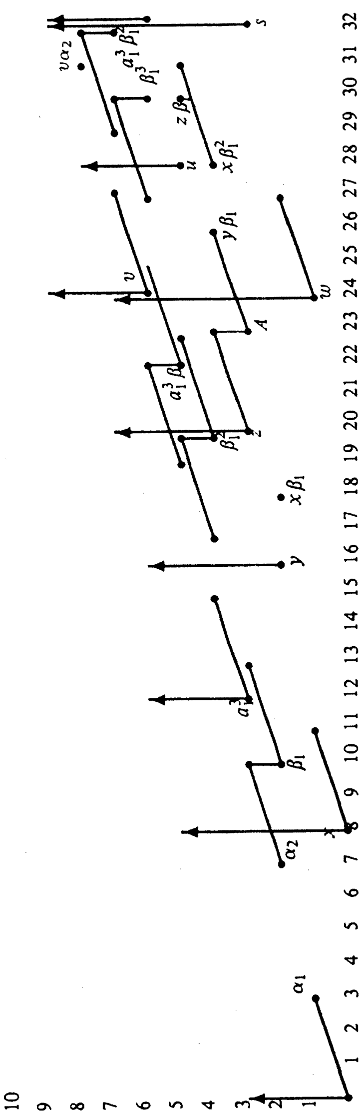

FIGURE 1

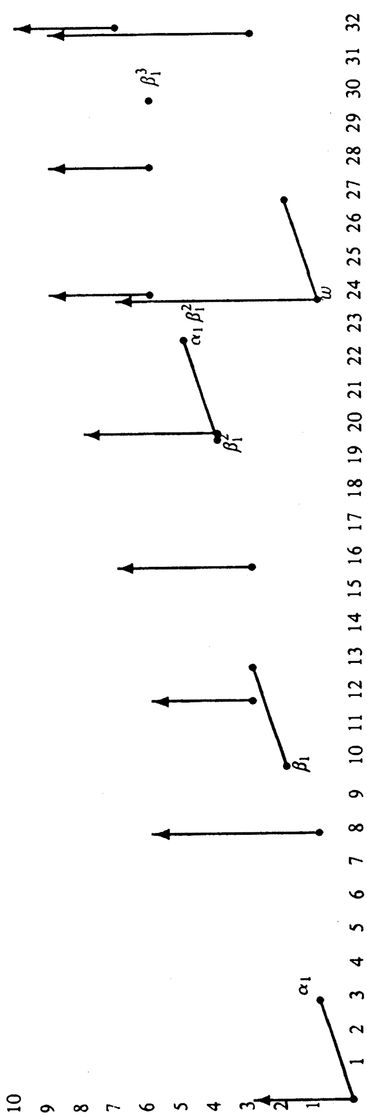

FIGURE 2

thin line, and $y\alpha_2 = z\alpha_1$. The multiplicative relations we need are:

$$z = xa_1^3.$$
$$v = (a_1^3)^2.$$
$$u = ya_1^3.$$
$$t = xv.$$

One must remember in these relations that most of these elements are not really well-defined. For example t is just some class in filtration 6 in the 32-stem which is not divisible by a_0. There are many such elements, because of the BP summands not shown on the chart. What we really mean is that we can choose t and v so that $t = xv$. Or, equivalently, that any choice for t and v will give $t \equiv xv$ modulo a_0.

The easiest way to get these extensions is to consider the differential d_2 simultaneously. The reader is encouraged to do this herself, as this kind of argument is easier to think of than to follow. The facts we need are that the torsion is all killed by 3 and any well-behaved v_1-element, that no v_1-element can be divisible by 3, that the product of 2 v_1-elements is again a v_1-element, and there have to be well-behaved v_1-elements in every dimension $4k$, where $k > 1$.

Note that $a_0\beta_1$ cannot survive the spectral sequence, since we know that β_1, and indeed all of the torsion in $\pi_*MO\langle 8\rangle$, is killed by 3. Thus $d_2(x\alpha_1) = a_0\beta_1$, which implies $d_2 x = \alpha_2$. The same argument implies that $d_2(z\alpha_1) = a_0a_1^3\beta_1$, which implies that $d_2(z) = a_1^3\alpha_2$. It follows that $z = xa_1^3$, at least modulo a_0^2, which is all we need.

Now consider the class $a_1^3\alpha_1\beta_1$. This class cannot survive the spectral sequence, since a_1^3 is a v_1-element. Thus we must have $d_2(y\beta_1) = a_1^3\alpha_1\beta_1$. It follows that $d_2 y = a_1^3\alpha_1$. It also follows that $d_2(A) = a_1^3\beta_1$, from which we get the extensions $a_0A = z\alpha_1$ and $y\alpha_2 = z\alpha_1$.

Whatever $(a_1^3)^2$ is, it must be a v_1-element of filtration at least 6. It follows that it must be v, again modulo a_0. Then $d_2(ya_1^3) = v\alpha_1$, so $ya_1^3 = u$, modulo a_0. Finally, $d_2(xv) = v\alpha_2$, so we must have $t = xv$ modulo a_0.

These extensions and the d_2's calculated above determine all the possible d_2's in this range, and the resulting E_3 term is shown in Figure 2.

In this range, there is only one higher differential, $d_4(w) = \alpha_1\beta_1^2$. This differential gives rise to an extension $\alpha_1 w\alpha_1 = \beta_1^3$, just as one gets the corresponding α_1 extension to β_1^4 in π_*S^0. In any case, β_1^3 certainly survives the spectral sequence, so the Adams-Novikov spectral sequence cannot collapse at E_6.

At this point, we present some speculations about the behavior of this Adams spectral sequence in larger dimensions. These conjectures are based on speculative calculations.

Conjecture 4.1. (1) *$H_*MO\langle 8\rangle$ splits into a direct sum of suspensions of M, $N = M \otimes \mathbf{F}_3[\zeta_1]/(\zeta_1^2)$, and $\mathscr{P}_*$. The first N summand begins in dimension* 28.

(2) *There are differentials $d_3a_2^3 = w\alpha_1\beta_1^2$, and $d_6a_2^3\alpha_1 = \beta_1^5$.*

(3) *The Adams spectral sequence collapses after d_2, d_3, d_4, and d_6.*

(4) *The algebraic Novikov spectral sequence* [Rav, Theorem 4.4.4] *collapses after d_1, which corresponds to the Adams d_2, so that the Adams E_3-term is a reindexed form of the Adams-Novikov E_2-term.*

(5) *The Adams-Novikov* d_5 *corresponds to the Adams* d_3 *and* d_4, *and the Adams-Novikov* d_9 *corresponds to the Adams* d_6.

Of course, there are many other questions to ask and answer here. For example, a_2^9 should survive the spectral sequence and represent a v_2-element. One would expect, in analogy to MSU at $p = 2$, that the higher a_i survive as well and represent v_i-elements. One would also expect a splitting of $MO\langle 8\rangle$, in analogy to the splitting of MSU. And certainly one should determine the image of the unit in homotopy. In our range, the only classes in that image are α_1, β_1, $\alpha_1\beta_1$, β_1^2, and β_1^3. However, just outside our range, the class $w\beta_1$ is also in the image. It is usually denoted ϵ.

5. Generalizations

In this section, we point out that our methods do apply to more spectra than $MO\langle 8\rangle$ at $p = 3$ and $MU\langle 6\rangle$ at $p = 2$ and $p = 3$, though they do not apply to other $MO\langle k\rangle$. Recall that we identified $BO\langle 8\rangle$ at $p = 3$ with $\mathbf{BP}\langle \mathbf{1}\rangle_8$ and used knowledge of Hopf rings. The problem is that for $n > 2p + 2$, the map

$$\mathbf{BP}_n \longrightarrow \mathbf{BP}\langle \mathbf{1}\rangle_n$$

is not onto in homology, so we do not know that the homology of $\mathbf{BP}\langle \mathbf{1}\rangle_n$ is evenly graded, nor do we know how to compute the coaction of the Steenrod algebra.

However, as long as $n \le 2(p^r + p^{r-1} + \cdots p + 1)$, the map

$$\mathbf{BP}_n \longrightarrow \mathbf{BP}\langle \mathbf{r}\rangle_n$$

will be onto in homology [Wil]. So we should build Thom spectra over these spaces. We must assume that n is even, and that n is divisible by 4 if p is odd. At $p = 2$, write $n = 2t + 2$. For p odd, write $n = tq + s$, where $4 \le s \le q$. If p is odd, consider the composite of infinite loop maps

$$\mathbf{BP}\langle \mathbf{r}\rangle_n \longrightarrow \mathbf{BP}\langle \mathbf{1}\rangle_n \xrightarrow{[v_1^t]} \mathbf{BP}\langle \mathbf{1}\rangle_s \longrightarrow BO_{(p)}.$$

For $p = 2$, consider the analogous map

$$\mathbf{BP}\langle \mathbf{r}\rangle_n \longrightarrow \mathbf{BP}\langle \mathbf{1}\rangle_n \xrightarrow{[v_1^t]} \mathbf{BP}\langle \mathbf{1}\rangle_2 \longrightarrow BU_{(2)}.$$

In both cases, the last map comes from the splitting of infinite loop spaces of Corollary 1.5.

These maps do not give us vector bundles over $\mathbf{BP}\langle \mathbf{r}\rangle_n$, because they are p-local. So we must build Thom spectra over maps $X \longrightarrow BO_{(p)}$. We adopt a naive approach to this problem: surely there is a better way that would involve equivariant homotopy theory. In our cases, X is always simply connected, and the map actually comes from a map to $BSO_{(p)}$. We build Thom spectra by putting together Thom spaces as usual. So assume we have a map $Y \xrightarrow{f} BSO(r)_{(p)}$, where Y is simply connected. We can assume r is at least 3, since we are really only interested in spectra. These assumptions get rid of the fundamental group, which can be a problem in localizations. What one would normally do is to pull back the universal bundle to Y, take its disk bundle, and mod out by its sphere bundle. But p-localization is really only defined on the homotopy category of spaces, where one cannot see the $SO(r)$-action very well. But we can localize the universal sphere bundle $S(\xi_r)$ over $BSO(r)$ to get a fibration $S(\xi_r)_{(p)} \longrightarrow BSO(r)_{(p)}$ with fiber $S^{r-1}_{(p)}$. Then we can define $S(f)$ to be the induced fibration over Y. Then define the Thom space $T(f)$ to be the cofibre of $S(f) \longrightarrow Y$. We will leave it to the reader to verify the usual properties of Thom spaces and Thom spectra, which all hold.

Let us denote the resulting Thom spectrum over $\mathbf{BP}\langle\mathbf{r}\rangle_n$ by $MBP(r, n)$. The B is still there because it refers to Brown, not to classifying space. The $MBP(r, n)$ are only defined when n is divisible by 4 if p is odd and when n is even if $p = 2$. They are commutative ring spectra. We will concentrate on the case $n = tq$. Corollary 1.5 shows that $MBP(r, tq)$ admits a ring spectrum map to $MO\langle k\rangle$ and to $MU\langle k\rangle$ when $k \leq tq$. The proof of Rosen's theorem applies without change to the $MBP(r, tq)$, since the proof actually showed that the image of $H_* \mathbf{BP}_{tq}$ in H_*BP is at least as large as $(P//P(t-2))_*$, and that the image of $H_*MO\langle k\rangle$ is at least as small as $(P//P(t-2))_*$ when $k \geq (t-1)q + 2$. Thus we get the following theorem.

Theorem 5.1. *Let f denote the map*

$$MBP(r, tq) \longrightarrow MSO_{(p)} \longrightarrow BP.$$

Then the kernel of H^*f is the left ideal generated by the augmentaion ideal of $P(t-2)$.

We then get the following corollary.

Corollary 5.2. *Let X denote a p-local finite spectrum whose cohomology is evenly graded and free over $P(t-2)$. Then, if $tq \leq 2(p^r + \cdots + p + 1)$, $MBP(r, tq) \wedge X$ is a wedge of suspensions of BP.*

Then methods of Section 2 then apply, and we get

Theorem 5.3. *Suppose $tq \leq 2(p^r + \cdots + p + 1)$. Then the torsion in $\pi_*MBP(r, tq)$ is bounded and the Bousfield class of $MBP(r, tq)$ coincides with the Bousfield class of BP.*

This theorem does allow us to deduce the Bousfield class of the $MU\langle k\rangle$.

Corollary 5.4. (1) *For any k, the Bousfield class of $MU\langle k\rangle_{(p)}$ is the same as that of BP. If p is odd, the same is true for the Bousfield class of $MO\langle k\rangle_{(p)}$.*

(2) *At $p = 2$, the Bousfield class of $MO\langle\phi(r)\rangle$ is less than or equal to that of $BP\langle r-1\rangle$.*

Here $\phi(r)$ is the dimension of the rth nonzero homotopy group of BSO as before.

Proof. If r and t are large enough, there is an orientation $MBP(r, tq) \longrightarrow MU\langle k\rangle$, so the Bousfield classes of $MU\langle k\rangle$ and $MO\langle k\rangle$ are bounded above by that of BP. On the other hand, there is an orientation $MU\langle k\rangle \longrightarrow BP$ and if p is odd, an orientation $MO\langle k\rangle \longrightarrow BP$. Thus the Bousfield classes of $MU\langle k\rangle$ and, if p is odd, $MO\langle k\rangle$, are also bounded below by that of BP.

To examine the Bousfield class of $MO\langle\phi(r)\rangle$ at $p = 2$, we must recall the usual notation for Bousfield class that we have been avoiding, $\langle X\rangle$. The results of Section 2 together with the work of Bahri and Mahowald previously mentioned [BM] imply that $MO\langle\phi(r)\rangle \wedge X$ is a wedge of suspensions of $H\mathbf{F}_2$ if X is a finite spectrum whose homology which is free over $A(r-1)$. Any such spectrum will have type r. We now apply the fundamental Bousfield class decomposition used in [Hov1]:

$$\langle S^0\rangle = \langle \mathrm{Tel}(0)\rangle \vee \cdots \vee \langle \mathrm{Tel}(r-1)\rangle \vee \langle F(r)\rangle.$$

Here $\mathrm{Tel}(i)$ is the telescope of a v_i-self-map on a type i finite spectrum, and $F(r)$ is a finite spectrum of type r. The Bousfield classes are independent of the specific spectra chosen.

If we smash this decomposition with $MO\langle\phi(r)\rangle$, the last term will be $\langle H\mathbf{F}_2\rangle$. Also, since $MO\langle\phi(r)\rangle$ is a module over $MU\langle\phi(r)\rangle$, which has the same Bousfield class as BP, we have $\langle MO\langle\phi(r)\rangle \wedge \mathrm{Tel}(i)\rangle = \langle MO\langle\phi(r)\rangle \wedge K(i)\rangle$. Thus

$$\langle MO\langle\phi(r)\rangle\rangle \leq \langle K(0)\rangle \vee \cdots \vee \langle K(r-1)\rangle \vee \langle H\mathbf{F}_2\rangle = \langle BP\langle r-1\rangle\rangle,$$

as required. □

It is natural to conjecture that the Bousfield class of $MO\langle\phi(r)\rangle$ at $p=2$ is precisely that of $BP\langle r-1\rangle$. To prove this, it would suffice to show that $K(i)_*MO\langle\phi(r)\rangle \neq 0$ for $i<r$.

Now we will show that, just as the 3-torsion in $\pi_*MO\langle 8\rangle_{(3)} = \pi_*MBP(1, 2q)$ is all killed by 3, the p-torsion in $\pi_*MBP(2, 2q)$ is all killed by p. To do this we first must examine how the image of J behaves in $\pi_*MBP(r, tq)$.

Lemma 5.5. *If p is an odd prime, the composite*

$$\mathrm{Im}J \longrightarrow \pi_*S^0 \longrightarrow \pi_*MBP(r, tq)$$

is injective in dimensions $\leq tq-2$ and 0 in dimensions $\geq tq-1$. At $p=2$ the same theorem is true for the image of the complex J homomorphism.

Proof. The proof is just like the proof of the analogous theorem for $MO\langle k\rangle$ and $MU\langle k\rangle$ in [Hov]. First assume p is odd. Suppose $x \in \mathrm{Im}\, J$ in dimension k. Then the mapping cone on x, $C(x)$, is the Thom complex of a map $S^{k+1} \longrightarrow BO$. Because the image of J is concentrated in dimensions congruent to -1 mod q, this map will lift to $S^{k+1} \longrightarrow \mathbf{BP}\langle\mathbf{1}\rangle_q$. Assume $k > tq-2$. Then it will lift further to $S^{k+1} \longrightarrow \mathbf{BP}\langle\mathbf{1}\rangle_{tq}$. Since the map of spectra $BP\langle r\rangle \longrightarrow BP\langle 1\rangle$ is onto on homotopy groups, it will lift even further to $S^{k+1} \longrightarrow \mathbf{BP}\langle\mathbf{r}\rangle_{tq}$. Hence x maps to 0 in $\pi_*MBP(r, tq)$. Injectivity for smaller values of k just follows from the high connectivity of $\mathbf{BP}\langle\mathbf{1}\rangle_{tq}$. The case $p=2$ is similar. □

Theorem 5.6. *The torsion in $\pi_*MBP(2, 2q)$ is all killed by p.*

Proof. We follow the same outline as the proof of the analogous theorem for $\pi_*MO\langle 8\rangle$. We know that $MBP(2, 2q) \wedge X$ is a wedge of suspensions of BP for an evenly graded X whose cohomology is a free P^1-module. So we can take X to be the $(p-1)q$ skeleton of BP, a finite spectrum with p cells. If we denote by $\overline{X}$ the fiber of the inclusion of the bottom cell $S^0 \longrightarrow X$, we must calculate the image of $[\overline{X}, S^0]$ in $[\overline{X}, MBP(2, 2q)]$. As before, $[\overline{X}, S^0] \cong \mathbf{Z}/p^{p-1}$, with generator α_1 on the bottom cell. The α_i for $1 \leq i \leq p-1$ all pile up to make the $\mathbf{Z}/p^{p-1}$, in the sense that $p\alpha_1$ is 0 on the bottom cell but α_2 on the second cell, etc. But all of these α_i go to 0 in $MBP(2, 2q)$ except α_1. So the image in $[\overline{X}, MBP(2, 2q)]$ has order p. □

One can also find v_1-elements in $\pi_* MBP(r, tq)$, show that the v_1-torsion and the p-torsion in $\pi_* MBP(2, 2q)$ coincide if p is odd, and show that any such torsion element is killed by any well-behaved v_1-element. The proofs are all analogous to the corresponding theorems for $MO\langle 8\rangle$.

One also gets results analogous to those of Section 3 for $MBP(2, 2q)$. That is, the Adams-Novikov spectral sequence collapses with a flat vanishing line as soon as a power of β_1 is killed. Note that sparseness tells us the only non-zero differentials are d_{kq+1}, since $H_* MBP(2, 2q)$ is concentrated in degrees congruent to 0 mod q. Again $\alpha_1 \beta_1^p$ cannot survive the spectral sequence, since it does not survive the Adams-Novikov spectral sequence for the sphere. (It is killed by the Toda differential [Rav].) Once again, the only way this can happen is if there is a $w \in E_2^{0, pq(p-1)}$ with $d_{q+1} w = \alpha_1 \beta^{p-1}$. The fact that β_1 is the p-fold Toda bracket of α_1 then induces a differential $d_{(p-1)q+1}(w^{p-1}\alpha) = \beta_1^{p^2-2p+2}$. So the Adams-Novikov spectral sequence for $MBP(2, 2q)$ collapses after at most $p-1$ differentials. This differential may of course be pre-empted by a shorter one, though we would be very surprised if that happened.

This behavior is very similar to that of Adams-Novikov spectral sequence for EO_{p-1}. The first author has shown [Hov] that no $MO\langle k\rangle$ can admit an orientation to EO_{p-1} if $p > 3$. That proof does not apply to the $MBP(2, 2q)$, so it is possible that they do in fact admit orientations to EO_{p-1}.

References

[AM] J. F. Adams and H. Margolis, *Modules over the Steenrod algebra*, Topology **10** (1971), 271–282.

[BM] A. Bahri and M. Mahowald, *Stiefel-Whitney classes in $H^* BO\langle\phi(r)\rangle$*, Proc. Amer. Math. Soc. **83** (1981), 653–655.

[Br] R. Bruner, *Ext in the nineties*, Algebraic Topology (Oaxtepec, 1991), Contemp. Math., vol. 146, Amer. Math. Soc., Providence, RI, 1993, pp. 71–90.

[Bot] B. Botvinnik, *The structure of the ring MSU_**, Math. USSR-Sb. **69** (1991), 581–596.

[Bou] A. K. Bousfield, *The localization of spectra with respect to homology theories*, Topology **18** (1979), 257–281.

[BMMS] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ Ring spectra and their applications*, Lecture Notes in Math., vol. 1176, Springer-Verlag, New York, 1986.

[CF] P. E. Conner and E. E. Floyd, *Torsion in SU-bordism*, Mem. Amer. Math. Soc. **60** (1966).

[DHS] E. Devinatz, M. Hopkins, and J. Smith, *Nilpotence and stable homotopy theory* I, Ann. of Math. **128** (1988), 207–242.

[DM] D. Davis and M. Mahowald, *v_1- and v_2-periodicity in stable homotopy theory*, Amer. J. Math. **103** (1981), 615–659.

[Giam] V. Giambalvo, *The* mod p *cohomology of $BO\langle 4k\rangle$*, Proc. Amer. Math. Soc. **20** (1969), 593–597.

[GM] V. Gorbunov and M. Mahowald, *Some homotopy of the cobordism spectrum $MO\langle 8\rangle$*, preprint (1993).

[HM] M. Hopkins and H. Miller, *Enriched multiplication on the cohomology theory E_n*, to appear.

[Hov] M. Hovey, *v_n-elements in ring spectra and applications to bordism theory*, preprint (1993).

[Hov1] ———, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, to appear in the Proceedings of the Cech Centennial Conference on Homotopy Theory.

[LMM] L. G. Lewis, Jr., J. P. May, and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Math., vol. 1213, Springer-Verlag, New York, 1986.

[MS] M. Mahowald and H. Sadofsky, *v_n-telescopes and the Adams spectral sequence*, preprint (1992).

[Mar] H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland, Amsterdam, 1983.

[May] J. P. May, *A general algebraic approach to Steenrod operations*, The Steenrod Algebra and Its Applications, Lecture Notes in Math., vol. 168, Springer-Verlag, New York, 1970, pp. 153–231.

[MW] H. Miller and C. Wilkerson, *Vanishing lines for modules over the Steenrod algebra*, J. Pure Appl. Algebra **22** (1981), 293–307.

[Mit] S. Mitchell, *Finite complexes with $A(n)$-free cohomology*, Topology **24** (1985), 227–248.

[MP] J. Moore and F. Peterson, *Modules over the Steenrod algebra*, Topology **11** (1972), 387–395.

[Nov] S. P. Novikov, *The methods of algebraic topology from the viewpoint of cobordism theories*, (in Russian) Izv. Akad. Nauk SSR Ser. Mat. **31** (1967), 855–951; translation, Math. USSR-Izv. (1967), 827–913.

[Pen] D. Pengelley, *The homotopy type of MSU*, Amer. J. Math. **104** (1982), 1101–1123.

[Rav] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, Orlando, 1986.

[RW] D. C. Ravenel and W. S. Wilson, *The Hopf ring for complex cobordism*, J. Pure Appl. Algebra **9** (1977), 241–280.

[Ros] S. Rosen, *On torsion in connective cobordism*, Ph. D. thesis, Northwestern University, 1972.

[Sin] W. Singer, *Connective fiberings over BU and U*, Topology **7** (1968), 271-303.

[Sm] J. H. Smith, *Finite complexes with vanishing lines of small slope*, preprint (1992).

[St] R. Stong, *Determination of $H^*(BO(k, \dots, \infty); \mathbf{Z}_2)$ and $H^*(BU(k, \dots, \infty); \mathbf{Z}_2)$*, Trans. Amer. Math. Soc. **107** (1963), 526–544.

[Ver] V. Vershinin, *Cobordisms and spectral sequences*, Transl. Math. Mono., Amer. Math. Soc., vol. 130, Providence, RI, 1993.

[Wil] W. S. Wilson, *The Ω-spectrum for Brown-Peterson cohomology*, II, Amer. J. Math. **97** (1975), 101–123.

[Wit] E. Witten, *The index of the Dirac operator in loop space*, Lecture Notes in Math., vol. 1326, Springer-Verlag, 1988, pp. 161–181.

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