The periodicity theorem in the solution to the Arf-Kervaire invariant problem

Harvard-MIT Summer Seminar on the Kervaire Invariant

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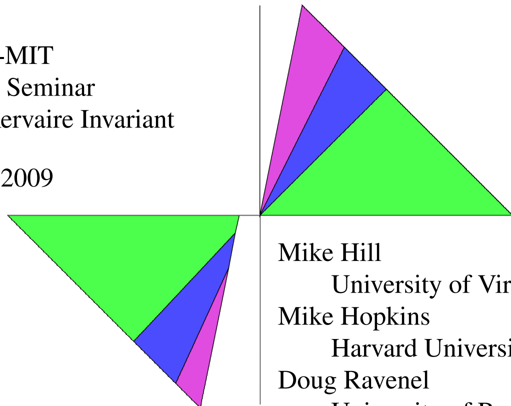

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1 Review of our strategy

Review of our strategy

Our goal is to prove

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ *do not exist for* $j \geq 7$.

Our strategy is to find a map $S^0 \to M$ to a nonconnective spectrum M with the following properties.

(i) It has an Adams-Novikov spectral sequence in which the image of each θ_j is nontrivial. This is the Detection Theorem discussed by Hopkins here on July 8.

(ii) $\pi_{-2}(M) = 0$. This is the Gap Theorem discussed by Hill here on July 15.

(iii) It is 256-periodic, meaning $\Sigma^{256} M \cong M$. This is the Periodicity Theorem. 1.2

Our strategy (continued)

(ii) and (iii) imply that $\pi_{254}(M) = 0$.

If θ_7 exists, (i) implies it has a nontrivial image in this group, so it cannot exist.

The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$. 1.3

2 The spectrum M

The spectrum M

As explained previously, there is an action of the cyclic group C_8 on the 4-fold smash product $MU^{(4)}$. It is derived using a norm induction from the action of C_2 on MU by complex conjugation.

We show that its homotopy fixed point set $(MU^{(4)})^{hC_8}$ and its actual fixed point set $(MU^{(4)})^{C_8}$ are equivalent. It is an E_∞-ring spectrum, and M is obtained from it by inverting an element $D \in \pi_{256}$ which we will identify below.

The homotopy of $(MU^{(4)})^{hC_8}$ can be computed using the homotopy fixed point spectral sequence, for which

$$E_2 = H^*(C_8; \pi_*(MU^{(4)}))$$

In this case it concides with the Adams-Novikov spectral sequence for $\pi_*((MU^{(4)})^{hC_8})$ The algebraic methods described by Hopkins can be used to show that it detects the θ_js. D has to be chosen so that this is still true after we invert it. 1.4

The spectrum M (continued)

The homotopy of $(MU^{(4)})^{C_8}$ and $M = D^{-1}(MU^{(4)})^{C_8}$ can be also computed using the *slice spectral sequence* described by Hill. It has the convenient property that π_{-2} vanishes in the E_2-term. In fact π_k vanishes for $-4 < k < 0$.

This is our main motivation for developing the slice spectral sequence. We do not know how to show this vanishing using the other spectral sequence.

In order to identify D we need to study the slice spectral sequence in more detail. 1.5

3 The slice spectral sequence

The slice spectral sequence

Recall that for $G = C_8$ we have a *slice tower*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_G^{n+1}MU^{(4)} & \longrightarrow & P_G^{n}MU^{(4)} & \longrightarrow & P_G^{n-1}MU^{(4)} & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & {}^{G}P_{n+1}^{n+1}MU^{(4)} & & {}^{G}P_{n}^{n}MU^{(4)} & & {}^{G}P_{n-1}^{n-1}MU^{(4)} & & \end{array}$$

in which

- the inverse limit is $MU^{(4)}$,
- the direct limit is contractible and
- ${}^{G}P_n^nMU^{(4)}$ is the fiber of the map $P_G^nMU^{(4)} \to P_G^{n-1}MU^{(4)}$.

${}^{G}P_n^nMU^{(4)}$ is the *nth slice* and the decreasing sequence of subgroups of $\pi_*(MU^{(4)})$ is the *slice filtration.* We also get slice filtrations of the $RO(G)$-graded homotopy $\pi_\star(MU^{(4)})$ and the homotopy groups of fixed point sets $\pi_*((MU^{(4)})^H)$ for each subgroup H. 1.6

The slice spectral sequence (continued)

This means the slice filtration leads to a *slice spectral sequence* converging to $\pi_*(MU^{(4)})$ and its variants.

One variant has the form

$$E_2^{s,t} = \pi_{t-s}^G({}^{G}P_t^tMU^{(4)}) \implies \pi_{t-s}^G(MU^{(4)}).$$

Recall that $\pi_*^G(MU^{(4)})$ is by definition $\pi_*((MU^{(4)})^G)$, the homotopy of the fixed point set.

Slice Theorem . *In the slice tower for $MU^{(4)}$, every odd slice is contractible and $P_{2n}^{2n} = \hat{W}_n \wedge H\mathbf{Z}$, where $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum and $\hat{W}_n$ is a certain wedge of the following three types of finite G-spectra:*

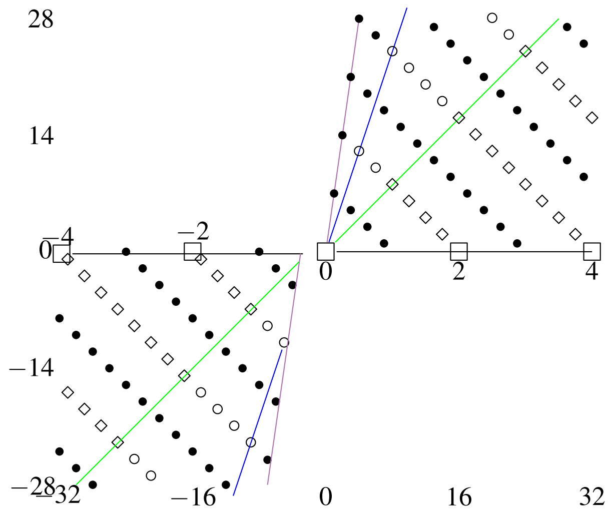
- $S^{(n/4)\rho_8}$*, where ρ_g denotes the regular real representation of C_g,*
- $C_8 \wedge_{C_4} S^{(n/2)\rho_4}$ *and*
- $C_8 \wedge_{C_2} S^{n\rho_2}$.

The same holds after we invert D, in which case negative values of n can occur. 1.7

3.1 Slices of the form $S^{m\rho_8} \wedge H\mathbf{Z}$

Slices of the form $S^{m\rho_8} \wedge H\mathbf{Z}$

Here is a picture of some slices $S^{m\rho_8} \wedge H\mathbf{Z}$.



1.8

Slices of the form $S^{m\rho_8} \wedge H\mathbf{Z}$ (continued)

- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7, and are concentrated on diagonals where t is divisible by 8.
- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.
- A similar picture for $S^{m\rho_4} \wedge H\mathbf{Z}$ would be confined to the regions between the black lines and blue lines with slope 3 and concentrated on diagonals where t is divisible by 4.
- A similar picture for $S^{m\rho_2} \wedge H\mathbf{Z}$ would be confined to the regions between the black lines and green lines with slope 1 and concentrated on diagonals where t is divisible by 2.

1.9

3.2 Implications for the slice spectral sequence

Implications for the slice spectral sequence

These calculations imply the following.

- The slice spectral sequence for $MU^{(4)}$ is concentrated in the first quadrant and confined by the same vanishing lines.
- Later we will invert elements in $\pi_{m\rho_8}(MU^{(4)})$. The fact that

$$S^{-\rho_8} \wedge (C_8 \wedge_H S^{m\rho_h}) = C_8 \wedge_H S^{(m-8/h)\rho_h}$$

  means that the resulting slice spectral sequence is confined to the regions of the first and third quadrants shown in the picture.

1.10

4 Geometric fixed points

Geometric fixed points

In order to proceed further, we need another concept from equivariant stable homotopy theory.

Unstably a G-space X has a *fixed point set*,

$$X^G = \{x \in X : \gamma(x) = x \, \forall \gamma \in G\}.$$

This is the same as $F(S^0, X_+)^G$, the space of based equivariant maps $S^0 \to X_+$, which is the same as the space of unbased equivariant maps $* \to X$.

The *homotopy fixed point set* X^{hG} is the space of based equivariant maps $EG_+ \to X_+$, where EG is a contractible free G-space. The equivariant homotopy type of X^{hG} is independent of the choice of EG. 1.11

Geometric fixed points (continued)

Both of these definitions have stable analogs, but the fixed point functor is awkward for two reasons:

- it fails to commute with smash products and
- it fails to commute with infinite suspensions.

The *geometric fixed set* $\Phi^G X$ is a convenient substitute that avoids these difficulties. In order to define it we need the *isotropy separation sequence*, which in the case of a finite cyclic 2-group G is

$$EC_{2+} \to S^0 \to \tilde{E}C_2.$$

Here $E\mathbf{Z}/2$ is a G-space via the projection $G \to \mathbf{Z}/2$ and S^0 has the trivial action, so $\tilde{E}C_2$ is also a G-space. 1.12

Geometric fixed points (continued)

Under this action EC_2^G is empty while for any proper subgroup H of G, $EC_2^H = EC_2$, which is contractible. For an arbitrary finite group G it is possible to construct a G-space with the similar properties.

Definition. *For a finite cyclic 2-group G and G-spectrum X, the geometric fixed point spectrum is*

$$\Phi^G X = (X \wedge \tilde{E}C_2)^G.$$

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Geometric fixed points (continued)

This functor has the following properties:

- For G-spectra X and Y, $\Phi^G(X \wedge Y) = \Phi^G X \wedge \Phi^G Y$.
- For a G-space X, $\Phi^G \Sigma^\infty X = \Sigma^\infty(X^G)$.
- A map $f : X \to Y$ is a G-equivalence iff $\Phi^H f$ is an ordinary equivalence for each subgroup $H \subset G$.

From the suspension property we can deduce that

$$\Phi^{C_8} MU^{(4)} = MO,$$

the unoriented cobordism spectrum.

Geometric Fixed Point Theorem. *Let σ denote the sign representation. Then for any G-spectrum X, $\pi_\star(\tilde{E}C_2 \wedge X) = a_\sigma^{-1}\pi_\star(X)$, where $a_\sigma : S^0 \to S^\sigma$ is the element defined in Hill's lecture.*

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Geometric fixed points (continued)

Recall that $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. It is not hard to show that

$$\pi_*(MU^{(4)}) = \mathbf{Z}[r_i, \gamma(r_i), \gamma^2(r_i), \gamma^3(r_i) : i > 0]$$

where $|r_i| = 2i$, γ is a generator of G and $\gamma^4(r_i) = (-1)^i r_i$. In $\pi_{i\rho_8}(MU^{(4)})$ we have the element

$$Nr_i = r_i \gamma(r_i) \gamma^2(r_i) \gamma^3(r_i).$$

Applying the functor Φ^G to the map $Nr_i : S^{i\rho_8} \to MU^{(4)}$ gives a map $S^i \to MO$.

Lemma. *The generators r_i and y_i can be chosen so that*

$$\Phi^G Nr_i = \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise.} \end{cases}$$

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5 Some slice differentials

Some slice differentials

It follows from the above that the slice spectral sequence for $MU^{(4)}$ has a vanishing line of slope 7. We will describe the subring of elements lying on it.

Let $f_i \in \pi_i(MU^{(4)})$ be the composite

$$S^i \xrightarrow{a_{i\rho_8}} S^{i\rho_8} \xrightarrow{Nr_i} MU^{(4)}.$$

The following facts about f_i are easy to prove.

- It appears in the slice spectral sequence in $E_2^{7i,8i}$, which is on the vanishing line.
- The subring of elements on the vanishing line is the polynomial algebra on the f_i.

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Some slice differentials (continued)

- Under the map
$$\pi_*(MU^{(g/2)}) \to \pi_*(\Phi^G MU^{(g/2)}) = \pi_*(MO)$$
we have
$$f_i \mapsto \begin{cases} 0 & \text{for } i = 2^k - 1 \\ y_i & \text{otherwise} \end{cases}$$
- Any differential landing on the vanishing line must have a target in the ideal $(f_1, f_3, f_7, \ldots)$. A similar statement can be made after smashing with $S^{2^k\sigma}$.

1.17

Some slice differentials (continued)

Recall that for an oriented representation V there is a map $u_V : S^{|V|} \to \Sigma^V H\mathbf{Z}$, which lies in $\pi_{V-|V|}(H\mathbf{Z})$.

Slice Differentials Theorem. *In the slice spectral sequence for $\Sigma^{2^k\sigma} MU^{(4)}$ (for $k > 0$) we have $d_r(u_{2^k\sigma}) = 0$ for $r < 1 + 8(2^k - 1)$, and*

$$d_{1+8(2^k-1)}(u_{2^k\sigma}) = a_\sigma^{2^k} f_{2^k-1}.$$

Inverting a_σ in the slice spectral sequence will make it converge to $\pi_*(MO)$. This means each f_{2^k-1} must be killed by some power of a_σ. The only way this can happen is as indicated in the theorem.

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Some slice differentials (continued)

Let
$$\overline{\Delta}_k^{(8)} = Nr_{2^k-1} \in \pi_{(2^k-1)\rho_8}(MU^{(4)}).$$

We want to invert this element and study the resulting slice spectral sequence. As explained previously, it is confined to the first and third quadrants with vanishing lines of slopes 0 and 7.

The differential d_r on $u_{2^{k+1}\sigma}$ described in the theorem is the last one possible since its target, $a_\sigma^{2^{k+1}} f_{2^{k+1}-1}$, lies on the vanishing line. If we can show that this target is killed by an earlier differential after inverting $\overline{\Delta}_k^{(8)}$, then $u_{2^{k+1}\sigma}$ will be a permanent cycle.

1.19

Some slice differentials (continued)

We have

$$\begin{aligned} f_{2^{k+1}-1}\overline{\Delta}_k^{(8)} &= a_{(2^{k+1}-1)\rho_8} Nr_{2^{k+1}-1} Nr_{2^k-1} \\ &= a_{2^k\rho_8}\overline{\Delta}_{k+1}^{(8)} f_{2^k-1} \\ &= \overline{\Delta}_{k+1}^{(8)} d_{r'}(u_{2^k\sigma}) \text{ for } r' < r. \end{aligned}$$

Corollary. *In the $RO(G)$-graded slice spectral sequence for $\left(\overline{\Delta}_k^{(8)}\right)^{-1} MU^{(4)}$, the class $u_{2\sigma}^{2^k}$ is a permanent cycle.*

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6 The proof of the Periodicity Theorem

The proof of the Periodicity Theorem

The corollary shows that inverting a certain element makes a power of $u_{2\sigma}$ a permanent cycle. We need a similar statement about a power of $u_{2\rho_8}$.

We will get this by using the norm property of u, namely that if V is an oriented representation of a subgroup $H \subset G$ with $V^H = 0$ and induced representation V', then the norm functor N_h^g from H-spectra to G-spectra satisfies $N_h^g(u_V)u_{2\rho_{G/H}}^{|V|/2} = u_{V'}$.

From this we can deduce that $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$, where σ_m denotes the sign representation on C_{2^m}.

1.21

The proof of the Periodicity Theorem (continued)

We have $u_{2\rho_8} = u_{8\sigma_3} N_4^8(u_{4\sigma_2}) N_2^8(u_{2\sigma_1})$.

By the Corollary we can make a power of each factor a permanent cycle by inverting some $\overline{\Delta}_{k_m}^{(2^m)}$ for $1 \le m \le 3$. If we make k_m too small we will lose the detection property, that is we will get a spectrum that does not detect the θ_j. It turns out that k_m must be chosen so that $8|2^m k_m$.

- Inverting $\overline{\Delta}_4^{(2)}$ makes $u_{32\sigma_1}$ a permanent cycle.
- Inverting $\overline{\Delta}_2^{(4)}$ makes $u_{8\sigma_2}$ a permanent cycle.
- Inverting $\overline{\Delta}_1^{(8)}$ makes $u_{4\sigma_3}$ a permanent cycle.
- Inverting the product D of the norms of all three makes $u_{32\rho_8}$ a permanent cycle.

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The proof of the Periodicity Theorem (continued)

Let

$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)}).$$

The we define $\tilde{M} = D^{-1}MU^{(4)}$ and $M = \tilde{M}^{C_8}$.

Since the inverted element is represented by a map from $S^{m\rho_8}$, the slice spectral sequence for $\pi_*(M)$ has the usual properties:

- It is concentrated in the first and third quadrants and confined by vanishing lines of slopes 0 and 7.
- It has the gap property, i.e., no homotopy between dimensions -4 and 0.

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The proof of the Periodicity Theorem (continued)

Preperiodicity Theorem. *Let* $\Delta_1^{(8)} = u_{2\rho_8}(\overline{\Delta}_1^{(8)})^2 \in E_2^{16,0}(D^{-1}MU^{(4)})$. *Then* $(\Delta_1^{(8)})^{16}$ *is a permanent cycle.*

To prove this, note that $(\Delta_1^{(8)})^{16} = u_{32\rho_8}\left(\overline{\Delta}_1^{(8)}\right)^{32}$. Both $u_{32\rho_8}$ and $\overline{\Delta}_1^{(8)}$ are permanent cycles, so $(\Delta_1^{(8)})^{16}$ is also one.

Thus we have an equivariant map $\Sigma^{256}D^{-1}MU^{(4)} \to D^{-1}MU^{(4)}$ and a similar map on the fixed point set. The latter one is invertible because $u_{2\rho_8}^{32}$ restricts to the identity.

Thus we have proved

Periodicity Theorem. *Let* $M = (D^{-1}MU^{(4)})^{C_8}$. *Then* $\Sigma^{256}M$ *is equivalent to* M.

1.24