The Microstable Adams-Novikov Spectral Sequence

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Abstract. In the Adams–Novikov spectral sequence one considers Ext groups over the Hopf algebroid $\Gamma = BP_*(BP)$. There are spectra $T(m)$ with $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$, which leads one to replace Γ by $\Gamma(m+1) = \Gamma/(t_1, \ldots, t_m)$. The corresponding Ext groups have certain structural features that are independent of m. In this paper we set up an algebraic framework for studying the limit as $m \to \infty$. In particular there is an analog of the chromatic spectral sequence in which the Morava stabilizer group gets replaced by an infinitesimal analog, hence the title.

1. Introduction

For a fixed prime p, recall the spectra $T(m)$ (introduced in [**Rav86**, §6.5]) with

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m] \subset BP_*(BP).$$

It is a p-local summand of the Thom spectrum associated with the map

$$\Omega SU(k) \to \Omega SU = BU$$

for any k satisfying $p^m \leq k < p^{m+1}$. These Thom spectra figure in the proof of the nilpotence theorem of [**DHS88**]. The $T(m)$ themselves figure in the method of infinite descent, the technique for calculating the stable homotopy groups of spheres described in [**Rav86**, Chapter 7] and [**Ravb**].

Very briefly, there are maps

$$S^0 = T(0) \to T(1) \to T(2) \to \cdots \to BP$$

with $T(m)$ homotopy equivalent to BP below dimension $|v_{m+1}| - 1$. Interpolating between $T(m)$ and $T(m+1)$ are $T(m)$-module spectra $T(m)_h$ for $h \geq 0$ with

$$BP_*(T(m)_h) = BP_*[t_1, \ldots, t_m]\{1, t_{m+1}, t_{m+1}^2, \ldots, t_{m+1}^h\}.$$

There are maps

$$T(m) = T(m)_0 \to T(m)_1 \to T(m)_2 \to \cdots \to T(m+1)$$

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with $T(m)_h$ homotopy equivalent to $T(m+1)$ below dimension $(h+1)|v_{m+1}|-1$. For each m and i there is a spectral sequence converging to $\pi_*(T(m)_{p^i-1})$ with

$$E_1 = \pi_*(T(m)_{p^{i+1}-1}) \otimes E(h_{m+1,i+1}) \otimes P(b_{m+1,i+1})$$

where

$$h_{m+1,i+1} \in E_1^{1,2p^{i+1}(p^{m+1}-1)} \qquad \text{and} \qquad b_{m+1,i+1} \in E_1^{2,2p^{i+2}(p^{m+1}-1)}.$$

Thus in a given range of dimensions, a finite number of applications of this spectral sequence will get us from $\pi_*(T(m+1))$ to $\pi_*(T(m))$ and hence from $\pi_*(BP)$ to $\pi_*(S^0)$. This is discussed in more detail in [**Ravb**].

Empirical evidence suggests that $\pi_*(T(m))$ for roughly $2p^{m+1} < * < 2p^{2m+2}$ is the same (up to a suitable regrading) as that of $\pi_*(T(m+1))$ for roughly $2p^{m+2} < * < 2p^{2m+3}$. *The purpose of this note is to set up an algebraic framework that allows us to study the limit of this behavior as m goes to infinity.* We will define a limiting Ext group which would be the E_2-term for the conjectural spectral sequence of the title; see Conjecture 4 below.

This will entail defining a bigraded Hopf algebroid $(\widehat{A}, \widehat{\Gamma})$. The grading is over $\mathbf{Z} \oplus \mathbf{Z}\omega$ where ω becomes p^m when we specialize to $T(m)$. We call the corresponding Ext group the *microstable* Adams-Novikov E_2-term for the following reason. For each spectrum $T(m)$ one can set up a chromatic spectral sequence as in [**Rav86**, Chapter 5]. Each Morava stabilizer group S_n gets replaced by a certain open subgroup which shrinks as m increases. Thus in the limit each S_n gets replaced by an infinitesimal version of itself. We conjecture that this Ext group is the E_2-term of a trigraded spectral sequence.

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2. Empirical evidence: similarities among the groups $\pi_*(T(m))$

In this section we will quote several theorems about the Adams–Novikov spectral sequence for $T(m)$ that are proved elsewhere.

Let (A, Γ) denote the Hopf algebroid $(BP_*, BP_*(BP))$; see [**Rav86**, A1] for more information. A change-of-rings isomorphism identifies the Adams-Novikov E_2-term for $T(m)$ with $\mathrm{Ext}_{\Gamma(m+1)}(A, A)$ where

$$\Gamma(m+1) = \Gamma/(t_1, \dots, t_m) = BP_*[t_{m+1}, t_{m+2}, \dots]$$

This Hopf algebroid is cocommutative below the dimension of t_{2m+2}, so its Ext group (and the homotopy of $T(m)$) in this range is relatively easy to deal with. We will denote this Ext group by $\mathrm{Ext}_{\Gamma(m+1)}$ for short.

The following was proved in [**Rav86**, 6.5.9 and 6.5.12].

Theorem A. *For each $m \geq 0$ and each prime p,*

$$\mathrm{Ext}^0_{\Gamma(m+1)} = \mathbf{Z}_{(p)}[v_1, \dots, v_m],$$

and we denote this ring by $A(m)$. Each of these generators is a permanent cycle, and there are no higher Ext groups below dimension $|v_{m+1}|-1$. Hence $\pi_*(T(m)) \cong A(m)$ in this range.

More generally, for each $n \geq 0$

$$\mathrm{Ext}^0_{\Gamma(m+1)}(A, A/I_n) = A(m+n)/I_n,$$

where

$$I_n = (p, v_1, v_2, \ldots, v_{n-1}).$$

Our next result concerns Ext^1 and increases the range of dimensions by a factor of p. Before stating it we need some chromatic notation. Consider the short exact sequence of Γ-comodules (and hence of $\Gamma(m+1)$-comodules)

$$0 \longrightarrow N^0 \longrightarrow M^0 \longrightarrow N^1 \longrightarrow 0 \tag{1}$$

where

$$\begin{aligned} N^0 &= BP_*, \\ M^0 &= p^{-1}BP_* = \mathbf{Q} \otimes BP_*, \\ \text{and} \qquad N^1 &= BP_*/(p^\infty) = \mathbf{Q}/\mathbf{Z}_{(p)} \otimes BP_*. \end{aligned}$$

We write elements in N^1 as fractions

$$\frac{x}{p^e}$$

where $e > 0$ and $x \in BP_*$ is not divisible by p. The long exact sequence of Ext groups associated with (1) has a surjective connecting homomorphism

$$\mathrm{Ext}^0_{\Gamma(m+1)}(N^1) \to \mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$$

and we will identify elements in $\mathrm{Ext}^0_{\Gamma(m+1)}(N^1)$ with their images in Ext^1. The algebraic statement in the following was proved in [**Rav86**, 6.5.11] while the topological part is proved in [**Ravb**].

Theorem B. *In all cases except* $m = 0$ *and* $p = 2$, $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$ *is isomorphic to the* $A(m)$*-submodule of* N^1 *generated by the set*

$$\left\{ \frac{v_{m+1}^i}{ip} : i > 0 \right\}.$$

Each of these elements is a permanent cycle, and there are no higher Ext groups below dimension $p|v_{m+1}| - 2$.

For the 2-line and above, we have the following, essentially proved as Theorem 7.1.13 in [**Rav86**].

Theorem C. *For* $m > 0$, $\mathrm{Ext}^{2,t}(BP_*(T(m)))$ *for* $t \leq 2p^2 - 2p + p^2|v_{m+1}|$ *is the* $A(m)$*-module generated by*

$$\left\{ \frac{v_{m+2}^p}{pv_1^p} \right\} \cup E(h_{m+1,0}) \otimes P(b_{m+1,0}) \otimes \left\{ \frac{v_{m+1}^j v_{m+2}^i}{ipv_1} : 0 < i \leq p,\ 0 \leq j \leq p^2 - pi \right\},$$

where

$$h_{m+1,0} = \frac{v_{m+1}}{p} \quad \textit{and} \quad b_{m+1,0} = \frac{v_{m+2}}{pv_1}.$$

We also let

$$\begin{aligned} b_{m+1,1} &= \frac{v_{m+1}^p}{pv_1^p}, \\ b_{m+2,0} &= \frac{v_{m+3}}{pv_1} - \frac{v_2 v_{m+2}^p}{pv_1^{1+p}} + \frac{v_2^{p^{m+1}} v_{m+1}}{p^2 v_1} \\ \text{and} \quad v_{m+1} b_{m+2,0} &= \frac{v_{m+1} v_{m+3}}{pv_1} - \frac{v_2 v_{m+1} v_{m+2}^p}{pv_1^{1+p}} + \frac{v_2^{p^{m+1}} v_{m+1}^2}{2p^2 v_1}. \end{aligned}$$

Our next result concerns the first differential in the Adams–Novikov spectral sequence for $T(m)$ and is proved in [**Rava**]. The differential occurs slightly beyond the range of Theorem C. Recall that for an odd prime, the first nontrivial differential in the Adams–Novikov spectral sequence for $T(0) = S^0$ is

$$d_{2p-1}(b_{1,1}) = h_{1,0} b_{1,0}^p.$$

Theorem D. *The first nontrivial differential in the Adams–Novikov spectral sequence for the spectrum $T(1)$ at an odd prime p is*

$$d_{2p-1}(b_{3,0}) = h_{2,0} b_{2,0}^p$$

where $b_{3,0} \in E_2^{2,2p^4-2p}$.

For $m > 1$ the first nontrivial differential in the Adams–Novikov spectral sequence for the spectrum $T(m)$ at an odd prime p is

$$d_{2p-1}(v_{m+1} b_{m+2,0}) = v_2 h_{m+1,0} b_{m+1,0}^p$$

where $v_{m+1} b_{m+2,0} \in E_2^{2,2p^{m+3}+2p^{m+1}-2p-2}$. In this case there is also a nontrivial group extension in $\pi_*(T(m))$, namely

$$pb_{m+2,0} = v_2 b_{m+1,0}^p.$$

For $p = 3$ this is illustrated for $m = 1$ and $m = 2$ in Figures 1 and 2 respectively.

3. The bigraded Hopf algebroid $(\widehat{A}, \widehat{\Gamma})$

Recall that $(A, \Gamma) = (BP_*, BP_*(BP))$ is defined by

$$\begin{aligned} A &= \mathbf{Z}_{(p)}[v_i : i > 0] \qquad \text{with } |v_i| = 2p^i - 2; \\ \Gamma &= A[t_i : i > 0] \qquad \text{with } |t_i| = 2p^i - 2. \end{aligned}$$

The generators v_i are related to the coefficients ℓ_i of the logarithm associated with the universal p-typical formal group law by Araki's formula

$$p\ell_n = \sum_{0 \le i \le n} \ell_i v_{n-i}^{p^i},$$

where $\ell_0 = 1$ and $v_0 = p$. The right unit and coproduct are defined by

$$\begin{aligned} \eta_R(\ell_n) &= \sum_{0 \le i \le n} \ell_i t_{n-i}^{p^i} \\ \text{and} \quad \sum_{0 \le i \le n} \ell_i \Delta(t_{n-i}^{p^i}) &= \sum_{0 \le i+j \le n} \ell_i t_j^{p^i} \otimes t_{n-i-j}^{p^{i+j}}, \end{aligned}$$

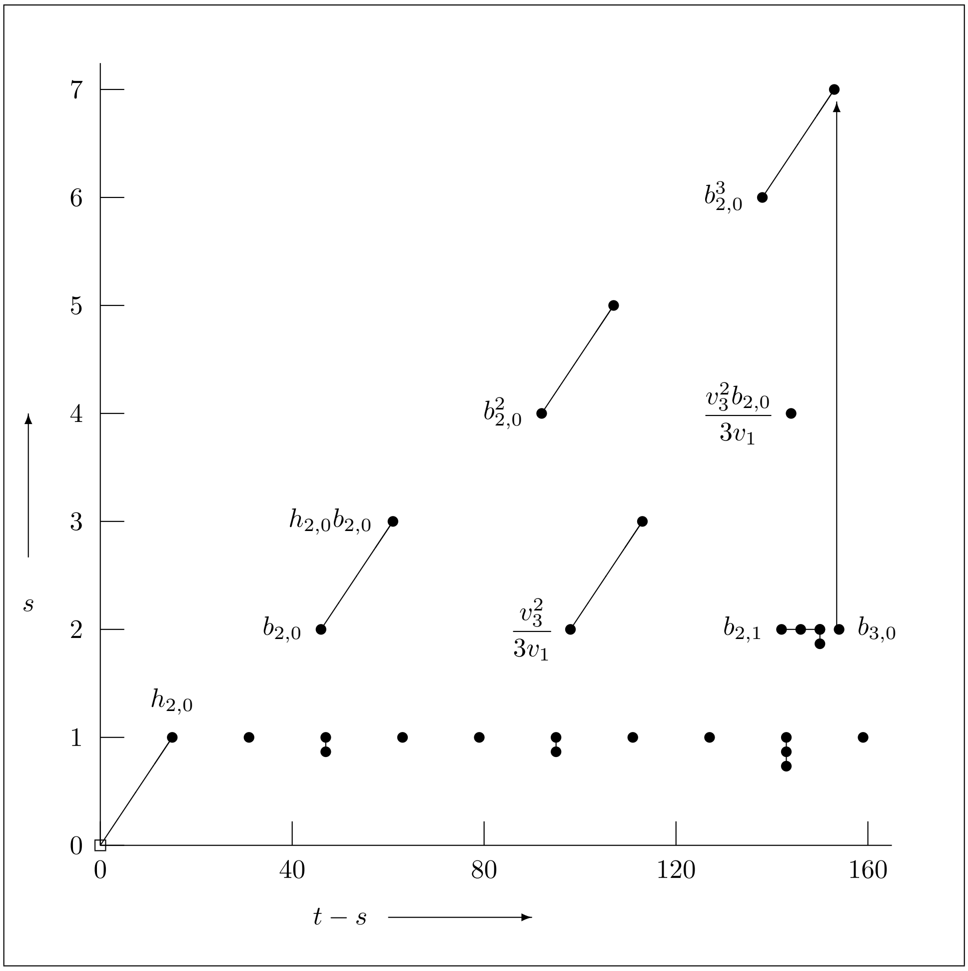

FIGURE 1. The Adams-Novikov E_2-term for $T(1)$ at $p = 3$ in dimensions ≤ 154, showing the first nontrivial differential. Elements on the 0- and 1-lines divisible by v_1 are not shown. Elements on the 2-line and above divisible by v_2 are not shown.

where $t_0 = 1$. These formulas determine the right unit and coproduct in $\Gamma \otimes \mathbf{Q}$, but are known to come from similar (but more complicated) ones in Γ itself. For more details see [**Rav86**, §4.3] or [**Ada74**, Part II].

The right unit formula can be rewritten as

$$\sum_{0 \leq j+k \leq i} \ell_{i-j-k} v_j^{p^{i-j-k}} t_k^{p^{i-k}} = \sum_{0 \leq j+k \leq i} \ell_{i-j-k} t_j^{p^{i-j-k}} \eta_R(v_k^{p^{i-k}}) \tag{2}$$

(where j and k are always nonnegative) for each $i \geq 0$, or equivalently

$$\sum_{i,j}{}^F v_i t_j^{p^i} = \sum_{i,j}{}^F t_i \eta_R(v_j)^{p^i}; \tag{3}$$

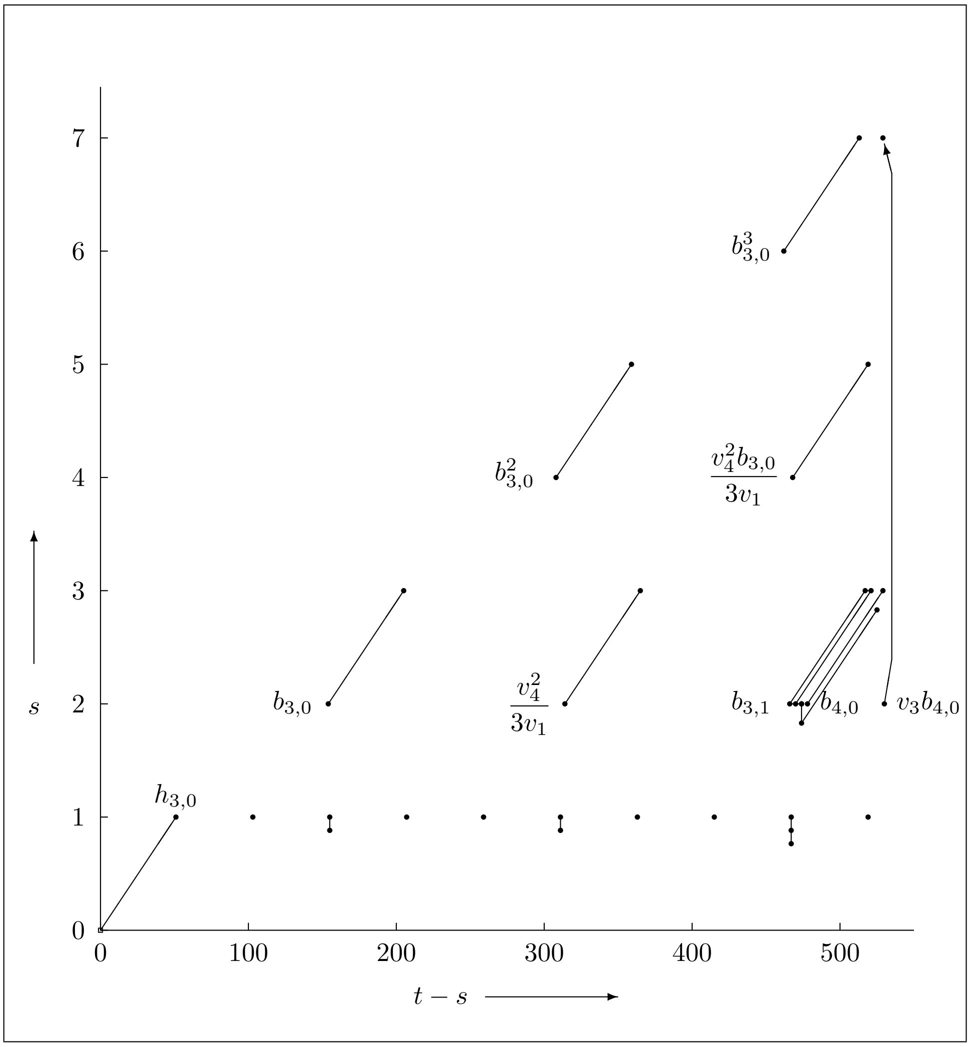

FIGURE 2. The Adams-Novikov E_2-term for $T(2)$ at $p = 3$ in dimensions ≤ 530. Elements on the 0- and 1-lines divisible by v_1 or v_2 are not shown. Elements on the 2-line and above divisible by v_2 or v_3 are not shown except for $v_3 b_{4,0}$ and $v_2 h_{3,0} b_{3,0}^3$, the source and target of the first differential.

see [**Rav86**, A2.2.5] or [**Rav76**]. The sums here are with respect to the formal group law F, i.e.,

$$x +_F y = F(x, y)$$

which is determined recursively by

$$\sum_{i \geq 0} \ell_i F(x, y)^{p^i} = \sum_{i \geq 0} \ell_i x^{p^i} + \sum_{i \geq 0} \ell_i y^{p^i}.$$

These formulas determine the structure of

$$\Gamma(m+1) = \Gamma/(t_1, \ldots, t_m).$$

The coporoduct and right unit are particularly simple on the generators t_{m+i} and v_{m+i} for $0 < i < m+2$. The coproduct formula in this range simplifies to

$$\sum_{0\le j<i} \ell_j \Delta(t_{m+i-j}^{p^j}) = \sum_{0\le j<i} \ell_j (t_{m+i-j}^{p^j} \otimes 1 + 1 \otimes t_{m+i-j}^{p^j}), \tag{4}$$

or equivalently

$$\sum_{0<i<m+2}{}^F \Delta(t_{m+i}) = \sum_{0<i<m+2}{}^F F(t_{m+i} \otimes 1, 1 \otimes t_{m+i}). \tag{5}$$

The right unit formula (2) when projected to $\Gamma(m+1)$ implies (by induction on i) that v_i for $i \le m$ has trivial right unit in $\Gamma(m+1)$, i.e., that

$$\eta_R(v_i) = v_i.$$

With this in mind we can rewrite (2) as

$$\begin{aligned} &\sum_{0\le j\le m+i} \ell_j v_{m+i-j}^{p^j} + \sum_{0\le j+k<i} \ell_j v_k^{p^j} t_{m+i-j-k}^{p^{j+k}} \\ &\qquad = \sum_{0\le j\le m+i} \ell_j \eta_R(v_{m+i-j}^{p^j}) + \sum_{0\le j+k<i} \ell_j t_{m+i-j-k}^{p^j} v_k^{p^{m+i-k}}, \end{aligned} \tag{6}$$

for $i \le m+1$, or equivalently in this range

$$\sum_{i>0}{}^F v_{m+i} + \sum_{i\ge 0, j>0}{}^F v_i t_{m+j}^{p^i} = \sum_{i>0}{}^F \eta_R(v_{m+i}) + \sum_{i>0, j\ge 0}{}^F t_{m+i} v_j^{p^{m+i}}. \tag{7}$$

We wish to study the "limiting behavior" as m approaches ∞; the precise nature of this limit will be discussed below.

Theorem 1. *There is a Hopf algebroid $(\widehat{A}, \widehat{\Gamma})$ over $\mathbf{Z}_{(p)}$, graded over $\mathbf{Z} \oplus \mathbf{Z}\omega$, with*

$$\begin{aligned} \widehat{A} &= BP_*[c_{i,m}, \widehat{v}_i : 0 \le i \le m]/(c_{i,m} - v_i^{(p-1)p^m} c_{i,m+1}) \\ &\qquad \text{with } v_0 = p,\ |c_{i,m}| = (\omega - p^m)|v_i|, \text{ and } |\widehat{v}_i| = 2p^i\omega - 2; \\ \widehat{\Gamma} &= \widehat{A}[\widehat{t}_i : i > 0] \qquad \text{with } |\widehat{t}_i| = 2p^i\omega - 2. \end{aligned}$$

(The notation for $\widehat{A}$ means that it includes elements $c_{i,m}$ for all $m \ge 0$ as well as the indicated values of i.)

The right unit on the elements v_i and $c_{i,m}$ are trivial (meaning that they are invariant) while the ones on the $\widehat{v}_i$ are given by

$$\sum_{i>0}{}^F \widehat{v}_i + \sum_{i\ge 0, j>0}{}^F v_i \widehat{t}_j^{p^i} = \sum_{i>0}{}^F \eta_R(\widehat{v}_i) + \sum_{i>0, j\ge 0}{}^F \widehat{t}_i v_j^{\omega p^i}. \tag{8}$$

The coproduct is given by

$$\sum_{i>0}{}^F \Delta(\widehat{t}_i) = \sum_{i>0}{}^F F(\widehat{t}_i \otimes 1, 1 \otimes \widehat{t}_i); \tag{9}$$

equivalently the element

$$\sum_{0\le j\le i} \ell_j \widehat{t}_{i-j}^{p^j} \in \mathbf{Q} \otimes \widehat{\Gamma}$$

is primitive for each $i > 0$.

Note that the coproduct in $\widehat{\Gamma}$ is cocommutative.

We will denote the element $v_i^{p^m} c_{i,m}$ by v_i^ω for $0 \leq i \leq m$. Because of the relations in $\widehat{A}$, this element is independent of m and is infinitely divisible by v_i. This includes the 0-dimensional element v_0^ω, which is infinitely divisible by p.

Let

$$V = BP_*[c_{i,m} : m \geq i \geq 0]/(c_{i,m} - v_i^{(p-1)p^m} c_{i,m+1}), \tag{10}$$

so that

$$\widehat{A} = V[\widehat{v}_i \colon i > 0].$$

and

$$BP_*[v_0^\omega, v_1^\omega, \dots] \subset V$$

with v_i^ω infinitely divisible by v_i in V. It follows that for $i < n$, $c_{i,m}$ and v_i^ω are trivial in V/I_n.

Since the right unit on V is trivial, $\widehat{\Gamma}$ is a Hopf algebroid over V. Similarly $\Gamma(m+1)$ is a Hopf algebroid over $A(m)$.

Proof of Theorem 1. We need to show that the right unit and coproduct satisfy the Hopf algebroid axioms (see [**Rav86**, A1.1.1]). The structure of $(\widehat{A}, \widehat{\Gamma})$ is obtained from that of $(A, \Gamma(m+1))$ in the following heuristic way. The elements $\widehat{v}_i$ and wt_i in the former correspond to v_{m+i} and and t_{m+i} for large m in the latter. Whenever the symbol p^m appears in the latter, either in an exponent or in the dimension of a generator, we replace it by the symbol ω. In this way (8) and (9) are derived from (7) and (5) respectively.

To verify that they satisfy the necessary axioms, it suffices to work in $\mathbf{Q} \otimes \widehat{\Gamma}$ since $\widehat{\Gamma}$ is torsion free. The coproduct there is coassociative because it is primitively generated.

To verify the coassociativity of the right unit, we will work in $K \otimes_{\mathbf{Z}_{(p)}[v_0^\omega]} \widehat{\Gamma}$ where

$$\begin{aligned} K &= \mathbf{Z}_{(p)}[v_0^\omega][(p - v_0^{\omega p^i})^{-1} \colon i > 0] \\ &= \mathbf{Q}[v_0^\omega][(1 - v_0^{\omega p^i - 1})^{-1} \colon i > 0] \end{aligned}$$

There we can define elements $\widehat{\ell}_i$ for $i > 0$ recursively by

$$p\widehat{\ell}_i = \sum_{0 < j \leq i} \widehat{\ell}_j v_{i-j}^{\omega p^j} + \sum_{0 \leq j < i} \ell_j \widehat{v}_{i-j}^{p^j}. \tag{11}$$

This gives

$$\widehat{\ell}_i \equiv \frac{\widehat{v}_i}{p - v_0^{\omega p^i}} \qquad \bmod (\widehat{v}_1, \dots, \widehat{v}_{i-1})$$

so

$$K \otimes \widehat{A} = K \otimes V[\widehat{\ell}_1, \widehat{\ell}_2, \dots],$$

and it suffices to show that the right unit on the $\widehat{\ell}_i$ is coassociative.

We can derive $\eta_R(\widehat{\ell}_i)$ from (11). In the following calculation, each expression is to be summed over all nonnegative values of the indices with the understanding that $\widehat{v}_0 = \widehat{\ell}_0 = \widehat{t}_0 = 0$. We have

$$\begin{aligned} p\eta_R(\widehat{\ell}_i) &= \eta_R(\widehat{\ell}_i)v_j^{\omega p^i} + \ell_i\eta_R(\widehat{v}_j^{p^i}) \\ &= \eta_R(\widehat{\ell}_i)v_j^{\omega p^i} + \ell_i\widehat{v}_j^{p^i} + \ell_i v_j^{p^i}\widehat{t}_k^{p^{i+j}} - \ell_i\widehat{t}_j^{p^i}v_k^{\omega p^{i+j}} \\ &= \eta_R(\widehat{\ell}_i)v_j^{\omega p^i} + \ell_i\widehat{v}_j^{p^i} + p\ell_i\widehat{t}_j^{p^i} - \ell_i\widehat{t}_j^{p^i}v_k^{\omega p^{i+j}} \\ &= \eta_R(\widehat{\ell}_i)v_j^{\omega p^i} + p\widehat{\ell}_i - \widehat{\ell}_i v_j^{\omega p^i} + p\ell_i\widehat{t}_j^{p^i} - \ell_i\widehat{t}_j^{p^i}v_k^{\omega p^{i+j}} \\ &= p(\widehat{\ell}_i + \ell_i\widehat{t}_j^{p^i}) + (\eta_R(\widehat{\ell}_i) - \widehat{\ell}_i - \ell_i\widehat{t}_j^{p^i})v_k^{\omega p^{i+j}}. \end{aligned}$$

Without the summation convention, this can be rewritten as

$$p\eta_R(\widehat{\ell}_i) = p\widehat{\ell}_i + \sum_{0\le j<i}\ell_j\widehat{t}_{i-j}^{p^j} + p\sum_{0<j<i}\left(\eta_R(\widehat{\ell}_j) - \widehat{\ell}_j - \sum_{0\le k<j}\ell_k\widehat{t}_{j-k}^{p^k}\right)v_{i-j}^{\omega p^j}$$

for each $i > 0$. Using induction on i one can deduce that the second sum vanishes, so

$$\eta_R(\widehat{\ell}_i) = \widehat{\ell}_i + \sum_{0\le j<i}\ell_j\widehat{t}_{i-j}^{p^j},$$

which is coassociative since $\eta_R(\widehat{\ell}_i) - \widehat{\ell}_i$ is primitive. □

4. Maps from subalgebras of $\widehat{\Gamma}$ to the $\Gamma(m+1)$

Now we will be more precise about the relation between $\widehat{\Gamma}$ and $\Gamma(m+1)$. There is no map from one to the other in either direction. There is a rather for each m a sub-Hopf algebroid of $(\widehat{A}, \widehat{\Gamma})$ that maps to $(A, \Gamma(m+1))$ (with a change of grading), and $(\widehat{A}, \widehat{\Gamma})$ itself is the union of all of these subobjects. *This is the sense in which* $\widehat{\Gamma}$ *is the limit of the* $\Gamma(m+1)$ *as* $m \to \infty$.

Specifically let

$$\begin{aligned} \left(\widehat{A}(m),\, \widehat{G}(1,m)\right) &\subset \left(\widehat{A},\, \widehat{\Gamma}\right) \\ \text{and} \quad \left(\widehat{A}(m+n)/I_n,\, \widehat{G}(1,m,n)\right) &\subset \left(\widehat{A}/I_n,\, \widehat{\Gamma}/I_n\right) \end{aligned}$$

for $m, n > 0$ by

$$\begin{aligned} \widehat{A}(m) &= \mathbf{Z}_{(p)}[v_1, \ldots, v_m; v_0^{\omega-p^m}, v_1^{\omega-p^m}, \ldots, v_m^{\omega-p^m}; \widehat{v}_1, \ldots, \widehat{v}_{m+1}] \\ \widehat{G}(1,m) &= \widehat{A}(m)[\widehat{t}_1, \ldots, \widehat{t}_{m+1}] \\ \widehat{A}(m,n) &= \mathbf{Z}_{(p)}[v_1, \ldots, v_{m+n}; v_0^{\omega-p^m}, v_1^{\omega-p^m}, \ldots, v_{m+n}^{\omega-p^m}; \widehat{v}_1, \ldots, \widehat{v}_{m+n+1}]/I_n \\ \widehat{G}(1,m,n) &= \widehat{A}(m,n)[\widehat{t}_1, \ldots, \widehat{t}_{m+1}]. \end{aligned}$$

Then the following is straightforward.

Proposition 2. *Let*

$$\begin{array}{rcl} A(k) & = & \mathbf{Z}_{(p)}[v_1, \dots, v_k], \\ G(m+1, m) & = & A(2m+1)[t_{m+1}, \dots, t_{2m+1}] \qquad \textit{as in } [\mathbf{Rav86}, §7.1], \\ \textit{and} \quad G(m+1, k, n) & = & A(m+1+k+n)/I_n[t_{m+1}, \dots, t_{m+1+k}]. \end{array}$$

There are maps

$$\widehat{G}(1, m) \xrightarrow{\theta_m} G(m+1, m) \quad \subset \quad \Gamma(m+1) \tag{12}$$

and

$$\widehat{G}(1, m, n) \xrightarrow{\theta_m} G(m+1, m, n) \quad \subset \quad \Gamma(m+1)/I_n \tag{13}$$

given by

$$\begin{array}{rcl} v_i & \mapsto & v_i \\ \widehat{v}_i & \mapsto & v_{i+m} \\ v_i^\omega & \mapsto & v_i^{p^m} \\ \widehat{t}_i & \mapsto & t_{i+m}. \end{array}$$

The indexing set $\mathbf{Z} \oplus \mathbf{Z}\omega$ *is mapped to* $\mathbf{Z}$ *by sending* ω *to* p^m.

Thus we have a diagram of Hopf algebroids

$$\begin{array}{ccccccc} \widehat{G}(1,0) & \longrightarrow & \widehat{G}(1,1) & \longrightarrow & \widehat{G}(1,2) & \longrightarrow \cdots \longrightarrow & \widehat{\Gamma} \\ \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \\ \Gamma(1) & & \Gamma(2) & & \Gamma(3) & & \end{array}$$

Remark 3. *For* $i \leq m+1$, *we have* $\widehat{\ell}_i \in K \otimes \widehat{A}(m)$ *as defined by (11). We can extend* θ_m *uniquely to* $K \otimes_{\mathbf{Z}_{(p)}[v_0^\omega]} \widehat{A}(m)$; *it sends* K *to* $\mathbf{Q}$. *Hence*

$$\theta_m(\widehat{\ell}_i) \in \mathbf{Q} \otimes A(2m+1)$$

satisfies

$$p\theta_m(\widehat{\ell}_i) = \sum_{0<j\leq i} \theta_m(\widehat{\ell}_j) v_{i-j}^{p^{j+m}} + \sum_{0\leq j<i} \ell_j v_{m+i-j}^{p^j},$$

so

$$\theta_m(\widehat{\ell}_i) \equiv \frac{v_{m+i}}{p - p^{i+m}} \qquad \bmod (v_1, \dots, v_{m+i-1}).$$

We also have

$$\eta_R(\theta_m(\widehat{\ell}_i)) = \theta_m(\widehat{\ell})_i) + \sum_{0\leq j<i} \ell_j t_{m+i-j}^{p^j}$$

so $\ell_{m+i} - \theta_m(\widehat{\ell})_i$ *is invariant. One can show that it is the sum of all terms in* ℓ_{m+i} *that are monomials in the* v_j *with* $1 \leq j \leq m$.

Then each element of $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\,\widehat{A})$ can be pulled back to $\mathrm{Ext}_{\widehat{G}(1,m)}(\widehat{A}(m),\,\widehat{A}(m))$ for $m \gg 0$, and hence mapped via θ_m to $\mathrm{Ext}_{\Gamma(m+1)}(A,\,A)$, which is the Adams–Novikov spectral sequence E_2-term for the spectrum $T(m)$.

Conjecture 4. *There is a spectral sequence with*

$$E_2 = \mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\,\widehat{A})$$

which is compatible in a range of dimensions with the Adams–Novikov spectral sequence for $T(m)$. We call this the **microstable** *Adams–Novikov spectral sequence.*

Remark 5. *The map θ_m is onto below dimension $|t_{2m+2}|$, and $T(m)$ is equivalent to BP below dimension $|t_{m+1}|$. We believe the behavior of the Adams–Novikov spectral sequence in this range is essentially isomorphic (up to regrading) to that of the Adams–Novikov spectral sequence for $T(m+1)$ between dimensions $|t_{m+2}|$ and $|t_{2m+3}|$. Theorem D is evidence that the behavior of differentials and group extensions in "low" dimensions is independent of m for m sufficiently large. It indicates that the first differential in this spectral sequence would be*

$$d_{2p-1}(\widehat{v}_1\widehat{b}_{2,0}) = v_2\widehat{h}_{1,0}\widehat{b}_{1,0}^p$$

and that there would be a group extension of the form

$$p\widehat{b}_{2,0} = v_2\widehat{b}_{1,0}^p.$$

This is the rationale for the conjecture.

5. The microchromatic spectral sequence

The chromatic spectral sequence converging to $\mathrm{Ext}_\Gamma(A,\,A)$ is obtained from the resolution

$$0 \to BP_* \to M^0 \to M^1 \to M^2 \to \dots$$

where

$$M^n = v_n^{-1}BP_*/(p^\infty, v_1^\infty, \dots, v_{n-1}^\infty).$$

More details can be found in [**Rav86**, Chapter 5].

We also define

$$M_i^{n-i} = v_n^{-1}BP_*/(p, \dots, v_{i-1}, v_i^\infty, \dots, v_{n-1}^\infty).$$

so for each $i > 0$ there is a resolution

$$0 \to BP_*/I_i \to M_i^0 \to M_i^1 \to M_i^2 \to \dots,$$

and there are short exact sequences

$$0 \longrightarrow M_{i+1}^{n-i-1} \longrightarrow \Sigma^{|v_i|}M_i^{n-i} \xrightarrow{v_i} M_i^{n-i} \longrightarrow 0$$

which lead to Bockstein spectral sequences. In particular there is a chain of n Bockstein spectral sequences leading from $\mathrm{Ext}_\Gamma(A,\,v_n^{-1}BP_*/I_n)$ to $\mathrm{Ext}_\Gamma(A,\,M^n)$. There is a change-of-rings isomorphism

$$\mathrm{Ext}_\Gamma(A,\,M^n) \cong \mathrm{Ext}_{\Sigma(n)}(K(n)_*,\,K(n)_*) \tag{14}$$

where

$$\begin{aligned} K(n)_* &= \mathrm{Ext}^0_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n) \\ &= \mathbf{Z}/(p)[v_n, v_n^{-1}] \\ \text{and} \qquad \Sigma(n) &= K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_* \\ &= K(n)_*[t_i : i > 0]/(v_n t_i^{p^n} - v_n^{p^i} t_i) \end{aligned}$$

as an algebra, with coproduct inherited from $BP_*(BP)$. The formula (3) is pivotal in the proof of this result. Details can be found in [**Rav86**, §6.1] or [**MR77**].

The comodule M_i^{n-i} be tensored over A with $\widehat{A}$, leading in the same way to a spectral sequence converging to $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ which we call the *microchromatic spectral sequence.* Let

$$\widehat{M}_i^{n-i} = M_i^{n-1} \otimes_A \widehat{A}$$

Then the microchromatic spectral sequence converging to $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A}, \widehat{A})$ is the resolution spectral sequence based on

$$0 \to \widehat{A} \to \widehat{M}^0 \to \widehat{M}^1 \to \widehat{M}^2 \to \cdots .$$

The microstable analog is of (14) is

Theorem 6. *There is a change-of-rings isomorphism*

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, v_n^{-1}\widehat{A}/I_n) = \mathrm{Ext}_{\widehat{\Sigma}(n)}(\widehat{K}(n)_*,\, \widehat{K}(n)_*)$$

where

$$\begin{aligned} \widehat{K}(n)_* &= \mathrm{Ext}^0_{\widehat{\Gamma}}(\widehat{A}, v_n^{-1}\widehat{A}/I_n) \\ &= v_n^{-1}V/I_n[\widehat{v}_1, \ldots, \widehat{v}_n] \\ &\qquad \text{where } V \text{ is as in (10)} \\ \textit{and} \qquad \widehat{\Sigma}(n) &= \widehat{K}(n)_* \otimes_{\widehat{A}} \widehat{\Gamma} \otimes_{\widehat{A}} \widehat{K}(n)_* \\ &= \widehat{K}(n)_*[\widehat{t}_i : i > 0]/(v_n \widehat{t}_i^{p^n} - v_n^{\omega p^i} \widehat{t}_i). \end{aligned}$$

Proof of Theorem 6. The change-of-rings-isomorphism theorem [**Rav86**, A1.3.12] says that given a Hopf algebroid map $f : (A, \Gamma) \to (B, \Sigma)$ satisfying certain conditions, one has

$$\mathrm{Ext}_\Gamma(A,\, (\Gamma \otimes_A B)\square_\Sigma B) \cong \mathrm{Ext}_\Sigma(B, B).$$

Applying this to the map

$$\left(\widehat{A}, \widehat{\Gamma}\right) \xrightarrow{\ f\ } \left(\widehat{K}(n)_*, \widehat{\Sigma}(n)\right) \tag{15}$$

we get

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, (\widehat{\Gamma} \otimes_{\widehat{A}} \widehat{K}(n)_*)\square_{\widehat{\Sigma}(n)}\widehat{K}(n)_*) \cong \mathrm{Ext}_{\widehat{\Sigma}(n)}(\widehat{k}(n)_*,\, \widehat{K}(n)_*).$$

Thus we have to verify that the map of (15) satisfies the relevant hypotheses and then identify $(\widehat{\Gamma} \otimes_{\widehat{A}} \widehat{K}(n)_*)\square_{\widehat{\Sigma}(n)}\widehat{K}(n)_*$ with $v_n^{-1}\widehat{A}/I_n$.

The hypotheses required of f are [**Rav86**, A1.1.19]

(i) the induced map $\widehat{\Gamma} \otimes_{\widehat{A}} B \to \widehat{\Sigma}(n)$ is onto, and

(ii) $(\widehat{\Gamma}\otimes_{\widehat{A}}\widehat{K}(n)_*)\Box_{\widehat{\Sigma}(n)}\widehat{K}(n)_*$ is a $\widehat{K}(n)_*$-module and a $\widehat{K}(n)_*$-summand of

$$\widehat{\Gamma}\otimes_{\widehat{A}}\widehat{K}(n)_*.$$

The first of these follows from the definition of $\widehat{\Sigma}(n)$. For the second we have

$$\begin{aligned}\widehat{\Gamma}\otimes_{\widehat{A}}\widehat{K}(n)_* &\cong \widehat{K}(n)_*[\widehat{t}_i \colon i>0],\\ (\widehat{\Gamma}\otimes_{\widehat{A}}\widehat{K}(n)_*)\Box_{\widehat{\Sigma}(n)}\widehat{K}(n)_* &\cong \widehat{K}(n)_*[v_n\widehat{t}_i^{p^n}-v_n^{\omega p^i}\widehat{t}_i \colon i>0],\end{aligned}$$

and the latter is a $\widehat{K}(n)_*$-summand of the former.

Finally we have

$$v_n^{-1}\widehat{A}/I_n\cong\widehat{K}(n)_*[\widehat{v}_{n+i}\colon i>0]$$

and there is a $\widehat{\Sigma}(n)$-comodule isomorphism

$$\begin{aligned} v_n^{-1}\widehat{A}/I_n &\to (\widehat{\Gamma}\otimes_{\widehat{A}}\widehat{K}(n)_*)\Box_{\widehat{\Sigma}(n)}\widehat{K}(n)_*\\ \text{defined by}\qquad \widehat{v}_{n+i} &\mapsto v_n\widehat{t}_i^{p^n}-v_n^{\omega p^i}\widehat{t}_i. \quad\Box\end{aligned}$$

THEOREM 7. *The Ext group of Theorem 6 is*

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, v_n^{-1}\widehat{A}/I_n)=\widehat{K}(n)_*\otimes E(\widehat{h}_{i,j}:0<i\le n,\ j\in\mathbf{Z}/(n))$$

where $\widehat{h}_{i,j}$ corresponds to $\widehat{t}_i^{p^j}$.

It is also true [**Rav86**, 6.5.6] that

$$\mathrm{Ext}_{\Gamma(m+1)}(A,\, v_n^{-1}A/I_n)\cong K(n)_*[v_{n+1},\dots,v_{2n}]\otimes E(h_{i+m,j}:0<i\le n,\ j\in\mathbf{Z}/(n))$$

for $m+1>\frac{pn}{2(p-1)}$ (but not for smaller values of m), where $h_{i+m,j}$ corresponds to $t_{i+m}^{p^j}$. Thus the microchromatic spectral sequence is simpler than the chromatic spectral sequence for the sphere spectrum.

Proof of Theorem 7. We mimic the methods of [**Rav86**, §6.3] and [**Rav77**]. As in [**Rav86**, 6.3.1] we can define an increasing filtration on $\widehat{\Sigma}(n)$ with

$$||\widehat{t}_i^{p^j}||=\begin{cases} i & \text{if } i\le n\\ p||\widehat{t}_{i-n}^{p^j}|| & \text{if } i>n.\end{cases}$$

Then $E^0\widehat{\Sigma}(n)$ is the universal enveloping algebra of a restricted abelian Lie algebra $\widehat{L}(n)$ over $\widehat{K}(n)_*$ with basis $\{x_{i,j}: i>0, j\in\mathbf{Z}/(n)\}$ and restriction given by

$$\xi(x_{i,j})=\begin{cases} 0 & \text{if } i\le n\\ -v_nx_{i-n,j+1} & \text{otherwise.}\end{cases}$$

Then as in [**Rav86**, 6.3.4] we have two spectral sequences. The first is

$$\begin{aligned} E_2 &= H^*(\widehat{L}(n))\otimes P(b_{i,j})\\ &= \widehat{K}(n)_*\otimes E(h_{i,j})\otimes P(b_{i,j}) \implies H^*(E_0\widehat{\Sigma}(n)).\end{aligned}$$

with differentials

$$h_{i,j}\mapsto -v_nb_{i-n,j+1},$$

leaving

$$E_\infty=\widehat{K}(n)_*\otimes E(\widehat{h}_{i,j}:0<i\le n, j\in\mathbf{Z}/(n)).$$

The second spectral sequence is

$$E_2 = H^*(E^0\widehat{\Sigma}(n)) \implies H^*(\widehat{\Sigma}(n)).$$

It collapses from E_2 since each $\widehat{t}_i^{p^j}$ with $i \le n$ is primitive. □

6. The microstable 0- and 1-lines

We can use the microchromatic spectral sequence to compute $\mathrm{Ext}^s_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ for $s = 0$ and 1 in the same way that we use the chromatic spectral sequence to compute $\mathrm{Ext}^s_{\Gamma}(A,\, A)$. The following can proved in the same way as [**Rav86**, 5.2.1].

THEOREM 8.

$$\begin{aligned} \mathrm{Ext}^s_{\widehat{\Gamma}}(\widehat{A}, \widehat{M}^0) &= \begin{cases} \mathbf{Q} \otimes V & \textit{if } s = 0 \\ 0 & \textit{otherwise} \end{cases} \\ \mathrm{Ext}^0_{\widehat{\Gamma}}(\widehat{A}, \widehat{A}) &= V. \end{aligned}$$

THEOREM 9. $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A}, \widehat{A})$ *is the* V*-module generated by the set*

$$\left\{ \frac{\widehat{v}_1^i}{ip} : i > 0 \right\}$$

PROOF. We need to analyze the Bockstein spectral sequence going from

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{M}^0_1) = \widehat{K}(1)_* \otimes E(\widehat{h}_{1,0})$$

to $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{M}^1_0)$. This behaves in much the same way as the stable analog, i.e., the one going from

$$\mathrm{Ext}_{\Gamma}(A,\, M^0_1) = K(1)_* \otimes E(h_{1,0})$$

to $\mathrm{Ext}_{\Gamma}(A,\, M^1_0)$.

For odd primes the relevant fact about the right unit is that for all $i > 0$,

$$\eta_R(\widehat{v}_1^i) \equiv \widehat{v}_1^i + pi\widehat{v}_1^{i-1}\widehat{t}_1 \qquad \mathrm{mod}\ (p^2 i).$$

From this we deduce that $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ is the V-module generated by the set

$$\left\{ \frac{\widehat{v}_1^i}{pt} : i > 0 \right\}.$$

For $p = 2$ let

$$w_{1,1} = \widehat{v}_1^2 + 2v_1^{2\omega-1}\widehat{v}_1 + 4v_1^{-1}\widehat{v}_2.$$

Then for all $j > 0$ we have

$$\begin{aligned} \eta_R(\widehat{v}_1^{2j-1}) &\equiv \widehat{v}_1^{2j-1} + 2\widehat{v}_1^{2j-2}\widehat{t}_1 \quad \mathrm{mod}(4) \\ \text{and} \qquad \eta_R(w_{1,1}^s) &\equiv w_{1,1}^j + 4j\widehat{v}_1^{2j-1}\widehat{t}_1 \quad \mathrm{mod}(8j). \end{aligned}$$

From this we deduce that $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ is the V-module generated by the set

$$\left\{ \frac{\widehat{v}_1^{2j-1}}{2}, \frac{w_{1,1}^j}{4j} : j > 0 \right\}. \tag{16}$$

Now a simple calculation shows that

$$\frac{w_{1,1}^{s}}{2j} = \begin{cases} \dfrac{\widehat{v}_1^{2j}}{4} + \dfrac{v_1^{2\omega-1}\widehat{v}_1^{2j-1}}{2} & \text{for } j \text{ odd} \\[2ex] \dfrac{\widehat{v}_1^{2j}}{4j} + \dfrac{v_1^{2\omega-1}\widehat{v}_1^{2j-1}}{2} + \dfrac{v_1^{4\omega-2}\widehat{v}_1^{2j-2}}{2} & \text{for } j \text{ even,} \end{cases}$$

so the V-module of (16) is the same as the one stated in the theorem. □

For all primes the calculation above also shows that

$$\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{M_0^1}) = 0,$$

unlike the stable case where $\mathrm{Ext}^1_{\Gamma}(A,\, M_0^1) \supset \mathbf{Q}/\mathbf{Z}$.

Note that for odd primes each element in Ext^1 can be pulled back to

$$\mathrm{Ext}_{\widehat{G}(1,0)}\left(\widehat{A}(0),\, \widehat{A}(0)\right),$$

so we can map them via the map θ_m of (12) to

$$\mathrm{Ext}^1_{\Gamma(m)}(A,\, A)$$

for $m \geq 0$. For $p = 2$ we can only do this for $m \geq 1$. This is to be expected since the structure of $\mathrm{Ext}^1_{\Gamma(1)}(A,\, A)$ for $p = 2$ differs from that of $\mathrm{Ext}^1_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ in that for $j > 1$, $\frac{v_1^{2j}}{2}$ is divisible by $4j$ while $\frac{\widehat{v}_1^{2j}}{2}$ is only divisible by $2j$.

7. The Thom reduction

One can ask about the image of $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A})$ in $\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A}/I)$, where $I = (p, v_1, v_2, \dots)$, since the latter can be computed explicitly. Each $\widehat{t}_i$ is primitive mod I, so we have

$$\mathrm{Ext}_{\widehat{\Gamma}}(\widehat{A},\, \widehat{A}/I) \quad = \quad \widehat{A}/I \otimes E(\widehat{h}_{i,j} : i > 0, j \geq 0) \otimes P(\widehat{b}_{i,j} : i > 0, j \geq 0)$$

where $\widehat{h}_{i,j} \in \mathrm{Ext}^{1,2p^j(p^i\omega-1)}$ corresponds to $\widehat{t}_i^{p^j}$, and $\widehat{b}_{i,j} \in \mathrm{Ext}^{2,2p^{j+1}(p^i\omega-1)}$ is its transpotent.

Let ρ denote the mod I reduction in Ext. Then we have

$$\begin{aligned}
\rho\left(\frac{\widehat{v}_1^t}{pt}\right) &= \begin{cases} \widehat{v}_1^{t-1}\widehat{h}_{1,0} & \text{for } p \text{ odd}\\ \widehat{v}_1^{t-1}\widehat{h}_{1,0} + (t-1)\widehat{v}_1^{t-2}\widehat{h}_{1,1} & \text{for } p=2.\end{cases}\\
\rho\left(\frac{\widehat{v}_1^s\widehat{v}_2^t}{pv_1}\right) &= st\widehat{v}_1^{s-1}\widehat{v}_2^{t-1}\widehat{h}_{1,1}\widehat{h}_{1,0} + t\widehat{v}_1^s\widehat{v}_2^{t-1}\widehat{b}_{1,0}\\
&\quad + t(t-1)\widehat{v}_1^s\widehat{v}_2^{t-2}\widehat{h}_{1,1}\widehat{h}_{2,0}\\
\rho\left(\frac{\widehat{v}_1^s\widehat{v}_2^{p^jt}}{pv_1^{p^j}}\right) &= st\widehat{v}_1^{s-1}\widehat{v}_2^{(t-1)p^j}\widehat{h}_{1,j+1}\widehat{h}_{1,0}\\
&\quad + t\widehat{v}_1^s\widehat{v}_2^{p^j(t-1)}\widehat{b}_{1,j} \qquad \text{for } j>0\\
\rho\left(\frac{\widehat{v}_3^t}{pv_1v_2}\right) &= t(t-1)\widehat{v}_3^{t-2}(\widehat{h}_{1,2}\widehat{b}_{2,0} - \widehat{h}_{2,1}\widehat{b}_{1,1})\\
&\quad + t(t-1)(t-2)\widehat{v}_3^{t-3}\widehat{h}_{1,2}\widehat{h}_{2,1}\widehat{h}_{3,0}
\end{aligned}$$

Hence the image appears to be rather complicated.

On the other hand, it appears likely that all of the $\widehat{b}_{i,j}$ are in the image. Given $x \in BP_*[\widehat{t}_1,\dots] \otimes \mathbf{Q}$, let $x^{(j)}$ denote the expression obtained from x by replacing each v_k and $\widehat{t}_k$ by its p^jth power. Using chromatic notation, we conjecture that

$$A_{i,j} = \sum_{0\le k<i} \frac{(p^{i-1}\ell_k\widehat{t}_{i-k}^{p^k})^{(j+1)}}{p^i}$$

is a cocycle that reduces to $\widehat{b}_{i,j}$ mod I. For example we have

$$A_{1,j} = \frac{\widehat{t}_1^{p^{j+1}}}{p}$$

which is cohomologous to

$$\sum_{0<k<p^{j+1}} p^{-1}\binom{p^{j+1}}{k}\widehat{t}_1^k \otimes \widehat{t}_1^{p^{j+1}-k} \equiv \sum_{0<k<p} p^{-1}\binom{p}{k}\widehat{t}_1^{kp^j} \otimes \widehat{t}_1^{(p-k)p^j} \bmod (p),$$

which is the usual definition of $\widehat{b}_{1,j}$.

Next we consider $A_{2,j}$. Araki's definition of the v_i gives

$$v_1 \equiv p\ell_1 \bmod (p^2),$$

so the primitivity of $\widehat{t}_2 + \ell_1\widehat{t}_1^p$ implies that the coproduct on $\widehat{t}_2$ is congruent to

$$\widehat{t}_2 \otimes 1 + 1 \otimes \widehat{t}_2 - v_1 \sum_{0<k<p} p^{-1}\binom{p}{k}\widehat{t}_1^k \otimes \widehat{t}_1^{p-k}$$

modulo p. Now let d denote the differential in the cobar complex we have

$$\begin{aligned} d(\widehat{t}_2) &\equiv -v_1 \sum_{0<k<p} p^{-1}\binom{p}{k}\widehat{t}_1^k \otimes \widehat{t}_1^{p-k} \qquad \bmod (p) \\ \text{so} \qquad d(\widehat{t}_2^{p^{j+1}}) &\equiv -v_1^{p^{j+1}} \sum_{0<k<p} p^{-1}\binom{p}{k}\widehat{t}_1^{kp^{j+1}} \otimes \widehat{t}_1^{(p-k)p^{j+1}} \\ \text{and} \qquad d(p\widehat{t}_2^{p^{j+1}} + v_1^{p^{j+1}}\widehat{t}_1^{p^{j+2}}) &\equiv 0 \qquad \bmod (p^2). \end{aligned}$$

It follows that

$$A_{2,j} = \frac{p\widehat{t}_2^{p^{j+1}} + v_1^{p^{j+1}}\widehat{t}_1^{p^{j+2}}}{p^2}$$

is a cocycle, and it is easily seen that it is cohomologous to

$$\sum_{0<k<p} p^{-1}\binom{p}{k}\widehat{t}_2^{kp^j} \otimes \widehat{t}_2^{(p-k)p^j}$$

modulo (p, v_1).

References

[Ada74] J. F. Adams. *Stable Homotopy and Generalised Homology.* University of Chicago Press, Chicago, 1974.

[DHS88] E. Devinatz, M. J. Hopkins, and J. H. Smith. Nilpotence and stable homotopy theory. *Annals of Mathematics*, 128:207–242, 1988.

[MR77] H. R. Miller and D. C. Ravenel. Morava stabilizer algebras and the localization of Novikov's E_2-term. *Duke Mathematical Journal*, 44:433–447, 1977.

[MW76] H. R. Miller and W. S. Wilson. On Novikov's Ext^1 modulo an invariant prime ideal. *Topology*, 15:131–141, 1976.

[Rava] D. C. Ravenel. The first differential in the Adams-Novikov spectral sequencefor the spectrum $T(m)$. `http://www.math.rochester.edu:8080/u/drav/preprints.html`.

[Ravb] D. C. Ravenel. The method of infinite descent in stable homotopy theory. `http://www.math.rochester.edu:8080/u/drav/preprints.html`.

[Rav76] D. C. Ravenel. The structure of BP_*BP modulo an invariant prime ideal. *Topology*, 15:149–153, 1976.

[Rav77] D. C. Ravenel. The cohomology of Morava stabilizer algebras. *Mathematische Zeitschrift*, 152:287–297, 1977.

[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, New York, 1986.

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