A solution to the Arf-Kervaire invariant problem

Mike Hill and Doug Ravenel

(joint work with Mike Hopkins)

For more information on this topic, including links to our preprint and detailed notes for our talks, we refer the reader to the second author's website

`http://www.math.rochester.edu/u/faculty/doug/kervaire.html`

We gave a series of four lectures, the first by the second author and the rest by the first author. The latter was followed by a 2 hour question and answer session.

The problem in question is nearly 50 years old and began with Kervaire's paper [Ker60] of 1960 in which he defined a $\mathbf{Z}/2$-valued invariant $\phi(M)$ on certain manifolds M of dimension $4m+2$. He showed that for $m=2$ and M aclosed smooth manifolds, it must vanish. He also constructed a topological 10-manifold M on which it is nontrivial. This was one of the earliest examples of a nonsmoothable manifold. Milnor's paper on exotic 7-spheres [Mil56] had appeared four years earlier. In their subsequent joint work [KM63] they gave a complete classification of exotic spheres in dimensions ≥ 5 in terms of the stable homotopy groups of spheres, modulo a question about manifolds which they left unanswered:

> For which m is there a smooth framed manifold of dimension $4m+2$ with nontrivial Kervaire invariant?

Such manifolds were known to exist for $m=0$, 1 and 3, and Kervaire had shown there are none for $m=2$. A pivotal step in answering the question was the following result of Browder [Bro69] published in 1969.

Browder's Theorem. *The Kervaire invariant $\phi(M)$ of a smooth framed manifold M of dimension $4m+2$ is trivial unless $m=2^{j-1}-1$ for some $j>0$. In that case such an M with $\phi(M)\neq 0$ exists if and only if the element h_j^2 in the Adams spectral sequence is a permanent cycle.*

The Adams spectral sequence referred to in the theorem was first introduced in [Ada58], and we refer the reader to [Rav04] for more information. The relation between framed manifolds and stable homotopy groups of spheres had been established decades earlier by Pontryagin.

This result raised the stakes considerably and brought the problem into the realm of stable homotopy thoery. The name θ_j was given to the hypothetical element in the stable homotopy group $\pi_{2^{j+1}-2}(S^0)$ representing the permanent cycle h_j^2. It was known to exist for $j=1$, 2 and 3. In the next few years its existence was established for $j=4$ ([BMT70] and [Jon78]) and $j=5$ [BJM84]. It was widely believed that such framed manifolds existed for *all* values of j. In the ensuing decade there were many unsuccessful (and unpublished) attempts to construct them. *We now know that they were trying to prove the wrong theorem.*

In [Mah67] Mahowald described a beautiful pattern in the unstable homotopy groups of spheres based on the assumption that the θ_j exist for all j. It was so compelling that the possibility that they did not all exist was later called the Doomsday Hypothesis. After 1985 the problem faded into the background because it was thought to be inaccessible. In early 2009 Snaith published a book [Sna09] on it "to stem the tide of oblivion."

Soon after we announced the following.

Main Theorem. *The element $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ (representing h_j^2 in the Adams spectral sequence and corresponding to a framed manifold in the same dimension with nontrivial Kervaire invariant) does not exist for $j \geq 7$.*

Our proof relies heavily on equivariant stable homotopy theory and complex cobordism theory. Neither was available in the 1970s. Here is our strategy.

We construct a nonconnective ring spectrum Ω with the following properties:

(i) **Detection Theorem.** *If θ_j exists, its composition with the unit map $S^0 \to \Omega$ is nontrivial.*

(ii) **Periodicity Theorem.** *$\pi_k\Omega$ depends only on k modulo 256.*

(iii) **Gap Theorem.** $\pi_{-2}\Omega = 0$.

Note that (ii) and (iii) imply that $\pi_{254}\Omega = 0$, and 254 is the dimension of θ_7. But (i) says that if θ_7 exists it has nontrivial image in this group, so it cannot exist. The argument for larger j is similar.

Our spectrum Ω is the fixed point set of a C_8-equivariant spectrum $\tilde{\Omega}$, i.e., $\Omega = \tilde{\Omega}^{C_8}$. We will describe $\tilde{\Omega}$ below. It also has a homotopy fixed point $\tilde{\Omega}^{hC_8}$. We show that it has properties (i) and (ii), while the actual fixed point set satisfies (iii). Thus we need a fourth result,

(iv) **Fixed Point Theorem.** *The map $\tilde{\Omega}^{C_8} \to \tilde{\Omega}^{hC_8}$ is an equivalence.*

The starting point for constructing $\tilde{\Omega}$ is the observation, originally due to Landweber [Lan68], that the complex cobordism spectrum MU has a C_2-equivariant structure defined in terms of complex conjugation. Recall that MU is defined in terms of Thom spaces of certain complex vector bundles over complex Grassmannians. Complex conjugation acts on everything in sight and commutes with the relevant structure maps. The resulting equivariant spectrum is known as real cobordism theory and is denoted by $MU_{\mathbf{R}}$.

Next there is a formal construction which we call the norm for inducing up from an H-equivariant spectrum X to form a G-equivariant spectrum $N_H^G X$ for any finite group G containing H. The underlying spectrum (meaning the one we get by forgetting the equivariant structure) of $N_H^G X$ is the $|G/H|$-fold smash power of X. G then acts by permuting the factors, each of which is invariant under H. The case of interest to us is $H = C_2$, $X = MU_{\mathbf{R}}$ and $G = C_8$. The underlying spectrum of $N_H^G MU_{\mathbf{R}}$ is $MU^{(4)}$, the 4-fold smash power of MU.

Let V be a real representation of G and let S^V be its one point compactification. For a G-equivariant space or spectrum X we denote the group of equivariant maps from S^V to X by $\pi_V^G X$. In this way a G-equivariant spectrum X has homotopy groups indexed by $RO(G)$, the real representation ring of G. These are denoted collectively by $\pi_\star^G X$.

We can now describe our C_8-equivariant spectrum $\tilde{\Omega}$. We choose a certain element $D \in \pi_{19\rho}^{C_8} MU_{\mathbf{R}}^{(4)}$, where ρ denotes the real regular representation of C_8.

There are many choices of D that would lead to Periodicity (possible with periods other than 256) and Gap Theorems. Ours is the simplest one that also gives the Detection Threorem. Since $MU_{\mathbf{R}}^{(4)}$ is a ring spectrum, we get a map

$$MU_{\mathbf{R}}^{(4)} \xrightarrow{\quad D \quad} \Sigma^{-19\rho} MU_{\mathbf{R}}^{(4)}.$$

This can be iterated, and we define $\tilde{\Omega}$ to be the resulting telescope,

$$\tilde{\Omega} = D^{-1} MU_{\mathbf{R}}^{(4)}.$$

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