A solution to the
Arf-Kervaire invariant
problem II

Middle East
Technical University
Topology Seminar

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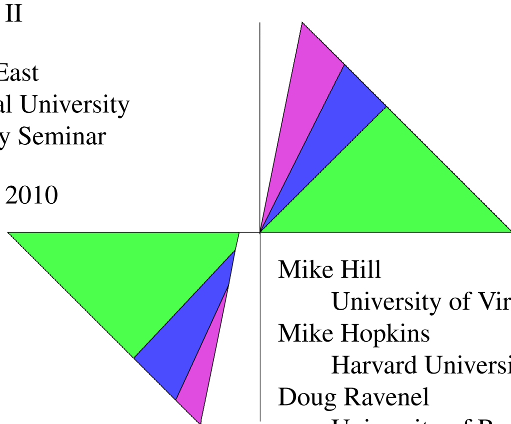

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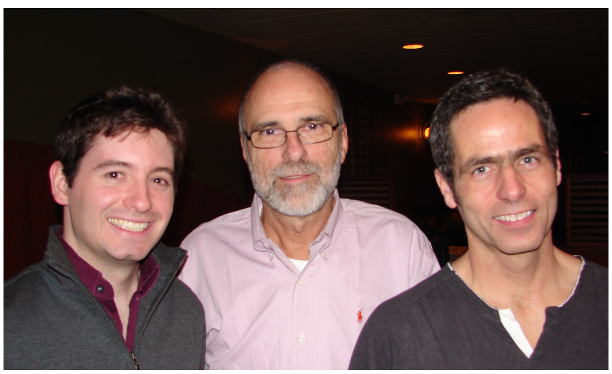

Mike Hill, myself and Mike Hopkins
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February 11, 2010 2.2

1 Our strategy

1.1 The main theorem

The main theorem

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2+n}(S^n)$ *for large n do not exist for* $j \geq 7$. 2.3

The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It has long been known that such things can exist only in dimensions that are 2 less than a power of 2.

θ_j is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62.

Our theorem says θ_j does not exist for $j \geq 7$.

The case $j = 6$ is still open.

1.2 The spectrum Ω

The spectrum Ω

We will produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. This means that if θ_j exists, we will see its image in $\pi_*(\Omega)$.
(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.
(iii) Gap Theorem. $\pi_{-2}(\Omega) = 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.

2.4

The spectrum Ω (continued)

Here again are the properties of Ω

(i) Detection Theorem. If θ_j exists, it has nontrivial image in $\pi_*(\Omega)$.
(ii) Periodicity Theorem. $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.
(iii) Gap Theorem. $\pi_{-2}(\Omega) = 0$.

(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

Ω will be the fixed point set associated with a C_8-equivariant spectrum $\tilde{\Omega}$ related to the complex cobordism spectrum. As we will explain below, a G-equivariant spectrum is more than just a spectrum with a G-action.

2.5

1.3 How we construct Ω

How we construct Ω

Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum MU. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The notation $MU_{\mathbf{R}}$ (real complex cobordism) is used to denote MU regarded as a C_2-spectrum.

To get a C_8-spectrum, we use the following general construction for getting from a space X acted on by a group H to one acted on by a larger group G containing H as a subgroup. Let

$$Y = \mathrm{Map}_H(G, X),$$

the space of H-equivariant maps from G to X. Here the action of H on G is by right multiplication, and the resulting object has an action of G by left multiplication. As a space, $Y = X^{|G/H|}$, the $|G/H|$-fold Cartesian power of X. A general element of G permutes these factors, each of which is left invariant by the subgroup H.

2.6

How we construct Ω (continued)

There is an analogous construction for an H-spectrum X, and the result is denoted by $N_H^G X$, the norm of X along the inclusion $H \subset G$. Its underlying spectrum is $X^{(|G/H|)}$. When G and H are cyclic, we denote their orders by g and h, and the norm functor by N_h^g.

In particular we get a C_8-spectrum

$$MU_{\mathbf{R}}^{(4)} = N_2^8 MU_{\mathbf{R}}.$$

This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is. 2.7

2 Spectra and equivariant spectra

2.1 Ordinary spectra

Ordinary spectra

In order to construct the slice spectral sequence, we need some notions from equivariant stable homotopy theory. Before describing them it will be useful to recall some notions from ordinary stable homotopy theory.

A prespectrum D is a collection of spaces D_n with maps $\Sigma D_n \to D_{n+1}$. The adjoint of the structure map is a map $D_n \to \Omega D_{n+1}$. Here ΩX denotes the loop space of X, the space of base point preserving maps from the circle S^1 to X.

We get a spectrum E from the prespectrum D by defining

$$E_n = \varinjlim_{k} \Omega^k D_{n+k}$$

This makes E_n homeomorphic to ΩE_{n+1}. Here $\Omega^k X$ denotes the k-fold loop space of X, the space of base point preserving maps from S^k to X. 2.8

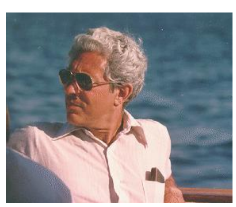
Ordinary spectra (continued)

Example 1. For a space X, let $D_n = \Sigma^n X$ with the obvious maps. The resulting spectrum, $\Sigma^\infty X$, is called the suspension spectrum of X .

Example 2. For an abelian group A, let D_n be the Eilenberg-Mac Lane space $K(A,n)$ with the obvious maps. The resulting spectrum, HA, is called the Eilenberg-Mac Lane spectrum for A. 2.9

Ordinary spectra (continued)

For technical reasons it is convenient to replace the collection $\{E_n\}$ by a collection $\{E_V\}$ indexed by finite dimensional subspaces V of a countably infinite dimensional real Euclidean space $\mathscr{U}$ called a universe. This theory is due to Peter May.



The homotopy type of E_V depends only on the dimension of V and there are homeomorphisms

$$E_V \to \Omega^{|W|-|V|} E_W \qquad \text{for } V \subset W \subset \mathscr{U}.$$

A map of spectra $f : E \to E'$ is a collection of maps of based spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps. 2.10

2.2 Equivariant spectra

Equivariant spectra

Let G be a finite group. Experience has shown that in order to do equivariant stable homotopy theory, one needs G-spaces E_V indexed by finite dimensional orthogonal representations V sitting in a countably infinite dimensional orthogonal representation $\mathscr{U}$, the direct sum of countably many copies of the regular real representation of G, which we denote by ρ_G. 2.11

G-equivariant spectra (continued)

A G-equivariant spectrum (G-spectrum for short) consists of a based G-space E_V for each finite dimensional invariant subspace $V \subset \mathscr{U}$, together with a transitive system of based G-homeomorphisms

$$E_V \xrightarrow[\cong]{\tilde{\sigma}_{V,W}} \Omega^{W-V} E_W$$

for $V \subset W \subset \mathscr{U}$. Here $\Omega^V X = F(S^V, X)$, the G-space of maps to X from S^V, the one point compactification of V. $W - V$ denotes the orthogonal complement of V in W. As in the classical case, the G-homotopy type of E_V depends only on the isomorphism class of V. 2.12

G-equivariant spectra (continued)

A map of G-spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps.

Dropping the requirement that the structure maps be homeomorphisms gives us a G-prespectrum as in the ordinary case.

The structure map $\tilde{\sigma}_{V,W}$ is adjoint to a map

$$\sigma_{V,W} : \Sigma^{W-V} E_V \to E_W,$$

where $\Sigma^V X$ is defined to be $S^V \wedge X$.

A suspension G-prespectrum is a G-prespectrum in which the maps above are G-equivalences for V sufficiently large. 2.13

2.3 $RO(G)$-graded homotopy groups

$RO(G)$-graded homotopy groups

Given a representation V one has a suspension G-spectrum $\Sigma^\infty S^V$, which is often denoted abusively (as in the nonequivariant case) by S^V.

We define S^{-V} by saying its Wth space for $V \subset W$ is S^{W-V}. This is the analog of formal desuspension in the nonequivariant case. 2.14

$RO(G)$-graded homotopy groups (continued)

Given a virtual representation $W = V' - V$, we define $S^W = \Sigma^{V'} S^{-V}$. Hence we have a collection of sphere spectra graded over the orthogonal representation ring $RO(G)$.

We define

$$\pi_W^G(X) = [S^W, X]_G,$$

the group of G-equivariant homotopy classes of maps from S^W to X. These are the $RO(G)$-graded homotopy groups of the G-spectrum X, denoted by $\pi_\star^G(X)$.

For an integer n,

$$\pi_n^G(X) = [S^n, X]_G = [S^n, X^G] = \pi_n(X^G),$$

the ordinary nth homotopy group of the fixed point spectrum X^G. 2.15

3 The slice spectral sequence

3.1 Postnikov towers

The classical Postnikov tower

The slice spectral sequence is based an equivariant analog of the Postnikov tower. First we need to recall some things about the classical Postnikov tower.

The nth Postnikov section P^nX of a space or spectrum X is obtained from X by attaching cells to kill all homotopy groups of X above dimension n. The fiber of the map $X \to P^nX$ is P_nX, the n-connected cover of X.

These two functors have some universal properties. Let $\mathscr{S}$ and $\mathscr{S}_{>n}$ denote the categories of spectra and n-connected spectra.

Then the functor $P_n : \mathscr{S} \to \mathscr{S}$ satisfies

- For all spectra X, $P_nX \in \mathscr{S}_{>n}$.
- For all $A \in \mathscr{S}_{>n}$ and $X \in \mathscr{S}$, the map of function spectra $\mathscr{S}(A, P_nX) \to \mathscr{S}(A, X)$ is a weak equivalence.

In other words, the map $P_nX \to X$ is universal among maps from n-connected spectra to X. 2.16

The classical Postnikov tower (continued)

Similarly the map $X \to P^nX$ is universal among maps from X to spectra which are $\mathscr{S}_{>n}$-null in the sense that all maps to them from n-connected spectra are null. In other words,

- The spectrum P^nX is $\mathscr{S}_{>n}$-null.
- For any $\mathscr{S}_{>n}$-null spectrum Z, the map $\mathscr{S}(P^nX, Z) \to \mathscr{S}(X, Z)$ is an equivalence.

Since $\mathscr{S}_{>n} \subset \mathscr{S}_{>n-1}$, there is a natural transformation $P^n \to P^{n-1}$, whose fiber is denoted by P_n^nX.

The Postnikov tower for X is the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n+1}X \arrow[r] & P^nX \arrow[r] & P^{n-1}X \arrow[r] & \cdots \\
& P_{n+1}^{n+1}X \arrow[u] & P_n^nX \arrow[u] & P_{n-1}^{n-1}X \arrow[u] &
\end{tikzcd}$$

Here the inverse limit is X and the direct limit is contractible. 2.17

The classical Postnikov tower (continued)

Suppose we replace the subcategory $\mathscr{S}_{>n} \subset \mathscr{S}$ with a subcategory $\mathscr{C}$ having similar formal properties, namely it is closed under

- mapping cones
- arbitrary wedges
- smash products with suspension spectra.

Using formal machinery developed by Emmanuel Dror-Farjoun, one can define functors $P^{\mathscr{C}}$ and $P_{\mathscr{C}}$ similar to P^n and P_n with a functorial cofiber sequence

$$P_{\mathscr{C}}X \to X \to P^{\mathscr{C}}X.$$

2.18

3.2 An equivariant version

An equivariant Postnikov tower

In what follows G will be an arbitrary finite cyclic 2-group, and $g = |G|$. For a subgroup $H \subset G$, let $h = |H|$ and let ρ_h denote its regular real representation. For $m \in \mathbf{Z}$, let

$$\widehat{S}(m\rho_h) = G_+ \wedge_H S^{m\rho_h}.$$

The underlying spectrum here is a wedge of g/h copies of S^{mh}.

Let $\mathscr{S}^G$ denote the category of G-equivariant spectra. We need an equivariant analog of $\mathscr{S}_{>n}$. Our choice for this is somewhat novel.

Recall that $\mathscr{S}_{>n}$ is the category of spectra built up out of spheres of dimension $> n$ using arbitrary wedges, mapping cones and smash products with suspension spectra. 2.19

An equivariant Postnikov tower (continued)

We will replace the set of sphere spectra by

$$\mathscr{A} = \left\{ \widehat{S}(m\rho_h), \Sigma^{-1}\widehat{S}(m\rho_h) \colon H \subset G, m \in \mathbf{Z} \right\}.$$

We will refer to the elements in this set as slice cells. Note that $\Sigma^{-2}\widehat{S}(m\rho_H)$ (and larger desuspensions) are not slice cells. A free cell is one of the form $\widehat{S}(m\rho_1)$, a wedge of g m-spheres permuted by G. Its desuspension is $\widehat{S}((m-1)\rho_1)$.

In order to define $\mathscr{S}^G_{>n}$, we need to assign a dimension to each slice cell. We do this in terms of the underlying spheres, namely

$$\dim \widehat{S}(m\rho_h) = mh \quad \text{and} \quad \dim \Sigma^{-1}\widehat{S}(m\rho_h) = mh - 1.$$

2.20

An equivariant Postnikov tower (continued)

Then $\mathscr{S}^G_{>n}$ is the category built up out of slice cells of dimension $> n$ using arbitrary wedges, mapping cones and smash products with equivariant suspension spectra.

With this definition it is possible to construct functors P^G_n and P^n_G with the same formal properties as in the classical case. Thus we get a tower

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n+1}_G X \arrow[r] & P^n_G X \arrow[r] & P^{n-1}_G X \arrow[r] & \cdots \\
& {}^G P^{n+1}_{n+1} X \arrow[u] & {}^G P^n_n X \arrow[u] & {}^G P^{n-1}_{n-1} X \arrow[u] &
\end{tikzcd}$$

in which the inverse limit is X and the direct limit is contractible. 2.21

3.3 The slice spectral sequence

The slice spectral sequence

We call this the slice tower. ${}^G P^n_n X$ is the nth slice and the decreasing sequence of subgroups of $\pi^G_*(X)$ is the slice filtration. We also get slice filtrations of the $RO(G)$-graded homotopy $\pi^G_\star(X)$ and the homotopy groups of fixed point sets $\pi_*(X^H)$.

There is an important difference between this tower and the classical one. In the classical case the map $X \to P^n X$ does not change homotopy groups in dimensions $\leq n$. This is not true in this equivariant case.

Equivalently, in the classical case, $P^n_n X$ is an Eilenberg-Mac Lane spectrum whose nth homotopy group is that of X. In our case, $\pi_*({}^G P^n_n X)$ need not be concentrated in dimension n. We will discuss some computational specifics below. 2.22

The slice spectral sequence (continued)

This means the slice filtration leads to a slice spectral sequence converging to $\pi_*^G(X)$ and its variants.

One variant has the form

$$E_2^{s,t} = \pi_{t-s}^G({}^G P_t^t X) \implies \pi_{t-s}^G(X) = \pi_{t-s}\left(X^G\right).$$

This is the spectral sequence we will use to study $MU_{\mathbf{R}}^{(4)}$ and its relatives. 2.23

4 *MU*

4.1 Basic properties

The complex cobordism spectrum

MU is the Thom spectrum for the universal complex vector bundle, which is defined over the classifying space of the stable unitary group, BU.

- MU has an action of the group C_2 via complex conjugation.
- $H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_i : i > 0]$ where $|b_i| = 2i$.
- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.

2.24

4.2 MU as a C_2-spectrum

MU as a C_2-spectrum

Let $\rho = \rho_2$ denote the real regular representation of C_2. It is isomorphic to the complex numbers $\mathbf{C}$ with conjugation.

We define a C_2-prespectrum mu by $mu_{k\rho} = MU(k)$, the Thom space of the universal $\mathbf{C}^k$-bundle over $BU(k)$, which is a direct limit of complex Grassmannian manifolds. The action of C_2 is by complex conjugation.

Since any finite dimensional orthogonal representation V of C_2 is contained in $k\rho$ for $k > 0$, we can define the C_2-spectrum $MU_{\mathbf{R}}$ by

$$(MU_{\mathbf{R}})_V = \lim_{\substack{\rightarrow \\ k}} \Omega^{k\rho - V} MU(k).$$

2.25

4.3 Norming up from $MU_{\mathbf{R}}$

Norming up from $MU_{\mathbf{R}}$

We will now construct a spectrum acted on by a larger cyclic 2-group. We apply the norm construction to the case $H = C_2$, $G = C_{2^{n+1}}$ and $X = MU_{\mathbf{R}}$. The underlying spectrum of $N_H^G MU_{\mathbf{R}}$ is the 2^n-fold smash power $MU_{\mathbf{R}}^{(2^n)}$.

We will need to identify the slices associated with $N_H^G MU_{\mathbf{R}}$. The following notion is helpful.

Definition. *Suppose X is a G-spectrum such that its underlying homotopy group $\pi_k^u(X)$ is free abelian. A refinement of $\pi_k^u(X)$ is an equivariant map*

$$c : \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cell of dimension k whose underlying spheres represent a basis of $\pi_k^u(X)$.

2.26

4.4 Refining homotopy

The refinement of $\pi_*^u(MU_{\mathbf{R}}^{(4)})$

Recall that $\pi_*(MU)$ is concentrated in even dimensions and is free abelian. $\pi_{2k}^u(MU_{\mathbf{R}})$ is refined by an map from a wedge of copies of $\widehat{S}(k\rho_2)$.

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ is a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension $2i$ by $r_i(j)$ for $1 \le j \le 4$. The action of a generator $\gamma \in G = C_8$ is given by

$$\gamma(r_i(j)) = \begin{cases} r_i(j+1) & \text{for } 1 \le j \le 3 \\ (-1)^i r_i(1) & \text{for } j = 4. \end{cases}$$

We will explain how $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ can be refined.

$\pi_2^u(MU_{\mathbf{R}}^{(4)})$ has 4 generators $r_1(j)$ that are permuted up to sign by G. It is refined by an equivariant map

$$\widehat{W}_1 = \widehat{S}(\rho_2) \to MU_{\mathbf{R}}^{(4)}.$$

Recall that the underlying spectrum of $\widehat{W}_1$ is a wedge of 4 copies of S^2. 2.27

The refinement of $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

In $\pi_4^u(MU_{\mathbf{R}}^{(4)})$ there are 14 monomials that fall into 4 orbits (up to sign) under the action of G, each corresponding to a map from a $\widehat{S}(m\rho_h)$.

$$\begin{array}{rcl} \widehat{S}(2\rho_2) & \longleftrightarrow & \{r_1(1)^2, r_1(2)^2, r_1(3)^2, r_1(4)^2\} \\ \widehat{S}(2\rho_2) & \longleftrightarrow & \{r_1(1)r_1(2), r_1(2)r_1(3), r_1(3)r_1(4), r_1(4)r_1(1)\} \\ \widehat{S}(2\rho_2) & \longleftrightarrow & \{r_2(1), r_2(2), r_2(3), r_2(4)\} \\ \widehat{S}(\rho_4) & \longleftrightarrow & \{r_1(1)r_1(3), r_1(2)r_1(4)\} \end{array}$$

(Recall that $\widehat{S}(\rho_4)$ is underlain by $S^4 \vee S^4$.) It follows that $\pi_4^u(MU_{\mathbf{R}}^{(4)})$ is refined by an equivariant map from

$$\widehat{W}_2 = \widehat{S}(2\rho_2) \vee \widehat{S}(2\rho_2) \vee \widehat{S}(\rho_4) \vee \widehat{S}(2\rho_2).$$

2.28

The refinement of $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

A similar analysis can be made in any even dimension and for any cyclic 2-group G. G always permutes monomials up to sign. In $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ the first case of a singleton orbit occurs in dimension 8, namely

$$\widehat{S}(\rho_8) \quad \longleftrightarrow \quad \{r_1(1)r_1(2)r_1(3)r_1(4)\}.$$

Note that the free cell $\widehat{S}(k\rho_1)$ never occurs as a wedge summand of $\widehat{W}_k$.

A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties. 2.29

The slice spectral sequence (continued)

From now on we will drop the symbol G from the functors P^n, P_n and P_n^n.

Slice Theorem . *In the slice tower for $MU_{\mathbf{R}}^{(g/2)}$, every odd slice is contractible and $P_{2n}^{2n} = \widehat{W}_n \wedge H\mathbf{Z}$, where $\widehat{W}_n$ is the wedge of slice cells indicated above and $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum. $\widehat{W}_n$ never has any free summands.*

2.30

Thus we need to find the groups

$$\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z}) = \pi_*^H(S^{m\rho_h} \wedge H\mathbf{Z}).$$

We need this for all integers m because eventually we will obtain the spectrum $\tilde{\Omega}$ by inverting a certain element in $\pi_{19\rho_8}^G(MU_{\mathbf{R}}^{(4)})$. Here is what we will learn.

Computing $\pi_*^G(W(m\rho_h)\wedge H\mathbf{Z})$

Vanishing Theorem . • *For $m \geq 0$, $\pi_k^H(S^{m\rho_h}\wedge H\mathbf{Z}) = 0$ unless $m \leq k \leq mh$.*

- *For $m < 0$ and $h > 1$, $\pi_k^H(S^{m\rho_h}\wedge H\mathbf{Z}) = 0$ unless $mh \leq k \leq m-2$. The upper bound can be improved to $m-3$ except in the case $(h,m) = (2,-2)$ when $\pi_{-4}^H(S^{-2\rho_2}\wedge H\mathbf{Z}) = \mathbf{Z}$.*

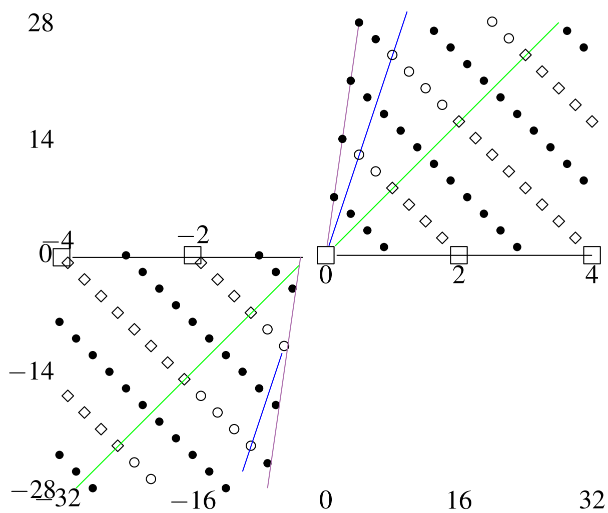
2.31

Gap Corollary. *For $h > 1$ and all integers m, $\pi_k^H(S^{m\rho_h}\wedge H\mathbf{Z}) = 0$ for $-4 < k < 0$.*

This means a similar statement must hold for $\pi_*^{C_8}(\tilde{\Omega}) = \pi_*(\Omega)$, which gives the Gap Theorem.

Computing $\pi_*^G(W(m\rho_h)\wedge H\mathbf{Z})$ (continued)

Here is a picture of some slices $S^{m\rho_8}\wedge H\mathbf{Z}$.



2.32

5 Proof of Gap Theorem

The proof of the Gap Theorem

Assuming the Slice Theorem, the Gap Theorem (the statement that $\pi_{-2}(\Omega) = 0$) follows immediately from the Gap Corollary above.

The proofs of the Vanishing Theorem and Gap Corollary are surprisingly easy.

We begin by constructing an equivariant cellular chain complex $C(m\rho_g)_*$ for $S^{m\rho_g}$, where $m \geq 0$. In it the cells are permuted by the action of G. It is a complex of $\mathbf{Z}[G]$-modules and is uniquely determined by fixed point data of $S^{m\rho_g}$.

For $H \subset G$ we have

$$(S^{m\rho_g})^H = S^{mg/h}$$

This means there is a G-CW-complex with one cell in dimension m, two cells in each dimension from $m+1$ to $2m$, four cells in each dimension from $2m+1$ to $4m$, and so on. 2.33

The proof of the Gap Theorem (continued)

In other words,

$$C(m\rho_g)_k = \begin{cases} 0 & \text{unless } m \le k \le gm \\ \mathbf{Z} & \text{for } k = m \\ \mathbf{Z}[G/H] & \text{for } mg/2h < k \le mg/h \text{ and } h < g. \end{cases}$$

Each of these is a cyclic $\mathbf{Z}[G]$-module. The boundary operator is uniquely determined by the fact that $H_*(C(m\rho_g)) = H_*(S^{gm})$.

Then we have

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))).$$

These groups are nontrivial only for $m \le k \le gm$, which gives the Vanishing Theorem for $m \ge 0$. 2.34

The proof of the Gap Theorem (continued)

We will look at the bottom three groups in the complex $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g)_*)$. Since $C(m\rho_g)_k$ is a cyclic $\mathbf{Z}[G]$-module, the Hom group is always $\mathbf{Z}$.

For $m > 1$ our chain complex $C(m\rho_g)$ has the form

$$\begin{array}{ccccccccc} & & C(m\rho_g)_m & & C(m\rho_g)_{m+1} & & C(m\rho_g)_{m+2} & & \\ & & \| & & \| & & \| & & \\ 0 & \longleftarrow & \mathbf{Z} & \xleftarrow{\varepsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \cdots \end{array}$$

Applying $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ to this gives (in dimensions $\le 2m$ for $m > 4$)

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \longleftarrow & \cdots \\ m & & m+1 & & m+2 & & m+3 & & m+4 & & \end{array}$$

2.35

The proof of the Gap Theorem (continued)

Again, $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))$ in low dimensions is

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \longleftarrow & \cdots \\ m & & m+1 & & m+2 & & m+3 & & m+4 & & \end{array}$$

It follows that for $m \le k < 2m$,

$$\pi_k^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \begin{cases} \mathbf{Z}/2 & k \equiv m \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

2.36

The proof of the Gap Theorem (continued)

For negative multiples of ρ_g, $S^{-m\rho_g}$ (with $m > 0$) is the equivariant Spanier-Whitehead dual of $S^{m\rho_g}$. This means that

$$\pi_*^G(S^{-m\rho_g} \wedge H\mathbf{Z}) = H^*(\mathrm{Hom}_{\mathbf{Z}[G]}(C(m\rho_g), \mathbf{Z})).$$

Applying the functor $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ to our chain complex $C(m\rho_g)$

$$\begin{array}{ccccccccc} \mathbf{Z} & \xleftarrow{\varepsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \mathbf{Z}[C_2 \text{ or } C_4] & \xleftarrow{1-\gamma} & \cdots \\ m & & m+1 & & m+2 & & m+3 & & \end{array}$$

gives a cochain complex beginning with

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \longrightarrow & \cdots \\ -m & & -m-1 & & -m-2 & & -m-3 & & -m-4 & & \end{array}$$

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The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case $m \geq 4$.

$$
\begin{tikzcd}
m & m+1 & m+2 & m+3 & m+4 & \\
\mathbf{Z} & \mathbf{Z} \arrow[l, "2"'] & \mathbf{Z} \arrow[l, "0"'] & \mathbf{Z} \arrow[l, "2"'] & \mathbf{Z} \arrow[l, "0"'] & \cdots \arrow[l] \\
\mathbf{Z} & \mathbf{Z}[C_2] \arrow[l, "\varepsilon"'] & \mathbf{Z}[C_2] \arrow[l, "1-\gamma"'] \arrow[u, Rightarrow, "\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z},\cdot)"'] \arrow[d, Rightarrow, "\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot,\mathbf{Z})"] & \mathbf{Z}[C_2] \arrow[l, "1+\gamma"'] & \mathbf{Z}[C_2] \arrow[l, "1-\gamma"'] & \cdots \arrow[l, "1-\gamma"'] \\
\mathbf{Z} \arrow[r, "1"] & \mathbf{Z} \arrow[r, "0"] & \mathbf{Z} \arrow[r, "2"] & \mathbf{Z} \arrow[r, "0"] & \mathbf{Z} \arrow[r] & \cdots \\
-m & -m-1 & -m-2 & -m-3 & -m-4 &
\end{tikzcd}
$$

Note the difference in behavior of the map $\varepsilon : \mathbf{Z}[C_2] \to \mathbf{Z}$ under the functors $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z},\cdot)$ and $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot,\mathbf{Z})$. They convert it to maps of degrees 2 and 1 respectively. This difference is responsible for the Gap. 2.38