A solution to the Arf-Kervaire invariant problem II

Middle East
Technical University
Topology Seminar

May 18, 2010

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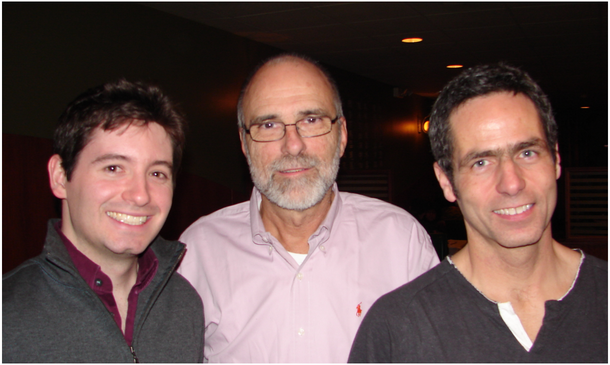

Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010

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The case $j = 6$ is still open.

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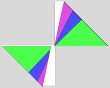

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Ω will be the fixed point set associated with a C_8-equivariant spectrum $\tilde{\Omega}$ related to the complex cobordism spectrum. As we will explain below, a G-equivariant spectrum is more than just a spectrum with a G-action.

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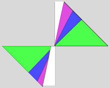

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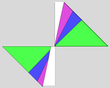

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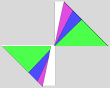
There is an analogous construction for an H-spectrum X, and the result is denoted by $N_H^G X$, the norm of X along the inclusion $H \subset G$.



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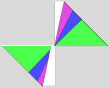

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This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.

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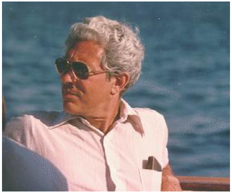

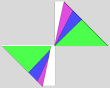

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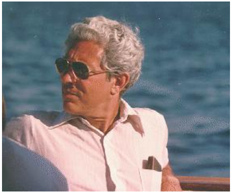

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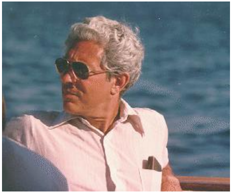

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A map of spectra $f : E \to E'$ is a collection of maps of based spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps.

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Equivariant spectra

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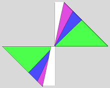

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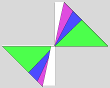

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A G-equivariant spectrum (G-spectrum for short) consists of a based G-space E_V for each finite dimensional invariant subspace $V \subset \mathcal{U}$, together with a transitive system of based G-homeomorphisms

$$E_V \xrightarrow[\cong]{\tilde{\sigma}_{V,W}} \Omega^{W-V} E_W$$

for $V \subset W \subset \mathcal{U}$. Here $\Omega^V X = F(S^V, X)$, the G-space of maps to X from S^V, the one point compactification of V. $W - V$ denotes the orthogonal complement of V in W. As in the classical case, the G-homotopy type of E_V depends only on the isomorphism class of V.

G-equivariant spectra (continued)

A map of G-spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps.

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A suspension G-prespectrum is a G-prespectrum in which the maps above are G-equivalences for V sufficiently large.

$RO(G)$-graded homotopy groups

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We define S^{-V} by saying its Wth space for $V \subset W$ is S^{W-V}. This is the analog of formal desuspension in the nonequivariant case.

$RO(G)$-graded homotopy groups (continued)

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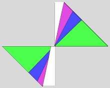

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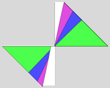
For an integer n,

$$\pi_n^G(X) = [S^n, X]_G = [S^n, X^G] = \pi_n(X^G),$$

the ordinary nth homotopy group of the fixed point spectrum X^G.



The classical Postnikov tower

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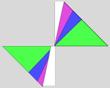

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In other words, the map $P_nX \to X$ is universal among maps from n-connected spectra to X.

The classical Postnikov tower (continued)

Similarly the map $X \to P^n X$ is universal among maps from X to spectra which are $\mathcal{S}_{>n}$-null in the sense that all maps to them from n-connected spectra are null. In other words,

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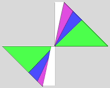

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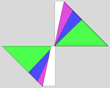

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A solution to the Arf-Kervaire invariant problem II

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Recall that $\mathcal{S}_{>n}$ is the category of spectra built up out of spheres of dimension $> n$ using arbitrary wedges, mapping cones and smash products with suspension spectra.

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We will replace the set of sphere spectra by

$$\mathcal{A} = \left\{ \widehat{S}(m\rho_h),\ \Sigma^{-1}\widehat{S}(m\rho_h) \colon H \subset G,\ m \in \mathbf{Z} \right\}.$$

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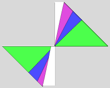

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With this definition it is possible to construct functors P_n^G and P_G^n with the same formal properties as in the classical case. Thus we get a tower

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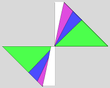

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We call this the slice tower. ${}^{G}P_n^n X$ is the nth slice and the decreasing sequence of subgroups of $\pi_*^G(X)$ is the slice filtration.

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We call this the slice tower. ${}^{G}P_{n}^{n}X$ is the nth slice and the decreasing sequence of subgroups of $\pi_{*}^{G}(X)$ is the slice filtration. We also get slice filtrations of the $RO(G)$-graded homotopy $\pi_{\star}^{G}(X)$ and the homotopy groups of fixed point sets $\pi_{*}(X^{H})$.

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Equivalently, in the classical case, $P_n^n X$ is an Eilenberg-Mac Lane spectrum whose nth homotopy group is that of X.

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The slice spectral sequence

We call this the slice tower. ${}^G P_n^n X$ is the nth slice and the decreasing sequence of subgroups of $\pi_*^G(X)$ is the slice filtration. We also get slice filtrations of the $RO(G)$-graded homotopy $\pi_\star^G(X)$ and the homotopy groups of fixed point sets $\pi_*(X^H)$.

There is an important difference between this tower and the classical one. In the classical case the map $X \to P^n X$ does not change homotopy groups in dimensions $\leq n$. This is not true in this equivariant case.

Equivalently, in the classical case, $P_n^n X$ is an Eilenberg-Mac Lane spectrum whose nth homotopy group is that of X. In our case, $\pi_*({}^G P_n^n X)$ need not be concentrated in dimension n.

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Equivalently, in the classical case, $P_n^n X$ is an Eilenberg-Mac Lane spectrum whose nth homotopy group is that of X. In our case, $\pi_*({}^G P_n^n X)$ need not be concentrated in dimension n. We will discuss some computational specifics below.

The slice spectral sequence (continued)

This means the slice filtration leads to a slice spectral sequence converging to $\pi_*^G(X)$ and its variants.

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This is the spectral sequence we will use to study $MU_{\mathbf{R}}^{(4)}$ and its relatives.

The complex cobordism spectrum

MU is the Thom spectrum for the universal complex vector bundle, which is defined over the classifying space of the stable unitary group, *BU*.

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- MU has an action of the group C_2 via complex conjugation.
- $H_*(MU; \mathbf{Z}) = \mathbf{Z}[b_i : i > 0]$ where $|b_i| = 2i$.
- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.

MU as a C_2-spectrum

Let $\rho = \rho_2$ denote the real regular representation of C_2.

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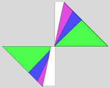
Since any finite dimensional orthogonal representation V of C_2 is contained in $k\rho$ for $k > 0$, we can define the C_2-spectrum $MU_{\mathbf{R}}$ by

$$(MU_{\mathbf{R}})_V = \lim_{\vec{k}} \Omega^{k\rho - V} MU(k).$$

Norming up from $MU_{\mathbf{R}}$

We will now construct a spectrum acted on by a larger cyclic 2-group.



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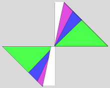

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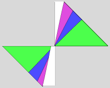

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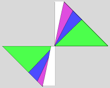

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Definition

Suppose X is a G-spectrum such that its underlying homotopy group $\pi_k^u(X)$ is free abelian. A refinement of $\pi_k^u(X)$ is an equivariant map

$$c : \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cell of dimension k whose underlying spheres represent a basis of $\pi_k^u(X)$.

The refinement of $\pi_*^u(MU_{\mathbf{R}}^{(4)})$

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Recall that the underlying spectrum of $\widehat{W}_1$ is a wedge of 4 copies of S^2.

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In $\pi_4^u(MU_{\mathbf{R}}^{(4)})$ there are 14 monomials that fall into 4 orbits (up to sign) under the action of G, each corresponding to a map from a $\widehat{S}(m\rho_h)$.

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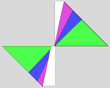

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A similar analysis can be made in any even dimension and for any cyclic 2-group G. G always permutes monomials up to sign. In $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ the first case of a singleton orbit occurs in dimension 8, namely

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A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties.

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We need this for all integers m because eventually we will obtain the spectrum $\tilde{\Omega}$ by inverting a certain element in $\pi_{19\rho_8}^G(MU_{\mathbf{R}}^{(4)})$. Here is what we will learn.

Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$

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- *For* $m < 0$ *and* $h > 1$, $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ *unless* $mh \leq k \leq m - 2$.

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For $h > 1$ *and all integers* m, $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ *for* $-4 < k < 0$.

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This means a similar statement must hold for $\pi_*^{C_8}(\tilde{\Omega}) = \pi_*(\Omega)$, which gives the Gap Theorem.

Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$ (continued)

Here is a picture of some slices $S^{m\rho_8} \wedge H\mathbf{Z}$.

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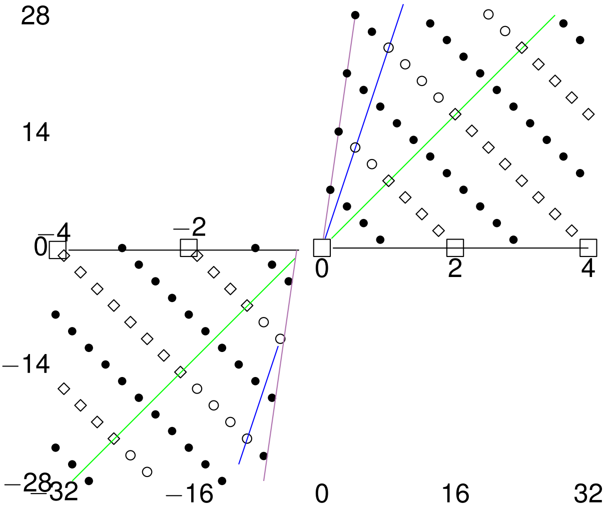

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The proof of the Gap Theorem (continued)

In other words,

$$C(m\rho_g)_k = \begin{cases} 0 & \text{unless } m \leq k \leq gm \\ \mathbf{Z} & \text{for } k = m \\ \mathbf{Z}[G/H] & \text{for } mg/2h < k \leq mg/h \text{ and } h < g. \end{cases}$$

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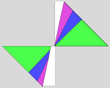
Each of these is a cyclic $\mathbf{Z}[G]$-module.



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Then we have

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))).$$

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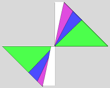
Then we have

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))).$$

These groups are nontrivial only for $m \leq k \leq gm$, which gives the Vanishing Theorem for $m \geq 0$.

The proof of the Gap Theorem (continued)

We will look at the bottom three groups in the complex $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g)_*)$.



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For $m > 1$ our chain complex $C(m\rho_g)$ has the form

$$\begin{array}{ccccccccc} & & C(m\rho_g)_m & & C(m\rho_g)_{m+1} & & C(m\rho_g)_{m+2} & & \\ & & \| & & \| & & \| & & \\ 0 & \longleftarrow & \mathbf{Z} & \xleftarrow{\ \epsilon\ } & \mathbf{Z}[C_2] & \xleftarrow{\ 1-\gamma\ } & \mathbf{Z}[C_2] & \xleftarrow{\ 1+\gamma\ } & \cdots \end{array}$$

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Applying $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ to this gives (in dimensions $\leq 2m$ for $m > 4$)

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xleftarrow{\ 2\ } & \mathbf{Z} & \xleftarrow{\ 0\ } & \mathbf{Z} & \xleftarrow{\ 2\ } & \mathbf{Z} & \xleftarrow{\ 0\ } & \mathbf{Z} & \longleftarrow & \cdots \\ m & & m+1 & & m+2 & & m+3 & & m+4 & & \end{array}$$

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Again, $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))$ in low dimensions is

$$\begin{array}{ccccccccccc}
\mathbf{Z} & \xleftarrow{\;2\;} & \mathbf{Z} & \xleftarrow{\;0\;} & \mathbf{Z} & \xleftarrow{\;2\;} & \mathbf{Z} & \xleftarrow{\;0\;} & \mathbf{Z} & \longleftarrow & \cdots \\
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\end{array}
$$

It follows that for $m \leq k < 2m$,

$$
\pi_k^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \begin{cases} \mathbf{Z}/2 & k \equiv m \bmod 2 \\ 0 & \text{otherwise.} \end{cases}
$$

The proof of the Gap Theorem (continued)

For negative multiples of ρ_g, $S^{-m\rho_g}$ (with $m > 0$) is the equivariant Spanier-Whitehead dual of $S^{m\rho_g}$.

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$$\pi_*^G(S^{-m\rho_g} \wedge H\mathbf{Z}) = H^*(\mathrm{Hom}_{\mathbf{Z}[G]}(C(m\rho_g), \mathbf{Z})).$$

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$$\begin{array}{ccccccccc} \mathbf{Z} & \xleftarrow{\epsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \mathbf{Z}[C_2 \text{ or } C_4] & \xleftarrow{1-\gamma} & \cdots \\ m & & m+1 & & m+2 & & m+3 & & \end{array}$$

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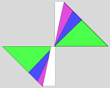
gives a cochain complex beginning with

$$\begin{array}{cccccccccc} \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \longrightarrow \cdots \\ -m & & -m-1 & & -m-2 & & -m-3 & & -m-4 & \end{array}$$



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m & & m+1 & & m+2 & & m+3 & & m+4 & & \\
\mathbf{Z} & \xleftarrow[2]{} & \mathbf{Z} & \xleftarrow[0]{} & \mathbf{Z} & \xleftarrow[2]{} & \mathbf{Z} & \xleftarrow[0]{} & \mathbf{Z} & \longleftarrow & \cdots \\
 & & & & \Big\Uparrow \mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot) & & & & & & \\
\mathbf{Z} & \xleftarrow{\epsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \cdots \\
 & & & & \Big\Downarrow \mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z}) & & & & & & \\
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\end{array}
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\mathbf{Z} & \xleftarrow[2]{} & \mathbf{Z} & \xleftarrow[0]{} & \mathbf{Z} & \xleftarrow[2]{} & \mathbf{Z} & \xleftarrow[0]{} & \mathbf{Z} & \longleftarrow & \cdots \\
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Note the difference in behavior of the map $\epsilon : \mathbf{Z}[C_2] \to \mathbf{Z}$ under the functors $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ and $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$.

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The proof of the Gap Theorem (continued)

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\begin{array}{ccccccccccc}
m & & m+1 & & m+2 & & m+3 & & m+4 & & \\
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