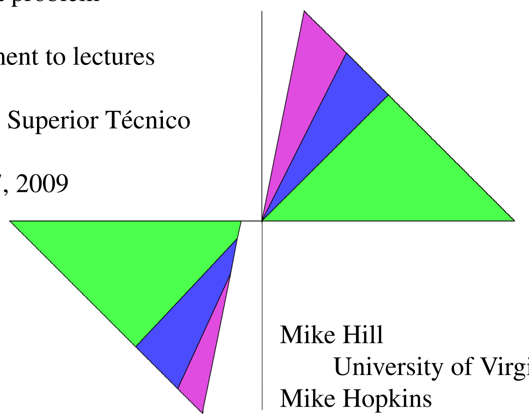
The detection theorem

A solution to the Arf-Kervaire invariant problem

Supplement to lectures given at Instituto Superior Técnico Lisbon May 5-7, 2009



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S.1

1 The Detection Theorem

1.1 θ_j in the Adams-Novikov spectral sequence

θ_j in the Adams-Novikov spectral sequence

Browder's theorem says that θ_j is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2).$$

This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \mathrm{Ext}_{MU_*(MU)}^{2,6i-2j}(MU_*, MU_*)$$

for certain values of of i and j. When $j=1$, it is customary to omit it from the notation. The definition of these elements can be found in Chapter 5 of the third author's book *Complex Cobordism and Stable Homotopy Groups of Spheres*. S.2

θ_j in the Adams-Novikov spectral sequence (continued)

Here are the first few of these in the relevant bidegrees.

$$\begin{array}{ll} \theta_5: & \beta_{8/8} \text{ and } \beta_{6/2} \\ \theta_6: & \beta_{16/16}, \beta_{12/4} \text{ and } \beta_{11} \\ \theta_7: & \beta_{32/32}, \beta_{24/8} \text{ and } \beta_{22/2} \\ \theta_8: & \beta_{64/64}, \beta_{48/16}, \beta_{44/4} \text{ and } \beta_{43} \end{array}$$

and so on. In the bidegree of θ_j, only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely h_j^2) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1\alpha_{2^j-1}$.

We need to show that any element mapping to h_j^2 in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for M. S.3

θ_j in the Adams-Novikov spectral sequence (continued)

Detection Theorem. *Let $x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$ be any element whose image in $\mathrm{Ext}^{2,2^{j+1}}_{A}(\mathbf{Z}/2, \mathbf{Z}/2)$ is h_j^2 with $j \geq 6$. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_*(M))$ is nonzero.*

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, *the theory of formal A-modules*, where A is the ring of integers in a suitable field.

S.4

1.2 Formal A-modules

Formal A-modules

Recall the a formal group law over a ring R is a power series

$$F(x,y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x,y]]$$

with certain properties.

For positive integers m one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers m, and we get a homomorphism τ from $\mathbf{Z}$ to $End(F)$, the endomorphism ring of F.

S.5

Formal A-modules (continued)

If the ground ring R is an algebra over the p-local integers $\mathbf{Z}_{(p)}$ or the p-adic integers $\mathbf{Z}_p$, then we can make sense of $[m](x)$ for m in $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$.

Now suppose R is an algebra over a larger ring A, such as the ring of integers in a number field or a finite extension of the p-adic numbers. We say that the formal group law F is a *formal A-module* if the homomorphism τ extends to A in such a way that

$$[a](x) \equiv ax \bmod (x^2) \text{ for } a \in A.$$

The theory of formal A-modules is well developed. Lubin-Tate used them to do local class field theory.

S.6

Formal A-modules (continued)

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity. The maximal ideal of A is generated by $\pi = \zeta_8 - 1$, and π^4 is a unit multiple of 2. There is a formal A-module G over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) satisfying

$$\log_G(G(x,y)) = \log_G(x) + \log_G(y)$$

where

$$\log_G(x) = \sum_{n \geq 0} \frac{w^{2^n - 1} x^{2^n}}{\pi^n}.$$

The classifying map $\lambda : MU_* \to R_*$ for G factors through BP_*, where the logarithm is

$$\log_F(x) = \sum_{n \geq 0} \ell_n x^{2^n}.$$

S.7

Formal A-modules (continued)

Recall that $BP_* = \mathbf{Z}_{(2)}[v_1, v_2, \dots]$ with $|v_n| = 2(2^n - 1)$. The v_n and the ℓ_n are related by Hazewinkel's formula,

$$\begin{aligned} \ell_1 &= \frac{v_1}{2} \\ \ell_2 &= \frac{v_2}{2} + \frac{v_1^3}{4} \\ \ell_3 &= \frac{v_3}{2} + \frac{v_1 v_2^2 + v_2 v_1^4}{4} + \frac{v_1^7}{8} \\ \ell_4 &= \frac{v_4}{2} + \frac{v_1 v_3^2 + v_2^5 + v_3 v_1^8}{4} + \frac{v_1^3 v_2^4 + v_1^9 v_2^2 + v_2 v_1^{12}}{8} + \frac{v_1^{15}}{16} \\ &\vdots \end{aligned}$$

S.8

1.3 $\pi_*(MU^{(4)})$ and R_*

The relation between $MU^{(4)}$ and formal A-modules

What does all this have to do with our spectrum $M = D^{-1}MU^{(4)}$? Recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$. We saw earlier that inverting a product of this sort is needed to get the Periodicity Theorem, but we did not explain the choice of subscripts of $\overline{\Delta}$. They are the smallest ones that satisfy the second part of the following.

Lemma. *The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for G factors through $\pi_*(MU^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU^{(4)}) \to R_*$ is equivariant, where C_8 acts on $\pi_*(MU^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8.*
- *The element $D \in \pi_*(MU^{(4)})$ that we invert to get M goes to a unit in R_*.*

We will prove this later.

S.9

1.4 The proof of the Detection Theorem

The proof of the Detection Theorem

It follows that we have a map

$$H^*(C_8; \pi_*(D^{-1}MU^{(4)})) = H^*(C_8; \pi_*(M)) \to H^*(C_8; R_*).$$

The source here is the E_2-term of the homotopy fixed point spectral sequence for M, and the target is easy to calculate. We will use it to prove the Detection Theorem, namely

Detection Theorem. *Let $x \in \operatorname{Ext}^{2,2^{j+1}}_{MU_*(MU)}(MU_*, MU_*)$ be any element whose image in $\operatorname{Ext}^{2,2^{j+1}}_{A}(\mathbf{Z}/2, \mathbf{Z}/2)$ is h_j^2 with $j \geq 6$. (Here A denotes the mod 2 Steenrod algebra.) Then the image of x in $H^{2,2^{j+1}}(C_8; \pi_*(M))$ is nonzero.*

We will prove this by showing that the image of x in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero. S.10

The proof of the Detection Theorem (continued)

We will calculate with BP-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \quad \text{where } |t_n| = 2(2^n - 1).$$

We will abbreviate $\operatorname{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*)$ by $\operatorname{Ext}^{s,t}$.

There is a map from this Hopf algebroid to one associated with $H^*(C_8;R_*)$ in which t_n maps to an R_*-valued function on C_8 (regarded as the group of 8th roots of unity) determined by

$$[\zeta](x) = \sum_{n\geq 0}^{F} \langle t_n,\, \zeta\rangle x^{2^n}.$$

An easy calculation shows that the function t_1 sends a primitive root in C_8 to a unit in R_*. S.11

The proof of the Detection Theorem (continued)

Let

$$b_{1,j-1} = \frac{1}{2}\sum_{0<i<2^j} \binom{2^j}{i} \left[t_1^i | t_1^{2^j-i}\right] \in \operatorname{Ext}^{2,2^{j+1}}$$

It is is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2. We will show that its image in $H^{2,2^{j+1}}(C_8;R_*)$ is nontrivial for $j\geq 2$.

$H^*(C_8;R_*)$ is the cohomology of the cochain complex

$$R_*[C_8] \xrightarrow{\gamma-1} R_*[C_8] \xrightarrow{\text{Trace}} R_*[C_8] \xrightarrow{\gamma-1} \cdots$$

where Trace is multiplication by $1+\gamma+\cdots+\gamma^7$. S.12

The proof of the Detection Theorem (continued)

The cohomology groups $H^s(C_8;R_*)$ for $s>0$ are periodic in s with period 2. We have

$$\begin{aligned} H^1(C_8;R_{2m}) &= \ker(1+\zeta_8^m+\cdots+\zeta_8^{7m})/\operatorname{im}(\zeta_8^m-1) \\ &= \begin{cases} w^m A/(\pi) & \text{for } m \text{ odd} \\ w^m A/(\pi^2) & \text{for } m\equiv 2 \bmod 4 \\ w^m A/(2) & \text{for } m\equiv 4 \bmod 8 \\ 0 & \text{for } m\equiv 0 \bmod 8 \end{cases} \\ H^2(C_8;R_{2m}) &= \ker(\zeta_8^m-1)/\operatorname{im}(1+\zeta_8^m+\cdots+\zeta_8^{7m}) \\ &= \begin{cases} w^m A/(8) & \text{for } m\equiv 0 \bmod 8 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

An easy calculation shows that $b_{1,j-1}$ maps to $4w^{2^j}$, which is the element of order 2 in $H^2(C_8;R_{2^{j+1}})$. S.13

The proof of the Detection Theorem (continued)

To finish the proof we need to show that the other βs in the same bidegree map to zero. We will do this for $j\geq 6$. The set of these is

$$\left\{\beta_{c(j,k)/2^{j-1-2k}} : 0\leq k< j/2\right\}$$

where $c(j,k)=2^{j-1-2k}(1+2^{2k+1})/3$. Note that $\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}}$, so we need to show that the elements with $k>0$ map to zero.

We will see in the proof of the Lemma below that v_1 and v_2 map to unit multiples of $\pi^3 w$ and $\pi^2 w^3$ respectively. This means we can define a valuation on BP_* compatible with the one on A in which $||2||=1$, $||\pi||=1/4$, $||v_1||=3/4$ and $||v_2||=1/2$. We extend the valuation on A to R_* by setting $||w||=0$. S.14

The proof of the Detection Theorem (continued)

Hence for $k \geq 1$ and $j \geq 6$ we have

$$
\begin{aligned}
||\beta_{c(j,k)/2^{j-1-2k}}|| &= \left|\left|\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}}\right|\right| \\
&= \frac{c(j,k)}{2} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\
&= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\
&= (2^{j-1} - 7\cdot 2^{j-3-2k})/3 - 1 \\
&\geq 5.
\end{aligned}
$$

This means $\beta_{c(j,k))/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero. S.15

The proof of the Detection Theorem (continued)

We have to make a similar computation with the element $\alpha_1\alpha_{2^j-1}$. We have

$$
\begin{aligned}
||\alpha_{2^j-1}|| &= \left|\left|\frac{v_1^{2^j-1}}{2}\right|\right| \\
&= \frac{3(2^j-1)}{4} - 1 \\
&\geq \frac{21}{4} - 1 \geq 4 \quad \text{for } j \geq 3.
\end{aligned}
$$

This completes the proof of the Detection Theorem modulo the Lemma. S.16

1.5 The proof of the Lemma

The proof of the Lemma

Here it is again.

Lemma. *The classifying homomorphism $\lambda : \pi_*(MU) \to R_*$ for G factors through $\pi_*(MU^{(4)})$ in such a way that*

- *the homomorphism $\lambda^{(4)} : \pi_*(MU^{(4)}) \to R_*$ is equivariant, where C_8 acts on $\pi_*(MU^{(4)})$ as before, it acts trivially on A and $\gamma w = \zeta_8 w$ for a generator γ of C_8.*
- *The element $D \in \pi_*(MU^{(4)})$ that we invert to get M goes to a unit in R_*.*

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The proof of the Lemma (continued)

To prove the first part, consider the following diagram for an arbitrary ring K.

$$
\begin{tikzcd}
& MU_*(MU) \arrow[d, equal] & \\
\pi_*(MU) \arrow[ur, "\eta_L"] \arrow[dr, "\lambda_1"'] & \pi_*(MU^{(2)}) \arrow[d, dotted, "\lambda^{(2)}"'] & \pi_*(MU) \arrow[ul, "\eta_R"'] \arrow[dl, "\lambda_2"] \\
& K &
\end{tikzcd}
$$

The maps λ_1 and λ_2 classify two formal group laws F_1 and F_2 over K. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence of $\lambda^{(2)}$ is equivalent to that of a compatible strict isomorphism between F_1 and F_2. S.18

The proof of the Lemma (continued)

Similarly consider the diagram

$$\begin{tikzcd}
 & & \pi_*(MU^{(4)}) \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*(MU) \arrow[urr] \arrow[drr, "\lambda_1"'] & \pi_*(MU) \arrow[ur] \arrow[dr, "\lambda_2"] & & \pi_*(MU) \arrow[ul] \arrow[dl, "\lambda_3"'] & \pi_*(MU) \arrow[ull] \arrow[dll, "\lambda_4"] \\
 & & K & &
\end{tikzcd}$$

The existence of $\lambda^{(4)}$ is equivalent to that of compatible strict isomorphisms between the formal group laws F_j classified by the λ_j. S.19

The proof of the Lemma (continued)

$$\begin{tikzcd}
 & & \pi_*(MU^{(4)}) \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*(MU) \arrow[urr] \arrow[drr, "\lambda_1"'] & \pi_*(MU) \arrow[ur] \arrow[dr, "\lambda_2"] & & \pi_*(MU) \arrow[ul] \arrow[dl, "\lambda_3"'] & \pi_*(MU) \arrow[ull] \arrow[dll, "\lambda_4"] \\
 & & K & &
\end{tikzcd}$$

Now suppose that K has a C_8-action and that $\lambda^{(4)}$ is equivariant with respect to the previously defined C_8-action on $MU^{(4)}$. Then the isomorphism induced by the fourth power of a generator $\gamma \in C_8$ is the isomorphism sending x to its formal inverse on each of the F_j.

This means that the existence of an equivariant $\lambda^{(4)}$ is equivalent to that of a formal $\mathbf{Z}[\zeta_8]$-module structure on each of the F_j, which are all isomorphic. This proves the first part of the Lemma. S.20

The proof of the Lemma (continued)

For the second part, recall that $D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$, where

$$\overline{\Delta}_k^{(g)} = \begin{cases} x_{2^k-1} & \text{for } g = 2 \\ N_4^g(r_{2^k-1}) & \text{otherwise.} \end{cases}$$

Since our formal A-module is 2-typical we can do the calculations using BP in place of MU. Hence we can replace x_{2^k-1} by v_k and r_{2^k-1} by t_k. We have $\overline{\Delta}_k^{(2)} = v_k$. Using Hazewinkel's formula we find that

$$\begin{array}{lcl}
v_1 & \mapsto & (-\pi^3 - 4\pi^2 - 6\pi - 4)w \\
v_2 & \mapsto & (4\pi^3 + 11\pi^2 + 6\pi - 6)w^3 \\
v_3 & \mapsto & (40\pi^3 + 166\pi^2 + 237\pi + 100)w^7 \\
v_4 & \mapsto & (-15754\pi^3 - 56631\pi^2 - 63495\pi - 9707)w^{15}.
\end{array}$$

so v_4 (but not v_n for $n < 4$) and therefore $N_2^8(\overline{\Delta}_4^{(2)})$ maps to a unit. S.21

The proof of the Lemma (continued)

We have $\overline{\Delta}_k^{(2)} = t_k$. We consider the equivariant composite

$$BP_*^{(2)} \to BP_*^{(4)} \to R_*$$

under which

$$\eta_R(\ell_n) \mapsto \frac{\zeta_8^2 w^{2^n-1}}{\pi^n}.$$

Using the right unit formula we find that

$$\begin{array}{rcl} t_1 & \mapsto & (\pi+2)w \\ t_2 & \mapsto & (\pi^3+5\pi^2+9\pi+5)w^3. \end{array}$$

This means t_2 (but not t_1) and therefore $N_4^8(\overline{\Delta}_2^{(4)})$ maps to a unit. S.22

The proof of the Lemma (continued)

Finally, we have $\overline{\Delta}_n^{(8)} = t_n(1) \in BP_*^{(4)}$, where $t_n(1)$ is the analog of $r_{2^n-1}(1)$. Then we find

$$\begin{array}{rcl} \ell_n(1) & \mapsto & \dfrac{w^{2^n-1}}{\pi^n} \\ \ell_n(2) & \mapsto & \dfrac{(\zeta_8 w)^{2^n-1}}{\pi^n}. \end{array}$$

This implies

$$\overline{\Delta}_1^{(8)} = \ell_1(2) - \ell_1(1) \mapsto w.$$

Thus we have shown that each factor of

$$D = \overline{\Delta}_1^{(8)} N_4^8(\overline{\Delta}_2^{(4)}) N_2^8(\overline{\Delta}_4^{(2)})$$

and hence D itself maps to a unit in R_*, thus proving the lemma. S.23