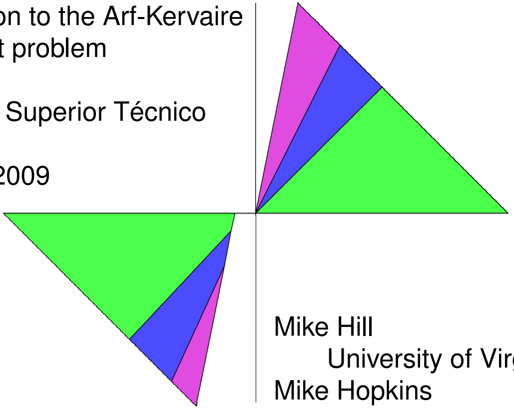
Lecture 2

A solution to the Arf-Kervaire invariant problem

Instituto Superior Técnico
Lisbon
May 6, 2009



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University of Virginia
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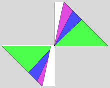

The spectrum M

Our goal is to prove

Main Theorem

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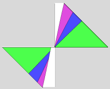

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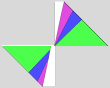

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(iii) $\pi_{-2}(M) = 0$.

The spectrum M (continued)

We will construct an equivariant C_8-spectrum $\tilde{M}$ and show that its homotopy fixed point set $\tilde{M}^{hC_*}$ (to be defined below) and its actual fixed point set $\tilde{M}^{C_8}$ are equivalent.

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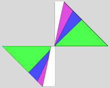
- The homotopy of $\tilde{M}^{hC_*}$ can be computed using a spectral sequence similar to that of Hopkins-Miller. Twenty year old algebraic methods can be used to show that it detects the θ_js.
- In order to establish (ii) and (iii), we will use equivariant methods to construct a new spectral sequence (the *slice spectral sequence*) converging to the homotopy of the actual fixed point set $\tilde{M}^{C_8}$.

The complex cobordism spectrum

MU is the Thom spectrum for the universal complex vector bundle, which is defined over the classifying space of the stable unitary group, *BU*.



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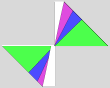

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- $\pi_*(MO) = \mathbf{Z}/2[y_i : i > 0, i \neq 2^k - 1]$ where $|y_i| = i$. This is the unoriented cobordism ring.

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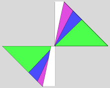

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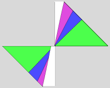

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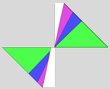
(iii) (Associativity) $F(x,F(y,z)) = F(F(x,y),z)$. This implies more complicated relations among the $a_{i,j}$.

Examples of formal group laws

- $x + y$, the additive formal group law.



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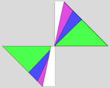

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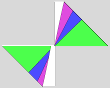

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  It is originally due to Euler, see *De integratione aequationis differentialis* $(m\mathrm{d}x)/\sqrt{1-x^4} = (n\mathrm{d}y)/\sqrt{1-x^4}$, 1753.

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Let

$$L = \mathbf{Z}[a_{i,j}]/(\text{relations})$$

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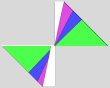

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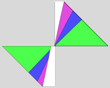
where $\lambda(G(x, y))$ is the formal group law over R obtained from G by applying λ to each of the $a_{i,j}$.



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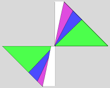

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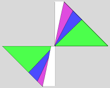
There is a cohomology theory associated with MU under which

$$\begin{aligned} MU^*(\mathbf{C}P^\infty) &= \pi_*(MU)[[x]] \\ \text{and} \qquad MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) &= \pi_*(MU)[[x \otimes 1, 1 \otimes x]]. \end{aligned}$$



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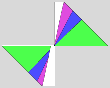
The map $\mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ (corresponding to tensor product of complex line bundles) induces a homomorphism

$$MU^*(\mathbf{C}P^\infty) \to MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty)$$

that sends x to a power series in $x \otimes 1$ and $1 \otimes x$ which is a formal group law over $\pi_*(MU)$.



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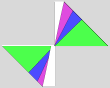
The homomorphism $\theta : L \to \pi_*(MU)$ induced by the formal group law over $\pi_*(MU)$ defined above is an isomorphism.



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The homomorphism $\theta : L \to \pi_*(MU)$ *induced by the formal group law over* $\pi_*(MU)$ *defined above is an isomorphism.*

This means that the internal structure of *MU*, and the associated homology and cohomology theories, is intimately related to the structure of formal group laws.

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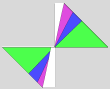
Some relatives of *MU*

Here is an example of this connection.



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After localizing at a prime p, MU splits into a wedge of suspensions of smaller spectra (Brown-Peterson) BP with

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The Brown-Peterson splitting is the topological analog of Cartier's theorem.

More relatives of *MU*

The *Morava spectrum* E_n (for a positive integer n) is an E_∞-ring spectrum such that $\pi_*(E_n)$ obtained from $\pi_*(BP)$ as follows:

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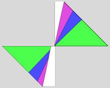

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The ring $\pi_*(E_n)$ was studied by Lubin-Tate. They showed that it classifies liftings (to Artinian rings) of a certain formal group law F_n over $\mathbf{F}_{p^n}$, the *Honda formal group law*.

The Morava stabilizer group S_n

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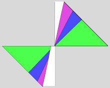

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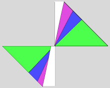

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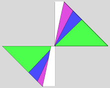

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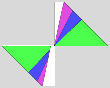

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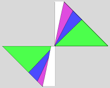

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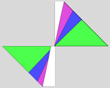

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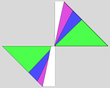
We will be interested in some finite subgroups of S_n.

The Hopkins-Miller theorem

The algebraically defined action of S_n on $\pi_*(E_n)$ leads to action on E_n itself, but it is defined only up to homotopy.



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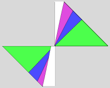

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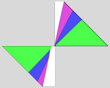

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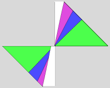
It coincides with the Adams-Novikov spectral sequence for E_n^{hG}.



Finite subgroups of S_n

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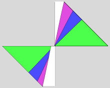

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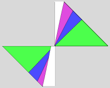
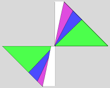
- $C_8 \subset S_4$. The division algebra D_4 contains eighth roots of unity for similar reasons.



A first attempt to define the magic spectrum M

- The spectrum $E_4^{hC_8}$ can be shown to satisfy the first condition required of M, namely its Adams-Novikov spectral sequence detects all of the θ_js.



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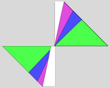

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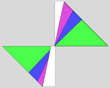
- The one for $E_4^{hC_8}$ is too complicated for us to use it to prove that $\pi_{-2} = 0$.



A C_8-equivariant substitute for E_4

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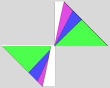

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Property (ii) (periodicity) involves some differentials in the slice spectral sequence.

There is an analogous construction for $E_{2^{k-1}}$ as a C_{2^k}-spectrum for any k. The slice spectral sequence for $k = 1$ was the subject of Dan Dugger's thesis, and we will illustrate at at the end of the third lecture.

G-spaces

Before we can describe any of this, we need to introduce *equivariant stable homotopy theory.*

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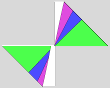

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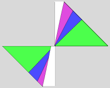

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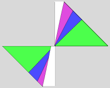
The product $X \times Y$ of two G-spaces is a G-space under the diagonal action, as is the smash product of two based G-spaces.

Maps of G-spaces

The space $F(X, Y)$ of based maps $X \to Y$ is itself a G-space with G-action defined by $(\gamma f)(x) = \gamma f(\gamma^{-1} x)$ for $\gamma \in G$.



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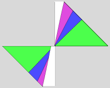

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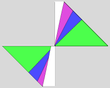

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A map of G-spaces $f : X \to Y$ is said to be a *weak G-equivalence* if for each subgroup $H \subset G$, the induced map $f : X^H \to Y^H$ is a weak equivalence in the nonequivariant sense.

G-CW complexes via orbits

There are two ways to generalize the construction of CW-complexes to the equivariant world, one based on orbits and a second based on representations.

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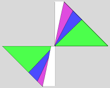
For the orbit construction, given any subgroup H of G we may form the homogeneous space G/H and its based counterpart, G/H_+.



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These are treated as 0-dimensional cells, and they play a role in equivariant theory analogous to the role of points in nonequivariant theory.

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G-CW complexes via orbits (continued)

We form the n-dimensional cells from these homogeneous spaces. In the unbased context, the cell-sphere pair is

$$(G/H \times D^n, G/H \times S^{n-1})$$

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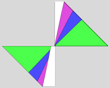

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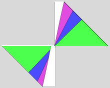

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Starting from these cell-sphere pairs, we form G-CW complexes exactly as nonequivariant CW-complexes are formed from the cell-sphere pairs (D^n, S^{n-1}). In such a complex, an element $\gamma \in G$ acts on a cell either by mapping it homeomorphically to another cell or by fixing it.

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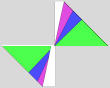
G-CW complexes via representations

Let V be an orthogonal representation of G. Denote its one-point compactification by S^V, with ∞ as the basepoint.



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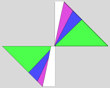

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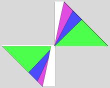

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We can use these objects to build G-CW complexes as well. In this case G can act on an individual cell by "rotating" it via the representation V.

More general G-CW complexes

We can also mix these two constructions by considering cell-sphere pairs such as

$$(G \times_H D(V), G \times_H S(V))$$

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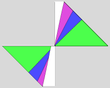
In such a complex, individual cells may be either permuted or rotated by an element of G.

Toward equivariant spectra

Before defining equivariant spectra, we need to recall the definition of an ordinary spectrum.



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We get a spectrum E from the prespectrum D by defining

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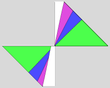

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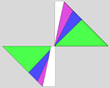
For technical reasons it is convenient to replace the collection $\{E_n\}$ by $\{EV\}$ indexed by finite dimensional subspaces V of a countably infinite dimensional real vector space U called a *universe.*

Toward equivariant spectra (continued)

The homotopy type of EV depends only on the dimension of V and there are homeomorphisms

$$EV \to \Omega^{|W|-|V|}EW \qquad \text{for } V \subset W \subset U.$$



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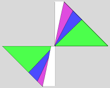
A map of spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : EV \to E'V$ which commute with the respective structure maps.



G-equivariant spectra

Let G be a finite group. Experience has shown that in order to do equivariant stable homotopy theory, one needs G-spaces EV indexed by finite dimensional orthogonal representations V sitting in a countably infinite dimensional orthogonal representation U.

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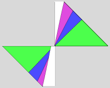

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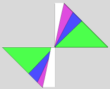

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This universe U is said to be *complete* if it contains infinitely many copies of each irreducible representation of G. A canonical example of a complete universe for finite G is the direct sum of countably many copies of the regular real representation of G.

G-equivariant spectra (continued)

A *G-equivariant spectrum* (G-spectrum for short) indexed on U consists of a based G-space $\mathrm{E}V$ for each finite dimensional subspace $V \subset U$ together with a transitive system of based G-homeomorphisms

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for $V \subset W \subset U$. Here $\Omega^V X = F(S^V, X)$ and $W - V$ is the orthogonal complement of V in W. As in the classical case, the G-homotopy type of EV depends only on the isomorphism class of V.

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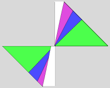

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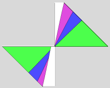

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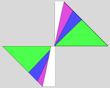

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A *suspension G-prespectrum* is a G-prespectrum in which the maps above are G-equivalences for V sufficiently large.

$RO(G)$-graded homotopy groups

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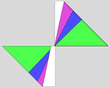

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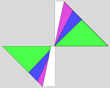
We define S^{-V} by saying its Wth space for $V \subset W$ is S^{W-V}. This is the analog of formal desuspension in the nonequivariant case.



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Given a virtual representation $\nu = W - V$, we define $S^{\nu} = \Sigma^{W} S^{-V}$. *Hence we have a collection of sphere spectra graded over the orthogonal representation ring* $RO(G)$*.*

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We define

$$\pi_{\nu}^{G}(X) = [S^{\nu}, X]_{G}$$

the $RO(G)$-graded homotopy groups of the G-spectrum X.

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MU as a C_2-spectrum

Let ρ denote the real regular representation of C_2.

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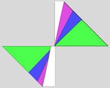

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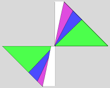

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Since any orthogonal representation V of C_2 is contained in $k\rho$ for $k \gg 0$, we can define the C_2-spectrum *MU* by

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This spectrum in known as real cobordism theory and has been studied by Landweber, Araki, Hu-Kriz and Kitchloo-Wilson.

Inducing and coinducing up to a larger group

Let $H \subset G$ be groups and let X be a H-space. There are two ways to get a G-space from it. The corresponding functors are the left and right adjoints to the forgetful functor from G-spaces to H-spaces.

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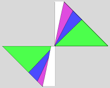

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$$(G/H \times D^n, G/H \times S^{n-1}).$$

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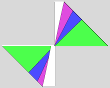
There is the *coinduced G-space*

$$\mathrm{map}_H(G, X) \quad = \quad \{f \in \mathrm{map}(G, X) \colon f(\gamma\eta^{-1}) = \eta f(\gamma) \\ \forall \eta \in H \text{ and } \gamma \in G\}$$



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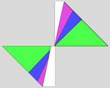

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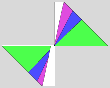

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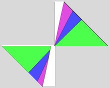
It extends to H-spectra. For a H-spectrum X we denote the coinduced G-spectrum by $N_H^G X$, the *norm of X along the inclusion $H \subset G$*.



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We apply this construction to the case $H = C_2$, $G = C_{2^{n+1}}$ and $X = MU$.

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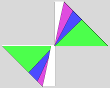

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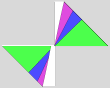

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Let $\gamma \in G$ be a generator and let z_i be a point in MU. Then the action of G on $MU^{(2^n)}$ is given by

$$\gamma(z_1 \wedge \cdots \wedge z_{2^n}) = \overline{z}_{2^n} \wedge z_1 \wedge \cdots \wedge z_{2^n-1},$$

where $\overline{z}_{2^n}$ is the complex conjugate of z_{2^n}.

Our spectrum M

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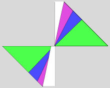

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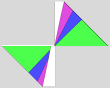
(i) It is a C_8-equivariant spectrum, while E_4 was merely an ordinary spectrum with a C_8 "action" for which a homotopy fixed point set could be defined.



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In particular this makes $MU^{(4)}$ into a C_8-spectrum. *Our spectrum $\tilde{M}$ is obtained from it by equivariantly inverting a certain element in its homotopy. Them $M = \tilde{M}^{C_8}$, which we will show to be equivalent to $\tilde{M}^{hC_8}$.*

The spectrum $MU^{(4)}$ has two advantages over our earlier candidate E_4.

(i) It is a C_8-equivariant spectrum, while E_4 was merely an ordinary spectrum with a C_8 “action” for which a homotopy fixed point set could be defined.

(ii) The action of C_8 on $\pi_*(MU^{(4)})$ is transparent, unlike its mysterious action on $\pi_*(E_4)$.