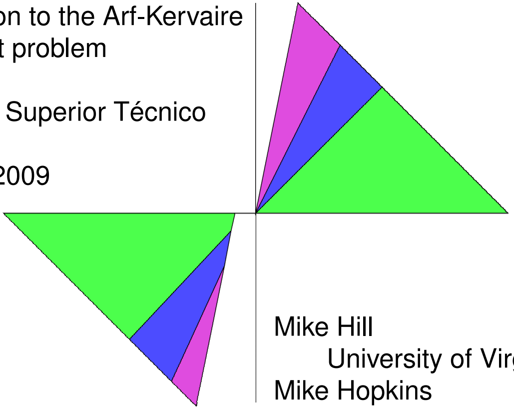
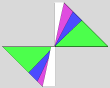
Lecture 1

A solution to the Arf-Kervaire invariant problem

Instituto Superior Técnico
Lisbon
May 5, 2009



Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester



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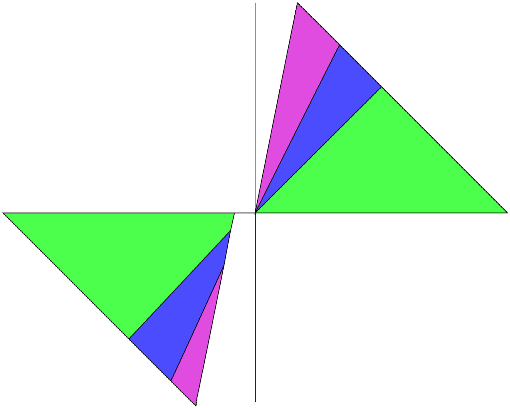

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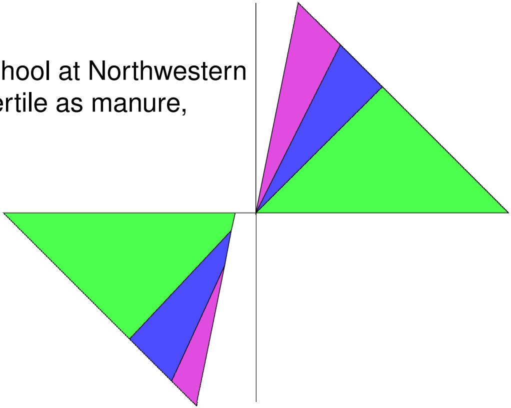

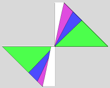

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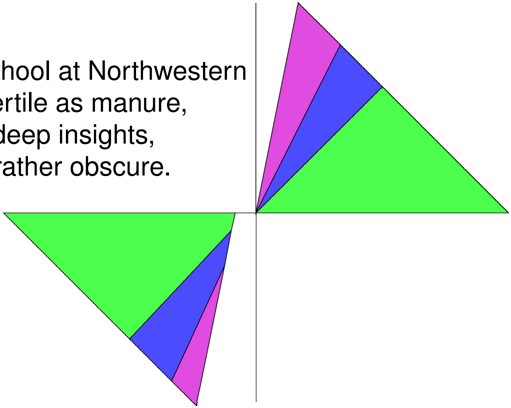

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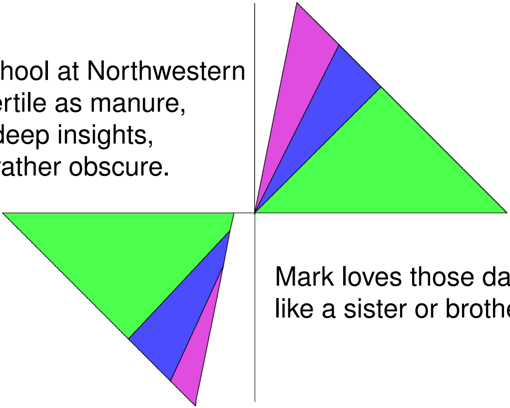

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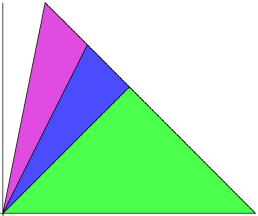

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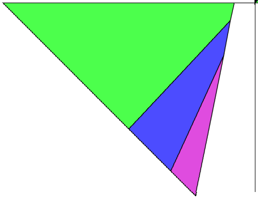

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Our main result

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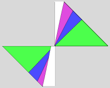

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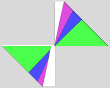

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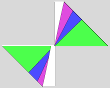
- For $(z_1, z_2) \in S^3$, define

$$\eta(z_1, z_2) = \begin{cases} z_1/z_2 & \text{for } z_2 \neq 0 \\ \infty & \text{for } z_2 = 0 \end{cases}$$



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- Maps $\nu : S^7 \to S^4$ and $\sigma : S^{15} \to S^8$ can be defined in a similar way using quaternions and Cayley numbers or octonions.

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- $\theta_4 \in \pi_{30}$ and $\theta_5 \in \pi_{62}$ were constructed in the '60s and '70s. The latter is the subject of a paper by Barratt-Jones-Mahowald published in 1984.

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- The status of $\theta_6 \in \pi_{126}$ is still open.

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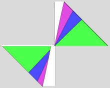
These theorems are opposite sides of the same coin.



The classification of exotic spheres

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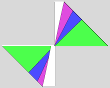

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Such a Σ^k, when embedded in Euclidean space, has a (nonunique) framing on its normal bundle and thus represents an element in the framed cobordism group Ω_k^{framed}. By the Pontrjagin-Thom construction, Ω_k^{framed} is isomorphic to the stable k-stem $\pi_k(S^0)$.

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The Pontrjagin-Thom construction

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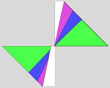

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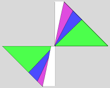
The resulting map $\tilde{f}_M : S^{n+k} \to S^n$ represents an element in the homotopy group $\pi_{n+k}(S^n)$, which for large n is isomorphic to the stable k-stem π_k. Pontrjagin showed that a cobordism between M_1 and M_2 leads to a homotopy between $\tilde{f}_{M_1}$ and $\tilde{f}_{M_2}$.

The Pontrjagin-Thom construction (continued)

He also showed the converse:



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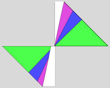

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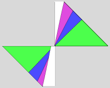

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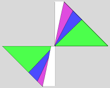

The classification of exotic spheres (continued)

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$$J : \pi_k(SO(n)) \to \pi_{n+k}(S^n).$$

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The classification of exotic spheres (continued)

Two framings on a framed k-manifold $M^k \subset \mathbf{R}^{n+k}$ differ by a map $M^k \to SO(n)$, the special orthogonal group of $n \times n$ real matrices. When M^k is a sphere (or a homotopy sphere) we get an element in $\pi_k(SO(n))$. This leads to the definition of the Hopf-Whitehead J-homomorphism

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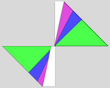

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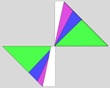

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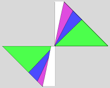

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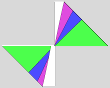

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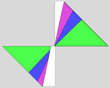
Such matrices have been classified over the real numbers up to the appropriate equivalence relation. The key invariant is the signature $\sigma(B)$, the difference between the number of positive and negative eigenvalues over $\mathbf{R}$, which is always divisible by 8.

An interesting matrix

Here is a symmetric matrix with even diagonal entries and signature 8.



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$$B = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

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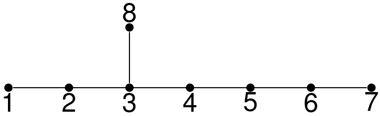

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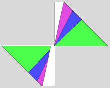

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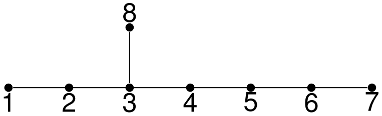

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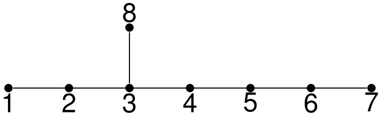

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Nodes i and j are connected by an edge iff $b_{i,j} \neq 0$.

Implications of the Hirzebruch signature theorem

The Hirzebruch signature theorem relates the signature of a smooth oriented closed 4ℓ-manifold M to its Pontrjagin numbers.

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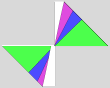
If our $M^{4\ell}$ is bounded by a sphere diffeomorphic to $S^{4\ell-1}$, then we can close M by attached a 4ℓ-ball. We get a new manifold $N^{4\ell}$ that is framed at every point except the center of that ball.



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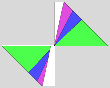
Such a manifold is said to be *almost framed*.

The signature of an almost framed manifold

Hirzebruch's formula implies that our signature $\sigma(N^{4\ell})$ is divisible by a certain integer related to the numerator of a Bernoulli number.



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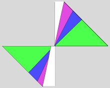

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The kernel of $\tau_{4\ell-1}$ is a large cyclic group whose order was determined by Kervaire-Milnor.

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To recap, we have a homomorphism

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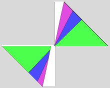

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It turns out that the two groups are related. For each ℓ, one is trivial iff the other is $\mathbf{Z}/2$.

Framed $4\ell + 2$-manifolds

We have a framed $4\ell + 2$-manifold, possibly bounded by a sphere $\Sigma^{4\ell+1}$. We can surger it into a 2ℓ-connected manifold.

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There is a map (not a homomorphism) $\mu : H^{2\ell+1} \to \mathbf{Z}/2$ such that

$$\lambda(x, y) = \mu(x) + \mu(y) + \mu(x + y).$$

The Arf invariant

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The *Arf-Kervaire invariant* $\Phi(M)$ of a framed $(4\ell + 2)$-manifold is defined to be the Arf invariant of its quadratic form.

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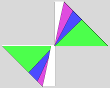

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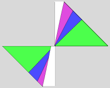

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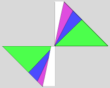

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Algebraic topologists attacked this question vigorously in the 1960s. The best result was the following.

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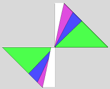

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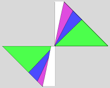
Relation to the Adams spectral sequence. *A framed* $(4\ell + 2)$*-manifold with nontrivial Arf-Kervaire invariant can exist only when* $\ell = 2^{j-1} - 1$ *for some integer* j*.*



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Relation to the Adams spectral sequence. *A framed $(4\ell+2)$-manifold with nontrivial Arf-Kervaire invariant can exist only when $\ell = 2^{j-1}-1$ for some integer j. In that case it exists iff the Adams spectral sequence element*

$$h_j^2 \in E_2^{2,2^{j+1}} = \mathrm{Ext}_A^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2)$$

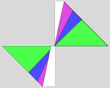
is a permanent cycle.

The classical Adams spectral sequence

Here A denotes the mod 2 Steenrod algebra and

$$h_j \in E_2^{2,2^j} = \mathrm{Ext}_A^{1,2^j}(\mathbf{Z}/2, \mathbf{Z}/2)$$

is the element corresponding to Sq^{2^j}.



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For $j \geq 4$ there is a nontrivial differential

$$d_2(h_j) = h_0 h_{j-1}^2.$$

The classical Adams spectral sequence (continued)

Here is a picture of the Adams spectral sequence for the prime 2 in low dimensions.

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Mike Hopkins
Doug Ravenel

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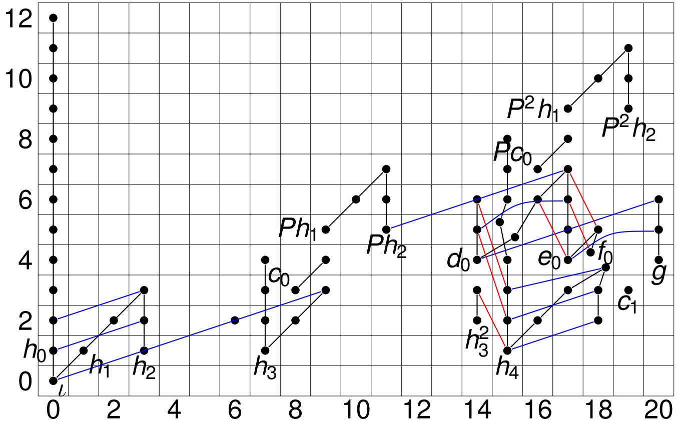

Spectral sequences in stable homotopy theory

Spectral sequences have been used to study the stable homotopy groups of spheres for the past 50 years.

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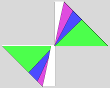

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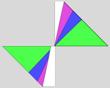

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Complex cobordism theory has proven to be extremely useful. The corresponding spectral sequence was first studied by Novikov in 1967 and is known as the *Adams-Novikov spectral sequence*.

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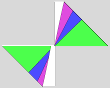
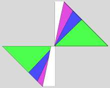


The Adams-Novikov spectral sequence for $p = 2$ in low dimensions

It is helpful to separate it into two parts having to do with v_1-periodic and v_1-torsion elements. These are related to the image and cokernel of J.



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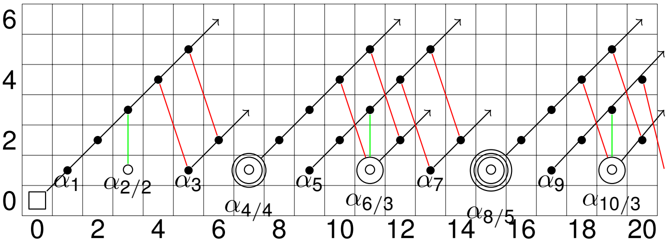

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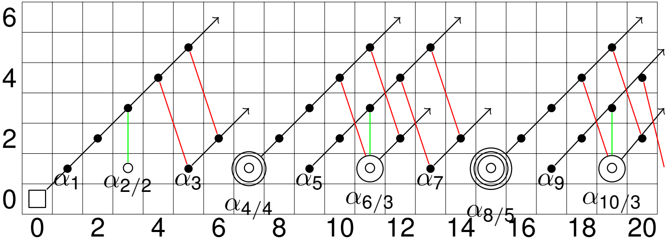

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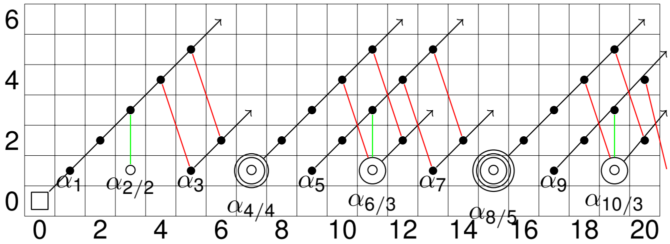

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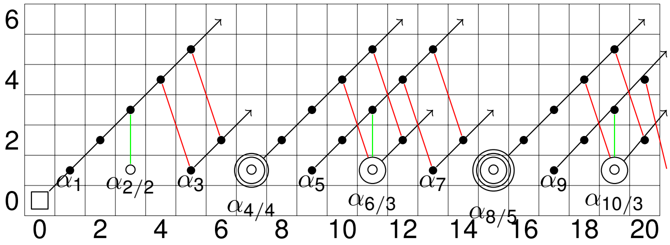

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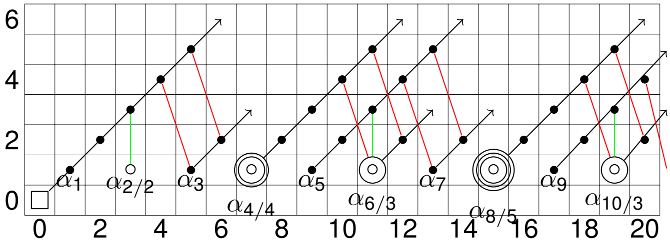

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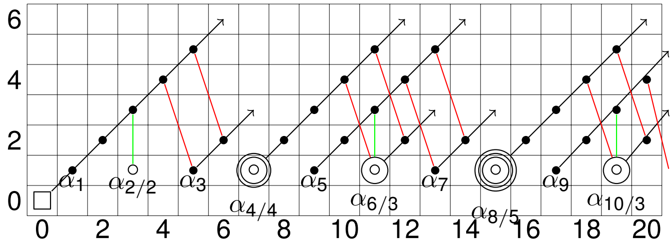

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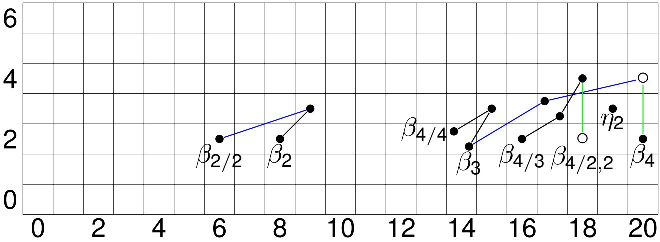

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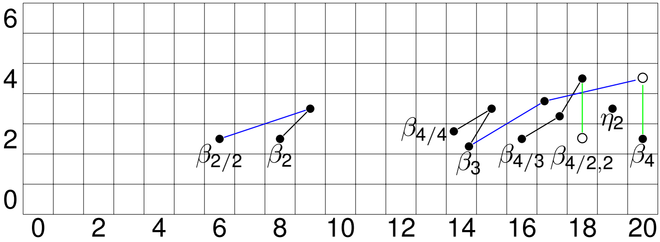
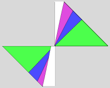


$\beta_{2/2} = h_2^2 = \theta_2$ and $\beta_{4/4} = h_3^2 = \theta_3$.



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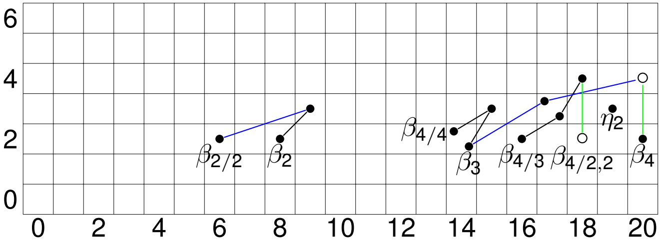

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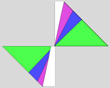

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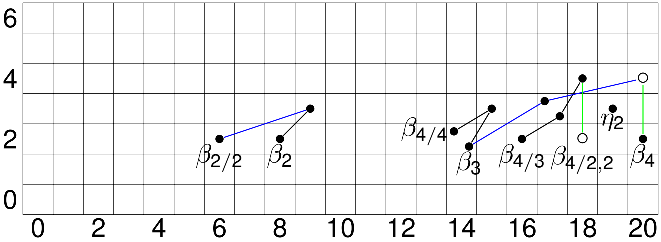

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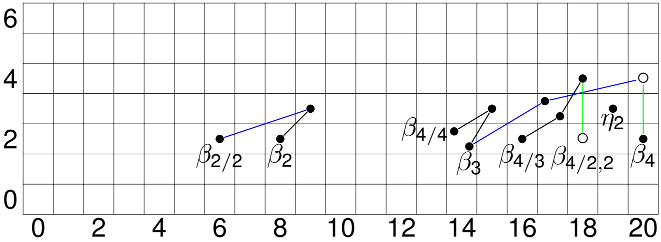

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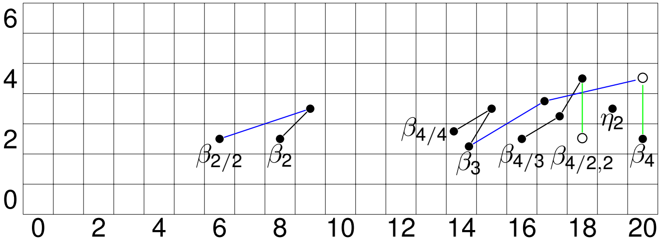

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The first differential in this spectral sequence occurs in dimension 26.

The Arf-Kervaire invariant in the Adams-Novikov spectral sequence

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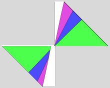

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It cannot be the target of a differential because its filtration is too low. We will show that it is the source of a nontrivial differential for $j \geq 7$.

Our strategy

We will produce a map $S^0 \to M$, where M is a nonconnective spectrum with the following properties.

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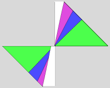

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We will produce a map $S^0 \to M$, where M is a nonconnective spectrum with the following properties.

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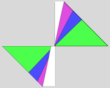

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The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.