THE SLICE SPECTRAL SEQUENCE FOR $RO(C_{p^n})$-GRADED SUSPENSIONS OF $H\underline{\mathbf{Z}}$ I

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WORK IN PROGRESS

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1. Introduction

A key ingredient to our solution to the Kervaire invariant one problem [HHR] was a new equivariant tool: the slice filtration. Generalizing the C_2-equivariant work of Dugger [Dug05] and modeled on Voevodsky's motivic slice filtration [Voe02], this is an exhaustive, strongly convergent filtration on genuine G-spectra for a finite group G. While quite powerful, the filtration quotients are quite difficult to determine, and much of our solution revolved around determining them for various spectra built out of MU.

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Classically one has a map from a spectrum X to its n^{th} Postinikov section $P^n X$. The latter is obtained from X by attaching cells to kill off all homotopy groups above dimension n, and the tower can be thought of as assembling X by putting in the homotopy groups one at a time. This process can also be described as a localization or nullification functor, where we localize by killing the subcategory of n-connected spectra. Though this is a very big subcategory, it is generated by a much smaller set: the spheres of dimension at least $(n+1)$.

In the equivariant context, there are two axes along which we can vary this construction:

(1) there are more spheres, namely representation spheres for real representations of G and
(2) there are subgroups of G, the behavior for each of which can be controlled.

The equivariant Postnikov filtration largely ignores the first option, choosing instead to build a localization tower killing the subcategories generated by all spheres (with a trivial action!) of dimension at least $(n+1)$ for all subgroups of G. Equivalently, we nullify the subcategory generated by all spectra of the form

$$G_+ \wedge_H S^m$$

for $m > n$.

The slice filtration takes more seriously the option of blending the representation theory and the homotopy theory.

Definition 1.1. *For each integer n, let $\tau_{\geq n}$ denote the localizing subcategory of G-spectra generated by*

$$G_+ \underset{H}{\wedge} S^{k\rho_H - \epsilon},$$

where H ranges over all subgroups of G, ρ_H is the regular representation of H, $k|H| - \epsilon \geq n$ and $\epsilon = 0, 1$.

If X is an object of $\tau_{\geq n}$, we say that X is slice greater than or equal to n and that X is slice $(n-1)$-connected.

The associated localization tower:

$$X \to P^* X$$

is the slice tower of X.

The slice filtration and the slice tower refine the non-equivariant Postnikov tower, in the sense that forgetting the G-action takes the slice tower to the Postnikov tower, but the equivariant layers of the slice tower are in general much more complicated. In particular, they need not be Eilenberg-Mac Lane spectra.

This paper is the start of a short series analyzing several curious points of the slice filtration for cyclic p-groups applied to a very simple yet interesting family: the suspensions of the Eilenberg-Mac Lane spectrum $H\underline{\mathbf{Z}}$ by virtual representation spheres. This paper will discuss the suspensions by multiples of the irreducible faithful representation λ; subsequent ones building on the thesis work of Yarnall will address the remaining cases.

We make a huge, blanket assumption and a slight abuse of notation. *Everything* we consider is localized at p, where p is the prime dividing the order of the group. Thus when we write things like $\mathbf{Z}$, we actually mean $\mathbf{Z}_{(p)}$. This being said, we have tried as much as possible to make integral statements which are prime independent (specifying actual representations, rather than J-equivalence classes whenever possible). Additionally, essentially everything we say holds for odd primes. The case of $p = 2$ behaves quite differently, given the existence of a non-trivial 1-dimensional real representation, the sign representation.

1.1. **Names of representations.** Fix an identification of C_{p^n} with the p^nth roots of unity μ_{p^n}. For all $k \in \mathbf{Z}$, let $\lambda(k)$ denote the composite of the inclusion of these roots of unity with the degree k-map on S^1. This is a representation of complex dimension 1.

The real regular representation will be denoted ρ. We have a splitting

$$\rho = 1 + \bigoplus_{j=1}^{\frac{p^n-1}{2}} \lambda(j)$$

for $p > 2$. We will also let $\bar{\rho}$ denote the reduced regular representation, the quotient of the regular representation by the trivial summand.

1.2. **Representation Spheres.** Representation spheres for C_{p^n} have an exceptionally simple cell structure. This renders computations much more tractable than one might expect. The key feature is that the subgroups of C_{p^n} are linearly ordered, and we can use this to build significantly smaller cell structures that might be expected. Our analysis follows [HHR] and [HHR11].

We first consider the sphere $S^{\lambda(p^k)}$. A cell structure is given by rays from the origin through the roots of unity, together with the sectors between these one cells. This gives a cell structure

$$S^0 \cup C_{p^n}/C_{p^k +} \wedge e^1 \cup C_{p^n}/C_{p^k +} \wedge e^2.$$

A picture is given in Figure 1 for $\lambda(1)$ and C_8.

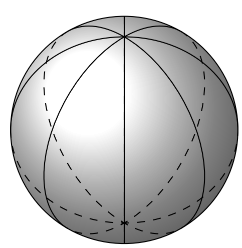

FIGURE 1.

We can smash these together to get a cell structure on S^V for any [virtual] representation V. The simplification arises from the different stabilizers which arise. For all $m \leq k$, the restriction $i^*_{C_{p^m}} \lambda(p^k)$ is trivial. Thus

$$\begin{aligned} S^{\lambda(p^m)+\lambda(p^k)} &\cong S^{\lambda(p^m)} \wedge S^{\lambda(p^k)} \\ &\cong S^{\lambda(p^m)} \wedge S^{\lambda(p^k)} \\ &\cong S^{\lambda(p^k)} \cup C_{p^k}/C_{p^m +} \wedge e^3 \cup C_{p^k}/C_{p^m +} \wedge e^4. \end{aligned}$$

By induction, this shows how to build any representation sphere: decompose V into irreducibles, sort these in order of decreasing stabilizer subgroup, and repeat the above trick. As an aside, this also allows a determination of a cell structure for virtual representation spheres, where we approach them the same way.

Our discussion of cell structure focused primarily on that of $\lambda(p^k)$. This is because equivariant homotopy groups are actually more honestly called $JO(G)$-graded: they depend not on the representation but rather on the homotopy type of the associated sphere. In the p-local context, most of the irreducible representations for C_{p^n} have equivalent one point compactifications: if r is prime to p, then

$$S^{\lambda(rp^k)} \simeq S^{\lambda(p^k)}.$$

For this reason, we can single out a single JO-equivalence class: let $\lambda_k = \lambda(p^k)$. *We will denote* λ_0 *simply by* λ. As k varies, the associated representation spheres hit every JO-equivalence class.

We summarize the above discussion in a simple proposition.

Proposition 1.2. *If*

$$V = k_n + k_{n-1}\lambda_{n-1} + \cdots + k_0\lambda_0,$$

then

$$\begin{aligned} S^V = S^{k_n} \cup {C_{p^n}/C_{p^{n-1}}}_+ \wedge e^{k_n+1} \underset{1-\gamma}{\cup} {C_{p^n}/C_{p^{n-1}}}_+ \wedge e^{k_n+2} \cup \cdots \cup \\ {C_{p^n}/C_{p^{n-1}}}_+ \wedge e^{k_n+2k_{n-1}-1} \underset{1-\gamma}{\cup} {C_{p^n}/C_{p^{n-1}}}_+ \wedge e^{k_n+2k_{n-1}} \cup \\ {C_{p^n}/C_{p^{n-2}}}_+ \wedge e^{k_n+2k_{n-1}+1} \cup \dots \underset{1-\gamma}{\cup} {C_{p^n}}_+ \wedge e^{\dim V}, \end{aligned}$$

where γ *is a generator of* C_{p^n}.

1.3. **The slice filtration.** We quickly recall several important facts about the slice filtration. This section contains no new results, and all proofs can be found in [Hil] and in [HHR, §3-4].

We first connect the slice filtration and the Postnikov filtration.

Proposition 1.3. *If* X *is* $(n-1)$*-connected for* $n \geq 0$*, then* X *is in* $\tau_{\geq n}$*.*

The converse of this is visibly not true, as generators of the form $S^{k\rho_G}$ are not $(k|G|-1)$-connected for $G \neq \{e\}$.

A slight elaboration on this lets us generalize this for smash products.

Proposition 1.4. *If* X *is* (-1)*-connected and* Y *is in* $\tau_{\geq n}$*, then* $X \wedge Y$ *is in* $\tau_{\geq n}$*.*

Unfortunately, smashing reduced regular representation spheres for G shows that results of this form are the best possible.

The form of the generating spectra for $\tau_{\geq n}$ shows that the slice tower commutes with suspensions by regular representation spheres.

Proposition 1.5. *For any spectrum* X*,*

$$P^k\Sigma^\rho X \cong \Sigma^\rho P^{k-|G|}X,$$

and identically for slices.

The final feature we will need is a recipe for determining for which n a particular representation sphere S^V is in $\tau_{\geq n}$. This is restatement of [Hil, Corollary 3.9], specializing the Corollary there to the case $Y = S^0$.

Proposition 1.6. *If W is a representation of G such that S^W is in $\tau_{\geq \dim W}$ and V is a sub representation such that*

(1) *the inclusion induces an equality $V^G = W^G$ and*
(2) *for all proper subgroups H, the restriction $i_H^* S^V$ is in $\tau_{\geq \dim V}$,*

then S^V is in $\tau_{\geq \dim V}$.

The conditions are very easy to check, by induction on the order of the group. In general, we will alway choose W to be a regular representation or a reduced regular representation, as these are guaranteed to be in the correct localizing categories.

2. Dramatis personae: some special $\underline{\mathbf{Z}}$-modules

A large number of Mackey functors will show up in our analysis. They are all variants of the constant Mackey functor $\underline{\mathbf{Z}}$ and the Mackey functor $\underline{B}$ that is the "Bredon homology Mackey functor". Many of our discussions are facilitated by pictures of Mackey functors, and following Lewis, we draw them vertically. A generic Mackey functor $\underline{M}$ for C_p will be drawn

$$\begin{tikzcd} \underline{M}(G/G) \arrow[d, bend right, "res"'] \\ \underline{M}(G/e) \arrow[u, bend right, "tr"'] \arrow[loop below, "\gamma"] \end{tikzcd}$$

where res is the restriction map, tr is the transfer, and γ generates the Weyl group. For larger cyclic groups, we will use the obvious extension of this notation. Moreover, if the Weyl action is trivial or obvious, then we will suppress the map γ.

2.1. Forms of $\underline{\mathbf{Z}}$.

The constant Mackey functor $\underline{\mathbf{Z}}$ (in which all restriction maps are the identity are the transfers are multiplication by the index) and its dual (in which the roles of restriction and transfer are reversed) are just two members of a family of distinct Mackey functors which take the value $\underline{\mathbf{Z}}$ on each orbit G/H. For C_{p^n}, there are 2^n distinct Mackey functors, corresponding to the n choices "is the restriction 1 or p" for adjacent subgroups.

A large subfamily of these occur in our discussion, and we give some notation here.

Definition 2.1. *Let $0 \leq j < k \leq n$ be integers. For each pair, let $\underline{\mathbf{Z}}(k,j)$ denote the Mackey functor with constant value $\mathbf{Z}$ for which the restriction maps are*

$$res^{p^{s+1}}_{p^s} = res^{C_{p^{s+1}}}_{C_{p^s}} = \begin{cases} 1 & s < j, \\ p & j \leq s < k, \\ 1 & k \leq s. \end{cases}$$

From now on, when a cyclic group appears as an index, we will abbreviate it by the order of the group as we did above.

Thus the restriction of $\underline{\mathbf{Z}}(k,j)$ to C_{p^j} is just the constant Mackey functor $\underline{\mathbf{Z}}$. For subgroups of C_{p^k} which properly contain C_{p^j}, it looks like the dual to the constant Mackey functor $\underline{\mathbf{Z}}$, and for those which properly contain C_{p^k}, it looks again like $\underline{\mathbf{Z}}$.

Equivalently, $\underline{\mathbf{Z}}(k,j)$ is the unique $\mathbf{Z}$-valued Mackey functor C_{p^n} in which $res_1^{p^j}$, $tr_{p^j}^{p^k}$ and $res_{p^k}^{p^n}$ are isomorphisms.

While $\underline{\mathbf{Z}}^* = \underline{\mathbf{Z}}(n,0)$, $\underline{\mathbf{Z}}$ does not occur in our list. If we allow $j=k$, then $\underline{\mathbf{Z}} = \underline{\mathbf{Z}}(j,j)$ for any choice of j. For the reader's convenience, we draw out the four variants of $\underline{\mathbf{Z}}$ that occur for C_{p^2}.

$\underline{\mathbf{Z}}$

$$\begin{tikzcd} \mathbf{Z} \arrow[d, bend right, "1"'] \\ \mathbf{Z} \arrow[u, bend right, "p"'] \arrow[d, bend right, "1"'] \\ \mathbf{Z} \arrow[u, bend right, "p"'] \end{tikzcd}$$

$\underline{\mathbf{Z}}(1,0)$

$$\begin{tikzcd} \mathbf{Z} \arrow[d, bend right, "1"'] \\ \mathbf{Z} \arrow[u, bend right, "p"'] \arrow[d, bend right, "p"'] \\ \mathbf{Z} \arrow[u, bend right, "1"'] \end{tikzcd}$$

$\underline{\mathbf{Z}}(2,1)$

$$\begin{tikzcd} \mathbf{Z} \arrow[d, bend right, "p"'] \\ \mathbf{Z} \arrow[u, bend right, "1"'] \arrow[d, bend right, "1"'] \\ \mathbf{Z} \arrow[u, bend right, "p"'] \end{tikzcd}$$

$\underline{\mathbf{Z}}(2,0)$

$$\begin{tikzcd} \mathbf{Z} \arrow[d, bend right, "p"'] \\ \mathbf{Z} \arrow[u, bend right, "1"'] \arrow[d, bend right, "p"'] \\ \mathbf{Z} \arrow[u, bend right, "1"'] \end{tikzcd}$$

In our analysis of the slices, we will need some elementary computations with $\underline{\mathbf{Z}}(k,j)$. We will eventually produce a projective resolution of all of these, but for now, we shall content ourselves to $\underline{\mathbf{Z}}(n,k)$.

First a brief digression on induced and restricted Mackey functors.

Definition 2.2. *Let $\underline{M}$ and $\underline{N}$ be Mackey functors for abelian groups G and $H \subset G$ respectively, and let i_H^* denote the forgetful functor from G-sets to H-sets. Then the induced Mackey functor $\uparrow_H^G(\underline{N})$ on G and the restricted Mackey functor $\downarrow_H^G(\underline{M})$ on H are given by*

$$\begin{aligned} \uparrow_H^G \underline{N}(G/K) &= \underline{N}(i_H^* G/K) \\ &= \begin{cases} \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} \underline{N}(H/K) & \text{for } K \subseteq H \\ (\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} \underline{N}(H/H))^K & \text{for } H \subseteq K \end{cases} \\ \downarrow_H^G \underline{M}(H/K) &= \underline{M}(G/K) \qquad \text{for } K \subseteq H. \end{aligned}$$

The Weyl action of G in $\uparrow_H^G \underline{N}$ is induced by the H-action on $\underline{N}$ and the G-action on G/K. The Weyl action of H in $\downarrow_H^G \underline{M}$ is the restriction of the the G-action on $\underline{M}$.

Note that the second description of $\uparrow_H^G \underline{N}$ is not complete for general G since there are subgroups K that neither contain nor are contained in H. However it is complete when G is a cyclic p-group, the case of interest here.

In particular if $\underline{N}$ is the fixed point Mackey functor for a $\mathbf{Z}[H]$-module N defined by $\underline{N}(H/K) = N^K$, then

$$\uparrow_H^G \underline{N} = \underline{\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} N},$$

the fixed point Mackey functor for the $\mathbf{Z}[G]$-module $\mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} N$.

We now return to cyclic p-groups. For the constant Mackey functor $\underline{\mathbf{Z}}$ on $G = C_{p^n}$, the composite of induction and the restriction to C_{p^k} is the fixed point Mackey functor for the C_{p^n}-module $\mathbf{Z}[C_{p^n}/C_{p^k}]$:

$$\uparrow_{p^k}^{p^n} \downarrow_{p^k}^{p^n} (\underline{\mathbf{Z}}) \cong \underline{\mathbf{Z}[C_{p^n}/C_{p^k}]}.$$

As such, it is very easy to describe maps out of this Mackey functor in the category of $\underline{\mathbf{Z}}$-modules:

$$\mathrm{Hom}_{\underline{\mathbf{Z}}}(\uparrow_{p^k}^{p^n}\downarrow_{p^k}^{p^n}(\underline{\mathbf{Z}}),\underline{M})\cong\mathrm{Hom}_{\downarrow_{p^k}^{p^n}\underline{\mathbf{Z}}}(\downarrow_{p^k}^{p^n}\underline{\mathbf{Z}},\downarrow_{p^k}^{p^n}\underline{M})\cong\underline{M}(C_{p^n}/C_{p^k})$$

for a Mackey functor $\underline{M}$ on G. In particular, these are all projective objects in the category of $\underline{\mathbf{Z}}$-modules.

These observations determine the maps in the following proposition.

Proposition 2.3. *The sequence*

$$0\to\underline{\mathbf{Z}}\to\underline{\mathbf{Z}[C_{p^n}/C_{p^k}]}\xrightarrow{1-\gamma}\underline{\mathbf{Z}[C_{p^n}/C_{p^k}]}\to\underline{\mathbf{Z}}(n,k)\to 0$$

is exact, where γ denotes a generator of C_{p^n}. The map

$$\underline{\mathbf{Z}[C_{p^n}/C_{p^k}]}\to\underline{\mathbf{Z}}(n,k)$$

is left-adjoint to the identity map from $\underline{\mathbf{Z}}$ to the restriction to C_{p^k} of $\underline{\mathbf{Z}}(n,k)$.

The first three terms are a projective resolution of $\underline{\mathbf{Z}}(n,k)$ in the category of $\underline{\mathbf{Z}}$-modules.

Proof. It is obvious that $\underline{\mathbf{Z}}$ is the kernel of the map $1-\gamma$. For subgroups of C_{p^k}, the fixed point Mackey functors are a direct sum of copies of $\underline{\mathbf{Z}}$, and γ acts by permuting the summands. Thus the quotient Mackey functor is also the constant Mackey functor $\underline{\mathbf{Z}}$ for subgroups of C_{p^k}.

When we look at subgroups of C_{p^n} which contain C_{p^k} then we are actually looking at the Mackey functor for the integral regular representation of the group C_{p^n}/C_{p^k}. The fixed points are given by various transfers, and the fixed points are the obvious inclusions. Passing to the quotient by $(1-\gamma)$ then sets all these inclusion maps to multiplication by the index. The transfer maps are the identity. $\square$

A useful way to understand these is that we have for C_{p^k} the value $\mathbf{Z}^{p^{n-k}}$. The restriction maps are diagonals, and the transfers are fold maps. Then the statement about passing to a quotient is obvious.

2.2. **Forms of $\underline{B}$.** The Hom groups between various $\underline{\mathbf{Z}}(k,j)$ are all easy to work out (the associated Hom groups are all $\mathbf{Z}$). We shall encounter maps from $\underline{\mathbf{Z}}(k,j)$ to $\underline{\mathbf{Z}}$, and it is not difficult to see that the maps are parameterized by where the element 1 in $\underline{\mathbf{Z}}(k,j)(G/e)$ goes.

Definition 2.4. *Let $\underline{B}_{k,j}$ denote the quotient Mackey functor associated to the inclusion $\underline{\mathbf{Z}}(k+j,j)\to\underline{\mathbf{Z}}$ which is an isomorphism when evaluated on G/e. Equivalently, $\underline{B}_{k,j}$ is the quotient of the unique map*

$$\underline{\mathbf{Z}}(k+j,0)\to\underline{\mathbf{Z}}(j,0)$$

which is the identity when restricted to C_{p^j}.

Let $\underline{B}^*_{k,j}$ be the quotient of the unique map

$$\underline{\mathbf{Z}}(n,j)\to\underline{\mathbf{Z}}(n,k+j)$$

which is the identity when restricted to C_{p^j}.

It will be helpful to allow $k=0$ in the above definition, in which case $\underline{B}_{0,j}$ is the zero Mackey functor.

The Mackey functor $\underline{B}_{k,j}$ is simple to describe:

$$\underline{B}_{k,j}(C_{p^n}/C_{p^m}) = \begin{cases} \mathbf{Z}/p^k & m \geq k+j, \\ \mathbf{Z}/p^{m-j} & j < m < k+j, \\ 0 & m \leq j. \end{cases}$$

All restriction maps are the canonical quotients, and the transfers are multiplication by p. Thus j refers to the largest subgroup for which $\underline{B}_{k,j}$ restricts to 0, while k indicates the order of the maximal p-torsion present.

Here is a partial Lewis diagram illustrating the definitions of $\underline{B}_{k,j}$ and $\underline{B}^*_{k,j}$ for $G = C_{p^n}$, where $m = n-k-j$, $0 \leq s \leq k$ and $s' = k-s$. The horizontal sequences are short exact.

$$\begin{tikzcd}
\underline{M} & \underline{\mathbf{Z}}(k+j,0) \arrow[r] & \underline{\mathbf{Z}}(j,0) \arrow[r] & \underline{B}_{k,j} & \underline{B}^*_{k,j} & \underline{\mathbf{Z}}(n,k+j) \arrow[l] & \underline{\mathbf{Z}}(n,j) \arrow[l] \\
\underline{M}(G/G) & \mathbf{Z} \arrow[r, "p^k"] \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^m"] & \mathbf{Z} \arrow[r] \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^m"] & \mathbf{Z}/p^k \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^m"] & \mathbf{Z}/p^k \arrow[d, bend right, "p^m"'] \arrow[d, <-, bend left, "1"] & \mathbf{Z} \arrow[l] \arrow[d, bend right, "p^m"'] \arrow[d, <-, bend left, "1"] & \mathbf{Z} \arrow[l, "p^k"'] \arrow[d, bend right, "p^m"'] \arrow[d, <-, bend left, "1"] \\
\underline{M}(G/C_{p^{k+j}}) & \mathbf{Z} \arrow[r, "p^k"] \arrow[d, bend right, "p^{s'}"'] \arrow[d, <-, bend left, "1"] & \mathbf{Z} \arrow[r] \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^{s'}"] & \mathbf{Z}/p^k \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^{s'}"] & \mathbf{Z}/p^k \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^{s'}"] & \mathbf{Z} \arrow[l] \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^{s'}"] & \mathbf{Z} \arrow[l, "p^k"'] \arrow[d, bend right, "p^{s'}"'] \arrow[d, <-, bend left, "1"] \\
\underline{M}(G/C_{p^{j+s}}) & \mathbf{Z} \arrow[r, "p^s"] \arrow[d, bend right, "p^s"'] \arrow[d, <-, bend left, "1"] & \mathbf{Z} \arrow[r] \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^s"] & \mathbf{Z}/p^s \arrow[d, bend right] \arrow[d, <-, bend left] & \mathbf{Z}/p^s \arrow[d, bend right] \arrow[d, <-, bend left] & \mathbf{Z} \arrow[l] \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^s"] & \mathbf{Z} \arrow[l, "p^s"'] \arrow[d, bend right, "p^s"'] \arrow[d, <-, bend left, "1"] \\
\underline{M}(G/C_{p^j}) & \mathbf{Z} \arrow[r, "1"] \arrow[d, bend right, "p^j"'] \arrow[d, <-, bend left, "1"] & \mathbf{Z} \arrow[r] \arrow[d, bend right, "p^j"'] \arrow[d, <-, bend left, "1"] & 0 \arrow[d, bend right] \arrow[d, <-, bend left] & 0 \arrow[d, bend right] \arrow[d, <-, bend left] & \mathbf{Z} \arrow[l] \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^j"] & \mathbf{Z} \arrow[l, "1"'] \arrow[d, bend right, "1"'] \arrow[d, <-, bend left, "p^j"] \\
\underline{M}(G/e) & \mathbf{Z} \arrow[r, "1"] & \mathbf{Z} \arrow[r] & 0 & 0 & \mathbf{Z} \arrow[l] & \mathbf{Z} \arrow[l, "1"']
\end{tikzcd}$$

We will also need the following hybrid of $\underline{B}_{k,j}$ and $\underline{B}^*_{k,j}$.

Definition 2.5. *For $0 \leq \ell$, let $\underline{B}^\ell_{k,j}$ be the Mackey functor which agrees with $\underline{B}_{k,j}$ when restricted to $C_{p^{\ell+k+j}}$ and which for groups containing $C_{p^{\ell+k+j}}$, agrees with $\underline{B}^*_{k,j}$.*

For $-k \leq \ell \leq 0$, let $\underline{B}^\ell_{k,j}$ be $\underline{B}_{k+\ell,j}$.

In other words, $\underline{B}^\ell_{k,j}$ becomes the dual once we reach $C_{p^n}/C_{p^{\ell+k+j}}$ while moving up the Lewis diagram. In particular,

$$\underline{B}^0_{k,j} = \underline{B}^*_{k,j} \qquad \text{and} \qquad \underline{B}^m_{k,j} = \underline{B}_{k,j},$$

where again $m = n-k-j$. For $0 < \ell < m$, $\underline{B}^\ell_{k,j}$ is the quotient of a map between $\mathbf{Z}$-valued Mackey functors, but not between the ones defined above. Here is another partial Lewis diagram illustrating this definition.

$\underline{M}$	$\underline{B}_{k,j}=\underline{B}^m_{k,j}$	$\underline{B}^\ell_{k,j}$	$\underline{B}^*_{k,j}=\underline{B}^0_{k,j}$
$\underline{M}(G/G)$	$\mathbf{Z}/p^k$	$\mathbf{Z}/p^k$	$\mathbf{Z}/p^k$
	$1 \downarrow \quad \uparrow p^{m-\ell}$	$p^{m-\ell} \downarrow \quad \uparrow p^{\ell}$	$p^{m-\ell} \downarrow \quad \uparrow 1$
$\underline{M}(G/C_{p^{\ell+k+j}})$	$\mathbf{Z}/p^k$	$\mathbf{Z}/p^k$	$\mathbf{Z}/p^k$
	$1 \downarrow \quad \uparrow p^{\ell}$	$1 \downarrow \quad \uparrow 1$	$p^{\ell} \downarrow \quad \uparrow 1$
$\underline{M}(G/C_{p^{k+j}})$	$\mathbf{Z}/p^k$	$\mathbf{Z}/p^k$	$\mathbf{Z}/p^k$
	$1 \downarrow \quad \uparrow p^{k-s}$	$1 \downarrow \quad \uparrow p^{k-s}$	$1 \downarrow \quad \uparrow p^{k-s}$
$\underline{M}(G/C_{p^{j+s}})$	$\mathbf{Z}/p^s$	$\mathbf{Z}/p^s$	$\mathbf{Z}/p^s$
	$\downarrow \quad \uparrow$	$\downarrow \quad \uparrow$	$\downarrow \quad \uparrow$
$\underline{M}(G/C_{p^j})$	0	0	0
	$\downarrow \quad \uparrow$	$\downarrow \quad \uparrow$	$\downarrow \quad \uparrow$
$\underline{M}(G/e)$	0	0	0

In our analysis of $S^{k\lambda}\wedge H\underline{\mathbf{Z}}$, we will only need those $\underline{B}_{k,j}$ with $j=0$. To simplify notation, let $\underline{B}_k$ be the Mackey functor $\underline{B}_{k,0}$.

3. The $RO(C_{p^n})$-graded homotopy of $H\underline{\mathbf{Z}}$

3.1. **Realizing forms of $\underline{\mathbf{Z}}$ topologically.** A nice feature of the Mackey functor $\underline{\mathbf{Z}}(k,j)$ is that $H\underline{\mathbf{Z}}(k,j)$ is a virtual representation sphere smashed with $H\underline{\mathbf{Z}}$. We will see this via a direct chain complex computation.

Theorem 3.1. *For $j<k$, we have an equivalence*

$$\Sigma^{\lambda_k-\lambda_j}H\underline{\mathbf{Z}}\simeq H\underline{\mathbf{Z}}(k,j).$$

Proof. We use the simplified cell structures described above. If we smash S^{λ_k} with the cell structure for $S^{-\lambda_j}$, then we have a complex of the form

$$\big(C_{p^n}/C_{p^j}\big)_+\wedge S^0\cup\big(C_{p^n}/C_{p^j}\big)_+\wedge e^1\cup e^0\wedge S^{\lambda_k}.$$

Smashing with $H\underline{\mathbf{Z}}$ and applying $\underline{\pi_*}$ gives a chain complex of Mackey functors

$$\begin{tikzcd}
\underline{\mathbf{Z}[C_{p^n}/C_{p^k}]} \arrow[d, "1-\gamma"'] \arrow[drr, "\uparrow_{p^k}^{p^n} res^{p^k}_{p^j}"] & & \\
\underline{\mathbf{Z}[C_{p^n}/C_{p^k}]} \arrow[d] \arrow[drr, "-\uparrow_{p^k}^{p^n} res^{p^k}_{p^j}"'] & & \underline{\mathbf{Z}[C_{p^n}/C_{p^j}]} \arrow[d, "1-\gamma"] \\
\underline{\mathbf{Z}} & & \underline{\mathbf{Z}[C_{p^n}/C_{p^j}]}
\end{tikzcd}$$

The map labeled $\uparrow_{p^k}^{p^n} res^{p^k}_{p^j}$ is actually the canonical inclusion of fixed points for the subgroups. Thus there is no homology in the top degree. Similarly, the kernel of the left most $1-\gamma$ is $\underline{\mathbf{Z}}$, which maps isomorphically to the kernel of the second $1-\gamma$. Thus the homology of our chain complex is the homology of the simpler complex:

$$\begin{tikzcd}
\underline{\mathbf{Z}}(n,k) \arrow[d] \arrow[drr, "-\uparrow_{p^k}^{p^n} res^{p^k}_{p^j}"] & & \\
\underline{\mathbf{Z}} & & \underline{\mathbf{Z}}(n,j)
\end{tikzcd}.$$

Now we make a few observations. For subgroups of C_{p^k}, the map

$$\underline{\mathbf{Z}}(n,k) \to \underline{\mathbf{Z}}$$

is an isomorphism, and therefore the pushout looks like $\underline{\mathbf{Z}}(n,j)$. For subgroups of C_{p^n} which contain C_{p^k}, the map

$$\underline{\mathbf{Z}}(n,k) \to \underline{\mathbf{Z}}(n,j)$$

is an equivalence, and therefore the pushout looks like $\underline{\mathbf{Z}}$. This is the description of $\underline{\mathbf{Z}}(k,j)$. □

Corollary 3.2. *As a Mackey functor,*

$$\underline{\pi}_{\lambda_k} S^{\lambda_j} \wedge H\underline{\mathbf{Z}} = \underline{\mathbf{Z}}(k,j).$$

Using the multiplicative structure of $H\underline{\mathbf{Z}}$, we can extend this to a map of $H\underline{\mathbf{Z}}$-module spectra. Let

$$v_{j/k} \colon S^{\lambda_k} \wedge H\underline{\mathbf{Z}} \to S^{\lambda_j} \wedge H\underline{\mathbf{Z}}$$

be a generator of $\underline{\mathbf{Z}}(k,j)(C_{p^n}/C_{p^n})$ which restricts to a chosen orientation of the underlying 2-spheres.

Since we are working at an odd prime, all representations of C_{p^n} are orientable. In the equivariant context, this means that

$$\underline{\pi}_{\dim V}(S^V \wedge H\underline{\mathbf{Z}}) \cong \underline{H}_{\dim V}(S^V; \underline{\mathbf{Z}}) \cong \underline{\mathbf{Z}}.$$

A choice of generator for $\underline{\mathbf{Z}}(C_{p^n}/C_{p^n})$ determines one for all other values of the Mackey functor and, since all restriction and transfer maps are injective, this is detected in the underlying homotopy. Let

$$u_V \colon S^{\dim V} \to S^V \wedge H\underline{\mathbf{Z}}$$

be such a choice of generator. We will refer to these as "orientation classes". By choosing one for each irreducible representation of C_{p^n}, we can ensure this is compatible with the multiplicative structure in the sense that

$$u_{V\oplus W} = u_V \cdot u_W.$$

Proposition 3.3. *The composite*

$$S^2 \xrightarrow{u_{\lambda_k}} S^{\lambda_k} \wedge H\underline{\mathbf{Z}} \xrightarrow{v_{j/k}} S^{\lambda_j} \wedge H\underline{\mathbf{Z}},$$

is homotopic to u_{λ_j}. Thus

$$v_{j/k} = \frac{u_{\lambda_j}}{u_{\lambda_k}}.$$

Proof. The composite is detected in $\underline{\pi}_2 S^{\lambda_j} \wedge H\underline{\mathbf{Z}} \cong \underline{\mathbf{Z}}$, and it is determined by the underlying map. By assumption, this is a map of degree 1. □

3.2. **The gold (or *au*) relation.** The orientation classes u_{λ_k} carve out a portion of the $RO(C_{p^n})$-graded homotopy of $H\underline{\mathbf{Z}}$. They do not tell a complete picture, however. We need some classes in the Hurewicz image.

Definition 3.4. *For a representation V, let*

$$a_V \colon S^0 \to S^V$$

denote the inclusion of the origin and the point at infinity into S^V.

We will often refer to the classes a_V as “Euler classes”. It is obvious that if $V^G \neq \{0\}$, then a_V is null-homotopic (as the map factors through the inclusion $S^{V^G} \to S^V$). It follows that the restriction of a_V to any subgroup for which $V^H \neq \{0\}$ is also null-homotopic. For groups for which $V^H = \{0\}$, the class a_V and all of its suspensions and powers are essential. We will also let a_V denote the Hurewicz image. Our chain complex models show that this generates $H_0(S^V; \underline{\mathbf{Z}})$.

The classes a_{λ_k} can be generalized in an interesting way. If $j < k < n$, there are unstable maps

$$\frac{a_{\lambda_k}}{a_{\lambda_j}} \colon S^{\lambda_j} \to S^{\lambda_k}.$$

This can be formed most easily by observing that the p^{k-j}th power map extends to the one point compactifications.

Composing with the Euler class a_{λ_j} obviously results in S^{λ_k}, and by our assumptions on k, this map is essential (and in fact, torsion free).

Smashing this with $H\underline{\mathbf{Z}}$ produces an essential map which we will also call by the same name

$$\frac{a_{\lambda_k}}{a_{\lambda_j}} \colon S^{\lambda_j} \wedge H\underline{\mathbf{Z}} \to S^{\lambda_k} \wedge H\underline{\mathbf{Z}}.$$

Our chain complex models show that the classes $u_{\lambda_0}, \dots, u_{\lambda_{n-1}}$ and $a_{\lambda_0}, \dots, a_{\lambda_{n-1}}$ generate as an algebra the portion of the $RO(C_{p^n})$-graded homotopy of $H\underline{\mathbf{Z}}$ in dimensions of the form $k - V$. There is a relation which is immediate from our fractional classes.

Theorem 3.5. *In the $RO(C_{p^n})$-graded homotopy of $H\underline{\mathbf{Z}}$, we have a relation*

$$a_{\lambda_k} u_{\lambda_j} = p^{k-j} a_{\lambda_j} u_{\lambda_k}.$$

Proof. If we compose $a_{\lambda_k}/a_{\lambda_j}$ with the map $u_{\lambda_j}/u_{\lambda_k}$, then we get a map

$$\frac{u_{\lambda_j}}{u_{\lambda_k}} \frac{a_{\lambda_k}}{a_{\lambda_j}} \colon S^{\lambda_j} \wedge H\underline{\mathbf{Z}} \to S^{\lambda_j} \wedge H\underline{\mathbf{Z}}.$$

This map is completely determined by the effect in underlying homotopy, and this is obviously the map of degree p^{k-j}. Clearing denominators gives the desired result. □

Remark 3.6. *There is only one other kind of relation in this sector of the $RO(C_{p^n})$-graded homotopy:*

$$p^{n-j} a_{\lambda_j} = 0.$$

This follows from the above remarks, as the restriction to C_{p^j} of a_{λ_j} is null-homotopic. Composing with the transfer yields multiplication by p^{n-j} in homology. The cellular models for representation spheres show that no other relations are possible: all homology groups

$$H_{k-V}(S^0; \underline{\mathbf{Z}})$$

are cyclic.

3.3. **Fiber sequences realizing $H\underline{B}_{k,j}$.** If we apply the Eilenberg-Mac Lane functor to the short exact sequence

$$\underline{\mathbf{Z}}(k+j,j) \to \underline{\mathbf{Z}} \to \underline{B}_{k,j},$$

then we get a fiber sequence of spectra. Combing this with equivalence from Theorem 3.1 yields a fiber sequence

$$\Sigma^{\lambda_k-\lambda_j} H\underline{\mathbf{Z}} \to H\underline{\mathbf{Z}} \to H\underline{B}_{k,j}.$$

Since $\underline{B}_{k,j}$ is zero on cells induced up from subgroups of C_{p^j}, for all $i \leq j$, the map a_{λ_i} induces an equivalence

$$H\underline{B}_{k,j} \simeq \Sigma^{\lambda_i} H\underline{B}_{k,j}.$$

Smashing our fiber sequence with S^{λ_j} then gives a more readily interpreted form.

Proposition 3.7. *There is a fiber sequence*

$$\Sigma^{\lambda_k} H\underline{\mathbf{Z}} \to \Sigma^{\lambda_j} H\underline{\mathbf{Z}} \to H\underline{B}_{k,j}.$$

Proposition 3.7 is the heart of our analysis of the slice filtration for these $RO(G)$-graded suspensions: we can strip away copies of S^{λ_j}, replacing them with copies of S^{λ_k} for $k > j$.

Before continuing, we note two more natural fiber sequences which tie in the dual Mackey functors.

Proposition 3.8. *We have fiber sequences*

$$\Sigma^{\lambda_k-\lambda_0} H\underline{\mathbf{Z}} \to \Sigma^{\lambda_j-\lambda_0} H\underline{\mathbf{Z}} \to \underline{B}_{k,j}$$

and

$$\Sigma^{\lambda_n-\lambda_j} H\underline{\mathbf{Z}} \to \Sigma^{\lambda_n-\lambda_k} H\underline{\mathbf{Z}} \to \underline{B}^*_{k,j}.$$

4. Bredon homology computations

We will see in the §5 below that each slice for $s^{\infty\lambda} \wedge H\underline{\mathbf{Z}}$ has the smash product of a certain representation sphere with some $H\underline{B}_k$. The relevant homotopy groups will be given in Theorem 4.7. §4.1 gives some preparatory calculations for this result. §4.2 gives some results that will help us determine the slices in §5.

4.1. **Bredon homology and $\underline{B}_{k,j}$.**

4.1.1. *Exact Sequences involving $\underline{B}$.* A small initial observation is needed. The Mackey functors $B_{(k,j}$ do not really depend on C_{p^n} in the sense that once n is at least $(k+j)$, the value for the Mackey functors on orbits stabilizes. Thus in the short exact sequences that follow, it is helpful to pretend they are written as Mackey functors for the Prüfer group C_{p^∞} and then restrict the result to any chosen C_{p^n}. The net result is that if any index is greater than n, just set it equal to n in the formulas.

To compute the E_2-page of the slice spectral sequence for $S^{m\lambda}$, we need to determine $H_*(S^V; \underline{B}_{k,j})$ for various V. For this, we make several additional observations. If we let ϕ_{p^j} denote the quotient map $C_{p^n} \to C_{p^n}/C_{p^j}$, then there is an associated inflation (aka pullback) functor

$$\phi^*_{p^j} \colon C_{p^n}/C_{p^j}\text{-}\mathcal{M}ackey \longrightarrow C_{p^n}\text{-}\mathcal{M}ackey.$$

The Mackey functors $\underline{B}_{k,j}$ are all in the image of $\phi^*_{p^j}$, and this is an exact functor. For us, it simply inserts zeros at the bottom of a Lewis diagram.

Proposition 4.1. *We have natural isomorphisms*

$$\phi_{p^j}^*\underline{B}_{k,m}\cong\underline{B}_{k,m+j} \text{ and } \phi_{p^j}^*\underline{B}_{k,m}^{\ell}\cong\underline{B}_{k,m+j}^{\ell}.$$

It therefore suffices to consider the case of $j=0$. We then have the following generalization of Proposition 2.3, and the proof is a direct computation.

Proposition 4.2. *We have an exact sequence*

$$0\to\underline{B}_{\min(\ell,k),0}\to\uparrow_{p^\ell}^{p^n}\downarrow_{p^\ell}^{p^n}\underline{B}_{k,0}\xrightarrow{\gamma-1}\uparrow_{p^\ell}^{p^n}\downarrow_{p^\ell}^{p^n}\underline{B}_{k,0}\to\underline{B}_{k,0}^{\ell-k}\to 0.$$

Proposition 4.2 is the initial input for computing the Mackey functor cellular homology. The map $\gamma-1$ is the attaching map of the even cells in S^V to the odd cells when we use our minimal cell model. Thus the cellular homology is completely determined by looking at the maps from the cokernels in Proposition 4.2 to the kernels as ℓ varies. We have two intermediate lemmata which serve as the work-horse for any computations. These are immediate computations, and in both cases, the middle map is an isomorphism when we restrict to the subgroup for which both middle Mackey functors coincide.

Lemma 4.3. *For $\ell\geq -k$, we have an exact sequence*

$$0\to\underline{B}^*_{\min(k,k+\ell),\max(k,k+\ell)}\to\underline{B}_{k,0}^{\ell}\to\underline{B}_{k,0}\to\underline{B}_{k,k+\ell}\to 0$$

in which the middle map is the identity when restricted to $C_{p^{k+\ell}}$.

Pulling this back gives what happens for a general pair (k,j):

Corollary 4.4. *For any j and for $\ell\geq -k$, we have an exact sequence*

$$0\to\underline{B}^*_{\min(k,k+\ell),\max(k,k+\ell)+j}\to\underline{B}_{k,j}^{\ell}\to\underline{B}_{k,j}\to\underline{B}_{k,k+\ell+j}\to 0$$

in which the middle map is the identity when restricted to $C_{p^{k+\ell+j}}$.

4.1.2. *The Bredon homology of S^V with coefficients in $\underline{B}_{k,j}$.* We can splice all of the exact sequences from the previous subsection to determine the Bredon homology of any representation sphere with coefficients in $\underline{B}_{k,j}$. This section is only relevant for readers who wish to run the slice spectral sequence, as it explains how to completely determine the slice E_2-term for $S^{\infty\lambda}\wedge H\underline{\mathbf{Z}}$ and for $S^{m\lambda}\wedge H\underline{\mathbf{Z}}$.

First, we reduce to the cases most of interest.

Proposition 4.5. *Let W be a representation of C_{p^n}, let $W_j=W^{C_{p^{j+1}}}$, and let U_j be the orthogonal complement of W_j in W. Then the natural inclusion*

$$a_{U_j}\colon S^{W_j}\to S^W$$

induces an equivalence upon smashing with $H\underline{B}_{k,j}$.

Proof. By assumption, the stabilizer of every point of $U_j-\{0\}$ is properly contained in $C_{p^{j+1}}$. We therefore conclude that we can choose a cellular model for S^{U_j} such that the zero skeleton is S^0 and every cell of dimension greater than zero is induced from a proper subgroup of $C_{p^{j+1}}$. Since the restriction of $\underline{B}_{k,j}$ to any proper subgroup of $C_{p^{j+1}}$ is zero, we conclude that the Euler class a_{U_j} is an equivalence upon smashing with $H\underline{B}_{k,j}$. □

Thus the Bredon homology computation we care about cannot distinguish between S^W and its $C_{p^{j+1}}$-fixed points. It is also immediate that the chain complex computing the Bredon homology is in the image of the pullback functor, so without loss of generality, we may assume that $j = 0$.

It is also clear that we may, without loss of generality, restrict attention to fixed point free representations of C_{p^n}, since the inclusion of a trivial summand serves only to shift the homology groups around.

Notation 4.6. *Let V be a fixed-point free representation of C_{p^n} that restricts trivially to C_p. Then find a JO-equivalent representation W, and write*

$$W = \sum_{0<r\leq n} k_r \lambda_{n-r},$$

Define natural numbers K_i for $i > 0$ by

$$K_i = 2\sum_{r=1}^{i} k_r,$$

and let $K_0 = 0$.

Finally, let $0 < i_0 < \cdots < i_m$ denote those integers i such that $k_i \neq 0$, and let $h_r = n - i_r$.

Since the homology groups of any space are bound between its connectivity and its dimension, we need only determine the homology groups between dimensions 0 and K_m.

Theorem 4.7. *The Mackey functor valued Bredon homology groups of S^V with coefficients in $\underline{B}_k$ are given by*

$$\underline{H}_s(S^V;\underline{B}_k) = \begin{cases} 0 & s<0, \\ \underline{B}_{k,h_0} & s=0 \\ \underline{B}^*_{\min(h_0,k),\max(h_0,k)} & s=1 \\ \underline{B}_{\min(k,n-i),n-i} & K_{i-1}+2\leq s\leq K_i-2, i \textit{ even} \\ \underline{B}^*_{\min\,(k,n-i),n-i} & K_{i-1}+3\leq s\leq K_i-1, i \textit{ odd} \\ \underline{B}_{\min\,(k,h_r),h_{r+1}} & s=K_{i_r} \\ \underline{B}_{\min\,(k,h_{r+1}),\max(\min(k,h_r),h_{r+1})} & s=K_{i_r}+1. \end{cases}$$

Proof. This follows immediately from our cell structure of Proposition 1.2 and the short exact sequences above.

Applying the Mackey functor $\underline{B}_k$ to the cellular complex results in a chain complex of Mackey functors:

$$\underline{B}_k \leftarrow \uparrow_{h_0}\downarrow_{h_0}\underline{B}_k \xleftarrow{1-\gamma} \uparrow_{h_0}\downarrow_{h_0}\underline{B}_k \leftarrow \uparrow_{h_0}\downarrow_{h_0}\underline{B}_k \xleftarrow{1-\gamma} \uparrow_{h_0}\downarrow_{h_0}\underline{B}_k \leftarrow \cdots \leftarrow \uparrow_{h_m}\downarrow_{h_m}\underline{B}_k$$

(for ease of exposition, we have assumed that $k_{i_0} \geq 2$). By our assumptions on V, we know that the sole instance of $\underline{B}_k$ lies in dimension 0, and the complex is homologically graded. In particular, there is the map $1-\gamma$ from every positive even degree to the adjacent odd degree, up through $\dim V$. Proposition 4.2 allows us to identify the kernels and cokernels of the maps labeled $1-\gamma$, letting us replace the cellular chains with a simpler complex:

$$\begin{tikzcd}
\underline{B}_k & \uparrow_{h_0}\downarrow_{h_0}\underline{B}_k \arrow[l] \arrow[d, two heads] & \uparrow_{h_0}\downarrow_{h_0}\underline{B}_k \arrow[l, "1-\gamma"'] & \uparrow_{h_0}\downarrow_{h_0}\underline{B}_k \arrow[l] \arrow[d, two heads] & \uparrow_{h_0}\downarrow_{h_0}\underline{B}_k \arrow[l, "1-\gamma"'] & \cdots \arrow[l] \\
& \underline{B}_{k,0}^{h_0-k} \arrow[ul, dotted] & \underline{B}_{\min(h_0,k),0} \arrow[u, hook] & \underline{B}_{k,0}^{h_0-k} \arrow[l, dotted] & \underline{B}_{\min(h_0,k),0} \arrow[u, hook] & \cdots \arrow[l, dotted]
\end{tikzcd}$$

Having controlled the differential from even degrees to odd degrees, it remains only to understand the maps from the cokernels of the various $1-\gamma$s to the kernels. This is exactly what Lemma 4.3 records. The result is then a simple application of this lemma and counting. □

4.2. A criterion for slice codimension.

4.2. A criterion for slice codimension. In determining slice dimensions, we will need a fact about the Bredon cohomology for a few representation spheres.

In what follows, let $\underline{H}^k(X;\underline{M})$ denote the Mackey functor valued Bredon cohomology of X with coefficients in $\underline{M}$. This can be computed by choosing an equivariant cellular model for X and building the cellular co-chain complex which assigns to an n-cell with stabilizer H the Mackey functor $\uparrow_H^G\downarrow_H^G\underline{M}$. The maps are induced by the cellular attaching maps in the standard way. Equivalently, we can simply build a cellular model for DX, the Spanier-Whitehead dual of X, and compute the negative Bredon homology of DX.

We need this only for X a representation sphere and $\underline{M}$ a $\underline{\mathbf{Z}}$-module, and even there, we need very few specific computations.

Lemma 4.8. *Let V be a representation of C_{p^n}. If λ_m is a summand of V, then for every $\underline{M}$ for which $res^{p^\ell}_{p^m}$ is an isomorphism for $\ell \geq m$,*

$$\underline{H}^0(S^V;\underline{M}) = \underline{H}^1(S^V;\underline{M}) = 0.$$

Proof. If λ_n is a summand, then connectivity implies the result. The stabilizers of the complement of the origin in λ_j grows as j does, and our cell models show that the bottom skeleton of S^V is that of S^{λ_j} for j maximal amongst those summands that occur in V. Without loss of generality, we may assume $j = m$, and we need only compute $\underline{H}^\epsilon(S^{-\lambda_m};\underline{M})$.

Our standard model for $D(S^{\lambda_m}) = S^{-\lambda_m}$ is

$$(C_{p^n}/C_{p^m})_+ \wedge S^{-2} \cup (C_{p^n}/C_{p^m})_+ \wedge e^{-1} \cup e^0.$$

This gives us the following cochain complex for the cohomology of S^{λ_m} with coefficients in any Mackey functor $\underline{M}$:

$$\begin{array}{cccccc}
\underline{C}^*(S^{\lambda_m};\underline{M}): & \underline{M} & \xrightarrow{res^{p^n}_{p^m}} & \uparrow^{p^n}_{p^m}\downarrow^{p^n}_{p^m}(\underline{M}) & \xrightarrow{1-\gamma} & \uparrow^{p^n}_{p^m}\downarrow^{p^n}_{p^m}(\underline{M}), \\
\dim & 0 & & -1 & & -2
\end{array}$$

and the first map is

(1) the composite of the restriction map to C_{p^m} and the diagonal for subgroups of C_{p^n} which contain C_{p^m}, and
(2) the diagonal for subgroups of C_{p^m}.

For subgroups of C_{p^m} and for any $\underline{M}$, the sequence is obviously exact. This is really a restatement of the fact that λ_m restricts to the trivial representation for C_{p^m}.

For the remaining subgroups, we observe that the kernel of the map denoted $(1-\gamma)$ looks like the constant Mackey functor $\underline{M}(C_{p^n}/C_{p^m})$ (the value for subgroups of C_{p^m} is of course *a priori* different, but those are already understood). Thus we have an exact sequence for the cohomology of S^{λ_m} with coefficients in $\underline{M}$ evaluated on C_{p^n}/C_{p^ℓ} for $\ell \geq m$:

$$\begin{tikzcd} \underline{H}^0(S^{\lambda_m};\underline{M})(C_{p^n}/C_{p^\ell}) \arrow[r, hook] & \underline{M}(C_{p^n}/C_{p^\ell}) \arrow[d, "res^{p^\ell}_{p^m}"] & \\ & \underline{M}(C_{p^n}/C_{p^m}) \arrow[r, two heads] & \underline{H}^1(S^{\lambda_m};\underline{M})(C_{p^n}/C_{p^\ell}). \end{tikzcd}$$

□

Corollary 4.9. *If λ_m is a summand of V for $m \geq k$, then*

$$H^0(S^V;\underline{B}_k) = H^1(S^V;\underline{B}_k) = 0.$$

5. The slices of $S^{\infty\lambda} \wedge H\underline{\mathbf{Z}}$

Let $L = S^{\infty\lambda} \wedge H\underline{\mathbf{Z}}$. We will see that its slice tower is determined by a sequence of representations. The cofiber sequences in the previous section, together with a surprisingly simple induction argument, show that the naturally occurring tower is the slice tower.

From this section on, let $p > 2$.

5.1. A sequence of representations.

The basic argument is that we will strip away copies of λ from L, replacing them with other irreducible representations until we produce a copy of 2ρ. Since the slice tower commutes with ρ-suspensions, this will give us an iterative, periodic approach.

The sequence of representations is curiously simple.

Definition 5.1. *For all $j \geq 1$, let*

$$V_j = \bigoplus_{m=1}^{j} \lambda(2m-1),$$

and let $V_0 = 0$.

The first thing to observe is that $V_{p^n} = 2\rho$ (we run through a complete set of coset representatives). Since $\lambda(k + p^n) = \lambda(k)$ as representations of C_{p^n}, we therefore conclude that

$$V_{j+p^n} = 2\rho + V_j$$

for all $j \geq p^n$.

We also have $V_{(p^n\pm 1)/2} = \rho \pm 1$ and $V_{p^n-j} = 2\rho - V_j$ for $0 < j < p^n$.

The following formula for V_j may be useful.

Theorem 5.2. Floor function formula for V_j. *Let our group be C_{p^n} for p an odd prime. For $\ell \geq 0$, let $c_\ell = (p^\ell - 1)/2$. Then*

$$V_j = \sum_{0\leq \ell<n} \lambda_\ell \sum_{\substack{0\leq t\leq p-1\\ t\neq c_1}} \left\lfloor \frac{j + p^{\ell+1} - 1 - c_\ell - p^\ell t}{p^{\ell+1}} \right\rfloor + 2\left\lfloor \frac{j+c_n}{p^n} \right\rfloor.$$

Proof. By definition, $V_0 = 0$ and

$$V_j = V_{j-1} + \lambda(2j-1) = V_{j-1} + \lambda_{v_p(2j-1)}$$

We will illustrate with the case $p = 3$, for which $c_1 = 1$. The $\ell = 0$ term in our sum is

$$\lambda_0 \sum_{\substack{0 \le t \le 2 \\ t \ne 1}} \left\lfloor \frac{j+3-1-c_0-t}{3} \right\rfloor = \lambda_0 \left(\left\lfloor \frac{j+2}{3} \right\rfloor + \left\lfloor \frac{j}{3} \right\rfloor \right)$$

Increasing j by 1 increases the coefficient of λ_0 when j is congruent to 1 or 3 mod 3, namely the times when $v_3(2j-1) = 0$ and we are adding a copy of λ_0 to V_{j-1}.

Similarly the $\ell = 1$ term

$$\lambda_1 \sum_{\substack{0 \le t \le 2 \\ t \ne 1}} \left\lfloor \frac{j+9-1-c_1-3t}{9} \right\rfloor = \lambda_1 \left(\left\lfloor \frac{j+7}{9} \right\rfloor + \left\lfloor \frac{j+1}{9} \right\rfloor \right)$$

Increasing j by 1 increases the coefficient of λ_1 when j is congruent to 2 or 8 mod 9, namely the times when $v_3(2j-1) = 1$ and we are adding a copy of λ_1 to V_{j-1}.

The same thing happens for larger ℓ and larger primes.

When we get to $\ell = n$, we have $\lambda_n = 2$ by definition. It will be added to V_{j-1} when ever $2j-1$ is divisible by p^n, which is equivalent to j being congruent to $-c_n$ modulo p^n. The final coefficient, $\lfloor (j+c_n)/p^n \rfloor$ increases for precisely such j. □

Corollary 5.3. Coefficients in cases of interest.

Let $j = (ap-1)/2$ for odd $a > 0$, and let $a = 2b+1$. Then the coeffcient $k_{n-\ell}$ of λ_ℓ in the formula of Theorem 5.2 is

$$k_{n-\ell} = \begin{cases} (p-1)b + c_1 & \text{for } \ell = 0 \\ \displaystyle\sum_{\substack{0 \le t \le p-1 \\ t \ne c_1}} \left\lfloor \frac{b+p^\ell-1-c_{\ell-1}-p^{\ell-1}t}{p^\ell} \right\rfloor & \text{for } 0 < \ell < n \\ \left\lfloor \dfrac{bp+c_1+c_n}{p^n} \right\rfloor & \text{for } \ell = n. \end{cases}$$

In the case $0 < \ell < n$ we have

$$(p-1)\left\lfloor \frac{b}{p^\ell} \right\rfloor \le k_{n-\ell} \le (p-1)\left\lfloor \frac{b+p^\ell}{p^\ell} \right\rfloor.$$

Note that k_0 is the coefficient of λ_n, which by definition is the trivial representation of degree two, *Proof.* The coefficient k_n of λ_0 for $j = (ap-1)/2$ with a odd is

$$\begin{aligned}
k_n &= \sum_{\substack{0\le t\le p-1\\ t\ne c_1}} \left\lfloor \frac{j+p-1-t}{p} \right\rfloor \\
&= \sum_{\substack{0\le t\le p-1\\ t\ne c_1}} \left\lfloor \frac{ap-1+2p-2-2t}{2p} \right\rfloor \\
&= \sum_{\substack{0\le t\le p-1\\ t\ne c_1}} \left(\left\lfloor \frac{a-1}{2} \right\rfloor + \left\lfloor \frac{3p-3-2t}{2p} \right\rfloor \right) \\
&= \sum_{\substack{0\le t\le p-1\\ t\ne c_1}} \left(b+1+ \left\lfloor \frac{p-3-2t}{2p} \right\rfloor \right) \\
&= (p-1)(b+1) - c_1 = (p-1)b + c_1 = c_1 a.
\end{aligned}$$

That of λ_ℓ for $0 < \ell < n$ is

$$\begin{aligned}
k_{n-\ell} &= \sum_{\substack{0\le t\le p-1\\ t\ne c_1}} \left\lfloor \frac{(ap-1)/2 + p^{\ell+1} - 1 - c_\ell - p^\ell t}{p^{\ell+1}} \right\rfloor \\
&= \sum_{\substack{0\le t\le p-1\\ t\ne c_1}} \left\lfloor \frac{bp + c_1 + p^{\ell+1} - 1 - c_\ell - p^\ell t}{p^{\ell+1}} \right\rfloor \\
&= \sum_{\substack{0\le t\le p-1\\ t\ne c_1}} \left\lfloor \frac{b + p^\ell - 1 - c_{\ell-1} - p^{\ell-1} t}{p^{\ell}} \right\rfloor \qquad \text{since } c_1 - c_\ell = -pc_{\ell-1}.
\end{aligned}$$

Since

$$0 < 1 + c_{\ell-1} + p^{\ell-1}t < p^\ell \qquad \text{for each } t,$$

we have

$$(p-1)\left\lfloor \frac{b}{p^\ell} \right\rfloor \le k_{n-\ell} \le (p-1)\left\lfloor \frac{b+p^\ell}{p^\ell} \right\rfloor.$$

We leave the case of k_0 to the reader. □

This will let us simplify several arguments.

5.2. **Special slices.** We are now in a position where we can show that all of the $RO(G)$-graded suspensions of $H\underline{B}_j$ which arise are in fact slices. We begin with several simple theorems about V_j-fold suspensions.

Theorem 5.4. *For any Y which is (-1)-connected, $S^{V_j} \wedge Y$ is slice greater than or equal to $(2j)$.*

Proof. Since (-1)-connected spectra are slice non-negative and since smashing with a slice non-negative spectrum at worst preserves slice connectivity, it suffices to show that S^{V_j} is slice greater than or equal to $(2j)$. Here we can invoke Proposition 1.6. We first observe that the restriction of V_j to any subgroup of C_{p^n} is the corresponding V_j for that subgroup. Thus we may use induction on the order of the group in a very simple away. For $0 \leq j < p^n$ let

$$W_j = \begin{cases} \rho - 1 & j \leq \frac{p^n - 1}{2} \\ 3\rho - 1 & j \geq \frac{p^n+1}{2}, \end{cases}$$

then $V_j \subset W_j$ and $V_j^G = W_j^G$. By induction, for all proper subgroups H, $i_H^* S^{V_j}$ is slice greater than or equal to $(2j)$, and therefore Proposition 1.6 implies that desired result. □

Corollary 5.5. *For any Mackey functor $\underline{M}$, $S^{V_j} \wedge H\underline{M}$ is slice greater than or equal to $(2j)$.*

As is often the case, showing that the desired spectra are slice less than or equal to $(2j)$ is by direct computation. We will reduce the computation to Corollary 4.9.

Theorem 5.6. *For every $\underline{M}$ in which $res^{p^\ell}_{p^m}$ is an isomorphism for $m \geq k := v_p(2j+1)$, $S^{V_j} \wedge H\underline{M}$ is slice less than or equal to $(2j)$. Here $v_p(2j+1)$ denotes the number of powers of p dividing $2j+1$.*

Proof. We need to show that for all triples (r, H, ϵ) such that $r|H| - \epsilon > 2j$, we have

$$[G_+ \wedge_H S^{r\rho_H - \epsilon}, \Sigma^{V_j} H\underline{M}]^G = [S^{r\rho_H-\epsilon}, \Sigma^{V_j} H\underline{M}]^H = 0.$$

We can again appeal to the linear ordering of the subgroups, breaking them into two cases. First, observe that if $2j+1 \equiv 0 \mod p^k$, then $j \equiv \frac{p^k-1}{2} \mod p^k$. This means that

$$i^*_{C_{p^k}} V_j = b\rho + \bar{\rho},$$

for some b. Hence for any Mackey functor $\underline{M}$,

$$i^*_{C_{p^k}} \Sigma^{V_j} H\underline{M} \cong \Sigma^{b\rho+\bar{\rho}} H i^*_{C_{p^k}} \underline{M}$$

is a $2j$-slice for C_{p^k}. This is the essential feature of the argument, as it shows that the slice upper bound on this spectrum is determined by the slice upper bound for those subgroups of G which properly contain C_{p^k}. The condition that all restriction maps are injections is the same for all subgroups in this range, so without loss of generality, we need only show

$$[S^{r\rho_G - \epsilon}, \Sigma^{V_j} H\underline{M}]^G = 0.$$

For the rest of this proof we will denote ρ_G by simply ρ. It suffices to show this for $j \leq p^n - 1$, as larger values of j result in ρ-fold suspensions and these commute with slice dimension. We therefore only have to consider slice cells $S^{r\rho_G - \epsilon}$, where $rp^n - \epsilon > 2j$.

There are two cases, depending on whether or not we have passed the special value of j: $\frac{p^n-1}{2}$.

For $1 \leq j \leq \frac{p^n-1}{2}$, we know that $V_j \subset S^{\bar{\rho}}$, with equality iff $j = \frac{p^n-1}{2}$. We therefore must compute

$$[S^{r\rho-\epsilon}, \Sigma^{V_j} H\underline{M}] = [S^{(r-1)\bar{\rho} + V_j^{\perp} + (r-\epsilon)}, H\underline{M}],$$

where $V_j^{\perp}$ is the orthogonal complement of V_j in $\bar{\rho}$. If $r - \epsilon > 0$, then the connectivity of the domain exceeds the coconnectivity of the range, and therefore all homotopy classes are zero. We pause here to note that this includes the exceptional value of j, as here $r = \epsilon = 1$ results in $S^{r\rho-\epsilon}$ being a $2j$-slice cell.

For the remaining cases, we assume that $r = \epsilon = 1$ and $j < \frac{p^n-1}{2}$. By assumption on j, now, the representation $\lambda(2j+1) \subset V_j^{\perp}$. The corresponding sphere is JO-equivalent to the sphere of $\lambda_{v_p(2j+1)}$. We are therefore computing

$$H^0\big(S^{V_j^{\perp}};\underline{M}\big),$$

and by Lemma 4.8, this group is zero.

For $\frac{p^n-1}{2} < j < p^n$, we have a similar analysis. Here $\rho \subset V_j \subset 2\rho$. Assume that $rp^n - \epsilon > 2j$, as before. In particular, we see that $r \geq 2$. We again consider

$$[S^{r\rho-\epsilon}, \Sigma^{V_j} H\underline{M}] = [S^{(r-2)\bar{\rho}+V_j^{\perp}+(r-2)-\epsilon}, H\underline{M}],$$

where $V_j^{\perp}$ is the orthogonal complement of V_j in 2ρ. If $(r-2)-\epsilon$ is positive, then connectivity finishes the proof. We need to consider $(r-2)-\epsilon$ being 0 or -1 (the latter only occurring for $j < p^n$), and this reduces the computation to that of

$$H^{\epsilon}(S^{V_j^{\perp}+(r-2)\bar{\rho}};\underline{M}).$$

If $r \geq 3$, then all λ_i are summands of $\bar{\rho}$, and Lemma 4.8 immediately implies these groups are zero. If $r = 2$, then just as before, we observe that $V_j^{\perp}$ contains the representation $\lambda(2j+1)$, and so Lemma 4.8 again implies that these groups are zero.

Finally we observe that since $v_p(2p^n+1) = 0$, the conditions of the theorem require that all restrictions maps be injections. This means that $H\underline{M}$ is a zero slice, and hence

$$\Sigma^{V_j} H\underline{M} = \Sigma^{2\rho} H\underline{M}$$

is a $2p^n$-slice. □

Combining these theorems yields a number of slices.

Corollary 5.7. *For all j, the spectrum $\Sigma^{V_j} H\underline{B}_{v_p(2j+1)}$ is a $(2j)$-slice.*

Corollary 5.8. *For all j, the spectra $\Sigma^{V_j} H\underline{\mathbf{Z}}$ and $\Sigma^{V_j} H\underline{\mathbf{Z}}(k,\ell)$ for $k \leq v_p(2j+1)$ or $\ell \geq v_p(2j+1)$ are $(2j)$-slices.*

The condition on k and ℓ above is that $v_p(2j+1) \notin (\ell, k)$, the open interval from ℓ to k.

5.3. **The 2ρ-periodic slice tower.** The results of the previous section actually determine for us the slice tower. It is easiest to observe this via the 2ρ-suspensions that showed up. As a corollary to the previous section, we have a tower

$$\begin{array}{ccccc}
\Sigma^{2\rho}L & \longrightarrow & \Sigma^{2\rho-\lambda}L & \longrightarrow & \Sigma^{2\rho-\lambda}H\underline{B}_0 \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & \Sigma^{V_j}L & \longrightarrow & \Sigma^{V_j}H\underline{B}_{v_p(2j+1)} \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & L & \longrightarrow & \Sigma^0 H\underline{B}_0
\end{array}$$

Recall that $H\underline{B}_0$ is contractible, $V_{p^n} = 2\rho$ and $V_{p^n-1} = 2\rho - \lambda$. All of the layers on the right-hand side of the tower are slices by Corollary 5.7. In particular, they are all simultaneously less than or equal to and greater than or equal to the dimension of the associated representation sphere. The final one (the one in the upper right corner) is then less than or equal to $(2p^n - 2)$. Now the 2ρth suspension of L is greater than or equal to $2p^n$, and we therefore conclude that all of the cofiber sequences are exactly the cofiber sequences

$$P_n(X) \to X \to P^{n-1}(X)$$

for various spectra X of the form $S^V \wedge L$. Splicing them all together, using the obvious 2ρ-periodicity of the tower, we see that we have determined the slice co-tower of L. We group this together in the following theorem.

Theorem 5.9. *All odd slices and all slices in dimensions not congruent to -1 modulo p of L are contractible. The $(ap^k - 1)$-slice, where a is odd and prime to p, is given by*

$$\Sigma^{V_j} H\underline{B}_k \qquad \textit{where } j = (ap^k - 1)/2,$$

and the maps in the tower are all determined by the cofiber sequences

$$\Sigma^{V_{j+1}} H\underline{\mathbf{Z}} \xrightarrow{u_\lambda/u_{\lambda(2j+1)}} \Sigma^{V_j+\lambda} H\underline{\mathbf{Z}} \to \Sigma^{V_j} H\underline{B}_{v_p(2j+1)}.$$

We leave it to the interested reader to state a corollary to Theorem 4.7 as it applies to the $\Sigma^{V_j} H\underline{B}_k$ here. In particular it would say that the top dimension for the homology of the $ap^k - 1$-slice is $ap^{k-1} - 1$. This means that in the usual slice spectral sequence chart all nontrivial elements occur between lines of slopes $p - 1$ and $p^n - 1$ meeting at $(s, t) = (-1, 0)$.

Remark 5.10. *Yarnall's thesis shows that the slice sections of $S^n \wedge H\underline{\mathbf{Z}}$ are all of the form $S^V \wedge H\underline{\mathbf{Z}}$. This result is of a different flavor. Our result here is that the slice* connective covers *$P_n L$ are all representations spheres smashed with $H\underline{\mathbf{Z}}$. This is a curious and confusing fact.*

We will now apply Theorem 4.7 to determine the homotopy groups of the slices described by Theorem 5.9.

Figure 2 shows the spectral sequence for C_{27}, subect to the following regrading convention. Under the usual convention, meaning the point (s, t) shows the Mackey functor $\underline{E}_2^{s,t+s}$, all nontrivial elements would lie between lines of slopes 2 and 26 intersecting at $(s, t) = (0, -1)$, meaning $(x, y) = (-1, 0)$. In order to save space we rescale in such a way that the vanishing lines have slope 0 and 8 and the horizontal coordinate is unchanged. In the general case this means

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t - s \\ s - \frac{p-1}{p}(t+1) \end{bmatrix}.$$

This change rescales differentials as well, converting d_{1+2pr} for $r > 0$ (the only ones that can occur dues to sparseness) to d_{1+2r}. The figure makes use of Mackey functor symbols indicated in Table 1. The dashed lines are non-trivial extensions determined by hidden transfers.

The reader may construct similar charts for other cyclic groups using the information in Corollary 5.3.

Table 1. Mackey functors appearing in Figures 2 and 3.

$\underline{B}_1 = \underline{B}_{1,0}$	$\underline{B}_{1,1}$	$\underline{B}^*_{1,1}$	$\underline{B}_{1,2} = \underline{B}^*_{1,2}$
$\bullet$	$\underline{\bullet}$	$\underline{\bullet}^*$	$\underline{\underline{\bullet}} = \underline{\underline{\bullet}}^*$
$\mathbf{Z}/3 \underset{0}{\overset{1}{\rightleftarrows}} \mathbf{Z}/3 \underset{0}{\overset{1}{\rightleftarrows}} \mathbf{Z}/3 \rightleftarrows 0$	$\mathbf{Z}/3 \underset{0}{\overset{1}{\rightleftarrows}} \mathbf{Z}/3 \rightleftarrows 0 \rightleftarrows 0$	$\mathbf{Z}/3 \underset{1}{\overset{0}{\rightleftarrows}} \mathbf{Z}/3 \rightleftarrows 0 \rightleftarrows 0$	$\mathbf{Z}/3 \rightleftarrows 0 \rightleftarrows 0 \rightleftarrows 0$
$\underline{B}_2 = \underline{B}_{2,0}$	$\underline{B}_{2,1}$	$\underline{B}_3 = \underline{B}_{3,0}$	$\underline{\mathbf{Z}}$
$\circ$	$\underline{\circ}$	$\circledcirc$	$\square$
$\mathbf{Z}/9 \underset{3}{\overset{1}{\rightleftarrows}} \mathbf{Z}/9 \underset{3}{\overset{1}{\rightleftarrows}} \mathbf{Z}/3 \rightleftarrows 0$	$\mathbf{Z}/9 \underset{3}{\overset{1}{\rightleftarrows}} \mathbf{Z}/3 \rightleftarrows 0 \rightleftarrows 0$	$\mathbf{Z}/27 \underset{3}{\overset{1}{\rightleftarrows}} \mathbf{Z}/9 \underset{3}{\overset{1}{\rightleftarrows}} \mathbf{Z}/3 \rightleftarrows 0$	$\mathbf{Z} \underset{3}{\overset{1}{\rightleftarrows}} \mathbf{Z} \underset{3}{\overset{1}{\rightleftarrows}} \mathbf{Z} \underset{3}{\overset{1}{\rightleftarrows}} \mathbf{Z}$

6. The slices of $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$

The slice tower for $L = S^{\infty\lambda} \wedge H\underline{\mathbf{Z}}$ almost completely determines the one for $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$. We first bound the slice tower of $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$ from above.

Theorem 6.1. *For all $m \geq 0$, $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$ is less than or equal to $2m$.*

Proof. Since $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$ restricts to the same spectrum for any subgroup, by induction on the order of G, we know that the restriction is less than or equal to $2m$. This means that we need only check that for all k and ϵ such that $k \cdot p^n - \epsilon > 2m$,

$$[S^{k\rho-\epsilon}, S^{m\lambda} \wedge H\underline{\mathbf{Z}}] = [S^0, S^{m\lambda-k\rho+\epsilon} \wedge H\underline{\mathbf{Z}}] = 0.$$

We again can show this using a cell decomposition of $S^{m\lambda}$ and then smashing it with $S^{-k\rho+\epsilon}$. This gives

$$\begin{aligned}\big(S^0 \cup (C_{p^n+} \wedge e^1) \cup \cdots \cup (C_{p^n+} \wedge e^{2m})\big) \wedge S^{-k\rho+\epsilon} \wedge H\underline{\mathbf{Z}} \\ = \big(S^{-k\rho+\epsilon} \cup (C_{p^n+} \wedge e^{1-kp^n+\epsilon}) \cup \cdots \cup (C_{p^n+} \wedge e^{2m-kp^n+\epsilon})\big) \wedge H\underline{\mathbf{Z}}.\end{aligned}$$

The only ways this could have homotopy in dimension 0 are possibly from the first term (when $k = \epsilon = 1$) or from the final term in the decomposition. Since $2m - kp^n + \epsilon < 0$ by assumption, all cells except the first are 0-coconnected. The standard computations show that the first term also contributes no $\underline{\pi}_0$. $\square$

We therefore know we need only determine the slices up to the $2m^{\text{th}}$ slice. Our computation for L actually does most of this. Let F_m be the fiber of the natural inclusion $S^{m\lambda} \wedge H\underline{\mathbf{Z}} \hookrightarrow L$.

Theorem 6.2. *The spectrum F_m is greater than or equal to $2m$.*

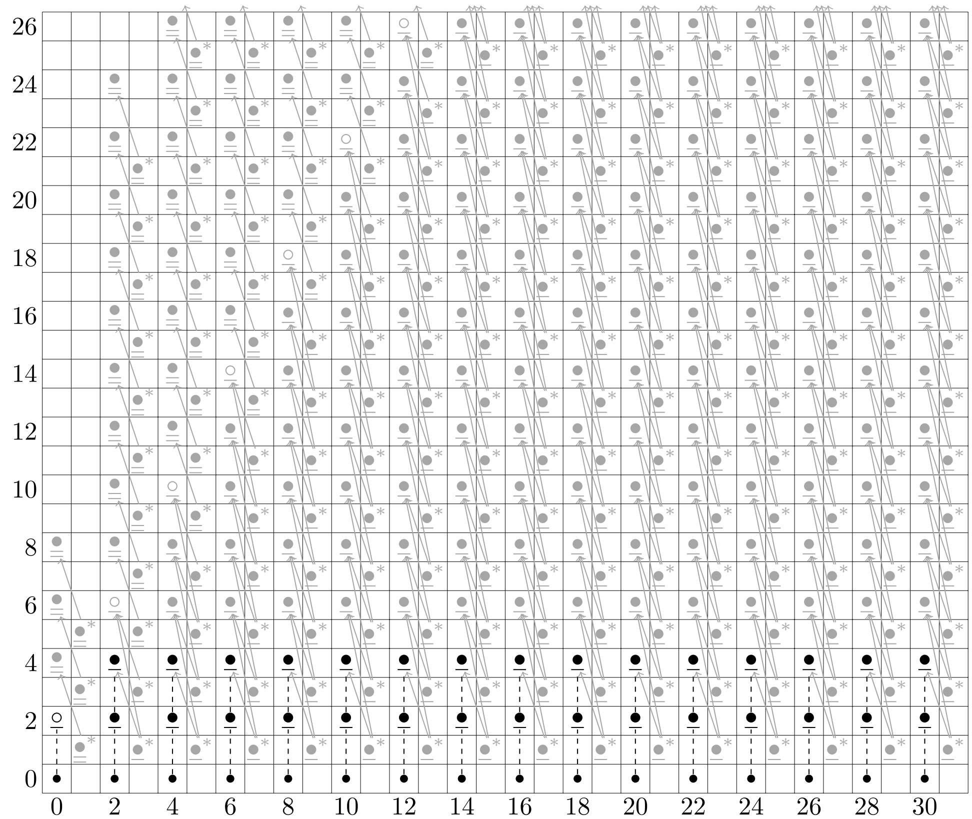

FIGURE 2. The slice spectral sequence for $S^{\infty\lambda} \wedge H\underline{\mathbf{Z}}$ for $G = C_{27}$.

Proof. The long exact sequence in homotopy for the fiber sequence

$$F_m \to S^{m\lambda} \wedge H\underline{\mathbf{Z}} \to L$$

shows that

$$\underline{\pi}_s(F_m) = \begin{cases} 0 & s < 2m, \\ \underline{\mathbf{Z}} & s = 2m, \\ 0 & s = 2k > 2m, \\ \underline{B}(n,0) & s = 2k+1 > 2m. \end{cases}$$

Since the spectrum F_m is $(2m-1)$-connected, we know that it is greater than or equal to $2m$. □

Corollary 6.3. *The natural map $S^{m\lambda} \wedge H\underline{\mathbf{Z}} \to S^{\infty\lambda} \wedge H\underline{\mathbf{Z}}$ induces an equivalence*

$$P^{2m-1}\big(S^{m\lambda} \wedge H\underline{\mathbf{Z}}\big) \to P^{2m-1}\big(L\big).$$

We therefore know all of the slices of $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$ below the $2m^{\text{th}}$ slice, and we also know that we have a single remaining slice. Moreover, we know the maps down to the slice sections, since they are the composites

$$S^{m\lambda} \wedge H\underline{\mathbf{Z}} \to L \to P^j\big(L\big).$$

We therefore have the slice tower. The fiber of the map

$$S^{m\lambda} \wedge H\underline{\mathbf{Z}} \to P^{2m-1}\big(S^{m\lambda} \wedge H\underline{\mathbf{Z}}\big)$$

is the $(2m)$-slice, since it is simultaneously greater than or equal to $2m$ and less than or equal to $2m$. The analysis of the slice tower for L also identifies this with a representation sphere for us, as determined by the analysis in the previous section.

Considering the actual fiber sequences for the slice cotower for L shows us what the $(2m)$-slice is for $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$. The determination of the map is immediate from the consideration of the layers of the tower (and the description of stripping off copies of λ and replacing them with $\lambda(2j-1)$ for j at most m). We simplify the presentation by taking advantage of the fact that $H\underline{\mathbf{Z}}$ is a commutative ring spectrum.

Theorem 6.4. *For all $m \geq 0$, the $(2m)$-slice of $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$ is $S^{V_m} \wedge H\underline{\mathbf{Z}}$.*
The map from $S^{V_m} \wedge H\underline{\mathbf{Z}}$ to $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$ is simply

$$\prod_{j=1}^{m} \frac{u_{\lambda(2j-1)}}{u_\lambda} = \frac{u_{V_m}}{u_{m\lambda}}$$

As a closing remark, we note a fact we found initially quite curious. If p is large relative to m, then $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$ is in fact a $(2m)$-slice. It takes a while before the slice tower of $S^{m\lambda} \wedge H\underline{\mathbf{Z}}$ becomes more complicated.

To illustrate how this plays out, we show in Table 2 the J-equivalence classes of V_m for $p = 3$, $n = 2$ and $1 \leq m \leq 10$.

TABLE 2. V_j for C_{27} and $1 \leq j \leq 27$. Note that $V_{27-j} = 2\rho - V_j$.

j	V_j	j	V_j	j	V_j
1	λ	2	$\lambda + \lambda_1$	3	$2\lambda + \lambda_1$
4	$3\lambda + \lambda_1$	5	$3\lambda + \lambda_1 + \lambda_2$	6	$4\lambda + \lambda_1 + \lambda_2$
7	$5\lambda + \lambda_1 + \lambda_2$	8	$5\lambda + 2\lambda_1 + \lambda_2$	9	$6\lambda + 2\lambda_1 + \lambda_2$
10	$7\lambda + 2\lambda_1 + \lambda_2$	11	$7\lambda + 3\lambda_1 + \lambda_2$	12	$8\lambda + 3\lambda_1 + \lambda_2$
13	$9\lambda + 3\lambda_1 + \lambda_2$	14	$9\lambda + 3\lambda_1 + \lambda_2 + 2$	15	$10\lambda + 3\lambda_1 + \lambda_2 + 2$
	$= \rho - 1$		$= \rho + 1$		$= \rho + \lambda + 1$
16	$11\lambda + 3\lambda_1 + \lambda_2 + 2$	17	$11\lambda + 4\lambda_1 + \lambda_2 + 2$	18	$12\lambda + 4\lambda_1 + \lambda_2 + 2$
19	$13\lambda + 4\lambda_1 + \lambda_2 + 2$	20	$13\lambda + 5\lambda_1 + \lambda_2 + 2$	21	$14\lambda + 5\lambda_1 + \lambda_2 + 2$
22	$15\lambda + 5\lambda_1 + \lambda_2 + 2$	23	$15\lambda + 5\lambda_1 + 2\lambda_2 + 2$	24	$16\lambda + 5\lambda_1 + 2\lambda_2 + 2$
25	$17\lambda + 5\lambda_1 + 2\lambda_2 + 2$	26	$17\lambda + 6\lambda_1 + 2\lambda_2 + 2$	27	$18\lambda + 6\lambda_1 + 2\lambda_2 + 2$
	$= 2\rho - \lambda - \lambda_1$		$= 2\rho - \lambda$		$= 2\rho$

As an example, we present the slice spectral sequence for $S^{8\lambda} \wedge H\underline{\mathbf{Z}}$ in Figure 3. In this, as before, a dot represents a copy of $\underline{B}_{1,0}$, a circle indicates $\underline{B}_{2,0}$, and underlining increases the second subscript. Here we also have circled circles, denoting $\underline{B}_{3,0}$, and the box indicates a copy of $\underline{\mathbf{Z}}$. Asterisks represent the dual to the named Mackey functor.

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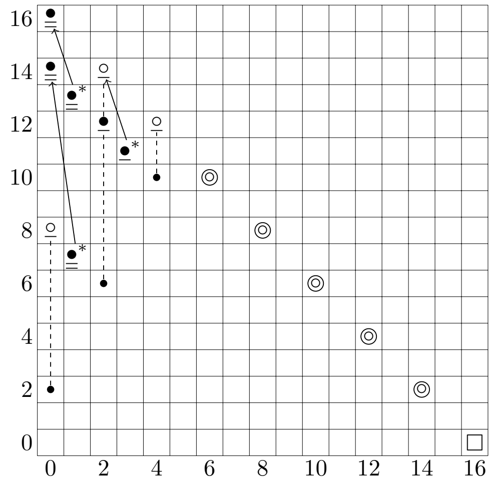

FIGURE 3. The slice spectral sequence for $S^{8\lambda} \wedge H\underline{\mathbf{Z}}$ and C_{27}.

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