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★Nilpotence and periodicity in stable homotopy theory.
Appendix C by Jeff Smith.
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Stable homotopy theory most traditionally concerns itself with the study of groups $\{Y,Z\}$, the group of homotopy classes of stable maps between spaces Y and Z, particularly when Y and Z are finite cell complexes. Letting $\{Y,Z\}_n$ denote $\{\Sigma^n Y,Z\}$, for a fixed X, $\{X,X\}_*$ becomes a graded ring under composition, and one can ask both qualitative and quantitative questions about this ring. The most famous and most studied example is when X is S^0, and the corresponding ring is denoted π_*^S. A qualitative result about this ring is G. Nishida's theorem from the early 1970s that positive degree elements are nilpotent, a theorem whose proof ultimately relies on some elegant observations about extended powers of spaces and spectra. Two major studies leading to explicit quantitative information about π_*^S were made by J. F. Adams in the 1960s: his organization of stable homotopy via the Adams spectral sequence based on ordinary cohomology, and his use of K-theory and K-theory operations to compute $\operatorname{Im} J \subseteq \pi_*^S$.

Adams' two studies can be linked together by replacing the classical Adams spectral sequence by the Adams-Novikov spectral sequence based on the complex bordism MU_*: the $\operatorname{Im} J$ elements are essentially the elements detected by filtration 1. Furthermore, in contrast to Nishida's theorem, localized at each prime p, a certain infinite family of elements in $\operatorname{Im} J$ can be constructed by using iterates of a nonnilpotent, positive degree element $\alpha \in \{M(p),M(p)\}_*$, where $M(p)$ is the mod p Moore space.

In some sense, the story told in the book under review starts with how one shows that all iterates of $\alpha\colon \Sigma^n M(p) \to M(p)$ are essential. One way is to note that on $MU_*(M(p))$, α induces multiplication by a power of an element $v_1 \in MU_*$, easily seen to be nonnilpotent. Equivalently, and perhaps more neatly, one can observe that α induces an isomorphism of the nonzero graded K^*-module $K^*(M(p))$.

In 1969 Quillen showed that the study of MU_* was intimately related to the theory of formal group laws, a subject perhaps studied most deeply by algebraic number theorists. It was the insight of J. Morava in the early 1970s that one should take this theory seriously to organize information about MU_* operations. In particular, he constructed, for each prime p, a sequence of periodic generalized homology theories (with products) $K(0)^*, K(1)^*, K(2)^*, \cdots$ with $K(0)$ the rational Eilenberg-Mac Lane spectrum $H\mathbf{Q}$, $K(1)$ a summand of mod p K-theory, and having coefficients, for $n \geq 1$, $K(n)_* = \mathbf{F}_p[v_n, v_n^{-1}]$, where v_n has degree $2p^n - 2$.

Work by the author and his collaborators H. Miller and W. S. Wilson in the mid-1970s demonstrated that Morava's ideas, formulated in terms of a "chromatic filtration", could be used to guide explicit calculations of stable homotopy groups: this is the subject of the author's previous book [*Complex cobordism and stable homotopy groups of spheres*, Academic Press, Orlando, FL, 1986; MR0860042]. However, by the late 1970s the author, inspired by these same ideas, had begun to formulate various global conjectures about the nature of periodicity and nilpotence in the stable homotopy category. Clarified using the language of Bousfield localization, these were published in 1984 by the author

[Amer. J. Math. **106** (1984), no. 2, 351–414; MR0737778].

As the author relates in the preface: "I had some vague ideas about how to approach the conjectures, but in 1982 when Waldhausen asked me if I expected to see them settled before the end of the century, I could offer him no assurances." Remarkably, all but one of these were proven by 1986 (with the remaining "telescope" conjecture shown to be false several years later, in a presumably typical case), in a mathematical tour de force by E. S. Devinatz, M. J. Hopkins and J. H. Smith [Ann. of Math. (2) **128** (1988), no. 2, 207–241; MR0960945] (and a 1992 preprint by Hopkins and Smith).

Ravenel states as his book's goals: "to make this material accessible to a general mathematical audience, and to provide algebraic topologists with a coherent and reasonably self-contained account of this material". Towards this end, he begins on page 1 with the definition of what it means for two maps to be homotopic, and by page 6 has stated one form of the nilpotence theorem: if X is a finite cell complex, then $f \in \{X, X\}_*$ is nilpotent if and only if $MU_*(f)$ is nilpotent. Two pages later, Chapter 1 ends with a statement of the periodicity theorem: if X is a p-local finite cell complex, and n is chosen smallest so that the reduced homology $K(n)^*(X)$ is not zero, then there exists $f \in \{X, X\}_*$ such that $K(n)^*(f)$ is an isomorphism (and f is unique after suitable iteration).

Chapter 2 includes previously known examples of nonnilpotent self-maps, a statement of Nishida's theorem, and introduces a first version of the chromatic filtration of homotopy.

Chapters 3 and 4 contain background material on formal group laws and complex bordism (with more information in Appendix B), as well as a statement of Hopkins and Smith's thick subcategory theorem: every proper thick subcategory of the category of p-local finite cell complexes is just the full subcategory of $K(n)_*$-acyclic complexes, for some n. Here a subcategory is "thick" (a translation of Gabriel's term "épaisse") if it is closed under cofibrations and retracts.

Chapter 5 contains the deduction of the thick subcategory theorem from the nilpotence theorem, following the arguments of Hopkins and Smith, while Chapter 6, supported by Appendices A and C, leads the reader through the deduction of the periodicity theorem. Smith's construction in Appendix C has not been available previously in published form.

Bousfield localization makes its appearance in Chapter 7, where two previously unpublished theorems due to Hopkins and the author are stated. Let L_n denote Bousfield localization with respect to the sum of the theories $K(0), \cdots, K(n)$. One has the smash product theorem—for all X, $L_n X \cong X \wedge L_n S^0$—and the chromatic convergence theorem—for all p-local finite cell complexes X, $X \cong \operatorname{holim} L_n X$. Sketch proofs of these make up Chapter 8.

Finally, the Devinatz-Hopkins-Smith proof of the nilpotence theorem is given in Chapter 9.

It is an amazing, but in some ways very accessible, tale that the author tells in less than 200 pages. Those in Ravenel's "general mathematical audience" should be warned, however, that a careful reader will find himself confronting the wide variety of tools used in the proofs, ranging from the modular representation theory of the symmetric groups, to Hopf algebroids, to Hopf invariants and Snaith's stable splitting of $\Omega^2 S^{2n+1}$ (in a reincarnation of Nishida's extended power argument). *N. J. Kuhn*