Progress Report on the Telescope Conjecture

Douglas C. Ravenel *
University of Rochester
Rochester, New York 14627

1991

The Telescope Conjecture (made public in a lecture at Northwestern University in 1977) says that the v_n–periodic homotopy of a finite complex of type n has a nice algebraic description. It also gives an explicit description of certain Bousfield localizations. In this paper we outline a proof that it is *false* for $n = 2$ and $p \geq 5$. A more detailed account of this work will appear in [Rav]. In view of this result, there is no longer any reason to think it is true for larger values of n or smaller primes p.

In Section 1 we will give some background surrounding the conjecture. In Section 2 we outline Miller's proof of it for the case $n = 1$ and $p > 2$. This includes a discussion of the localized Adams spectral sequence. In Section 3 we describe the difficulties in generalizing Miller's proof to the case $n = 2$. We end that section by stating a theorem (3.5) about some differentials in the Adams spectral sequence, which we prove in Section 4. This material is new; I stated the theorem in my lecture at the conference, but said nothing about its proof. In Section 5 we construct the parametrized Adams spectral sequence, which gives us a way of interpolating between the Adams spectral sequence and the Adams–Novikov spectral sequence. We need this new machinery to use Theorem 3.5 to disprove the Telescope Conjecture. This argument is sketched in Section 6.

1 Background

Recall that for each prime p there are generalized homology theories $K(n)_*$ (the Morava K–theories) for each integer $n \geq 0$ with the following properties:

(i) $K(0)_*$ is rational homology and $K(1)_*$ is one of $p-1$ isomorphic summands of mod p complex K–theory.

(ii) For $n > 0$, $K(n)_*(\text{pt.}) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ with $|v_n| = 2p^n - 2$.

(iii) There is a Künneth isomorphism

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes K(n)_*(Y).$$

*Partially supported by the National Science Foundation and MSRI

(iv) If X is a finite spectrum with $K(n)_*(X) = 0$, then $K(n-1)_*(X) = 0$.

(v) If the p–localization of X (as above) is not contractible, then

$$K(n)_*(X) \neq 0 \quad \text{for} \quad n \gg 0.$$

The last two properties imply that we can make the following.

Definition 1.1 *A noncontractible finite p–local spectrum X has* **type** *n if n is the smallest integer such that $K(n)_*(X) \neq 0$.*

Definition 1.2 *If X as above has type n then a v_n–***map** *on X is a map*

$$\Sigma^d X \xrightarrow{f} X$$

with $K(n)_*(f)$ an isomorphism and $K(m)_*(f) = 0$ for all $m \neq n$.

The Periodicity Theorem of Hopkins–Smith [HS] says that such a map always exists and is unique in the sense that if g is another such map then some iterate of f is homotopic to some iterate of g. The Telescope Conjecture concerns the telescope $\widehat{X}$, which is defined to be the homotopy direct limit of the system

$$X \xrightarrow{f} \Sigma^{-d} X \xrightarrow{f} \Sigma^{-2d} X \xrightarrow{f} \cdots$$

The Periodicity Theorem tells us that this is independent of the choice of the v_n–map f.

The motivation for studying $\widehat{X}$ is that the associated Adams–Novikov spectral sequence has nice properties. We will illustrate with some simple examples. Suppose

$$BP_*(X) = BP_*/I_n = BP_*/(p, v_1, \cdots v_{n-1}).$$

This happens when X is the Toda complex $V(n-1)$. These are known to exist for small n and large p. Then

$$BP_*(\widehat{X}) = v_n^{-1} BP_*/I_n.$$

The E_2–term of the associated Adams–Novikov spectral sequence is

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, v_n^{-1} BP_*/I_n),$$

which can be computed directly. For more details, see 5.1.14 and Chapter 6 of [Rav86]. It is a free module over $K(n)_*$. In particular when $n = 2$ and $p \geq 5$ (in which case the spectrum $V(1)$ is known to exist) it has total (for all values of s) rank 12 and vanishes for $s > 4$. This means that the Adams–Novikov spectral sequence collapses and there are no extension problems.

The computability of this Ext group was one of the original motivations for studying v_n–periodic homotopy theory.

However, we do not know that this Adams–Novikov spectral sequence converges to $\pi_*(\widehat{X})$. It is known [Rav87] to converge to $\pi_*(L_n X)$, where $L_n X$ denotes the Bousfield localization of X with respect to $E(n)$–theory. (When X

is a finite spectrum of type n, this is the same as the localization with respect to $K(n)$–theory.) Since $\widehat{X}$ is $K(n)_*$–equivalent to X, there are maps

$$X \xrightarrow{i} \widehat{X} \xrightarrow{\lambda} L_n X.$$

The Telescope Conjecture says that λ *is an equivalence*, or equivalently that the Adams–Novikov spectral sequence converges to $\pi_*(\widehat{X})$. This statement is trivial for $n = 0$, known to be true for $n = 1$ ([Mil81] and [Mah82]). The object of this paper is to sketch a counterexample for $n = 2$ and $p \geq 5$.

2 Miller's proof for $n = 1$ and $p > 2$

It is more or less a formality to reduce the Telescope Conjecture for a given value of n and p to proving it for one particular p–local finite spectrum of type n. We will outline Miller's proof for the mod p Moore spectrum $V(0)$. In that case the v_1–map

$$\Sigma^{2p-2} V(0) \xrightarrow{\alpha} V(0) \tag{2.1}$$

is the map discovered long ago by Adams in [Ada66]. There is a map

$$S^{2p-3} \longrightarrow S^0 \longrightarrow V(0)$$

which corresponds to an element in the Adams–Novikov spectral sequence called $h_{1,0}$. The Telescope Conjecture says that

$$\pi_*(\widehat{V(0)}) = K(1)_* \otimes E(h_{1,0}) \tag{2.2}$$

where $E(\cdot)$ denotes an exterior algebra.

Miller studies this problem by looking at the classical Adams spectral sequence for $\pi_*(V(0))$. In its E_2–term there is an element

$$v_1 \in E_2^{1,2p-1}$$

that corresponds to the Adams map α. One can formally invert this element and get a localized Adams spectral sequence converging to $\pi_*(\widehat{V(0)})$. (This convergence is not obvious, and is proved in [Mil81].)

We will describe the construction of this localized Adams spectral sequence. Recall that the classical Adams spectral sequence for the homotopy of spectrum X is constructed as follows. One has an *Adams resolution for* X, which is a diagram of the form

$$\begin{array}{ccccccc} X = X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

with the following properties.

(i) Each K_s is a wedge of suspensions of mod p Eilenberg–Mac Lane spectra.

(ii) Each map f_s induces a monomorphism in mod p homology.

(iii) X_{s+1} is the fibre of f_s.

The *canonical Adams resolution* for X is obtained by setting

$$K_s = X_s \wedge H/(p).$$

A map $g : X \to Y$ induces a map of Adams resolutions, i.e., a collection of maps $g_s : X_s \to Y_s$ with suitable properties. The map g has *Adams filtration* $\geq t$ if it lifts to a map $g' : X \to Y_t$. In this case it is automatic that g_s lifts to Y_{s+t}.

Now consider the example at hand, namely $X = V(0)$. The map α has Adams filtration 1, so we have maps

$$V(0) = X_0 \xrightarrow{\alpha_0'} \Sigma^{-q} X_1 \xrightarrow{\alpha_1'} \Sigma^{-2q} X_2 \xrightarrow{\alpha_2'} \cdots,$$

where $q = 2p - 2$. We define $\widehat{X}_s$ to be the limit of

$$X_s \xrightarrow{\alpha_s'} \Sigma^{-q} X_{s+1} \xrightarrow{\alpha_{s+1}'} \Sigma^{-2q} X_{s+2} \xrightarrow{\alpha_{s+2}'} \cdots,$$

and $\widehat{K}_s$ to be the cofibre of the map $\widehat{X}_{s+1} \to \widehat{X}_s$, or equivalently the limit of

$$K_s \longrightarrow \Sigma^{-q} K_{s+1} \longrightarrow \Sigma^{-2q} K_{s+2} \longrightarrow \cdots,$$

Like K_s, it is a bouquet of mod p Eilenberg–Mac Lane spectra. These spectra are defined for *all integers* s, not just for $s \geq 0$ as in the classical case.

Thus we get a *localized Adams resolution*, i.e., a diagram

$$\begin{array}{ccccccccc} \cdots & \longleftarrow & \widehat{X}_s & \longleftarrow & \widehat{X}_{s+1} & \longleftarrow & \widehat{X}_{s+2} & \longleftarrow & \cdots \\ & & {\scriptstyle \widehat{f}_s}\downarrow & & {\scriptstyle \widehat{f}_{s+1}}\downarrow & & {\scriptstyle \widehat{f}_{s+2}}\downarrow & & \\ & & \widehat{K}_s & & \widehat{K}_{s+1} & & \widehat{K}_{s+2} & & \end{array} \tag{2.3}$$

and a spectral sequence converging to the homotopy of the telescope $\widehat{V(0)}$, which is the limit of

$$\widehat{X}_0 \longrightarrow \widehat{X}_{-1} \longrightarrow \widehat{X}_{-2} \longrightarrow \cdots$$

To prove the spectral sequence converges, one must show that the inverse limit of the $\widehat{X}_s$ is contractible.

Unlike the classical Adams spectral sequence, which is confined to the first quadrant, the localized Adams spectral sequence is a full plane spectral sequence with $E_1^{s,t}$ conceivably nontrivial for all integers s and t. However, it can be shown that the E_2–term has a vanishing line of slope $1/q$, namely

$$E_2^{s,t} = 0 \quad \text{for} \quad s > \frac{t-s+1}{q}.$$

Fortunately the E_2–term of the localized Adams spectral sequence is far simpler than that of the usual Adams spectral sequence. In order to describe it we need to recall some facts about the Steenrod algebra A. Its dual is

$$A_* = E(\tau_0, \tau_1, \cdots) \otimes P(\xi_1, \xi_2, \cdots)$$

where $P(\cdot)$ denotes a polynomial algebra over $\mathbf{Z}/(p)$. We will denote these two factors by Q_* and P_* respectively.

We will use the homological (as opposed to cohomological) formulation of the Adams spectral sequence for $\pi_*(X)$, so the E_2–term is

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X)) \tag{2.4}$$

where $H_*(X)$ (the mod p homology of X) is regarded as a comodule over A_*.

There is an extension of Hopf algebras

$$P_* \longrightarrow A_* \longrightarrow Q_*$$

which leads to a Cartan–Eilenberg spectral sequence converging to (2.4) with

$$E_2 = \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}_{Q_*}(\mathbf{Z}/(p), H_*(X))).$$

The inner Ext group is easy to compute since Q_* is dual to an exterior algebra. For $X = V(0)$ it is

$$P(v_1, v_2, \cdots) \quad \text{with} \quad v_n \in \mathrm{Ext}^{1,2p^n-1}.$$

(The elements v_n correspond so closely to the generators of $\pi_*(BP)$ that we see no point in making a notational distinction between them.)

For odd primes the Cartan–Eilenberg spectral sequence collapses. (See [Rav86, 4.4.3]. It is stated there only for $X = S^0$, but the proof given will work for any X.) It follows that

$$\mathrm{Ext}^s_{A_*}(\mathbf{Z}/(p), H_*(X)) \cong \bigoplus_{i+j=s} \mathrm{Ext}^i_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}^j_{Q_*}(\mathbf{Z}/(p), H_*(X))). \tag{2.5}$$

We can pass to the telescope $\widehat{V(0)}$ by inverting v_1. Then we have the following very convenient change–of–rings isomorphism.

$$\begin{aligned} \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), v_1^{-1}P(v_1, v_2, \cdots)) &\cong \mathrm{Ext}_{B(1)_*}(\mathbf{Z}/(p), K(1)_*) \\ &\cong K(1)_* \otimes \mathrm{Ext}_{B(1)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \end{aligned} \tag{2.6}$$

where $K(1)_*$ as usual denotes the ring $v_1^{-1}P(v_1)$ and

$$B(1)_* = P(\xi_1, \xi_2, \cdots)/(\xi_i^p).$$

This Hopf algebra has a cocommutative coproduct, so its Ext group is easy to compute and we have

$$\mathrm{Ext}_{B(1)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \cong E(h_{1,0}, h_{2,0}, \cdots) \otimes P(b_{1,0}, b_{2,0}, \cdots)$$

where

$$\begin{aligned} h_{i,0} &\in \mathrm{Ext}^{1,2p^i-2} \\ b_{i,0} &\in \mathrm{Ext}^{2,2p^{i+1}-2p}. \end{aligned}$$

This should be compared with the localized form of the Adams–Novikov spectral sequence, in which the E_2–term is

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_1^{-1}BP_*/(p)).$$

One can get a spectral sequence converging to this called the algebraic Novikov spectral sequence by filtering BP_* by powers of the ideal

$$I = (p, v_1, v_2, \cdots). \tag{2.7}$$

The E_2–term of this spectral sequence is a regraded form of (2.6). We denote the r^{th} differential in this spectral sequence by δ_r. These can all be computed by algebraic methods coming from BP–theory. In this case we have

$$\delta_2(h_{i+1,0}) = v_1 b_{i,0} \quad \text{for} \quad i > 0.$$

Miller uses this to deduce that there are similar differentials in the localized Adams spectral sequence, namely

$$d_2(h_{i+1,0}) = v_1 b_{i,0}.$$

This gives

$$E_3 = E_\infty = K(1)_* \otimes E(h_{1,0}),$$

which proves the Telescope Conjecture for $n = 1$ and $p > 2$.

3 Difficulties for $n = 2$

One can mimic Miller's argument for $n = 2$ and $p \geq 5$. In that case one has the spectrum

$$V(1) = S^0 \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p},$$

which is the cofibre of the Adams map α of (2.1). There is a v_2–map

$$\Sigma^{2p^2-2}V(1) \xrightarrow{\beta} V(1)$$

constructed by Larry Smith [Smi71] and H. Toda [Tod71]. The Adams E_2–term is

$$\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), P(v_2, v_3, \cdots)).$$

We can use the map β to localize this Adams spectral sequence in the same way as Miller localized the one for $V(0)$. The resulting E_2–term is

$$K(2)_* \otimes \mathrm{Ext}_{B(2)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

where

$$B(2)_* = P(\xi_1, \xi_2, \cdots)/(\xi_i^{p^2}).$$

This does not have a cocommutative coproduct, so its Ext group is not as easy to compute as (2.6), but it is still manageable. It is a subquotient of the cohomology of the cochain complex

$$C^{*,*} = E(h_{1,0}, h_{2,0}, \cdots; h_{1,1}, h_{2,1}, \cdots) \otimes P(b_{1,0}, b_{2,0}, \cdots; b_{1,1}, b_{2,1}, \cdots)$$

where

$$\begin{array}{rcl} h_{i,j} & \in & C^{1,2p^j(p^i-1)} \\ b_{i,j} & \in & C^{2,2p^{j+1}(p^i-1)} \end{array}$$

and the coboundary ∂ is given by

$$\begin{array}{rcl} \partial(h_{i,0}) & = & \pm h_{1,0}h_{i-1,1} \\ \partial(h_{i,1}) & = & 0 \\ \partial(b_{i,0}) & = & \pm h_{1,1}b_{i-1,1} \\ \partial(b_{i,1}) & = & 0. \end{array} \tag{3.1}$$

There is also an algebraic Novikov spectral sequence with the following differentials.

$$\delta_2(h_{i,0}) = \pm v_2 b_{i-2,1} \tag{3.2}$$

$$\delta_{1+p^{i-1}}(h_{i,1}) = \pm v_2^{p^{i-1}} b_{i-2,0} \quad \text{for } i \geq 3 \tag{3.3}$$

The reader may object to (3.2) on the grounds that $h_{i,0}$ is not a cocycle in $C^{*,*}$, and he would be correct. It would be more accurate to say that the algebraic Novikov spectral sequence has differentials formally implied by (3.2), such as

$$\begin{array}{rcl} \delta_2(h_{1,0}h_{i,0}) & = & \pm v_2 h_{1,0} b_{i-2,1} \quad \text{and} \\ \delta_2(h_{i-1,1}h_{i,0}) & = & \pm v_2 h_{i-1,1} b_{i-2,1}. \end{array}$$

In any case these differentials kill the elements $b_{i,j}$ and $h_{i+2,j}$ for all $i > 0$, and the E_2–term of the Adams–Novikov spectral sequence is the cohomology of

$$K(2)_* \otimes E(h_{1,0}, h_{1,1}, h_{2,0}, h_{2,1})$$

with the coboundary given by (3.1), namely

$$\partial(h_{2,0}) = \pm h_{1,0}h_{1,1}.$$

This is a $K(2)_*$–module of rank 12 with basis

$$E(h_{2,1}) \otimes \{1,\ h_{1,0},\ h_{1,1},\ h_{1,0}h_{2,0},\ h_{1,1}h_{2,0},\ h_{1,0}h_{1,1}h_{2,0}\}. \tag{3.4}$$

This is the value of $\pi_*(\widehat{V(1)})$ predicted by the Telescope Conjecture.

The difficulty is that while Miller's methods allow us to translate the algebraic differentials implied by (3.2) into differentials in the localized Adams spectral sequence, they do *not* enable us to do so for those of (3.3). The latter would give us d_r's for arbitrarily large r, and such differentials could be interfered with by other shorter differentials not related to the algebraic Novikov spectral sequence.

The following result says that such interfering differentials *do* occur in the localized Adams spectral sequence.

Theorem 3.5 *In the localized Adams spectral sequence for $\widehat{V(1)}$ for $p \geq 5$,*

$$d_{2p}(h_{i,1}) = \pm v_2 b_{i-1,0}^p \quad \text{for } i \geq 2$$

modulo nilpotent elements.

The proof of this will be sketched below in Section 4. For $i = 2$ this can be deduced from the Toda differential [Rav86, 4.4.22] by direct calculation.

This result shows that the E_{2p}–term of the localized Adams spectral sequence is a subquotient of

$$E(h_{1,0}, h_{1,1}, h_{2,0}) \otimes P(b_{1,0}, b_{2,0}, \cdots)/(b_{i,0})^p.$$

Even though this is infinite dimensional, it is too small in the sense that it appears to have only two elements with Novikov filtration one (namely $h_{1,0}$ and $h_{1,1}$), while there are three such elements in (3.4).

4 Computing the differentials $d_{2p}(h_{i,1})$

The purpose of this section is to prove Theorem 3.5. The following is rationale for these differentials, which will be made more rigorous and precise below. In the appropriate form of the cobar complex, we have

$$-d(h_{i+2,0}) \equiv h_{1,0}h_{i+1,1} + v_2 b_{i,1} \tag{4.1}$$

modulo terms with higher Adams filtration. It follows that the target of this differential must be a permanent cycle in the localized Adams spectral sequence. Now suppose we knew that

$$d_{2p-1}(b_{i,1}) = h_{1,0} b_{i,0}^p. \tag{4.2}$$

Then combining this with (4.1) would determine the differential on $h_{i+1,1}$, giving Theorem 3.5 up to suitable indeterminacy.

The Toda differential

For $i = 1$, (4.2) is the Toda differential, first established in [Tod67]. The following is a reformulation of Toda's proof. The generator of $\pi_{2p-2}(BU)$ is represented by a map which extends (via the loop space structure of BU) to a map

$$\Omega S^{2p-1} \longrightarrow BU,$$

which can be composed with the map

$$\Sigma\Omega^2 S^{2p-1} \longrightarrow \Omega S^{2p-1}$$

(adjoint to the identity map) to give a vector bundle over $\Sigma\Omega^2 S^{2p-1}$. Its Thom spectrum is the cofibre of a stable map

$$\Omega^2 S^{2p-1} \xrightarrow{f} S^0.$$

Now $\Omega^2 S^{2p-1}$ splits stably into an infinite wedge of finite spectra B_i for $i > 0$ described explicitly by Snaith in [Sna74]. Localization at p makes B_i contractible except when $i \equiv 0$ or 1 mod p, and makes B_{pj+1} equivalent to $\Sigma^{2p-3} B_{pj}$ for $j > 0$. The best way to see this is to look at mod p homology. We have

$$H_*(\Omega^2 S^{2p-1}; \mathbf{Z}/(p)) = E(x_0, x_1, \cdots) \otimes P(y_1, y_2, \cdots)$$

where the dimensions of x_j and y_j are $2p^j(p-1)-1$ and $2p^j(p-1)-2$ respectively.

In order to describe the Snaith splitting homologically, it is conveneient to assign a *weight* to each monomial. We do this by defining the weight of both x_j and y_j to be p^j. This leads to a direct sum decomposition of the homology corresponding to the Snaith splitting of the suspension spectrum, i.e., $H_*(B_i)$ is spanned by the monomials of weight i.

Now observe that the only generator whose weight is not divisible by p is x_0, which is an exterior generator. It follows that multiplication by x_0 gives an isomorphism from the subspace spanned by monomials with weight divisible by p to the that spanned by the ones with weight congruent to 1 mod p. This isomorphism can be realized by a p–local equivalence $\Sigma^{2p-3} B_{pi} \to B_{pi+1}$. Moreover, every monomial has weight congruent to 0 or 1 mod p.

Also note that the first monomial of weight pi is y_1^i, which has dimension $2i(p^2 - p - 1)$. It follows that

$$(B_{pi})_{(p)} = \Sigma^{2i(p^2-p-1)} D_i$$

for some (-1)–connected finite spectrum D_i.

Thus the Snaith splitting (after localizing at p) has the form

$$\Omega^2 S_+^{2p-1} \simeq (S^0 \vee S^{2p-3}) \wedge \bigvee_{i \geq 0} \Sigma^{2i(p^2-p-1)} D_i.$$

In particular the resulting map

$$S^{2p-3} \longrightarrow S^0$$

is α_1, the generator of the $(2p-3)$–stem corresponding to the element $h_{1,0}$ in the Adams spectral sequence.

D_1 is the mod p Moore spectrum and the map

$$\Sigma^{2(p^2-p-1)} D_1 \xrightarrow{f_1} S^0$$

is β_1 on the bottom cell, i.e., the generator of the $2(p^2 - p - 1)$–stem, which corresponds to $b_{1,0}$ in the Adams spectral sequence. In general, the bottom cell of D_i is mapped in by β_1^i.

D_p is a 4–cell complex of the form

$$D_p = S^0 \cup_p e^1 \cup_{\alpha_1} e^{2p-2} \cup_p e^{2p-1},$$

where the third cell is attached to the bottom cell by α_1.

The restriction of f_p to the bottom cell is β_1^p. The fact that this extends over the third cell means that $\alpha_1 \beta_1^p = 0$ in $\pi_*(S^0)$. This means that $h_{1,0} b_{1,0}^p$ must be the target of a differential in the Adams spectral sequence for the sphere spectrum.

If one computes the Adams E_2–term through the relevant range of dimensions, one finds that the only possible source for this differential is $b_{1,1}$. However, one can also deduce this by studying the map f_p more closely. Let

$$S^0 = Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow \cdots$$

be an Adams resolution for S^0. Then routine calculations show that the restrictions of f_p to various skeleta of D_p lift to various Y_i. Let the relevant suspensions of these skeleta be denoted for brevity by

$$S^{2p(p^2-p-1)} = D_p^{(0)} \longrightarrow D_p^{(1)} \longrightarrow D_p^{(2)} \longrightarrow D_p^{(3)} = \Sigma^{2p(p^2-p-1)} D_p.$$

Then we have liftings

$$\begin{array}{ccccccc} D_p^{(0)} & \longrightarrow & D_p^{(1)} & \longrightarrow & D_p^{(2)} & \longrightarrow & D_p^{(3)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{2p} & \longrightarrow & Y_{2p-1} & \longrightarrow & Y_2 & \longrightarrow & Y_1 \end{array}$$

In each case the corresponding map to Y_s/Y_{s+1} (the generalized Eilenberg–Mac Lane spectrum whose homotopy is $E_1^{s,*}$) factors through the top cell of the finite complex. The four resulting elements in the Adams E_1–term are $b_{1,0}^p$, $b_{1,0}^{p-1} h_{1,1}$, $b_{1,1}$ and $h_{1,2}$.

This, along with the fact that the third cell of D_p is attached to the first by α_1, gives the Toda differential

$$d_{2p-1}(b_{1,1}) = h_{1,0} b_{1,0}^p.$$

Generalizing the Toda differential to $i > 1$

One might hope to generalize Toda's proof of (4.2) for $i = 1$ to larger values of i by constructing a map

$$\Omega^2 S^{2p^i-1} \xrightarrow{\ f\ } S^0$$

with suitable properties. In particular, the bottom cell, S^{2p^i-3} would have to represent $h_{i,0}$. However, this is impossible since the latter is not a permanent cycle.

We can get around this difficulty by replacing S^0 by the spectrum $T(i-1)$, which is a connective p–local ring spectrum characterized by

$$BP_*(T(i-1)) = BP_*[t_1, t_2, \cdots t_{i-1}].$$

In particular, $T(0) = S^0$. To construct these spectra for $i > 0$, recall that $\Omega SU \simeq BU$ by Bott periodicity, so for each $n > 0$ we have a map

$$\Omega SU(n) \longrightarrow BU$$

which induces a stable vector bundle over $\Omega SU(n)$. We denote the resulting Thom spectrum by $X(n)$. After localizing at p, each of these admits a splitting generalizing the Brown–Peterson splitting of $MU_{(p)}$, which is the case $n = \infty$. This is proved in [Rav86, 6.5.1]. The resulting minimal summand is $T(i)$ for $p^i \leq n \leq p^{i+1} - 1$.

Now consider the commutative diagram of spaces

$$\begin{array}{ccccc} \Omega^2 SU(p^i) & \longrightarrow & \text{pt.} & \longrightarrow & \Sigma\Omega^2 SU(p^i) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^2 S^{2p^i-1} & \longrightarrow & \Omega SU(p^i - 1) & \longrightarrow & \Omega SU(p^i) \end{array}$$

where the top row is a cofibre sequence and the bottom row is a fibre sequence, and the left most map is induced by the usual projection of $SU(p^i)$ onto S^{2p^i-1}. There is a natural stable vector bundle over every space in sight, and Thomification leads to a diagram of p–local spectra

$$\begin{array}{ccccc} \Omega^2 SU(p^i) & \longrightarrow & S^0 & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^2 S^{2p^i-1} & \longrightarrow & X(p^i - 1) & \longrightarrow & X(p^i) \\ \| & & \downarrow & & \downarrow \\ \Omega^2 S^{2p^i-1} & \xrightarrow{f} & T(i-1) & \longrightarrow & T(i) \end{array}$$

where T is the Thom spectrum of the bundle over $\Sigma\Omega^2 SU(p^i)$

This gives us the map f we are looking for. We could use it to prove a statement similar to (4.2) in the Adams spectral sequence for $T(i-1)$. However, for $i > 1$ the element $h_{1,0}$ is trivial in this setting, and $b_{i,1}$ is actually a permanent cycle. (The latter can be seen by observing that the first element of Novikov filtration $2p + 1$ is $h_{i,0} b_{i,0}^p$, whose dimension exceeds that of $b_{i,1}$ when $i > 1$.)

Fortunately, all is not lost. $T(i-1)$ is a split ring spectrum, and $T(i-1) \wedge T(i-1)$ is a wedge of suspensions of $T(i-1)$, indexed by the monomials in the t_j's for $j < i$. Thus for each such monomial we get a map from $T(i-1)$ to an appropriate suspension of itself. These maps induce cohomology operations in $T(i-1)$–theory. The Quillen operations in BP–theory are constructed in the same way.

We are interested in the map

$$T(i-1) \xrightarrow{r_1} \Sigma^{2p-2} T(i-1)$$

corresponding to the monomial t_1. This is analogous to the first Steenrod reduced power operation $\mathcal{P}^1$. The induced map in homotopy, which we also denote by r_1, lowers degree by $2p - 2$.

Using arguments similar to Toda's one can use the map f to prove the following.

Theorem 4.3 *For $i > 1$,*

$$r_1(b_{i,1}) = b_{i,0}^p$$

in $\pi_*(T(i-1))$.

Completing the proof of Theorem 3.5

Theorem 4.3 can be used to prove an analog of Theorem 3.5 in the localized Adams spectral sequence for $V(1) \wedge T(i-1)$. This determines $d_{2p}(h_{i+1,1})$ in the localized Adams spectral sequence for $V(1)$ itself modulo the kernel of the map from $\widehat{V(1)}$ to $\widehat{V(1)} \wedge T(i-1)$. This is good enough because disproving the Telescope Conjecture requires only that $d_{2p}(h_{i+1,1})$ be an element which is not nilpotent.

Let

$$\cdots \longleftarrow \widehat{X}_{-1} \longleftarrow \widehat{X}_0 \longleftarrow \widehat{X}_1 \longleftarrow \widehat{X}_2 \longleftarrow \cdots \tag{4.4}$$

be a localized Adams resolution (as defined in (2.3)) for $V(1) \wedge T(i-1)$. Then we have

$$h_{i+1,0},\ h_{i,1} \in \pi_*(\widehat{X}_1/\widehat{X}_2).$$

The resolution of (4.4) can be obtained by smashing $T(i-1)$ with a similar resolution for $V(1)$. It forllows that for $i > 1$, the map r_1 is defined on the entire resolution and (4.1) implies that

$$r_1(h_{i+1,0}) = -h_{i,1}.$$

Differentials on these elements in the localized Adams spectral sequence correspond to their images under the map δ induced by

$$\widehat{X}_1/\widehat{X}_2 \longrightarrow \Sigma \widehat{X}_2.$$

One can also deduce from (4.1) that

$$\delta(h_{i+1,0}) = -v_2 b_{i,1}.$$

The following diagram commutes

$$\begin{array}{ccc} \hat{X}_1/\hat{X}_2 & \xrightarrow{\delta} & \Sigma\hat{X}_2 \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_1} \\ \Sigma^{q+1}\hat{X}_1/\hat{X}_2 & \xrightarrow{\delta} & \Sigma^{q+1}\hat{X}_2 \end{array}$$

so we have

$$\begin{aligned} \delta(h_{i,1}) &= -\delta(r_1(h_{i+1,0})) \\ &= -r_1(\delta(h_{i+1,0})) \\ &= r_1(v_2 b_{i,1}) \\ &= v_2 r_1(b_{i,1}) \\ &= v_2 b_{i,0}^p, \end{aligned}$$

which is the desired result.

5 A parametrized Adams spectral sequence

In this section we will describe a variant of the Adams spectral sequence that we need to disprove the Telescope Conjecture.

Again we need to recall how the Adams spectral sequence is set up. Let F denote the mod p Eilenberg–Mac Lane spectrum H/p, let $\overline{F}$ denote the fibre of the map

$$\overline{F} \longrightarrow S^0 \longrightarrow F$$

and let $\overline{F}^{(s)}$ denote the s^{th} smash power of $\overline{F}$.

Then for any spectrum X we have a tower

$$X \longleftarrow X \wedge \overline{F} \longleftarrow X \wedge \overline{F}^{(2)} \longleftarrow \cdots.$$

This gives us a collection of cofibre sequences

$$X \wedge \overline{F}^{(s+1)} \longrightarrow X \wedge \overline{F}^{(s)} \longrightarrow X \wedge \overline{F}^{(s)} \wedge F,$$

which in turn give long exact sequences of homotopy groups. These form an exact couple which gives the Adams spectral sequence. If X is a connective p–torsion spectrum, then the spectral sequence converges to $\pi_*(X)$, and for any X the E_2–term is

$$E_2^{s,t} = \operatorname{Ext}_{A_*}^{s,t}(\mathbf{Z}/(p), H_*(X)).$$

The Adams–Novikov spectral sequence is constructed in the same way, replacing $\overline{F}$ by $\overline{E}$, the fibre of the map

$$\overline{E} \longrightarrow S^0 \longrightarrow BP.$$

Let

$$X_{i,j} = X \wedge \overline{E}^{(i)} \wedge \overline{F}^{(j)}$$

so we have a diagram

$$\begin{array}{ccccccc}
X & \longleftarrow & X_{1,0} & \longleftarrow & X_{2,0} & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
X_{0,1} & \longleftarrow & X_{1,1} & \longleftarrow & X_{2,1} & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
X_{0,2} & \longleftarrow & X_{1,2} & \longleftarrow & X_{2,2} & \longleftarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array} \tag{5.1}$$

The left edge of this diagram is the tower giving the classical Adams spectral sequence, while the top edge gives the Adams–Novikov spectral sequence.

Question 5.2 *Is there an algebraic structure that exploits the diagram (5.1) the way a spectral sequence exploits a tower?*

Such a structure would be very useful. For example, the computations of [Rav86, 4.4] indicate that the 2–component of the stable homotopy groups of spheres can be computed through a respectable range of dimensions simply by comparing the E_2–terms of the Adams spectral sequence and the Adams–Novikov spectral sequence. The structure of the two spectral sequences each imply the existence of nontrivial differentials in the other. It would be nice to have a more systematic way of doing this.

We will construct some more spectral sequences associated with (5.1), but we do not think this is the definitive answer to 5.2. The situation is still like the parable of the blind men and the elephant. (When I first used this metaphor in a lecture, I actually saw an elephant on the Rochester campus the next day.)

We can assume that all maps in (5.1) are inclusions (see [Rav84, 3.1] for a proof), so it makes sense to speak of unions and intersections of the various $X_{i,j}$ as subspectra of X.

Now fix a pair of relatively prime, nonnegative integers m and n, and define

$$W_s = \bigcup_{mi+nj\geq s} X_{i,j}.$$

This gives a tower

$$X = W_0 \longleftarrow W_1 \longleftarrow W_2 \longleftarrow \cdots \tag{5.3}$$

from which we can derive a generalization of the Adams spectral sequence. It is the classical Adams spectral sequence when $m = 0$ and $n = 1$, and the Adams–Novikov spectral sequence when $m = 1$ and $n = 0$.

The fact that the map

$$\overline{E} \longrightarrow S^0$$

factors through $\overline{F}$ can easily be seen to imply the following.

Proposition 5.4 *The spectral sequence described above is the classical Adams spectral sequence whenever $n \geq m$.*

In view of this result, the case $n > m$ is superfluous and we may as well assume that $m \geq n$. Let ϵ denote the rational number n/m. Then we have $0 \leq \epsilon \leq 1$ and the extreme values of ϵ give the Adams–Novikov spectral sequence and the classical Adams spectral sequence respectively.

Definition 5.5 *For a rational number $\epsilon = n/m$ (with m and n relatively prime) between 0 and 1, the* **Adams spectral sequence parametrized by** *ϵ is the homotopy spectral sequence based on the exact couple associated with the tower*

$$X = W_0 \longleftarrow W_{1/m} \longleftarrow W_{2/m} \longleftarrow \cdots$$

with

$$W_s = \bigcup_{i+\epsilon j\geq s} X \wedge \overline{E}^{(i)} \wedge \overline{F}^{(j)}$$

where the union is over nonnegative integers i and j and it is understood that $\overline{E}^{(0)} = \overline{F}^{(0)} = S^0$. (This notation is not the same as in (5.3); W_s here is W_{ms} there.)

Notice that there are inclusion maps

$$X \wedge \overline{E}^{\lceil s \rceil} \longrightarrow W_s \longrightarrow X \wedge \overline{F}^{\lceil s \rceil}$$

(where $\lceil s \rceil$ denotes the smallest integer $\geq s$) which induce maps

$$\begin{array}{c}\text{Adams–Novikov spectral sequence}\\ \downarrow \\ \text{parametrized Adams spectral sequence}\\ \downarrow \\ \text{classical Adams spectral sequence}\end{array}$$

once suitable indexing conventions have been adopted. The composite is the usual reduction map. Thus the parametrized Adams spectral sequences interpolate between the Adams–Novikov spectral sequence and the classical Adams spectral sequence.

We have adopted a convention that allows the filtration grading s to be any nonnegative multiple of $1/m$; the same will be true of the differential index r. (The reader who is uncomfortable with these fractional indices is free to replace the rational numbers r, s and t with the integers mr, ms and $t + (m-1)s$ throughout the discussion.) With this understanding, we have the usual

$$E_r^{s,t} \xrightarrow{d_r} E_r^{s+r,t+r-1}. \tag{5.6}$$

The difference $t - s$ is still an integer, i.e., $E_r^{s,t}$ vanishes when $t - s$ is not an integer.

The usual E_2–term is replaced by the $E_{1+\epsilon}$–term, at least when ϵ is a reciprocal integer. Recall (2.5) that in the classical (i.e., $\epsilon = 1$) case we have

$$E_2^{s,t} \cong \bigoplus_{i+j=s} \operatorname{Ext}_{P_*}^{i,t-j}(\mathbf{Z}/(p), \operatorname{Ext}_{Q_*}^{j}(\mathbf{Z}/(p), H_*(X))).$$

Theorem 5.7 *In the parametrized Adams spectral sequence with $\epsilon = 1/m$ (and $m > 1$ if $p = 2$), if X is such that $E_2 = E_m$ in the classical Adams spectral sequence for $X \wedge BP$,*

$$E_{1+\epsilon}^{s,t} = \bigoplus_{i+j\epsilon=s} \operatorname{Ext}_{P_*}^{i,t-\epsilon j}(\mathbf{Z}/(p), \operatorname{Ext}_{Q_*}^{j}(\mathbf{Z}/(p), H_*(X))).$$

In other words, the element v_n, which has filtration 0 in the Adams–Novikov spectral sequence and filtration 1 in the classical Adams spectral sequence, has filtration ϵ in the parametrized Adams spectral sequence.

The hypothesis on X is satisfied when X is S^0, $V(0)$, $V(1)$, or any spectrum for which the Adams spectral sequence for $X \wedge BP$ collapses. In this case there is an isomorphism

$$\operatorname{Ext}_{Q_*}^{j}(\mathbf{Z}/(p), H_*(X)) \cong I^j BP_*(X)/I^{j+1} BP_*(X)$$

of P_*–comodules, where I is as in (2.7). It follows that the $E_{1+\epsilon}$–term of the parametrized Adams spectral sequence is isomorphic (up to reindexing) to the E_1–term of the algebraic Novikov spectral sequence. Then we have

Theorem 5.8 *Let $\epsilon = 1/m$ and suppose X is such that the classical Adams spectral sequence for $BP \wedge X$ collapses from E_2. In the parametrized Adams spectral sequence for X, let $x \in E_r^{s,t}$ be represented by an element $\tilde{x} \in E_{1+\epsilon}^{s,t}$ which corresponds to an element $\hat{x}$ in the algebraic Novikov spectral sequence which is* not *a permanent cycle. Then for m sufficiently large (depending on x), $d_r(x)$ in the parametrized Adams spectral sequence corresponds to $\delta_{mr}(\hat{x})$ in the algebraic Novikov spectral sequence.*

In other words, in many cases the differential on x can be computed by BP–theoretic methods when ϵ is sufficiently small.

A similar statement can be made about permanent cycles in the Adams–Novikov spectral sequence.

Theorem 5.9 *In the parametrized Adams spectral sequence, let ϵ and X be as in 5.8. Let $x \in E_r^{s,t}$ be represented by an element $\tilde{x} \in E_{1+\epsilon}^{s,t}$ which correpsonds to a permanent cycle in both the algebraic Novikov spectral sequence and the Adams–Novikov spectral sequence. Then for m sufficiently large (depending on the dimension $t-s$), x is a permanent cycle in the parametrized Adams spectral sequence.*

6 Disproving the Telescope Conjecture

Now we can outline our disproof of the Telescope Conjecture. As noted above (3.4), the predicted value of $\pi_*(\widehat{V(1)})$ is

$$K(2)_* \otimes E(h_{2,1}) \otimes \{1,\, h_{1,0},\, h_{1,1},\, h_{1,0}h_{2,0},\, h_{1,1}h_{2,0},\, h_{1,0}h_{1,1}h_{2,0}\}.$$

This can be shown to imply that

$$\pi_*(\widehat{V(1)} \wedge T(2)) \cong K(2)_* \otimes P(v_3, v_4) \otimes E(h_{3,0},\, h_{3,1},\, h_{4,0},\, h_{4,1}).$$

We will disprove the Telescope Conjecture by showing that $h_{4,1}$ is *not* in $\pi_*(\widehat{V(1)} \wedge T(2))$. If it were, then for some $N \gg 0$, $v_2^N h_{4,1}$ would be a permanent cycle in the Adams–Novikov spectral sequence for $\pi_*(V(1) \wedge T(2))$. Using 5.9, this means that a similar element would be a permanent cycle in the parametrized Adams spectral sequence for sufficiently large m. It follows that $h_{4,1}$ would be a permanent cycle in the localized parametrized Adams spectral sequence for all sufficiently small $\epsilon > 0$.

Here is what we know about the localized parametrized Adams spectral sequence for the spectrum $V(1) \wedge T(2)$ for $\epsilon = 1/m$. We have

$$\begin{aligned} E_{1+\epsilon} \;=\; & K(2)_* \otimes P(v_3, v_4) \otimes E(h_{3,0}, h_{4,0}, \cdots; h_{3,1}, h_{4,1}, \cdots) \\ & \otimes P(b_{3,0}, b_{4,0}, \cdots; b_{3,1}, b_{4,1}, \cdots). \end{aligned}$$

and

$$d_{1+\epsilon}(h_{i+2,0}) = \pm v_2 b_{i,1} \qquad \text{for } i \geq 3,$$

which gives

$$\begin{aligned} E_{1+2\epsilon} \;=\; & K(2)_* \otimes P(v_3, v_4) \otimes E(h_{3,0}, h_{4,0}; h_{3,1}, h_{4,1}, \cdots) \\ & \otimes P(b_{3,0}, b_{4,0}, \cdots). \end{aligned}$$

Now we combine 3.5 and (3.3) to get the heuristic formula

$$d(h_{i,1}) = \pm v_2 b_{i-1,0}^p \pm v_2^{p^{i-1}} b_{i-2,0}, \tag{6.1}$$

where the second term is understood to be trivial for $i = 4$ and both terms are trivial for $i = 3$. The second term has lower filtration when

$$\epsilon < \frac{2p-2}{p^{i-1}-1},$$

i.e., for small values of i, while the first term has lower filtration for $i \gg 0$.

Suppose for example that $\epsilon = 1/p^3$. Then (6.1) gives

$$\begin{aligned} d(h_{4,1}) &= \pm v_2 b_{3,0}^p \\ d(h_{5,1}) &= \pm v_2 b_{4,0}^p \pm v_2^{p^4} b_{3,0} \end{aligned}$$

Hence

$$d_{1+p}(h_{5,1}) = \pm v_2^{p^4} b_{3,0}$$

and $b_{3,0}^p$ is dead in E_{2p-1}, so the expected differential on $h_{4,1}$ is trivial. However (6.1) also gives

$$\begin{aligned} d(h_{5,1}(d(h_{5,1})^{p-1})) &= (d(h_{5,1}))^p \\ &= \pm v_2^p b_{4,0}^{p^2} \pm v_2^{p^5} b_{3,0}^p, \end{aligned}$$

from which we get

$$d(h_{4,1} \pm v_2^{1-p^5} h_{5,1}(d(h_{5,1})^{p-1})) = \pm v_2^{1+p-p^5} b_{4,0}^{p^2},$$

which gives the differential

$$d_{2p^2-1+(1+p-p^5)\epsilon}(h_{4,1}) = \pm v_2^{1+p-p^5} b_{4,0}^{p^2}.$$

We need to be sure that this differential is nontrivial, i.e., that $b_{4,0}^{p^2}$ has not been killed earlier by another differential. For this value of ϵ, the first term of (6.1) is the dominant one for all $i \geq 6$. It follows that

$$\begin{aligned} E_{2p-1+2\epsilon} &= K(2)_* \otimes P(v_3, v_4) \otimes E(h_{3,0}, h_{4,0}, h_{3,1}, h_{4,1}) \\ &\qquad \otimes P(b_{4,0}, b_{5,0}, \cdots)/(b_{5,0}^p, b_{6,0}^p, \cdots). \end{aligned}$$

The first three exterior generators can be shown to be permanent cycles by studying the Adams–Novikov spectral sequence for $V(1) \wedge T(2)$ in low dimensions. The elements $b_{i,0}$ for $i \geq 5$ can be ignored here because they have (after being multiplied by a suitable negative power of v_2 to get them in roughly the same dimension as $h_{4,1}$) lower filtration than $h_{4,1}$.

It follows that the indicated differential on $h_{4,1}$ is nontrivial as claimed for $\epsilon = 1/p^3$.

Similar computations can be made for smaller positive values of ϵ. For example when $\epsilon = p^{-j}$ for $j \geq 2$, we get

$$d_r(h_{4,1}) = \pm v_2^e b_{j+1,0}^{p^{j-1}}$$

for suitable values of r and e. It follows that $h_{4,1}$ is not a permanent cycle in the localized parametrized Adams spectral sequence for any positive value of ϵ, and the Telescope Conjecture for n = 2 and $p \geq 5$ is false.

References

[Ada66] J. F. Adams. On the groups J(X), IV. *Topology*, 5:21–71, 1966.

[HS] M. J. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory II. To appear.

[Mah82] M. E. Mahowald. The image of J in the EHP sequence. *Annals of Mathematics*, 116:65–112, 1982.

[Mil81] H. R. Miller. On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space. *Journal of Pure and Applied Algebra*, 20:287–312, 1981.

[Rav] D. C. Ravenel. A counterexample to the Telescope Conjecture. To appear.

[Rav84] D. C. Ravenel. The Segal conjecture for cyclic groups and its consequences. *American Journal of Mathematics*, 106:415–446, 1984.

[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Academic Press, New York, 1986.

[Rav87] D. C. Ravenel. The geometric realization of the chromatic resolution. In W. Browder, editor, *Algebraic topology and algebraic K–theory*, pages 168–179, 1987.

[Smi71] L. Smith. On realizing complex cobordism modules, IV, Applications to the stable homotopy groups of spheres. *American Journal of Mathematics*, 99:418–436, 1971.

[Sna74] V. Snaith. Stable decomposition of $\Omega^n\Sigma^n X$. *Journal of the London Mathematical Society*, 7:577–583, 1974.

[Tod67] H. Toda. An important relation in the homotopy groups of spheres. *Proceedings of the Japan Academy*, 43:893–942, 1967.

[Tod71] H. Toda. On spectra realizing exterior parts of the Steenrod algebra. *Topology*, 10:53–65, 1971.