THE CHROMATIC EXT GROUPS $\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*, M_2^1)$ (DRAFT VERSION)

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1. Introduction

Let BP be the Brown-Peterson spectrum for a fixed prime p. In [Rav86, §6.5], the third author has introduced the spectrum $T(m)$ which has BP_*-homology

$$BP_*(T(m)) = BP_*[t_1, \cdots\cdots, t_m],$$

and is homotopy equivalent to BP below dimension $2p^{m+1} - 3$. Then the Adams-Novikov E_2-term converging to the homotopy groups of $T(m)$

$$E_2^{*,*}(T(m)) = \mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m)))$$

is isomorphic by [Rav86] Corollary 7.1.3 to

$$\mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*),$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \ldots, t_m) = BP_*[t_{m+1}, t_{m+2}, \ldots].$$

In particular $\Gamma(1) = BP_*(BP)$ by definition. To get the structure of this Ext group, we can use the chromatic method introduced in [MRW77].

Define the chromatic module M_n^s by

$$M_n^s = v_{n+s}^{-1} BP_*/(p, v_1, \cdots, v_{n-1}, v_n^\infty, \cdots, v_{n+s-1}^\infty)$$

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and consider the chromatic spectral sequence converging to $\mathrm{Ext}_{\Gamma(m+1)}(BP_*/I_n)$ with

$$E_1^{s,t} = \mathrm{Ext}^t_{\Gamma(m+1)}(M_n^s).$$

Shimomura calls this Ext group the **general chromatic E_1-term**. In this paper we will determine the module structure of

$$\mathrm{Ext}^0_{\Gamma(m+1)}(M_2^1).$$

The analogous result for $m = 0$ was obtained long ago by Miller-Wilson in [MW76], and is as follows.

Theorem 1.1. [Miller-Wilson] *As a $k(2)_*$-module, $\mathrm{Ext}^0_{\Gamma(1)}(M_2^1)$ is the direct sum of*

(a) *the cyclic submodules generated by $x_k^s/v_2^{a(k)}$ for $k \geq 0$ and $s \in \mathbf{Z} - p\mathbf{Z}$ where*

$$\begin{aligned} x_0 &= v_3, \\ x_1 &= v_3^p - v_2^p v_3^{-1} v_4, \\ \text{and for } k \geq 2 \quad x_k &= \begin{cases} x_{k-1}^p & \text{for } k \text{ even} \\ x_{k-1}^p - v_2^{(p^{k-1}-1)(p^3-1)/(p^2-1)} v_3^{p^k - p^{k-1}+1} & \text{for } k \text{ odd} \end{cases} \end{aligned}$$

and

$$\begin{aligned} a(0) &= 1, \\ a(1) &= p, \\ \text{and for } k \geq 2 \quad a(k) &= \begin{cases} pa(k-1) & \text{for } k \text{ even} \\ pa(k-1) + p - 1 & \text{for } k \text{ odd}; \end{cases} \end{aligned}$$

and

(b) *$K(2)_*/k(2)_*$, generated by $1/v_2^j$ for $j \geq 1$.*

Before this result was proved, the naive conjecture about this group would have had the exponents $a(k)$ being p^k for all $k \geq 0$. It was clear that

$$\frac{v_3^{sp^k}}{v_2^{p^k}} \in \mathrm{Ext}^0_{\Gamma(1)}(BP_*, M_2^1),$$

but the existence of "deeper" elements such as

$$\frac{x_3}{v_1^{a(3)}} = \frac{v_3^{p^2} - v_2^{p^3} v_3^{-p^2} v_4^{p^2} - v_2^{p^3-1} v_3^{p^3-p^2+1}}{v_2^{p^3+p-1}}$$

came as a surprise, as did the fact that the limiting value (as $k \to \infty$) of $a(k)/p^k$ is $(p^2+p+1)/(p^2+p)$ instead of 1.

Our result (Theorem 6.9 below) has the same form as Theorem 1.1 but with x_k and $a(k)$ replaced by $\widehat{x}_k$ (4.2, 6.4, 6.5, 6.6 and 6.7) and $\widehat{a}(k)$ (4.1 and 6.3), and with $k(2)_*$ replaced by a larger ring $\widehat{k}(2)_*$ defined in (1.2). In order to avoid the excessive appearance of the index m, we will use the following notation.

$$(1.2) \qquad \begin{cases} \widehat{v}_i = v_{m+i}, & \widehat{t}_i = t_{m+i}, \\ \widehat{K}(n)_* = K(n)_*[v_{n+1}, \ldots, v_{n+m}], & \widehat{h}_{i,j} = h_{m+i,j}, \\ \widehat{k}(n)_* = k(n)_*[v_{n+1}, \ldots, v_{n+m}], \quad \text{and} & \omega = p^m. \end{cases}$$

In §7–10 we will define elements $\widehat{x}_k \in v_3^{-1}BP_*/I_2$ and integers $\widehat{a}(k)$ whenever the condition $3 < 2(p-1)(m+1)/p$ of Theorem 2.1 is satisfied, and compute the reduced right unit d_0 on $\widehat{x}_k$. These will depend on m.

Using these computations, we can obtain the structure of the target Ext group. In particular, for large m we have

Theorem 1.3. *Assume that $p = 2$ and $m \geq 6$ or that $p > 2$ and $m \geq 5$. As a $\widehat{k}(2)_*$-module, $\mathrm{Ext}_{\Gamma(m+1)}(M_2^1)$ is the direct sum of*

(a) *the cyclic submodules generated by $\widehat{x}_k^s/v_2^{\widehat{a}(k)}$ for $k \geq 0$ and $s \in \mathbf{Z}-p\mathbf{Z}$ where*

$$\widehat{x}_k = \begin{cases} \widehat{v}_3 & \text{for } k = 0,\\ \widehat{x}_2^p - v_2^{p^3} v_3^{-p^2} \widehat{v}_4^{p^2} - v_2^{p^3-1} v_3^{(p\omega-1)p^2}\widehat{v}_3 & \text{for } k = 3,\\ \widehat{x}_5^p + v_2^{(p+1)p^5} v_3^{-p^5} W & \text{for } k = 6,\\ \widehat{x}_8^p - v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X^{p^2} - v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6}\widehat{x}_6 & \text{for } k = 9,\\ \widehat{x}_{k-1}^p & \text{for } 3\nmid k \leq 8,\\ \widehat{x}_{k-1}^p + v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1}(\widehat{x}_{k-4} - \widehat{x}_{k-5}^p) & \text{for } k \geq 10; \end{cases}$$

(see Lemma 5.2 and Proposition 5.9 for definitions of W and X) and

$$\widehat{a}(k) = \begin{cases} p^k & \text{for } 0 \leq k \leq 2,\\ (p+1)p^{k-1} & \text{for } 3 \leq k \leq 5,\\ (p^2+p+1)p^{k-2} & \text{for } 6 \leq k \leq 8,\\ p^3\widehat{a}(6) + \widehat{a}(5) & \text{for } k = 9,\\ p^{k-9}(\widehat{a}(9) - \widehat{a}(5)) + \widehat{a}(k-4) & \text{for } k \geq 10; \end{cases}$$

and

(b) *$\widehat{K}(2)_*/\widehat{k}(2)_*$, generated by $1/v_2^j$ for $j \geq 1$.*

□

This is a part of our main result (Theorem 6.9). In fact, we have determined the structure of $\mathrm{Ext}_{\Gamma(m+1)}(M_2^1)$ for $p = 2$ and $m \geq 3$, or $p > 2$ and $m \geq 2$.

In JAMI conference held at Johns Hopkins University in March 2000, Shimomura reported that he extended our result and obtained the structure of

$$\mathrm{Ext}^0_{\Gamma(m+1)}(M_{n-1}^1) \quad \text{for } m \geq n^2 - n - 1.$$

For $n = 3$, this result coincides to our Theorem 1.3. Recently he also determined the structure of higher Ext groups ([Shi2]).

In the above case the assymptotic behavor of the exponents is given by

$$\lim_{k\to\infty} \frac{\widehat{a}(k)}{p^k} = \frac{p^4+p^3+p^2}{p^4-1},$$

a slightly larger value than for the case $m = 0$. In addition, there is a new form of periodicity in our statement with no precedent in Theorem 1.1, namely for all $k \geq 9$ we have

$$\widehat{x}_k - \widehat{x}_{k-1}^p = -v_2^{p^k+p^{k-1}+p^{k-2}} v_3^{p^{m+k}-p^{k-1}-p^{k-2}-p^{k-3}} \widehat{x}_{k-4}^{p-1}(\widehat{x}_{k-4} - \widehat{x}_{k-5}^p),$$

and
$$\widehat{a}(k) = p^k + p^{k-1} + p^{k-2} + \widehat{a}(k-4).$$
(For lower m the period is 6 instead of 4.) A similar result for the chromatic module M_1^1 is obtained in [NR] or [Shi1], in which the period is 3 instead of 4.

2. Preliminaries

For a $\Gamma(m+1)$-comodule M, consider the cobar complex $\{C^*_{\Gamma(m+1)}M, \partial_*\}_{*\geq 0}$, where
$$C^n_{\Gamma(m+1)}M = \Gamma(m+1) \otimes_{BP_*} \cdots\cdots \otimes_{BP_*} \Gamma(m+1) \otimes_{BP_*} M$$
with n factors of $\Gamma(m+1)$. Then $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ is the cohomology of this chain complex. We will abbreviate $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ to $\mathrm{Ext}_{\Gamma(m+1)}(M)$ as usual.

By the change-of-ring isomorphism ([Rav86, Theorem 6.1.1]), we have
$$\begin{aligned}\mathrm{Ext}_{\Gamma(m+1)}\left(M_n^0\right) &= \mathrm{Ext}_{BP_*(BP)}\left(M_n^0 \otimes BP_*(T(m))\right)\\ &= \mathrm{Ext}_{\Sigma(n)}\left(K(n)_*, K(n)_*(T(m))\right).\end{aligned}$$

This object is already known by [Rav86] Corollary 6.5.6:

Theorem 2.1. *If $n < 2(p-1)(m+1)/p$, then*
$$\mathrm{Ext}_{\Gamma(m+1)}\left(M_n^0\right) = \widehat{K}(n)_* \otimes E\left(\widehat{h}_{i,j} : 1 \leq i \leq n, j \in \mathbb{Z}/(n)\right).$$
□

In particular, we have
$$\mathrm{Ext}_{\Gamma(m+1)}\left(M_3^0\right) = \widehat{K}(3)_* \otimes E\left(\widehat{h}_{i,j} : 1 \leq i \leq 3,\ j \in \mathbf{Z}/(3)\right). \tag{2.2}$$

From this information we can get the structure of $\mathrm{Ext}_{\Gamma(m+1)}\left(M_2^1\right)$ using the Bockstein spectral sequence.

Lemma 2.3. (cf. [MRW77] Remark 3.11) *Assume that there exists a $\widehat{k}(2)_*$-submodule B^t of $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right)$ for each $t < N$, such that the following sequence is exact:*

$$0 \longrightarrow \mathrm{Ext}^0_{\Gamma(m+1)}\left(M_3^0\right) \xrightarrow{1/v_2} B^0 \xrightarrow{v_2} B^0 \xrightarrow{\delta} \mathrm{Ext}^1_{\Gamma(m+1)}\left(M_3^0\right) \xrightarrow{1/v_2} \cdots$$
$$\cdots \xrightarrow{1/v_2} B^{N-1} \xrightarrow{v_2} B^{N-1} \xrightarrow{\delta} \mathrm{Ext}^N_{\Gamma(m+1)}\left(M_3^0\right)$$

where δ is the restriction of the coboundary map $\delta : \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right) \to \mathrm{Ext}^{t+1}_{\Gamma(m+1)}\left(M_3^0\right)$. Then the inclusion map $i_t : B^t \to \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right)$ is an isomorphism between $\widehat{k}(2)_*$-modules for each $t < N$.

Proof. Because $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right)$ is a v_2-torsion module, we can filter B^t and $\mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right)$ as
$$P_t(j) = \left\{x \in B^t : v_2^j x = 0\right\} \quad \text{and} \quad Q_t(j) = \left\{x \in \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right) : v_2^j x = 0\right\}.$$

Assume that the inclusion i_k is an isomorphism for $k \leq t-1$ (the $t = 0$ case is obvious), and consider the following commutative ladder diagram:

$$\begin{array}{ccccccccc}
B^{t-1} & \xrightarrow{\delta} & \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right) & \xrightarrow{1/v_2} & P_t(j) & \xrightarrow{v_2} & P_t(j-1) & \xrightarrow{\delta} & \mathrm{Ext}^{t+1}_{\Gamma(m+1)}\left(M_2^1\right) \\
\cong\downarrow i_{t-1} & & \| & & \downarrow i_t & & \downarrow i_t & & \| \\
\mathrm{Ext}^{t-1}_{\Gamma(m+1)}\left(M_1^2\right) & \xrightarrow{\delta} & \mathrm{Ext}^t_{\Gamma(m+1)}\left(M_2^1\right) & \xrightarrow{1/v_2} & Q_t(j) & \xrightarrow{v_2} & Q_t(j-1) & \xrightarrow{\delta} & \mathrm{Ext}^{t+1}_{\Gamma(m+1)}\left(M_2^1\right).
\end{array}$$

Using the Five Lemma, we can show that $P_t(j)$ is isomorphic to $Q_t(j)$ $(j \geq 1)$ by induction on j. □

In (4.2), (6.4), (6.5), (6.6) and (6.7) we will define elements $\widehat{x}_k$ of $v_3^{-1}BP_*/I_2)$ which is congruent to $\widehat{v}_3^{sp^k}$ modulo (v_2), and integers $\widehat{a}(k)$ in (4.1) and (6.3) such that each $\widehat{x}_k^s/v_2^\ell$ is a cycle of $\mathrm{Ext}^0_{\Gamma(m+1)}\left(M_2^1\right)$ for all $1 \leq \ell \leq \widehat{a}(k)$.

Then the structure of B^0 is expressed as follows:

Lemma 2.4. *As a $\widehat{k}(2)_*$-module,*

$$B^0 = \widehat{k}(2)_* \left\{ \frac{\widehat{x}_k^s}{v_2^{\widehat{a}(k)}} :\ k \geq 0, s > 0,\ \textit{and } p \nmid s \right\} \oplus \widehat{K}(2)_*/\widehat{k}(2)_*,$$

is isomorphic to $\mathrm{Ext}^0_{\Gamma(m+1)}\left(M_2^1\right)$, if the set

$$\left\{ \delta\left(\frac{\widehat{x}_k^s}{v_2^{\widehat{a}(k)}} \right) :\ k \geq 0, s > 0,\ \textit{and } p \nmid s \right\} \subset \mathrm{Ext}^1_{\Gamma(m+1)}\left(M_2^0\right)$$

is linearly independent, where δ is a coboundary map in Lemma 2.3.

Proof. We will show that the following sequence is exact.

$$0 \longrightarrow \mathrm{Ext}^0_{\Gamma(m+1)}\left(M_3^0\right) \xrightarrow{1/v_2} B^0 \xrightarrow{\quad v_2 \quad} B^0 \xrightarrow{\delta} \mathrm{Ext}^1_{\Gamma(m+1)}\left(M_3^0\right)$$

The only part of this that is not obvious is that Ker $\delta \subset$ Im v_2. To show this, separate the $\mathbf{Z}/(p)$-basis of B^0 into two parts,

$$A = \left\{ \frac{\widehat{x}_k^s}{v_2^{\widehat{a}(k)}} :\ k \geq 0, s > 0,\ \text{and } p \nmid s \right\} \qquad \text{and}$$

$$B = \left\{ \frac{\widehat{x}_k^s}{v_2^\ell} :\ k \geq 0, s > 0,\ p \nmid s,\ \text{and } 1 \leq \ell < \widehat{a}(k) \right\} \cup \left\{ v_2^{-j} : j > 0 \right\}.$$

Then it is obvious that $\delta(x_\lambda) \neq 0 \in \mathrm{Ext}^1_{\Gamma(m+1)}\left(M_3^0\right)$ for $x_\lambda \in A$, and that $\delta(y_\mu) = 0 \in \mathrm{Ext}^1_{\Gamma(m+1)}\left(M_3^0\right)$ for $y_\mu \in B$. Thus for any element $z = \sum_\lambda a_\lambda x_\lambda + \sum_\mu b_\mu y_\mu$ of B^0 $(a_\lambda, b_\mu \in \mathbb{Z}/p)$, we have $\delta(z) = \sum_\lambda a_\lambda \delta(x_\lambda)$. The condition implies that all a_λ are zero when $\delta(z) = 0$, and so $v_2 \sum_\mu b_\mu y_\mu / v_2 = z$. This completes the proof. □

3. Elementary calculations

In this section we will introduce elements $\widehat{w}_4$ and $\widehat{w}_5$, which we need to define our $\widehat{x}_k$. First we recall the right unit on $\widehat{v}_i$.

Lemma 3.1. *Assume that $p \geq 2$ and $m \geq 1$. In $\Gamma(m+1)/(p, v_1)$ the right unit map $\eta_R : \widehat{A} \to \widehat{\Gamma}$ on the element $\widehat{v}_i$ is given by*

$$\begin{array}{rrcll} & \eta(\widehat{v}_i) & = & \widehat{v}_i & \text{for } i \leq 2, \\ & \eta(\widehat{v}_3) & = & \widehat{v}_3 + v_2 \widehat{t}_1^{p^2} - v_2^{p\omega} \widehat{t}_1, & \\ & \eta(\widehat{v}_4) & = & \widehat{v}_4 + v_3 \widehat{t}_1^{p^3} + v_2 \widehat{t}_2^{p^2} - v_3^{p\omega} \widehat{t}_1 - v_2^{p^2\omega} \widehat{t}_2, & \\ \text{and} & \eta(\widehat{v}_5) & \equiv & \widehat{v}_5 + v_4 \widehat{t}_1^{p^4} + v_3 \widehat{t}_2^{p^3} + v_2 \widehat{t}_3^{p^2} & \\ & & & \quad - v_4^{p\omega} \widehat{t}_1 - v_3^{p^2\omega} \widehat{t}_2 - v_2^{p+1} \widehat{v}_3^{p(p-1)} \widehat{t}_1^{p^3} \quad \bmod \left(v_2^{2p+1}\right) & \\ & & & (\text{add } v_2^4 \widehat{t}_1^{17} \text{ when } (p, m) = (2, 1)). & \end{array}$$

Moreover, when $p \geq 2$ and $m \geq 2$, we have

$$\eta(\widehat{v}_6) \equiv \widehat{v}_6 + v_5 \widehat{t}_1^{p^5} + v_4 \widehat{t}_2^{p^4} + v_3 \widehat{t}_3^{p^3} - v_5^{p\omega} \widehat{t}_1 - v_4^{p^2\omega} \widehat{t}_2 - v_3^{p^3\omega} \widehat{t}_3 \quad \bmod (v_2).$$

□

Define elements $\widehat{w}_i$ for $4 \leq i \leq 5$ by

$$\begin{array}{rrcl} & \widehat{w}_4 & = & v_3^{-1} \widehat{v}_4 \\ \text{and} & \widehat{w}_5 & = & v_3^{-1}(\widehat{v}_5 - v_4 \widehat{w}_4^p + v_2^{p+1} \widehat{v}_3^{(p-1)p} \widehat{w}_4). \end{array}$$

Then we have the following lemma.

Lemma 3.2. *For any prime p, we have*

$$\begin{array}{rcl} d(\widehat{w}_4) & = & \widehat{t}_1^{p^3} + v_2 v_3^{-1} \widehat{t}_2^{p^2} - v_3^{p\omega-1} \widehat{t}_1 - v_2^{p^2\omega} v_3^{-1} \widehat{t}_2 \qquad \textit{for } m \geq 1, \\ d(\widehat{w}_5) & \equiv & \widehat{t}_2^{p^3} + v_2 v_3^{-1} \widehat{t}_3^{p^2} - v_3^{-1} v_4^{\omega p} \widehat{t}_1 - v_3^{p^2\omega-1} \widehat{t}_2 - v_2^p v_3^{-p-1} v_4 \widehat{t}_2^{p^3} + v_3^{p^2\omega-p-1} v_4 \widehat{t}_1^p \\ & & \quad + v_2^{p+2} v_3^{-2} \widehat{v}_3^{(p-1)p} \widehat{t}_2^{p^2} - v_2^{p+1} v_3^{p\omega-2} \widehat{v}_3^{(p-1)p} \widehat{t}_1 \quad \bmod \left(v_2^{2p+1}\right) \quad \textit{for } m \geq 2 \\ & & (\text{add } v_2^4 v_3^{-1} \widehat{t}_1^{17} \text{ when } (p, m) = (2, 1)). \end{array}$$

Proof. $d(\widehat{w}_4)$ is easily computed using Lemma 3.1. For $\widehat{w}_5$ we find that

$$\begin{array}{rcl} d(\widehat{v}_5) & \equiv & v_4 \widehat{t}_1^{p^4} + v_3 \widehat{t}_2^{p^3} + v_2 \widehat{t}_3^{p^2} - v_4^{p\omega} \widehat{t}_1 - v_3^{p^2\omega} \widehat{t}_2 - v_2^{p+1} \widehat{v}_3^{p(p-1)} \widehat{t}_1^{p^3} \quad \bmod \left(v_2^{2p+1}\right) \\ & & (\text{add } v_2^4 \widehat{t}_1^{17} \text{ for } (p, m) = (2, 1)). \end{array}$$

We can read off the fact that $d(v_4) = 0$ for $m \geq 2$ and $d(\widehat{v}_3) \equiv 0 \bmod (v_2)$ from Lemma 3.1. We have

$$\begin{cases} d(-v_4\widehat{w}_4^p) & \equiv & -v_4(\widehat{t}_1^{p^4} + v_2^p v_3^{-p}\widehat{t}_2^{p^3} - v_3^{(p\omega-1)p}\widehat{t}_1^p) & \bmod \left(v_2^{p^3\omega}\right), \\ d(v_2^{p+1}\widehat{v}_3^{p(p-1)}\widehat{w}_4) & \equiv & v_2^{p+1}\widehat{v}_3^{p(p-1)}(\widehat{t}_1^{p^3} + v_2v_3^{-1}\widehat{t}_2^{p^2} - v_3^{p\omega-1}\widehat{t}_1) & \bmod \left(v_2^{2p+1}\right). \end{cases}$$

Summing these congruences and multiplying v_3^{-1} gives $d(\widehat{w}_5)$. □

4. $d(\widehat{x}_k)$ for $0 \le k \le 5$

For all p and m we can construct $\widehat{x}_k$ $(0 \le k \le 5)$ and compute differentials on these one at a time (Lemma 4.3). Define integers $\widehat{a}(k)$ (as in Theorem 1.3) and $\widehat{b}(k)$ for $0 \le k \le 5$ by

$$(4.1)\qquad \widehat{a}(k) = \begin{cases} p^k & \text{if } 0 \le k \le 2 \\ (p+1)p^{k-1} & \text{if } 3 \le k \le 5 \end{cases} \quad\text{and}\quad \widehat{b}(k) = \begin{cases} 0 & \text{if } 0 \le k \le 2 \\ -p^{k-1} & \text{if } 3 \le k \le 5. \end{cases}$$

Define elements (as in Theorem 1.3) $\widehat{x}_k$ for $0 \le k \le 5$ by

$$(4.2)\qquad \begin{cases} \widehat{x}_0 & = & \widehat{v}_3, \\ \widehat{x}_3 & = & \widehat{x}_2^p - v_2^{p^3}\widehat{w}_4^{p^2} - v_2^{p^3-1}v_3^{(p\omega-1)p^2}\widehat{x}_0, \\ \widehat{x}_k & = & \widehat{x}_{k-1}^p \quad \text{for } k \not\equiv 0 \bmod (3). \end{cases} \quad\text{and}$$

Lemma 4.3. *Assume that $p \ge 2$ and $m \ge 2$. In the module $M_2^1 \otimes \Gamma(m+1)$ we have*

$$d(\widehat{x}_k) \equiv \begin{cases} v_2^{\widehat{a}(k)}\widehat{t}_1^{p^{k+2}} & \bmod \left(v_2^{p^{k+1}\omega}\right) & \text{for } 0 \le k \le 2, \\ -v_2^{\widehat{a}(k)}v_3^{\widehat{b}(k)}\widehat{t}_2^{p^{k+1}} & \bmod \left(v_2^{p^{k-3}(p^3-p\omega-1)}\right) & \text{for } 3 \le k \le 5. \end{cases}$$

In particular these equivalences hold modulo $(v_2^{1+\widehat{a}(k)})$.

Proof. We obtain $d(\widehat{x}_k)$ for $0 \le k \le 2$ from $\eta_R(\widehat{v}_3)$ (Lemma 3.1). For $d(\widehat{x}_3)$, we have

$$(4.4)\qquad \begin{cases} d(\widehat{x}_2^p) \equiv v_2^{p^3}\widehat{t}_1^{p^5} & \bmod \left(v_2^{p^4\omega}\right), \\ d(-v_2^{p^3}\widehat{w}_4^{p^2}) & \\ \qquad \equiv -v_2^{p^3}(\widehat{t}_1^{p^5} + v_2^{p^2}v_3^{-p^2}\widehat{t}_2^{p^4} - v_3^{(p\omega-1)p^2}\widehat{t}_1^{p^2}) & \bmod \left(v_2^{(p\omega+1)p^3}\right), \\ d(-v_2^{p^3-1}v_3^{(p\omega-1)p^2}\widehat{x}_0) & \\ \qquad = -v_2^{p^3}v_3^{(p\omega-1)p^2}(\widehat{t}_1^{p^2} - v_2^{p\omega-1}\widehat{t}_1). & \end{cases}$$

Notice that

$$p^3 + p\omega - 1 < \min\{p^4\omega, (p\omega+1)p^3\} = p^4\omega$$

for all p and m. Summing congruences in (4.4), we have

$$d(\widehat{x}_3) \equiv -v_2^{p^3+p^2}v_3^{-p^2}\widehat{t}_2^{p^4} + v_2^{p^3+p\omega-1}v_3^{(p\omega-1)p^2}\widehat{t}_1 \quad \bmod \left(v_2^{p^4\omega}\right).$$

Noticing that the inequality $p^3+p\omega-1>p^3+p^2$ holds only if $m\geq 2$, we find that

$$d(\widehat{x}_3)\equiv\begin{cases} v_2^{p^3+p^2-1}v_3^{(p^2-1)p^2}\widehat{t}_1 & \mod \left(v_2^{p^3+p^2}\right) & \text{for } m=1,\\ -v_2^{p^3+p^2}v_3^{-p^2}\widehat{t}_2^{p^4} & \mod \left(v_2^{p^3+p\omega-1}\right) & \text{for } m\geq 2.\end{cases}$$

By assumption, we may consider only for the case that $m\geq 2$, and set $\widehat{a}(3)=p^3+p^2$. The formulas for $4\leq k\leq 5$ are obvious. □

To define $\widehat{x}_k$ for higher k, we will prepare some lemmas in the next section. The definitions of $\widehat{x}_k$ $(k\geq 6)$ and computations of the chromatic differential d_0 on $\widehat{x}_k$ are separated into 4 sections (§7–10) according to the value of m. The results are stated in section 6.

5. Some lemmas

Here we will prove some lemmas for later use. In the rest of this paper we will treat $\mathrm{Ext}^0_{\Gamma(m+1)}M_2^1$ whenever the condition in Theorem 2.1

$$3<2(p-1)(m+1)/p \tag{5.1}$$

is satisfied. This is equivalent to

$$\begin{cases} m\geq 2 & \text{for } p\geq 3\\ m\geq 3 & \text{for } p=2.\end{cases}$$

Lemma 5.2. *There is an element W such that*

$$\begin{aligned} d(W) \quad\equiv\quad & \widehat{t}_2^{p^7}+v_2^{p^4}v_3^{-p^4}\widehat{t}_3^{p^6}-v_2^{p^4}v_3^{-p^4}\widehat{v}_3^{(p-1)p^5}d(\widehat{x}_5)\\ & -v_2^{p\omega-p^2-1}v_3^{(p^2\omega-1)p^4+p^3\omega}\widehat{t}_1 \qquad \mod \left(v_2^{e_1(p,m)}\right),\end{aligned}$$

where

$$e_1(p,m)=\begin{cases} (p^3-1)p^2 & \textit{if } m=2\\ (p^4-1)p^2 & \textit{if } (p,\,m)=(2,\,3)\\ (2p^3+p^2-1)p^2 & \textit{otherwise.}\end{cases}$$

Proof. We find the following congruences:

$$\begin{cases} d(\widehat{w}_5^{p^4}) \equiv \widehat{t}_2^{p^7} + v_2^{p^4} v_3^{-p^4} \widehat{t}_3^{p^6} - v_3^{-p^4} v_4^{p^5\omega} \widehat{t}_1^{p^4} - v_3^{(p^2\omega-1)p^4} \widehat{t}_2^{p^4} \\ \qquad -v_2^{p^5} v_3^{(-p-1)p^4} v_4^{p^4} \widehat{t}_2^{p^7} + v_3^{(p^2\omega-p-1)p^4} v_4^{p^4} \widehat{t}_1^{p^5} \\ \qquad +v_2^{(p+2)p^4} v_3^{-2p^4} \widehat{v}_3^{(p-1)p^5} \widehat{t}_2^{p^6} - v_2^{(p+1)p^4} v_3^{(p\omega-2)p^4} \widehat{v}_3^{(p-1)p^5} \widehat{t}_1^{p^4} \\ \qquad\qquad \mod \left(v_2^{(2p+1)p^4}\right), \\ d(v_2^{-\widehat{a}(2)} v_3^{-p^4} v_4^{p^5\omega} \widehat{x}_2) \\ \qquad \equiv v_3^{-p^4} v_4^{p^5\omega} \widehat{t}_1^{p^4} \qquad \mod \left(v_2^{(p\omega-1)p^2}\right), \\ d(-v_2^{-\widehat{a}(3)} v_3^{(p^2\omega-1)p^4+p^2} \widehat{x}_3) \\ \qquad \equiv v_3^{(p^2\omega-1)p^4} \widehat{t}_2^{p^4} - v_2^{p\omega-p^2-1} v_3^{(p^2\omega-1)p^4+p^3\omega} \widehat{t}_1 \quad \mod \left(v_2^{(p^2\omega-p-1)p^2}\right), \\ d(-v_2^{p^5-p\widehat{a}(5)} v_3^{-p^4} v_4^{p^4} \widehat{x}_5^{p}) \\ \qquad \equiv v_2^{p^5} v_3^{(-p-1)p^4} v_4^{p^4} \widehat{t}_2^{p^7} \qquad \mod \left(v_2^{(p\omega-1)p^3}\right), \\ d(-v_2^{-p\widehat{a}(2)} v_3^{(p^2\omega-p-1)p^4} v_4^{p^4} \widehat{x}_2^{p}) \\ \qquad \equiv -v_3^{(p^2\omega-p-1)p^4} v_4^{p^4} \widehat{t}_1^{p^5} \qquad \mod \left(v_2^{(p\omega-1)p^3}\right), \\ d(v_2^{(p+1)p^4-\widehat{a}(2)} v_3^{(p\omega-2)p^4} \widehat{v}_3^{(p-1)p^5} \widehat{x}_2) \\ \qquad \equiv v_2^{(p+1)p^4} v_3^{(p\omega-2)p^4} \widehat{v}_3^{(p-1)p^5} \widehat{t}_1^{p^4} \qquad \mod \left(v_2^{(2p+1)p^4-p^2}\right). \end{cases}$$

Summing these congruences gives the desired formula. The integer $e_1(p,m)$ is the minimum exponent of v_2 in these indeterminacies. □

Lemma 5.3. *There is an element Y such that*

$$d(Y) \equiv v_3^{p\omega(p^3+1)} \widehat{t}_1 - v_2^{p^2+1} \widehat{t}_3^{p^4} + v_2^{p^2+1} \widehat{v}_3^{(p-1)p^3} d(\widehat{x}_3) + v_2^{p\omega} v_4^{p^3\omega} \widehat{t}_1 \quad \mod \left(v_2^{e_2(p,m)}\right),$$

where

$$e_2(p,m) = \begin{cases} (p-1)(p^2-1) & \text{for } (p,\, m) = (3,\, 2), \\ (2p+1)p^2 & \text{otherwise.} \end{cases}$$

Proof. We find the following congruences:

$$\begin{cases} d(\widehat{w}_4) & \equiv \widehat{t}_1^{p^3} + v_2 v_3^{-1}\widehat{t}_2^{p^2} - v_3^{p\omega-1}\widehat{t}_1 \qquad \mod \left(v_2^{p^2\omega}\right), \\ d(-v_2^{-p}\widehat{x}_1) & \equiv -\widehat{t}_1^{p^3} \qquad \mod \left(v_2^{(p\omega-1)p}\right), \\ d(v_2 v_3^{-1-(p^2\omega-1)p^2}\widehat{w}_5^{p^2}) & \\ & \equiv v_2 v_3^{-1-(p^2\omega-1)p^2}\Big(\widehat{t}_2^{p^5} + v_2^{p^2} v_3^{-p^2}\widehat{t}_3^{p^4} - v_3^{-p^2} v_4^{p^3\omega}\widehat{t}_1^{p^2} - v_3^{(p^2\omega-1)p^2}\widehat{t}_2^{p^2} \\ & \quad - v_2^{p^3} v_3^{(-p-1)p^2} v_4^{p^2}\widehat{t}_2^{p^5} + v_3^{(p^2\omega-p-1)p^2} v_4^{p^2}\widehat{t}_1^{p^3} \\ & \quad + v_2^{(p+2)p^2} v_3^{-2p^2}\widehat{v}_3^{(p-1)p^3}\widehat{t}_2^{p^4} - v_2^{(p+1)p^2} v_3^{(p\omega-2)p^2}\widehat{v}_3^{(p-1)p^3}\widehat{t}_1^{p^2}\Big) \\ & \qquad \mod \left(v_2^{1+(2p+1)p^2}\right), \\ d(v_2^{1-\widehat{a}(4)} v_3^{p^3-1-(p^2\omega-1)p^2}\widehat{x}_4) & \\ & \equiv -v_2 v_3^{-1-(p^2\omega-1)p^2}\widehat{t}_2^{p^5} \qquad \mod \left(v_2^{1+(p\omega-p^2-1)p}\right), \\ d(v_3^{-1-p^4\omega} v_4^{p^3\omega}\widehat{x}_0) & \\ & = v_3^{-1-p^4\omega} v_4^{p^3\omega}(v_2\widehat{t}_1^{p^2} - v_2^{p\omega}\widehat{t}_1), \\ d(v_2^{1+p^3-\widehat{a}(4)} v_3^{-1-p^4\omega} v_4^{p^2}\widehat{x}_4) & \\ & \equiv v_2^{1+p^3} v_3^{-1-(p\omega+1)p^3} v_4^{p^2}\widehat{t}_2^{p^5} \qquad \mod \left(v_2^{1+(p\omega-1)p}\right), \\ d(-v_2^{1-p} v_3^{-1-p^3} v_4^{p^2}\widehat{x}_1) & \\ & \equiv -v_2 v_3^{-1-p^3} v_4^{p^2}\widehat{t}_1^{p^3} \qquad \mod \left(v_2^{1+(p\omega-1)p}\right), \\ d(v_2^{(p+1)p^2} v_3^{p^3\omega(1-p)-1-p^2}\widehat{v}_3^{(p-1)p^3}\widehat{x}_0) & \\ & \equiv v_2^{1+(p+1)p^2} v_3^{p^3\omega(1-p)-1-p^2}\widehat{v}_3^{(p-1)p^3}\widehat{t}_1^{p^2} \qquad \mod \left(v_2^{(2p+1)p^2}\right). \end{cases}$$

The sum of the right sides is

$$-v_3^{p\omega-1}\widehat{t}_1 + v_2^{p^2+1} v_3^{-1-p^4\omega}\widehat{t}_3^{p^4} + v_2^{(p+2)p^2+1} v_3^{-1-(p^2\omega+1)p^2}\widehat{v}_3^{(p-1)p^3}\widehat{t}_2^{p^4} - v_2^{p\omega} v_3^{-1-p^4\omega} v_4^{p^3\omega}\widehat{t}_1.$$

Multiplying $-v_3^{p^4\omega+1}$ gives the desired formula. The integer $e_2(p, m)$ is the minimium exponent of v_2 in these indeterminacies. □

Now we define $\widehat{w}_6$ by

$$\widehat{w}_6 = \begin{cases} v_3^{-1}(\widehat{v}_6 - v_2^{-p\widehat{a}(2)} v_5\widehat{x}_2^p + v_2^{-\widehat{a}(3)} v_3^{-\widehat{b}(3)} v_4\widehat{x}_3) & \text{for } m \geq 3, \\ \text{(above expression)} + v_2^{1-p^3} v_3^{-1-p^3\omega(p^3+1)}\widehat{x}_2^p Y^{p^2} - v_3^{-1-p\omega(p^3+1)}\widehat{x}_2^p Y & \text{for } m = 2. \end{cases}$$

Lemma 5.4. *In the module $v_3^{-1}(BP_*/I_2 \otimes \Gamma(m+1))$ we have*

$$d(\widehat{w}_6) \equiv \widehat{t}_3^{p^3} - v_3^{-1} v_5^{p\omega}\widehat{t}_1 - v_3^{-1} v_4^{p^2\omega}\widehat{t}_2 - v_3^{p^3\omega-1}\widehat{t}_3 \mod (v_2).$$

Proof. We find that

$$(5.5)\qquad \begin{cases} d(\widehat{v}_6) & \equiv v_5\widehat{t}_1^{p^5} + v_4\widehat{t}_2^{p^4} + v_3\widehat{t}_3^{p^3} - v_5^p\widehat{t}_1 - v_4^{p^2\omega}\widehat{t}_2 - v_3^{p^3\omega}\widehat{t}_3 \\ & \qquad \mod (v_2), \\ d(v_2^{-\widehat{a}(3)} v_3^{-\widehat{b}(3)} v_4\widehat{x}_3) & \equiv -v_4\widehat{t}_2^{p^4} \qquad \mod \left(v_2^{p\omega-p^2-1}\right). \end{cases}$$

Notice that $\eta_R(v_5) = v_5$ for $m \geq 3$, and so we have

$$\begin{aligned} d(-v_2^{-p\widehat{a}(2)} v_5 \widehat{x}_2^p) &= -v_2^{-p\widehat{a}(2)} (d(v_5)\widehat{x}_2^p + \eta(v_5) d(\widehat{x}_2^p)) \\ &= -v_2^{-p\widehat{a}(2)} v_5 d(\widehat{x}_2^p) \\ &\equiv -v_5 \widehat{t}_1^{p^5} \qquad \mod \left(v_2^{(p\omega-1)p^3}\right). \end{aligned} \tag{5.6}$$

Multiplying the sum of (5.5) and (5.6) by v_3^{-1}, we obtain the desired formula for $m \geq 3$. For $m = 2$, we know that $\eta_R(v_5) = \eta_R(\widehat{v}_3) = v_5 + v_2\widehat{t}_1^{p^2} - v_2^{p^3}\widehat{t}_1$, and so we have

$$\begin{aligned} d(-v_2^{-p^3} v_5 \widehat{x}_2^p) &\equiv -v_2^{-p^3}\left((v_2\widehat{t}_1^{p^2} - v_2^{p^3}\widehat{t}_1)\widehat{x}_2^p + (v_5 + v_2\widehat{t}_1^{p^2} - v_2^{p^3}\widehat{t}_1)(v_2^{p^3}\widehat{t}_1^{p^5})\right) \\ &\equiv -v_2^{-p^3}\left((v_2\widehat{t}_1^{p^2} - v_2^{p^3}\widehat{t}_1)\widehat{x}_2^p + v_2^{p^3} v_5 \widehat{t}_1^{p^5}\right) \\ &\equiv -v_2^{1-p^3}\widehat{x}_2^p\widehat{t}_1^{p^2} + \widehat{x}_2^p\widehat{t}_1 - v_5\widehat{t}_1^{p^5} \qquad \mod (v_2) \end{aligned} \tag{5.7}$$

We also find that

$$\begin{cases} d(v_2^{1-p^3} v_3^{-p^3\omega(p^3+1)}\widehat{x}_2^p Y^{p^2}) \equiv v_2^{1-p^3}\widehat{x}_2^p\widehat{t}_1^{p^2}, \\ d(-v_3^{-p\omega(p^3+1)}\widehat{x}_2^p Y) \equiv -\widehat{x}_2^p\widehat{t}_1 \qquad \mod (v_2). \end{cases} \tag{5.8}$$

Multiplying the sum of (5.5), (5.7) and (5.8) by v_3^{-1} gives the desired formula for $m = 2$. ☐

Using this $\widehat{w}_6$ we define X by

$$\begin{aligned} X = \ &\widehat{w}_6^{p^4} - v_2^{-\widehat{a}(2)} v_3^{-p^4} v_5^{p^5\omega}\widehat{x}_2 - v_2^{-\widehat{a}(3)} v_3^{-p^4+p^2} v_4^{p^6\omega}\widehat{x}_3 \\ &+ v_2^{p\omega-p^2-1} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega} Y. \end{aligned}$$

Then we have

Proposition 5.9. *For $m \geq 3$,*

$$d(X) \equiv \widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4}\widehat{t}_3^{p^4} \quad \mod \left(v_2^{p^4}\right).$$

Proof. We find the following congruences

$$
(5.10)\quad \begin{cases}
d(\widehat{w}_6^{p^4}) \equiv \widehat{t}_3^{p^7} - v_3^{-p^4} v_5^{p^5\omega}\widehat{t}_1^{p^4} - v_3^{-p^4} v_4^{p^6\omega}\widehat{t}_2^{p^4} - v_3^{(p^3\omega-1)p^4}\widehat{t}_3^{p^4} \\
\qquad\qquad \mod \left(v_2^{p^4}\right), \\
d(v_2^{-\widehat{a}(2)} v_3^{-p^4} v_5^{p^5\omega}\widehat{x}_2) \\
\qquad \equiv v_3^{-p^4} v_5^{p^5\omega}\widehat{t}_1^{p^4} \qquad \mod \left(v_2^{(p\omega-1)p^2}\right), \\
d(-v_2^{-\widehat{a}(3)} v_3^{-p^4+p^2} v_4^{p^6\omega}\widehat{x}_3) \\
\qquad \equiv -v_2^{-\widehat{a}(3)} v_3^{-p^4+p^2} v_4^{p^6\omega}(-v_2^{\widehat{a}(3)} v_3^{-p^2}\widehat{t}_2^{p^4} + v_2^{p^3+p\omega-1} v_3^{(p\omega-1)p^2}\widehat{t}_1) \\
\qquad\qquad \mod \left(v_2^{(p^2\omega-p-1)p^2}\right), \\
d(v_2^{p\omega-p^2-1} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega} Y) \\
\qquad \equiv v_2^{p\omega-p^2-1} v_3^{(\omega-p)p^3} v_4^{p^6\omega}\widehat{t}_1 \qquad \mod (v_2^{p\omega})
\end{cases}
$$

Summing these congruences gives the desired formula. □

In case that $m = 2$ we have the following analogy instead.

Proposition 5.11. *For $m = 2$, there is an element $\widetilde{X}$ such that*

$$d(\widetilde{X}) \equiv \widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4}\widehat{t}_3^{p^4} - v_2^{p^3} v_3^{p\omega(p^2-p^3-1-p^6)} v_4^{p^6\omega}\widehat{t}_3^{p^7} \quad \mod \left(v_2^{2p^3-p^2-1}\right).$$

Proof. Instead of the last congruence of 5.10 we find that

$$
\begin{aligned}
& d(v_2^{p\omega-p^2-1} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega} Y) \\
& \quad \equiv v_2^{p^3-p^2-1} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega}(v_3^{p\omega(p^3+1)}\widehat{t}_1 - v_2^{p^2+1}\widehat{t}_3^{p^4}) \\
& \qquad\qquad \mod \left(v_2^{2p\omega-p^2-1}\right).
\end{aligned}
$$

This gives

$$d(X) \equiv \widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4}\widehat{t}_3^{p^4} - v_2^{p\omega} v_3^{(\omega-p)p^3-p\omega(p^3+1)} v_4^{p^6\omega}\widehat{t}_3^{p^4} \quad \mod \left(v_2^{2p^3-p^2-1}\right).$$

Then, define $\widetilde{X}$ by

$$\widetilde{X} = X - v_2^{p^3} v_3^{p\omega(p^2-p^3-1-p^6)} v_4^{p^6\omega} X.$$

This gives the desired formula. □

Define integers $a_i(6)$ and $b_i(6)$ for $i = 1, 2$ by

$$
\begin{cases}
a_1(6) = p^4(p^2+p+1), \\
a_2(6) = (p+1)p^5 + p\omega, \\
b_1(6) = -(p+1)p^4, \\
b_2(6) = -(p+1)p^4 + p\omega(p^2-1)(p^3+1).
\end{cases}
$$

and the element M by

$$
\begin{aligned}
M \;=\; & \widehat{x}_5^p + v_2^{(p+1)p^5} v_3^{-p^5} W + v_2^{a_2(6)-p^2-1} v_3^{b_2(6)} Y \\
& - v_2^{a_2(6)+p\omega-p^2-1} v_3^{b_2(6)-p\omega(p^3+1)} v_4^{p^3\omega} Y
\end{aligned}
$$

Proposition 5.12. *For $m \geq 2$, we have*

$$\begin{aligned} d(M) \equiv\ & v_2^{a_1(6)} v_3^{b_1(6)} \left(\widehat{t}_3^{p^6} - \widehat{x}_5^{p-1} d(\widehat{x}_5)\right) - v_2^{a_2(6)} v_3^{b_2(6)} \left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)\right) \\ & + v_2^{a_2(6)+p\omega} v_3^{b_2(6)-p\omega(p^3+1)} v_4^{p^3\omega} \left(\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)\right) \\ & - v_2^{a_2(6)+2p\omega-p^2-1} v_3^{b_2(6)-p\omega(p^3+1)} v_4^{2p^3\omega} \widehat{t}_1 \qquad \mod (v_2^{e(p,m)}), \end{aligned}$$

where

$$e(p,m) = (p+1)p^5 + \min \left\{ \begin{array}{l} (p\omega - p^2 - 1)p^3 \\ e_1(p,m) \\ p\omega - p^2 - 1 + e_2(p,m) \end{array} \right\}$$

Proof. We find that

$$\left\{ \begin{array}{l} d(\widehat{x}_5^p) \equiv -v_2^{(p+1)p^5} v_3^{-p^5} \widehat{t}_2^{p^7} \qquad \mod \left(v_2^{(p^3+p\omega-1)p^3}\right), \\ d(v_2^{(p+1)p^5} v_3^{-p^5} W) \\ \qquad \equiv v_2^{(p+1)p^5} v_3^{-p^5} \widehat{t}_2^{p^7} + v_2^{a_1(6)} v_3^{b_1(6)} \left(\widehat{t}_3^{p^6} - \widehat{v}_3^{(p-1)p^5} d(\widehat{x}_5)\right) \\ \qquad\quad - v_2^{a_2(6)-p^2-1} v_3^{(p^2\omega-p-1)p^4+p^3\omega} \widehat{t}_1 \\ \qquad\qquad \mod \left(v_2^{(p+1)p^5+e_1(p,m)}\right), \\ d(v_2^{a_2(6)-p^2-1} v_3^{b_2(6)} Y) \\ \qquad \equiv v_2^{a_2(6)-p^2-1} v_3^{b_2(6)+p\omega(p^3+1)} \widehat{t}_1 \\ \qquad\quad - v_2^{a_2(6)} v_3^{b_2(6)} \left(\widehat{t}_3^{p^4} - \widehat{v}_3^{(p-1)p^3} d(\widehat{x}_3)\right) + v_2^{a_2(6)+p\omega-p^2-1} v_3^{b_2(6)} v_4^{p^3\omega} \widehat{t}_1 \\ \qquad\qquad \mod \left(v_2^{a_2(6)-p^2-1+e_2(p,m)}\right), \\ d(-v_2^{a_2(6)+p\omega-p^2-1} v_3^{b_2(6)-p\omega(p^3+1)} v_4^{p^3\omega} Y) \\ \qquad \equiv -v_2^{a_2(6)+p\omega-p^2-1} v_3^{b_2(6)} v_4^{p^3\omega} \widehat{t}_1 \\ \qquad\quad + v_2^{a_2(6)+p\omega} v_3^{b_2(6)-p\omega(p^3+1)} v_4^{p^3\omega} \left(\widehat{t}_3^{p^4} - \widehat{v}_3^{(p-1)p^3} d(\widehat{x}_3)\right) \\ \qquad\quad - v_2^{a_2(6)+2p\omega-p^2-1} v_3^{b_2(6)-p\omega(p^3+1)} v_4^{2p^3\omega} \widehat{t}_1 \\ \qquad\qquad \mod \left(v_2^{a_2(6)+p\omega-p^2-1+e_2(p,m)}\right). \end{array} \right.$$

Summing these congruences gives the desired formula. □

6. $d(\widehat{x}_k)$ FOR $k \geq 6$

Define integers c_i and d_i for $i = 1, 2$ by

$$\text{(6.1)} \qquad \left\{ \begin{array}{lcl} c_1 & = & a_1(6) + \widehat{a}(5), \\ c_2 & = & a_2(6) + \widehat{a}(3), \\ d_1 & = & b_1(6) + \widehat{b}(5), \\ d_2 & = & b_2(6) + \widehat{b}(3), \end{array} \right.$$

and integers $\ell(i)$ for $i = 1, 2$ by

$$\ell(1) = \begin{cases} 1 & \text{if } a_1(6) \le a_2(6), \\ 2 & \text{if } a_1(6) > a_2(6); \end{cases} \tag{6.2}$$
$$\ell(2) = \begin{cases} 1 & \text{if } c_1 \le c_2, \\ 2 & \text{if } c_1 > c_2. \end{cases}$$

Then we define $\widehat{a}(k)$ and $\widehat{b}(k)$ for $k \ge 6$ (they were defined for $0 \le k \le 5$ in (4.1)) by

$$\widehat{a}(k) = \begin{cases} p^{k-6} a_{\ell(1)}(6) & \text{for all } m \text{ and } 6 \le k \le 8, \\ (p^3-1)\widehat{a}(6) + c_{\ell(2)} & \text{for all } m \text{ and } k = 9, \\ p^{k-9}(\widehat{a}(9) - \widehat{a}(5)) + \widehat{a}(k-4) & \text{for } m \ge 5 \text{ and } k \ge 10, \\ p^{k-9}(\widehat{a}(9) - \widehat{a}(3)) + \widehat{a}(k-6) & \text{for } 2 \le m \le 4 \text{ and } k \ge 10; \end{cases} \tag{6.3}$$
$$\widehat{b}(k) = \begin{cases} p^{k-6} b_{\ell(1)}(6) & \text{for } m \ge 3 \text{ and } 6 \le k \le 8, \\ (p^3-1)\widehat{b}(6) + (p^3\omega - 1)p^6 + d_{\ell(2)} & \\ & \text{for } m \ge 3 \text{ and } k = 9, \\ p^{k-6}(b_2(6) - (p^3\omega-1)p^4) & \text{for } m = 2 \text{ and } 6 \le k \le 7, \\ p^2 b_2(6) & \text{for } m = 2 \text{ and } k = 8, \\ p^3\widehat{b}(6) + (p^3\omega-1)p^4 + \widehat{b}(3) & \text{for } m = 2 \text{ and } k = 9, \\ p^{k-9}(\widehat{b}(9) - \widehat{b}(5)) + \widehat{b}(k-4) & \text{for } m \ge 5 \text{ and } k \ge 10, \\ p^{k-9}(\widehat{b}(9) - \widehat{b}(3)) + \widehat{b}(k-6) & \text{for } 2 \le m \le 4 \text{ and } k \ge 10; \end{cases}$$

Define $\widehat{y}_i$ for $1 \le i \le 4$ by

$$\begin{cases} \widehat{y}_1 = -v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X^{p^2} - v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6} \widehat{x}_6, \\ \widehat{y}_2 = v_2^{(p^3-1)\widehat{a}(6)+a_2(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6+b_2(6)-(p^3\omega-1)p^4} (X - v_2^{-\widehat{a}(7)} v_3^{-\widehat{b}(7)} \widehat{x}_7), \\ \widehat{y}_3 = v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} X, \\ \widehat{y}_4 = -v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^4} \widehat{x}_6. \end{cases}$$

For $m \ge 5$, define $\widehat{x}_k \in v_3^{-1}BP_*$ for $k \ge 6$ by

$$\widehat{x}_k = \begin{cases} M^{p^{k-6}} & \text{for } 6 \le k \le 8, \\ \widehat{x}_8^p + \widehat{y}_1 & \text{for } k = 9 \text{ unless } (p, m) = (2, 5), \\ \widehat{x}_8^p + \widehat{y}_1 + \widehat{y}_2 & \text{for } k = 9 \text{ and } (p, m) = (2, 5), \\ \widehat{x}_{k-1}^p + v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1}(\widehat{x}_{k-4} - \widehat{x}_{k-5}^p) & \\ & \text{for } k \ge 10. \end{cases} \tag{6.4}$$

For $m = 4$, define $\widehat{x}_k$ for $k \ge 6$ by

$$\widehat{x}_k = \begin{cases} M^{p^{k-6}} & \text{for } 6 \le k \le 8, \\ \widehat{x}_8^p + \widehat{y}_1 + \widehat{y}_2 & \text{for } k = 9, \\ \widehat{x}_{k-1}^p - v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1}(\widehat{x}_{k-6} - \widehat{x}_{k-7}^p) & \\ & \text{for } k \ge 10. \end{cases} \tag{6.5}$$

For $m = 3$, define $\widehat{x}_k$ for $k \geq 6$ by

$$(6.6)\quad \widehat{x}_k = \begin{cases} M - v_2^{\widehat{a}(6)} v_3^{b_2(6)-(p^3\omega-1)p^4} X & \text{for } k = 6, \\ \widehat{x}_6^p & \text{for } k = 7, \\ M^{p^2} & \text{for } k = 8, \\ \widehat{x}_8^p + \widehat{y}_1 + \widehat{y}_3 & \text{for } k = 9, \\ \widehat{x}_{k-1}^p - v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1}(\widehat{x}_{k-6} - \widehat{x}_{k-7}^p) & \text{for } k \geq 10. \end{cases}$$

For $m = 2$, define $\widehat{x}_k$ for $k \geq 6$ by

$$(6.7)\quad \widehat{x}_k = \begin{cases} (M - v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)-(p^3\omega-1)p^4} \widetilde{X})(1 + v_2^{p^3} v_3^{p\omega(p^2-p^3-1-p^6)} v_4^{p^6\omega}) & \\ & \text{for } k = 6, \\ \widehat{x}_6^p & \text{for } k = 7, \\ M^{p^2} & \text{for } k = 8, \\ \widehat{x}_8^p + \widehat{y}_4 & \text{for } k = 9, \\ \widehat{x}_{k-1}^p - v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1}(\widehat{x}_{k-6} - \widehat{x}_{k-7}^p) & \text{for } k \geq 10. \end{cases}$$

Then we have

Lemma 6.8. *Assume that p and m satisfy (5.1). Modulo $\left(v_2^{1+\widehat{a}(k)}\right)$, the differential on $\widehat{x}_k$ $(k \geq 6)$ is expressed as*

$$\begin{cases} v_2^{\widehat{a}(k)} v_3^{\widehat{b}(k)} \widehat{t}_3^{p^k} & \text{for } m \geq 4 \text{ and } 6 \leq k \leq 8, \\ v_2^{\widehat{a}(k)} v_3^{\widehat{b}(k)} \widehat{t}_3^{p^k} - v_2^{\widehat{a}(k)} v_3^{p^{k-6}b_2(6)-(p^3\omega-1)p^{k-2}} \widehat{t}_3^{p^{k+1}} & \text{for } m = 3 \text{ and } 6 \leq k \leq 7, \\ v_2^{\widehat{a}(8)} v_3^{\widehat{b}(8)} \widehat{t}_3^{p^8} - v_2^{\widehat{a}(8)} v_3^{p^2 b_2(6)} \widehat{t}_3^{p^6} & \text{for } m = 3 \text{ and } k = 8, \\ -v_2^{\widehat{a}(k)} v_3^{\widehat{b}(k)} \widehat{t}_3^{p^{k+1}} & \text{for } m = 2 \text{ and } 6 \leq k \leq 7, \\ -v_2^{\widehat{a}(k)} v_3^{\widehat{b}(k)} \widehat{t}_3^{p^6} & \text{for } m = 2 \text{ and } k = 8, \\ v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-4}) & \text{for } m \geq 5 \text{ and } k \geq 9. \\ -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{x}_{k-6}) & \text{for } 2 \leq m \leq 4 \text{ and } k \geq 9. \end{cases}$$

□

We will prove this lemma in Sections 7, 8, 9 and 10. Then our main result is

Theorem 6.9. *As a $\widehat{k}(2)_*$-module, $\mathrm{Ext}_{\Gamma(m+1)}(M_2^1)$ is the direct sum of*

(a) *the cyclic submodules generated by $\widehat{x}_k^s / v_2^{\widehat{a}(k)}$ for $k \geq 0$ and $s \in \mathbf{Z} - p\mathbf{Z}$, where $\widehat{x}_k$ and $\widehat{a}(k)$ are elements defined in section 4 and this section.*
(b) *$\widehat{K}(2)_*/\widehat{k}(2)_*$, generated by $1/v_2^j$ for $j \geq 1$.*

□

7. Proof of Lemma 6.8 for $m \geq 5$

Notice that the exponent $e(p, m)$ in Proposition 5.12 is

$$(p+1)p^5 + \min \left\{ \begin{array}{l} (p\omega - p^2 - 1)p^3 \\ (2p+1)p^4 - p^2 \\ p\omega + 2p^3 - 1 \end{array} \right\} = (p+1)p^5 + (2p+1)p^4 - p^2,$$

which is always larger than

$$a_1(6) + \widehat{a}(5) + p^3 + p + 2 = (p+1)p^5 + (p^5 + 2p^4 + p^3 + p + 2)$$

for all primes p. By Proposition 5.12 we have

$$d(\widehat{x}_6) \equiv \begin{cases} v_2^{a_1(6)}v_3^{b_1(6)}(\widehat{t}_3^{p^6} - \widehat{x}_5^{p-1}d(\widehat{x}_5)) - v_2^{a_2(6)}v_3^{b_2(6)}\widehat{t}_3^{p^4} & \text{if } (p,m)=(2,5) \\ v_2^{a_1(6)}v_3^{b_1(6)}(\widehat{t}_3^{p^6} - \widehat{x}_5^{p-1}d(\widehat{x}_5)) & \text{otherwise,} \end{cases}$$

mod $(v_2^{a_1(6)+\widehat{a}(5)+p^3+p+2})$. In particular, we have

$$d(\widehat{x}_6) \equiv v_2^{\widehat{a}(6)}v_3^{\widehat{b}(6)}\widehat{t}_3^{p^6} \mod \left(v_2^{\widehat{a}(6)+\widehat{a}(5)}\right)$$

in both cases. For $d(\widehat{x}_9)$ we find that

$$d(\widehat{x}_8^p) \equiv v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}\widehat{t}_3^{p^9} \mod \left(v_2^{p^3(\widehat{a}(6)+\widehat{a}(5))}\right). \tag{7.1}$$

Assume that $(p,m)\neq(2,5)$. For $d(\widehat{y}_1)$ we find that

$$\begin{cases} d(-v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}X^{p^2}) \\ \quad \equiv -v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}(\widehat{t}_3^{p^9} - v_3^{(p^3\omega-1)p^6}\widehat{t}_3^{p^6}) \\ \qquad\qquad \mod \left(v_2^{p^3\widehat{a}(6)+p^6}\right), \\ d(-v_2^{(p^3-1)\widehat{a}(6)}v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6}\widehat{x}_6) \\ \quad \equiv -v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)+(p^3\omega-1)p^6}(\widehat{t}_3^{p^6} - \widehat{x}_5^{p-1}d(\widehat{x}_5)) \\ \qquad\qquad \mod \left(v_2^{\widehat{a}(9)+p^3+p+2}\right) \end{cases} \tag{7.2}$$

(Notice that the second formula fails if $(p,m)=(2,5)$). This gives

$$d(\widehat{y}_1) \equiv -v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)}\widehat{t}_3^{p^9} + v_2^{p^3\widehat{a}(6)}v_3^{p^3\widehat{b}(6)+(p^3\omega-1)p^6}\widehat{x}_5^{p-1}d(\widehat{x}_5). \tag{7.3}$$

mod $(v_2^{\widehat{a}(9)+p^3+p+2})$. Summing (7.1) and (7.2), we obtain

$$d(\widehat{x}_9) \equiv v_2^{\widehat{a}(9)-\widehat{a}(5)}v_3^{\widehat{b}(9)-\widehat{b}(5)}\widehat{x}_5^{p-1}d(\widehat{x}_5) \mod \left(v_2^{\widehat{a}(9)+p^3+p+2}\right)$$

unless $(p,m)=(2,5)$. We will see that a similar congruence holds even for $(p,m)=(2,5)$ after an appropriate change of $\widehat{x}_9$.

Define integers $n(k)$ by

$$n(k) = \begin{cases} p^3+p+2 & \text{for } k\equiv 1\ (4), \\ (p+2)p & \text{for } k\equiv 2\ (4), \\ (p+2)p^2 & \text{for } k\equiv 3\ (4), \\ (p+2)p^3 & \text{for } k\equiv 0\ (4). \end{cases}$$

Instead of (6.3), we may define integers $\widehat{a}(k)$ for $k\geq 9$ inductively on k by

$$\widehat{a}(k) = \begin{cases} p\widehat{a}(k-1)+p^5+p^4 & \text{for } k\equiv 1 \bmod (4), \\ p\widehat{a}(k-1)+p^4 & \text{for } k\equiv 2 \bmod (4), \\ p\widehat{a}(k-1) & \text{for } k\equiv 0 \text{ and } 3 \bmod (4). \end{cases}$$

This suggests that we may compute $d(\widehat{x}_k)$ modulo $(v_2^{\widehat{a}(k)+n(k)})$ inductively on k. Assume that the congruence

$$d(\widehat{x}_{k-1}) \equiv v_2^{p^{k-10}(\widehat{a}(9)-\widehat{a}(5))} v_3^{p^{k-10}(\widehat{b}(9)-\widehat{b}(5))} \widehat{x}_{k-5}^{p-1} d(\widehat{x}_{k-5}) \tag{7.4}$$

holds modulo $(v_2^{\widehat{a}(k-1)+n(k-1)})$. For $10 \le k \le 14$ it follows by direct calculations. Moreover, this gives $d(\widehat{x}_k)$ whenever $11 \le k \equiv 0$ or $3 \bmod (4)$, since $\widehat{x}_k = \widehat{x}_{k-1}^p$. In other cases we denote $\widehat{x}_k - \widehat{x}_{k-1}^p$ by $\widehat{z}_k$. Notice that $\widehat{z}_k$ is related with $\widehat{z}_{k-4}$ by

$$\widehat{z}_k = v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} \widehat{z}_{k-4}.$$

Then $d(\widehat{z}_k)$ is

$$\begin{aligned} d(\widehat{z}_k) &= v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} d\left(\widehat{x}_{k-4}^{p-1}\widehat{z}_{k-4}\right) \\ &= v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \left(d\left(\widehat{x}_{k-4}^{p-1}\right)\widehat{z}_{k-4} + \eta_R\left(\widehat{x}_{k-4}^{p-1}\right) d\left(\widehat{z}_{k-4}\right)\right). \end{aligned}$$

Notice that $\widehat{z}_{k-4}$ divided by $\widehat{a}(k-4)-\widehat{a}(k-8)$, so that we can ignore $d(\widehat{x}_{k-4}^{p-1})$, because

$$\begin{aligned} &\widehat{a}(k)+n(k)-(\widehat{a}(k)-\widehat{a}(k-4))-(\widehat{a}(k-4)-\widehat{a}(k-8)) \\ &\qquad = n(k)+\widehat{a}(k-8) \\ &\qquad < \widehat{a}(k-4). \end{aligned}$$

Thus we have

$$d\left(\widehat{z}_k\right) \equiv v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{z}_{k-4}) \qquad \mathrm{mod}\ \left(v_2^{\widehat{a}(k)+n(k)}\right). \tag{7.5}$$

On the other hand, by assumption (7.4) we have

$$\begin{aligned} d(\widehat{x}_{k-1}^p) &\equiv v_2^{p^{k-9}(\widehat{a}(9)-\widehat{a}(5))} v_3^{p^{k-9}(\widehat{b}(9)-\widehat{b}(5))} \widehat{x}_{k-5}^{(p-1)p} d(\widehat{x}_{k-5}^p) \\ &\equiv v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-5}^p) \qquad \mathrm{mod}\ \left(v_2^{\widehat{a}(k)+n(k)}\right). \end{aligned} \tag{7.6}$$

Summing (7.5) and (7.6), we obtain

$$d(\widehat{x}_k) \equiv v_2^{\widehat{a}(k)-\widehat{a}(k-4)} v_3^{\widehat{b}(k)-\widehat{b}(k-4)} \widehat{x}_{k-4}^{p-1} d(\widehat{x}_{k-4}) \qquad \mathrm{mod}\ \left(v_2^{\widehat{a}(k)+n(k)}\right)$$

as desired. □

Now we consider the $(p,m)=(2,5)$ case. We have the following congruence instead of the second one in (7.2):

$$\left\{\begin{aligned} &d(-v_2^{(p^3-1)\widehat{a}(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6} \widehat{x}_6) \\ &\quad \equiv -v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)+(p^3\omega-1)p^6} \left(\widehat{t}_3^{p^6} - \widehat{x}_5^{p-1} d(\widehat{x}_5)\right) \\ &\qquad\quad + v_2^{(p^3-1)\widehat{a}(6)+a_2(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6+b_2(6)} \widehat{t}_3^{p^4} \\ &\qquad\qquad\qquad \mathrm{mod}\ \left(v_2^{\widehat{a}(9)+p^3+p+2}\right) \end{aligned}\right. \tag{7.7}$$

Define $\widehat{y}_2$ by

$$\widehat{y}_2 = v_2^{(p^3-1)\widehat{a}(6)+a_2(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6+b_2(6)-(p^3\omega-1)p^4} (X - v_2^{-\widehat{a}(7)} v_3^{-\widehat{b}(7)} \widehat{x}_7).$$

We see that the differential on $X - v_2^{-\widehat{a}(7)} v_3^{-\widehat{b}(7)} \widehat{x}_7$ is expressed as

$$d(X - v_2^{-\widehat{a}(7)} v_3^{-\widehat{b}(7)} \widehat{x}_7) \equiv -v_3^{(p^3\omega-1)p^4} \widehat{t}_3^{p^4} \qquad \bmod \left(v_2^{p^4}\right).$$

This gives

$$d(\widehat{y}_2) \equiv -v_2^{(p^3-1)\widehat{a}(6)+a_2(6)} v_3^{(p^3-1)\widehat{b}(6)+(p^3\omega-1)p^6+b_2(6)} \widehat{t}_3^{p^4} \tag{7.8}$$

mod $(v_2^{(p^3-1)\widehat{a}(6)+a_2(6)+p^4})$. Summing (7.7) and (7.8), we have the same one as the second of (7.2). □

8. Proof of Lemma 6.8 for $m = 4$

Notice that the exponent $e(p, m)$ in Proposition 5.12 is

$$(p+1)p^5 + \min \left\{ \begin{array}{l} (p^5 - p^2 - 1)p^3 \\ (2p+1)p^4 - p^2 \\ p^5 + 2p^3 - 1 \end{array} \right\} = (p+1)p^5 + p^5 + 2p^3 - 1,$$

which is always greater than or equal to

$$a_2(6) + \widehat{a}(3) + p + 1 = (p+1)p^5 + p^5 + p^3 + p^2 + p + 1$$

for all primes p. By Proposition 5.12 we have

$$d(\widehat{x}_6) \equiv v_2^{a_1(6)} v_3^{b_1(6)} \widehat{t}_3^{p^6} - v_2^{a_2(6)} v_3^{b_2(6)} (\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3))$$

mod $(v_2^{a_2(6)+\widehat{a}(3)+p+1})$. In particular we have

$$d(\widehat{x}_6) \equiv v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \widehat{t}_3^{p^6} \quad \bmod \left(v_2^{a_2(6)}\right).$$

for any prime p. For $d(\widehat{x}_9)$ we have

$$d(\widehat{x}_8^p) \equiv v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \widehat{t}_3^{p^9} \qquad \bmod \left(v_2^{p^3 a_2(6)}\right). \tag{8.1}$$

Instead of (7.2) we have

$$d(\widehat{y}_1) \equiv -v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \widehat{t}_3^{p^9} + v_2^{\widehat{a}(9)-\widehat{a}(3)} v_3^{\widehat{b}(9)-\widehat{b}(3)} (\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)) \tag{8.2}$$

mod $(v_2^{\widehat{a}(9)+p+1})$. Moreover, as same as (7.8) we have

$$d(\widehat{y}_2) \equiv -v_2^{\widehat{a}(9)-\widehat{a}(3)} v_3^{\widehat{b}(9)-\widehat{b}(3)} \widehat{t}_3^{p^4} \qquad \bmod \left(v_2^{\widehat{a}(9)-\widehat{a}(3)+p^4}\right). \tag{8.3}$$

Summing (8.1), (8.2) and (8.3), we obtain

$$d(\widehat{x}_9) \equiv -v_2^{\widehat{a}(9)-\widehat{a}(3)} v_3^{\widehat{b}(9)-\widehat{b}(3)} \widehat{x}_3^{p-1} d(\widehat{x}_3) \qquad \mathrm{mod}\ \left(v_2^{\widehat{a}(9)+p+1}\right).$$

for any prime p.

Define integers $n(k)$ by

$$n(k) = \begin{cases} p+1 & \text{for } k \equiv 3\ (6), \\ (p+1)p^2 & \text{for } k \equiv 4\ (6), \\ (p+1)p^3 & \text{for } k \equiv 5\ (6), \\ p^2-p+2 & \text{for } k \equiv 0\ (6), \\ (p^2-p+2)p & \text{for } k \equiv 1\ (6), \\ (p^2-p+2)p^2 & \text{for } k \equiv 2\ (6). \end{cases} \tag{8.4}$$

Instead of (6.3), we may define integers $\widehat{a}(k)$ for $k \geq 9$ inductively on k by

$$\widehat{a}(k) = \begin{cases} p\widehat{a}(k-1) + p^5 - p^4 + p^3 + p^2 & \text{for } k \equiv 3 \bmod (6), \\ p\widehat{a}(k-1) + p^4 & \text{for } k \equiv 0 \bmod (6), \\ p\widehat{a}(k-1) & \text{for } k \not\equiv 0 \bmod (3). \end{cases}$$

This suggests that we may compute $d(\widehat{x}_k)$ modulo $(v_2^{\widehat{a}(k)+n(k)})$ inductively on k.

Assume that the congruences

$$d(\widehat{x}_{k-1}) \equiv -v_2^{p^{k-10}(\widehat{a}(9)-\widehat{a}(3))} v_3^{p^{k-10}(\widehat{b}(9)-\widehat{b}(3))} \widehat{x}_{k-7}^{p-1} d(\widehat{x}_{k-7}) \tag{8.5}$$

holds modulo $(v_2^{\widehat{a}(k-1)+n(k-1)})$. For $10 \leq k \leq 16$ case it follows by direct calculations. Moreover, this gives $d(\widehat{x}_k)$ whenever $10 \leq k \not\equiv 0 \bmod (3)$, since $\widehat{x}_k = \widehat{x}_{k-1}^p$. In other cases we denote $\widehat{x}_k - \widehat{x}_{k-1}^p$ by $\widehat{z}_k$. Notice that $\widehat{z}_k$ is related with $\widehat{z}_{k-6}$ by

$$\widehat{z}_k = -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} \widehat{z}_{k-6}.$$

Then $d(\widehat{z}_k)$ is expressed as

$$\begin{aligned} d(\widehat{z}_k) &= -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} d\left(\widehat{x}_{k-6}^{p-1} \widehat{z}_{k-6}\right) \\ &= -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \left\{ d\left(\widehat{x}_{k-6}^{p-1}\right) \widehat{z}_{k-6} + \eta_R\left(\widehat{x}_{k-6}^{p-1}\right) d\left(\widehat{z}_{k-6}\right) \right\}. \end{aligned}$$

Notice that $\widehat{z}_{k-6}$ is divided by $v_2^{\widehat{a}(k-6)-\widehat{a}(k-12)}$, so that we can ignore $d\left(\widehat{x}_{k-6}^{p-1}\right)$, because

$$\begin{aligned} &\widehat{a}(k) + n(k) - (\widehat{a}(k) - \widehat{a}(k-6)) - (\widehat{a}(k-6) - \widehat{a}(k-12)) \\ &\quad = n(k) + \widehat{a}(k-12) \\ &\quad < \widehat{a}(k-6). \end{aligned}$$

Thus we have

$$d(\widehat{z}_k) \equiv -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{z}_{k-6}) \qquad \mathrm{mod}\ \left(v_2^{\widehat{a}(k)+n(k)}\right). \tag{8.6}$$

On the other hand, by assumption (8.5) we have

$$
\begin{aligned}
d(\widehat{x}_{k-1}^{p}) &\equiv -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-7}^{(p-1)p} d(\widehat{x}_{k-7}^{p}) \\
&\equiv -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{x}_{k-7}^{p}) \qquad \bmod \left(v_2^{\widehat{a}(k)+n(k)}\right).
\end{aligned} \tag{8.7}
$$

Summing (8.6) and (8.7), we obtain

$$
d(\widehat{x}_k) \equiv -v_2^{\widehat{a}(k)-\widehat{a}(k-6)} v_3^{\widehat{b}(k)-\widehat{b}(k-6)} \widehat{x}_{k-6}^{p-1} d(\widehat{x}_{k-6}) \qquad \bmod \left(v_2^{\widehat{a}(k)+n(k)}\right)
$$

as desired. □

9. Proof of Lemma 6.8 for $m = 3$

Notice that the exponent $e(p, m)$ in Proposition 5.12 is

$$
(p+1)p^5 + \min \left\{ \begin{array}{l} (p^4 - p^2 - 1)p^3 \\ g(p, 3) \\ p^4 + 2p^3 - 1 \end{array} \right\} = (p+1)p^5 + p^4 + 2p^3 - 1,
$$

which is always greater than or equal to

$$
a_2(6) + \widehat{a}(3) + p + 1 = (p^6 + p^5 + p^4) + (p^3 + p^2) + p + 1
$$

for all primes p. By Proposition 5.12 and 6.8 we have

$$
\left\{ \begin{array}{l}
d(M) \equiv v_2^{a_1(6)} v_3^{b_1(6)} \widehat{t}_3^{p^6} - v_2^{a_2(6)} v_3^{b_2(6)} (\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)) \\
\qquad\qquad \bmod \left(v_2^{a_2(6)+\widehat{a}(3)+p+1}\right), \\
d(-v_2^{a_2(6)} v_3^{b_2(6)-(p^3\omega-1)p^4} X) \\
\qquad \equiv -v_2^{a_2(6)} v_3^{b_2(6)-(p^3\omega-1)p^4} (\widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4} \widehat{t}_3^{p^4}) \\
\qquad\qquad \bmod \left(v_2^{\widehat{a}(6)+p^4}\right).
\end{array} \right. \tag{9.1}
$$

This gives

$$
d(\widehat{x}_6) \equiv v_2^{a_1(6)} v_3^{b_1(6)} \widehat{t}_3^{p^6} + v_2^{a_2(6)} v_3^{b_2(6)} \widehat{x}_3^{p-1} d(\widehat{x}_3) - v_2^{a_2(6)} v_3^{b_2(6)-(p^3\omega-1)p^4} \widehat{t}_3^{p^7}
$$

mod $(v_2^{a_2(6)+\widehat{a}(3)+p+1})$. In particular we have

$$
d(\widehat{x}_6) \equiv v_2^{\widehat{a}(6)} (v_3^{\widehat{b}(6)} \widehat{t}_3^{p^6} - v_3^{b_2(6)-(p^3\omega-1)p^4} \widehat{t}_3^{p^7})
$$

mod $(v_2^{\widehat{a}(6)+\widehat{a}(3)})$. Using the first congruence in (9.1), we find that

$$
d(\widehat{x}_8) \equiv v_2^{p^2\widehat{a}(6)} (v_3^{p^2\widehat{b}(6)} \widehat{t}_3^{p^8} - v_3^{p^2 b_2(6)} \widehat{t}_3^{p^6})
$$

mod $(v_2^{p^2(\widehat{a}(6)+\widehat{a}(3))})$. For $d(\widehat{x}_9)$ we have

$$
\left\{ \begin{array}{l}
d(\widehat{x}_8^{p}) \equiv v_2^{p^3\widehat{a}(6)} (v_3^{p^3\widehat{b}(6)} \widehat{t}_3^{p^9} - v_3^{p^3 b_2(6)} \widehat{t}_3^{p^7}) \quad \bmod \left(v_2^{p^3(\widehat{a}(6)+\widehat{a}(3))}\right) \\
d(\widehat{y}_1) \equiv -v_2^{p^3\widehat{a}(6)} v_3^{p^3\widehat{b}(6)} \widehat{t}_3^{p^9} + v_2^{\widehat{a}(9)-\widehat{a}(3)} v_3^{\widehat{b}(9)-\widehat{b}(3)} (\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)) \\
\qquad\qquad \bmod \left(v_2^{\widehat{a}(9)+p+1}\right).
\end{array} \right. \tag{9.2}
$$

Moreover, we find that

$$d(\widehat{y}_3) \equiv v_2^{p^3\widehat{a}(6)} v_3^{p^3 b_2(6)} (\widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4} \widehat{t}_3^{p^4}) \mod \left(v_2^{p^3\widehat{a}(6)+p^4}\right). \tag{9.3}$$

Summing (9.2) and (9.3), we have

$$d(\widehat{x}_9) \equiv -v_2^{\widehat{a}(9)-\widehat{a}(3)} v_3^{\widehat{b}(9)-\widehat{b}(3)} \widehat{x}_3^{p-1} d(\widehat{x}_3) \mod \left(v_2^{\widehat{a}(9)+p+1}\right).$$

for all primes p.

Define integers $n(k)$ by

$$n(k) = \begin{cases} p+1 & \text{for } k \equiv 3 \ (6), \\ (p+1)p & \text{for } k \equiv 4 \ (6), \\ (p+1)p^2 & \text{for } k \equiv 5 \ (6), \\ p^3 & \text{for } k \equiv 0 \ (6), \\ p^4 & \text{for } k \equiv 1 \ (6), \\ p^5 & \text{for } k \equiv 2 \ (6). \end{cases} \tag{9.4}$$

Instead of (6.3), we may define integers $\widehat{a}(k)$ for $k \geq 9$ inductively on k by

$$\widehat{a}(k) = \begin{cases} p\widehat{a}(k-1) + p^3 + p^2 & \text{for } k \equiv 3 \mod (6), \\ p\widehat{a}(k-1) + p^4 & \text{for } k \equiv 0 \mod (6), \\ p\widehat{a}(k-1) & \text{for } k \not\equiv 0 \mod (3). \end{cases}$$

This suggests that we may compute $d(x_k)$ modulo $(v_2^{\widehat{a}(k)+n(k)})$ inductively on k. We can prove this proposition in the same fashion as in case that $m = 4$. □

10. Proof of Lemma 6.8 for $m = 2$

Notice that the exponent $e(p, m)$ in Proposition 5.12 is

$$(p+1)p^5 + \min \left\{ \begin{array}{l} (p^3 - p^2 - 1)p^3 \\ (p^3 - 1)p^2 \\ p^3 - p^2 - 1 + f(p, 2) \end{array} \right\}$$

which is larger than

$$a_2(6) + \widehat{a}(3) + 2 = (p+1)p^5 + p^3 + (p^3 + p^2) + 2$$

only if $p > 2$. We define N by

$$N = M - v_2^{a_2(6)} v_3^{b_2(6)-(p^3\omega-1)p^4} \widetilde{X}.$$

By Proposition 5.12 we have

$$\left\{ \begin{array}{l} d(M) \equiv -v_2^{a_2(6)} v_3^{b_2(6)} (\widehat{t}_3^{p^4} - \widehat{x}_3^{p-1} d(\widehat{x}_3)) \mod \left(v_2^{a_2(6)+\widehat{a}(3)+2}\right) \\ d(-v_2^{a_2(6)} v_3^{b_2(6)-(p^3\omega-1)p^4} \widetilde{X}) \\ \qquad \equiv -v_2^{a_2(6)} v_3^{b_2(6)-(p^3\omega-1)p^4} \\ \qquad\qquad (\widehat{t}_3^{p^7} - v_3^{(p^3\omega-1)p^4} \widehat{t}_3^{p^4} - v_2^{p^3} v_3^{p\omega(p^2-p^3-1-p^6)} v_4^{p^6\omega} \widehat{t}_3^{p^7}) \\ \qquad\qquad\qquad \mod \left(v_2^{a_2(6)+2p^3-p^2-1}\right) \end{array} \right. \tag{10.1}$$

This gives

$$\begin{aligned} d(N) &\equiv v_2^{a_2(6)} v_3^{b_2(6)} \widehat{x}_3^{p-1} d(\widehat{x}_3) \\ &\qquad - v_2^{a_2(6)} v_3^{b_2(6)-(p^3\omega-1)p^4} (\widehat{t}_3^{p^7} - v_2^{p^3} v_3^{p\omega(p^2-p^3-1-p^6)} v_4^{p^6\omega} \widehat{t}_3^{p^7}) \\ &\qquad\qquad \mathrm{mod}(v_2^{a_2(6)+\widehat{a}(3)+2}). \end{aligned}$$

Then it is easy to see that

$$d(\widehat{x}_6) \equiv v_2^{a_2(6)} v_3^{b_2(6)} \left(\widehat{x}_3^{p-1} d(\widehat{x}_3) - v_3^{-(p^3\omega-1)p^4} \widehat{t}_3^{p^7} \right) \quad \mathrm{mod} \left(v_2^{a_2(6)+\widehat{a}(3)+2} \right)$$

for $p > 2$. In particular we have

$$d(\widehat{x}_6) \equiv -v_2^{\widehat{a}(6)} v_3^{\widehat{b}(6)} \widehat{t}_3^{p^7} \quad \mathrm{mod} \left(v_2^{\widehat{a}(6)+\widehat{a}(3)} \right).$$

for $p > 2$. Using the first congruence in (10.1), we find that

$$d(\widehat{x}_8) \equiv -v_2^{p^2\widehat{a}(6)} v_3^{p^2 b_2(6)} \widehat{t}_3^{p^6} \quad \mathrm{mod} \left(v_2^{p^2(\widehat{a}(6)+p^3)} \right).$$

For $d(\widehat{x}_9)$ we have

$$(10.2) \qquad \begin{cases} d(\widehat{x}_8^p) \equiv -v_2^{p^3\widehat{a}(6)} v_3^{p^3 b_2(6)} \widehat{t}_3^{p^7} & \mathrm{mod} \left(v_2^{p^3(\widehat{a}(6)+p^3)} \right), \\ d(\widehat{y}_4) \equiv -v_2^{\widehat{a}(9)-\widehat{a}(3)} v_3^{\widehat{b}(9)-\widehat{b}(3)} \left(\widehat{x}_3^{p-1} d(\widehat{x}_3) - v_3^{-(p^3\omega-1)p^4} \widehat{t}_3^{p^7} \right) & \\ \qquad\qquad \mathrm{mod} \left(v_2^{\widehat{a}(9)+2} \right). \end{cases}$$

Summing (10.2), we obtain

$$d(\widehat{x}_9) \equiv -v_2^{\widehat{a}(9)-\widehat{a}(3)} v_3^{\widehat{b}(9)-\widehat{b}(3)} \widehat{x}_3^{p-1} d(\widehat{x}_3) \quad \mathrm{mod} \left(v_2^{p^3\widehat{a}(6)+\widehat{a}(3)+2} \right)$$

for $p > 2$.

Define integers $n(k)$ by

$$(10.3) \qquad n(k) = \begin{cases} 2 & \text{for } k \equiv 3 \ (6), \\ 2p & \text{for } k \equiv 4 \ (6), \\ 2p^2 & \text{for } k \equiv 5 \ (6), \\ p^3 & \text{for } k \equiv 0 \ (6), \\ p^4 & \text{for } k \equiv 1 \ (6), \\ p^5 & \text{for } k \equiv 2 \ (6). \end{cases}$$

Instead of (6.3), we may define integers $\widehat{a}(k)$ for $k \geq 9$ inductively on k by

$$\widehat{a}(k) = \begin{cases} p\widehat{a}(k-1) + p^3 + p^2 & \text{for } k \equiv 3 \bmod (6), \\ p\widehat{a}(k-1) + p^3 & \text{for } k \equiv 0 \bmod (6), \\ p\widehat{a}(k-1) & \text{for } k \not\equiv 0 \bmod (3). \end{cases}$$

This suggests that we may compute $d(x_k)$ modulo $(v_2^{\widehat{a}(k)+n(k)})$ inductively on k. We can prove this proposition in the same fashion as in case that $m = 4$. □

11. Proof of the main theorem

Here we will prove Theorem 6.9. First we consider the case that $m \geq 5$ (Theorem 1.3 included).

By Lemma 2.4 it suffices to show that the set

$$\left\{ \delta\left(\widehat{x}_k^s / v_2^{\widehat{a}(k)}\right) : \ k \geq 0, \ s > 0 \text{ and } p \nmid s \right\} \subset \mathrm{Ext}^1_{\Gamma(m+1)}\left(M_3^0\right)$$

in linearly independent over

$$\widehat{k}(2)_*/(v_2) = \mathbf{Z}/(p)[v_2, \ldots, v_m, \widehat{v}_1].$$

It follows from (2.2) that this group is the free $\widehat{K}(2)_*$-module on the nine classes represented by

$$\left\{ \widehat{t}_1, \widehat{t}_1^p, \widehat{t}_1^{p^2}, \widehat{t}_2, \widehat{t}_2^p, \widehat{t}_2^{p^2}, \widehat{t}_3, \widehat{t}_3^p, \widehat{t}_3^{p^2} \right\}. \tag{11.1}$$

In $\Gamma(m+1)/I_3$ we have

$$\begin{aligned} \eta_R(\widehat{v}_4) &= \widehat{v}_4 + v_3 \widehat{t}_1^{p^3} - v_3^{p\omega} \widehat{t}_1, \\ \eta_R(\widehat{v}_5) &= \widehat{v}_5 + v_4 \widehat{t}_1^{p^4} + v_3 \widehat{t}_2^{p^3} - v_4^{p\omega} \widehat{t}_1 - v_3^{p^2\omega} \widehat{t}_2, \\ \text{and} \qquad \eta_R(\widehat{v}_6) &= \widehat{v}_6 + v_5 \widehat{t}_1^{p^5} + v_4 \widehat{t}_2^{p^4} + v_3 \widehat{t}_3^{p^3} - v_5^{p\omega} \widehat{t}_1 - v_4^{p^2\omega} \widehat{t}_2 - v_3^{p^3\omega} \widehat{t}_3. \end{aligned}$$

so in $\mathrm{Ext}^1_{\Gamma(m+1)}(M_3^0)$ we have

$$\begin{aligned} \widehat{t}_1^{p^3} &= v_3^{p\omega-1} \widehat{t}_1, \\ \widehat{t}_2^{p^3} &= v_3^{p^2\omega-1} \widehat{t}_2 + v_3^{-1}\left(v_4^{p\omega} \widehat{t}_1 - v_4 \widehat{t}_1^{p^4}\right), \\ \text{and} \qquad \widehat{t}_3^{p^3} &= v_3^{p^3\omega-1} \widehat{t}_3 + v_3^{-1}\left(v_4^{p^2\omega} \widehat{t}_2 + v_5^{p\omega} \widehat{t}_1 - v_4 \widehat{t}_2^{p^4} - v_5 \widehat{t}_1^{p^5}\right). \end{aligned}$$

This means that for $m \geq 2$ we can replace the $\widehat{K}(2)_*$-basis of Ext^1 of (11.1) with

$$\left\{ \widehat{t}_1^{p^2}, \widehat{t}_1^{p^3}, \widehat{t}_1^{p^4}, \widehat{t}_2^{p^4}, \widehat{t}_2^{p^5}, \widehat{t}_2^{p^6}, \widehat{t}_3^{p^6}, \widehat{t}_3^{p^7}, \widehat{t}_3^{p^8} \right\},$$

so its basis over $\widehat{k}(2)_*/(v_2)$ is

$$\left\{ \widehat{v}_3^t \widehat{t}_1^{p^2}, \widehat{v}_3^t \widehat{t}_1^{p^3}, \widehat{v}_3^t \widehat{t}_1^{p^4}, \widehat{v}_3^t \widehat{t}_2^{p^4}, \widehat{v}_3^t \widehat{t}_2^{p^5}, \widehat{v}_3^t \widehat{t}_2^{p^6}, \widehat{v}_3^t \widehat{t}_3^{p^6}, \widehat{v}_3^t \widehat{t}_3^{p^7}, \widehat{v}_3^t \widehat{t}_3^{p^8} : t \geq 0 \right\}.$$

Now define integers $\widehat{c}(k)$ for $k \geq 0$ by

$$\widehat{c}_k = \begin{cases} 0 & \text{for } 0 \leq k \leq 8 \\ (p-1)p^{k-4} + \widehat{c}(k-4) & \text{for } k \geq 9. \end{cases}$$

For $k > 4$, write $k = k_0 + 4k_1$ with $5 \leq k_0 \leq 8$. Then the above is equivalent to

$$\widehat{c}(k) = (p-1)p^{k_0}\left(\frac{p^{4k_1}-1}{p^4-1}\right). \tag{11.2}$$

Then Lemmas 4.3 and 6.8 imply that

$$d(\widehat{x}_k) \equiv \pm v_2^{\widehat{a}(k)} v_3^{\widehat{b}(k)} \widehat{v}_3^{\widehat{c}(k)} \begin{cases} \widehat{t}_1^{p^{2+k}} & \text{for } 0 \le k \le 2 \\ \widehat{t}_2^{p^{1+k}} & \text{for } 3 \le k \le 4 \\ \widehat{t}_2^{p^6} & \text{for } k > 4 \text{ and } k \equiv 1 \bmod 4 \\ \widehat{t}_3^{p^6} & \text{for } k > 4 \text{ and } k \equiv 2 \bmod 4 \\ \widehat{t}_3^{p^7} & \text{for } k > 4 \text{ and } k \equiv 3 \bmod 4 \\ \widehat{t}_3^{p^8} & \text{for } k > 4 \text{ and } k \equiv 4 \bmod 4. \end{cases}$$

modulo $\left(v_2^{1+a(k)}\right)$. This and the multiplicative property of the right unit imply that

$$(11.3)\quad \delta\left(\frac{\widehat{x}_k^s}{v_3^{\widehat{a}(k)}}\right) = \pm s v_3^{\widehat{b}(k)} \widehat{v}_3^{(s-1)p^k+\widehat{c}(k)} \begin{cases} \widehat{t}_1^{p^{2+k}} & \text{for } 0 \le k \le 2 \\ \widehat{t}_2^{p^{1+k}} & \text{for } 3 \le k \le 4 \\ \widehat{t}_2^{p^6} & \text{for } k > 4 \text{ and } k \equiv 1 \bmod 4 \\ \widehat{t}_3^{p^6} & \text{for } k > 4 \text{ and } k \equiv 2 \bmod 4 \\ \widehat{t}_3^{p^7} & \text{for } k > 4 \text{ and } k \equiv 3 \bmod 4 \\ \widehat{t}_3^{p^8} & \text{for } k > 4 \text{ and } k \equiv 4 \bmod 4. \end{cases}$$

By Lemma 2.4 it suffices to show that these elements (with $k \ge 0$ and $s > 0$ not divisible by p) are linearly independent over $\widehat{k}(1)_*$. The ones for $0 \le k \le 4$ are clearly independent of those for $k > 0$, so it suffices to consider the exponents of $\widehat{v}_3$ above for $k > 4$, i.e., to show that the set

$$(11.4)\qquad \left\{\widehat{v}_3^{(s-1)p^k+\widehat{c}(k)} : k > 4,\ s > 0,\ p \nmid s\right\}$$

is linearly independent over $\widehat{k}(2)_*/(v_2)$. For a fixed value of k, the exponents appearing in (11.4) are congruent to $\widehat{c}(k)$ modulo p^k but not congruent (since $p \nmid s$) to $-p^k + \widehat{c}(k)$ modulo p^{k+1}.

Now (11.2) implies that for $k > 4$,

$$\begin{aligned} \widehat{c}(k) &\equiv p^k - \frac{(p-1)p^{k_0}}{p^4-1} && \bmod \left(p^{k+1}\right) \\ \text{so}\qquad (s-1)p^k + \widehat{c}(k) &\equiv sp^k - \frac{(p-1)p^{k_0}}{p^4-1} && \bmod \left(p^{k+1}\right). \end{aligned}$$

Hence our condition is that the exponents associated with k are congruent to $-\frac{(p-1)p^{k_0}}{p^4-1}$ modulo p^k but not modulo p^{k+1}, and these conditions are mutually exclusive for differing k. □

Proof for $2 \le m \le 4$. The argument is the same subject to the following changes. The integers $\widehat{c}(k)$ are defined by

$$\widehat{c}(k) = \begin{cases} 0 & \text{for } 0 \le k \le 8, \\ (p-1)p^{k-6} + \widehat{c}(k-6) & \text{for } k \ge 9. \end{cases}$$

For $k > 2$, write $k = k_0 + 6k_1$ with $3 \le k_0 \le 8$. Then the above is equivalent to

$$\widehat{c}(k) = (p-1)p^{k_0}\left(\frac{p^{6k_1}-1}{p^6-1}\right).$$

Then (11.3) gets replaced by

$$\delta\left(\frac{\widehat{x}_k^s}{v_3^{\widehat{a}(k)}}\right) = \pm s v_3^{\widehat{b}(k)} \widehat{v}_3^{(s-1)p^k+\widehat{c}(k)} \begin{cases} \widehat{t}_1^{p^{2+k}} & \text{for } 0 \le k \le 2 \\ \widehat{t}_2^{p^{1+k_0}} & \text{for } 3 \le k_0 \le 5 \text{ and} \\ \widehat{t}_3^{p^{k_0}} & \text{for } 6 \le k_0 \le 8 \text{ and} \end{cases}$$

(Notice that there is another term for $m = 3$ and $6 \le k \le 8$. We can ignore it because it is linear independent with the above coboundary.) We can argue for linear independence as before.

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