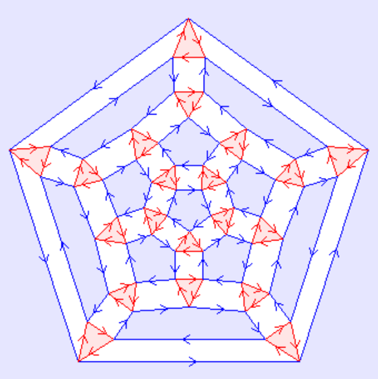
HKR again: Kuhn's work on generalized group characters

Mid-Atlantic Topology Conference 2015
University of Virginia

April 26, 2015



Doug Ravenel
University of Rochester

The paper

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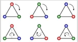

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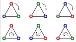

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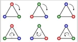

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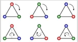

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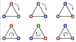

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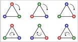

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- Atiyah's theorem implies that the rank of $K(1)^*BG$ is the number of conjugacy classes of elements of order p^i (for some i) in G.

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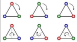

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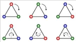

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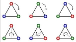

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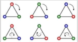

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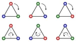

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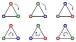

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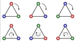

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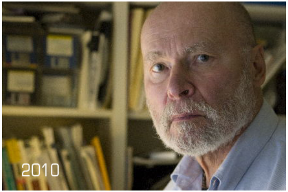
We wish to generalize the field L, the ring $Cl(G, L)$ and the Galois group $\widehat{\mathbf{Z}}^\times$ to E^*.

The Lubin-Tate construction



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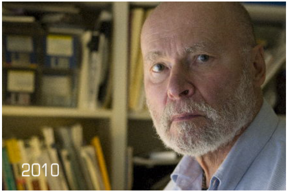

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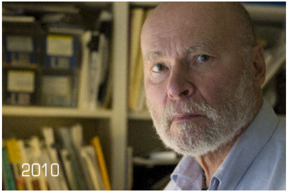

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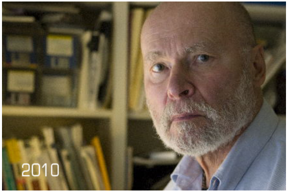

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John Tate

In 1965 Lubin and Tate gave a construction of the maximal abelian totally ramified extension of a finite extension K of the p-adic numbers $\mathbf{Q}_p$. It led them to prove some interesting things about formal groups laws that are very useful in stable homotopy theory.

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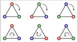

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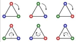

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$$\begin{aligned} & [p](x) \equiv px && \bmod (x)^2 \\ \text{and} \quad & [p](x) \equiv u^{p^n-1} x^{p^n} && \bmod I_n. \end{aligned}$$

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The generalized character theorem (continued)

The role of the Lubin-Tate extension here is the following.

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$$\begin{array}{ccc} x & & E_n^* B\mathbf{Z}/(p^i) \xrightarrow{\varphi_i} A[[x]]/(f^{(i)}(x)) \longrightarrow L_i \\ \big\downarrow & & \big\downarrow \qquad\qquad\qquad \big\downarrow f \qquad\qquad \big\downarrow \\ [p](x) & & E_n^* B\mathbf{Z}/(p^{i+1}) \xrightarrow{\varphi_{i+1}} A[[x]]/(f^{(i+1)}(x)) \longrightarrow L_{i+1} \end{array}$$

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where the left vertical map is induced by the surjection $\mathbf{Z}/(p^{i+1}) \to \mathbf{Z}/(p^i)$ and the right one is the inclusion of L_i, the splitting field of $f^{(i)}(x)$, into L_{i+1}.

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Next note that since $A = W(\mathbf{F}_{p^n})$ is a free $\mathbf{Z}_p$-module of rank n, we have an isomorphism $\alpha : A \to \mathbf{Z}_p^n$. We also have a ring homomorphism $\mu : \mathbf{Z}/(p^i) \to A/(p^i)$.

The generalized character theorem (continued)

Putting all this together gives, for each $\gamma \in G_{n,p}$, a diagram

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E_n^* BG \arrow[r, "\gamma^*"] & \varinjlim_i E_n^* B\mathbf{Z}/(p^i)^n \arrow[d, equal, "\alpha^*"] & \\
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This leads to the character map χ_G in the theorem.

HAPPY BIRTHDAY NICK!