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FORMAL A-MODULES AND THE ADAMS-NOVIKOV SPECTRAL SEQUENCE

Douglas C. RAVENEL*
Department of Mathematics, University of Washington, Seattle, WA 98195, USA

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Introduction

In recent years the Adams-Novikov spectral sequence has proven to be a useful tool in stable homotopy theory. Its E_2-term is an Ext group derived from a certain universal formal group law. In this paper we show how to derive a similar Ext group from a universal formal A-module, where A is the ring of integers in a number field or its p-adic completion. We do not know if this group has any topological significance, i.e. we do not know if it is the E_2-term of any spectral sequence.

In Section 1 we sketch the relevant theory behind the ANSS E_2-term. In Section 2 we generalize this theory from formal group laws to formal A-modules using results from §21 of Hazewinkel [3]. In Section 3 we make some calculations and discuss some open questions. In particular we state a conjecture about Ext^1 generalizing certain well known connections between the order of the image of the J-homomorphism and Bernoulli numbers.

1. Formal group laws and the ANSS E_2-term

Proofs and references for most results in this section can be found in [3] and [11] unless otherwise stated.

1.1. Definition. A *formal group law* (FGL) over a commutative unitary ring R is a power series $F(x,y)\in R[[x,y]]$ satisfying three conditions,

(i) $F(0,x)=F(x,0)=x$,
(ii) $F(x,y)=F(y,x)$, and
(iii) $F(F(x,y),z)=F(x,F(y,z))$.

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These correspond to the existence of an identity, commutativity and associativity in an abelian group. A thorough and lucid treatment of this theory can be found in [3].

1.2. Definition. A *universal* FGL $U(x,y)$ over a ring L is one having the property that for any FGL F over R there is a homomorphism $\theta: L \to R$ such that $\theta(U(x,y)) = F(x,y)$.

The existence of this object is easy to establish. Write

$$F(x,y) = \sum_{i,j\geq 0} a_{ij}x^i y^j.$$

Then 1.1 implies certain relations among the coefficients a_{ij}, e.g. (i) says $a_{i0} = a_{0i} = 0$ for $i > 1$, and (ii) says $a_{ij} = a_{ji}$. We let $L = Z[a_{ij}]/I$, where I is the ideal generated by these relations, and $U(x,y) = \sum a_{ij}x^i y^j$.

The explicit structure of this L is more difficult and was first determined by Lazard [4]. A crucial step in his argument is

1.3. Comparison Lemma. *Let F and G be two FGL's over R which agree modulo $(x,y)^n$. Then*

$$G(x,y) = F(x,y) + aC_n(x,y) \mod (x,y)^{n+1}$$

where

$$a \in R \quad \text{and} \quad C_n(x,y) = \frac{(x+y)^n - x^n - y^n}{v(n)}$$

with $v(n) = 1$ unless n is a power of a prime p in which case $v(n) = p$.

To describe L it is convenient to introduce a grading on it by setting $\deg a_{ij} = 2(i+j-1)$. Hence if $\deg x = \deg y = -2$, then $F(x,y)$ is homogeneous of degree -2.

1.4. Theorem. $L = Z[x_1, x_2, \ldots]$ *where* $\deg x_i = 2i$.

Proofs of 1.3 and 1.4 can be found in [3] and [11].

Quillen's theorem establishes an isomorphism between L and $\pi_*(MU)$ as follows. One has $MU^*(CP^\infty) = \pi_*(MU)[[x]]$ with $x \in MU^2(CP^\infty)$; since we are in cohomology the coefficient ring $\pi_*(MU)$ is negatively graded. The H-space map $CP^\infty \times CP^\infty \to CP^\infty$ sends x to a power series in $x \otimes 1$ and $1 \otimes x$ which is easily seen to be a FGL. Hence we get a homomorphism $\theta: L \to \pi_*(MU)$ which is Quillen's isomorphism.

1.5. Definition. Let $\Gamma \subset Z[[x]]$ be the group of power series of the form $\sum_{i\geq 0} b_i x^{i+1}$ with $b_0 = 1$ under composition. □

Γ acts on L as follows. Let $U(x, y)$ be the universal FGL and $\gamma \in \Gamma$. Then $\gamma^{-1}U(\gamma(x), \gamma(y))$ is another FGL over L and is therefore induced by some ring endomorphism $\theta_\gamma : L \to L$. Since γ is invertible θ_γ is an automorphism and one easily checks that this defines a Γ-action on L.

The ANSS E_2-term for the sphere can be shown to be isomorphic to $H^*(\Gamma; L)$. It is usually identified as $\mathrm{Ext}_{MU_*(MU)}(\pi_*(MU), \pi_*(MU))$ where $\pi_*(MU) = L$ and $MU_*(MU) = L(b_1, b_2, \ldots)$ with a certain coproduct. By a standard argument this Ext group is isomorphic to $\mathrm{Ext}_B(Z, L)$ where $B = Z[b_1, b_2, \ldots]$. This ring B can be identified with the ring of integer valued functions on Γ. An element $\gamma \in \Gamma$ can be written as $\gamma = \sum_{i\geq 0} b_i x^{i+1}$ with $b_0 = 1$ and $b_i \in Z$. Then a polynomial in B associates an integer to each γ. If $\gamma' = \sum b_i' x^{i+1}$ and $\gamma'' = \sum b_i'' x^{i+1}$, then

$$\gamma = \gamma''\gamma' = \sum b_k x^{k+1} = \sum b_i''\left(\sum b_j' x^{j+1}\right)^{i+1}.$$

By equating the coefficients of x^k in this equation one obtains the usual coproduct in $MU_*(MU)$, leading to our description of the E_2-term.

If one localizes at a prime p one can replace $MU_*(MU)$ by a smaller, more manageable object, $BP_*(BP)$.

1.6. Definition. Let F be a FGL over a torsion free ring R. Then $\log x$ is the power series over $\mathbb{Q} \otimes R$ given by

$$\log x = \int_0^x \frac{\mathrm{d}t}{F_2(t, 0)}$$

where $F_2(x, y) = \partial F / \partial y$ and one has

$$\log F(x, y) = \log x + \log y,$$

i.e. the logarithm is a isomorphism over $R \otimes \mathbb{Q}$ between F and the additive FGL $x + y$. F is *p-typical* if $\log x = \sum_{i \geq 0} \lambda_i x^{p^i}$.

The definition of p-typical can be generalized to an arbitrary R. More importantly it can be shown (Cartier [12]) that any FGL over a $Z_{(p)}$-algebra R (here $Z_{(p)}$ is the ring of integers localized at p) is canonically isomorphic to a p-typical one.

1.7. Theorem. *There is a universal p-typical FGL over a ring* $V = Z_{(p)}[v_1, v_2, \ldots]$ *with* $\deg v_n = 2(p^n - 1)$. *The logarithm of this FGL is* $f(x)$ *given recursively by*

$$f(x) = x + \sum_{i \geq 0} \frac{v_i}{p} f^{(p^i)}(x^{p^i})$$

where $f^{(p^i)}$ *is obtained from* $f(x) \in \mathbb{Q} \otimes V[[x]]$ *by substituting* $v_n^{p^i}$ *for* v_n *for each* n. $\square$

For example

$$\lambda_1 = \frac{v_1}{p}, \qquad \lambda_2 = \frac{v_1^{1+p}}{p^2} + \frac{v_2}{p}$$

and

$$\lambda_3 = \frac{v_3}{p} + \frac{v_2 v_1^{p^2} + v_1 v_2^p}{p^2} + \frac{v_1^{1+p+p^2}}{p^3}.$$

The ring V is isomorphic to $\pi_*(BP)$ in the same way that L is isomorphic to $\pi_*(MU)$. The canonical isomorphism referred to above leads to a splitting of the p-localization of MU into a wedge of suspensions of BP.

Unfortunately there is no reasonable subgroup of Γ which acts on V because most power series do not preserve p-typicality. Instead one has

1.8. Lemma. *Let F be a p-typical FGL over a torsion free ring R, $\gamma \in \Gamma$ an invertible power series and $G(x, y) = \gamma F(\gamma^{-1}(x), \gamma^{-1}(y))$. Then G is p-typical iff*

$$\log \gamma^{-1}(x) = \sum_{i \geq 0} \log t_i x^{p^i}.$$

This is Lemma 1.26 of [11].

Recall $MU_*(MU) = L[b_1, b_2, \ldots] \equiv LB$. A homomorphism θ from this ring to any R corresponds to an FGL F over R (given by the restriction of θ to L) and a power series $f(x) = \sum \theta(b_i)x^i$ in $R[[x]]$. Equivalently, θ is determined by F, $G = fF(f^{-1}(x), f^{-1}(y))$, and an isomorphism between them. Similarly a homomorphism from

$$BP_*(BP) = V[t_1, t_2, \ldots] \equiv VT \quad \text{to} \quad R$$

is determined by an isomorphism between two p-typical FGL's over R. These observations are due to Landweber [13]. Hence the sets $\mathrm{Hom}(V, R)$ and $\mathrm{Hom}(VT, R)$ constitute the objects and morphisms in the category of p-typical FGL's over R and isomorphisms between them. This category is a groupoid, i.e. a small category in which every morphism is invertible. It follows that VT is a cogroupoid object in the category of commutative rings. Such objects have been christened Hopf algebroids by Haynes Miller, since a commutative Hopf algebra is a cogroup object in the same category. Accordingly there are various structure maps between V and VT corresponding to the structure of the groupoid. In particular there are maps $\eta_L, \eta_R : V \to VT$ (known as the left and right units) corresponding to the source and target of a morphism, and $\Delta : VT \to VT \otimes_V VT$ (known as the coproduct), corresponding to composition of morphisms. Here the tensor product is with respect to the bimodule structure given by η_R and η_L.

1.9. Lemma. *In $BP_*(BP) = VT$, $\eta_L : V \to VT$ is the standard inclusion and η_R is given by $\eta_R(\lambda_n) = \sum \lambda_i t_{n-i}^{p^i}$ in $VT \otimes \mathbb{Q}$. Moreover we have*

$$\sum^F v_i t_j^{p^i} = \sum^F \eta_R(v_i)^{p^j} t_i \quad \mathrm{mod}(p)$$

where $\sum^F$ denotes summation using the FGL instead of ordinary addition, e.g. we write $F(x, y) = x +_F y$. The coproduct is given by $\sum \lambda_i \Delta(t_j)^{p^i} = \sum \lambda_i t_j^{p^i} \otimes t_k^{p^{i+j}}$ or equivalently $\sum^F \Delta(t_i) = \sum^F t_i \otimes t_j^{p^i}$.

The Ext group we want is defined in the category of VT-comodules, i.e. V-modules M equipped with suitable structure maps $\psi: M \to VT \otimes_V M$. LB comodules are similarly defined and Ext is the derived functor Hom. For any connective spectrum X, $BP_*(X)$ is a VT-comodule and $\mathrm{Ext}_{VT}(V, BP_*(X))$ is the E_2-term of the ASS converging to the p-localization of $\pi_*(X)$. We also have the following local-global result.

1.10. Theorem. *For an LB-comodule M,*

$$Z_{(p)} \otimes \mathrm{Ext}_{LB}(L, M) = \mathrm{Ext}_{VT}(V, V \otimes_L M).$$

For a VT-comodule M we will abbreviate $\mathrm{Ext}_{VT}(V, M)$ by $\mathrm{Ext}(M)$. Of particular interest is $\mathrm{Ext}(V)$, the ANSS E_2-term for the sphere. One approach to it is the chromatic spectral sequence (CSS) of [7] which we now describe.

Define comodules M^i and N^i inductively as follows. $N^0 = V$, $M^n = v_n^{-1} V \otimes N^n$ (where $v_0 = p$) and $N^{n+1} = M^n / N^n$. Then the short exact sequences (SES)

$$0 \to N^n \to M^n \to N^{n+1} \to 0$$

splice together to make a long exact sequence (LES)

$$0 \to V \to M^0 \to M^1 \to M^2 \to \cdots$$

called the *chromatic resolution*. Standard homological algebra gives

1.11. Lemma. *There is a spectral sequence converging to* $\mathrm{Ext}(V)$ *with* $E_1^{n,s} = \mathrm{Ext}^s(M^n)$ *and* $d_r: E_r^{n,s} \to E_r^{n+r, s-r+1}$.

Each element in M^n is annnihilated by some power of $I_n = (p, v_1, \ldots, V_{n-1}) \subset V$ and multiplication by v_i is surjective for $i \le n$ and an isomorphism for $i = n$. We define comodules M_i^{n-i} inductively by $M_0^n = M^n$ and M_i^{n-i} is the kernel of multiplication by v_{i-1} in M_{i-1}^{n-i+1}. Hence we have SES's

$$0 \to M_i^{n-i} \to M_{i-1}^{n-i+1} \xrightarrow{v_{i-1}} M_{i-1}^{n-i+1} \to 0$$

leading to LES of Ext groups which in principle reduce the problem of computing $\mathrm{Ext}(M^n)$ to that of finding $\mathrm{Ext}(M_n^0)$, where $M_n^0 = v_n^{-1} V/I_n$. This group is surprisingly accessible, thanks to some profound insights of Jack Morava; indeed the CSS was constructed in order to exploit this. It is very closely related to the cohomology of the automorphism group of a certain FGL over the field with p^n elements. This theory is developed in [6], [8] and [9] and we will give a brief account of it now.

Let $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ and make it a V-module by defining multiplication by v_i to be trivial if $i \ne n$. Then let

$$\Sigma(n) = K(n)_* \otimes_V VT \otimes_V K(n)_*.$$

This is a Hopf algebroid (called the nth Morava stabilizer algebra) corresponding to the category of p-typical FGL's over $\mathbb{F}_p$-algebras R for which the map $\theta: V \to R$ factors through $K(n)_*$. (Such FGL's are said to have *height n*. The height of an FGL over a field of characteristic p determines it up to isomorphism over the algebraic closure.) From 1.9 one can deduce that

$$\Sigma(n) = K(n)_*[t_1, t_2, \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i)$$

as a ring, and it inherits the coproduct from VT. The relevance of $\Sigma(n)$ to the problem at hand is given by

1.12. Change of Rings Theorem [6].

$$\mathrm{Ext}(M_n^0) = \mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*).$$

To relate this to group cohomology, make $\mathbb{F}_{p^n}$ a $K(n)_*$-module by sending v_n to 1 and let F_n be the corresponding FGL over $\mathbb{F}_{p^n}$. Its endomorphism ring E_n is an algebra of rank n^2 over the p-adic integers generated by a primitive (p^n-1)th root of unit ω (the endomorphism sending x to $\bar{\omega}x$ where $\bar{\omega}$ is the mod(p) reduction of ω) and an element S (sending x to x^p) subject to the relations $S\omega = \omega^p S$ and $S^n = p$. (The endomorphism corresponding to a natural number k sends x to a power series $[k](x)$ defined inductively to be $F(x, [k-1](x))$ as in 2.1 below. In F_n one has $[p](x) = x^{p^n}$.) This ring is a maximal order in $D_n = E_n \otimes \mathbb{Q}_p$, a central division algebra over the p-adic numbers. It is also a complete local ring with maximal ideal (S) and residue field $\mathbb{F}_{p^n}$. Let $S_n \subset E_n^\times$ denote the group of units congruent to 1 mod(S). It is a nilpotent pro p-group, the inverse limit of its (finite) images in $E_n/(S^k)$. Hence it is a compact topological group and we have

1.13. Theorem. *$\Sigma(n) \otimes_{K(n)_*} \mathbb{F}_{p^n}$ is a Hopf algebra isomorphic to the continuous linear dual of the $\mathbb{F}_{p^n}$-group algebra of S_n.*

This result and 1.12 imply a close relation between $\mathrm{Ext}(M^n)$ and the continuous mod(p) cohomology of S_n. In particular $E_1 = Z_p$ so $S_1 \subset Z_p^\times$ is the group of limits congruent to 1 mod(p). It is known that any degree n extension K of $\mathbb{Q}$ can be embedded in the division algebra D_n. By letting K be the pth cyclotonic extension we find that S_{p-1} has a subgroup of order p. Its cohomology is useful for detecting elements in $\mathrm{Ext}(V)$, the ANSS E_2-term for the sphere; see [10].

2. Formal A-modules

The reference for all results in this section unless otherwise stated is §21 of [3]. A formal A-module (FAM) is a certain type of FGL defined over an A-algebra R. Before defining it we need

2.1. Definition. An endomorphism of an FGL F is a power series $f(x) \in R[[x]]$ satisfying $f(F(x, y)) = F(f(x), f(y))$. For each integer n an endomorphism $[n](x)$ is given as follows. $[0](x) = 0$ and $[n+1](x) = F(x, [n](x))$ for $n \geq 0$. We define $[-n](x)$ by $F([n](x), [-n](x)) = 0$.

Note that $[n](x) \equiv nx \bmod(x^2)$ and $F(x, [-1](x)) = 0$ so $[-1](x)$ is the formal group inverse of x. It is easy to verify that

$$F([m](x), [n](x)) = [m+n](x) \quad \text{and} \quad [m]([n](x)) = [mn](x).$$

Hence we have a homomorphism from Z to the endomorphism ring of F. It is known that if R is a finite field of characteristic p, then $[p](x)$ determines F.

2.2. Definition. Let A be the ring of integers in a number field or its p-adic completion and let R be an A-algebra. An FGL over R is a *formal A-module* (FAM) if the homomorphism $Z \to \mathrm{End}_F$ above extends to A, i.e. if there are power series $[a](x)$ for each $a \in A$ having suitable properties.

If R has characteristic p, then an FGL over R is automatically a formal Z_p-module, where Z_p denotes the p-adic integers. This follows from the fact that $[p](x) \equiv 0 \bmod(x^2)$, so $[p^i](x) \equiv 0 \bmod(x^{2^i})$, so a power series in p (i.e. a p-adic integer) will lead to a power series with coefficient in R. A similar argument works if R is a Z_p-algebra.

For A the ring of integers in a finite extension K of the p-adic numbers $\mathbb{Q}_p$, an important example of a formal A-module over A was given by Lubin–Tate [5]. They used to construct explicit abelian extensions of K.

The FGL defined above over $K(n)_*$ is closely related to the mod(p) reduction of the Lubin–Tate FAM for A the unramified degree n extension of Z_p. However, the relation between the Ext groups 1.12 and the corresponding FAM Ext group (to be defined below) is more remote. There is a homomorphism from the former to the latter analogous (via 1.13 and 2.12) to the restriction map from the cohomology of S_n to the subgroup of $A^\times$ (with A as above) consisting of units in A congruent to 1 mod(p).

We wish to generalize the theory of Section 1 to FAM's. The definition of a universal FAM is obvious and we denote its ground ring by L_A. Its existence is easy to show; in addition to the coefficients of F one needs the coefficients of $[a](x)$ for all $a \in A$, and these must satisfy certain relations. Then L_A is simply the polynomial ring over A on all these indeterminants modulo the ideal generated by all the relations.

There is a FAM comparison lemma, stated and proved in [3] as 21.2.4.

2.3. FAM Comparison Lemma. *Let A be a torsion free ring and let F and G be two FAM's which agree modulo terms of degree n. Then*

$$F(x, y) \equiv G(x, y) + d\, C_n(X, y) \bmod(x, y)^{n+1}$$

where

$$d(a-a^n)\in(v(n)) \quad \textit{for all } a\in A \quad \textit{and} \quad v(n)=1$$

unless n is a power of a prime p, in which case $v(n)=p$.

The condition on d is vacuous if n is not a prime power. If A is the ring of integers in a number field K in which the prime p splits completely, then the condition is vacuous when n is a power of p. Using the grading on L_A introduced above, it follows that the indecomposable quotient of L_A^{n-1} is the ideal P_n^A generated by all such d. If K has class number greater than one, then P_n^A may not be principal and L_A will not be a polynomial ring. The argument of 21.3.5 in [3] indicates that L_A is polynomial if all of the ideals P_n^A are principal.

If K is a Galois extension of the rationals we can describe the P_n^A more explicitly. In A we have the ideal decomposition

$$(p)=\prod_{i=1}^{k}\pi_i^e$$

where each π_i is a distinct prime ideal and e is the ramification degree. Moreover A/π_i is the field with p^f elements for each i and efk is the degree of the extension. Now let $n=p^j$. If f divides j, then $a^n-a\in\pi_i$ for each i, but it need not be in π_i^2, so we have $P_n^A=\prod_{i=1}^k\pi_i^{e-1}$. If f does not divide j, then a^n-a need not be in any prime ideal containing (p), so $P_n^A=(p)$. Hence we have

2.4. Lemma. *With notation as above and K a Galois extension of the rationals with*

$$(p)=\prod_{i=1}^{k}\pi_i^e \quad \textit{in } A,$$

then

$$P_n^A=\begin{cases}(1) & \textit{if } n \textit{ is a not a prime power,}\\ (p) & \textit{if } n \textit{ is a power of } p \textit{ but not a power of } p^f,\\ \prod_{i=1}^{k}\pi_i^{e-1} & \textit{if } n \textit{ is a power of } p^f.\end{cases}$$

Of course P_n^A can fail to be principal only if n is a suitable power of a prime at which K is ramified. For example $K=\mathbb{Q}(\sqrt{-5})$ is ramified at the prime 2 and $P_{2^j}^A=(2,1+\sqrt{-5})$, i.e. it is not a principal ideal. If each P_n^A is principal, then L_A is polynomial. In any case L_A can be embedded in a polynomial ring as follows.

If K is an infinite field it is known [3, 21.2.10] that L_K, the ground ring for the universal formal K-module, is $K[m_1,m_2,\dots]$ with $\log F(x,y)=\log x+\log y$ where $\log x=\sum_{i\geq 0}m_ix^{i+1}$. In other words the power series $\log x$ defines an isomorphism from the universal FKM to the additive one, $G(x,y)=x+y$. Moreover this FKM gives an FAM over $R_A=A[m_1,m_2,\dots]$ and hence a homomorphism $L_A\to R_A$. 2.3 can be used to show this map is injective in the same way that 1.3 is used in the case $A=Z$.

The generalization of the group Γ is easy. Let Γ_A denote the corresponding group of power series over A.

2.5. Lemma. *Let F be a FAM and $\gamma \in \Gamma_A$. Then $G(x,y)=\gamma F(\gamma^{-1}(x),\gamma^{-1}(y))$ is also a FAM.*

Proof. Let $[a]_F(x)$ and $[a]_G(x)$ denote the endomorphism of F and G corresponding to $a \in A$. Then it is easy to check that $[a]_G(x)$ can be taken to be $\gamma([a]_F(\gamma^{-1}(x)))$. □

It follows that we can define a Hopf algebroid $L_A B = L_A[b_1, b_2, \ldots]$ as in Section 1. If F is defined over a torsion free ring, it has a logarithm as in 1.6, which brings us to the notion of p-typicality. In Section 1 the theory works best for FGL's defined over $Z_{(p)}$-algebras. Such an FGL is easily seen to be a formal $Z_{(p)}$-module.

2.5. Definition. Let A be a discrete valuation ring with maximal ideal (π), finite residue field $\mathbb{F}_q$ and quotient field K. If K has characteristic 0 and R is an A-algebra mapping injectively to $R \otimes_A K$, then a FAM over R is A-typical if its logarithm has the form $\sum \lambda_i^A x^{q^i}$ with $\lambda_i^A \in R \otimes_A K$.

Again this definition can be generalized and any FAM is canonically isomorphic to an A-typical one [3, 21.5.6]. *For the rest of this section A will be as in* 2.5. The analogue of 1.7 is

2.6. Theorem. *There is a universal A-typical FAM over a ring $V_A = A[v_1^A, v_2^A, \ldots]$ with* $\deg v_n^A = 2(q^n-1)$. *The logarithm of this FAM is $f_A(x)$ given recursively by*

$$f_A(x) = x + \sum_{i>0} \frac{v_i^A}{\pi} f_A^{(q^i)}(x^{q^i})$$

where $f_A^{(q^i)}$ is obtained from f_A by substituting $(v_n^A)^{q^i}$ for v_n^A for all $n>0$.

Note that the generators v_n^A depend on the choice of π, the logarithm $f_A(x)$ being fixed.

If A is a $Z_{(p)}$-algebra, then an A-typical FAM is also a p-typical FGL, so there is a homomorphism $\theta_A : V \to V_A$. It must send the logarithm $f(x)$ of 1.7 to $f_A(x)$ above and this enables us to calculate it explicitly. For example

2.7. Corollary. *Let K be a degree f extension of $\mathbb{Q}$ in which p is unramified and does not split, and let A be the ring of integers in K localized at (p). Using p as a uniformizing parameter in A gives*

$$\theta_A(v_i) = \begin{cases} 0 & \text{if } f \nmid i, \\ v_{i/f}^A & \text{if } f \mid i. \end{cases}$$

Proof. We have $q=p^f$ so $f_A(x)=\sum_{i\geq 0}\lambda_i^A x^{p^{if}}=\sum \theta(\lambda_i)x^{p^i}$. The result follows by equating coefficients of these two power series. $\square$

The analogue of 1.8 is

2.8. Lemma. *Let F be an A-typical FAM, $\gamma\in\Gamma_A$ and*

$$G(x,y)=\gamma F(\gamma^{-1}(x),\gamma^{-1}(y)).$$

Then G is A-typical if

$$\gamma^{-1}(x)=\sum\nolimits^F t_i^A x^{q^i}\quad\textit{for some } t_i^A\in R.$$

This leads to the Hopf algebroid $V_AT=V_A[t_1^A,t_2^A,\ldots]$ corresponding to the groupoid of isomorphisms among A-typical FAM's. Here $\deg t_n^A=2(q^n-1)$.

2.9. Theorem. *The maps*

$$\eta_{\mathrm{R}}:V_A\to V_AT\quad\textit{and}\quad \Delta:V_AT\to V_AT\otimes_{V_A}V_AT$$

are given by

$$\eta_{\mathrm{R}}(\lambda_n^A)=\sum\lambda_i^A(t_{n-i}^A)^{q^i}\quad\textit{with}\quad \sum\nolimits^F v_i^A(t_j^A)^{q^i}\equiv\sum\nolimits^F\eta_{\mathrm{R}}(v_i^A)^{q^j}t_j^A \mod(\pi)$$

and

$$\sum\nolimits^F\Delta(t_i^A)=\sum\nolimits^F t_i^A\otimes(t_j^A)^{q^i}.$$

Proof. All but the mod(π) formula for η_{R} can be easily deduced from results in [3]. We will prove the formula on the nose for a different set of generators w_i^A and then show they agree with the v_i^A mod(π). The v_i^A are defined recursively by

$$\pi\lambda_n^A=\sum_{0\leq i<n}\lambda_i^A(v_{n-i}^A)^{q^i}$$

which can be rewritten as

$$\pi f_A(x)=\pi x+\sum_{i>0}f_A(v_i^A x^{q^i}).$$

The formula for the w_i^A is nearly identical, namely

$$\text{(1)}\qquad \pi\lambda_n^A=\sum_{0\leq i\leq n}\lambda_i^A(w_{n-i}^A)^{q^i}\quad\text{or}\quad [\pi](x)=\sum_{i\geq 0}{}^F w_i^A x^{q^i}\quad\text{where } w_0^A=\pi.$$

Hence the w_i^A are integral and we now show they agree with the v_i^A mod(π) and hence generate V_A. Comparing the two defining formulae gives

$$\pi x=\sum_{i\geq 0}f_A(w_i^A x^{q^i})-\sum_{i>0}f_A(v_i^A x^{q^i}).$$

Let g_A be the functional inverse of f_A. Applying it to both sides gives

$$g_A(\pi x)=\sum_{i\geq 0}{}^F w_i^A x^{q^i}-_F\sum_{i>0}{}^F v_i^A x^{q^i}.$$

If the left hand side is integral and divisible by π, then the desired congruence will follow by induction on i. To show that $g_A(\pi x)/\pi$ is integral note that its functional inverse is

$$f_A(\pi x)/\pi = \sum \lambda_i(\pi x)^{q^i}/\pi$$

which is integral since $\pi^i\lambda_i$ is.

To prove the right unit formula we reindex (1) and get

$$\sum \pi\lambda_i^A = \sum \lambda_i(w_j^A)^{q^i} \tag{2}$$

applying η_{R} gives

$$\sum \pi\lambda_i^A(t_j^A)^{q^i} = \sum \lambda_i^A(t_j^A)^{q^i}\eta_{\mathrm{R}}(w_k^A)^{q^{i+j}}$$

and substituting (1) on the left hand side gives

$$\sum \lambda_i^A(w_j^A)^{q^i}(t_k^A)^{q^{i+j}} = \sum \lambda_i^A(t_j^A)^{q^i}\eta_{\mathrm{R}}(w_k^A)^{q^{i+j}}$$

or

$$\sum f_A(w_i^A(t_j^A)^{q^i}) = \sum f_A(t_i^A\eta_{\mathrm{R}}(w_j^A)^{q^i}).$$

Applying the inverse of f_A to both sides gives

$$\sum^F w_i^A(t_j^A)^{q^i} = \sum t_i^A\eta_{\mathrm{R}}(w_j^A)^{q^i}$$

as desired. □

Now we consider the chromatic spectral sequence for FAM's with A a discrete valuation ring as in 2.5. Its construction is quite straightforward. We have the chromatic resolution

$$0 \to V_A \to M_A^0 \to M_A^1 \to \cdots$$

obtained from SES's

$$0 \to N_A^n \to M_A^n \to N_A^{n+1} \to 0$$

where $N_A^0 = V_A$ and $M_A^n = (v_n^A)^{-1}N_A^n$.

2.10. Lemma. *There is a spectral sequence converging to* $\mathrm{Ext}_{V_AT}(V_A, V_A)$ *as in* 1.11 *with* $E_1^{n,s} = \mathrm{Ext}_{V_AT}^s(V_A, M_A^n)$.

As in Section 1 we abbreviate $\mathrm{Ext}_{V_AT}(V_A, M)$ by $\mathrm{Ext}(M)$ for a V_AT-comodule M. As before the problem of computing $\mathrm{Ext}(M_A^n)$ reduces in principle to finding $\mathrm{Ext}((v_n^A)^{-1}V_A/I_n^A)$ where $I_n^A = (\pi, v_1^A, \ldots, v_{n-1}^A) \subset V_A$.

We let

$$K_A(n) = \mathbb{F}_q[v_n^A, (v_n^A)^{-1}] \quad \text{and} \quad \Sigma_A(n) = K_A(n)_* \otimes_{V_A} V_AT \otimes_{V_A} K_A(n)_*.$$

The ring structure of $\Sigma_A(n)$ is

$$\Sigma_A(n) = K_A(n)_*[t_1^A, t_2^A, \ldots]/(v_n^A(t_i^A)^{q^n} - (v_n^A)^{q^i}t_i^A)$$

by 2.9. The proof of 1.12 given in [6] generalizes easily and we have

2.11. Changes of Rings Theorem.

$$\mathrm{Ext}((v_n^A)^{-1}V_AT/I_n^A) = \mathrm{Ext}_{\Sigma_A(n)}(K_A(n)_*, K_A(n)_*).$$

The proof of 1.13 given in [8] also generalizes easily once we have the appropriate formulation. Let F_n^A be the FAM over $\mathbb{F}_{q^n}$ given by the composite $V_A \to K_A(n) \to \mathbb{F}_{q^n}$ sending v_n^A to 1. Now any FAM over a finite field is also a formal $\hat{A}$-module, where $\hat{A}$ is the completion of A, so *we assume now that A is complete and that its quotient field K is a finite extension of $\mathbb{Q}_p$*. We will describe the endomorphism ring E_n^A of F_n^A (see [3, 21.8.15]). It is an A-algebra of rank n^2 generated by a primitive (q^n-1)th root of unity ω_A (sending x to $\bar{\omega}_A x$) and S_A (send x to x^q) with $S_A^n = \pi$ and $S_A\omega_A = \omega_A^q S_A$. $D_n^A = E_n^A \otimes_A K$ is a central division algebra over K. E_n^A is a complete local ring with maximal ideal (S_A) and residue field $\mathbb{F}_{q^n}$. We let $S_n^A \subset (E_n^A)^\times$ be the group of units congruent to $1 \bmod (S_A)$.

2.12. Theorem. *$\Sigma_A(n) \otimes_{K_A(n)_*} \mathbb{F}_{q^n}$ is a Hopf algebra isomorphic to the continuous linear dual of the $\mathbb{F}_{q^n}$-group algebra of S_n^A.*

3. Some applications and open questions

In this section A will be either the ring of integers in a finite extension of $\mathbb{Q}$ (global case) or the localization or completion of same at some prime (local case). In the local case for a V_AT-comodule M, $\mathrm{Ext}_A(M)$ will denote $\mathrm{Ext}_{V_AT}(V_A, M)$ and $\mathrm{Ext}_A(V_A)$ will be abbreviated by Ext_A. In the global case similar abbreviations will be made for Ext groups defined over L_AB.

The attentive reader no doubt notice our omission in Section 2 of an analogue of 1.10, the relation between local and global Ext groups. We do not know if such an isomorphism holds in general, so we will merely formulate a conjecture. Suppose A is the ring of integers in a number field, i.e. a finite extension of $\mathbb{Q}$. Then Ext_A is an A-module and can therefore be localized at any prime ideal π in A. Let A_π denote the localization of A. This ring is a DVR as in 2.5 so we have A_π-typical Ext groups.

3.1. Local-global Conjecture. For an L_AB-comodule M

$$A_\pi \otimes_A \mathit{Ext}_{L_AB}(A, M) = \mathrm{Ext}_{V_{A_\pi}T}(A_\pi, V_{A_\pi} \otimes_{L_A} M).$$

3.2. Theorem. $\mathrm{Ext}_A^0 = A$, *concentrated in dimension zero and* Ext_A^s *for $s>0$ is all torsion.*

Proof. Let K be the quotient field of A. The torsion free part of Ext_A injects into $K \otimes_A \mathrm{Ext}_A$, so it suffices to compute the latter. One easily sees that it is isomorphic to $\mathrm{Ext}_A(K \otimes_A V_A)$ or $\mathrm{Ext}_A(K \otimes_A L_A)$. In the local case this group is $\mathrm{Ext}_{\Sigma_A(0)}(K_A(0)_*, K_A(0)_*)$ and $\Sigma_A(0) = K_A(0)_* = K$.

In the global case one has $K \otimes_A L_A = L_K = K[m_1, m_2, \ldots]$ by [3, 21.4.1] where the m_i are the coefficients in the log series. The calculation of $\mathrm{Ext}_A(K \otimes L_A) = \mathrm{Ext}_K$ proceeds as in the case $A = Z$ to give the desired result. $\square$

The determination of Ext_A^1 is more difficult and we only have partial results. We begin by recalling the calculation in the classical local case, $A = Z_{(p)}$. From the CSS we have a SES

$$0 \to E_2^{1,0} \to \mathrm{Ext}^1 \to E_3^{0,1} \to 0.$$

The chromatic theory gives $E_1^{0,1} = 0$ so we need to compute $E_1^{1,0}$ and $E_2^{1,0}$ and for the former we need $\mathrm{Ext}^s_{Z_{(p)}}(v_1^{-1}V/(p))$ for $s = 0, 1$. Using 1.11 and 1.12 (see [9] for details) we have

3.3. Lemma. *For $p > 2$, $\mathrm{Ext}_{Z_{(p)}}(v_1^{-1}V/(p)) = K(1)_* \otimes E(h_0)$ where $h_0 \in \mathrm{Ext}^{1,2p-2}$ corresponds to the primitive $t_1 \in VT/(p)$. For $p = 2$ the group is*

$$K(1)_* \otimes P(h_0) \otimes E(\varrho_1)$$

where $\varrho_1 \in \mathrm{Ext}^{1,0}$ is represented by $v_1^{-3}(t_2 + t_1^3) + v_1^{-4}v_2 t_1 \in v_1^{-1}VT/(2)$.

Here $E(\ )$ and $P(\ )$ denote exterior and polynomial algebras.

To compute $\mathrm{Ext}^0(M^1)$ we use the SES

$$0 \to M_1^0 \xrightarrow{i} M^1 \xrightarrow{p} M^1 \to 0$$

giving

$$0 \to \mathrm{Ext}^0(M_1^0) \xrightarrow{i} \mathrm{Ext}^0(M^1) \xrightarrow{p} \mathrm{Ext}^0(M^1) \xrightarrow{\delta} \mathrm{Ext}^1(M_1^0) \to \cdots \tag{3.4}$$

where δ is the connecting homomorphism and $\mathrm{Ext}(M_1^0)$ is described in 3.3. The image of i is generated by $\{v_1^k/p : k \in Z\}$ and this is the subgroup of exponent p. We need to determine how many times each generator is divisible by p. An element in $\mathrm{Ext}^0(M^1)$ is not divisible by p iff it has a nontrivial image under δ. To compute this image one divides by p and applies $\eta_R - \eta_L$. For $p > 2$ and $p \nmid j$ we have

$$(\eta_R - \eta_L)\,\frac{v_1^{jp^i}}{p^{i+2}} = j\,\frac{v_1^{jp^i - 1}t_1}{p}$$

so $\delta(v_1^{jp^i}/p^{i+1}) = v_1^{jp^i - 1}h_0$ and we get

3.5. Theorem. *For $p>2$*

$$\mathrm{Ext}^{1,t}_{Z_{(p)}}=\begin{cases} Z/(p^{i+1}) & \text{if } t=2(p-1)jp^i \text{ for } p\nmid j,\ j>0,\\ 0 & \text{otherwise,}\end{cases}$$

generated by $v_1^{jp^i}/p^{i+1}$.

The above argument breaks down for $p=2$. For example

$$(\eta_R-\eta_L)\frac{v_1^2}{8}=\frac{v_1t_1+t_1^2}{2},$$

which represents a trivial element in $\mathrm{Ext}^1(M_1^0)$ so $v_1^2/2$ is divisible by 4, contrary to what one would expect by analogy with 3.4. One finds that

$$(\eta_R-\eta_L)\frac{(v_1^2+4v_1^{-1}v_2)}{16}=\frac{v_1^2\varrho_1}{2},$$

so $(v_1^2+4v_1^{-1}v_2)/8$ is in $\mathrm{Ext}^0 M^1$ and is not divisible by 2. Since it involves a negative power of v_1, it supports a nontrivial d_1 in the CSS, hitting $v_2/2v_1\in\mathrm{Ext}^0(M^2)$. More generally for odd j and $i\geq 1$ we have

$$(\eta_R-\eta_L)\frac{(v_1^2+4v_1^{-1}v_2)^{j2^{i-1}}}{2^{i+3}}=\frac{v_1^{j2^i}\varrho_1}{2}$$

and for $j2^i>1$ this expression does not involve a negative power of v_1.

3.6. Theorem. *For $p=2$*

$$\mathrm{Ext}^{1,t}_{Z_{(2)}}=\begin{cases} 0 & \text{if } t\leq 0 \text{ or } t \text{ is odd},\\ Z/(2) & \text{if } t \text{ is even but } 4\nmid t,\\ Z/(4) & \text{if } t=4,\\ Z/(2^{i+2}) & \text{if } t=j2^{i+1} \text{ for } i\geq 1 \text{ and } j \text{ odd},\end{cases}$$

the generators being

$$\frac{v_1^j}{2},\quad \frac{v_1^2}{4}\quad\text{and}\quad \frac{v_1^{j2^i}+2^{i+1}v_1^{j2^i-3}v_2}{2^{i+2}}.$$

We will see below that the extra factor of p in 3.6 is caused by the presence of a pth root of unity.

Now we will try to generalize the argument of 3.5. From 2.12 we see that the analogue of 3.3 depends on the group S_1^A, which is abelian and, if A has no pth root of unity, torsion free. In any case the element h_0 will be present in its H^1, which is all we need. In the analogue of 3.4, im i is generated by $\{(v_1^A)^k/\pi : k\in Z\}$.

In order to proceed further we need the π-adic valuation $v: A\to\mathbb{Q}\cup\{\infty\}$ satisfying

$$v(a)=\infty \quad\text{iff}\quad a=0,\qquad v(ab)=v(a)+v(b),$$

$$\upsilon(a+b)\geq \min\{\upsilon(a), \upsilon(b)\}, \qquad \upsilon(a)\geq 0,$$

$$\upsilon(a)=0 \quad \text{iff} \quad a \text{ is a unit} \quad \text{and} \quad \upsilon(p)=1.$$

Hence if $a\in(\pi^k)$, then $\upsilon(a)=k\upsilon(\pi)$. It is known that each local A has such a valuation and that $\upsilon(\pi)=1/e$ where e is the ramification index of p in A, i.e. the number such that $(\pi^e)=(p)$,

3.7. Theorem. *If A is local with $\upsilon(\pi)>1/(p-1)$. Then*

$$\operatorname{Ext}_A^{1,t}=\begin{cases} A/(p^i\pi) & \text{if } t=2(q-1)jp^i \text{ with } p\nmid j \text{ and } j>0,\\ 0 & \text{otherwise,}\end{cases}$$

generated by $(\upsilon_1^A)^{jp^i}/(\pi p^i)$, where q is the cardinality of the residue field $A/(\pi)$.

[Note that 3.5 is a special case of 3.7, but that $Z_{(2)}$ *does not satisfy the hypothesis* $\upsilon(\pi)>1/(p-1)$.]

Proof. We proceed as in 3.5 and show that

$$(\eta_R-\eta_L)\frac{(\upsilon_1^A)^{jp^i}}{\pi^2 p^i}=\frac{j(\upsilon_1^A)^{jp^i-1}}{\pi}t_1^A,$$

from which the result follows. To make this calculation we have $\eta_R(\upsilon_1^A)=\upsilon_1^A+\pi t_1^A$, so

$$(\eta_R-\eta_L)(\upsilon_1^A)^{jp^i}=\sum_{k>0}\binom{jp^i}{k}\pi^k(\upsilon_1^A)^{jp^i-k}(t_1^A)^k$$

and we need to know the valuation of each of the coefficients $\binom{jp^i}{k}\pi^k$ for $k>1$ is greater than that for $k=1$. For $k=1$ this valuation is clearly $i+\upsilon(\pi)$.

Now we need two simple facts which the reader can verify:

$$\upsilon\binom{jp^i}{k}=i-\upsilon(k) \quad \text{and} \quad \text{for } k>1\text{:} \quad \frac{\upsilon(k)}{k-1}\leq\frac{1}{p-1}.$$

Hence the valuation of the kth coefficient is $i-\upsilon(k)+k\upsilon(\pi)$. Subtracting $i+\upsilon(\pi)$ gives $(k-1)\upsilon(\pi)-\upsilon(k)$, which is positive for all $k>1$ if $\upsilon(\pi)>1/(p-1)$. □

What happens when $\upsilon(\pi)\leq 1/(p-1)$? We will look at two examples, the rings of integers in $\mathbb{Q}_3(\sqrt{3})$ and $\mathbb{Q}_3(\sqrt{3})$. The latter is the cyclotomic extension of $\mathbb{Q}_3$ obtained by adjoining cube roots of unity. We can take $\pi=\sqrt{\pm 3}$ and in both cases $\upsilon(\pi)=1/2$. Now consider the expansion

$$(\eta_R-\eta_L)(\upsilon_1^A)^{j3^i}=\sum_{k>0}\binom{j3^i}{k}(\upsilon_1^A)^{j3^i-k}(\pi t_1^A)^k.$$

Since $\upsilon(k)/(k-1)<1/2$ unless $k=3$, we only have to consider the first and third terms. We have

$$(\eta_R-\eta_L)((\upsilon_1^A)^3-\pi^3(\upsilon_1^A)^{-1}\upsilon_2^A)$$

$$\equiv (v_1^A+\pi t_1^A)^3-(v_1^A)^3-\pi^3(v_1^A)^{-1}(v_1^A(t_i^A)^3-(v_1^A)^3 t_1^A) \bmod(\pi^4)$$
$$\equiv 3(v_1^A)^2\pi t_1^A+\pi^3(t_i^A)^3-\pi^3(t_1^A)^3+\pi^3(v_1^A)^2 t_1^A$$
$$\equiv (3\pi+\pi^3)(v_1^A)^2 t_1^A.$$

The number $3\pi+\pi^3$ is zero or nonzero depending on which of the two examples we are considering. Hence for $A=Z_3(\sqrt{3})$ we get a result similar to 3.7, while for $A=Z_3(\sqrt{-3})$ we get larger Ext^1 groups as in 3.6. In general when $v(\pi)=1/(p-1)$ we get the extra factor of π when π can be chosen to satisfy $\pi^p+p\pi\equiv 0 \bmod(\pi^{p+1})$, which is the case if the field has pth roots of unity.

Now we will discuss the global Ext^1 groups. In the classical case $A=Z$, Ext^1 can be read off from 3.5 and 3.6 using 1.10. The result is that $\mathrm{Ext}_Z^{1,2m}=Z/(j_m)$ for certain numbers j_m having some interesting properties, e.g.

3.8. Theorem. *Up to a factor of 2 the number j_m is the greatest common division of the numbers $k^N(k^m-1)$ for $k\in Z$ and N sufficiently large.*

A proof and further discussion can be found in [1]. For example, $j_2=12$, $k^2\equiv 1 \bmod(8)$ for $2\nmid k$, $k^2\equiv \bmod(3)$ for $3\nmid k$ and no similar relations exist for larger primes, so the number given by the theorem is 24.

3.9. Global Conjecture. For global A, $\mathrm{Ext}_A^{1,2m}=A/J_m^A$ where J_m^A is, up to some small factor, the ideal generated by $a^N(a^m-1)$ for $a\in A$ and N sufficiently large.

The numbers j_m of 3.8 are also related to Bernoulli numbers and the values of the Riemann zeta function at negative integers, but these properties do not appear to generalize to other number fields. For example if the field is not totally real its Dedekind zeta function vanishes at all negative integers.

Our evidence for this conjecture is purely local (although the similarity between J_n^A and the P_n^A of 2.4 could have some significance), so we are assuming 3.1. The local form of the conjecture would have J_m^A be the ideal generated by a^m-1 for all units $a\in A$. Then we have $J_m^A\subset(\pi)$ iff $(q-1)$ divides m, so let $m=(q-1)n$. If A is complete, then any unit congruent to $1 \bmod(\pi)$ is a $(q-1)$th power, and we can reformulate 3.9 as

3.10. Local Conjecture. If A is the ring of integers in a finite extension of $\mathbb{Q}_p$ with maximal ideal (π) and residue field $\mathbb{F}_q$, then

$$\mathrm{Ext}_A^{1,2n(q-1)}=A/J_{n(q-1)}^A$$

where $J_{n(q-1)}^A$ is the ideal generated by elements of the form a^n-1 for units $a\in A$ congruent to $1 \bmod(\pi)$.

The $\mathrm{Ext}_A^{1,t}$ for other t must vanish since $V_A T$ is concentrated in dimensions divisible by $2(q-1)$. If A satisfies the hypothesis of 3.7, then 3.10 is true since

$$a^n - 1 = (1+\pi b)^n - 1 = \sum_{k>0} \binom{n}{k} \pi^k b^k$$

for some $b \in A$ and we can analyze this expansion in the same we analyzed $(v_1^A + \pi t_1^A)^n$.

Next we will consider some general properties of the Hopf algebras $\Sigma_A(n)$ of 2.11. let A be the ring of integers in a finite separable extension K of the p-adic numbers $\mathbb{Q}_p$. Let $(\pi) \subset A$ be the maximal ideal with $A/(\pi) = \mathbb{F}_q$ where $q = p^f$. Let $e = 1/v(\pi)$. Then ef is the degree of the extension [2, Proposition 1.5.3]. Recall the homomorphism $\theta_A : V \to V_A$. We extend it to a Hopf algebroid map $\theta_A : VT \to V_A T$ as follows. A map θ from $V_A T$ to an A-algebra R corresponds to an isomorphism between two A-typical FAM's over R, and in particular to an isomorphism between two p-typical FGL's over R. Hence we have a natural transformation of functors $[V_A T, -] \to [VT, -]$ which must be represented by a homomorphism θ_A as above.

3.11. Lemma. (a) *With notation as above*

$$\theta_A(t_i) = \begin{cases} t_{i/f}^A & \text{if } f \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *Let L be the unramified extension of $\mathbb{Q}_p$ of degree f and $B \subset L$ its ring of integers. Then K is a totally ramified extension of L of degree e and we denote by $\theta_{A/B}$ the map from V_B to V_A. We have*

$$\theta_B(v_i) = \begin{cases} v_{i/f}^B & \text{if } f \mid i, \\ 0 & \text{otherwise,} \end{cases}$$

and $\theta_{AB}^{-1}(I_7^A) = I_{ne}^A$ where $I_n^A = (\pi, v_1^A, \ldots, v_{n-1}^A) \subset V_A$ and $I_{ne}^B = (\pi, v_1^B, \ldots, v_{ne-1}^B)$. Moreover $\theta_A(v_{nef}) \equiv c(v_n^A)^k \bmod I_n^A$ where $k = (p^{nef}-1)/(p^{nf}-1)$ and $c \in \mathbb{F}_q$ is the mod(π) *reduction of p/π^e.*

Proof. For (a) the elements t_i and t_i^A are defined by 1.8 and 2.8. An isomorphism γ from an A-typical FAM F to another one must satisfy

$$\gamma^{-1}(x) = \textstyle\sum^F t_i^A x^{q^i} = \sum^F t_i x^{p^i} \quad \text{with } t_0 = t_0^A = 1$$

by 2.8 and 1.8 and the result follows.

For (b) we take p as our generator of the maximal ideal in B. Then we have $f_B(x) = \theta_B f(x)$, so

$$px + \sum \theta_B(v_i)\theta_B(f^{(p^i)}(x^{p^i})) = px + \sum v_i^B f^{(q^i)}(x^{q^i})$$

from which we can compute $\theta_B(v_i)$ by induction on i.

For the statement about $\theta_{A/B}$ we use the generators w_i^A of V_A used in the proof of 2.9 defined by

$$\pi f_A(x) = \sum_{i \ge 0} f_A(w_i^A x^{q^i}) \quad \text{with } w_0^A = \pi.$$

This is equivalent to

$$[\pi](x)=\sum^F w_i^A x^{q^i}.$$

Now $(\pi^e)=(p)$ and we want a formula for $[\pi^e](x)$. We have for example

$$[\pi^2]x=\sum^F w_i^A([\pi](x))^{q^i}\equiv\sum_{i,j>0} w_i^A(w_j^A)^{q^i}x^{q^{i+j}} \mod(\pi),$$

3.12
$$[\pi^e](x)\equiv\sum^F_{i_1,i_2,\dots,i_e>0} w_{i_1}^A(w_{i_2}^A)^{a_1}(w_{i_2}^A)^{a_2}\cdots(w_{i_e}^A)^{a_{e-1}}x^{a_e} \mod(\pi)$$

where $a_j=q^{i_1+i_2+\cdots i_j}$. Observe that the coefficient of x^{q^m} is nonzero mod $I_n^A=(\pi, w_1^A,\dots,w_{n-1}^A)$. Now $p\equiv\pi^e \mod(\pi^{e+1})$, so the same is true of the coefficients of $[p](x)=\sum^F w_i^B x^{q^i}$. Since we saw in the proof of 2.9 that $w_i^B\equiv v_i^B \mod(p)$, the result follows.

For the mod I_n^A reduction of $\theta_A(v_{nef})$, consider the reduction of 3.12. The leading term on the right is $(w_n^A)^k x^{q^{en}}$, so equating the coefficients of $x^{q^{en}}$ gives

$$c^{-1}\theta_A(v_{nef})=(v_n^A)^k. \qquad \square$$

3.13. Corollary. *There is a Hopf algebroid homomorphism*

$$\theta_A:\Sigma(nef)\to\Sigma_A(n)$$

with

$$\theta_A(t_i)=\begin{cases} t_{i/f}^A & \text{if } f \text{ divides } i,\\ 0 & \text{otherwise,}\end{cases}$$

and $\theta_A(v_{nef})=c(v_n^A)^k$ *where* $k=(p^{nef}-1)/(p^{nf}-1)$, $c\in\mathbb{F}_q$ *is the* $\mathrm{mod}(\pi)$ *reduction of* p/π^e, *e is the ramification index and* p^f *is the cardinality of the residue field.*

For $n=1$ this is related (via 1.13 and 2.12) to the fact that the field K (an extension of $\mathbb{Q}_p$ of degree ef) embeds in the division algebra D_{ef}.

N.B. A proof of 3.1 has recently been found by my student A. Pearlman. His account of it will include a proof of 1.10, which does not seem to exist currently in the literature.

Since $L_AB=L_A\otimes_L LB$, L_A is an LB-comodule and standard arguments show $\mathrm{Ext}_{L_AB}(L_A,L_A)=\mathrm{Ext}_{LB}(L,L_A)$. Hence if there exists a spectrum S_A with $MU_*(S_A)=L_A$ as an LB-comodule, then our Ext group would be the E_2-term for the ANSS converging to $\pi_*(S_A)$. Apparently $BP_*(S_A)\neq V_A$ except possibly when p splits completely in A.

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