Topology Vol. 21, No. 4, pp. 373–391, 1982
Printed in Great Britain.

0040-9383/82/040373–19$03.00/0
Pergamon Press Ltd.

# EHP SEQUENCES IN BP THEORY

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(Received 22 December 1980)

## §1. INTRODUCTION

The *EHP* sequences in the homotopy groups of spheres arise as follows. A prime $p$ is fixed, and the spaces under consideration are to be localized at $p$. For each positive integer $n$, there is the James map

$$h_{2n+1}:\Omega S^{2n+1} \longrightarrow \Omega S^{2np+1}.$$

The homotopy fibre of $h_{2n+1}$ will be denoted $\hat{S}^{2n}$. There is another James map

$$h_{2n}:\Omega \hat{S}^{2n} \longrightarrow \Omega S^{2np-1}.$$

The homotopy fibre of $h_{2n}$ turns out to be $S^{2n-1}$. Each of these fibrations gives a long exact sequence of homotopy groups. Using the adjointness isomorphism $\pi_q(\Omega X) \approx \pi_{q+1}(X)$ to replace some of these groups, we obtain the two *EHP* exact sequences

$$\ldots \pi_q(\hat{S}^{2n}) \xrightarrow{E} \pi_{q+1}(S^{2n+1}) \xrightarrow{H} \pi_{q+1}(S^{2np+1}) \xrightarrow{P} \pi_{q-1}(\hat{S}^{2n}) \xrightarrow{E} \cdots \tag{1.1}$$

$$\ldots \pi_{q-1}(S^{2n-1}) \xrightarrow{E} \pi_q(\hat{S}^{2n}) \xrightarrow{H} \pi_q(S^{2np-1}) \xrightarrow{P} \pi_{q-1}(S^{2n-1}) \xrightarrow{E} \cdots \tag{1.2}$$

We have given the homomorphisms in these sequences their usual names: $E$ for suspension, $H$ for Hopf invariant, and $P$ for Whitehead product. When the prime $p=2$, it also happens that $\hat{S}^{2n} = S^{2n}$, and the *EHP* sequences provide a way of calculating the groups $\pi_q(S^n)$, inductive on the sphere dimension $n$, and on the stem dimension $q-n$. When $p$ is an odd prime, the space $\hat{S}^{2n}$ (which has cells in each dimension $2n, 4n, \ldots, 2n(p-1)$) replaces the even dimensional sphere in the *EHP* induction. The homotopy groups of the even-dimensional spheres localized at $p$ may be obtained from the fibration

$$S^{2n-1} \longrightarrow \Omega S^{2n} \longrightarrow \Omega S^{4n-1}$$

which splits as a product.

In [4], we have constructed for each simply-connected *CW* space $X$, a spectral sequence $\{E_r^{s,t}(X)\}$ which is called the unstable Adams–Novikov spectral sequence for $X$. This construction will be summarized in §5. This spectral sequence is constructed from $X$ using the Brown–Peterson spectrum $BP$ (associated with the prime $p$), and converges to the homotopy groups of $X$ localized at $p$. In this paper, we shall show that there are long exact sequences for the $E_2$ terms as follows.

$$\cdots \longrightarrow E_2^{s,t}(\hat{S}^{2n}) \longrightarrow E_2^{s,t+1}(S^{2n+1}) \longrightarrow E_2^{s,t+1}(S^{2np+1}) \xrightarrow{\partial} E_2^{s+1,t}(\hat{S}^{2n}) \longrightarrow \cdots \quad (1.3)$$

$$\cdots \longrightarrow E_2^{s,t-1}(S^{2n-1}) \longrightarrow E_2^{s,t}(\hat{S}^{2n}) \longrightarrow E_2^{s-1,t-1}(S^{2np-1}) \xrightarrow{\partial} E_2^{s+1,t-1}(S^{2n-1}) \longrightarrow \quad (1.4)$$

In [4, §6], a certain non-abelian category $G$ is constructed (in [4], this category is denoted $M(G)$, and is called the category of unstable coalgebras over the $BP$ Steenrod algebra). For each $CW$ space $X$, we denote the homology of $X$ with coefficients in the spectrum $BP$ by $H_*(X; BP)$. In the literature, this is sometimes denoted $BP_*(X)$. If $X$ is a $CW$ space for which $H_*(X; BP)$ is free over the coefficient ring $\pi_*(BP)$, then it is shown in [4, 6.17)], that

$$E_2^{s,t}(X) \approx \operatorname{Exp}_G^{s,t}(H_*(X; BP)) .$$

Here, and throughout this paper, we use the notation

$$\operatorname{Ext}_G^{s,t}(-)$$

to stand for the $s$th derived functor of the functor

$$\operatorname{Hom}_G(H_*(S^t; BP), -).$$

In §2, we give brief exposition of the theory of [1, 2, 5] concerning derived functors on non-abelian categories.

The sequence (1.3) is obtained from the James fibration

$$\hat{S}^{2n} \longrightarrow \Omega S^{2n+1} \longrightarrow \Omega S^{2np+1}.$$

The $BP$-homology of these spaces form an injective extension sequence of coalgebras (see §3). By adapting the theory of Moore–Smith[12] and Bousfield[5] to this situation, we show in §4 and §7 that this leads to a long exact sequence of Ext groups, which when identified as the $E_2$ terms of the spaces in the James fibration becomes

$$\cdots \longrightarrow E_2^{s,t}(\hat{S}^{2n}) \longrightarrow E_2^{s,t}(\Omega S^{2n+1}) \longrightarrow E_2^{s,t}(\Omega S^{2np+1}) \xrightarrow{\partial} \cdots . \quad (1.5)$$

In §6, we use special properties of unstable $BP_*$-resolutions to show that

$$E_2^{s,t}(\Omega S^{2n+1}) \approx E_2^{s,t+1}(S^{2n+1}).$$

Using this isomorphism (for $S^{2n+1}$ and for $S^{2np+1}$) to substitute into (1.5) gives the EHP sequence (1.3).

The other $EHP$ sequence (1.4) arises from a composite functor spectral sequence (abbreviated $CFSS$) which will be constructed in §5. For each $M$ in $G$, the $CFSS$ in a purely algebraic spectral sequence converging to $\operatorname{Ext}_G(M)$, and for which

$$E_2^{i,j} \approx \operatorname{Ext}_U{}^j(R^iP(M)).$$

We are suppressing the second index $t$ in these Ext groups. Here $R^iP$ stands for the $i$th derived functors of the primitive element functor $P$ on the category of coalgebras (see §3 and [5]). $U$ is the abelian category defined in [4, §7] (in [4, §7], $U$ is denoted $A(U)$). When $M$ is in $G$, and is *nice* as a coalgebra (i.e. $R^iP(M) = 0$ for $i > 1$), the *CFSS* has only two non-zero columns, and gives an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}_U{}^s(PM) \longrightarrow \operatorname{Ext}_G{}^s(M) \longrightarrow \operatorname{Ext}_U^{s-1}(R^1PM) \xrightarrow{\partial} \cdots. \tag{1.6}$$

For the space $\hat{S}^{2n}$, $H_*(\hat{S}^{2n}; BP) = M$ is nice as a coalgebra; $PM$ and $R^1P(M)$ are easily determined. Using isomorphisms

$$\operatorname{Ext}_U{}^s(PM) \approx E_2{}^s(S^{2n-1})$$

$$\operatorname{Ext}_U{}^{s-1}(R^1PM) \approx E_2{}^{s-1}(S^{2np-1})$$

to replace the terms in (1.6) gives (1.4).

In §8, we also use the *CFSS* to study the double suspension

$$E_2{}^{s,t-1}(S^{2n-1}) \longrightarrow E_2{}^{s,t+1}(S^{2n+1}).$$

As a coalgebra $H_*(\Omega S^{2n+1}; BP)$ is nice, and we call

$$R^1PH_*(\Omega S^{2n+1}; BP) = W(n).$$

It is shown in §3 that $W(n)$ is a free $\pi_*(BP)/(p)$-module with generators in degrees $2np, 2np^2, 2np^3, \ldots$. The *CFSS* again has but two columns, which after identifying the terms, becomes

$$\cdots \longrightarrow E_2{}^{s,t-1}(S^{2n-1}) \longrightarrow E_2{}^{s,t+1}(S^{2n+1}) \longrightarrow E_2{}^{s-1,t-1}(W(n)) \xrightarrow{\partial} \cdots \tag{1.7}$$

where $\partial$ has a bidegree $(2, 0)$. It follows easily that for $(s, t) \neq (0, 2n+1)$, the groups $E_2{}^{s,t}(S^{2n+1})$, and thus all succeeding $E_r{}^{s,t}(S^{2n+1})$, have exponent $p^n$. We also use this sequence to establish vanishing lines, zones of stability, and zones of exponent $p^n$ for the groups $E_2{}^{s,t}(S^{2m+1})$, and for the stable groups $E_2{}^{s,t}(S^0)$. These exact sequences are used by one of us [18] to make extensive calculations in $E_2{}^{s,t}(S^{2n+1})$.

There are analogous *EHP* sequences for the unstable Adams $E_2$ terms based on mod-$p$ homology, which are obtained in [4, 6, 7] by somewhat special methods. We leave it to the reader to see that each sequence may also be obtained by a suitable *CFSS*.

The notations and conventions of this paper are similar to those of [3, 4]. Space means Hausdorff topological space with base point. The homotopy relation for maps or for spaces is indicated by $\simeq$. In an algebraic situation, a homomorphism is to preserve the structure; isomorphism is indicated by $\approx$. For a space $X$, $\Omega X$ stands for the loop-space. In any category, $I$ stands for the identity functor; the identity map of an object with itself is $1$ or $1_X$ when we wish to specify the object $X$. If $\theta: S \to T$ is a natural transformation of functors, we write

$$\theta = \theta_X : S(X) \to T(X)$$

to stand for the natural map. Where the meaning is clear, we may omit parentheses.

The ring of integers is denoted by $Z$, the rationals by $Q$, the field with $p$ elements by $Z_p$, and the ring of integers localized at $p$ is $Z_{(p)}$. We also occasionally abbreviate the words "long exact sequence" by *LES*, and "composite functor spectral sequence" by *CFSS*.

## §2. COTRIPLES AND DERIVED FUNCTORS

In this section, for the convenience of the reader, and to establish notation, we summarize some facts from [1, 2, 5] about cotriples, triples, and non-abelian derived functors.

A *cotriple* $(F, \delta, \epsilon)$ on a category $C$ consists of a covariant functor $F: C \to C$, together with natural transformations $\delta: F \to F^2$ and $\epsilon: F \to I$ such that the following diagrams commute.

$$\begin{array}{ccccc} F & \xleftarrow{\epsilon F} & F^2 & \xrightarrow{F\epsilon} & F \\ & \nwarrow{\scriptstyle =} & \uparrow & \nearrow{\scriptstyle =} & \\ & & F & & \end{array} \qquad\qquad \begin{array}{ccc} F & \xrightarrow{\delta} & F^2 \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta F} \\ F^2 & \xrightarrow[F\delta]{} & F^3 \end{array}$$

When $(F, \delta, \epsilon)$ is a cotriple on $C$, an $F$-structure on an object $X$ in $C$ is defined to be a map $\psi: X \to FX$ such that the following diagrams commute.

$$\begin{array}{ccc} X & \xrightarrow{\psi} & FX \\ & \searrow{\scriptstyle =} & \downarrow{\scriptstyle \epsilon} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\psi} & FX \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle F\psi} \\ FX & \xrightarrow[\delta]{} & F^2X \end{array} .$$

A map $f:(X, \psi) \to (X', \psi')$ between objects with $F$-structure is a map $f: X \to X'$ in $C$ for which $\psi' f = F(f)\psi$. The category whose objects are $\{(X, \psi)\}$ where $X$ is in $C$ with $F$-structure $\psi$, and whose maps are maps of objects with $F$-structure will be denoted $F$ (in [4], this category is denoted $C(F)$). Notice that each object of the form $FX$ has a canonical $F$-structure, namely $\delta: FX \to F^2X$. The objects of the form $(FX, \delta)$ will be called the models in $F$.

We next observe that $F$ is also the functor of a *triple* $(F, \mu, \eta)$ on $F$. That is, there are natural transformations $\mu : F^2 \to F$ and $\eta : I \to F$ on $F$ which are defined by

$$\mu = F\epsilon : (F^2X, \delta) \longrightarrow (FX, \delta)$$

$$\eta = \psi : (X, \psi) \longrightarrow (FX, \delta).$$

It is easily verified that $\mu$ and $\eta$ satisfy the following commutative diagrams.

$$\begin{array}{ccccc} F & \xrightarrow{F\eta} & F^2 & \xleftarrow{\eta F} & F \\ & \searrow{\scriptstyle =} & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle =} & \\ & & F & & \end{array} \qquad\qquad \begin{array}{ccc} F^3 & \xrightarrow{F\mu} & F^2 \\ {\scriptstyle \mu F}\downarrow & & \downarrow{\scriptstyle \mu} \\ F^2 & \xrightarrow[\mu]{} & F \end{array} .$$

These are the properties which assert that $(F, \mu, \eta)$ is a triple on $F$.

In this situation, suppose that $T$ is a functor from $F$ to an abelian category $A$ (it is enough that $T$ be defined on the full subcategory of models in $F$). Then the right derived functors

$$R^qT_F : F \longrightarrow A$$

are defined as follows. The triple $(F, \mu, \eta)$ on $F$ defines a functor $\mathbf{K}_F$ from $F$ to the category of augmented cosimplicial complexes over $F$: for each $(X, \psi)$ in $F$, and for each non-negative integer $q$,

$$\mathbf{K}_F(X)^q = F^{q+1}X.$$

The coface and codegeneracy operators in $\mathbf{K}_F(X)$ are given by

$$d^i = F^{q-i}\eta F^i : F^q X \longrightarrow F^{q+1}X$$

$$s^i = F^{q-1}\mu F^i : F^{q+2}X \longrightarrow F^{q+1}X$$

for each $0 \leq i \leq q$. The augmentation

$$d^{-1}: \mathbf{K}_F(X)^{-1} \longrightarrow \mathbf{K}_F(X)^0$$

is the map $\psi: X \to FX$. Let $T\tilde{\mathbf{K}}_F(X)$ be the cosimplicial complex which results from applying the functor $T$ to each degree of the unaugmented cosimplicial complex $\tilde{\mathbf{K}}_F(X)$, and to each coface and codegeneracy operator. The cochain complex

$$(\text{ch } T\tilde{\mathbf{K}}_F(X), \partial)$$

is obtained by taking

$$\partial = \sum (-1)^i T(d^i).$$

*Definition* (2.1). For each $(X, \psi)$ in $F$ and each non-negative integer $q$, the right derived functors are defined by

$$R^q T_F(X) = H^q(\text{ch } T\tilde{\mathbf{K}}_F(X), \partial).$$

As it will be of use to us later, we give the fundamental property of derived functors: namely, that they may be computed from more general resolutions as follows.

*Definition* (2.2). For $X$ in $F$, a cosimplicial resolution of $X$ by models consists of an augmented cosimplicial complex $\mathbf{Y}$ over $F$ such that

(1) $\mathbf{Y}^{-1} = X$
(2) For each $q \geq 0$, $\mathbf{Y}^q$ is a model in $F$.
(3) For each model $M$ in $F$, the chain complex

$$\text{Hom}_{F^+}(\text{ch}^+(\mathbf{Y}), M)$$

is acyclic. Here $F^+$ refers to the category whose objects are the same as those of $F$, and whose maps are to be formal finite sums of maps in $F$.

We remark that if the cosimplical complex $\mathbf{Y}$ is a cosimplicial complex of abelian groups, then condition (3) may be replaced by

(3′) The cochain complex $(\text{ch}(\mathbf{Y}), \partial)$ is acyclic

THEOREM (2.3). *If* $Y$ *is a cosimplicial resolution of* $X$ *by models, then*

$$R^q T_F(X) \approx H^q(\text{ch } T\tilde{Y}, \partial)$$

The proof is in [1, 5].

## §3. COALGEBRAS AND PRIMITIVES

In this section we recall the theory of Bousfield [5] and Moore–Smith [12] concerning the primitive element functor $P$ and its right derived functors $R^iP$ on the category of coalgebras. At the same time, we make some minor modifications so that the theory applies in our situation. We then use this to make calculations of $R^iP(C)$ for certain coalgebras $C$ which will occur later.

Let $A$ be a commutative ring with a unit element. In the applications, $A$ will be the coefficient ring $A = \pi_*(BP)$. Let $M$ be the category of positively graded $A$-modules which are free of finite type. For each $M$ in $M$, let $S(M)$ be the cofree coassociative cocommutative coalgebra without counit generated by $M$. $S$ is the functor of a cotriple $(S, \delta, \epsilon)$ on $M$. A module $M$ with an $S$-structure will be called a coalgebra. Equivalently, a coalgebra $M$ is a module $M$ in $M$ with a diagonal map

$$\Delta : M \longrightarrow M \otimes_A M$$

which makes $M$ a coassociative cocommutative coalgebra (without counit) in the usual sense. Using the notation of §2, $S$ will denote the category of modules in $M$ with $S$-structure; in other words, $S$ is the category of coalgebras over $A$.

For each coalgebra $C$ in $S$, the group of primitives $P(C)$ is defined by

$$P(C) = \ker \Delta : C \longrightarrow C \otimes_A C.$$

Thus $P$ is a functor from $S$ to the category of abelian groups. The theory of [1, 5] which was summarized in §2 yields right derived functors $R^iP$ on $S$. Our situation differs from that of [5] in two ways. The first difference is that [5] considers coalgebras over a field $k$, while we consider coalgebras which are free of finite type over the ring $A$. This causes no change in the theory. The second difference is that [5] considers connected homology coalgebras $B$ with counit (that is, there is a counit map $\epsilon : B \to A$ with $\epsilon : B_0 \approx A$). For such a coalgebra with counit, the primitives are defined by

$$P(B) = \{x \quad \text{in} \quad B : \Delta(x) = 1 \otimes x + x \otimes 1\}.$$

On the other hand we consider positively graded coalgebras $C$ without counit (that is, $C_0 = 0$), and the primitives are taken to be $\ker \Delta$. The two theories are seen to be equivalent as follows. Let $S_0$ be the category consisting of connected homology coalgebras over $A$, which as $A$-modules are free of finite type. There are natural equivalences of categories

$$S \underset{\beta}{\overset{\alpha}{\rightleftarrows}} S_0$$

as follows. For $C$ in $S$, let $\alpha(C) = A \oplus C$, where the extra $A$-summand is given degree zero. For $B$ in $S_0$, let $\beta(B) = \ker \epsilon$. Then $\alpha$ and $\beta$ are easily seen to be natural equivalences of categories. Furthermore, for $C$ in $S$, $B$ in $S_0$,

$$P(\alpha(C)) \approx P(C)$$
$$P(B) \approx P(\beta(B)).$$

We also observe that the canonical cosimplicial resolutions of §2 in the categories $S$ and $S_0$ are related by

$$\mathbf{K}_{S_0}(\alpha(C)) \approx \alpha\mathbf{K}_S(C)$$
$$\mathbf{K}_S(\beta(B)) \approx \beta\mathbf{K}_{S_0}(B).$$

Therefore, we also have

$$R^qP_S(C) \approx R^qP_{S_0}(\alpha(C)).$$

We may suppress the subscript $S$ or $S_0$ and write simply $R^qP(C)$ for these derived functors. We also note from [5, (3.2)] that

$$R^0P(C) \approx P(C).$$

Following [5], we adopt the definition:

*Definition* (3.1). An injective extension sequence is a sequence of maps in $S$

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$

such that
(1) $g$ is an epimorphism (of $A$-modules).
(2) The map $f$ is the inclusion

$$C \,\square_{C''} A \longrightarrow C$$

(Here $\square$ stands for cotensor product.)
(3) $C$ is injective as a $C''$-comodule.

PROPOSITION 3.2. *Let $C' \to C \to C''$ be an injective extension sequence in $S$. Then there is a long exact sequence of abelian groups*:

$$0 \longrightarrow P(C') \longrightarrow P(C) \longrightarrow P(C'') \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} R^iP(C') \longrightarrow R^iP(C) \longrightarrow R^iP(C'') \xrightarrow{\delta} \cdots$$

*where $\delta$ raises derived degree $i$ by* 1.

*Proof.* Using the natural equivalence of $S$ with $S_0$, the proof is exactly as in [5].

We conclude this section with some calculations of $R^iP(C)$ which will be of use later. A prime number $p$ is fixed and $Z_{(p)}$ refers to the ring of integers localized at $p$.

$BP$ is the Brown–Peterson spectrum for the prime $p$, and $A = \pi_*(BP)$, which is the polynomial algebra

$$A = Z_{(p)}[v_1, v_2, \ldots]$$

with degree $(v_i) = 2(p^i - 1)$. We shall consider certain coalgebras over $A$, for which we need some notation. First, let $M(n_1, n_2, \ldots)$ stand for the free $A$-module with generators of degree $n_1, n_2, \ldots$.

(1) For each positive integer $m$, let $C(x_m)$ be the coalgebra which as an $A$-module is freely generated by $x_m$, and with $x_m$ primitive. Evidently $PC(x_m) \approx M(m)$.

(2) For each positive integer $n$, let $D(x_{2n})$ be the coalgebra which as an $A$-module is freely generated by $\{x_{2ns}, s \geq 1\}$. The diagonal map $\Delta$ is given by

$$\Delta(x_{2ns}) = \sum_{i+j=s} x_{2ni} \otimes x_{2nj}.$$

It is immediate that $PD(x_{2n}) \approx M(2n)$. The coalgebra $D(x_{2n})$ is the $A$-dual of the polynomial algebra on a generator of degree $2n$; thus $D(x_{2n})$ is cofree as a coalgebra. $D(x_{2n})$ also has a Hopf algebra structure; as an algebra $D(x_{2n})$ is a divided power algebra, which explains the notation.

(3) For each positive integer $n$, let $T(x_{2n})$ be the coalgebra, which as an $A$-module is freely generated by $\{x_{2ns}, s \geq 1\}$. The diagonal $\Delta$ is given by

$$\Delta(x_{2ns}) = \sum_{i+j=s} \binom{s}{i} x_{2ni} \otimes x_{2nj}.$$

Then $PT(x_{2n}) \approx M(2n)$. $T(x_{2n})$ also has a Hopf algebra structure, the algebra structure being that of a polynomial algebra on one generator $x_{2n}$.

(4) For each positive integer $n$, let $T_1(x_{2n})$ be the sub-coalgebra of $T(x_{2n})$, which as an $A$-module is freely generated by $\{x_{2ns}, 1 \leq s \leq p-1\}$. Then also $PT_1(x_{2n}) \approx M(2n)$.

(5) For each positive integer $n$, let $B(x_{2n})$ be the coalgebra of the bipolynomial algebra over $A$ defined as follows. As an algebra, $B(x_{2n})$ is the polynomial algebra over $A$ generated by $\{x_{2np^s}, x \geq 0\}$. For each $s \geq 0$, the element

$$x_{2n}^s + p(x_{2np})^{s-1} + \cdots + p^s x_{2np^s}$$

is to be primitive. It can be verified (see [10]) that this well-defines the map $\Delta$ making $B(x_{2n})$ a Hopf algebra. The $A$-dual of $B(x_{2n})$ is another Hopf algebra of the same form which explains the notation. From the definition

$$P(B(x_{2n})) \approx M(2n, 2np, \ldots, 2np^s, \ldots)$$

PROPOSITION (3.3). *For these coalgebras, the right derived functors $R^iP$ are as follows.*

(*i*) $R^1PC(x_{2n-1}) = 0$
(*ii*) $R^1PC(x_{2n}) \approx M(4n)$
(*iii*) $R^1PD(x_{2n}) = 0$
(*iv*) $R^1PT(x_{2n}) = M(2np, 2np^2, \ldots) \otimes_Z Z_p$
(*v*) $R^1PB(x_{2n}) = 0$
(*vi*) $R^1PT_1(x_{2n}) \approx M(2np)$

*Also, each of these coalgebras $C(x_m)$, $D(x_{2n})$, $T(x_{2n})$, $B(x_{2n})$ is nice; that is, $R^iP$ vanishes for $i > 1$.*

*Proof.* First, $D(x_{2n})$, $B(x_{2n})$ and $C(x_{2n-1})$ are cofree as coalgebras (because their $A$-duals are free (anti-commutative) algebras). For $C(x_{2n})$, there is an injective extension sequence:

$$C(x_{2n}) \xrightarrow{f} D(y_{2n}) \xrightarrow{g} D(z_{4n})$$

where $f(s_{2n}) = y_{2n}$ and $g(y_{2ns}) = z_{2ns}$ for $s$ even. Then the $LES$ of (3.2) becomes a four term exact sequence:

$$0 \longrightarrow PC(x_{2n}) \longrightarrow PD(y_{2n}) \longrightarrow PD(z_{4n}) \longrightarrow R^1PC(x_{2n}) \longrightarrow 0.$$

We have

$$PC(x_{2n}) \approx M(2n)$$

$$PD(y_{2n}) \approx M(2n)$$

$$PD(z_{4n}) \approx M(4n).$$

Hence

$$R^1PC(x_{2n}) \approx M(4n)$$

$$R^iPC(x_{2n}) = 0, \quad \text{for } i > 1.$$

For $T_1(x_{2n})$, there is an injective extension sequence:

$$T_1(x_{2n}) \xrightarrow{f} D(y_{2n}) \xrightarrow{g} D(z_{2np})$$

where $f(x_{2n}) = y_{2n}$, and $g(y_{2np^s}) = z_{2np^s}$ for $s \geq 1$. The $LES$ of (3.2) becomes a four term sequence:

$$0 \longrightarrow PT_1(x_{2n}) \longrightarrow PD(y_{2n}) \longrightarrow PD(z_{2np}) \longrightarrow R^1PT_1(x_{2n}) \longrightarrow 0$$

We have

$$PT^1(x_{2n}) \approx M(2n)$$

$$PD(y_{2n}) \approx M(2n)$$

$$PD(z_{2np}) \approx M(2np).$$

Hence

$$R^1PT_1(x_{2n}) \approx M(2np)$$

$$R^iPT_1(x_{2n}) = 0, \quad \text{for } i > 1.$$

For $T(x_{2n})$, there is an injective extension sequence:

$$T(x_{2n}) \xrightarrow{f} B(y_{2n}) \xrightarrow{g} B(z_{2np})$$

where $f(x_{2n}) = y_{2n}$, and $g(y_{2np^s}) = z_{2np^s}$ for $s \geq 1$. The *LES* of (3.2) becomes a four term sequence

$$0 \longrightarrow PT(x_{2n}) \xrightarrow{f_*} PB(y_{2n}) \xrightarrow{g_*} PB(z_{2np}) \longrightarrow R^1PT(x_{2n}) \longrightarrow 0$$

We have

$$PT(x_{2n}) \approx M(2n)$$
$$PB(y_{2n}) \approx M(2n, 2np, 2np^2, \ldots)$$
$$PB(z_{2np}) \approx M(2np, 2np^2, 2np^3, \ldots).$$

The map $g_*$ is easily seen to be multiplication by $p$ on the primitives of degrees $2np$, $2np^2, \ldots$. Hence

$$R^1PT(x_{2n}) \approx M(2np, 2np^2, \ldots) \otimes_Z Z_p.$$

In §8, we shall write $W(n)$ for $R^1PT(x_{2n})$.

## §4. THE UNSTABLE ADAMS-NOVIKOV SPECTRAL SEQUENCE

In this section, we summarize the results of [4], which gives the construction and properties of the unstable Adams–Novikov spectral sequence $\{E_r^{s,t}(X; BP)\}$ for a space $X$. When $H_*(X; BP)$ is free over the coefficient ring $\pi_*(BP)$, the $E_2$-term is isomorphic to an Ext group in a certain non-abelian category $G$. We show that an injective extension sequence in $G$ gives rise to a long exact sequence of Ext groups. In §7, this will be used to give one of the *EHP* sequences.

As in §3, $BP$ refers to the Brown–Peterson spectrum for a fixed prime $p$, and $A = \pi_*(BP)$ is the coefficient ring. For each space $X$, the (reduced) homology of $X$ with coefficients in the spectrum $BP$ is denoted $H_*(X; BP)$. We consider spaces $X$ for which $H_*(X; BP)$ is $A$-free of finite type. The category of positively graded free $A$-modules of finite type will be denoted $M$. Of great importance to the theory in [4] is the construction of a certain cotriple $(G, \delta, \epsilon)$ on $M$. As in §2, the category of modules in $M$ with $G$-structure will be denoted $G$ (in [5], this category $G$ is denoted $M(G)$, and is called the category of unstable $\Gamma$-coalgebras, where $\Gamma$ is the $BP$-analogue of the dual of the Steenrod algebra). Let $A[t] = H_*(S^t; BP)$ which is the free $A$-module on one generator of degree $t$, considered in the category $G$, with the trivial $G$-structure. As in §2, the derived functors of $\mathrm{Hom}_G(A[t], -)$ will be denoted $\mathrm{Ext}_G^{s,t}(-)$. One of the main theorems in [4] is the following.

THEOREM (4.1). *Let $X$ be a simply connected CW space which is p-local. Then there is a spectral sequence $E_r^{s,t}(X; BP)$ which converges to $\pi_*(X)$. If $H_*(X; BP)$ is A-free of finite type, then*

$$E_2^{s,t}(X; BP) \approx \mathrm{Ext}_G^{s,t}(H_*(X; BP))$$

The proof is in [4, §6]. This spectral sequence is called the unstable Adams-Novikov spectral sequence for $X$, and for the remainder of this paper we simplify the notation to $E_2^{s,t}(X)$, with $BP$ understood.

It sometimes happens that we are led to consider groups $\mathrm{Ext}_U(M)$, where $U$ is a simpler (abelian) category as follows. For each $M$ in $M$, let $U(M) = PG(M)$, where the functor $G$ is as above, and $P$ is the primitive element functor as in §2. In [4], $U$ is shown to be the functor of a cotriple $(U, \delta, \epsilon)$ on $M$ which is then extended to a cotriple on the abelian category $A$ of all positively graded $A$-modules (free or not). On $A$, $U$ is an exact functor. The category of modules in $A$ with $U$-structure will be denoted $U$ (in [4], this category $U$ is denoted $A(U)$, and is called the category of unstable $\Gamma$-comodules). We also note from [4], that if $M$ is in $G$, then $M$ is in particular a coalgebra, and the group of primitives $P(M)$ is in $U$. Thus $P$ may be considered as a functor from $G$ to $U$. Another of the main results of [4] is the following.

Proposition (4.2). *Let $M$ be in $G$, with $M$ cofree as a coalgebra. Then*

$$\mathrm{Ext}_G{}^{s,t}(M) \approx \mathrm{Ext}_U{}^{s,t}(PM)$$

This is proven in [4, §7] by means of the CFSS, which will be discussed more fully in §5 of this paper.

We also point out for future reference that for $M$ in $U$, the groups $\mathrm{Ext}_U^{**}(M)$ may be computed as the homology groups of the unstable cobar complex $C_U^{**}(M)$ which is the cochain complex of the cosimplicial complex

$$\mathrm{Hom}_M(A[t], \tilde{\mathbf{K}}_U(M)).$$

Thus, for each pair of non-negative integers $(q, t)$,

$$C_U{}^{q,t}(M) \approx U^q(M)_t.$$

We proceed with a general result concerning $\mathrm{Ext}_G(-)$ groups, which is very analogous to (3.2) and to [5, (3.6)].

Theorem 4.3. *Let $C' \xrightarrow{f} C \xrightarrow{g} C''$ be a sequence of maps in $G$, which, considered as a sequence in $S$ (coalgebras), forms an injective extension sequence. Then there is a long exact sequence:*

$$\cdots \longrightarrow \mathrm{Ext}_G{}^{s,t}(C') \xrightarrow{f_*} \mathrm{Ext}_G{}^{s,t}(C) \xrightarrow{g_*} \mathrm{Ext}_G{}^{s,t}(C'') \xrightarrow{\partial} \cdots$$

*where $\partial$ has bidegree* $(1, 0)$.

*Proof.* We prove this theorem by adapting the theory of [5] and §2 to the category $G$. First, we construct a category $G_0$ naturally equivalent to $G$, just as $S_0$ is naturally equivalent to $S$. The objects in $G_0$ are to be of the form $A \oplus M$, where $M$ is in $G$; the generator of the summand $A$ is given degree zero. Observe that if both $M$ and $M'$ are models in $G_0$, then so is $M \otimes_A M'$. Also, if $M$ is a model in $G_0$ then the diagonal map $\Delta : M \to M \otimes_A M$ is a map in $G_0$. Thus, the theory of [5] applies. Let $f : \mathbf{X} \to \mathbf{Y}$ be a map of cosimplicial objects over $G_0$. Then the mapping cone $\mathbf{M}(f)$ is the cosimplicial object over $G_0$ defined by the formulas of [5, (3.7)]. Using also the notation of [5, §3], we obtain a sequence of mapping cones

$$\mathbf{M}(\eta) \to \mathbf{M}(\tilde{\mathbf{K}}j) \to \mathbf{M}(\alpha) = \tilde{\mathbf{K}}C.$$

After application of the primitive element functor $P : G \to U$, we obtain a short exact sequence of cosimplicial objects over $U$:

$$0 \longrightarrow P\mathbf{M}(\eta) \longrightarrow P\mathbf{M}(\tilde{\mathbf{K}}j) \longrightarrow P\tilde{\mathbf{K}}C \longrightarrow 0.$$

The functor $\mathrm{Hom}_U(A, -)$ applied to this sequence gives another sequence which is also short exact because $PM(\eta)$ is injective in $U$. Using the isomorphism

$$\mathrm{Hom}_G(A, -) \approx \mathrm{Hom}_U(A, P(-))$$

we then obtain a short exact sequence of cosimplicial abelian groups:

$$0 \longrightarrow \mathrm{Hom}_G(A, \mathbf{M}(\eta)) \longrightarrow \mathrm{Hom}_G(A, \mathbf{M}(\tilde{\mathbf{K}}j)) \longrightarrow \mathrm{Hom}_G(A, \tilde{\mathbf{K}}C) \longrightarrow 0$$

$$H^s(\mathrm{ch}\,\mathrm{Hom}_G(A, \mathbf{M}(\eta))) \approx \mathrm{Ext}_G^{s-1}(C'')$$

$$H^s(\mathrm{ch}\,\mathrm{Hom}_G(A, \mathbf{M}(\tilde{\mathbf{K}}j))) \approx \mathrm{Ext}_G^s(C')$$

$$H^s(\mathrm{ch}\,\mathrm{Hom}_G(A, \tilde{\mathbf{K}}C)) \approx \mathrm{Ext}_G^s(C).$$

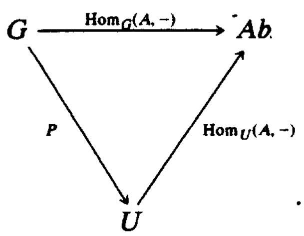
Passing to the homology groups of the chain complexes of the cosimplicial groups in the short exact sequence above, and using these isomorphisms, we obtain the long exact sequence of (4.3) as asserted.

## §5. THE COMPOSITE FUNCTOR SPECTRAL SEQUENCE

As noted in §5, if $M$ is in $G$, then the group of primitives $P(M)$ is in $U$, and

$$\mathrm{Hom}_G(A, M) \approx \mathrm{Hom}_U(A, P(M)).$$

Thus the functor $\mathrm{Hom}_G(A, -)$ from $G$ to the category $Ab$ of abelian groups factors as the composite




Let $A$ be the category of positively graded $A$-modules, not necessarily free. As will be shown below, the functor $U$ is an exact functor on $A$, and the category $U$ is an abelian category. By construction, the models in $U$ (that is, the objects of the form $U(M)$ for some $M$ in $A$) are injective in $U$. Therefore, for each $M$ in $G$, there is a composite functor spectral sequence (abbreviated *CFSS*) converging to $\mathrm{Ext}_G(M)$, and for which

$$E_2^{p,q} \approx \mathrm{Ext}_U^p(R^p P_G(M)).$$

In each of these Ext groups, the notation $\mathrm{Ext}^{s,t}(-)$ refers to the $s$-th right derived functor of $\mathrm{Hom}(A[t], -)$. For convenience, we suppress the second index $t$.

For completeness, and because our situation has some special features not present in the general case, we show how the *CFSS* comes about.

First, we show that $U$ is an exact functor on $A$. To see this, let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence in $A$. Then

$$0 \longrightarrow \Gamma \otimes_A M' \longrightarrow \Gamma \otimes_A M \longrightarrow \Gamma \otimes_A M'' \longrightarrow 0$$

is also short exact, because $\Gamma$ is flat over $A$. Here $\Gamma = \pi_*(BP \wedge BP)$ is the $BP$ analogue of the dual of the Steenrod algebra. In [4], it is shown that $U(M)$ is isomorphic to a certain subgroup of $\Gamma \otimes_A M$ in such a way that

$$0 \longrightarrow U(M') \longrightarrow U(M) \longrightarrow U(M'') \longrightarrow 0$$

is also short exact.

Next, we form a double complex $D^{*,*}(M)$ as follows. For each $p \geq 0$, $q \geq 0$, let

$$\begin{aligned} D^{p,q}(M) &= U^q P G^{p+1}(M) \\ &= U^{q+1} G^p(M). \end{aligned}$$

The latter equality occurs because $U = PG$. For each fixed $q \geq 0$, we have

$$D^{*,q}(M) \approx U^q P(\bar{\mathbf{K}}_G^*(M))$$

which gives $D^{*,q}(M)$ the structure of a cosimplicial complex. For each fixed integer $p \geq 0$, we also have

$$D^{p,*}(M) \approx C_U^*(UG^p M)$$

which gives $D^{p,*}(M)$ the structure of a cosimplicial complex. The differentials of $D^{*,q}(M)$ and $D^{p,*}(M)$ commute, and $D^{*,*}(M)$ becomes a double complex. There are two spectral sequences converging to the homology of the total complex of $D^{*,*}(M)$.

(1) Filter $D^{*,*}(M)$ by the first index $p$. Using the exactness of the functor $U^q$, we have

$$E_1^{p,q} \approx U^q R^p P_G(M).$$

Then for each fixed integer $p \geq 0$,

$$(E_1^{p,*}, d_1) \approx (C_U^*(R^p P_G(M)), \partial)$$

and therefore,

$$E_2^{p,q} \approx \operatorname{Ext}_U{}^q(R^p P_G(M)).$$

(2) Filter $D^{*,*}(M)$ by the second index $q$. This spectral sequence will be distinguished from (1) by tildes. From the fact that $UG^p(M)$ is injective in $U$, we have

$$\begin{aligned} \tilde{E}_1{}^{p,q} &= 0, \quad \text{for } q > 0 \\ \tilde{E}_1{}^{p,0} &\approx G^p(M). \end{aligned}$$

Then

$$\begin{aligned} (\tilde{E}_1{}^{*,0}, \tilde{d}_1) &\approx (\operatorname{ch} \operatorname{Hom}_G(A, \bar{\mathbf{K}}_G^*(M)), \partial) \\ &\approx (C_G^*(M), \partial) \end{aligned}$$

and therefore

$$\tilde{E}_2^{p,0} \approx \mathrm{Ext}_G{}^p(M).$$

From the second (tilde) spectral sequence, we conclude that the homology of the total complex of $D^{*,*}(M)$ is $\mathrm{Ext}_G^*(M)$. The first spectral sequence (1) is called the composite functor spectral sequence.

*Remark* (5.1). As noted in ([4], §7), each model in $G$ is cofree as a coalgebra, and so is a model in $S$. Therefore, a cosimplicial resolution by models in $G$ is *a fortiori* a cosimplicial resolution by models in $S$. Thus, by (2.3), for each $M$ in $G$,

$$R^pP_G(M) \approx R^pP_S(M)$$

and we may write $R^pP(M)$ unambiguously for these derived functors.

THEOREM (5.2). (i) *Let $M$ be in $G$, and suppose that $M$ is cofree. Then*

$$\mathrm{Ext}_G{}^{s,t}(M) \approx \mathrm{Ext}_U{}^{s,t}(P(M))$$

(ii) Let $M$ be in $G$, and suppose that as a coalgebra, $M$ is nice (§2). Then there is a LES

$$\cdots \longrightarrow \mathrm{Ext}_U{}^{s,t}(PM) \longrightarrow \mathrm{Ext}_G{}^{s,t}(M) \longrightarrow \mathrm{Ext}_U{}^{s-1,t}(R^1PM) \xrightarrow{\partial} \cdots$$

where $\partial$ has bidegree $(2, 0)$.

*Proof.* (i) If $M$ is a cofree as a coalgebra, $R^iP(M) = 0$ for $i > 0$, so the *CFSS* collapses, and (i) follows.

(ii) The condition that coalgebra $M$ be nice means that $R^iP(M) = 0$ for $i > 1$. In this case, the *CFSS* has only two columns; that is, $E_2{}^{i,s} = 0$ for $i \neq 0, 1$. The abuttment of the *CFSS* to $\mathrm{Ext}_G{}^{*,*}(M)$ gives the *LES* as asserted.

In §7, for $M = H_*(\hat{S}^{2n}; BP)$, this *LES* will be interpreted as one of the *EHP* sequences. In §8, for $M = H_*(\Omega S^{2n+1}; BP)$, this *LES* will be used to study the double suspension.

## §6. $E_2(\Omega S^{2n+1})$

In this section, we prove the following theorem for the $E_2$-terms of the unstable Adams–Novikov spectral sequences.

THEOREM (6.1). For each odd-dimensional sphere $S^{2n+1}$,

$$E_2{}^{s,t-1}(\Omega S^{2n+1}) \approx E_2{}^{s,t}(S^{2n+1}).$$

*Proof.* We shall show that a complex which computes $E_2(S^{2n+1})$ is also a complex which computes $E_2(\Omega S^{2n+1})$. Let $M = M(2n+1)$ be the free $A$-module on one generator $x_{2n+1}$ of degree $2n+1$. We also regard $M$ with the trivial $U$-structure as an object in the category $U$. Recall that from §5 and §2

$$E_2^{*,*}(S^{2n+1}) \approx \mathrm{Ext}_U^{*,*}(M) \approx H^*(C_U^{\dagger *}(M))$$

where $C_U^{\dagger *}(M)$ is the canonical cobar complex for $M$ in $U$ (see §4).

For each module $N$ in $M$, let $\sigma^{-1}(N)$ denote the isomorphic $A$-module, but with degrees shifted downward by one: $\sigma^{-1}(N)_t = N_{t+1}$. The results of Wilson[17] imply that if $N$ is a free $A$-module on odd-dimensional generators, then

$$\sigma^{-1}(U(N)) \approx QG(\sigma^{-1}(N))$$

where $Q(-)$ stands for the indecomposables under the $*$-product (i.e. loop-product). It then follows that for $M = M(2n+1)$,

$$\sigma^{-1}(U^q(N)) \approx QG(\sigma^{-1}U^{q-1}(N))$$

for each positive integer $q$.

From (4.1) we also know that

$$E_2^{s,t}(\Omega S^{2n+1}) \approx \mathrm{Ext}_G^{s,t}(H_*(\Omega S^{2n+1}; BP)).$$

Recall that

$$H_*(\Omega S^{2n+1}; BP) \approx T(x_{2n})$$

where $T(x_{2n})$ is the Hopf algebra over $A$, which as an algebra is the polynomial algebra over $A$ generated by $x_{2n}$, with $x_{2n}$ primitive. Let $Y$ be the cosimplicial complex where for each $q \geq 0$,

$$Y^q = G(\sigma^{-1}U^q(M))$$

$Y$ is augmented by

$$T(x_{2n}) = Y^{-1} \xrightarrow{\epsilon} Y^0 = G(\sigma^{-1}(M))$$

where $\epsilon$ is the algebra homomorphism which sends $x_{2n}$ in $T(x_{2n})$ to $\sigma^{-1}(x_{2n+1})$ in $G(\sigma^{-1}(M))$. Then $\epsilon$ is also a coalgebra map, and is a $\Gamma$-comodule map; by ([4], (7.3)), $\epsilon$ is a map in $G$. By construction, each $Y^q$ is a model in $G$. To show that $Y$ is a cosimplicial resolution of $T(x_{2n})$, it remains only to show that the cochain complex of $Y$ is acyclic. Note that each $Y^q$ is an algebra under the $*$-product, and the coface operators are algebra homomorphisms. Each $Y^q$ is filtered by powers of its augmentation ideal, and the quotients of this filtration are denoted $E_0(Y^q)$. Then

$$E_0(Y^q) \approx T(Q(\sigma^{-1}U^qM)) \approx T(\sigma^{-1}U^{q+1}M)$$

Thus

$$E_0(Y) \approx T(\sigma^{-1}\mathbf{K}_U(M)).$$

That is, $E_0(Y)$ is the result of applying the polynomial algebra functor $T(-)$ to the acyclic cosimplicial complex $\sigma^{-1}\mathbf{K}_U(M)$. Hence $E_0(Y)$ is acyclic, and then $Y$ itself is acyclic also.

### §7. THE *EHP* SEQUENCES OF $E_2$-TERMS

THEOREM (7.1). *For each positive integer n, there are long exact sequences*:

$$\text{(i)}\quad \cdots \longrightarrow E_2^{s,t}(\hat{S}^{2n}) \longrightarrow E_2^{s,t+1}(S^{2n+1}) \longrightarrow E_2^{s,t+1}(S^{2np+1}) \xrightarrow{\partial} E_2^{s+1,t}(\hat{S}^{2n}) \longrightarrow \cdots$$

$$\text{(ii)}\quad \cdots \longrightarrow E_2^{s,t-1}(S^{2n-1}) \longrightarrow E_2^{s,t}(\hat{S}^{2n}) \longrightarrow E_2^{s-1,t-1}(S^{2np-1}) \xrightarrow{\partial} E_2^{s+1,t-1}(S^{2n-1}) \longrightarrow \cdots$$

*Proof.* For (i), we consider the fibration

$$\hat{S}^{2n} \xrightarrow{i} \Omega S^{2n+1} \xrightarrow{h} \Omega S^{2np+1}$$

where $h = h_{2n+1}$ is the James map as in §1, and $i$ is the inclusion of the homotopy fibre of $h$. The $BP_*$-homology of these spaces are as follows.

$$\begin{array}{c} H_*(\hat{S}^{2n}) \approx T_1(x_{2n}) \\ \downarrow i_* \\ H_*(\Omega S^{2n+1}) \approx T(x_{2n}) \\ \downarrow h_* \\ H_*(\Omega S^{2np+1}) \approx T(x_{2np}) \end{array}$$

where $T_1(x_{2n})$, $T(x_{2n})$ and $T(x_{2np})$ are the coalgebras of §3. The maps $i_*$ and $h_*$ are induced from maps of spaces, and are therefore maps in $G$. Considered as maps in $S$, $i_*$ and $h_*$ form an injective extension sequence (recall (3.1)). Hence by (4.3), there is a *LES*

$$(7.2)\quad \cdots \longrightarrow \mathrm{Ext}_G^{s,t}(T_1(x_{2n})) \longrightarrow \mathrm{Ext}_G^{s,t}(T(x_{2n})) \longrightarrow \mathrm{Ext}_G^{s,t}(T(x_{2np})) \longrightarrow \cdots$$

From (4.1) and (6.1), we have

$$\begin{aligned} \mathrm{Ext}_G^{s,t}(T(x_{2n})) &\approx E_2^{s,t}(\Omega S^{2n+1}) \\ &\approx E_2^{s,t+1}(S^{2n+1}). \end{aligned}$$

Similarly,

$$\mathrm{Ext}_G^{s,t} T(x_{2np}) \approx E_2^{s,t+1}(S^{2np+1})$$

and

$$\mathrm{Ext}_G^{s,t}(T_1(x_{2n})) \approx E_2^{s,t}(\hat{S}^{2n}).$$

Using these isomorphisms to replace the groups in the sequence (7.2), we obtain the first of the *EHP* sequences (7.1(i)).

For (7.1(ii)), we consider $H_*(\hat{S}^{2n}; BP)$ in the category $G$, and as a coalgebra. As a coalgebra,

$$H_*(\hat{S}^{2n}; BP) \approx T_1(x_{2n}) ,$$

From (4.1) we also have

$$\begin{aligned}\mathrm{Ext}_G^{s,t}(T(x_{2n})) &\approx E_2^{s,t}(\Omega S^{2n+1})\\ &\approx E_2^{s,t+1}(S^{2n+1}).\end{aligned}$$

Using these isomorphisms to replace the terms in (8.2) gives the sequence (8.1).

*Remark* (8.3). In [18], it will be shown that the $U$-structure (that is, the unstable $\Gamma$-coaction) on $W(n)$ is the following. Let $h_i$ be the elements of $\Gamma$ defined in [3, (2.9)]. Then, the coaction

$$\psi\colon W(n) \longrightarrow UW(n)$$

is given by

$$\psi(y_k) = \sum_{j=1}^{k} p^{k-j} h_{k-y}^{np^j} \otimes y_j.$$

This formula for the unstable $\Gamma$-coaction in $W(n)$ will not be needed in this paper.

An abelian group is said to have exponent $m$ if $mx = 0$ for each element $x$ in the group.

COROLLARY (8.4). *For $(s, t) \neq (0, 2n+1)$, the group $E_2^{s,t}(S^{2n+1})$ has exponent $p^n$.*

*Proof.* The group $W(n)$ has exponent $p$. Then the cobar complex for computing $\mathrm{Ext}_U^{*,*}(W(n))$ has exponent $p$, and thus $\mathrm{Ext}_U^{s,t}(W(n))$ has exponent $p$ also. From the exactness of (8.2), both the kernel and cokernel of double suspension have exponent $p$. The statement of (8.4) follows from the fact that $E_2^{s,t}(S^1) = 0$ for $(s, t) \neq (0, 1)$, and simple induction on $n$.

Let functions $f(s)$ be defined as follows. For each positive integer $k$,

$$\begin{aligned}f(2k) &= 2(p-1)pk\\ f(2k+1) &= 2(p-1)(pk+1).\end{aligned}$$

COROLLARY (8.5). *The double suspension map*

$$E_2^{s,t-1}(S^{2n-1}) \longrightarrow E_2^{s,t+1}(S^{2n+1})$$

*is an isomorphism for $t < f(s-2) + 2np$, and is onto for $t < f(s-1) + 2np$.*

*Proof.* Recall from [3, (5.8)] that for $m \geq 2p-1$, $\mathrm{Ext}_U^{s,t}(M(m))$ has a vanishing line:

$$\mathrm{Ext}_U^{s,t}(M(m)) = 0, \quad \text{for } t < f(s) + m.$$

If $M$ is any module in $U$ with $M_i = 0$ for $i < 2p-1$, then $\mathrm{Ext}_U^{s,t}(M)$ also has a vanishing line (from the lower non-zero degree in $M$). In particular, $W(n)_i = 0$ for $i < 2pn$, so

$$\mathrm{Ext}_U^{s,t}(W(n)) = 0, \quad \text{for } t < f(s) + 2np.$$

The assertions in the corollary then follow from the *LES* (8.1).

which is nice; that is, $R^iPT(x_{2n}) = 0$ for $i > 1$. From §3,

$$PT_1(x_{2n}) \approx M(2n)$$

$$R^1PT_1(x_{2n}) \approx M(2np).$$

In this case, the *LES* of (5.2) becomes

$$\cdots \longrightarrow \mathrm{Ext}_U{}^{s,t}(M(2n)) \longrightarrow \mathrm{Ext}_G{}^{s,t}(T_1(x_{2n})) \longrightarrow \mathrm{Ext}_U{}^{s-1,t}(M(2np)) \longrightarrow \cdots \quad (7.3)$$

From [4], we have isomorphisms

$$\mathrm{Ext}_U{}^{s,t}(M(2n)) \approx \mathrm{Ext}_U{}^{s,t-1}(M(2n-1))$$
$$\approx E_2{}^{s,t-1}(S^{2n-1}).$$

Similarly,

$$\mathrm{Ext}_U{}^{s-1,t}(M(2np)) \approx E_2{}^{s-1,t-1}(S^{2np-1}).$$

Substituting these terms into the *LES* (7.3) gives the second *EHP* sequence (7.1(ii)).

## §8. THE DOUBLE SUSPENSION

THEOREM (8.1). *For each positive integer n, there is a long exact sequence*:

$$\cdots \longrightarrow E_2{}^{s,t-1}(S^{2n-1}) \longrightarrow E_2{}^{s,t+1}(S^{2n+1}) \longrightarrow \mathrm{Ext}_U{}^{s-1,t}(W(n)) \xrightarrow{\partial} \cdots$$

*where* $W(n) = R^1PT(x_{2n})$. *The boundary homomorphism* $\partial$ *has bidegree* (2, 0).

*Proof.* We consider (5.2(ii)) applied to $M = H_*(\Omega S^{2n+1}; BP)$ in the category $G$. As a coalgebra, in the notation of §3,

$$H_*(\Omega S^{2n+1}; BP) \approx T(x_{2n}).$$

Recall from (3.3) that

$$PT(x_{2n}) = M(2n)$$

$$R^1PT(x_{2n}) = W(n)$$

$$R^iPT(x_{2n}) = 0, \quad \text{for } i > 1.$$

Here, $W(n)$ is the free $A \otimes_Z \dot{Z}_p$-module generated by elements $y_k$ of degree $2np^k$, for $k = 1, 2, \ldots$. By (5.2(ii)), we have

$$\cdots \longrightarrow \mathrm{Ext}_U{}^{s,t}(M(2n)) \longrightarrow \mathrm{Ext}_G{}^{s,t}(T(x_{2n})) \longrightarrow \mathrm{Ext}_U{}^{s-1,t}(W(n)) \xrightarrow{\partial} \cdots \quad (8.2)$$

From [4, (8.6)], (4.1) and (4.2), we have

$$\mathrm{Ext}_U{}^{s,t}(M(2n)) \approx \mathrm{Ext}_U{}^{s,t-1}(M(2n-1))$$
$$\approx \mathrm{Ext}_G{}^{s,t-1}(M(2n-1))$$
$$\approx E_2{}^{s,t-1}(S^{2n-1}).$$

COROLLARY (8.6). *For $2n+1 \neq t < f(s-1) + 2kp + 2n + 1$, the group $E_2^{s,t}(S^{2n+1})$ has exponent $p^k$. Also, for the stable groups (indexed in the usual way: $t - s$ = stem degree) the group $E_2^{s,t}(S^0)$ has exponent $p^k$ for $0 \neq t < f(s-1) + 2kp$.*

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