6/29/09

Akhmet'ev's compression theorem

In his lecture at Princeton on May 19, Akhmetiev stated the following, which he referred to as the Desuspension Theorem.

Compression Theorem. *For an arbitrary q there exists an integer $l_0 = l_0(q)$, such that an arbitrary element $X \in Imm^{sf}(2^l - 3, 1)$ admits a desuspension of order q for each $l \geq l_0$.*

We will discuss the value of $l_0(q)$ for small q at the end of this document.

Here $Imm^{sf}(2^l - 3, 1)$ denotes the cobordism group of codimension 1 immersions in $\mathbf{R}^{2^l-2}$, which is isomorphic to the stable group $\pi_{2^l-2}(\mathbf{R}P^\infty)$. Such an immersion of a manifold M^{2^l-3} leads to a map $M \to \mathbf{R}P^{2^l-3}$ classifying the normal bundle of the immersion. After suspending a few times (to convert the immersion to an embedding in a higher dimensional Euclidean space) we get a map

$$S^{2^l-2+t} \to \Sigma^t MO(1) = \Sigma^t \mathbf{R}P^\infty,$$

which is an element in the indicated stable group. The stable map $S^{2^l-2} \to \mathbf{R}P^\infty$ factors through $\mathbf{R}P^{2^l-2}$. A desuspension of order q is a factorization through $\mathbf{R}P^{2^l-2-q}$. Akhmetiev defines this in terms of a map from M, which is the preimage of a hyperplane in $\mathbf{R}P^{2^l-2-q}$, so M maps to $\mathbf{R}P^{2^l-3-q}$.

We will sketch a proof. Consider the diagram

$$\begin{array}{ccccc}
 & & S^{2^l-2} & & \\
 & \swarrow\!\cdot\cdot & \downarrow{\scriptstyle \kappa_l} & \searrow & \\
\mathbf{R}P^{2^l-2-q} & \longrightarrow & \mathbf{R}P^{2^l-2} & \longrightarrow & \mathbf{R}P^{2^l-2}_{2^l-1-q} \\
\downarrow & & \downarrow{\scriptstyle j_q} & & \| \\
S^{2^l-2-q} & \longrightarrow & \mathbf{R}P^{2^l-2}_{2^l-2-q} & \longrightarrow & \mathbf{R}P^{2^l-2}_{2^l-1-q}
\end{array}$$

where the bottom two rows are cofiber sequences. The map κ_l is determined by the immersion, and the map to the stunted projective space $\mathbf{R}P^{2^l-2}_{2^l-1-q}$ is the obstruction to the desuspension.

Before proceeding further, we need to recall some facts about James periodicity. The stable homotopy type of the stunted projective space $\Sigma^{-n}\mathbf{R}P_n^{n+k}$ for a fixed $k > 0$ depends only on the congruence class of n modulo a power of 2 determined by k. Since $\mathbf{R}P_n^{n+k}$ is the Thom space for n times the canonical line bundle λ over $\mathbf{R}P^k$, it suffices to determine the order of λ in $KO(\mathbf{R}P^k)$. This was done by Adams in his vector field paper.

Adams' Lemma. Order of the canonical real line bundle. *The order of the canonical real line bundle λ in $KO(\mathbf{R}P^k)$ is $2^{\rho(k)}$ where*

$$\begin{aligned}\rho(k) &= \begin{cases} 4m & \text{for } k = 8m \\ 4m+1 & \text{for } k = 8m+1 \\ 4m+2 & \text{for } k = 8m+2 \text{ or } 8m+3 \\ 4m+3 & \text{for } k = 8m+4,\ 8m+5,\ 8m+6 \text{ or } 8m+7 \end{cases} \\ &= k/2 + \begin{cases} 0 & \text{for } k = 0 \text{ or } 6 \bmod 8 \\ 1/2 & \text{for } k = 1,\ 3 \text{ or } 5 \bmod 8 \\ 1 & \text{for } k = 2 \text{ or } 4 \bmod 8 \\ -1/2 & \text{for } k = 7 \bmod 8. \end{cases}\end{aligned}$$

Corollary. James periodicity. *The stable homotopy type of the stunted projective space $\Sigma^{-n}\mathbf{R}P_n^{n+k}$ for a fixed $k > 0$ depends only on the congruence class of n modulo $2^{\rho(k)}$.*

Next we need the following consequence of Lin's Theorem.

Inverse System Lemma. *Let $\mathbf{R}P_{-\infty}^n$ denote the homotopy inverse limit*

$$holim_{s\to-\infty}\mathbf{R}P_s^n.$$

Then $\pi_{-2}(\mathbf{R}P_{-\infty}^{-2}) = 0$ and the corresponding inverse system of groups has the Mittag-Leffler property, namely for each $q > 0$ there is an integer $r_0(q) \geq q$ such that the homomorphism $\pi_{-2}(\mathbf{R}P_{-2-r}^{-2}) \to \pi_{-2}(\mathbf{R}P_{-1-q}^{-2})$ is trivial for each $r \geq r_0(q)$.

Proof. Lin's Theorem says that $\mathbf{R}P_{-\infty}^{\infty}$ is the 2-adic completion of S^{-1}, which we denote by S_2^{-1}. Consider the cofiber sequence

$$\mathbf{R}P_{-\infty}^{-2} \to \mathbf{R}P_{-\infty}^{\infty} \to \mathbf{R}P_{-1}^{\infty}.$$

We can use Lin's Theorem to study the long exact sequence of homotopy groups. We get

$$\begin{array}{ccccccccccc} \pi_0(\mathbf{R}P_{-1}^{\infty}) & \to & \pi_{-1}(\mathbf{R}P_{-\infty}^{-2}) & \to & \pi_{-1}(S^{-1}) & \to & \pi_{-1}(\mathbf{R}P_{-1}^{\infty}) & \to & \pi_{-2}(\mathbf{R}P_{-\infty}^{-2}) & \to & \pi_{-2}(S^{-1}) \\ \| & & \| & & \| & & \| & & \| & & \| \\ 0 & & \mathbf{Z}_2 & \xrightarrow{2} & \mathbf{Z}_2 & \longrightarrow & \mathbf{Z}/2 & & 0 & & 0 \end{array}$$

where the indicated homotopy groups of S_2^{-1} and $\mathbf{R}P_{-1}^{\infty}$ can be computed directly. Similarly ewe find that $\pi_{-3}(\mathbf{R}P_{-\infty}^{-2}) = 0$.

We also have short exact sequences

$$\begin{array}{c} 0 \longrightarrow \lim^1 \pi_{-1}(\mathbf{R}P_{-s}^{-2}) \longrightarrow \pi_{-2}(\mathbf{R}P_{-\infty}^{-2}) \longrightarrow \lim \pi_{-2}(\mathbf{R}P_{-s}^{-2}) \longrightarrow 0 \\ \| \\ 0 \end{array}$$

and

$$\begin{array}{c} 0 \longrightarrow \lim^1 \pi_{-2}(\mathbf{R}P_{-s}^{-2}) \longrightarrow \pi_{-3}(\mathbf{R}P_{-\infty}^{-2}) \longrightarrow \lim \pi_{-3}(\mathbf{R}P_{-s}^{-2}) \longrightarrow 0 \\ \| \\ 0 \end{array}$$

Hence we have

$$\lim{}^1 \pi_{-2}(\mathbf{R}P_{-s}^{-2}) = \lim \pi_{-2}(\mathbf{R}P_{-s}^{-2}) = 0.$$

The vanishing of $\lim^1$ for a system of finitely generated abelian groups is equivalent to the Mittag-Leffler condition. □

Proof of the Comporession Theorem. We have a diagram

$$\begin{array}{ccccc} & & S^n & & \\ & \swarrow & \downarrow f & \searrow f' & \\ \mathbf{R}P^{n-q} & \longrightarrow & \mathbf{R}P^n & \longrightarrow & \mathbf{R}P^{n}_{n+1-q}\end{array}$$

The bottom row is a cofiber sequence, so the map to $\mathbf{R}P^{n-q}$ exists iff the map f' is null. Let r_0 be the integer provided by the Lemma, and consider the diagram.

$$\begin{array}{ccccc} & & S^n & & \\ & \swarrow & \downarrow g & \searrow f' & \\ \mathbf{R}P^{n-q}_{n-r_0} & \longrightarrow & \mathbf{R}P^n_{n-r_0} & \longrightarrow & \mathbf{R}P^{n}_{n+1-q}\end{array}$$

where the map to f' is the same as before. For $j+1 \geq \rho(r_0)$, James periodicity makes this equivalent to

$$\begin{array}{ccccc} & & S^{-2} & & \\ & \swarrow & \downarrow f & \searrow f' & \\ \mathbf{R}P^{-2-q}_{-2-r_0} & \longrightarrow & \mathbf{R}P^{-2}_{-2-r_0} & \longrightarrow & \mathbf{R}P^{-2}_{-1-q}\end{array}$$

Now the Inverse System Lemma tells us that f' is null. □

Finding the integers $r_0(q)$ in the Inverse System Lemma and $l_0(q)$ in the Compression Theorem.

Here we compute $\pi_{-2}(\mathbf{R}P^{-2}_{-1-q})$ for small values of q and the homomorphisms between them. We start by finding $J_{-2}(\mathbf{R}P^{-2}_{-1-q})$. Recall that the spectrum J at the prime 2 is defined as follows. The map

$$\begin{array}{ccccc} J & \longrightarrow & bo & \longrightarrow & \Sigma^4 bsp \\ & & & \searrow{\scriptstyle \psi^3-1} & \downarrow \\ & & & & bo \end{array}$$

Here bo is the spectrum for real connective K-theory, $\Sigma^4 bsp$ is its 3-connected cover, and ψ^3 is the Adams operation. J is the fiber of the unique lifting of $\psi^3 - 1$ to $\Sigma^4 bsp$, where bsp denotes the connective delooping of the classifying space BSp of the stable symplectic group.

The cofiber sequence

$$\Sigma^3 bsp \to J \to bo$$

leads to a long exact sequence

$$bo_{-1}(P^{-2}_{-1-q}) \to bsp_{-5}(P^{-2}_{-1-q}) \to J_{-2}(P^{-2}_{-1-q}) \to bo_{-2}(P^{-2}_{-1-q}) \to bsp_{-6}(P^{-2}_{-1-q}).$$

Here is a table of these groups, which can be found using the Atiyah-Hirzebruch spectral sequence. A closely related calculation is described in the last few pages of §1.5 of the green book.

q	$bo_{-1}(P^{-2}_{-1-q})$	$bsp_{-5}(P^{-2}_{-1-q})$	$J_{-2}(P^{-2}_{-1-q})$	$bo_{-2}(P^{-2}_{-1-q})$	$bsp_{-6}(P^{-2}_{-1-q})$
1	$\mathbf{Z}/2$		$\mathbf{Z}$	$\mathbf{Z}$	
2	$\mathbf{Z}/4$		$\mathbf{Z}/2$	$\mathbf{Z}/2$	
3	$\mathbf{Z}/4$		$\mathbf{Z}/2$	$\mathbf{Z}/2$	
4	$\mathbf{Z}/8$	$\mathbf{Z}/2$	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	$\mathbf{Z}/2$	
5	$\mathbf{Z}/8$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}$	$\mathbf{Z}$
6	$\mathbf{Z}/8$	$\mathbf{Z}/2$	$\mathbf{Z}/2$		
7	$\mathbf{Z}/8$	$\mathbf{Z}/2$	$\mathbf{Z}/2$		
8	$\mathbf{Z}/2^4$	$\mathbf{Z}/4$	$\mathbf{Z}/4$		
9	$\mathbf{Z}/2^5$	$\mathbf{Z}/8$	$\mathbf{Z}/8$	$\mathbf{Z}$	$\mathbf{Z}$
10	$\mathbf{Z}/2^6$	$\mathbf{Z}/2^4$	$\mathbf{Z}/2^4 \oplus \mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
11	$\mathbf{Z}/2^4$	$\mathbf{Z}/2^4$	$\mathbf{Z}/2^4 \oplus \mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
12	$\mathbf{Z}/2^7$	$\mathbf{Z}/2^5$	$\mathbf{Z}/2^5 \oplus \mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$
13	$\mathbf{Z}/2^7$	$\mathbf{Z}/2^5$	$\mathbf{Z}/2^5$	$\mathbf{Z}$	$\mathbf{Z}$
14	$\mathbf{Z}/2^7$	$\mathbf{Z}/2^5$	$\mathbf{Z}/2^5$		
15	$\mathbf{Z}/2^7$	$\mathbf{Z}/2^5$	$\mathbf{Z}/2^5$		
16	$\mathbf{Z}/2^8$	$\mathbf{Z}/2^6$	$\mathbf{Z}/2^6$		
17	$\mathbf{Z}/2^9$	$\mathbf{Z}/2^7$	$\mathbf{Z}/2^7$	$\mathbf{Z}$	$\mathbf{Z}$

When an entry is blank, the group in question is trivial. For each q the five groups fit into a long exact sequence, so there are homomorphims from left to right. Within each column there are vertical homomorphisms going up induced by pinch maps.

Each vertical homomorphism in the first and second columns is onto, and the vertical map in the fourth column is nilpotent. Horizontal homomorphisms from the first to the second are trivial. Ones from the fourth to the fifth columns are multiplication by 2^4.

Thus we have

$$J_{-2}(\mathbf{R}P^2_{-1-q}) = bsp_{-5}(\mathbf{R}P^2_{-1-q}) \oplus \begin{cases} \mathbf{Z} & \text{for } q = 1 \\ \mathbf{Z}/2 & \text{for } q \equiv 2, 3 \text{ or } 4 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

and

$$bsp_{-5}(\mathbf{R}P^2_{-1-q}) = \mathbf{Z}/2^{\phi(q)}$$

where

$$\phi(q) = \begin{cases} 0 & \text{for } q < 4 \\ 1 & \text{for } 4 \le q \le 7 \\ 2 & \text{for } q = 8 \\ 3 & \text{for } q = 9 \\ 4 + \phi(q-8) & \text{for } q \ge 10 \end{cases}$$

ach element in $J_{-2}(\mathbf{R}P^2_{-1-q})$ in the image of $bsp_{-5}(\mathbf{R}P^2_{-1-q})$ is in the image of $J_{-2}(\mathbf{R}P^2_{-2-r})$ for all $r > q$ since

$$\lim_{\leftarrow} J_{-2}(\mathbf{R}P^2_{-1-q}) = \mathbf{Z}_2$$

Its preimage in $\pi_{-2}(\mathbf{R}P^2_{-1-q})$ does not have this property and hence supports a nontrivial differential in the Atiyah-Hirzebruch spectral sequence for $J_{-2}(\mathbf{R}P^2_{-2-r})$ for some $r > q$. Each of these comes from a generator of the image of J on a suitable cell. For the first few of them we have a diagram

$$\begin{array}{ccccccccc}
S^{-2} & \xrightarrow{2} & S^{-2} & \xrightarrow{2} & S^{-2} & \xrightarrow{2} & S^{-2} & \xrightarrow{2} & S^{-2} \\
\downarrow{\scriptstyle \overline{\alpha}_6} & & \downarrow{\scriptstyle \overline{\alpha}_5} & & \downarrow{\scriptstyle \overline{\alpha}_4} & & \downarrow{\scriptstyle \overline{\alpha}_3=\sigma} & & \downarrow{\scriptstyle \overline{\alpha}_2=\nu} \\
S^{-13} & & S^{-11} & & S^{-10} & & S^{-9} & & S^{-5} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{R}P^2_{-13} & \longrightarrow & \mathbf{R}P^2_{-11} & \longrightarrow & \mathbf{R}P^2_{-10} & \longrightarrow & \mathbf{R}P^2_{-9} & \longrightarrow & \mathbf{R}P^2_{-5}
\end{array}$$

Here $\overline{\alpha}_j$ denotes the jth generator of the image of the Hopf-Whitehead J-homomorphism, with

$$|\overline{\alpha}_j| = 2j + \begin{cases} 0 & \text{for } j \equiv 0 \bmod 4 \\ -1 & \text{for } j \equiv 1 \bmod 4 \\ -1 & \text{for } j \equiv 2 \bmod 4 \\ 1 & \text{for } j \equiv 3 \bmod 4 \end{cases}$$

For $2 \le j \le 5$ there is known to be a diagram

$$\begin{array}{ccccc}
S^{-2} & \xrightarrow{\overline{\alpha}_j} & S^{-2-|\overline{\alpha}_j|} & \longrightarrow & \mathbf{R}P^{-2}_{-2-|\overline{\alpha}_j|} \\
\downarrow{\scriptstyle \theta_j} & & & & \uparrow \\
S^{-2^{j+1}} & & \longleftarrow & & \mathbf{R}P^{-2}_{-2^{j+1}} \\
& & & & \uparrow \\
& & & & \mathbf{R}P^{-2}_{-2^{j+1}-1}
\end{array}$$

where the arrow pointing left is the cofiber of the pinch map $\mathbf{R}P^{-2}_{-2^{j+1}-1} \to \mathbf{R}P^{-2}_{-2^{j+1}}$. In other words θ_j is the obstruction to pulling the element back to $\pi_{-2}(\mathbf{R}P^{-2}_{-2^{j+1}-1})$. It has long been thought that there is similar diagram for any j for which θ_j exists.

This means that in the Inverse System Lemma, when $q = 1 + |\overline{\alpha}_j|$, $r_0(q) \ge 2^{j+1} - 2$. Computations described in Mahowald's 1967 Memoir indicate that for these q, this lower bound on $r_0(q)$ is its actual value. More explicitly,

$$r_0(q) = \begin{cases} 6 & \text{for } 4 \le q \le 7 \\ 14 & \text{for } q = 8 \\ 30 & \text{for } q = 9 \\ 62 & \text{for } 10 \le q \le 11 \\ 126 & \text{for } 12 \le q \le 15 \text{ if } \theta_6 \text{ exists.} \end{cases}$$

The integer $l_0(q)$ in the Compression Theorem is $\rho(r_0(q))$ and $\rho(8k+6) = 4k+3$, so we have

$$l_0(q) = \begin{cases} 3 & \text{for } 4 \le q \le 7 \\ 7 & \text{for } q = 8 \\ 15 & \text{for } q = 9 \\ 31 & \text{for } 10 \le q \le 11 \\ 63 & \text{for } 12 \le q \le 15 \text{ if } \theta_6 \text{ exists.} \end{cases}$$

I do not know what happens for $12 \le q \le 15$ if θ_6 does not exist. It seems likely that $l_0(q)$ would be smaller than indicated above, but it would have to be no less that $l_0(11) = 31$.

Update of 7/8/09

For the sake of argument, suppose the following opposite of the Doomsday Hypothesis (which implies that only finitely many θ_j exist) is true.

World Without End Hypothesis. *For each $j \geq 2$, θ_j exists and the diagram above relating it to $\overline{\alpha}_j$ commutes.*

This implies that for each integer $k \geq 0$,

$$r_0(q) \geq \begin{cases} 2^{4k+3} - 2 & \text{for } 8k+4 \leq q \leq 8k+7 \\ 2^{4k+4} - 2 & \text{for } q = 8k+8 \\ 2^{4k+5} - 2 & \text{for } q = 8k+9 \\ 2^{4k+6} - 2 & \text{for } 8k+10 \leq q \leq 8k+11, \end{cases}$$

and hence

$$l_0(q) \geq \begin{cases} 2^{4k+2} - 1 & \text{for } 8k+4 \leq q \leq 8k+7 \\ 2^{4k+3} - 1 & \text{for } q = 8k+8 \\ 2^{4k+4} - 1 & \text{for } q = 8k+9 \\ 2^{4k+5} - 1 & \text{for } 8k+10 \leq q \leq 8k+11. \end{cases}$$

In other words, the lower bound on $l_0(q)$ is roughly $2^{q/2}$.

For example, in Theorem 6 of Akhmet'ev's main paper, q is reqired to be at least 32. Under the above hypothesis, this means $l_0(q) \geq 2^{15} - 1 = 32767$.