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The Cohomology of the Morava Stabilizer Algebras

Douglas C. Ravenel*

Department of Mathematics, University of Washington, Seattle, Washington 98195, USA

In this paper we continue our study of the groups $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$. In [5] it was shown that these groups are essentially isomorphic to the cohomology of a certain Hopf algebra $S(n)$ which we called the Morava stabilizer algebra since it was implicitly introduced in [6]. The structure of $S(n)$ was analyzed in [8] where we defined a filtration on it and described the associated graded Hopf algebra $E_0S(n)$ explicitly. We will use the results of [8] in this paper extensively. Applications to the Novikov spectral sequence will appear in a forthcoming paper with Miller and Wilson.

In § 1 we show how the machinery developed by May in [3] can be applied to this situation. As a trivial corollary we show that for $n<p-1$, $H^*S(n)$ is the cohomology of a certain n-stage nilpotent Lie algebra of dimension n^2. In § 2 we describe $H^1S(n)$ in all cases, use Theorem 2.10 of [8] to get a splitting of $H^*S(n)$ when $p\nmid n$, and we obtain a general expression for $H^2S(n)$ for $n>2$. In § 3 we compute $H^*S(n)$ at all primes for $n\leqq 2$ and for $n=3$ with $p\geqq 5$.

§1. We begin by recalling the pertinent results of [5] and [8]. Recall $BP_* = Z_{(p)}[v_1, v_2, \ldots]$ with $|v_i|=2(p^i-1)$ (see [1] Part II) and let $K(n)_*=\mathbb{F}_p[v_n, v_n^{-1}]$ have the obvious BP_* module structure. Let $K(n)_*K(n)=K(n)_*\otimes_{BP_*}BP_*BP \otimes_{BP_*}K(n)_*$. Then the following was proved in [5].

(1.1) **Theorem.** $\mathrm{Ext}^*_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)\cong \mathrm{Ext}^*_{K(n)_*K(n)}(K(n)_*, K(n)_*)$. □

We then make $\mathbb{F}_p$ into a $K(n)_*$ module by sending v_n to 1 and define $S(n)_*=\mathbb{F}_p\otimes_{K(n)_*}K(n)_*K(n)$. $S(n)_*$ is a commutative Hopf algebra graded over $\mathbb{Z}/(2(p^i-1)$ and $S(n)$ is the appropriately defined (see [8]) linear dual of $S(n)_*$.

(1.2) **Proposition.**

$$\mathrm{Ext}^*_{K(n)_*K(n)}(K(n)_*, K(n)_*)\otimes_{K(n)_*}\mathbb{F}_p\cong \mathrm{Ext}^*_{S(n)_*}(\mathbb{F}_p, \mathbb{F}_p). \quad \square$$

(1.3) **Theorem.** *As an algebra* $S(n)_*\cong\mathbb{F}_p[t_1, t_2, \ldots]/(t_i^{p^n}-t_i)$ *and the coproduct is that inherited from* BP_*BP. *In particular for* $i\leqq n$, $\Delta(t_i)=\sum_{0\leqq j\leqq i} t_j\otimes t_{i-j}^{p^j}$ *and*

$$\Delta(t_{n+i})=\sum_{0\leqq j\leqq n+i} t_j\otimes t_{n+i-j}^{p^j}-C_{p^n}(t_i\otimes 1, t_{i-1}\otimes t_i^p, \ldots, 1\otimes t_i)$$

where $t_0=1$ *and* $C_{p^n}(x_1, \ldots, x_k)$ *is the* mod p *reduction of* $p^{-1}((\sum x_i)^{p^n}-\sum(x_i^{p^n}))$. □

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A more comprehensive formula for the coproduct is given in Theorem 1.8 of [8].

We now describe a certain increasing filtration on $S(n)_*$. Define integers d_i by $d_i=0$ for $i\leqq 0$ and $d_i=\max(i, pd_{i-n})$ for $i>0$. By Theorem 3.1 of [8], $S(n)_*$ has a unique filtration with deg $t_i^{p^j}=d_i$ for all i and j. Let $x_{i,j}\in E_0S(n)$ be the dual (with respect to the monomial basis) of the element $t_{i,j}\in E^0S(n)_*$ corresponding to $t_i^{p^j}$. The second subscript is an element of $\mathbb{Z}/(n)$.

(1.4) **Theorem.** *$E_0S(n)$ is the restricted enveloping algebra on primitives $x_{i,j}$ with bracket*

$$[x_{i,j}, x_{k,l}]=\begin{cases}\delta^l_{i+j}x_{i+k,j}-\delta^j_{k+l}x_{i+k,l} & \text{for } i+k\leqq m\\ 0 & \text{otherwise}\end{cases}$$

where m is the largest integer not exceeding $pn/(p-1)$, and $\delta^s_t=1$ iff $S\equiv t \bmod n$ and $\delta^s_t=0$ otherwise. The restriction ξ is given by

$$\xi(x_{i,j})=\begin{cases}0 & \text{if } i\leqq n/p-1\\ -x_{i+n,j+1} & \text{otherwise.}\end{cases}\qquad\square$$

Note that $x_{i,j}$ has internal dimension $2p^j(p^i-1)$.

Let $L(n)$ be the Lie algebra without restriction with basis $x_{i,j}$ and bracket as above. We now recall the main results of [3].

(1.5) **Theorem.** *There are spectral sequences*

a) $E_2=H^*L(n)\otimes P(b_{i,j})\Rightarrow H^*E_0S(n)$;
b) $E_2=H^*E_0S(n)\Rightarrow H^*S(n)$

where $b_{i,j}\in H^{2,pd_i}E_0S(n)$ with internal degree $2p^{j+1}(p^i-1)$ and $P(\cdot)$ is the polynomial algebra on the indicated generators. $\square$

Now let $L(n,k)$ be the quotient of $L(n)$ obtained by setting $x_{i,j}=0$ for $i>k$. Then our first result is

(1.6) **Theorem.** *The E_2 term of the first May spectral sequence (Theorem (5a)) may be replaced by $H^*L(n,m)\otimes P(b_{i,j}\colon i\leqq m-n)$ where $m=[pn/(p-1)]$ as before*

Proof. By Theorem 1.4, $L(n)$ is the product of $L(n,m)$ and an abelian Lie algebra, so

$$H^*L(n)\cong H^*L(m,n)\otimes E(h_{i,j}\colon i>m),$$

where $E(\cdot)$ denotes the exterior algebra on the indicated generators and $h_{i,j}\in H^1L(n)$ is the element corresponding to $x_{i,j}$. It also follows from Theorem 1.5 that the appropriate differential will send $h_{i,j}$ to $-b_{i-n,j-1}$ for $i>m$. It follows that the entire spectral sequence decomposes as a tensor product of two spectral sequences, one with the E_2 term indicated in the statement of the Theorem, and the other having $E_2=E(h_{i,j})\otimes P(b_{i-n,j})$ with $i>m$ and $E_\infty=\mathbb{F}_p$. $\square$

(1.7) **Theorem.** *The second May spectral sequence (Theorem (1.5b)) collapses for $n\leqq p-1$.*

Proof. The differentials in this spectral sequence are computed by comparing the resolution of $E^0S(n)_*$ with the cobar resolution of $S(n)_*$. The structure of the former is determined by the coproduct of $t_{i,j}$ for $i\leqq m$. Theorems 1.3 and 1.4 show that this coproduct corresponds precisely to that of $t_i^{p^i}$, i.e. the latter contains no terms of lower filtration. It follows that there are no nontrivial differentials in the spectral sequence. □

Note that Theorem 1.7 does not exclude the possibility of nontrivial extensions in the multiplicative structure of $H^* S(n)$.

(1.8) **Corollary.** *$E^0 H^* S(n) \simeq H^* L(n,n)$ for $n<p-1$.* □

The computation of $H^* L(n,k)$ for $k \leqq m$ may be carried out using the Koszul complex for a Lie algebra. A straightforward consequence of Theorem 1.4 is the following.

(1.9) **Theorem.** *$H^* L(n,k)$ for $k \leqq m$ is the cohomology of the exterior complex $E(h_{i,j})$ on one dimensional generators $h_{i,j}$ with $i \leqq k$ and $j \in \mathbb{Z}/(n)$, with coboundary*

$$dh_{i,j} = \sum_{0<s<i} h_{s,j} h_{i-s,s+j}.$$

The element $h_{i,j}$ corresponds to the element $x_{i,j}$ and therefore has filtration degree i and internal degree $2p^j(p^i-1)$.

Proof. This follows from standard facts about the cohomology of Lie algebras ([2] XIII, § 7). □

Since $L(n,k)$ is nilpotent its cohomology can be computed with a sequence of change of rings spectral sequences, i.e.

(1.10) **Theorem.** *There are spectral sequences with*

$$E_2 = E(h_{k,j}) \otimes H^* L(n,k-1) \Rightarrow H^* L(n,k)$$

and $E_3 = E_\infty$.

Proof. The spectral sequence is that of Hochschild-Serre (see [2], pp. 349–351) for the extension of Lie algebras

$$A_{n,k} \to L(n,k) \to L(n,k-1)$$

where $A_{n,k}$ is the abelian Lie algebra on $x_{k,j}$. Hence $H^* A_{n,k} = E(h_{k,j})$. The E_2-term, $H^*(L(n,k-1), H^* A_{n,k})$ is isomorphic to the indicated tensor product since the extension is central.

For the second statement, recall that the spectral sequence can be constructed by filtering the complex of Theorem 1.9 in the obvious way. Inspection of this filtered complex shows that $E_3 = E_\infty$. □

In addition to the spectral sequence of Theorem (1.5a), there is an alternative method of computing $H^* E_0 S(n)$. Define $\tilde{L}(n,k)$ for $k \leqq m$ to be the quotient of $PE_0 S(n)$ by the restricted sub-Lie algebra generated by the elements $x_{i,j}$ for $k<i \leqq m$, and define $F(n,k)$ to be the kernel of the extension

$$0 \to F(n,k) \to \tilde{L}(n,k) \to \tilde{L}(n,k-1) \to 0.$$

Let $H^* \tilde{L}(n,k)$ denote the cohomology of the restricted enveloping algebra of $\tilde{L}(n,k)$. Then we have

(1.11) **Theorem.** *There are change of rings spectral sequences converging to* $H^*\tilde{L}(n,k)$ *with*

$$E_2 = H^*F(n,k) \otimes H^*\tilde{L}(n,k-1)$$

where

$$H^*F(n,k) = \begin{cases} E(h_{k,j}) & \text{for } k > m-n \\ E(h_{k,j}) \otimes P(b_{k,j}) & \text{for } k \leqq m-n \end{cases}$$

and $H^*\tilde{L}(n,m) = H^*E_0S(n)$.

Proof. Again the spectral sequence is that given in Theorem XVI, 6.1 of [2]. As before, the extension is central, so the E_2-term is the indicated tensor product. The structure of $H^*F(n,k)$ follows from Theorem 1.4 and the last statement is a consequence of Theorem 1.6. □

§2. We begin the computation of $H^1S(n)$ with:

(2.1) **Lemma.** *$H^1E_0S(n)$ is generated by $\zeta_n = \sum_j h_{n,j}$; $\rho_n = \sum_j h_{2n,j}$ for $p=2$; and for $n>1$, $h_{1,j}$ for each $j \in \mathbb{Z}/(n)$.*

Proof. By Theorems 1.5a), and 1.6, $H^1E_0S(n) = H^1L(n,m)$. The indicated elements are nontrivial cycles by Theorem 1.9. It follows from Theorem 1.4 that $L(n,m)$ can have no other generators since $[x_{1,j}, x_{i-1,j+1}] = x_{i,j} - \delta^j_{i+j} x_{i,j+1}$. □

In order to pass to $H^1S(n)$ we need to produce primitive elements in $S(n)_*$ corresponding to ζ_n and ρ_n (the primitive $t_1^{p^j}$ corresponds to $h_{1,j}$). We will do this with the help of the determinant of a certain matrix. Recall that in Theorem 2.3 of [8] we showed that $S(n)_* \otimes \mathbb{F}_{p^n}$ was isomorphic to the dual group ring of a certain group which had a certain faithful representation over $W(\mathbb{F}_{p^n})$ ([8], Proposition 2.9). The determinant of this representation gave a homomorphism of $S(n)$ into $\mathbb{Z}_p^\times$, the multiplicative group of units in the p-adic integers. We will see that in H^1 this map gives us ζ_n and ρ_n.

More precisely let $M = (m_{i,j})$ be the n by n matrix over $\mathbb{Z}_p[t_1, t_2, \ldots]/(t_i - t_i^{p^n})$ given by

$$m_{i,j} = \begin{cases} \sum_{k \geqq 0} p^k t_{kn+j-i}^{p^i} & \text{for } i \leqq j \\ \sum_{k \geqq 1} p^k t_{kn+j-i}^{p^i} & \text{for } i > j \end{cases}$$

where $t_0 = 1$.

Now define $T_n \in S(n)_*$ to be the mod p reduction of p^{-1} (det $M - 1$) and for $p=2$ define $U_n \in S(n)_*$ to be the mod 2 reduction of $\frac{1}{8}$(det $M^2 - 1$). Then we have

(2.2) **Theorem.** *The elements $T_n \in S(n)_*$ and for $p=2$ $U_n \in S(n)_*$ are primitive and represent the elements ζ_n and $\rho_n + \zeta_n \in H^1S(n)$ respectively. Hence $H^1S(n)$ is generated by these elements and for $n>1$ by the $h_{1,j}$ for $j \in \mathbb{Z}/(n)$.*

Proof. The statement that T_n and U_n are primitive follows from Proposition 2.9 of [8]. That they represent ζ_n and $\rho_n+\zeta_n$ follows from the fact that $T_n \equiv \sum_j t_n^{p^j} \bmod (t_1, t_2, \ldots, t_{n-1})$ and $U_n \equiv \sum_j t_{2n}^{2^j} + t_n^{2^j} \bmod (t_1, t_2, \ldots, t_{n-1})$. □

Examples. $T_1=t_1$, $U_1=t_1+t_2$, $T_2=t_2+t_2^p-t_1^{1+p}$, $U_2=t_4+t_4^2+t_1t_3^2+t_1^2t_3+t_2+t_2^2t_1^3t_2+t_1^3t_2^2$, and $T_3=t_3+t_3^p+t_3^{p^2}+t_1^{1+p+p^2}-t_1t_2^p-t_1^p t_2^{p^2}-t_1^{p^2}t_2$.

Moreira ([7]) has found primitive elements in BP_*BP/I_n which reduce to our T_n. The following result is a corollary of Theorem 2.10 of [8].

(2.3) **Proposition.** *If $p \nmid n$, then $H^*S(n)$ decomposes as a tensor product of an appropriate subalgebra with $E(\zeta_n)$ for $p>2$ and $P(\zeta_n)\otimes E(\rho_n)$ for $p=2$.* □

We now turn to the computation of $H^2S(n)$ for $n>2$. We will compute all of $H^*S(n)$ for $n \leqq 2$ in §3.

(2.4) **Theorem.** *Let $n>2$*

a) *For $p=2$, $H^2S(n)$ is generated as a vector space by the elements ζ_n^2, $\rho_n\zeta_n$, $\zeta_n h_{1,j}$, $\rho_n h_{1,j}$, and $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, where $h_{1,i}h_{1,j}=h_{1,j}h_{,i}$ and $h_{1,i}^2 \neq 0$.*

b) *For $p>2$, $H^2S(n)$ is generated by the elements*

$$\zeta_n h_{1,i}, b_{1,i}, g_i=\langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle, \quad k_i=\langle h_{1,i+1}, h_{1,i+1}, h_{1,i}\rangle,$$

and $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, where $h_{1,i}h_{1,j}+h_{1,j}h_{1,i}=0$.

Both statements require a sequence of Lemmas. We treat the case $p=2$ first.

(2.5) **Lemma.** *Let $p=2$, $n>2$.*

a) *$H^1L(n,2)$ is generated by $h_{1,i}$ for $i \in \mathbb{Z}/(n)$.*

b) *$H^2L(n,2)$ is generated by the elements $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, g_i, k_i and $e_{3,i}=\langle h_{1,i}, h_{1,i+1}, h_{1,i+2}\rangle$. The latter elements are represented by $h_{1,i}h_{2,i}$, $h_{1,i+1}h_{2,i}$, and $h_{1,i}h_{2,i+1}+h_{2,i}h_{1,i+2}$ respectively.*

c) *$e_{3,i}h_{1,i+1}=h_{1,i}e_{3,i+1}+e_{3,i}h_{1,i}h_{1,i+3}=0$ and these are the only relations among the elements $h_{1,i}e_{3,j}$.*

Proof. We use the spectral sequence of Theorem 1.10, with $E_2=E(h_{1,i}, h_{2,i})$ and $d_2h_{2,i}=h_{1,i}h_{1,i+1}$. All three statements can be verified by inspection. □

(2.6) **Lemma.** *Let $p=2$, $n>2$, and $2<k \leqq 2n$*

a) *$H^1L(n,k)$ is generated by the elements $h_{1,i}$, along with ζ_n for $k \geqq n$ and ρ_n for $k=2n$.*

b) *$H^2L(n,k)$ is generated by products of elements in $H^1L(n,k)$ subject to $h_{1,i}h_{1,i+1}=0$, along with $g_i=\langle h_{1,i}, h_{1,i}, h_{1,i+1}\rangle$, $k_i=\langle h_{1,i}, h_{1,i+1}, h_{1,i+1}\rangle$, $\alpha_i=\langle h_{1,i}, h_{1,i+1}, h_{1,i+2}, h_{1,i+1}\rangle$ and $e_{k+1,i}=\langle h_{1,i}, h_{1,i+1}, \ldots, h_{1,i+k}\rangle$. The last two families of elements can be represented by $h_{3,i}h_{1,i+1}+h_{2,i}h_{2,i+1}$ and $\sum_s h_{s,i}h_{k+1-s,i+1}$ respectively.*

c) *$h_{1,i}e_{k+1,i+1}+e_{k+1,i}h_{1,i+1+k}=0$ and no other relations hold among products of the $e_{k+1,i}$ with elements of H^1.*

Proof. Again we use Theorem 1.10 and argue by induction on k, *using Lemma* 2.5 *to start the induction. We have* $E_2=E(h_{k,i})\otimes H^*L(n,k-1)$ with $d_2 h_{k,i}=e_{k,i}$. The existence of the α_i follows from the relation $e_{3,i}h_{1,i+1}=0\in H^3L(n,2)$ and that of $e_{k+1,i}$ from $h_{1,i}e_{k,i+1}+e_{k,i}h_{1,i+k}=0\in H^3L(n,k-1)$. The relation c) for $k>2$ is formal; it follows from a Massey product identity (Corollary 3.2 of [4]) or can be verified by direct calculation in the complex of Theorem 1.9. No combination of these products can be in the image of d_2 for degree reasons. □

(2.7) **Lemma.** *Let* $p=2$, $n>2$. *Then* $H^2E_0S(n)$ *is generated by the elements* $\rho_n\zeta_n$, $\rho_n h_{1,i}$, $\zeta_n h_{1,i}$, $h_{1,i}h_{1,j}$ for $i\neq j\pm 1$, α_i, *and* $h_{i,j}^2=b_{i,j}$ *for* $1\leqq i\leqq n$, $j\in\mathbb{Z}/(n)$.

Proof. We use the modified first May spectral sequence of Theorem 1.6. We have $m=2n$ and $H^2L(n,m)$ is given by Lemma 2.6. By easy direct computation one sees that $d_2 g_i=b_{1,i}h_{1,i+1}$ and $d_2 k_i=h_{1,i}b_{1,i+1}$. We will show that $d_2 e_{2n+1,i}=h_{1,i}b_{n,i}+h_{1,i+n}b_{n,i-1}$. We need a slight refinement of Theorem 1.3, i.e. that

$$\Delta(t_{2n+1})=\sum t_j\otimes t_{2n+1-j}^{p^j}+C_{2^n}(t_{n+1}\otimes 1,t_n\otimes t_1^{2^n},\ldots,1\otimes t_{n+1})$$

modulo terms of lower filtration. This can be derived from Theorem 1.9 of [8]. Then by direct computation in the cobar construction one can show that

$$dC_{2^n}(t_{n+1}\otimes 1,t_n\otimes t_1^{2^n},\ldots,1\otimes t_{n+1})=t_1\otimes C_{2^{n-1}}(t_n\otimes 1,t_{n-1}\otimes t_1^{2^{n-1}},\ldots,1\otimes t_n)$$
$$+C_{2^n}(t_n\otimes 1,\ldots,1\otimes t_n)\otimes t_1$$

modulo terms of lower filtration and the nontriviality of $d_2 e_{2b+1,i}$ follows. □

Proof of Theorem 2.4a). We now consider the second May spectral sequence (Theorem 1.5b)). By a direct computation in the filtered cobar construction similar to that of the above proof, one can show that $d_2 b_{i,j}=h_{1,j+1}b_{i-1,j+1}+h_{1,i+j}\neq 0$ for $i>1$. The remaining elements of $H^2E_0S(n)$ survive either for degree reasons or by Theorem 2.2. □

For $p>2$ we need an analogous sequence of Lemmas. We leave the proofs to the reader.

(2.8) **Lemma.** *Let* $n>2$ *and* $p>2$.

a) $H^1L(n,2)$ *is generated by* $h_{1,i}$.

b) $H^2L(n,2)$ *is generated by the elements* $h_{1,i}h_{1,j}$ (*with* $h_{1,i}h_{1,i+1}=0$), $g_i=h_{1,i}h_{2,i}$, $k_i=h_{1,i+1}h_{2,i}$, $e_{3,i}=h_{1,i}h_{2,i+1}+h_{2,i}h_{1,i+2}$.

c) *The only relations among the elements* $h_{1,i}e_{3,j}$ *are* $h_{1,i}e_{3,i+1}-e_{3,i}h_{1,i+3}=0$. □

(2.9) **Lemma.** *Let* $n>2$, $p>2$, *and* $2<k\leqq m$. *Then*

a) $H^1L(n,k)$ *is generated by* $h_{1,i}$ *and for* $k\geqq n$, ζ_n.

b) $H^2L(n,k)$ *is generated by* $h_{1,i}h_{1,j}$ (*with* $h_{1,i}h_{1,i+1}=0$),

$$g_i,k_i,e_{k+1,i}=\sum_{0<j<k+1}h_{j,i}h_{k+1-j,i+j},$$

and, for $k\geqq n$, $\zeta_n h_{1,i}$.

c) *The only relations among products of elements in* H^1 *with the* $e_{k+1,i}$ *are* $h_{1,i}e_{k+1,i+1}-e_{k+1,i}h_{1,k+1}=0$. □

(2.10) **Lemma.** *Let $n>2$ and $p>2$. Then $H^2E_0S(n)$ is generated by the elements $b_{i,j}$ for $i\leqq m-n$ and by the elements of $H^2L(n,m)$.* □

Proof of Theorem 2.4b. Again we look at the spectral sequence of Theorem 1.5b). By arguments similar to those for $p=2$ one can show that $d_p b_{i,j}=h_{1,i+j}b_{i-1,j}-h_{1,j+1}b_{i-1,j+1}$ for $i>1$ and $d_s e_{m+1,i}=h_{1,m+i-n}b_{m-n,i-1}-h_{1,i}b_{m-n,i}$ where $s=1+pn-(p-1)m$, and the remaining elements of $H^2E_0S(n)$ survive as before. □

§3. In this section we will compute $H^*S(n)$ at all primes for $n\leqq 2$ and at $p>3$ for $n=3$.

(3.1) **Theorem.**

a) $H^*S(1)=P(h_{1,0})\otimes E(\rho_1)$ *for* $p=2$;
b) $H^*S(1)=E(h_{1,0})$ *for* $p>2$

(*note that $S(1)$ is commutative and that $\zeta_1=h_{1,0}$*).

Proof. This follows immediately from Theorems 1.4, 1.6, and 1.7. □

(3.2) **Theorem.** *For $p>3$, $H^*S(2)$ is the tensor product of $E(\zeta_2)$ with the subalgebra with basis $\{1, h_{1,0}, h_{1,1}, g_0, g_1, g_0h_{1,1}\}$ where $g_i=\langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle$; $h_{1,0}g_1=g_0h_{1,1}$, $h_{1,0}g_0=h_{1,1}g_1=0$, and $h_{1,0}h_{1,1}=h_{1,0}^2=h_{1,1}^2=0$. In particular the Poincaré series is $(1+t)^2(1+t+t^2)$.*

Proof. The computation of $H^*L(2,2)$ by Propositions 1.9 or 1.10 is elementary, and there are no algebra extension problems for the spectral sequences of Proposition 1.10 or Theorem 1.5b). □

We will now compute $H^*S(2)$ for $p=3$. First we need some notation. Let

$$R=E(h_{1,0},h_{1,1})\otimes P(b_{1,0},b_{1,1})/(h_{1,0}h_{1,1},b_{1,0}^2+b_{1,1}^2,h_{1,0}b_{1,0}-h_{1,1}b_{1,1},h_{1,1}b_{1,0}+h_{1,0}b_{1,1})$$

and define a class $\xi\in H^2S(2)$ as a matric Massey product (see [4])

$$\xi=\left\langle (h_{1,0}\,h_{1,1}),\ \begin{pmatrix}h_{1,0} & -h_{1,1}\\ h_{1,1} & h_{1,0}\end{pmatrix},\ \begin{pmatrix}h_{1,1} & -h_{1,0}\\ h_{1,0} & h_{1,1}\end{pmatrix},\ \begin{pmatrix}h_{1,0}\\ h_{1,1}\end{pmatrix}\right\rangle .$$

That this class is well defined will become evident in the proof of the following result.

(3.3) **Theorem.** *For $p=3$, $H^*S(2)$ is isomorphic as an algebra to $E(\zeta_2,\xi)\otimes R$, where R and ξ are as defined above, and ζ_2 is as defined in Theorem 2.2. In particular the Poincaré series is*

$$(1+t)^2(1+t^2)/(1-t).$$

Proof. Our basic tools are the spectral sequences of Proposition 1.11 and some Massey product identities from [4]. We have $H^*\tilde{L}(2,1)\simeq E(h_{1,0},h_{1,1})\otimes P(b_{1,0},b_{1,1})$, and a spectral sequence converging to $H^*\tilde{L}(2,2)$ with $E_2=E(\zeta_2,\eta)\otimes H^*\tilde{L}(2,1)$, where $\zeta_2=h_{2,0}+h_{2,1}$ and $\eta=h_{2,1}-h_{2,0}$, $d_2\zeta_2=0$, $d_2\eta=h_{1,0}h_{1,1}$ and $E_3=E_\infty$. Hence E_∞ is a free module over $E(\zeta_2)\otimes P(b_{1,0},b_{1,1})$ on generators 1, $h_{1,0}$, $h_{1,1}$, g_0, g_1, and $h_{1,0}g_1=h_{1,1}g_0$, where $g_i=\langle h_{1,i},h_{1,i+1},h_{1,i}\rangle$. This determines the additive structure of $H^*\tilde{L}(2,2)$, but there are some nontrivial extensions in the multiplicative structure. We know by Proposition 2.3 that we can factor out $E(\zeta_2)$, and we can write $b_{1,i}$ as the Massey product $-\langle h_{1,i}, h_{1,i}, h_{1,i}\rangle$. Then by Corollary 3.2 of [4] we have $h_{1,i}g_i = -b_{1,i}h_{1,i+1}$, $h_{1,i}g_{i+1} = h_{1,i+1}b_{1,i}g_1^2 = -b_{1,i}g_{i+1}$, $g_i g_{i+1} = b_{1,i}b_{1,i+1}$. These facts along with the usual $h_{1,i}^2 = h_{1,0}h_{1,1} = 0$ determine $H^*\tilde{L}(2,2)$ as an algebra.

Next we have a spectral sequence converging to $H^*\tilde{L}(2,3) \cong H^*E^0S(2)$ (by Theorem 1.6) with $E_2 = E(h_{3,0}, h_{3,1}) \otimes H^*\tilde{L}(2.2)$ and $d_2(h_{3,i}) = g_i - b_{1,i+1}$ and $E_3 = E_\infty$. The computation of E_3 is essentially routine and there are no ambiguities in the algebra structure of $H^*\tilde{L}(2,3)$. The spectral sequence of Theorem 1.5b) collapses by Theorem 1.7, and the only ambiguity in the multiplicative structure of $H^*S(2)$ is the value of c in the expression $\xi^2 = cb_{1,0}b_{1,1}$. We will show below that $c=0$.

The computation of $H^*\tilde{L}(2,3)$ is clarified by the following construction. The subring of $H^*\tilde{L}(2,2)$ generated by $b_{1,0}, g_0, g_1$, and $b_{1,1}$ can be mapped isomorphically to the subring of $\mathbb{F}_3[s_0, s_1]$ generated by $-s_0^3, s_0^2 s_1, s_0 s_1^2$, and $-s_1^3$ respectively. Multiplication of these elements in $H^*\tilde{L}(2,2)$ by $h_{1,i}$ corresponds to multiplication of the corresponding polynomials by s_i. At this point it is convenient to tensor with $\mathbb{F}_9$ and perform a change of basis. Let $x = s_0 + is_1$, $y = s_0 - is_1$, where $i^2 = -1$. Then the elements x^3, x^2y, xy^2, and y^3 correspond to $-b_{1,0} + ib_{1,1}$, $-b_{1,0} + ig_0 + g_1 - ib_{1,1}$, $-b_{1,0} - ig_0 + g_1 + ib_{1,1}$, and $-b_{1,0} - ib_{1,1}$ respectively. In the spectral sequence for $H^*\tilde{L}(2,3) \otimes \mathbb{F}_9$, the elements $d_2(h_{3,0} + ih_{3,1})$ and $d_2(h_{3,0} - ih_{3,1})$ correspond to ix^2y and $-iy^2x$ respectively, and the element ξ is represented by $h_{1,0}h_{3,0} + h_{3,1}h_{1,1}$. Over $\mathbb{F}_9$, ξ is the ordinary Massey product $\langle u, v, u, v\rangle = \langle v, u, v, u\rangle$ where $u = h_{1,0} + ih_{1,1}$ and $v = h_{1,0} - ih_{1,1}$. By Proposition 2.9 of [4], ξ can be rewritten as

$$\frac{1}{2}\left\langle (u,v), \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} v \\ u \end{pmatrix} \right\rangle .$$

The appropriate change of coordinates in each matrix yields the expression indicated preceding the statement of the Theorem.

With these observations in mind it is easy to see that $H^*\tilde{L}(2,3) = H^*E_0S(2)$ has the indicated structure.

It remains to be shown that $\xi^2 = 0$ in $H^*S(2)$. For degree reasons we have $\xi^2 = cb_{1,0}b_{1,1}$. We will construct a Hopf algebra T and a map $f\colon T \to S(2) \otimes \mathbb{F}_9$ such that $f^*(\xi) = 0$ and $f^*(b_{1,0}b_{1,1}) \neq 0$. Recall (Theorem 2.3 of [8]) that $S(2) \otimes \mathbb{F}_9 \cong \mathbb{F}_9[S_2]$ where S_2 is a certain compact 3-adic Lie group. S_2 is the group of proper (congruent to 1 modulo the maximal ideal) units of the noncommutative degree 4 extension E_2 of $\mathbb{Z}_3$ obtained by adjoining i and S with $Si = -iS$, $i^2 = -1$, and $S^2 = 3$. S_2 has elements of order 3, e.g. $-\frac{1}{2} + \frac{(1+i)\sqrt{-2}}{4}S$, so we have a map f: $\mathbb{F}_9[\mathbb{Z}/(3)] \to S(2) \otimes \mathbb{F}_9$ and dually a map $f_*\colon S(2)_* \otimes \mathbb{F}_9 \to \mathbb{F}_9[z]/(z^3 - z)$ where z is primitive and $f_*(t_1) = (1+i)z$. Hence $f_*(t_1 - it_1^3) = 0$, so $f^*(\xi) = 0$. On the other hand it is easy to check that $f^*(b_{1,0}b_{1,1}) \neq 0$, so $c = 0$. □

We now turn to the case $n = p = 2$. We will only compute $E^0H^*S(2)$, so there will be some ambiguity in the multiplicative structure of $H^*S(2)$. In order to state our result we need to define some classes. Recall (Theorem 2.2) that $H^1 S(2)$ is the $\mathbb{F}_2$-vector space generated by $h_{1,0}, h_{1,1}$, ζ_2 and ρ_2. Let $\alpha_0 \in \langle \zeta_2, h_{1,0}, h_{1,1} \rangle$, $\beta \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle$, $g = \langle h, h^2, h, h^2 \rangle$ where $h = h_{1,0} + h_{1,1}$, $\tilde{x} + \langle x, h, h^2 \rangle$ for $x = \zeta_2, \alpha_0, \zeta_2^2, \alpha_0, \zeta_2$ (more precise definitions of α_0 and β will be given in the proof).

(3.4) **Theorem.** *$E^0 H^* S(2)$ for $p=2$ is a free module over $P(g) \otimes E(\rho_2)$ on 20 generators:* $1, h_{1,0}, h_{1,1}, h_{1,0}^2, h_{1,1}^2, h_{1,0}^3, \beta, \beta h_{1,0}, \beta h_{1,1}, \beta h_{1,0}^2, \beta h_{1,1}^2, \beta h_{1,0}^3, \zeta_2, \alpha_0, \zeta_2^2, \alpha_0 \zeta_2, \tilde{\zeta}_2, \tilde{\alpha}_0, \tilde{\zeta}_2^2, \widetilde{\alpha_0 \zeta_2}$, *where* $\alpha_0 \in H^2 S(2)$ *and has filtration degree* 4, $\beta \in H^3 S(2)$ *and has filtration degree* 8, $g \in H^4 S(2)$ *and has filtration degree* 8, *and the cohomological and filtration degrees of* $\tilde{x}$ *exceed those of* x *by* 2 *and* 4 *respectively. Moreover* $h_{10}^3 = h_{11}^3$, $\alpha_0^2 = \tilde{\zeta}_2^2$, *and all other products are zero. The Poincaré series is* $\dfrac{(1+t)^2(1-t^5)}{(1-t)^2(1+t^2)}$.

Proof. We will use the same notation for corresponding classes in the various cohomology groups we will be considering along the way.

Again our basic tool is Proposition 1.11. It follows from Remark 10 of [3] that $H^* E_0 S(2)$ is the cohomology of the complex $P(h_{1,0}, h_{1,1}, \zeta_2, h_{2,0}) \otimes E(h_{3,0}, h_{3,1}, \rho_2, h_{4,0})$ with $dh_{1,i} = d\zeta_2 = d\rho_2 = 0$, $dh_{3,i} = h_{1,i}\zeta_2$, $dh_{2,0} = h_{1,0} h_{1,1}$, and $dh_{4,0} = h_{1,0} h_{3,1} + h_{1,1} h_{3,0} + \zeta_2^2$. This fact will enable us to solve the algebra extension problems in the spectral sequences of Proposition 1.11.

For $H^* \tilde{L}(2,2)$ we have a spectral sequence with $E_2 = P(h_{1,0}, h_{1,1}, \zeta_2, h_{2,0})$ with $d_2 \zeta_2 = 0$ and $d_2 h_{2,0} = h_{1,0} h_{1,1}$. It follows easily that

$$H^* \tilde{L}(2,2) = P(h_{1,0}, h_{1,1}, \zeta_2, b_{2,0})/(h_{1,0} h_{1,1})$$

where $b_{2,0} = h_{2,0}^2 = \langle h_{1,0}, h_{1,1}, h_{1,0}, h_{1,1} \rangle$.

For $H^* \tilde{L}(2,3)$ we have a spectral sequence with $E_2 = E(h_{3,0}, h_{3,1}) \otimes H^* \tilde{L}(2,2)$ and $d_2 h_{3,i} = h_{1,i} \zeta_2$. Let $\alpha_i = h_{1,i+1} h_{3i} + \zeta_2 h_{2,i} \in \langle \zeta_2, h_{1,i}, h_{1,i+1} \rangle$. Then $H^* \tilde{L}(2,3)$ as a module over $H^* \tilde{L}(2,2)$ is generated by $1, \alpha_0$ and α_1 with $\zeta_2 h_{1,i} = \zeta_2(\alpha_0 + \alpha_1 + \zeta_2^2) = h_{1,i} \alpha_i = \zeta_2 h_{1,i+1} \alpha_i = 0$, and $\alpha_0^2 = \zeta_2^2 b_{2,0}, \alpha_1^2 = \zeta_2^2(\zeta_2^2 + b_{2,0})$, $\alpha_0 \alpha_1 = \zeta_2^2(\alpha_0 + b_{2,0})$. The Poincaré series for $H^* \tilde{L}(2,3)$ is $(1+t+t^2)/(1-t)^2$.

For $H^* \tilde{L}(2,4)$ we have a spectral sequence with $E_2 = E(h_{4,0}, \rho_2) \otimes H^* \tilde{L}(2,3)$ and $d_2 \rho_2 = 0$, $d_2 h_{4,0} = \alpha_0 + \alpha_1$. Define $\beta \in H^3 \tilde{L}(2,4)$ by $\beta = h_{4,0}(\alpha_0 + \alpha_1 + \zeta_2^2) + \zeta_2 h_{3,0} h_{3,1} \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1} \rangle$. Then $H^* \tilde{L}(2,4)$ is a free module over $E(\rho_2) \otimes P(b_{2,0})$ on generators 1, $h_{1,i}^t, \zeta_2, \zeta_2^2, \alpha_0, \alpha_0 \zeta_2, \beta$, and $\beta\ h_{1,i}^t$ where $t>0$. As a module over $H^* \tilde{L}(2,3) \otimes E(\rho_2)$ it is generated by 1 and β, with $(\alpha_0 + \alpha_1)1 = \zeta_2^3(1) = \alpha_0 \zeta_2^2(1) = 0$. To solve the algebra extension problem we observe that $\beta \zeta_2 = 0$ for degree reasons; $\beta \alpha_i = \beta \langle \zeta_2, h_{1,i}, h_{1,i+1} \rangle = \langle \beta, \zeta_2, h_{1,i} \rangle h_{1,i+1} = 0$ since $\langle \beta, \zeta_2, h_{1,i} \rangle = 0$ for degree reasons; and $E(\rho_2)$ splits off multiplicatively by the remarks at the beginning of the proof.

This completes the computation of $H^* E_0 S(2)$. Its Poincaré series is $(1+t)^2/(1-t)^2$. We now use the second May spectral sequence (Theorem (1.5b)) to pass to $E^0 H^* S(2)$. $H^* E_0 S(2)$ is generated as an algebra by the elements $h_{1,0}$, $h_{1,1}$, ζ_2, ρ_2, α_0, $b_{2,0}$ and β. The first four of these are premanent cycles by Theorem 2.2.

By direct computation in the cobar resolution we have

(3.4.1) $d(t_3 + t_1 t_2^2) = \zeta_2 \otimes t_1$, so the Massey product for α_0 is defined in $H^*S(2)$ and the α_0 is a permanent cycle. We also have $d(t_2\otimes t_2+t_1\otimes t_1^2 t_2+t_1 t_2\otimes t_1^2)=t_1\otimes t_1\otimes t_1+t_1^2\otimes t_1^2\otimes t_1^2$, so $d_2 b_{2,0}=h_{1,0}^3+h_{1,1}^3$. Inspection of the E_3 term shows that $b_{2,0}^2=\langle h, h^2, h, h^2\rangle$, (where $h=h_{1,0}+h_{1,1}$) is a permanent cycle for degree reasons.

We now show that $\beta=\langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1}\rangle$ is a permanent cycle by showing that its Massey product expression is defined in $E^0H^*S(2)$. The products $h_{1,0}\zeta_2$ and $\zeta_2^2 h_{1,1}$ are zero by (3.4.1), and we have

(3.4.2) $$d(\tilde{t}_3\otimes\tilde{t}_3^2+T_2\tilde{t}_3\otimes t_1^2+T_2\otimes t_4+T_2\otimes t_2^3+T_2\otimes t_1^3(1+t_2+t_2^2)=T_2\otimes T_2\otimes T_2,$$

where $\tilde{t}_3=t_3+t_1 t_2^2$ and $T_2=t_2+t_2^2+t_1^3$, so $\zeta_2^3=0\in H^*S(2)$. Inspection of $H^3E_0S(2)$ shows there are no elements of internal degree 2 or 4 and filtration degree >7, so the triple products $\langle h_{1,0}, \zeta_2, \zeta_2^2\rangle$ and $\langle \zeta_2, \zeta_2^2, h_{1,1}\rangle$ must vanish and β is a permanent cycle.

Now the E_3 term is a free module over $E(\rho_2)\otimes P(b_{2,0}^2)$ on 20 generators: $1, h_{1,0}, h_{1,1}, h_{1,0}^2, h_{1,1}^2, h_{1,0}^3=h_{1,1}^3, \beta, \beta h_{1,0}, \beta h_{1,1}, \beta h_{1,1}^2, \beta h_{1,0}^2, \beta h_{1,0}^3, \zeta_2, \alpha_0, \zeta_2^2, \alpha_0\zeta_2, \zeta_2 b_{2,0}, \alpha_0 b_{2,0}, \zeta_2^2 b_{2,0}$, and $\zeta_2\alpha_0 b_{2,0}$. The last 4 in the list now have Massey product expressions $\langle\zeta_2, h, h^2\rangle$, $\langle\alpha_0, h, h^2\rangle$, $\langle\zeta_2^2, h, h^2\rangle$, and $\langle\alpha_0\zeta_2, h, h^2\rangle$ respectively. These elements have to be permanent cycles for degree reasons, so $E_3=E_\infty$, and we have determined $E^0H^*S(2)$. □

We now describe an alternate method of computing $H^*S(2)\otimes\mathbb{F}_4$, which is quicker than the previous one, but yields less information about the multiplicative structure. By Corollary 2.7 of [8], this group is isomorphic to $H^*(S_2; \mathbb{F}_4)$, the continuous cohomology of certain 2-adic Lie group with trivial coefficients in $\mathbb{F}_4$. S_2 is the group of units in the degree 4 extension E_2 of $\mathbb{Z}_2$ obtained by adjoining ω and S with $\omega^2+\omega+1=0$, $S^2=2$, and $S\omega=\omega^2 S$.

Let Q denote the quaternion group, i.e. the multiplicative group (with 8 elements) of quaternionic integers of modulus 1.

(3.5) **Proposition.** *There is a split short exact sequence of groups*

(3.5.1) $$1\to G\xrightarrow{i} S_2\xrightarrow{j} Q\to 1.$$

The corresponding extension of dual group algebras over $\mathbb{F}_4$ *is*

$$\mathbb{F}_4\to Q_*\xrightarrow{j_*} S(2)_*\xrightarrow{i_*} G_*\to\mathbb{F}_4$$

where $Q_*\cong\mathbb{F}_4[x, y]/(x^4-x, y^2-y)$ *and* $G_*\cong S(2)_*/(t_1, t_2+\bar{\omega}t_2^2)$ *as algebras where* $j_*(x)=t_1$, $j_*(y)=\bar{\omega}t_2+\bar{\omega}^2 t_2^2$ *and* $\bar{\omega}$ *is the residue class of* ω.

Proof. The splitting follows the theory of division algebras over local fields ([9], pp. 137–138) which implies that $E_2\otimes\mathbb{Q}_2$ is isomorphic to the 2-adic quaternions. We leave the remaining datails to the reader. □

(3.6) **Proposition.** a) $H^*(Q; \mathbb{F}_2)=P[h_{1,0}, h_{1,1}, g]/(h_{1,0}h_{1,1}, h_{1,0}^3+h_{1,1}^3)$.

b) $H^*(G; \mathbb{F}_2)=E(\zeta_2, \rho_2, h_{3,0}, h_{3,1})$.

Proof. a) is an easy calculation with the change of rings spectral sequence ([2], p. 349) for $\mathbb{F}_2[x]/(x^4+x)\to Q_*\to\mathbb{F}_2[y]/(y^2+y)$. For b) the filtration on $S(2)_*$ induces one on G_*. It is easy to see that E^0G_* is cocommutative and the result follows with no difficulty. □

(3.7) **Proposition.** *In the Cartan-Eilenberg spectral sequence for* (3.5.1), $E_3=E_\infty$ *and we get the same additive structure for* $H^*S(2)$ *as in Theorem* 3.4.

Proof. We can take $H^*G\otimes H^*Q$ as our E_1-term. Each term is a free module over $E(\rho_2)\otimes P(g)$. We leave the evaluation of the differentials to the reader. □

Finally we consider the case $n=3$ and $p\geqq 5$. We will not make any attempt to describe the multiplicative structure as it is quite tedious and of little interest. An explicit basis of $E^0H^*S(3)$ will be given in the proof, from which the multiplication can be read off by the interested reader. It seems unlikely that there are any nontrivial multiplicative extensions.

(3.8) **Theorem.** *For* $p\geqq 5$, $H^*S(3)$ *has the following Poincaré series:* $(1+t)^3(1+t+6t^2+3t^3+6t^4+t^5+t^6)$.

Proof. We use the spectral sequences of Proposition 1.10 to compute $H^*L(3,2)$ and $H^*L(3,3)$. For the former the E_2-term is $E(h_{1,i})\otimes E(h_{2,i})$ with $i\in\mathbb{Z}/(3)$, $d_2h_{1,i}=0$ and $d_2h_{2,i}=h_{1,i}h_{1,i+1}$. The Poincaré series for $H^*L(3,2)$ is $(1+t)^2(1+t+5t^2+t^3+t^4)$ and it is generated as a vector space by the following elements and their Poincaré duals: 1, $h_{1,i}$, $g_i=h_{1,i}h_{2,i}$, $k_i=h_{2,i}h_{1,i+1}$, $e_{3,i}=h_{1,i}h_{2i+1}+h_{2,i}h_{1,i+2}$ (where $\sum_i e_{3,i}=0$), $g_ih_{1,i+1}=h_{1,i}k_i=h_{1,i}h_{2,i}h_{1,i+1}$ and $h_{1,i}e_{3,i}=g_ih_{1,i+2}=h_{1,i}h_{2,i}h_{1,i+2}$.

For $H^*L(3,3)$ we have $E_2=E(h_{3,i})\otimes H^*L(3,2)$ with $d_2h_{3,i}=e_{3,i}$, so $d_2\sum h_{3,i}=0$. $H^*L(3,3)$ has the indicated Poincare series and is a free module over $E(\zeta_3)$, where $\zeta_3=\sum h_{3,i}$, on the following 38 elements and the duals of their products with ζ_3: 1, $h_{1,i}$, g_i, k_i, $b_{1,i+2}=h_{1,i}h_{3,i+1}+h_{2,i}h_{2,i+2}+h_{3,i}h_{1,i}$, $g_ih_{1,i+1}=h_{1,i}k_i$, $h_{1,i}h_{2,i}h_{2,i+2}$, $h_{1,i}h_{2,i}h_{2,i+1}+h_{1,i}h_{1,i+1}h_{3,i}$, $h_{1,i}h_{2,i}h_{3,i}$, $h_{1,i}h_{2,i+2}h_{3,i+1}$, $\sum_i(h_{1,i}h_{2,i+1}-h_{1,i+1}h_{2,i+2})h_{3,i}$, $h_{1,i}k_ih_{3,j}$ (where $h_{1,i}k_i\sum_j h_{3,j}$ is divisible by ζ_3), and $h_{1,i+2}h_{1,i}h_{2,i}(h_{3,i}+h_{3,i+1})\pm h_{1,i}h_{2,0}h_{2,1}h_{2,2}$. □

References

1. Adams, J.F.: Stable homotopy and generalized homology. Chicago: University of Chicago Press 1974
2. Cartan, H., Eilenberg, S.: Homological Algebra. Princeton: Princeton University Press 1956
3. May, J.P.: The cohomology of restricted Lie algebras and of Hopf algebras. J. Algebra **3**, 123–146 (1966)
4. May, J.P.: Matric Massey products. J. Algebra **12**, 533–568 (1969)
5. Miller, H.R., Ravenel, D.C.: Morava stabilizer algebras and the localization of Novikov's E_2-term. Preprint
6. Morava, J.: Structure theorems for cobordism comodules (to appear)
7. Moreira, M.R.F.: Primitives of BP_*BP modulo an invariant prime ideal. M.I.T. Thesis, 1975
8. Ravenel, D.C.: The structure of Morava stabilizer algebras. To appear in Inventiones math.
9. Serre, J.-P.: Local class field theory. In: Algebraic number theory (Brighton 1965), pp. 128–161. New York-London: Academic Press 1967

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