A solution to the Arf-Kervaire invariant problem III: The Gap Theorem

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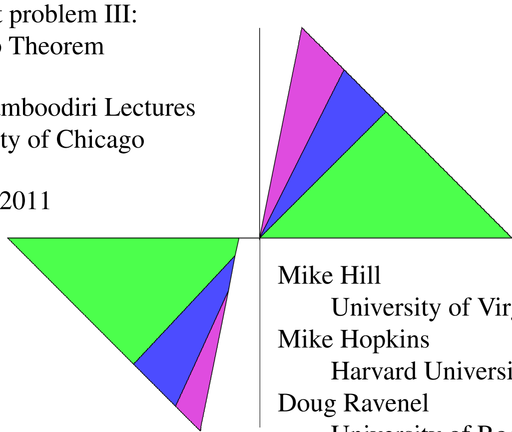

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University of Virginia
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1 Our strategy

1.1 The main theorem

The main theorem

Main Theorem. *The Arf-Kervaire elements* $\theta_j \in \pi_{2^{j+1}-2+n}(S^n)$ *for large n do not exist for* $j \geq 7$. 3.2

To prove this we produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. This means that if θ_j exists, we will see its image in $\pi_*(\Omega)$.

(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.

(iii) Gap Theorem. $\pi_{-2}(\Omega) = 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence and is the subject of this talk.

1.2 How we construct Ω

How we construct Ω

Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum MU. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The notation $MU_{\mathbf{R}}$ (real complex cobordism) is used to denote MU regarded as a C_2-spectrum.

MU is the Thom spectrum for the universal complex vector bundle, which is defined over the classifying space of the stable unitary group, BU.

- MU has an action of the group C_2 via complex conjugation. The resulting C_2-spectrum is denoted by $MU_{\mathbf{R}}$
- $H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_i : i > 0]$ where $|b_i| = 2i$.
- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring. 3.3

How we construct Ω (continued)

Given a spectrum X acted on by a group H of order h and a group G of order g containing H, there are two formal ways to construct a G-spectrum from X:

(i) The transfer. The spectrum

$$Y = G_+ \wedge_H X \qquad \text{underlain by} \qquad \bigvee_{g/h} X$$

has an action of G which permutes the wedge summands, each of which is invariant under H. This is used to construct our slice cells

$$\widehat{S}(m\rho_H) = G_+ \wedge_H S^{m\rho_H}.$$

(ii) The norm. The spectrum

$$N_H^G X \qquad \text{underlain by} \qquad \bigwedge_{g/h} X$$

has an action of G which permutes the smash factors, each of which is invariant under H. This was described in the last lecture.

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How we construct Ω (continued)

In particular for $G = C_8$ and $H = C_2$ we get a G-spectrum

$$MU_{\mathbf{R}}^{(4)} = N_H^G MU_{\mathbf{R}}.$$

It has homotopy groups $\pi_\star^G MU_{\mathbf{R}}^{(4)}$ indexed by the representation ring $RO(G)$.

Let ρ_G denote the regular representation of G. We form a G-spectrum $\tilde{\Omega}$ by inverting a certain element

$$D \in \pi_{19\rho_G} MU_{\mathbf{R}}^{(4)}.$$

Our spectrum Ω is its fixed point set,

$$\Omega = \tilde{\Omega}^G.$$

3.5

2 MU

2.1 Basic properties

The slice filtration on $N_H^G MU_{\mathbf{R}}$

We want to study

$$MU_{\mathbf{R}}^{(2^n)} = N_H^G MU_{\mathbf{R}} \qquad \text{where } H = C_2 \text{ and } G = C_{2^{n+1}}.$$

The homotopy of the underlying spectrum is

$$\pi_*^u MU_{\mathbf{R}}^{(2^n)} \mathbf{Z}[\gamma^j r_i \colon i > 0, 0 \leq j < 2^n] \qquad \text{where } |r_i| = 2i.$$

It has a slice filtration and we need to identify the slices. The following notion is helpful.

Definition. *Suppose X is a G-spectrum such that its underlying homotopy group $\pi_k^u(X)$ is free abelian. A refinement of $\pi_k^u(X)$ is an equivariant map*

$$c \colon \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cells of dimension k whose underlying spheres represent a basis of $\pi_k^u(X)$.

3.6

2.2 Refining homotopy

The refinement of $\pi_*^u(MU_{\mathbf{R}}^{(4)})$

Recall that $\pi_*(MU) = \pi_*^u(MU_{\mathbf{R}})$ is concentrated in even dimensions and is free abelian. $\pi_{2k}^u(MU_{\mathbf{R}})$ is refined by an map from a wedge of copies of $\widehat{S}(k\rho_2)$.

$\pi_*^u(MU_{\mathbf{R}}^{(4)})$ is a polynomial algebra with 4 generators in every positive even dimension. We will denote the generators in dimension $2i$ by $r_i(j)$ for $1 \le j \le 4$. The action of a generator $\gamma \in G = C_8$ is given by

$$r_i(1) \longrightarrow r_i(2) \longrightarrow r_i(3) \longrightarrow r_i(4) \xrightarrow{(-1)^j} r_i(1)$$

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The refinement of $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

$$r_i(1) \longrightarrow r_i(2) \longrightarrow r_i(3) \longrightarrow r_i(4) \xrightarrow{(-1)^j} r_i(1)$$

We will explain how $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ can be refined.

$\pi_2^u(MU_{\mathbf{R}}^{(4)})$ has 4 generators $r_1(j)$ that are permuted up to sign by G. It is refined by an equivariant map

$$\widehat{W}_1 = \widehat{S}(\rho_2) = C_{8+} \wedge_{C_2} S^{\rho_2} \to MU_{\mathbf{R}}^{(4)}.$$

Note that the slice cell $\widehat{S}(\rho_2)$ is underlain by a wedge of 4 copies of S^2. 3.8

The refinement of $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

$$r_i(1) \longrightarrow r_i(2) \longrightarrow r_i(3) \longrightarrow r_i(4) \xrightarrow{(-1)^j} r_i(1)$$

In $\pi_4^u(MU_{\mathbf{R}}^{(4)})$ there are 14 monomials that fall into 4 orbits (up to sign) under the action of G, each corresponding to a map from a $\widehat{S}(m\rho_h)$.

$$\begin{aligned}
\widehat{S}(2\rho_2) = C_{8+} \wedge_{C_2} S^{2\rho_2} &\quad\longleftrightarrow\quad \{r_1(1)^2, r_1(2)^2, r_1(3)^2, r_1(4)^2\} \\
\widehat{S}(2\rho_2) &\quad\longleftrightarrow\quad \{r_1(1)r_1(2), r_1(2)r_1(3), \\
&\qquad\qquad r_1(3)r_1(4), r_1(4)r_1(1)\} \\
\widehat{S}(2\rho_2) &\quad\longleftrightarrow\quad \{r_2(1), r_2(2), r_2(3), r_2(4)\} \\
\widehat{S}(\rho_4) = C_{8+} \wedge_{C_4} S^{\rho_4} &\quad\longleftrightarrow\quad \{r_1(1)r_1(3), r_1(2)r_1(4)\}
\end{aligned}$$

Note that the slice cells $\widehat{S}(2\rho_2)$ and$\widehat{S}(\rho_4)$ are underlain by wedges of 4 and 2 copies of S^4 respectively. 3.9

The refinement of $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ (continued)

$$r_i(1) \longrightarrow r_i(2) \longrightarrow r_i(3) \longrightarrow r_i(4) \xrightarrow{(-1)^j} r_i(1)$$

It follows that $\pi_4^u(MU_{\mathbf{R}}^{(4)})$ is refined by an equivariant map from

$$\widehat{W}_2 = \widehat{S}(2\rho_2) \vee \widehat{S}(2\rho_2) \vee \widehat{S}(2\rho_2) \vee \widehat{S}(\rho_4).$$

A similar analysis can be made in any even dimension and for any cyclic 2-group G. G always permutes monomials up to sign. In $\pi_*^u(MU_{\mathbf{R}}^{(4)})$ the first case of a singleton orbit occurs in dimension 8, namely

$$\widehat{S}(\rho_8) \quad \longleftrightarrow \quad \{r_1(1)r_1(2)r_1(3)r_1(4)\}.$$

Note that the free slice cell $\widehat{S}(m\rho_1)$ never occurs as a wedge summand of $\widehat{W}_m$.

A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties. 3.10

The slice spectral sequence (continued)

Slice Theorem . *In the slice tower for $MU_{\mathbf{R}}^{(g/2)}$, every odd slice is contractible, and the $2m$th slice is $\widehat{W}_m \wedge H\mathbf{Z}$, where $\widehat{W}_m$ is the wedge of slice cells indicated above and $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum. $\widehat{W}_m$ never has any free summands.* 3.11

This result is the technical heart of our proof.

Thus we need to find the groups

$$\pi_*^G(\widehat{S}(m\rho_h) \wedge H\mathbf{Z}) = \pi_*^H(S^{m\rho_h} \wedge H\mathbf{Z}) = \pi_*\left((S^{m\rho_h} \wedge H\mathbf{Z})^H\right).$$

We need this for all nontrivial subgroups H and all integers m because we construct the spectrum $\tilde{\Omega}$ by inverting a certain element in $\pi_{19\rho_8}^G(MU_{\mathbf{R}}^{(4)})$. Here is what we will learn.

Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$

Vanishing Theorem .

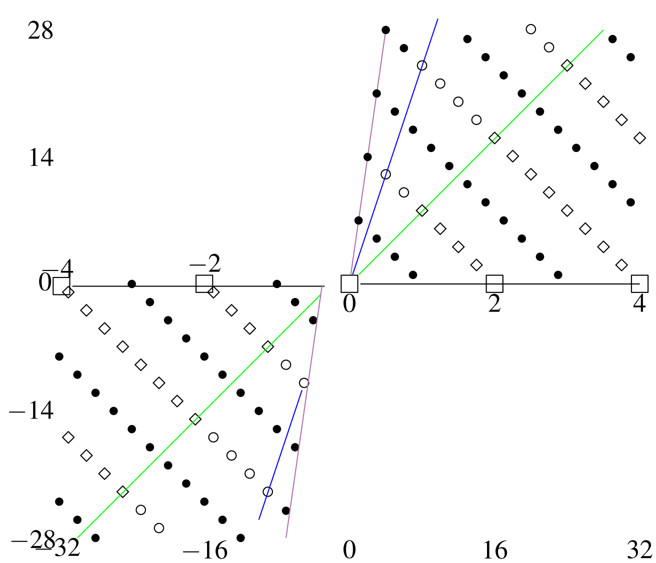
- *For $m \geq 0$, $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ unless $m \leq k \leq hm$.*
- *For $m < 0$ and $h > 1$, $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ unless $hm \leq k \leq m-2$. The upper bound can be improved to $m-3$ except in the case $(h,m) = (2,-2)$ when $\pi_{-4}^H(S^{-2\rho_2} \wedge H\mathbf{Z}) = \mathbf{Z}$.* 3.12

Gap Corollary. *For $h > 1$ and all integers m, $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ for $-4 < k < 0$.*

Given the Slice Theorem, this means a similar statement must hold for $\pi_*^{C_8}(\tilde{\Omega}) = \pi_*(\Omega)$, which gives the Gap Theorem.

Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$ (continued)

Here again is a picture showing $\pi_*^{C_8}(S^{m\rho_8} \wedge H\mathbf{Z})$ for small m.



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3 Proof of Gap Theorem

The proof of the Gap Theorem

The proofs of the Vanishing Theorem and Gap Corollary are surprisingly easy.

We begin by constructing an equivariant cellular chain complex $C(m\rho_g)_*$ for $S^{m\rho_g}$, where $m \geq 0$. In it the cells are permuted by the action of G. It is a complex of $\mathbf{Z}[G]$-modules and is uniquely determined by fixed point data of $S^{m\rho_g}$.

For $H \subset G$ we have

$$(S^{m\rho_g})^H = S^{mg/h}$$

This means that $S^{m\rho_g}$ is a G-CW-complex with

- one cell in dimension m,
- two cells in each dimension from $m+1$ to $2m$,
- four cells in each dimension from $2m+1$ to $4m$,

and so on. 3.14

The proof of the Gap Theorem (continued)

In other words,

$$C(m\rho_g)_k = \begin{cases} 0 & \text{unless } m \leq k \leq gm \\ \mathbf{Z} & \text{for } k = m \\ \mathbf{Z}[G/G'] & \text{for } m < k \leq 2m \text{ and } g \geq 2 \\ \mathbf{Z}[G/G''] & \text{for } 2m < k \leq 4m \text{ and } g \geq 4 \\ \vdots & \end{cases}$$

where G' and G'' are the subgroups of indices 2 and 4. Each of these is a cyclic $\mathbf{Z}[G]$-module. The boundary operator is uniquely determined by the fact that $H_*(C(m\rho_g)) = H_*(S^{gm})$.

Then we have

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))) = H_*(C(m\rho_g)^G).$$

These groups are nontrivial only for $m \leq k \leq gm$, which gives the Vanishing Theorem for $m \geq 0$. 3.15

The proof of the Gap Theorem (continued)

We will look at the bottom three groups in the complex $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g)_*)$. Since $C(m\rho_g)_k$ is a cyclic $\mathbf{Z}[G]$-module, the Hom group is always $\mathbf{Z}$.

For $m > 1$ our chain complex $C(m\rho_g)$ has the form

$$\begin{array}{ccccccccc} & & C(m\rho_g)_m & & C(m\rho_g)_{m+1} & & C(m\rho_g)_{m+2} & & \\ & & \| & & \| & & \| & & \\ 0 & \longleftarrow & \mathbf{Z} & \xleftarrow{\varepsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \cdots \end{array}$$

Applying $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ (taking fixed points) to this gives (in dimensions $\leq 2m$ for $m > 4$)

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \longleftarrow & \cdots \\ m & & m+1 & & m+2 & & m+3 & & m+4 & & \end{array}$$

3.16

The proof of the Gap Theorem (continued)

Again, $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))$ in low dimensions is

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \longleftarrow & \cdots \\ m & & m+1 & & m+2 & & m+3 & & m+4 & & \end{array}$$

It follows that for $m \leq k < 2m$,

$$\pi_k^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \begin{cases} \mathbf{Z}/2 & k \equiv m \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

3.17

The proof of the Gap Theorem (continued)

We can study the groups $\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z})$ for $m < 0$ in two different ways, topologically and algebraically.

For the topological approach, it is the same as the graded group

$$[S^{-m\rho_g}, H\mathbf{Z}]_*^G \qquad \text{where } m < 0.$$

Since G acts trivially on the target $H\mathbf{Z}$, equivariant maps to it are the same as ordinary maps from the orbit space $S^{-m\rho_g}/G$.

For simplicity, assume that $G = C_2$. Then the orbit space is $\Sigma^{-m+1}\mathbf{R}P^{-m-1}$, and we are computing its ordinary reduced cohomology with integer coefficients. We have

$$\begin{aligned} &\pi_{-k}^G(S^{m\rho_g} \wedge H\mathbf{Z}) \\ &\quad = \overline{H}^k(\Sigma^{-m+1}\mathbf{R}P^{-m-1}; \mathbf{Z}) \\ &\quad = 0 \begin{cases} \text{unless } k = -m+2 \text{ when } m = -2 \\ \text{unless } -m+3 \leq k \leq -2m \text{ when } m \leq -3. \end{cases} \end{aligned}$$

The increased lower bound is responsible for the gap.

3.18

The proof of the Gap Theorem (continued)

Alternatively, $S^{m\rho_g}$ (with $m < 0$) is the equivariant Spanier-Whitehead dual of $S^{-m\rho_g}$. This means that

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H^*(\mathrm{Hom}_{\mathbf{Z}[G]}(C(-m\rho_g), \mathbf{Z})).$$

Applying the functor $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ to our chain complex $C(-m\rho_g)$

$$\begin{array}{ccccccccc} \mathbf{Z} & \xleftarrow{\varepsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \mathbf{Z}[C_2 \text{ or } C_4] & \xleftarrow{1-\gamma} & \cdots \\ -m & & -m+1 & & -m+2 & & -m+3 & & \end{array}$$

gives a negative dimensional chain complex beginning with

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \longrightarrow & \cdots \\ m & & m-1 & & m-2 & & m-3 & & m-4 & & \end{array}$$

3.19

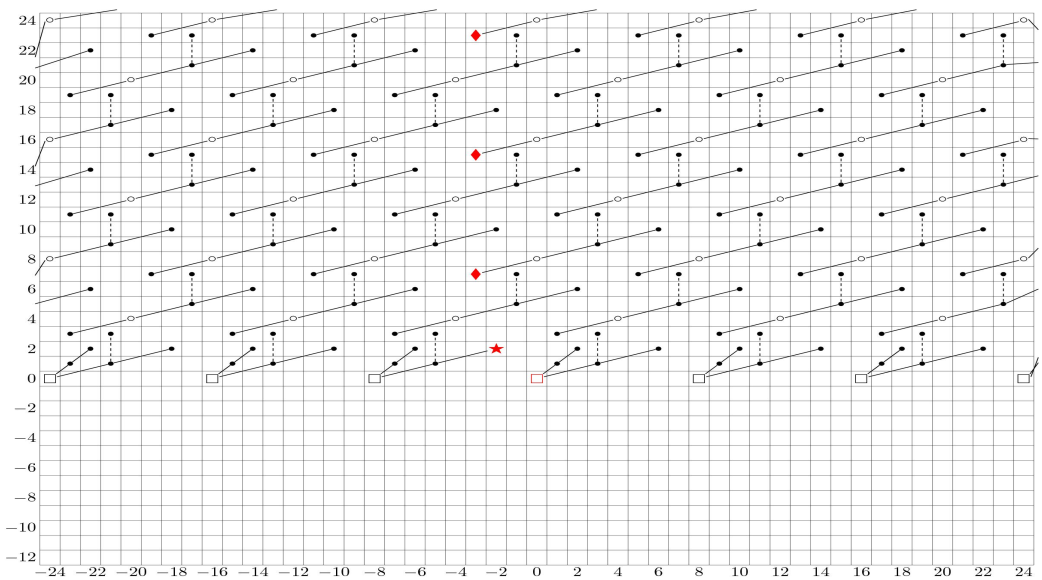
The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case $m \leq -4$.

$$
\begin{array}{ccccccccccc}
-m & & -m+1 & & -m+2 & & -m+3 & & -m+4 & & \\
\mathbf{Z} & \xleftarrow[2]{} & \mathbf{Z} & \xleftarrow[0]{} & \mathbf{Z} & \xleftarrow[2]{} & \mathbf{Z} & \xleftarrow[0]{} & \mathbf{Z} & \longleftarrow & \cdots \\
 & & & & \Big\Uparrow \mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z},\cdot) & & & & & & \\
\mathbf{Z} & \xleftarrow{\varepsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \cdots \\
 & & & & \Big\Downarrow \mathrm{Hom}_{\mathbf{Z}[G]}(\cdot,\mathbf{Z}) & & & & & & \\
\mathbf{Z} & \xrightarrow{1} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \longrightarrow & \cdots \\
m & & m-1 & & m-2 & & m-3 & & m-4 & &
\end{array}
$$

Note the difference in behavior of the map $\varepsilon : \mathbf{Z}[C_2] \to \mathbf{Z}$ under the functors $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z},\cdot)$ and $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot,\mathbf{Z})$. They convert it to maps of degrees 2 and 1 respectively. This difference is responsible for the gap. 3.20

A homotopy fixed point spectral sequence



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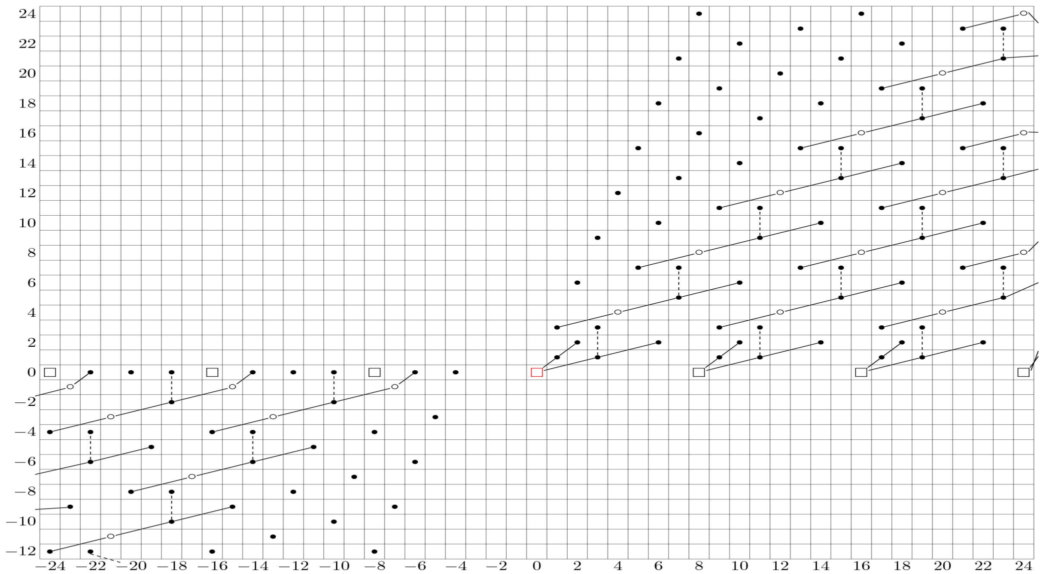
The corresponding slice spectral sequence



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