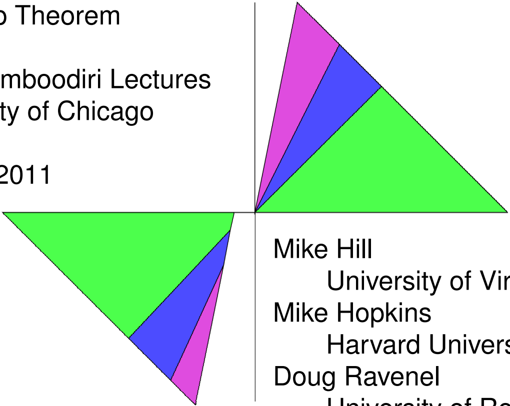
A solution to the Arf-Kervaire invariant problem III: The Gap Theorem

Unni Namboodiri Lectures
University of Chicago

April 5, 2011



Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester

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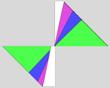

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- $H_*(MU; \mathbf{Z}) = \mathbf{Z}[b_i : i > 0]$ where $|b_i| = 2i$.
- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.

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Suppose X is a G-spectrum such that its underlying homotopy group $\pi_k^u(X)$ is free abelian. A refinement of $\pi_k^u(X)$ is an equivariant map

$$c : \widehat{W} \to X$$

in which $\widehat{W}$ is a wedge of slice cells of dimension k whose underlying spheres represent a basis of $\pi_k^u(X)$.

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Note that the slice cell $\widehat{S}(\rho_2)$ is underlain by a wedge of 4 copies of S^2.

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\widehat{S}(2\rho_2) = C_{8+} \wedge_{C_2} S^{2\rho_2} \quad &\longleftrightarrow \quad \{r_1(1)^2,\, r_1(2)^2,\, r_1(3)^2,\, r_1(4)^2\} \\
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A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties.

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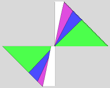

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We need this for all nontrivial subgroups H and all integers m because we construct the spectrum $\tilde{\Omega}$ by inverting a certain element in $\pi_{19\rho_8}^G(MU_{\mathbf{R}}^{(4)})$. Here is what we will learn.

Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$

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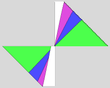

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Given the Slice Theorem, this means a similar statement must hold for $\pi_*^{C_8}(\tilde{\Omega}) = \pi_*(\Omega)$, which gives the Gap Theorem.

Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$ (continued)

Here again is a picture showing $\pi_*^{C_8}(S^{m\rho_8} \wedge H\mathbf{Z})$ for small m.

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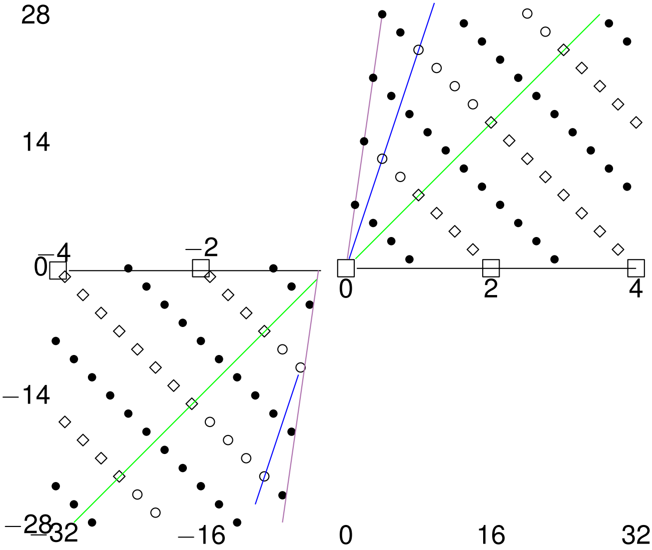

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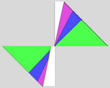

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These groups are nontrivial only for $m \le k \le gm$, which gives the Vanishing Theorem for $m \ge 0$.

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\end{tikzcd}$$

Applying $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ (taking fixed points) to this gives (in dimensions $\leq 2m$ for $m > 4$)

$$\begin{tikzcd}
\mathbf{Z} & \mathbf{Z} \arrow[l, "2"'] & \mathbf{Z} \arrow[l, "0"'] & \mathbf{Z} \arrow[l, "2"'] & \mathbf{Z} \arrow[l, "0"'] & \cdots \arrow[l] \\
m & m+1 & m+2 & m+3 & m+4 &
\end{tikzcd}$$

The proof of the Gap Theorem (continued)

Again, $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))$ in low dimensions is

$$\begin{tikzcd}
\mathbf{Z} & \mathbf{Z} \arrow[l, "2"'] & \mathbf{Z} \arrow[l, "0"'] & \mathbf{Z} \arrow[l, "2"'] & \mathbf{Z} \arrow[l, "0"'] & \cdots \arrow[l] \\
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\end{tikzcd}$$

It follows that for $m \leq k < 2m$,

$$\pi_k^G(S^{m\rho_g} \wedge H\mathbf{Z}) = \begin{cases} \mathbf{Z}/2 & k \equiv m \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

The proof of the Gap Theorem (continued)

We can study the groups $\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z})$ for $m < 0$ in two different ways, topologically and algebraically.

The proof of the Gap Theorem (continued)

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For the topological approach, it is the same as the graded group

$$[S^{-m\rho_g}, H\mathbf{Z}]_*^G \qquad \text{where } m < 0.$$

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$$\begin{aligned}\pi_{-k}^G(S^{m\rho_g} \wedge H\mathbf{Z}) & \\ &= \overline{H}^k(\Sigma^{-m+1}\mathbf{R}P^{-m-1}; \mathbf{Z})\end{aligned}$$

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The increased lower bound is responsible for the gap.

The proof of the Gap Theorem (continued)

Alternatively, $S^{m\rho_g}$ (with $m < 0$) is the equivariant Spanier-Whitehead dual of $S^{-m\rho_g}$.

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Applying the functor $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ to our chain complex $C(-m\rho_g)$

$$\begin{tikzcd}
\mathbf{Z} & \mathbf{Z}[C_2] \arrow[l, "\epsilon"'] & \mathbf{Z}[C_2] \arrow[l, "1-\gamma"'] & \mathbf{Z}[C_2 \text{ or } C_4] \arrow[l, "1+\gamma"'] & \cdots \arrow[l, "1-\gamma"'] \\
-m & -m+1 & -m+2 & -m+3 &
\end{tikzcd}$$

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\end{tikzcd}$$

gives a negative dimensional chain complex beginning with

$$\begin{tikzcd}
\mathbf{Z} \arrow[r, "1"] & \mathbf{Z} \arrow[r, "0"] & \mathbf{Z} \arrow[r, "2"] & \mathbf{Z} \arrow[r, "0"] & \mathbf{Z} \arrow[r] & \cdots \\
m & m-1 & m-2 & m-3 & m-4 &
\end{tikzcd}$$

The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case $m \leq -4$.

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\mathbf{Z} & \mathbf{Z} \arrow[l, "2"'] & \mathbf{Z} \arrow[l, "0"'] & \mathbf{Z} \arrow[l, "2"'] & \mathbf{Z} \arrow[l, "0"'] & \cdots \arrow[l] \\
\mathbf{Z} & \mathbf{Z}[C_2] \arrow[l, "\epsilon"'] & \mathbf{Z}[C_2] \arrow[l, "1-\gamma"'] \arrow[u, Rightarrow, "\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)"'] \arrow[d, Rightarrow, "\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})"'] & \mathbf{Z}[C_2] \arrow[l, "1+\gamma"'] & \mathbf{Z}[C_2] \arrow[l, "1-\gamma"'] & \cdots \arrow[l, "1-\gamma"'] \\
\mathbf{Z} \arrow[r, "1"] & \mathbf{Z} \arrow[r, "0"] & \mathbf{Z} \arrow[r, "2"] & \mathbf{Z} \arrow[r, "0"] & \mathbf{Z} \arrow[r] & \cdots \\
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\end{tikzcd}$$

Note the difference in behavior of the map $\epsilon : \mathbf{Z}[C_2] \to \mathbf{Z}$ under the functors $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ and $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$.

The proof of the Gap Theorem (continued)

Here is a diagram showing both functors in the case $m \leq -4$.

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-m & -m+1 & -m+2 & -m+3 & -m+4 & \\
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The proof of the Gap Theorem (continued)

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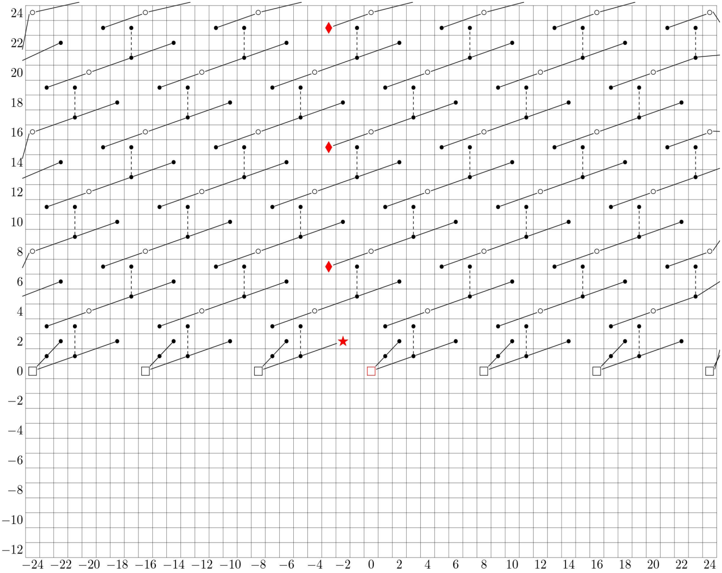
A homotopy fixed point spectral sequence



The corresponding slice spectral sequence