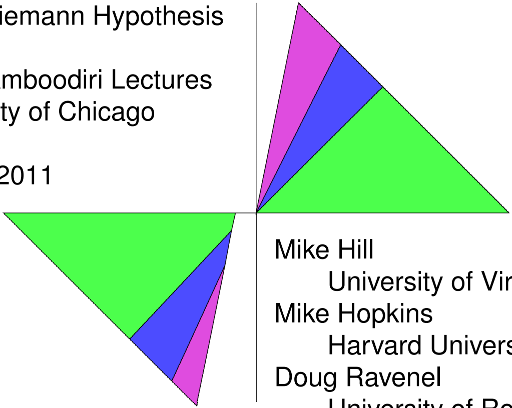
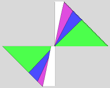
An equivariant approach to the Riemann Hypothesis

Unni Namboodiri Lectures
University of Chicago

April 1, 2011



Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester



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An equivariant approach to the Riemann Hypothesis (continued)

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Hence the problem is very similar to the Kervaire invariant question except that the group involved is not cyclic. The Slice Theorem (to be explained below) still holds, but the slices themselves are more complicated because of the bigger group. Using the techniques we have developed in the cyclic case, there is a good chance we can do the necessary calculations here and arrive at a similar proof.


APRIL
FOOL

A solution to the Arf-Kervaire invariant problem I: History and background

Unni Namboodiri Lectures
University of Chicago

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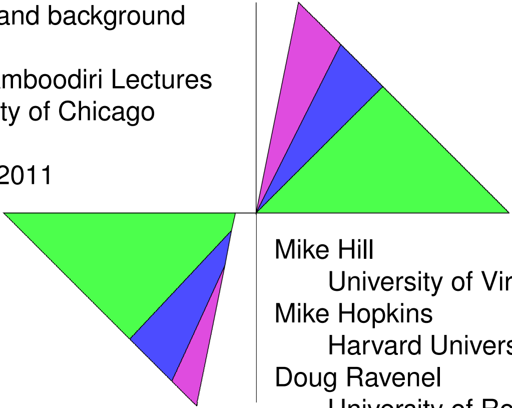

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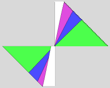
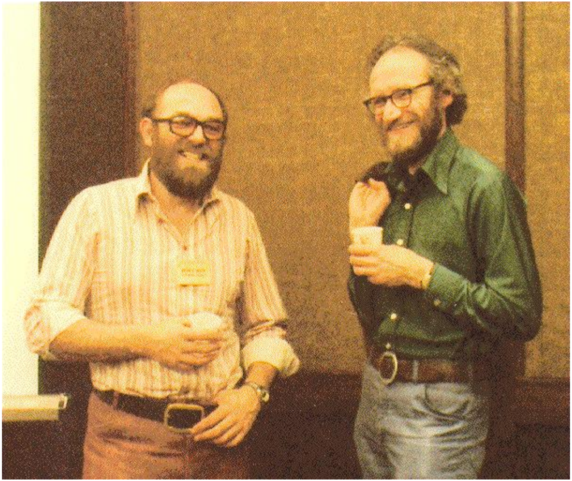
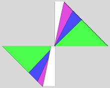


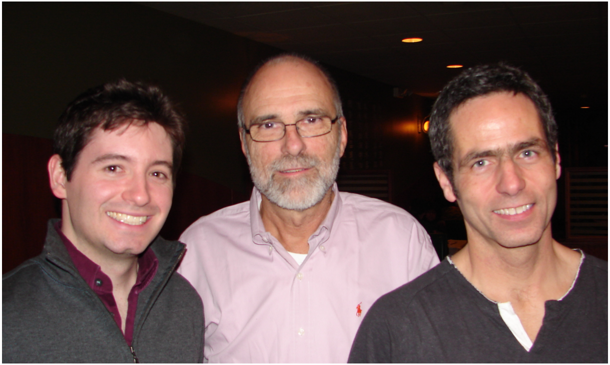


Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson



A wildly popular dance craze

Drawing by Carolyn Snaith 1981
London, Ontario



Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010

Our main result

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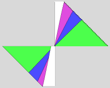

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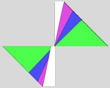

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Snaith's book (continued)

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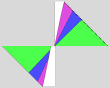

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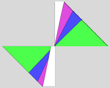

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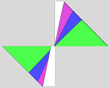

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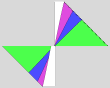

Pontryagin's early work on homotopy groups of spheres

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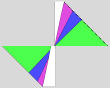

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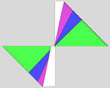

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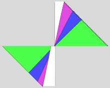

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- Pick a regular value $y \in S^n$. Its inverse image will be a smooth k-manifold M in S^{n+k}.
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.

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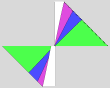

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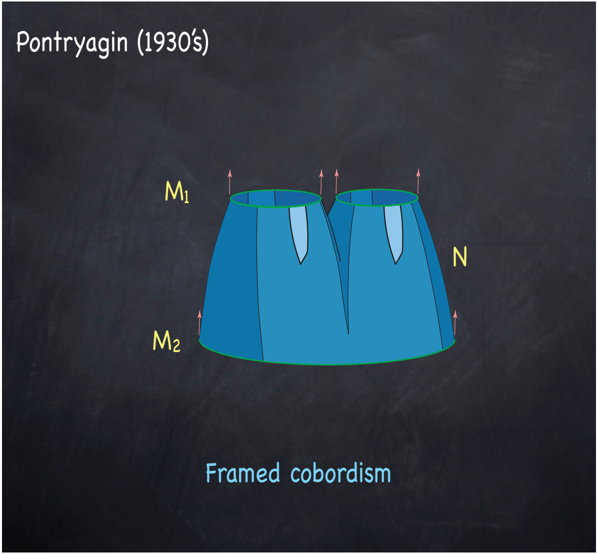
Pontryagin's early work (continued)

Here is an example of a framed cobordism for $n = k = 1$.



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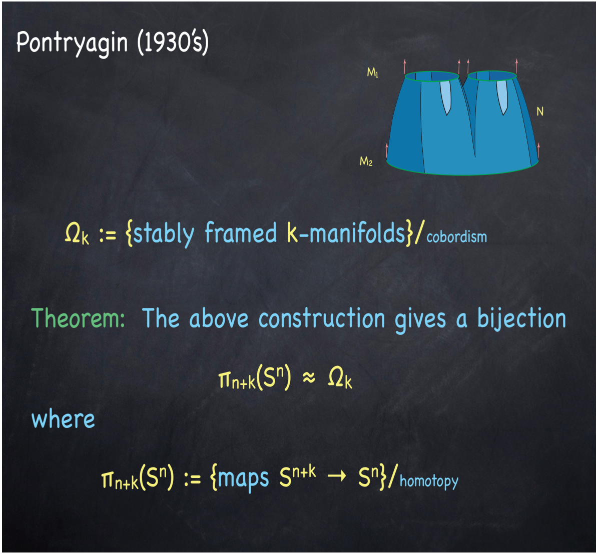

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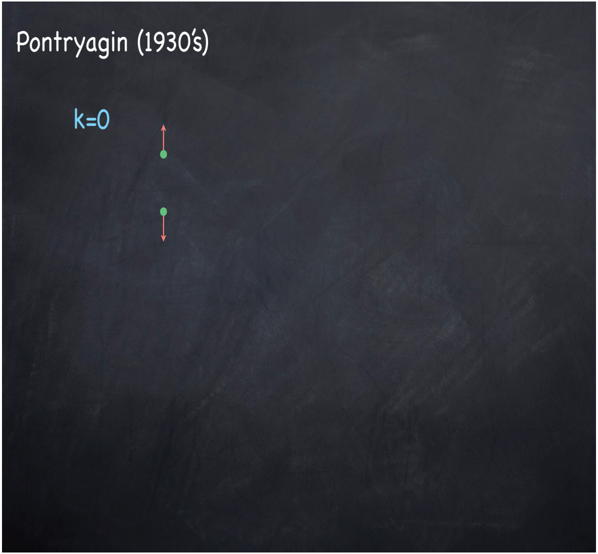

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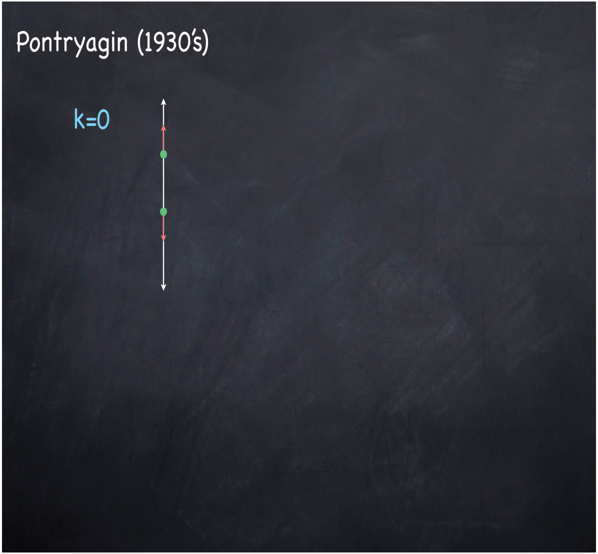

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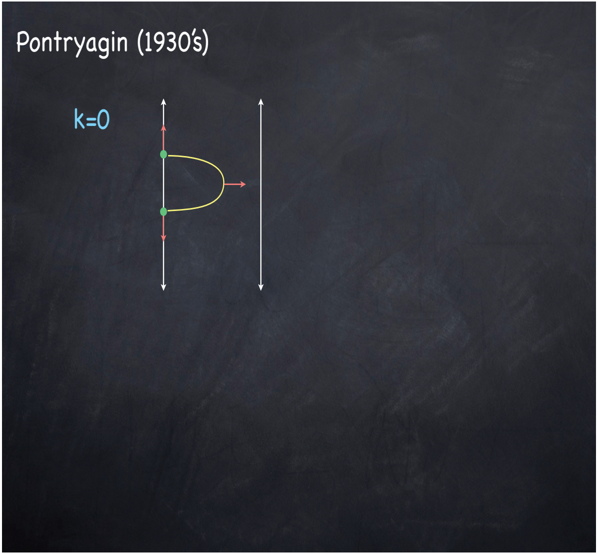

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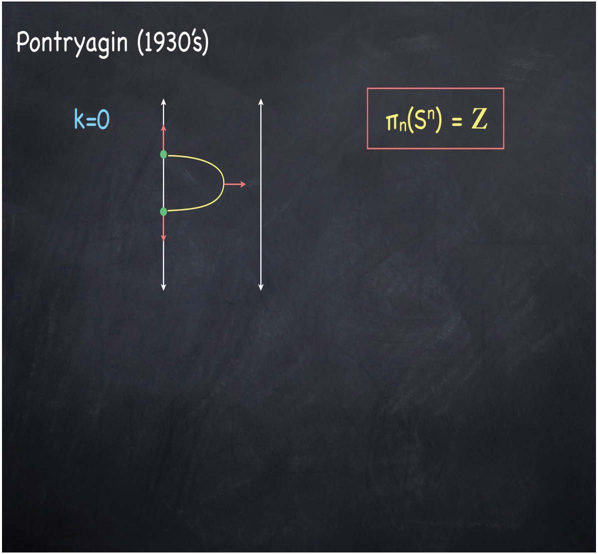

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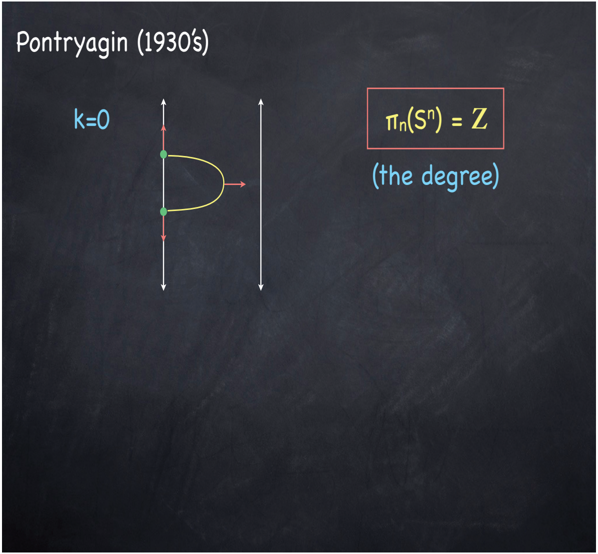

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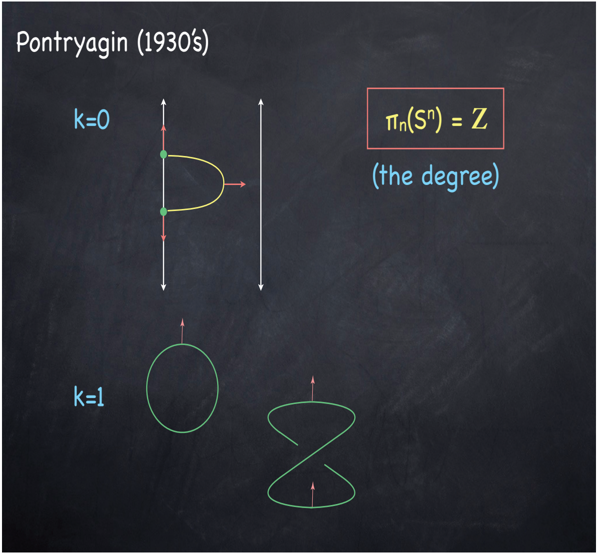

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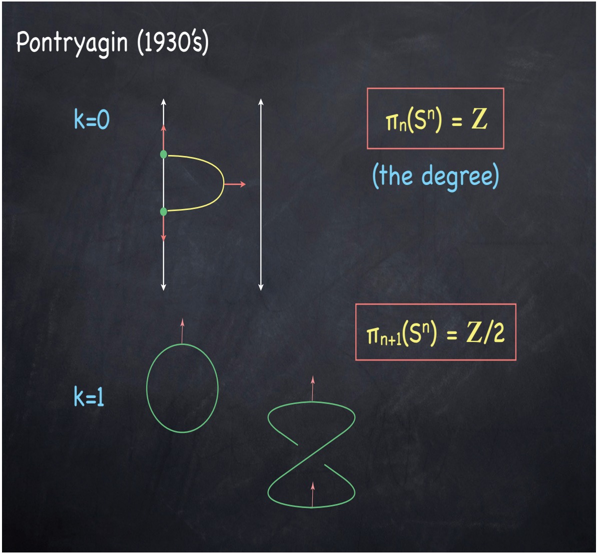

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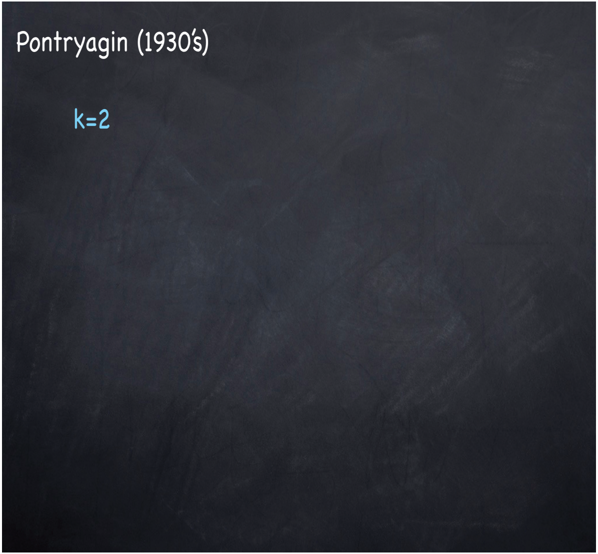

Pontryagin's early work (continued)

Pontryagin (1930's)

k=2 genus M = 0 ⇒ M is a boundary

(since S^2 bounds a disk and

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Pontryagin's early work (continued)

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$k=2$ genus $M = 0 \Rightarrow M$ is a boundary

(since S^2 bounds a disk and $\pi_2(GL_n(\mathbf{R}))=0$)

Suppose the genus of M is greater than 0.

Pontryagin's early work (continued)

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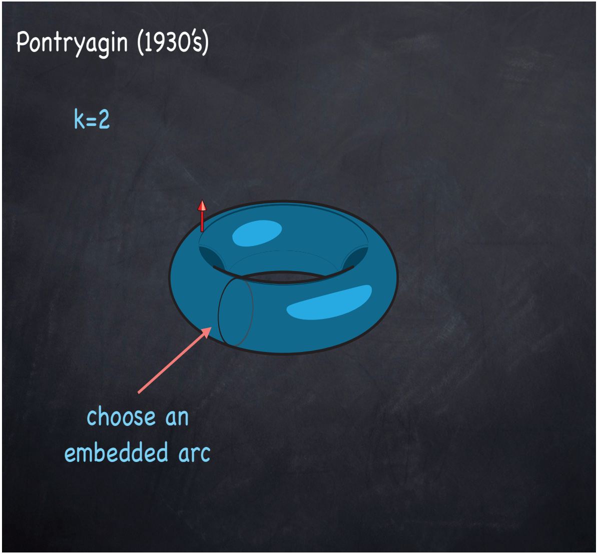

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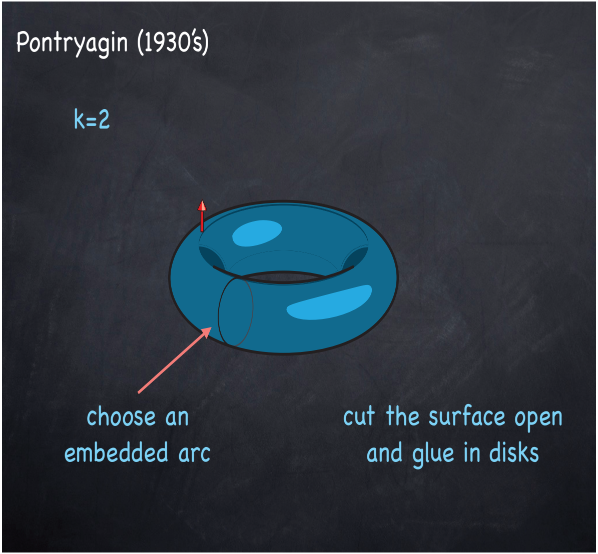

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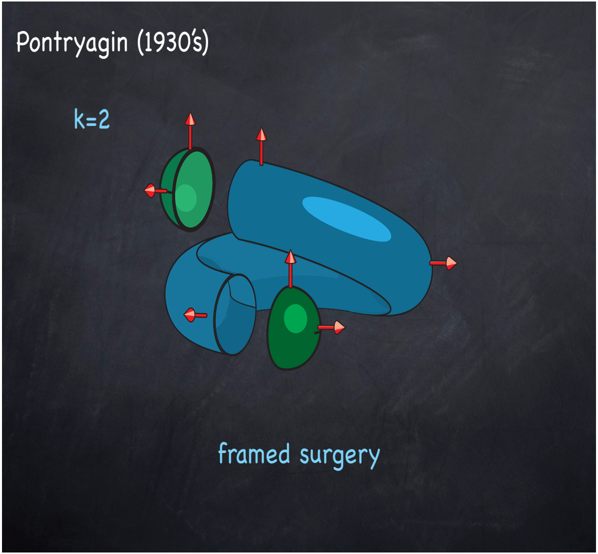

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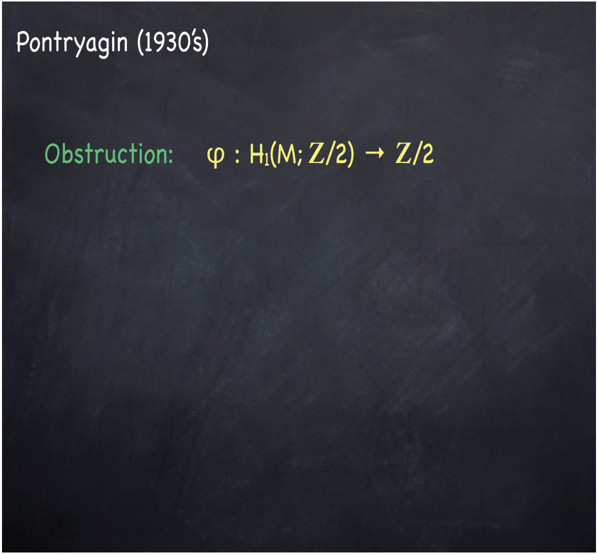

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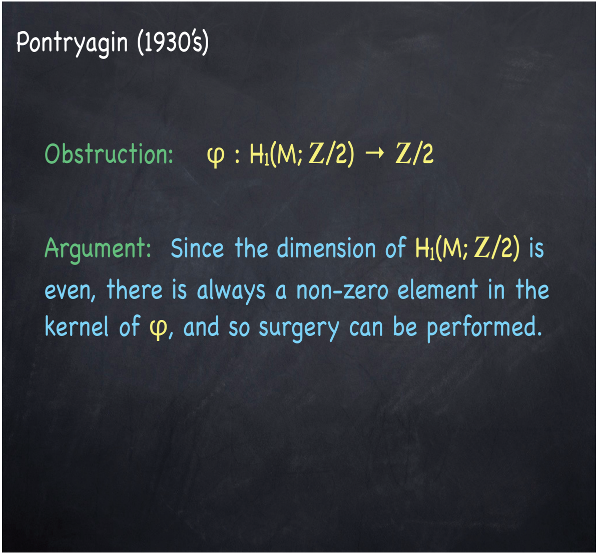

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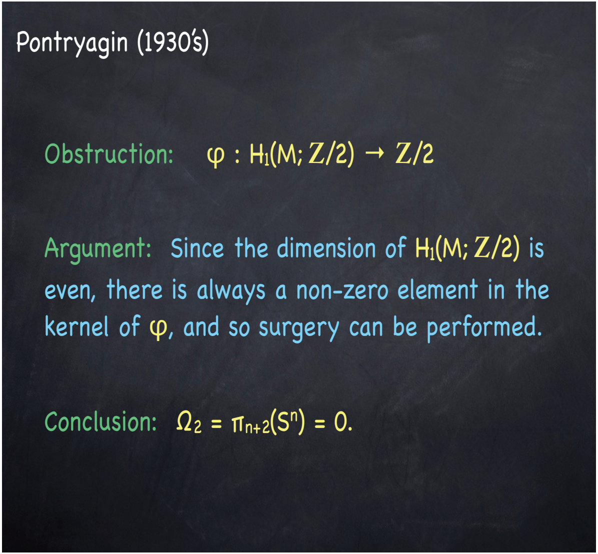

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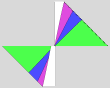
The map $\varphi : H_1(M; \mathbf{Z}/2) \to \mathbf{Z}/2$ is not a homomorphism!



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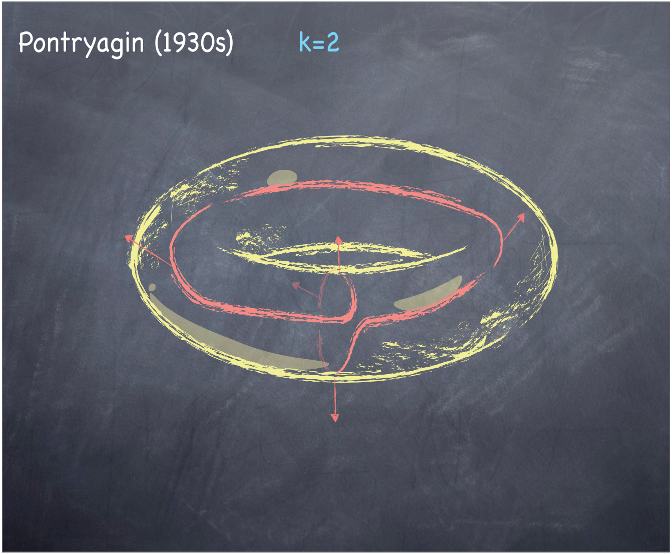

Tuesday, April 21, 2009

The Arf invariant of a quadratic form in characteristic 2

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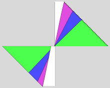

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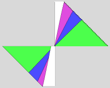
In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix}.$$

The Arf invariant of a quadratic form in characteristic 2 (continued)

A quadratic refinement of λ is a map $q : \overline{H} \to \mathbf{Z}/2$ satisfying



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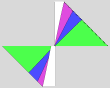

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Mike Hill
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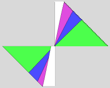
In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q.



Money talks: Arf's definition republished in 2009

Cahit Arf 1910-1997

The Kervaire invariant of a framed $(4m+2)$-manifold

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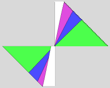

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Mike Hill
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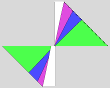

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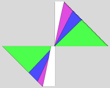

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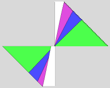
For $m = 0$, Kervaire's q coincides with Pontryagin's φ.



The Kervaire invariant of a framed $(4m + 2)$-manifold (continued)

What can we say about $\Phi(M)$?

Mike Hill
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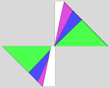

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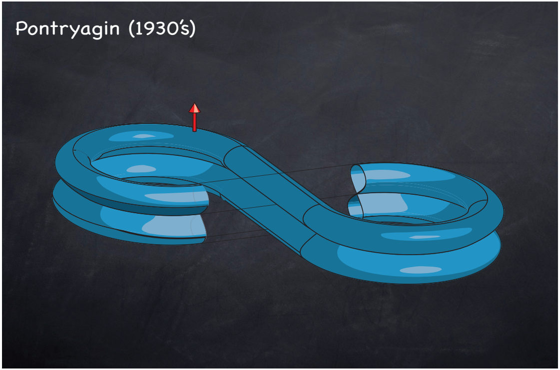

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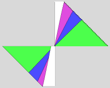

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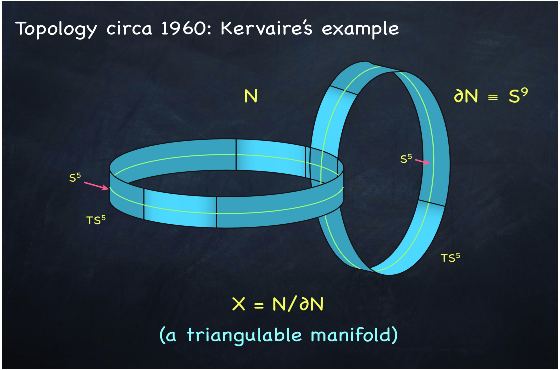

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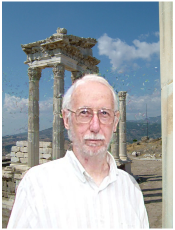
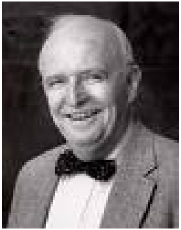
- 
  Ed Brown

  
  Frank Peterson
  1930-2000

  Brown-Peterson (1966) showed that it vanishes for all positive even m.

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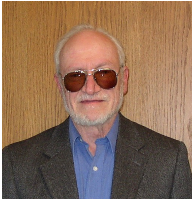
- 

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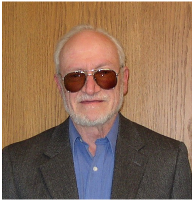
- 

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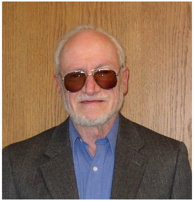
- 

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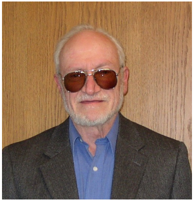
- 

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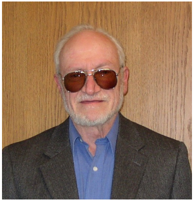
- 

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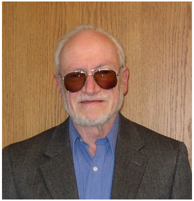
- 

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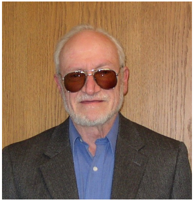
- 

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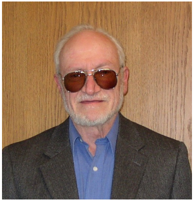
The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

- 

  Bill Browder

  Browder (1969) showed that it can be nontrivial only if $m = 2^{j-1} - 1$ for some positive integer j. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence. The corresponding element in $\pi_{n+2^{j+1}-2}(S^n)$ for large n is θ_j, the subject of our theorem. This is the stable homotopy theoretic formulation of the problem.
- θ_j is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62.
- In the decade following Browder's theorem, many topologists tried without success to construct framed manifolds with nontrivial Kervaire invariant in all dimensions 2 less than a power of 2.
- Our theorem says θ_j does not exist for $j \geq 7$. The case $j = 6$ is still open.

Questions raised by our theorem

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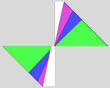

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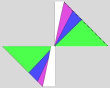

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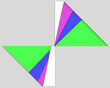

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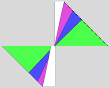

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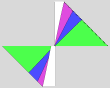

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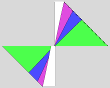

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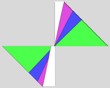
John Milnor

Sergei Novikov

Dan Quillen



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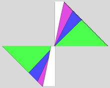

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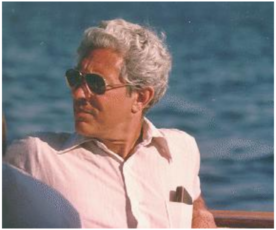
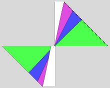


Peter May

John Greenlees

Gaunce Lewis
1949-2006



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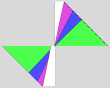

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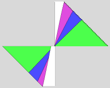

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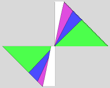

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Nitu Kitchloo

Steve Wilson

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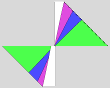

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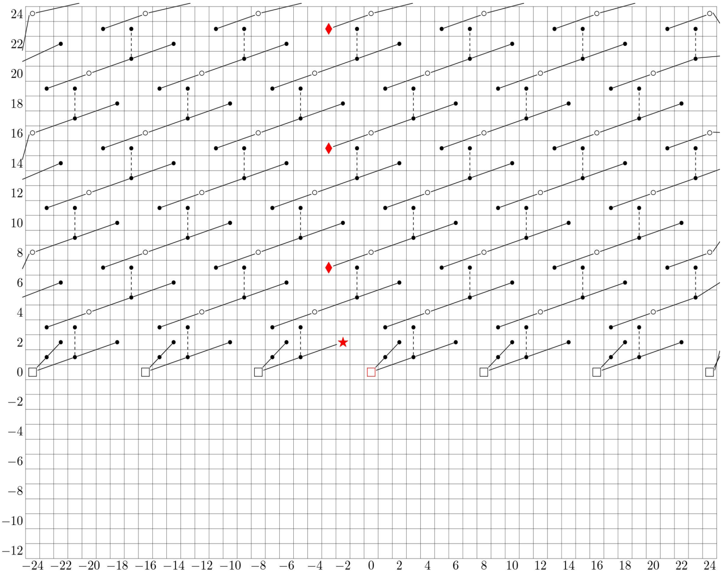
This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.

A homotopy fixed point spectral sequence



The corresponding slice spectral sequence