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MORAVA STABILIZER ALGEBRAS AND THE LOCALIZATION OF NOVIKOV'S E_2-TERM

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The E_2-term of the Adams–Novikov spectral sequence [2] for a spectrum X localized at the prime p has the form

$$\mathrm{Ext}_{BP_*BP}{}^* (BP_* , M) \tag{0.1}$$

where BP is the Brown–Peterson spectrum [2] at p and M is the "BP_*BP-comodule" [1] $BP_*(X)$. Recall [2] that $BP_* = \pi_*(BP) = \mathbf{Z}_{(p)}[v_1 , v_2 , \cdots]$, $|v_i| = 2p^i - 2$. The purpose of this paper is to identify (0.1) with an Ext group over a smaller "Hopf algebra" in case M is *v_n-local,* by which we mean that v_n acts on M bijectively.

The first theorem in this direction is due to Jack Morava [14]. Morava shows that if M is a v_n-local comodule which is killed by the ideal $I_n = (p, v_1 , \cdots , v_{n-1})$ and finitely generated over $v_n^{-1}BP_*/I_n$, then (0.1) may be computed in terms of the continuous cohomology of a certain p-adic Lie group with coefficients in a finite dimensional representation over $\mathbf{F}_{p^n}$ constructed out of M.

We prove the following "covariant" analogue of this theorem in Section 2. Let $K(0)_* = \mathbf{Q}$, and $K(n)_* = \mathbf{F}_p[v_n , v_n^{-1}]$ for $n > 0$, with the obvious BP_*-algebra structures. Let $K(n)_*K(n) = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$; it inherits from BP_*BP the structure of a Hopf algebra over the graded field $K(n)_*$.

THEOREM 2.10. *If M is v_n-local and $I_nM = 0$, then*

$$\mathrm{Ext}_{BP_*BP}{}^* (BP_* , M) \cong \mathrm{Ext}_{K(n)_*K(n)}{}^*(K(n)_* , K(n)_* \otimes_{BP_*} M)$$

under the natural map.

In Section 3 we strengthen Theorem 2.10 by dropping the requirement that $I_nM = 0$. Let $E(n)_* = \mathbf{Z}_{(p)}[v_1 , \cdots , v_n , v_n^{-1}]$ with the obvious BP_*-algebra structure, and let $E(n)_*E(n) = E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*$. Then we have

THEOREM 3.10. *If M is v_n-local, then*

$$\mathrm{Ext}_{BP_*BP}{}^* (BP_* , M) \cong \mathrm{Ext}_{E(n)_*E(n)}{}^* (E(n)_* , E(n)_* \otimes_{BP_*} M)$$

under the natural map.

Thus higher generators can be neglected, at the cost of introducing a rather complicated set of relations into BP_*BP.

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These change of rings theorems rest on a version of Shapiro's Lemma which we prove in Section 1. In this section we also review the definition and elementary properties of Hopf algebroids.

This paper forms the link between the work of the second author in [16] and [17] and our joint work with W. S. Wilson in [12] and [13]. The latter has resulted in proofs of the essentiality of a great many elements in the stable homotopy ring. Briefly stated, our program is to compute a portion of the Novikov E_2-term for the sphere, $\mathrm{Ext}_{BP_*BP}{}^*\,(BP_*\,,\, BP_*)$, by means of the "Bockstein" long exact sequences induced in $\mathrm{Ext}_{BP_*BP}{}^*\,(BP_*\,,\ \ )$ by the short exact sequences

$$0 \to BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0. \tag{0.2}$$

The foundation of the analysis of these long exact sequences is the study of

$$\varinjlim \mathrm{Ext}_{BP_*BP}{}^*\,(BP_*\,,\, BP_*/I_n),$$

where the maps in the directed system are induced by multiplication by v_n. It is easy to see that this limit is just $\mathrm{Ext}_{BP_*BP}{}^*\,(BP_*\,,\, v_n^{-1}BP_*/I_n)$, which Theorem 2.10 now identifies with $\mathrm{Ext}_{K(n)_*K(n)}{}^*\,(K(n)_*\,,\, K(n)_*)$. This is a substantial simplification: $K(n)_*K(n)$ is a *Hopf algebra* over a *field*, while BP_*BP is a (much larger) *Hopf algebroid* (see Section 1 below) over the (complicated) *ring* BP_*. Thus the computation of $\mathrm{Ext}_{K(n)_*K(n)}{}^*\,(K(n)_*\,,\, K(n)_*)$ falls within the scope of the methods of Peter May's thesis, and is carried out in certain cases in [17].

We wish to thank J. F. Adams, Peter Landweber, and Steve Wilson for useful conversations, and we gratefully acknowledge the deep influence Jack Morava has had on our work.

Section 1. Hopf algebroids and a Shapiro's Lemma.

We shall work here in somewhat greater generality than is necessary in the sequel. We begin by recalling Adams' algebraic context [1] for cooperation coalgebras for generalized homology theories.

Let K be a commutative ring. A *Hopf algebroid* (A, Γ) (over K) is a cogroupoid object in the category of commutative graded K-algebras. Thus we have structural K-algebra maps $\eta_R\,,\ \eta_L : A \to \Gamma$ (source, target), $\epsilon : \Gamma \to A$ (identity), $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$ (composition), $c : \Gamma \to \Gamma$ (inverse). Here Γ becomes a left A-module via η_L and a right A-module via η_R, and $\Gamma \otimes_A \Gamma$ is the usual tensor product of bimodules. We require of these maps that Δ and ϵ be A-bimodule maps and that the following diagrams commute.

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "\simeq"'] \arrow[d, "\Delta"] \arrow[dr, "\simeq"] & \\
A \otimes_A \Gamma & \Gamma \otimes_A \Gamma \arrow[l, "\epsilon \otimes_A \Gamma"] \arrow[r, "\Gamma \otimes_A \epsilon"'] & \Gamma \otimes_A A
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \Gamma \otimes_A \Gamma \arrow[d, "\Delta \otimes_A \Gamma"] \\
\Gamma \otimes_A \Gamma \arrow[r, "\Gamma \otimes_A \Delta"'] & \Gamma \otimes_A \Gamma \otimes_A \Gamma
\end{tikzcd}$$

$$\begin{tikzcd}
& \Gamma \arrow[dd, "c"] & \\
A \arrow[ur, "\eta_L"] \arrow[dr, "\eta_R"'] & & A \arrow[ul, "\eta_R"'] \arrow[dl, "\eta_L"] \\
& \Gamma &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "\eta_R"'] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[d] & A \arrow[d, "\eta_L"] \\
\Gamma & \Gamma \otimes_A \Gamma \arrow[l, "c \cdot \Gamma"] \arrow[r, "\Gamma \cdot c"'] & \Gamma
\end{tikzcd}$$

where $c \cdot \Gamma(\gamma_1 \otimes \gamma_2) = c(\gamma_1)\gamma_2$ and $\Gamma \cdot c(\gamma_1 \otimes \gamma_2) = \gamma_1 c(\gamma_2)$.

We leave to the reader the amusement of interpreting these diagrams as the axioms for a cogroupoid object. We will frequently let the coefficient algebra A be understood and write Γ for (A, Γ).

A (*left*) Γ-*comodule* is a left A-module M together with an A-linear map $\psi : M \to \Gamma \otimes_A M$ such that the following diagrams commute.

$$\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[dr, "\simeq"'] & \Gamma \otimes_A M \arrow[d, "\epsilon \otimes_A M"] \\
& A \otimes_A M
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma \otimes_A M \arrow[d, "\Delta \otimes_A M"] \\
\Gamma \otimes_A M \arrow[r, "\Gamma \otimes_A \psi"'] & \Gamma \otimes_A \Gamma \otimes_A M
\end{tikzcd}$$

A morphism of Γ-comodules is an A-linear map $f : M \to N$ such that $\psi_N f = (\Gamma \otimes_A f)\psi_M$.

Right comodules are defined analogously.

In general the category (Γ-(comod) of Γ-comodules may fail to have kernels; but if Γ is A-flat (as left or equivalently as right module) then the bottom row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, dashed, "\psi_L"] & M \arrow[r] \arrow[d, "\psi_M"] & N \arrow[d, "\psi_N"] \\
0 \arrow[r] & \Gamma \otimes_A L \arrow[r] & \Gamma \otimes_A M \arrow[r] & \Gamma \otimes_A N
\end{tikzcd}$$

is exact if the top row is, so ψ_L exists and defines a comodule structure on L. (Γ-comod) is then easily seen to be an Abelian category.

Our central example of a Hopf algebroid is (BP_*, BP_*BP); see [2], [10]. $BP_*BP = BP_*[t_1, t_2, \cdots]$ is flat over BP_*. For any spectrum X, $BP_*(X)$ is a BP_*BP-comodule.

For another example, note that if $\eta_L = \eta_R$ then Γ is just a commutative Hopf algebra over the graded commutative ring A.

A Γ-comodule M is a *relative injective* iff it is a summand of an *extended* comodule, i.e., one of the form $\Gamma \otimes_A X$ for some A-module X. A comodule map is a *relative monomorphism* iff it is split-mono as a map of A-modules. Using these notions we may build resolutions in the usual way, and define $\mathrm{Ext}_\Gamma^*(A, \ )$ as the relative right derived functor of the graded K-module-valued functor $\mathrm{Hom}_\Gamma(A, \ )$. We refer the reader to [4], section 4, for a discussion of relative homological algebra. Throughout the paper, "injective" will always mean "relative injective" in this sense.

In particular we have the *standard* [4] or *cobar resolution* $L^*(\Gamma; M)$. In degree n,

$$L^n(\Gamma; M) = \Gamma \otimes_A \cdots \otimes_A \Gamma \otimes_A M$$

with $(n + 1)$ factors of Γ, and differential

$$\begin{aligned} d(\gamma_0 \otimes \cdots \otimes \gamma_n \otimes m) = \sum_{i=0}^{n} (-1)^{\sigma(i)} \gamma_0 \otimes \cdots \otimes \gamma_i' \otimes \gamma_i'' \otimes \cdots \otimes \gamma_n \otimes m \\ + (-1)^{\sigma(n+1)} \sum \gamma_0 \otimes \cdots \otimes \gamma_n \otimes m' \otimes m'' \end{aligned}$$

where

$$\Delta\gamma_i = \sum \gamma_i' \otimes \gamma_i'', \qquad \psi m = \sum m' \otimes m'',$$

and

$$\sigma(i) = |\gamma_0| + \cdots + |\gamma_{i-1}| + i.$$

The usual contracting homotopy

$$s(\gamma_0 \otimes \cdots \gamma_n \otimes m) = \epsilon(\gamma_0)\gamma_1 \otimes \cdots \otimes \gamma_n \otimes m$$

shows that $L^*(\Gamma; M)$ is a relative injective resolution, so that $\mathrm{Ext}_\Gamma^*(A, M)$ is the homology of the *cobar complex*

$$\Omega^*(\Gamma; M) = \mathrm{Hom}_\Gamma(A, L^*(\Gamma; M)).$$

Our first job is to show that $\mathrm{Ext}_\Gamma^*(A, M)$ may be computed using a wider class of complexes than those described above.

LEMMA 1.1 *Let Γ be A-flat and let $0 \to M \to I^0 \to I^1 \to \cdots$ be a sequence of Γ-comodules which is exact (over K) and such that for each i, $\mathrm{Ext}_\Gamma^q(A, I^i) = 0$ for all $q > 0$. Then $\mathrm{Ext}_\Gamma^*(A, M) = H(\mathrm{Hom}_\Gamma(A, I^*))$.*

Proof. Consider the double complex

$$X^{**} = \Omega^*(\Gamma; I^*).$$

Both of the associated spectral sequences ([3], XV §6) collapse at E_2 to give the desired equality.

Now let $\pi = (\pi, \pi) : (A, \Gamma) \to (B, \Sigma)$ be a map of Hopf algebroids. For a Γ-comodule M, $B \otimes_A M$ is naturally a Σ-comodule via the B-linear extension of

$$M \xrightarrow{\psi} \Gamma \otimes_A M \xrightarrow{\pi \otimes_A M} \Sigma \otimes_A M = \Sigma \otimes_B (B \otimes_A M).$$

The usual lifting argument shows that π induces a map

$$\pi_* : \mathrm{Ext}_\Gamma{}^* (A, M) \to \mathrm{Ext}_\Sigma{}^* (B, B \otimes_A M).$$

Symmetrically, if M is a right Γ-comodule, then $M \otimes_A B$ is a right Σ-comodule. In particular $\Gamma \otimes_A B$ is a right Σ-comodule. It is also obviously a left Γ-comodule, and in a compatible way, so we have functors

$$(\Gamma\text{-comod}) \underset{(\Gamma \otimes_A B)\square_\Sigma}{\overset{B \otimes_A}{\rightleftarrows}} (\Sigma\text{-comod}). \tag{1.2}$$

Here $N' \square_\Sigma N''$ is the cotensor product of the right Σ-comodule N' and the left Σ-comodule N'', defined as the K-module kernel of

$$\psi_{N'} \otimes N'' - N' \otimes \psi_{N''} : N' \otimes_B N'' \to N' \otimes_B \Sigma \otimes_B N''.$$

Write $\pi_* = B \otimes_A$ and $\pi^* = (\Gamma \otimes_A B)\square_\Sigma$.

These functors are adjoint:

$$\mathrm{Hom}_\Gamma (M, \pi^*N) \cong \mathrm{Hom}_\Sigma (\pi_*M, N)$$

naturally in $M \in (\Gamma\text{-comod})$, $N \in (\Sigma\text{-comod})$. In particular

$$\mathrm{Hom}_\Gamma (A, \pi^*N) \cong \mathrm{Hom}_\Sigma (B, N).$$

Since they will be of use later we display the adjunction morphisms [4]. The front adjunction $\alpha_M : M \to \pi^*\pi_*M$ is defined as the composite of the morphisms in the top line of the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\simeq"] \arrow[dr, "\psi"'] & \Gamma \square_\Gamma M \arrow[r] \arrow[d, phantom, "\cap"] & (\Gamma \otimes_A B) \square_\Sigma (B \otimes_A M) \arrow[d, phantom, "\cap"] \\
& \Gamma \otimes_A M \arrow[r, "f"] & (\Gamma \otimes_A B) \otimes_B (B \otimes_A M)
\end{tikzcd}$$

where $f(\gamma \otimes m) = \gamma \otimes 1 \otimes 1 \otimes m$. The back adjunction $\beta_N : \pi_*\pi^*N \to N$ is the B-linear extension of the top composite in the commutative diagram in which $g(\gamma \otimes b \otimes n) = \pi(\gamma)\eta_R(b) \otimes n$.

$$\begin{tikzcd}
(\Gamma \otimes_A B) \square_\Sigma N \arrow[r] \arrow[d, phantom, "\cap"] & \Sigma \square_\Sigma N \arrow[d, phantom, "\cap"] & N \arrow[l, "\simeq"'] \arrow[dl, "\psi"] \\
(\Gamma \otimes_A B) \otimes_B N \arrow[r, "g"] & \Sigma \otimes_B N &
\end{tikzcd}$$

An ideal $I \subset A$ is *invariant* iff the ideal in Γ generated by $\eta_L(I)$ coincides with the ideal generated by $\eta_R(I)$. The most elementary change of rings theorem is now the following.

PROPOSITION 1.3. *Let $I \subset A$ be an invariant ideal, $\bar{A} = A/I$, $\bar{\Gamma} = \Gamma/I\Gamma = \Gamma/\Gamma I$. Then:*
(a) *There is a unique Hopf algebroid structure on $(\bar{A}, \bar{\Gamma})$ such that the projection $\pi : (A, \bar{\Gamma}) \to (\bar{A}, \bar{\Gamma})$ is a map of Hopf algebroids.*
(b) *For $M \in$ ($\bar{\Gamma}$-comod), β_M is an isomorphism. For $N \in$ (Γ-comod), α_N is the projection $N \to N/IN$.*
(c) *The adjoint pair* (1.2) *gives an equivalence between* ($\bar{\Gamma}$-comod) *and the full subcategory of* (Γ-comod) *generated by N such that $IN = 0$.*
(d) *If $IN = 0$ then*

$$\pi_* : \operatorname{Ext}_\Gamma{}^* (A, N) \to \operatorname{Ext}_{\bar{\Gamma}}{}^* (\bar{A}, N)$$

is an isomorphism.

Proof. (a) and (b) are immediate, and (b) implies (c). (d) follows from the fact that $\pi_* : \Omega(\Gamma; N) \to \Omega(\bar{\Gamma}; N)$ is an isomorphism.

PROPOSITION 1.4. ("*Shapiro's Lemma*"). *Let $\pi : (A, \Gamma) \to (B, \Sigma)$ be a morphism of Hopf algebroids. Let Γ be A-flat and assume that $\Gamma \otimes_A B$ is injective as a right Σ-comodule. Then*

$$\operatorname{Ext}_\Gamma{}^* (A, \pi^* N) \cong \operatorname{Ext}_\Sigma{}^* (B, N)$$

naturally in the Σ-comodule N, in such a way that for a Γ-comodule M the following diagram commutes.

$$\begin{tikzcd}
& \operatorname{Ext}_\Gamma{}^* (A, M) \arrow[dl, "{\operatorname{Ext}_\Gamma{}^* (A, \alpha_M)}"'] \arrow[dr, "\pi_*"] & \\
\operatorname{Ext}_\Gamma{}^* (A, \pi^*\pi_* M) \arrow[rr, phantom, "\cong"] & & \operatorname{Ext}_\Sigma{}^* (B, \pi_* M)
\end{tikzcd}$$

Proof. Let X be a right B-module such that $\Gamma \otimes_A B$ is a summand of $X \otimes_B \Sigma$. Let $N \to I^*$ be a resolution of N over Σ. Since I^* is B-split exact, $(X \otimes_B \Sigma) \square_\Sigma I^* \cong X \otimes_B I^*$ is exact, and thus the summand $\pi^* I^* \cong (\Gamma \otimes_A B) \square_\Sigma I^*$ is exact as well. Furthermore if I^q is a summand of $\Sigma \otimes_B Y$ for the left B-module Y then

$$(\Gamma \otimes_A B) \square_\Sigma (\Sigma \otimes_B Y) \cong (\Gamma \otimes_A B) \otimes_B Y \cong \Gamma \otimes_A Y$$

is an extended Γ-comodule. Thus its summand $\pi^* I^q = (\Gamma \otimes_A B) \square_\Sigma I^q$ is an injective Γ-comodule. The isomorphism now follows by adjointness and Lemma 1.1.

Commutativity of the diagram follows from commutativity of the corresponding diagram for Hom.

Section 2. The height n change of rings theorem.

We now turn our attention to the Hopf algebroid (BP_*, BP_*BP); thus a "comodule" will be a BP_*BP-comodule. Recall [7], [5], that the only invariant prime ideals in BP_* are $I_n = (p, v_1, \cdots, v_{n-1})$, $0 \leq n \leq \infty$. For $0 \leq n < \infty$ let $\underline{BP}_n$ denote the full subcategory of comodules M which are finitely presented over BP_* and killed by I_n. By Proposition 1.3 this is equivalent to the category of finitely presented BP_*BP/I_n (BP_*BP)-comodules.

The following generalizations of two theorems of P. S. Landweber will be useful. Their proofs are just as in [8], [9], once we observe that the invariant prime ideals of BP_*/I_n pull back to the ideals I_m, $m \geq n$, in BP_*.

PROPOSITION 2.1. *$M \in \underline{BP}_n$ has a finite filtration by subcomodules F_iM such that for each i, $F_iM/F_{i-1}M$ is a suspension of BP_*/I_k for some k with $n \leq k < \infty$.*

PROPOSITION 2.2. *Let G be a BP_*/I_n-module. $G \otimes_{BP_*}$ is exact on $\underline{BP}_n$ iff v_k is a nonzerodivisor on G/I_k for all k with $n \leq k < \infty$.*

Definition 2.3. A comodule M is of *height n* iff $I_nM = 0$ and $v_n \mid M$ is an isomorphism.

If M is a comodule killed by I_n then $v_n \mid M$ is a comodule map (because $\eta_R v_n \equiv v_n \pmod{I_n}$), so

$$v_n^{-1}M = \varinjlim_{v_n} M$$

is a comodule, of height n. In particular, let $B(n)_* = v_n^{-1}BP_*/I_n$. Then for any comodule M killed by I_n, $v_n^{-1}M = B(n)_* \otimes_{BP_*} M$.

Let $K(0)_* = \mathbf{Q}$ and $K(n)_* = \mathbf{F}_p[v_n, v_n^{-1}]$ for $n > 0$. There is an obvious ring-map $\pi : BP_* \to K(n)_*$. Using it, form

$$K(n)_* BP = K(n)_* \otimes_{BP_*} BP_*BP$$

$$BP_*K(n) = BP_*BP \otimes_{BP_*} K(n)$$

$$K(n)_*K(n) = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_* .$$

Clearly $K(n)_*K(n)$ is a commutative Hopf algebra over $K(n)_*$, and there is a natural map of Hopf algebroids $\pi : BP_*BP \to K(n)_*K(n)$.

Since I_n is invariant, we have a factorization

$$\begin{tikzcd}
B(n)_* \arrow[r, dashed, "\bar{\eta}_R"] & K(n)_* \,\square_{K(n)_*K(n)}\, K(n)_*BP \\
BP_* \arrow[u] \arrow[r, "\eta_R"'] & BP_*BP \arrow[r] & K(n)_*BP \arrow[ul, phantom, "\cap"]
\end{tikzcd}$$

of ring-maps. Thus $K(n)_*BP$ is a right $B(n)_*$-module. It is also obviously a left $K(n)_*K(n)$-comodule, and we have:

PROPOSITION 2.4. *There is a map*

$$K(n)_*BP \to K(n)_*K(n) \otimes_{K(n)_*} B(n)_*$$

which is an isomorphism of $K(n)_*K(n)$-comodules and of $B(n)_*$-modules, and which carries 1 *to* 1.

Proof. Our proof is a counting argument; and in order to meet requirements of connectivity and finiteness, we pass to suitable "valuation rings." Thus let

$$\begin{aligned} k(0)_* &= \mathbf{Z}_{(p)} \subset K(0)_* \\ k(n)_* &= \mathbf{F}_p[v_n] \subset K(n)_*\,, \qquad n > 0 \\ k(n)_*BP &= k(n)_* \otimes_{BP_*} BP_*BP \subset K(n)_*BP \\ b(n)_* &= k(n)_*[u_1\,, u_2\,, \cdots] \subset B(n)_* \end{aligned}$$

where $u_k = v_n^{-1}v_{n+k}$.

It follows from Theorem 1 of [15] that in $k(n)_*BP$,

$$\eta_R(v_{n+k}) \equiv v_n t_k^{p^n} - v_n^{p^k} t_k \bmod (\eta_R(v_{n+1}), \cdots \eta_R(v_{n+k-1})). \tag{2.5}$$

Hence $\eta_R : BP_* \to k(n)_*BP$ factors through an algebra map $b(n)_* \to k(n)_*BP$. It is clear from (2.5) that as a right $b(n)_*$-module, $k(n)_*BP$ is free on generators $t^\alpha = t_1^{\alpha_1}t_2^{\alpha_2} \cdots$ where $0 \le \alpha_i < p^n$ and all but finitely many α_i are 0; in particular it is of finite type over $b(n)_*$.

Now define

$$s(n)_* = k(n)_*BP \otimes_{b(n)_*} k(n)_* \subset K(n)_*K(n);$$

by the above remarks $s(n)_* = k(n)_*[t_1\,, t_2\,, \cdots]/(t_k^{p^n} - v_n^{p^k-1}t_k : k \ge 1)$ as an algebra. $(k(n)_*\,, s(n)_*)$ is clearly a sub Hopf algebroid of $(K(n)_*\,, K(n)_*K(n))$; so $s(n)_*$ is a Hopf algebra over the principal ideal domain $k(n)_*$.

The natural map $BP_*BP \to s(n)_*$ makes BP_*BP a left $s(n)_*$-comodule, and this induces a left $s(n)_*$-comodule structure on $k(n)_*BP$. We will show that the latter is an *extended* left $s(n)_*$-comodule.

Define a $b(n)_*$-linear map $f : k(n)_*BP \to b(n)_*$ by

$$f(t^\alpha) = \begin{cases} 1 & \text{if } \alpha = (0, 0, \cdots) \\ 0 & \text{otherwise.} \end{cases}$$

Then f satisfies the equations

$$\begin{aligned} f\bar{\eta}_R &= id. : b(n)_* \to b(n)_* \\ f \otimes_{b(n)_*} k(n)_* &= \epsilon : s(n)_* \to k(n)_*\,. \end{aligned}$$

Now let $\tilde{f}$ be the $s(n)_*$-comodule map lifting f:

$$\begin{tikzcd} k(n)_*BP \arrow[r, "\psi"] \arrow[dr, "\tilde{f}"'] & s(n)_* \otimes_{k(n)_*} k(n)_*BP \arrow[d, "1 \otimes f"] \\ & s(n)_* \otimes_{k(n)_*} b(n)_* \end{tikzcd} \tag{2.6}$$

Since $\psi\bar{\eta}_R(x) = 1 \otimes \bar{\eta}_R(x)$, ψ is $b(n)_*$-linear, so $\tilde{f}$ is too. We claim $\tilde{f}$ is an isomorphism. Since both sides are free of finite type over $b(n)_*$ it suffices to prove that $\tilde{f} \otimes_{b(n)_*} k(n)_*$ is an isomorphism. But (2.6) is then reduced to

$$\begin{tikzcd} s(n)_* \arrow[r, "\Delta"] \arrow[dr, "\tilde{f} \otimes_{b(n)_*} k(n)_*"'] & s(n)_* \otimes_{k(n)_*} s(n)_* \arrow[d, "1 \otimes \epsilon"] \\ & s(n)_* \otimes_{k(n)_*} k(n)_* \end{tikzcd}$$

so the claim follows from unitarity of Δ.

Now the map $K(n)_* \otimes_{k(n)_*} \tilde{f}$ satisfies the requirements of the proposition.

COROLLARY 2.7. *$\bar{\eta}_R : B(n)_* \to K(n)_* \square_{K(n)_*K(n)} K(n)_*BP$ is an isomorphism of $B(n)_*$-modules.*

Proof. The natural isomorphism

$$B(n)_* \to K(n)_* \square_{K(n)_*K(n)} (K(n)_*K(n) \otimes_{K(n)_*} B(n)_*)$$

is $B(n)_*$ -linear and carries 1 to 1. Hence

$$\begin{tikzcd} & K(n)_* \square_{K(n)_*K(n)} (K(n)_*K(n) \otimes_{K(n)_*} B(n)_*) \\ B(n)_* \arrow[ur, "\cong"] \arrow[dr, "\bar{\eta}_R"'] & \\ & K(n)_* \square_{K(n)_*K(n)} K(n)_*BP \arrow[uu, "\cong"'] \end{tikzcd}$$

commutes, and $\bar{\eta}_R$ is an isomorphism.

For any comodule M we have a natural factorization

$$\begin{tikzcd} M \arrow[r, "\alpha_M"] \arrow[d] & \pi^*\pi_*M \\ B(n)_* \otimes_{BP_*} M \arrow[ur, dashed, "\bar{\alpha}_M"'] & \end{tikzcd}$$

of the adjunction morphism α_M .

PROPOSITION 2.8. *$\bar{\alpha}_M$ is an isomorphism.*

Proof. We shall prove below (Lemma 2.11) that every comodule is a direct limit of finitely presented comodules. Thus we may assume that M is finitely presented. Replacing M by M/I_nM leaves source and target unaltered, so we may assume that $M \in \underline{BP}_n$.

Proposition 2.2 implies that $B(n)_* \otimes_{BP_*}$ and $K(n)_* \otimes_{BP_*}$ are exact on $\underline{BP}_n$; and by Proposition 2.4, $BP_*K(n) \square_{K(n)_*K(n)}$ is exact on $(K(n)_*K(n)$-comod).

Thus if $M \in \underline{BP}_n$ is filtered as in Proposition 2.1, then both source and target of $\bar{\alpha}_M$ are filtered, and by induction using the 5-lemma we are reduced to considering $M = BP_*/I_k$, $k \geq n$.

If $M = BP_*/I_k$ with $k > n$, then both source and target are 0. If $M = BP_*/I_n$, it is easy to see that $\bar{\alpha}_M$ is transformed by the conjugation c to

$$\bar{\eta}_R : B(n)_* \to K(n)_* \square_{K(n)_*K(n)} K(n)_*BP,$$

which is an isomorphism by Corollary 2.6.

THEOREM 2.9. *The category of BP_*BP-comodules of height n is naturally equivalent to the category of $K(n)_*K(n)$-comodules.*

Proof. The functors giving the equivalence are of course $\pi^* = BP_*K(n) \square_{K(n)_*K(n)}$ and $\pi_* = K(n)_* \otimes_{BP_*}$. Now $\beta_N : \pi_*\pi^*N \to N$ is clearly an isomorphism for any $K(n)_*K(n)$-comodule N. On the other hand we identified α_M with the map $M \to B(n)_* \otimes_{BP_*} M$, which is an isomorphism exactly when M is of height n.

THEOREM 2.10. *The natural map*

$$\mathrm{Ext}_{BP_*BP}{}^* (BP_*, M) \to \mathrm{Ext}_{K(n)_*K(n)}{}^* (K(n)_*, K(n)_* \otimes_{BP_*} M)$$

is an isomorphism if M is of height n.

Proof. By Proposition 2.4, our Shapiro's Lemma applies, and the result follows from Proposition 2.8.

We now turn to a proof of

LEMMA 2.11. *Every BP_*BP-comodule is a direct limit of finitely presented comodules.*

We begin with a result due to P. S. Landweber ([9], Prop. 2.4).

LEMMA 2.12. *Every element of a BP_*BP-comodule M is in the image of a comodule map from a comodule which is free and finitely generated over BP_*.*

Proof. First let N be a *right* BP_*BP-comodule which is free and finitely generated over BP_*, and write $\psi(x) = \sum_\alpha r_\alpha(x) \otimes t^\alpha \in N \otimes_{BP_*} BP_*BP$. $N^* = \mathrm{Hom}_{\mathbf{Z}_p} (N, \mathbf{Z}_p)$ is naturally a left BP_*-module, and in fact a left BP_*BP-comodule with $\psi(y) = \sum_\alpha z^\alpha \otimes r_\alpha{}^*(y)$, where z_i is the conjugate ct_i of t_i, and where $r_\alpha{}^* = \mathrm{Hom}_{\mathbf{Z}_{(p)}} (r_\alpha, \mathbf{Z}_{(p)})$.

In particular, for $d \geq 0$ let $N(d)$ be the right sub BP_*-module of BP_*BP generated by $\{z^\alpha : |\alpha| \leq d\}$, where $|\alpha| = 2\Sigma\alpha_i(p^i - 1)$ is the dimension of z^α. Since $\Delta(z^\alpha)$ is homogeneous, this is a subcomodule of BP_*BP.

Now let M be an arbitrary comodule, and let $x \in M$. There exists an integer d such that $\psi(x) \in N(d) \otimes_{BP_*} M$. By associativity, $\psi(x)$ corresponds under the isomorphism $N(d) \otimes_{BP_*} M \simeq \mathrm{Hom}_{BP_*} (N(d)^*, M)$ to a comodule map. If $y \in N(d)^*$ is such that

$$\langle z^{\alpha}, y\rangle = \begin{cases} 1 & \alpha = 0 \\ 0 & \alpha \neq 0 \end{cases}$$

then y hits x, and we are done.

Note that $N(d)^*$ is the BP-homology of the Spanier–Whitehead dual of the d-skeleton of BP.

COROLLARY 2.13. *Every BP_*BP-comodule is the direct limit of its finitely generated sub-comodules.*

COROLLARY 2.14. *Every BP_*BP-comodule M is a quotient of a comodule F which is BP_*-free. If M is finitely generated over BP_* then we may take F to be finitely generated over BP_*.*

Proof of Lemma 2.11. By Corollary 2.13 we may assume that M is finitely generated. By Corollary 2.14 there is a short exact sequence

$$0 \to R \to F \to M \to 0$$

of comodules such that F is free and finitely generated over BP_*. Let Σ be the directed set of finitely generated subcomodules S of R. Then by Corollary 2.13 and the exactness of $\lim_{\to}$,

$$M = \varinjlim_{S \in \Sigma} F/S;$$

and each F/S is finitely presented.

Section 3. The v_n-local change of rings theorem.

In this section we will generalize the change of rings theorem of Section 2 to a larger category of comodules. Instead of requiring that I_n annihilate the comodule, we demand only that each element be killed by some power of I_n.

Definition 3.1. Let A be a commutative ring and $I \subset A$ an ideal. An A-module M is *I-nil* iff for each $x \in M$ there is an integer k such that $I^k x = 0$. M is *I-nilpotent* iff for some k, $I^k M = 0$.

These notions coincide when M is finitely generated.

This condition is related to the invertibility of v_n by the following Lemma.

LEMMA 3.2. *If the BP_*BP-comodule M is I_n-nil, then there exists a unique BP_*BP-comodule structure on*

$$v_n^{-1}M = \varinjlim_{v_n} M$$

such that the localization map $M \to v_n^{-1}M$ is a map of comodules.

Proof. Let $M' \subset M$ be a finitely generated subcomodule. Then M' is I_n-nilpotent; say $I_n^k M' = 0$. Then ([11] : 3.6) multiplication by $v_n^{p^{k-1}}$ on M'

is a comodule map. Hence $v_n^{-1}M'$, regarded as the direct limit of the system

$$M' \xrightarrow{v_n^{p^{k-1}}} M' \xrightarrow{v_n^{p^{k-1}}} M' \longrightarrow \cdots ,$$

is a comodule in such a way that $M' \to v_n^{-1}M'$ is a comodule map. Also it is clear that if $M' \subset M'' \subset M$ with M'' finitely generated then $v_n^{-1}M' \to v_n^{-1}M''$ is a comodule map. Thus $v_n^{-1}M = \lim_{\to} v_n^{-1}M'$, the limit taken over finitely generated subcomodules of M, is a comodule. Uniqueness is clear.

Definition 3.3. A BP_*BP-comodule M is *v_n-local* iff v_n acts bijectively on M.

Example 3.4. Let $N_n^0 = BP_*/I_n$. Define comodules N_n^s, M_n^s inductively as follows. Suppose that N_n^s has been defined and is I_{n+s}-nil. Then $M_n^s = v_{n+s}^{-1}N_n^s$ is a comodule by Lemma 3.2, and the exact sequence

$$0 \to N_n^s \to M_n^s \to N_n^{s+1} \to 0$$

defines an I_{n+s+1}-nil comodule N_n^{s+1}. M_n^s is v_{n+s}-local. This example plays a central role in [13].

We postpone to the end of this section a proof of the following result of Peter Landweber. We are grateful to him for allowing us to include it.

PROPOSITION 3.5. *Any v_n-local comodule is I_n-nil.*

We next describe the replacement in this context for $K(n)_*$.

Definition 3.6. Let

$$E(n)_* = \mathbf{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$$

with the BP_*-algebra structure sending v_i to 0 for $i > n$.

Remark 3.7. Let E_* be any BP_*-module on which the sequence $p, v_1, \cdots$ is regular : that is, such that v_n acts injectively on E_*/I_nE_* for all $n \geq 0$. Then by Prop. 2.2 $E_* \otimes_{BP_*}$ is exact on $\underline{BP}_0$, so by Lemma 2.11 $E_* \otimes_{BP_*}$ is exact on the category of BP_*BP-comodules.

Now suppose E_* is a commutative BP_*-algebra on which $p, v_1, \cdots$ is regular. Then define

$$E_*E = E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_* .$$

E_*E is a Hopf algebroid by extension of the structure maps for BP_*BP. We claim that E_*E is E_*-flat. For an E_*-module M,

$$E_*E \otimes_{E_*} M = E_* \otimes_{BP_*} (BP_*BP \otimes_{BP_*} M).$$

By the above remarks and the flatness of BP_*BP, this is an exact functor of M, as desired.

Topological Remark 3.8. If E_* is a BP_*-module on which $p, v_1, \cdots$, acts regularly, then by the above remarks $E_*(X) = E_* \otimes_{BP_*} BP_*(X)$ defines an additive homology theory on spectra. Thus there is a spectrum E such that $E_*(X) \cong \pi_*(E \wedge X)$ naturally in X; see [2], Remark 6.5. Clearly

$$\pi_*(E \wedge E) \cong E_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_* \,. \tag{3.9}$$

In any case if $E_* = E(n)_*$, one can show from the work of [18] and [19] that $E(n)$ is an associative ring-spectrum in such a way that the map $BP_*(\ ) \to E(n)_*(\ )$ is multiplicative. By Remark 3.7, the right-hand side of (3.9) is $E(n)_*$-flat, so $\pi_*(E \wedge E)$ is the usual [1] Hopf algebroid of cooperations and (3.9) is an isomorphism of Hopf algebroids. Thus there is an Adams spectral sequence with

$$E_2^* = \mathrm{Ext}_{E(n)_*E(n)}{}^* \,(E(n)_*\,,\, E(n)_*(X)),$$

natural in the spectrum X.

For example, $E(0)_*(\ )$ is rational homology theory, and $E(1)_*(\ )$ is a factor of complex K-theory localized at p.

The main theorem of this section is

THEOREM 3.10. *If M is a v_n-local comodule then the natural map*

$$\mathrm{Ext}_{BP_*BP}{}^* \,(BP_*\,,\, M) \to \mathrm{Ext}_{E(n)_*E(n)}{}^* \,(E(n)_*\,,\, E(n)_* \otimes_{BP_*} M)$$

is an isomorphism.

The proof will use a couple of lemmas.

LEMMA 3.11. *Any v_n-local comodule is a direct limit of comodules $v_n^{-1}M$ where M is finitely presented and I_n-nilpotent.*

Proof. Write $M = \lim_\to M_\sigma$ as a direct limit of finitely presented comodules by Lemma 2.11. M_σ' Im $(i_\sigma : M_\sigma \to M)$ is finitely generated and I_n-nilpotent (since M is I_n-nil); let $k(\sigma)$ be the least integer k such that $I_n^k M_\sigma' = 0$. Then i_σ factors through $M_\sigma / I_n^{k(\sigma)} M_\sigma$. Note that if $\sigma \le \tau$ in Σ then $k(\sigma) \le k(\tau)$, so the comodules $M_\sigma / I_n^{k(\sigma)} M$ form a directed system with limit M. Finally, $M_\sigma / I_n^{k(\sigma)} M_\sigma$ is I_n-nil so $v_n^{-1} M_\sigma / I_n^{k(\sigma)} M_\sigma$ is a comodule. Now the system of these comodules has direct limit $M = v_n^{-1}M$ since localization commutes with direct limits.

LEMMA 3.12. *If M is a finitely presented I_n-nilpotent comodule then $v_n^{-1}M$ has a finite filtration with quotients isomorphic to suspensions of $B(n)_*$.*

Proof. Consider a Landweber filtration of M. Since any quotient of an I_n-nilpotent module is I_n-nilpotent, the associated quotients are suspensions of BP_*/I_k for $k \ge n$. Now invert v_n . Since localization is exact we obtain a filtration with quotients

$$v_n^{-1} BP_*/I_k = \begin{cases} B(n)_* & \text{if} \quad k = n \\ 0 & \text{if} \quad k > n. \end{cases}$$

Proof of Theorem 3.10. We may assume $M = v_n^{-1}M'$ with M' finitely presented and I_n-nilpotent, as the general case then follows from Lemma 3.11. By Lemma 3.12 and Prop. 2.1 it suffices to prove the theorem for $M' = BP_*/I_n$. So consider the commutative diagram

$$\begin{tikzcd} & \operatorname{Ext}^*_{BP_*BP}(BP_*, B(n)_*) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_3"] & \\ \operatorname{Ext}^*_{E(n)_*E(n)}(E(n)_*, K(n)_*) \arrow[rr, "\pi_2"] & & \operatorname{Ext}^*_{K(n)_*K(n)}(K(n)_*, K(n)_*). \end{tikzcd}$$

π_2 is an isomorphism by Prop. 1.3, and π_3 is an isomorphism by Theorem 2.10, so π_1 is an isomorphism.

We turn now to a proof of Proposition 3.5.

LEMMA 3.13. *Let $n > k \geq 0$ and let M be a BP_*BP-comodule such that $v_nM = M$ and $I_kM = 0$. Then each element of M is v_k-torsion.*

Proof. Suppose first that $x \in M$ is primitive—i.e. $\psi(x) = 1 \otimes x$—and let $x = v_ny$. Recall ([5] Lemma 1.7) that modulo I_k, $\eta_Rv_n \equiv v_kt_{n-k}^{p^k} + \cdots$, where the other terms involve t^α with $|\alpha| < |p^k\Delta_{n-k}|$; here Δ_{n-k} is the multiindex with 1 in the $(n-k)$th place and 0 elsewhere. Now let β_0 be a multiindex of maximal dimension such that $r_{\beta_0}(y) \neq 0$. Then the coefficient of $z^{p^k\Delta_{n-k}+\beta_0}$ in

$$1 \otimes x = \psi(v_ny) = \sum_{\alpha,\beta} z^{\alpha+\beta} \otimes r_\alpha(v_n)r_\beta(y) \tag{3.14}$$

is $v_kr_{\beta_0}(y)$, which is thus 0.

Next let β_1 be a multiindex of maximal dimension such that $v_kr_{\beta_1}(y) \neq 0$; thus $|\beta_1| < |\beta_0|$. Multiply (3.14) by v_k and observe that the coefficient of $z^{p^k\Delta_{n-k}+\beta_1}$ is $v_k^2r_{\beta_1}(y)$, which is thus 0.

Continuing, we find that $r_\beta(y)$ is v_k-torsion for all β, and in particular for $\beta = 0$. That is, y is v_k-torsion; so $x = v_ny$ is too, q.e.d.

Now let x be arbitrary. Let β_0 be a multiindex of maximal dimension such that $r_{\beta_0}(y) \neq 0$. Then for $\gamma \neq 0$, $r_\gamma r_{\beta_0}(x) = \sum_\alpha a_{\gamma\beta_0}{}^\alpha r_\alpha(x) = 0$—i.e., $r_{\beta_0}(x)$ is primitive, and hence v_k-torsion. Proceeding by induction, we find that x is primitive mod v_k-torsion, and hence is v_k-torsion.

Proof of Proposition 3.5. Since $I_0 = 0$, $I_0M = 0$, and Lemma 3.13 implies that M is I_1-nil. Suppose inductively that M is I_k-nil for some $k < n$. Let $F_1M = \{x \in M : I_kx = 0\}$; it is a subcomodule since I_k is invariant. Multiplication by v_n on F_1M is clearly monic, and we claim that it is epic as well. Let $x \in F_1M$; there exists $y \in M$ such that $x = v_ny$. Then $0 = I_kx = v_nI_ky$ implies $I_ky = 0$ since $v_n \mid M$ is monic. Now define a filtration of M inductively by means of the pull-back diagram

$$\begin{tikzcd} 0 \arrow[r] & F_iM \arrow[r] \arrow[d, equal] & F_{i+1}M \arrow[r] \arrow[d] & F_1(M/F_iM) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_iM \arrow[r] & M \arrow[r] & M/F_iM \arrow[r] & 0. \end{tikzcd}$$

By induction $F_{i+1}M/F_iM$ is v_n-local and killed by I_k, and hence by Lemma 3.13 is v_k-torsion. So $F_{i+1}M$ is I_{k+1}-nil for all i. Since M is I_k-nil, $M = \lim_{\to} F_iM$ and hence M is I_{k+1}-nil. The proposition now follows by induction on k.

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