lecture[1]Browder's work on the Arf-Kervaire invariant problemlecture-text

Browder's work on Arf-Kervaire invariant problem

Panorama of Topology
A Conference in Honor of William Browder

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1 Background and history

1.1 Browder's theorem and its impact

Browder's theorem and its impact

In 1969 Browder proved a remarkable theorem about the Kervaire invariant.

The Kervaire invariant of framed manifolds and its generalization*

By WILLIAM BROWDER

In 1960, Kervaire [11] introduced an invariant for almost framed $(4k+2)$-manifolds, $(k \neq 0, 1, 3)$, and proved that it was zero for framed 10-manifolds,

Browder's Theorem (1969). *The Kervaire invariant of a smooth framed $(4m+2)$-manifold M can be nontrivial only if $m = 2^{j-1} - 1$ for some $j > 0$. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence.*

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Browder's theorem and its impact (continued)

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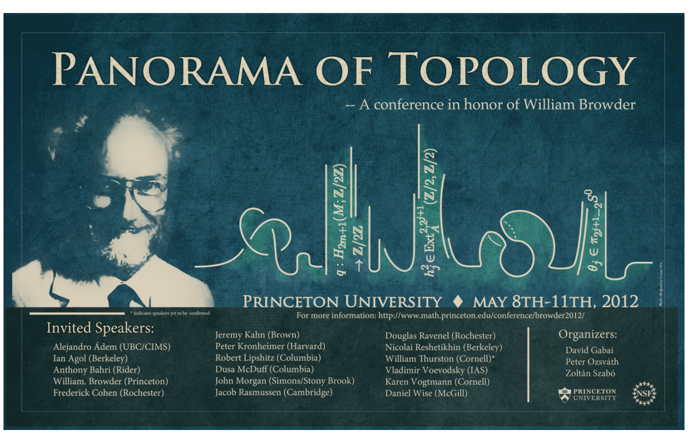

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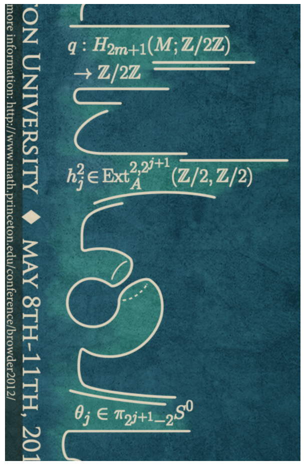

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This result established a link between surgery theory, specifically an unanswered question in the Kervaire-Milnor classification of exotic spheres, and stable homotopy theory, specifically the Adams spectral sequence.

This connection made the problem of constructing a smooth framed manifold with nontrivial Kervaire invariant in dimension $2^{j+1}-2$ a cause celebre in algebraic topology throughout the 1970s. For 40 years it was the definitive theorem on this subject.

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Browder's theorem and its impact (continued)

Browder's theorem says that there is a framed manifold with nontrivial Kervaire invariant in dimension $2^{j+1}-2$ iff a certain element in the Adams spectral sequence survives. This would correspond to an element $\theta_j \in \pi_{n+2^{j+1}-2}S^n$ for large n.

Mark Mahowald

Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j. He derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the Doomsday Hypothesis.

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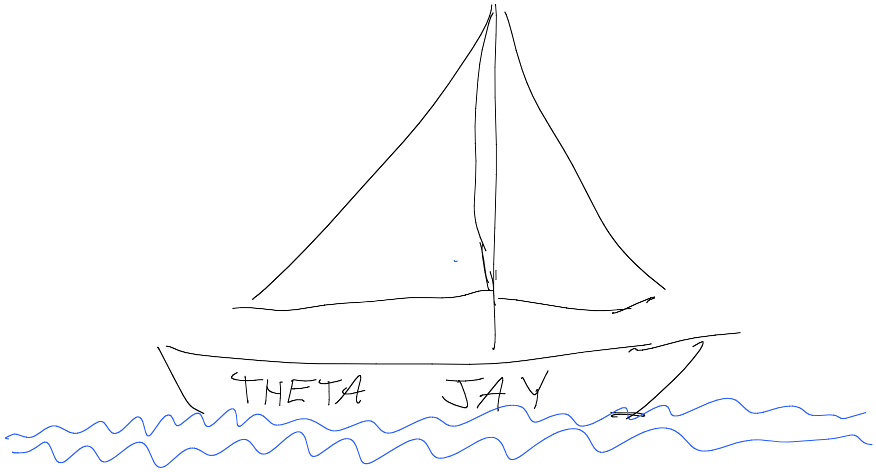
Mark Mahowald's sailboat



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Browder's theorem and its impact (continued)

Drawing by Carolyn Snaith
London, Ontario 1981

There were numerous attempts to construct such manifolds thoughout that decade. They all failed. We know now that they failed for good reason. After 1980 the problem faded into the background because it was thought to be too hard.

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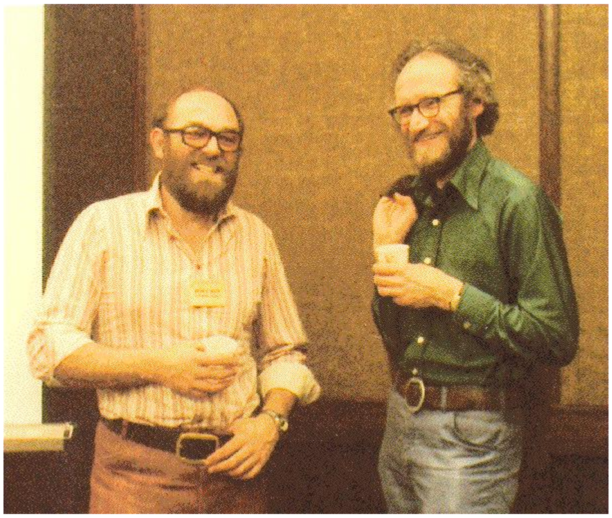
Browder's theorem and its impact (continued)



Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson .9

Browder's theorem and its impact (continued)

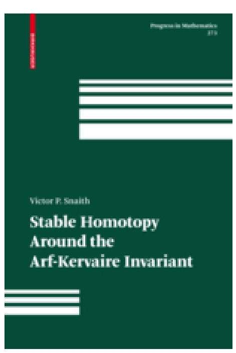
Snaith's book

Fast forward
to 2009

Stable Homotopy Around the Arf-Kervaire Invariant, published in early 2009.

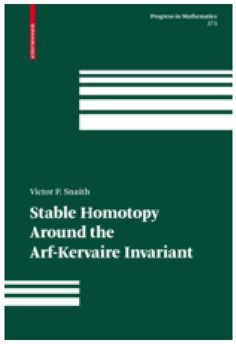
"As ideas for progress on a particular mathematics problem atrophy it can disappear. Accordingly I wrote this book to stem the tide of oblivion." .10

Browder's theorem and its impact (continued)



"For a brief period overnight we were convinced that we had the method to make all the sought after framed manifolds - a feeling which must have been shared by many topologists working on this problem. All in all, the temporary high of believing that one had the construction was sufficient to maintain in me at least an enthusiastic spectator's interest in the problem." .11

Browder's theorem and its impact (continued)

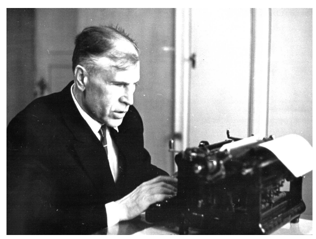


"In the light of the above conjecture and the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might turn out to be a book about things which do not exist. This [is] why the quotations which preface each chapter contain a preponderance of utterances from the pen of Lewis Carroll." .12

1.2 Some early homotopy theory

Pontryagin's early work on homotopy groups of spheres

Back to the 1930s



Lev Pontryagin 1908-1988

Pontryagin's approach to continuous maps $f : S^{n+k} \to S^k$ was

- Assume f is smooth. We know that any map f is homotopic to a smooth one.
- Pick a regular value $y \in S^k$. Its inverse image will be a smooth n-manifold M in S^{n+k}.
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.

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Pontryagin's early work on homotopy groups of spheres (continued)

$$\begin{tikzcd}
 & S^{n+k} \arrow[r, "f"] & S^k \\
M^n \times D^k \arrow[r, equal] & V^{n+k} \arrow[u, hook] \arrow[r] & D^k \arrow[u, hook] \\
 & M^n \arrow[u, hook] \arrow[r] & \{y\} \arrow[u, hook]
\end{tikzcd}$$

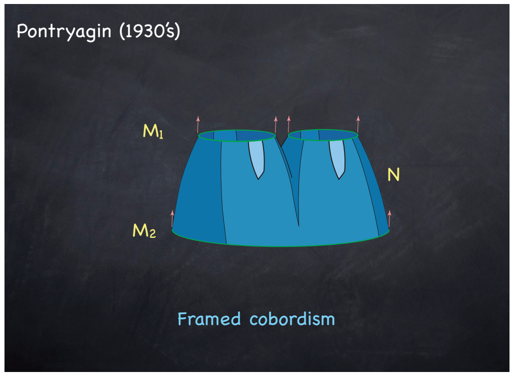
Let D^k be the closure of an open ball around a regular value $y \in S^k$. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^k) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^k$.

A local coordinate system around around the point $y \in S^k$ pulls back to one around M called a framing.

There is a way to reverse this procedure. A framed manifold $M^n \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^k$. .14

Pontryagin's early work (continued)

Suppose there is homotopy $h : S^{n+k} \times [0,1] \to S^k$ between two such maps $f_1, f_2 : S^{n+k} \to S^k$. If $y \in S^k$ is a regular value of h, then $h^{-1}(y)$ is a framed $(n+1)$-manifold $N \subset S^{n+k} \times [0,1]$ whose boundary is the disjoint union of $M_1 = f_1^{-1}(y)$ and $M_2 = f_2^{-1}(y)$. This N is called a framed cobordism between M_1 and M_2. When it exists the two closed manifolds are said to be framed cobordant.



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Pontryagin's early work (continued)

Let $\Omega^{fr}_{n,k}$ denote the cobordism group of framed n-manifolds in $\mathbf{R}^{n+k}$, or equivalently in S^{n+k}. Pontryagin's construction leads to a homomorphism

$$\Omega^{fr}_{n,k} \to \pi_{n+k} S^k.$$

Pontyagin's Theorem (1936). *The above homomorphism is an isomorphism in all cases.*

Both groups are known to be independent of k for $k > n$. We denote the resulting stable groups by simply Ω^{fr}_n and π^S_n.

The determination of the stable homotopy groups π^S_n is an ongoing problem in algebraic topology.

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1.3 Classifying exotic spheres

The Kervaire-Milnor classification of exotic spheres

Into the 60s again

About 50 years ago three papers appeared that revolutionized algebraic and differential topology.

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  John Milnor's *On manifolds homeomorphic to the 7-sphere*, 1956. He constructed the first "exotic spheres", manifolds homeomorphic but not diffeomorphic to the standard S^7. They were certain S^3-bundles over S^4.

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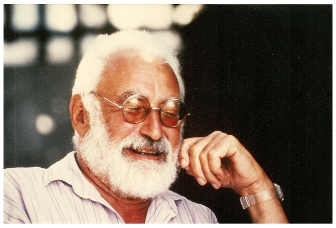
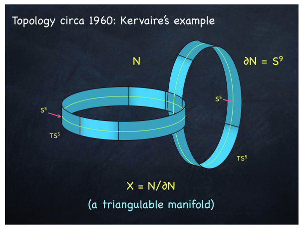
The Kervaire-Milnor classification of exotic spheres (continued)

- 
  

  Michel Kervaire 1927-2007

  Michel Kervaire's *A manifold which does not admit any differentiable structure*, 1960. His manifold was 10-dimensional. I will say more about it later. .18

The Kervaire-Milnor classification of exotic spheres (continued)

- Kervaire and Milnor's *Groups of homotopy spheres, I*, 1963.

  For example, for $n = 1, 2, 3, \cdots, 18$, it will be shown that the order of the group Θ_n is respectively:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$[\Theta_n]$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16.

  They gave a complete classification of exotic spheres in dimensions ≥ 5, with two caveats:

  (i) Their answer was given in terms of the stable homotopy groups of spheres, which remain a mystery to this day.

  (ii) There was an ambiguous factor of two in dimensions congruent to 1 mod 4. That problem is the subject of this talk. .19

1.4 Exotic spheres as framed manifolds

Exotic spheres as framed manifolds

Following Kervaire-Milnor, let Θ_n denote the group of diffeomorphism classes of exotic n-spheres Σ^n. The group operation here is connected sum.

Each Σ^n admits a framed embedding into some Euclidean space $\mathbf{R}^{n+k}$, but the framing is not unique. Thus we do not have a homomorphism from Θ_n to π_n^S, but we do get a map to a certain quotient.

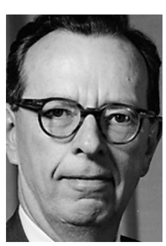
Two framings of an exotic sphere $\Sigma^n \subset S^{n+k}$ differ by a map to the special orthogonal group $SO(k)$, and this map does not depend on the differentiable structure on Σ^n. .20

Exotic spheres as framed manifolds (continued)

Varying the framing on the standard sphere S^n leads to a homomorphism

Heinz Hopf
1894-1971

$$\pi_n SO(k) \xrightarrow{J} \pi_{n+k} S^k$$



George Whitehead
1918-2004

called the Hopf-Whitehead J-homomorphism. It is well understood by homotopy theorists. .21

Exotic spheres as framed manifolds (continued)

Thus we get a homomorphism

$$\Theta_n \xrightarrow{p} \pi_n^S/\mathrm{Im}\,J.$$

The bulk of the Kervaire-Milnor paper is devoted to studying its kernel and cokernel using surgery. The two questions are closely related.

- The map p is onto iff every framed n-manifold is cobordant to a sphere, possibly an exotic one.
- It is one-to-one iff every exotic n-sphere that bounds a framed manifold also bounds an $(n+1)$-dimensional disk and is therefore diffeomorphic to the standard S^n.

They denote the kernel of p by bP_{n+1}, the group of exotic n-spheres bounding parallelizable $(n+1)$-manifolds. .22

Exotic spheres as framed manifolds (continued)

Hence we have an exact sequence

$$0 \longrightarrow bP_{n+1} \longrightarrow \Theta_n \xrightarrow{p} \pi_n^S/\mathrm{Im}\,J.$$

Kervaire-Milnor Theorem (1963).

- *The homomorphism p above is onto except possibly when $n = 4m+2$ for $m \in \mathbf{Z}$, and then the cokernel has order at most 2.*
- *Its kernel bP_{n+1} is trivial when n is even.*
- *bP_{4m} is a certain cyclic group. Its order is related to the numerator of the mth Bernoulli number. The key invariant here is the index of the $4m$-manifold.*
- *The order of bP_{4m+2} is at most 2.*
- *bP_{4m+2} is trivial iff the cokernel of p in dimension $4m+2$ is nontrivial.*

We now know the value of bP_{4m+2} in every case except $m = 31$. .23

Exotic spheres as framed manifolds (continued)

In other words have a 4-term exact sequence

$$0 \longrightarrow \Theta_{4m+2} \xrightarrow{p} \pi_{4m+2}^S/\mathrm{Im}\,J \longrightarrow \mathbf{Z}/2 \longrightarrow bP_{4m+2} \longrightarrow 0$$

The early work of Pontryagin implies that $bP_2 = 0$ and $bP_6 = 0$.

In 1960 Kervaire showed that $bP_{10} = \mathbf{Z}/2$.

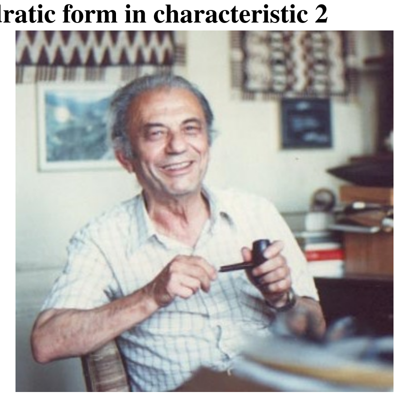
To say more about this we need to define the Kervaire invariant of a framed manifold. .24

2 The Arf-Kervaire invariant

2.1 The Arf invariant of a quadratic form in characteristic 2

The Arf invariant of a quadratic form in characteristic 2

Back to the 1940s



Cahit Arf 1910-1997

Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$. It is known that $\overline{H}$ has a basis of the form $\{a_i, b_i \colon 1 \le i \le n\}$ with

$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

.25

The Arf invariant of a quadratic form in characteristic 2 (continued)

In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

.26

The Arf invariant of a quadratic form in characteristic 2 (continued)

A quadratic refinement of λ is a map $q : \overline{H} \to \mathbf{Z}/2$ satisfying

$$q(x+y) = q(x) + q(y) + \lambda(x,y)$$

Its Arf invariant is

$$\mathrm{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbf{Z}/2.$$

In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q.

.27

Money talks: Arf's definition republished in 2009

Cahit Arf 1910-1997

.28

Bill's election year definition of the Arf invariant (1968)

The elements of $\overline{H}$ hold an election, using the function q to vote for 0 or 1. $\mathrm{Arf}(q)$ is the winner.

America is a democracy. If this is not an invariant, then I don't know what is.

.29

2.2 The Kervaire invariant of a framed $(4m+2)$-manifold

The Kervaire invariant of a framed $(4m+2)$-manifold

Into the 60s
a third time

Let M be a $2m$-connected smooth closed framed manifold of dimension $4m+2$. Let $H = H_{2m+1}(M;\mathbf{Z})$, the homology group in the middle dimension. Each $x \in H$ is represented by an embedding $i_x : S^{2m+1} \hookrightarrow M$ with a stably trivialized normal bundle. H has an antisymmetric bilinear form λ defined in terms of intersection numbers.

Here is a simple example. Let $M = T^2$, the torus, be embedded in S^3 with a framing. We define the quadratic refinement

$$q : H_1(T^2;\mathbf{Z}/2) \to \mathbf{Z}/2$$

as follows. An element $x \in H_1(T^2;\mathbf{Z}/2)$ can be represented by a closed curve, with a neighborhood V which is an embedded cylinder. We define $q(x)$ to be the number of its full twists modulo 2. .30

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

For $M = T^2 \subset S^3$ and $x \in H_1(T^2;\mathbf{Z}/2)$, $q(x)$ is the number of full twists in a cylinder V neighboring a curve representing x. This function is not additive!

.31

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

Again, let M be a $2m$-connected smooth closed framed manifold of dimension $4m+2$, and let $H = H_{2m+1}(M;\mathbf{Z})$. Each $x \in H$ is represented by an embedding $S^{2m+1} \hookrightarrow M$. H has an antisymmetric bilinear form λ defined in terms of intersection numbers.

Kervaire defined a quadratic refinement q on its mod 2 reduction $\overline{H}$ in terms of each sphere's normal bundle. The Kervaire invariant $\Phi(M)$ is defined to be the Arf invariant of q.

Recall the Kervaire-Milnor 4-term exact sequence

$$0 \longrightarrow \Theta_{4m+2} \xrightarrow{p} \pi^S_{4m+2}/\mathrm{Im}\, J \longrightarrow \mathbf{Z}/2 \longrightarrow bP_{4m+2} \longrightarrow 0$$

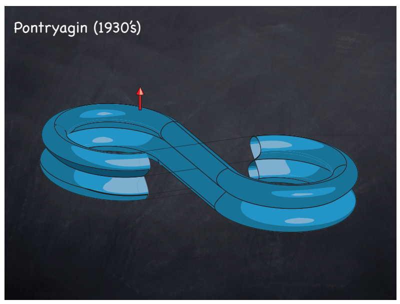
Kervaire-Milnor Theorem (1963). *$bP_{4m+2} = 0$ iff there is a smooth framed $(4m+2)$-manifold M with $\Phi(M)$ nontrivial.* .32

2.3 Some theorems about $\phi(M)$

Some theorems about $\phi(M)^{4m+2}$

What can we say about $\Phi(M)$?

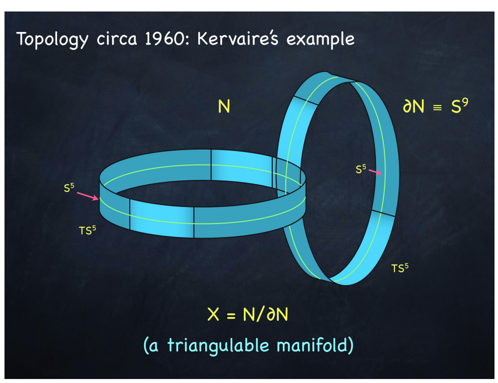
For $m = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant.



Pontryagin used it in 1950 (after some false starts in the 30s) to show $\pi_{k+2}(S^k) = \mathbf{Z}/2$ for all $k \geq 2$. There are similar framings of $S^3 \times S^3$ and $S^7 \times S^7$. This means that bP_2, bP_6 and bP_{14} are each trivial. .33

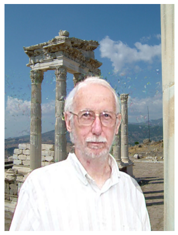
Some theorems about $\phi(M)^{4m+2}$ (continued)

Kervaire (1960) showed it must vanish when $m = 2$, so $bP_{10} = \mathbf{Z}/2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure.

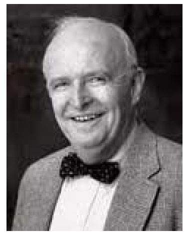


This construction generalizes to higher m, but Kervaire's proof that the boundary is exotic does not. .34

Some theorems about $\phi(M)^{4m+2}$ (continued)



Ed Brown



Frank Peterson
1930-2000

Brown-Peterson (1966) showed that it vanishes for all positive even m. This means $bP_{8\ell+2} = \mathbf{Z}/2$ for $\ell > 0$. .35

3 Browder's theorem

3.1 The quadratic operation

Browder's theorem

Browder's Theorem (1969). *The Kervaire invariant of a smooth framed $(4m+2)$-manifold M can be nontrivial only if $m = 2^{j-1} - 1$ for some $j > 0$. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence.*

This means that $bP_{4m+2} = \mathbf{Z}/2$ unless $m+1$ is a power of 2, and $bP_{2^{j+1}-2}$ vanishes only under the condition stated above.

Recall that the Kervaire invariant associated with a framing F is defined in terms of a quadratic map

$$H^{2m+1}M = H^{2m+1}(M;\mathbf{Z}/2) \xrightarrow{\quad\psi\quad} \mathbf{Z}/2$$

which Browder interprets this as follows. An element in H^nX is the same thing as a map from X to the Eilenberg-Mac Lane space

$$K_n = K(\mathbf{Z}/2, n).$$

.36

A sketch of Browder's proof

Now consider the diagram

$$\begin{tikzcd}
 & \Sigma K_{2m+1} \arrow[dl, dotted, "\widehat{i}"'] \arrow[d, "i"] \arrow[dr, "*"] & \\
F_{2m+2} \arrow[r] & K_{2m+2} \arrow[r, "Sq^{2m+2}"] & K_{4m+4}
\end{tikzcd}$$

Here the map i is adjoint to the equivalence $K_{2m+1} \to \Omega K_{2m+2}$, Sq^{2m+2} is the Steenrod squaring operation and F_{2m+2} is its fiber. This operation vanishes on the suspension of a $(2m+1)$-dimensional class, so $Sq^{2m+2}i$ is null and i lifts to F_{2m+2}.

The space F_{2m+2} has two nontrivial homotopy groups,

$$\pi_n F_{2m+2} = \begin{cases} \mathbf{Z}/2 & \text{for } n = 2m+2 \\ \mathbf{Z}/2 & \text{for } n = 4m+3 \\ 0 & \text{otherwise.} \end{cases}$$

The map $\widehat{i}$ is an equivalence thru dimension $4m+3$ and

$$\pi_{4m+2+k}\Sigma^k K_{2m+1} = \mathbf{Z}/2 \qquad \text{for } k > 0.$$

.37

A sketch of Browder's proof (continued)

A framed embedding of M in $\mathbf{R}^{k+4m+2}$ and a class $x \in H^{2m+1}M$ yields a diagram

$$S^{4m+2+k} \xrightarrow{p_F} \Sigma^k M_+ \xrightarrow{x} \Sigma^k K_{2m+1},$$

where the Pontryagin map p_F depends on the choice of framing F. The composite map represents an element in the homotopy group we just calculated, namely

$$\pi_{4m+2+k}\Sigma^k K_{2m+1} = \mathbf{Z}/2.$$

Browder showed that its value is the quadratic operation $\psi(x)$.

Browder's strategy:

Find the most general possible and simplest situation in which the Kervaire element can be defined and then study the place of framed manifolds in this situation. .38

3.2 Wu classes

Wu classes

This most general and simplest situation involves Wu classes.

Given a vector bundle ξ over a space X, let $w(\xi)$ denote its total Stiefel-Whitney class

$$w(\xi) = 1 + \sum_{i>0} w_i(\xi).$$

Let Sq denote the total Steenrod squaring operation

$$Sq = 1 + \sum_{i>0} Sq^i.$$

Both w and Sq are invertible, and we define the total Wu class $v(\xi)$ by

$$v(\xi) = \left(Sq^{-1}w(\xi)\right)^{-1}.$$

Hence $v_n(\xi)$ for each $n > 0$ is a certain polynomial in the Stiefel-Whitney classes. .39

Wu orientations

$$v(\xi) = \left(Sq^{-1}w(\xi)\right)^{-1}.$$

For a $(4m+2)$-manifold M we define $v_i(M) \in H^iM$ to be the ith Wu class of its normal bundle. It is known that for $x \in H^{4m+2-i}M$,

$$Sq^i x = v_i x \in H^{4m+2}M.$$

This implies via Poincaré duality that $v_i(M) = 0$ for $i > 2m+1$.

Consider the diagram

$$\begin{tikzcd}
 & M \arrow[dl, dotted, "\widehat{\nu}"'] \arrow[d, "\nu"] \arrow[dr, "*"] & \\
B\langle v_{2m+2}\rangle \arrow[r, "\pi"] & BO \arrow[r, "v_{2m+2}"] & K_{2m+2}
\end{tikzcd}$$

where BO is the classifying space of the stable orthogonal group O, ν is the map inducing the normal bundle, and $B\langle v_{2m+2}\rangle$ is the fiber of the map v_{2m+2}. Then the composite $v_{2m+2}\cdot\nu$ is null so the indicated lifting exists, but not uniquely. Browder calls $\widehat{\nu}$ a Wu orientation of M. .40

3.3 The Browder spectrum

The Browder spectrum

$$\begin{tikzcd}
 & & M \arrow[dl, dotted, "\widehat{\nu}"'] \arrow[d, "\nu"] \arrow[dr, "*"] & \\
K_{2m+1} \arrow[r] & B\langle v_{2m+2}\rangle \arrow[r, "\pi"] & BO \arrow[r, "v_{2m+2}"] & K_{2m+2}
\end{tikzcd}$$

We now consider the Thom spectra associated the universal bundle over BO and its pullbacks. The diagram becomes

$$\begin{tikzcd}
 & & T(\nu_M) \arrow[dl, dotted, "T\widehat{\nu}"'] \arrow[d, "T\nu"] \\
K_{2m+1} \arrow[r] & \mathrm{Br}_{2m+2} \arrow[r, "\overline{p}"] & MO
\end{tikzcd}$$

where $T(\nu_M)$ is the Thom spectrum for the normal bundle of M, K_{2m+1} here denotes the suspension spectrum of the space K_{2m+1} and Br_{2m+2}, the mth Browder spectrum, is the Thom spectrum associated with $B\langle v_{2m+2}\rangle$. .41

The Browder spectrum (continued)

$$\begin{tikzcd}
 & & T(\nu_M) \arrow[dl, dotted, "T\widehat{\nu}"'] \arrow[d, "T\nu"] \\
\Sigma^\infty K_{2m+1} \arrow[r] & \mathrm{Br}_{2m+2} \arrow[r, "\overline{p}"] & MO
\end{tikzcd}$$

The Spanier-Whitehead dual of $T(\nu_M)$ is $\Sigma^{-4m-2}M_+$, so we have a map

$$D\mathrm{Br}_{2m+2} \xrightarrow{\eta} \Sigma^{-4m-2}M_+.$$

Both of these spectra have no cells in positive dimensions and Sq^{2m+2} maps trivially to H^0. Now suppose we have an element $x \in H^{2m+1}M$ with $\eta^*(x) = 0$. Stably we have

$$\begin{tikzcd}
D\mathrm{Br}_{2m+2} \arrow[r, "\eta"] \arrow[d, equal] & \Sigma^{-4m-2}M_+ \arrow[d, "x"] & \\
X \arrow[r, "g"] & \Sigma^{-4m-2}K_{2m+1} \arrow[r, equal] & K
\end{tikzcd}$$

.42

The Browder spectrum (continued)

Let $q = 2m+1$, so our diagram reads

$$\begin{tikzcd}
D\mathrm{Br}_{q+1} \arrow[r, "\eta"] \arrow[d, equal] & \Sigma^{-2q}M_+ \arrow[d, "x"] & \\
X \arrow[r, "g"] & \Sigma^{-2q}K_q \arrow[r, equal] & K
\end{tikzcd}$$

Consider the following diagram with exact rows in black:

$$\begin{tikzcd}
0 & \iota_q \arrow[l, mapsto] \arrow[dddd, bend right, mapsto] & \alpha \arrow[l, mapsto] \arrow[dddd, bend right, mapsto] & & \\
H^{-q}X & H^{-q}K \arrow[l, "g^*"'] \arrow[d, "Sq^{q+1}"] & H^{-q}(K,X) \arrow[l] \arrow[d, "Sq^{q+1}"] & H^{-1-q}X \arrow[l] \arrow[d, "0"] & \\
 & H^1K & H^1(K,X) \arrow[l] & H^0X \arrow[l] & H^0K \arrow[l, "0"'] \\
 & & & 0 & Sq^q\iota_q \arrow[l, mapsto] \\
 & 0 & Sq^{q+1}\alpha \arrow[l, mapsto] & \psi(x) \arrow[l, mapsto] &
\end{tikzcd}$$

The diagram chase is shown in red. The element $\psi(x)$ is independent of the choice of α. Browder shows that the operation ψ is quadratic. .43

The Browder spectrum (continued)

If the manifold M has a framing F we get

$$\begin{tikzcd}
 & S^0 \arrow[d] & T(\nu_M) \arrow[l] \arrow[d, "T\nu"] \arrow[dl, dotted, "T\widehat{\nu}"'] \\
\Sigma^\infty K_{2m+1} \arrow[r] & \mathrm{Br}_{2m+2} \arrow[r, "\overline{p}"] & MO
\end{tikzcd}$$

This means we can replace $X = D\mathrm{Br}_{2m+2}$ by S^0, so the next diagram becomes

$$\begin{tikzcd}
S^0 \arrow[r, "p_F"] \arrow[d, equal] & \Sigma^{-4m-2}M_+ \arrow[d, "x"] \\
S^0 \arrow[r] & \Sigma^{-4m-2}K_{2m+1}
\end{tikzcd}$$

This is Browder's interpretation of the quadratic operation ψ described earlier. .44

3.4 The homotopy type of Br_{2m+2}

The homotopy type of Br_{2m+2}

A framed $(4m+2)$-manifold M with nontrivial Kervaire invariant represents, via Pontryagin's isomorphism, a nontrivial map

$$S^{4m+2} \xrightarrow{\theta} S^0.$$

Browder shows that the composite map to the Browder spectrum

$$S^{4m+2} \xrightarrow{\theta} S^0 \longrightarrow \mathrm{Br}_{2m+2}$$

must also be nontrivial.

He analyzes the homotopy type of Br_{2m+2} and gets a diagram

$$\begin{tikzcd}
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & \mathrm{Br}_{2m+2}{}^{(1)} \arrow[l] \arrow[d, "h"] & \mathrm{Br}_{2m+2}{}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
MO & K_{2m+1}\wedge MO & K_{4m+2} &
\end{tikzcd}$$

.45

The homotopy type of Br_{2m+2} (continued)

$$\begin{tikzcd}
S^0 \arrow[d] & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & \mathrm{Br}_{2m+2}{}^{(1)} \arrow[l] \arrow[d, "h"] & \mathrm{Br}_{2m+2}{}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
MO & K_{2m+1}\wedge MO & K_{4m+2} &
\end{tikzcd}$$

Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that MO is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra. This means that Br_{2m+2} is a 3-stage Postnikov system in the relevant range of dimensions.

It follows that θ must be detected by an element on the 2-line of the Adams spectral sequence. An explicit description of the map k rules out all elements other than h_j^2, which is shown to detect the Kervaire invariant in dimension $2^{j+1}-2$.

This completes the proof of the theorem. .46