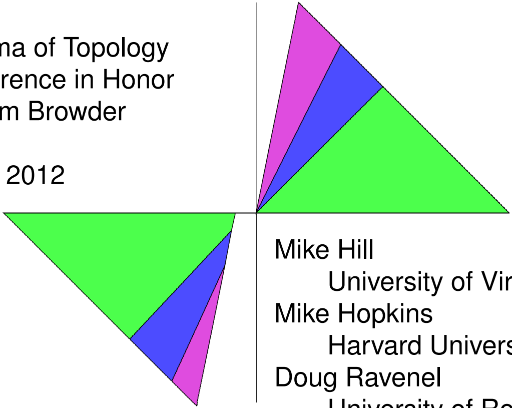
Browder's work on Arf-Kervaire invariant problem

Panorama of Topology
A Conference in Honor
of William Browder

May 10, 2012



Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester

Browder's theorem and its impact

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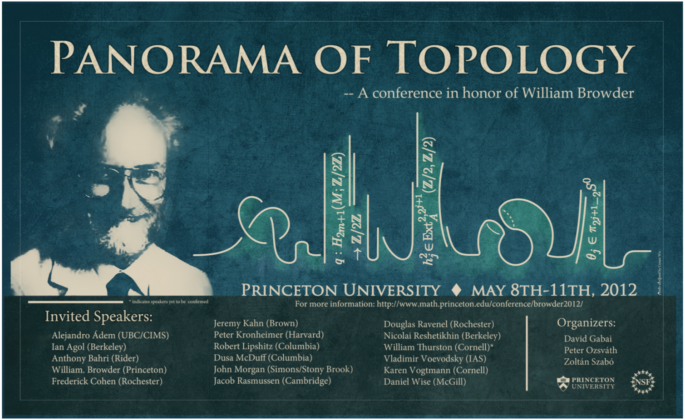

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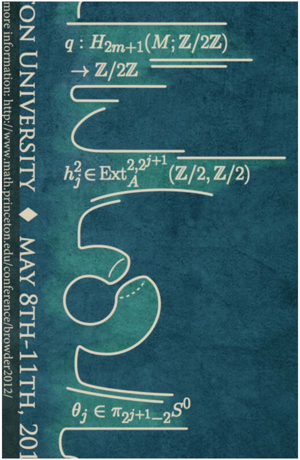

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Browder's theorem and its impact (continued)

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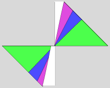

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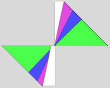

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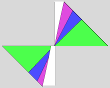
Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j. He derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the Doomsday Hypothesis.



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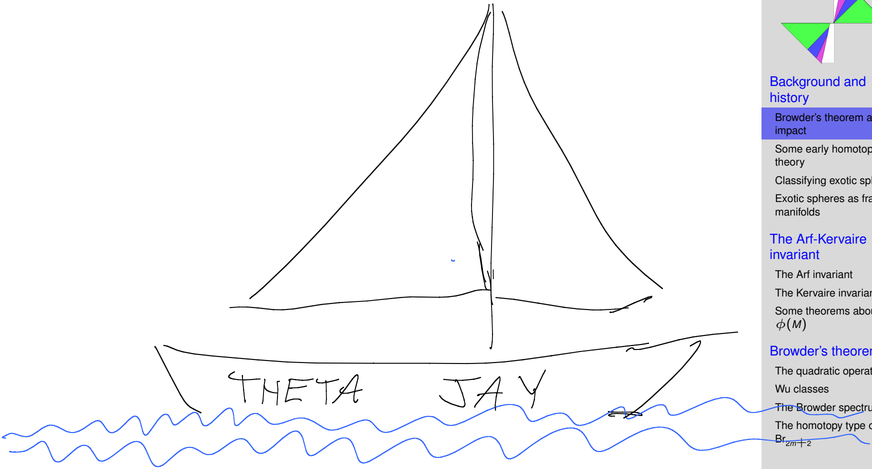
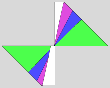


Browder's theorem and its impact (continued)

Drawing by Carolyn Snaith
London, Ontario 1981



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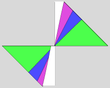

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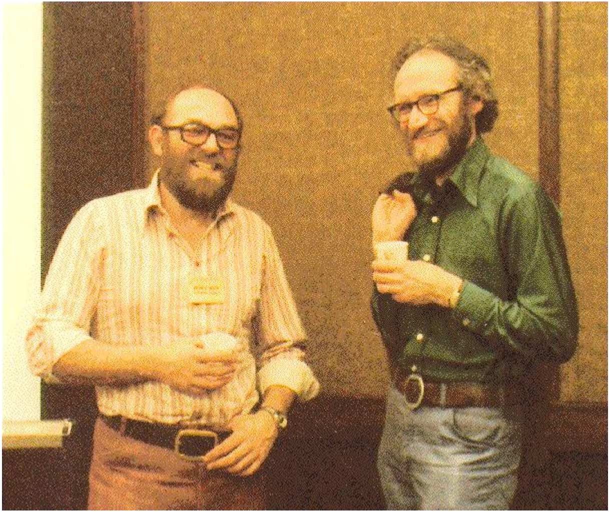

Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson

Browder's theorem and its impact (continued)

Fast forward
to 2009

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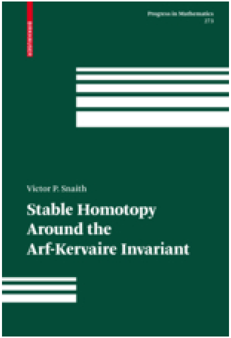

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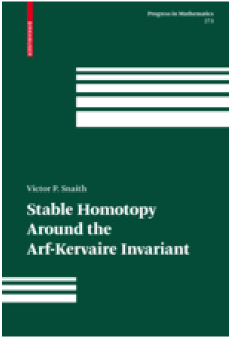

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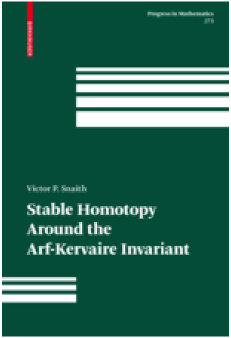

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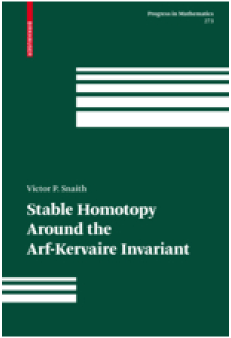

"For a brief period overnight we were convinced that we had the method to make all the sought after framed manifolds - a feeling which must have been shared by many topologists working on this problem. All in all, the temporary high of believing that one had the construction was sufficient to maintain in me at least an enthusiastic spectator's interest in the problem."

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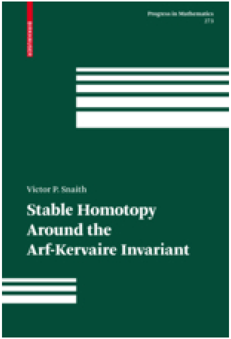

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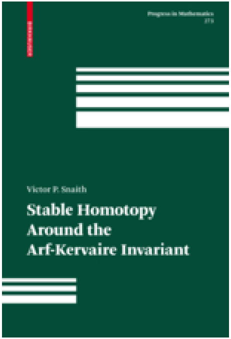

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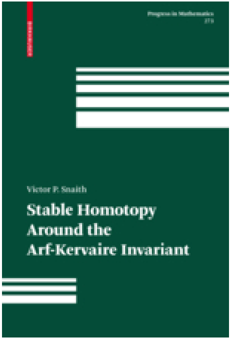

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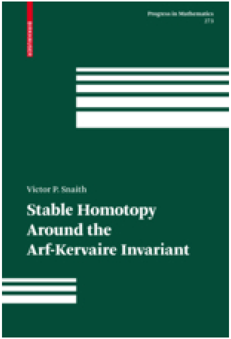

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Pontryagin's early work on homotopy groups of spheres

Back to the 1930s

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- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.

Pontryagin's early work on homotopy groups of spheres (continued)

$$
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 & & S^{n+k} & \xrightarrow{f} & S^k \\
 & & \uparrow & & \uparrow \\
M^n \times D^k & = & V^{n+k} & \longrightarrow & D^k \\
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There is a way to reverse this procedure. A framed manifold $M^n \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^k$.

Pontryagin's early work (continued)

Suppose there is homotopy $h : S^{n+k} \times [0,1] \to S^k$ between two such maps $f_1, f_2 : S^{n+k} \to S^k$.

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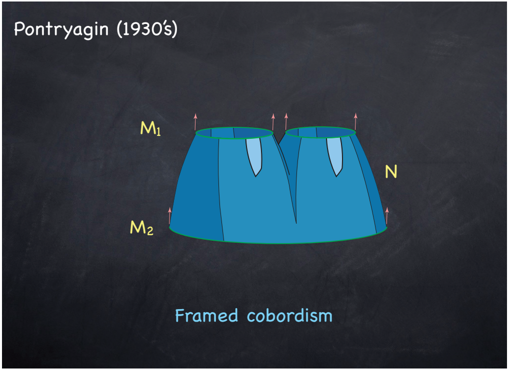

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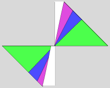

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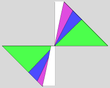

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Pontyagin's Theorem (1936)

The above homomorphism is an isomorphism in all cases.

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The determination of the stable homotopy groups π^S_n is an ongoing problem in algebraic topology.

The Kervaire-Milnor classification of exotic spheres

Into the 60s again

Mike Hill
Mike Hopkins
Doug Ravenel

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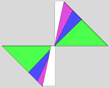

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Mike Hill
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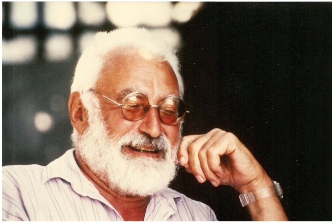
- 

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  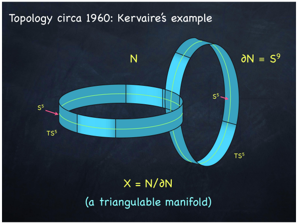

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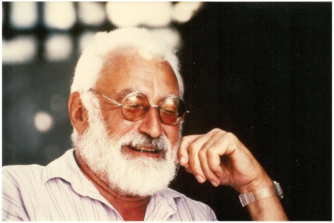
- 

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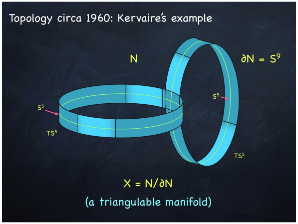

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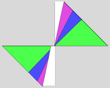

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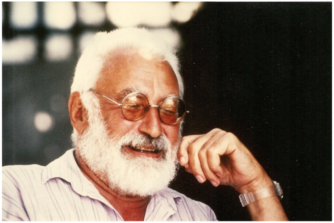
• 

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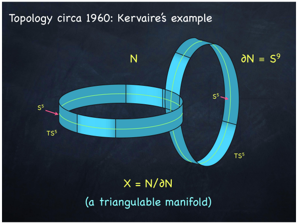

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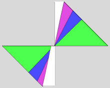

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For example, for $n = 1, 2, 3, \cdots, 18$, it will be shown that the order of the group Θ_n is respectively:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
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Exotic spheres as framed manifolds

Following Kervaire-Milnor, let Θ_n denote the group of diffeomorphism classes of exotic n-spheres Σ^n.

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Two framings of an exotic sphere $\Sigma^n \subset S^{n+k}$ differ by a map to the special orthogonal group $SO(k)$, and this map does not depend on the differentiable structure on Σ^n.

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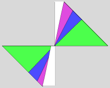

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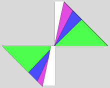
They denote the kernel of p by bP_{n+1}, the group of exotic n-spheres bounding parallelizable $(n+1)$-manifolds.



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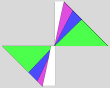

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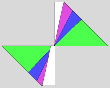

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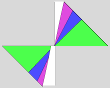

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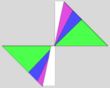

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We now know the value of bP_{4m+2} in every case except $m = 31$.

Exotic spheres as framed manifolds (continued)

In other words have a 4-term exact sequence

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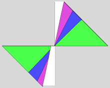

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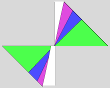
To say more about this we need to define the Kervaire invariant of a framed manifold.



The Arf invariant of a quadratic form in characteristic 2

Back to the 1940s

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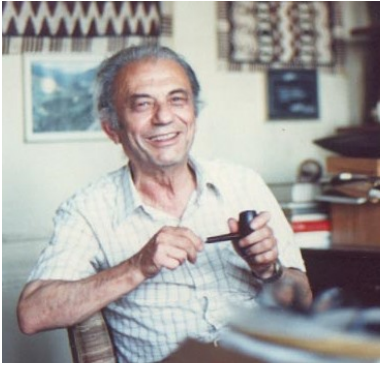

Cahit Arf 1910-1997

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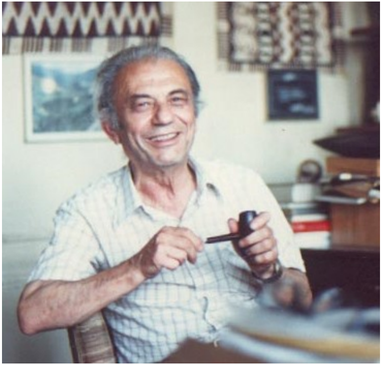

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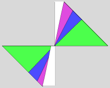

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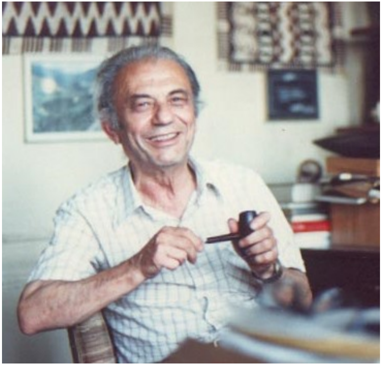

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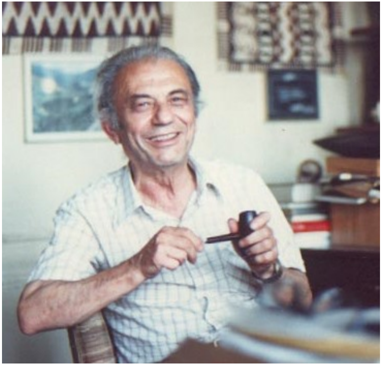

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$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

The Arf invariant of a quadratic form in characteristic 2 (continued)

In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix}.$$

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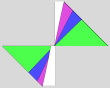
In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q.

Money talks: Arf's definition republished in 2009

Cahit Arf 1910-1997



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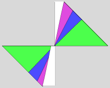

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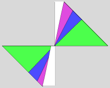

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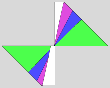

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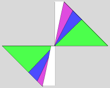
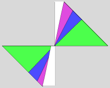


The Kervaire invariant of a framed $(4m+2)$-manifold

Into the 60s
a third time

Mike Hill
Mike Hopkins
Doug Ravenel



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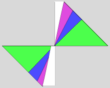

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For $M = T^2 \subset S^3$ and $x \in H_1(T^2; \mathbf{Z}/2)$, $q(x)$ is the number of full twists in a cylinder V neighboring a curve representing x.

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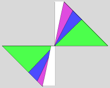

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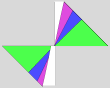

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Recall the Kervaire-Milnor 4-term exact sequence

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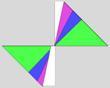
Kervaire-Milnor Theorem (1963)

$bP_{4m+2} = 0$ iff there is a smooth framed $(4m+2)$-manifold M with $\Phi(M)$ nontrivial.

Some theorems about $\phi(M)^{4m+2}$

What can we say about $\Phi(M)$?



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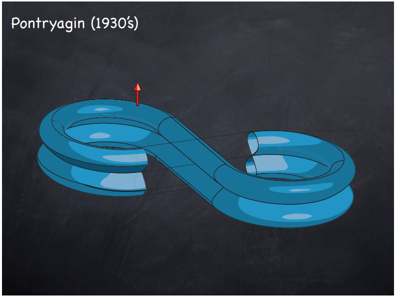

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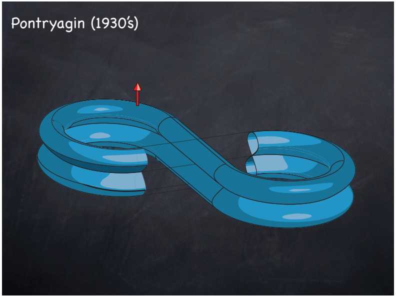

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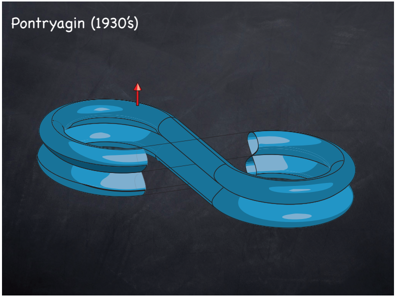

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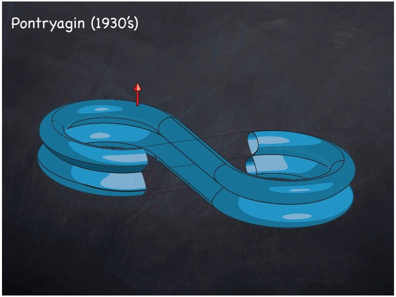

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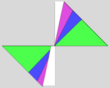
Kervaire (1960) showed it must vanish when $m = 2$, so $bP_{10} = \mathbf{Z}/2$.



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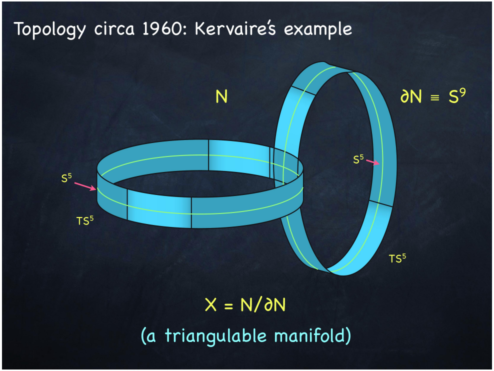

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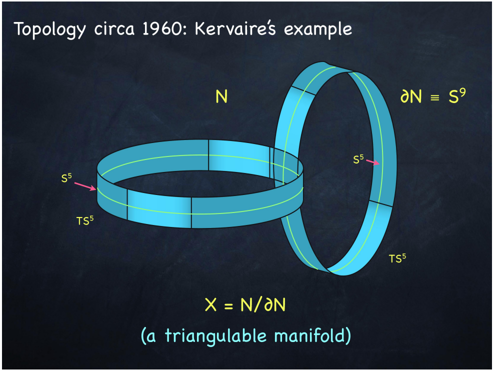

This construction generalizes to higher m, but Kervaire's proof that the boundary is exotic does not.

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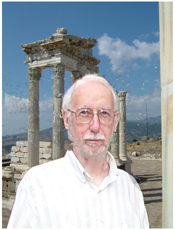
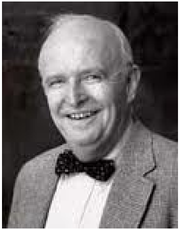


Ed Brown



Frank Peterson
1930-2000

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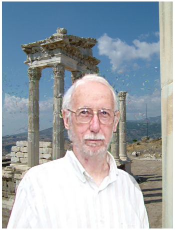

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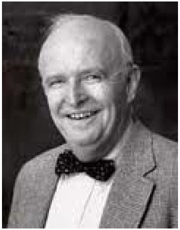

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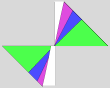

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The Kervaire invariant of a smooth framed $(4m+2)$-manifold M can be nontrivial only if $m = 2^{j-1} - 1$ for some $j > 0$. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence.

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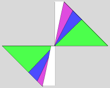

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Recall that the Kervaire invariant associated with a framing F is defined in terms of a quadratic map

$$H^{2m+1}M = H^{2m+1}(M; \mathbf{Z}/2) \xrightarrow{\quad\psi\quad} \mathbf{Z}/2$$

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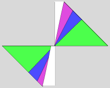
$$K_n = K(\mathbf{Z}/2, n).$$



A sketch of Browder's proof

Now consider the diagram

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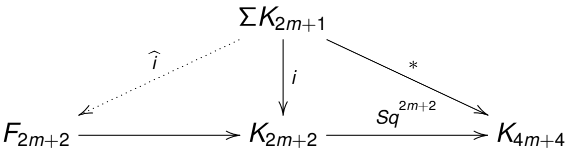

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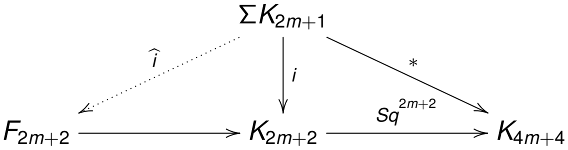

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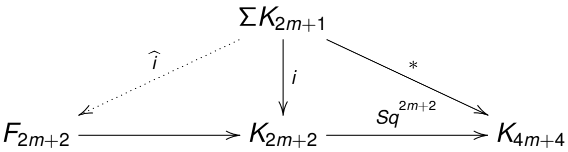

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The space F_{2m+2} has two nontrivial homotopy groups,

$$\pi_n F_{2m+2} = \begin{cases} \mathbf{Z}/2 & \text{for } n = 2m+2 \\ \mathbf{Z}/2 & \text{for } n = 4m+3 \\ 0 & \text{otherwise.} \end{cases}$$

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The map $\widehat{i}$ is an equivalence thru dimension $4m+3$ and

$$\pi_{4m+2+k} \Sigma^k K_{2m+1} = \mathbf{Z}/2 \qquad \text{for } k > 0.$$

A sketch of Browder’s proof (continued)

A framed embedding of M in $\mathbf{R}^{k+4m+2}$ and a class $x \in H^{2m+1}M$ yields a diagram

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A sketch of Browder's proof (continued)

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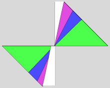

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Browder's strategy:

> *Find the most general possible and simplest situation in which the Kervaire element can be defined and then study the place of framed manifolds in this situation.*

Wu classes

This most general and simplest situation involves Wu classes.

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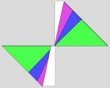

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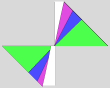

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Hence $v_n(\xi)$ for each $n > 0$ is a certain polynomial in the Stiefel-Whitney classes.

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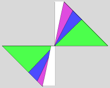

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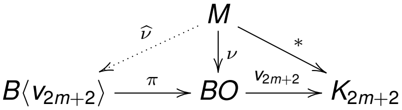

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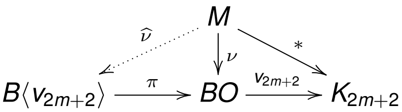

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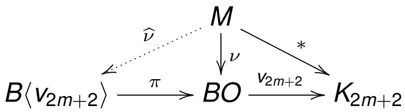

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The Browder spectrum

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\begin{array}{ccccccc}
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The Browder spectrum (continued)

$$\begin{array}{ccccc} & & & & T(\nu_M) \\ & & & \overset{T\widehat{\nu}}{\swarrow} & \Big\downarrow T\nu \\ \Sigma^{\infty} K_{2m+1} & \longrightarrow & \mathrm{Br}_{2m+2} & \xrightarrow{\;\overline{p}\;} & MO \end{array}$$

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$$\begin{array}{ccccc} D\mathrm{Br}_{2m+2} & \xrightarrow{\eta} & \Sigma^{-4m-2}M_+ & & \\ \| & & \downarrow x & & \\ X & \xrightarrow{g} & \Sigma^{-4m-2}K_{2m+1} & = & K \end{array}$$

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\end{array}$$

Consider the following diagram with exact rows in black:

$$\begin{array}{ccccccccc}
 & & 0 & \leftarrow & \iota_q & \leftarrow & \alpha & & \\
H^{-q}X & \xleftarrow{g^*} & H^{-q}K & \leftarrow & H^{-q}(K,X) & \leftarrow & H^{-1-q}X & & \\
 & & \big\downarrow Sq^{q+1} & & \big\downarrow Sq^{q+1} & & \big\downarrow 0 & & \\
 & & H^1K & \leftarrow & H^1(K,X) & \leftarrow & H^0X & \xleftarrow{0} & H^0K \\
 & & & & & & 0 & \leftarrow & Sq^q\iota_q \\
 & & 0 & \leftarrow & Sq^{q+1}\alpha & \leftarrow & \psi(x) & &
\end{array}$$

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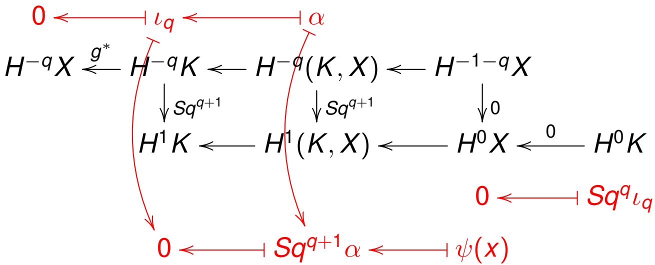

The diagram chase is shown in red.

The Browder spectrum (continued)

Let $q = 2m + 1$, so our diagram reads

$$\begin{array}{ccccc} D\mathrm{Br}_{q+1} & \xrightarrow{\eta} & \Sigma^{-2q}M_+ & & \\ \| & & \downarrow x & & \\ X & \xrightarrow{g} & \Sigma^{-2q}K_q & = & K \end{array}$$

Consider the following diagram with exact rows in black:

$$\begin{array}{ccccccc}
 & & {\color{red}0} & \longleftarrow\!\shortmid & {\color{red}\iota_q} & \longleftarrow\!\shortmid & {\color{red}\alpha} & & \\
H^{-q}X & \xleftarrow{g^*} & H^{-q}K & \longleftarrow & H^{-q}(K,X) & \longleftarrow & H^{-1-q}X & & \\
 & & \downarrow Sq^{q+1} & & \downarrow Sq^{q+1} & & \downarrow 0 & & \\
 & & H^1K & \longleftarrow & H^1(K,X) & \longleftarrow & H^0X & \xleftarrow{0} & H^0K \\
 & & & & & & {\color{red}0} & \longleftarrow\!\shortmid & {\color{red}Sq^q\iota_q} \\
 & & {\color{red}0} & \longleftarrow\!\shortmid & {\color{red}Sq^{q+1}\alpha} & \longleftarrow\!\shortmid & {\color{red}\psi(x)} & &
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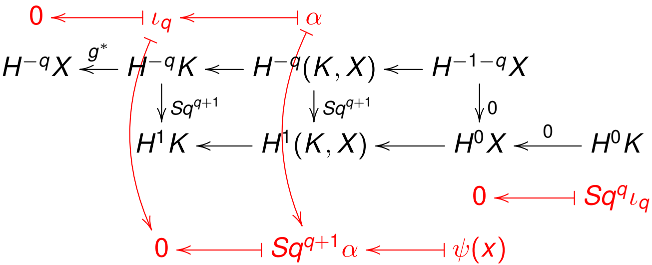
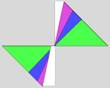


The diagram chase is shown in red. The element $\psi(x)$ is independent of the choice of α. Browder shows that the operation ψ is quadratic.

The Browder spectrum (continued)

If the manifold M has a framing F we get



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If the manifold M has a framing F we get

$$\begin{array}{ccccc} & & S^0 & \longleftarrow & T(\nu_M) \\ & & \downarrow & \swarrow{\scriptstyle T\widehat{\nu}} & \downarrow{\scriptstyle T\nu} \\ \Sigma^\infty K_{2m+1} & \longrightarrow & \mathrm{Br}_{2m+2} & \xrightarrow{\overline{p}} & MO \end{array}$$

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This means we can replace $X = D\mathrm{Br}_{2m+2}$ by S^0, so the next diagram becomes

$$\begin{array}{ccc} S^0 & \xrightarrow{p_F} & \Sigma^{-4m-2} M_+ \\ \| & & \downarrow x \\ S^0 & \longrightarrow & \Sigma^{-4m-2} K_{2m+1} \end{array}$$

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This is Browder's interpretation of the quadratic operation ψ described earlier.

The homotopy type of Br_{2m+2}

A framed $(4m+2)$-manifold M with nontrivial Kervaire invariant represents, via Pontryagin's isomorphism, a nontrivial map

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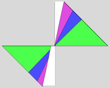
$$S^{4m+2} \xrightarrow{\theta} S^0.$$



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$$\begin{array}{ccccccc}
\mathrm{Br}_{2m+2} & \longleftarrow & \mathrm{Br}_{2m+2}{}^{(1)} & \longleftarrow & \mathrm{Br}_{2m+2}{}^{(2)} & \longleftarrow & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \\
\big\downarrow{\scriptstyle \overline{p}} & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle k} & & \\
MO & & K_{2m+1} \wedge MO & & K_{4m+2} & &
\end{array}$$

The homotopy type of Br_{2m+2} (continued)

$$\begin{array}{ccccccc}
S^0 & \xleftarrow{\quad\theta\quad} & & & S^{4m+2} & & \\
\downarrow & & & & \vdots & & \\
\mathrm{Br}_{2m+2} & \longleftarrow & \mathrm{Br}_{2m+2}{}^{(1)} & \longleftarrow & \mathrm{Br}_{2m+2}{}^{(2)} & \longleftarrow & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right)\\
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MO & & K_{2m+1} \wedge MO & & K_{4m+2} & &
\end{array}
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Here each horizontal map is the inclusion of the fiber of the following vertical map.

The homotopy type of Br_{2m+2} (continued)

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S^0 & \xleftarrow{\quad\theta\quad} & & & S^{4m+2} & & \\
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Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that MO is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra.

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S^0 & & \xleftarrow{\quad\quad\quad\theta\quad\quad\quad} & & S^{4m+2} & & \\
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It follows that θ must be detected by an element on the 2-line of the Adams spectral sequence.

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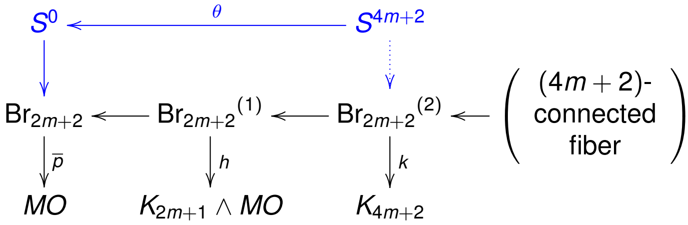

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This completes the proof of the theorem.