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THE NILPOTENCE AND PERIODICITY THEOREMS IN STABLE HOMOTOPY THEORY

by **Douglas C. RAVENEL**

1 Introduction and statement of theorems

In this paper we will outline some recent results in stable homotopy theory due to Devinatz, Hopkins and J. Smith ([DHS88], [Hop87] and [HS]) and conjectured by the author in [Rav84]. A more detailed account will appear in [Rav].

The theorems in question address one of the fundamental questions in algebraic topology: how to determine, by algebraic methods, when a continuous map from one topological space to another is homotopic to a constant map, i.e., a map that sends all of the source space to a single point in the target. This is the simplest case of the *homotopy classification problem*, which is to determine the set of homotopy classes of continuous maps from a space X to a space Y, and to classify spaces (satisfying various conditions) up to homotopy equivalence.

The basic strategy of algebraic topology is to study this problem by defining functors from the homotopy category (in which the objects are suitable topological spaces and the morphisms are homotopy classes of continuous maps) to various algebraic categories such as that of graded modules over graded rings. The most familiar examples are homotopy groups and ordinary homology and cohomology; in the latter case the functor is contravariant rather than covariant. A continuous map is essential (i.e., *not* homotopic to a constant map) if the functor carries it to a nontrivial homomorphism, but the converse is rarely true. In general there is a tradeoff between the computability of the functor and the amount of information it provides. When a functor that is easy to compute gives complete information about the homotopy class of a map, one considers it a great success.

Before stating the Nilpotence Theorem, we will indicate the modification of the homotopy classification problem that it addresses. First, the spaces under consideration are *finite CW-complexes*. The definition can be found in any textbook on algebraic topology. This class of spaces includes all of those one commonly studies geometrically, e.g. compact manifolds, algebraic varieties and simplicial complexes. Moreover, it is especially convenient for homotopy theory.

Second, we consider such spaces and maps between them *up to suspension.* The suspension ΣX of a space X is the topological of the cylinder

$$X \times [0,1]$$

obtained by collapsing each end of the cylinder (i.e., the subspaces $X \times \{0\}$ and $X \times \{1\}$) to a single point. For example, the suspension of the n–sphere S^n (the space of unit vectors in $\mathbf{R}^{n+1}$) is S^{n+1}.

The suspension Σf of a map $f: X \longrightarrow Y$ can be similarly defined, and suspension can be iterated.

A map f is *stably null homotopic* if some suspension $\Sigma^i f$ is homotopic to a constant map. Otherwise it is said to be *stably essential.* When the spaces in question are finite CW–complexes, there is an upper bound (depending on the connectivity and dimension of the spaces) on the number of suspensions needed to settle the question.

One can define stable homotopy and the stable homotopy classification problem in an obvious way. Experience has shown that stabilizing a problem in homotopy theory in this way makes it much easier to study in most cases. (There are a few exceptions to this statement, e.g. it is much easier to study maps to the circle S^1 up ordinary homotopy than up to stable homotopy.)

A map of the form

$$X \xleftarrow{f} \Sigma^d X$$

for $d \geq 0$ is called a *self-map* of X. It can be iterated by considering the maps

$$X \xleftarrow{\quad f \quad} \Sigma^d X \xleftarrow{\quad \Sigma^d f \quad} \Sigma^{2d} X \xleftarrow{\quad \Sigma^{2d} f \quad} \Sigma^{3d} X \longleftarrow \cdots$$

and one can ask if any of these composites is stably null homotopic. If the answer is yes, we say the map f is *stably nilpotent*; otherwise we say it is *periodic.*

When X is a finite CW–complex, the adjective 'stably' here is redundant. If one of the composites above is stably null homotopic, there is another one (obtained by moving further to the right) which is null homotopic without suspending.

Now we can state the simplest form of the Nilpotence Theorem of Devinatz–Hopkins–Smith [DHS88].

Theorem 1.1 (Nilpotence Theorem, Self–map Form) *There is a functor MU_* such that a self-map f of a finite CW-complex X is stably nilpotent if and only if some iterate of $MU_*(f)$ is trivial.*

This is the first of three equivalent forms of the Nilpotence Theorem. The others are 3.7 and 5.2.

The functor MU_*, known as complex bordism theory, takes values in the category of graded modules over a certain graded ring L, which is isomorphic to $MU_*(\text{pt.})$. These modules come equipped with an action by a certain infinite group Γ, which also acts on L. The ring L and the group Γ are closely related to the theory of formal group laws. $MU_*(X)$ was originally defined in terms of maps from certain manifolds to X, but this definition sheds little light on its algebraic structure. It is the algebra rather than the geometry which is central to our discussion. We will discuss this in more detail in Section 2 and more background can be found in [Rav86, Chapter 4]. In practice, $MU_*(X)$ is not difficult to compute.

We can also say something about periodic self-maps.

Before doing so we must discuss localization at a prime p. In algebra one does this by tensoring everything in sight by $\mathbf{Z}_{(p)}$, the integers localized at the prime p; it is the subring of the rationals consisting of fractions with denominator prime to p. If A is a finite abelian group, then $A \otimes \mathbf{Z}_{(p)}$ is the p–component of A. $\mathbf{Z}_{(p)}$ is flat as a module over the integers $\mathbf{Z}$; this means that tensoring with it preserves exact sequences.

There is an analogous procedure in homotopy theory. The definitive reference is [BK72]; a less formal account can be found in [Ada75]. For each CW–complex X there is a unique $X_{(p)}$ with the property that for most algebraic functors E_*, $E_*(X_{(p)}) \cong E_*(X) \otimes \mathbf{Z}_{(p)}$. We call $X_{(p)}$ the *p–localization* of X. If X is finite we say $X_{(p)}$ is a *p–local finite CW–complex.*

Proposition 1.2 *Suppose X is a simply connected CW–complex such that $\overline{H}_*(X)$ (the ordinary reduced homology of X) consists entirely of torsion.*

(i) If this torsion is prime to p then $X_{(p)}$ is contractible.

(ii) If it is all p–torsion then X is p–local, i.e., $X_{(p)}$ is homotopy equivalent to X. (In this case we say that X is a p–torsion complex.)

(iii) In general X is homotopy equivalent to the one–point union of its p–localizations for all the primes p in this torsion.

The most interesting periodic self–maps occur when X is a finite p–torsion complex. In these cases it is convenient to replace MU_* by the *Morava K–theories* $K(n)_*$. These were invented by Jack Morava in the early '70's, but he has yet to publish his work. Algebraic topologists are generally reluctant to give a precise definition of them in public because doing so would require a prohibitively long technical discussion. An axiomatic approach would be desirable, but has yet to be formulated.

Despite the difficulties with the definition, $K(n)_*(X)$ is generally easier to compute than $MU_*(X)$, and it makes certain properties of X (having to do with periodic self–maps) more readily apparent. The essential properties of Morava K–theory are contained in the following result, most of which is proved in [JW75]; a proof of (v) can be found in [Rav84].

Proposition 1.3 *For each prime p there is a sequence of functors $K(n)_*$ for $n \geq 0$ with the following properties. (We follow the standard practice of excluding p from the notation.)*

(i) *$K(0)_*(X) = H_*(X;\mathbf{Q})$ and $K(0)_*(X) = K(0)_*(\text{pt.})$ when X is a p–torsion complex.*

(ii) *$K(1)_*(X)$ is one of $p-1$ isomorphic summands of mod p complex K–theory.*

(iii) *$K(0)_*(\text{pt.}) = \mathbf{Q}$ and for $n > 0$, $K(n)_*(\text{pt.}) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ where the dimension of v_n is $2p^n - 2$. This ring is a graded field in the sense that every graded module over it is free. $K(n)_*(X)$ is a module over $K(n)_*(\text{pt.})$.*

(iv) *There is a Künneth isomorphism*

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*(\text{pt.})} K(n)_*(Y).$$

(v) *Let X be a p–local finite CW–complex. If*

$$K(n)_*(X) = K(n)_*(\text{pt.}),$$

then

$$K(n-1)_*(X) = K(n-1)_*(\text{pt.}).$$

(vi) *If X as above is not contractible then $K(n)_*(X)$ is nontrivial for n sufficiently large.*

Definition 1.4 *A p–local finite complex X has* **type** *n if n is the smallest integer such that $K(n)_*(X)$ is nontrivial.*

Because of the Künneth isomorphism, $K(n)_*(X)$ is easier to compute than $MU_*(X)$. Again there are still many interesting spaces for which this has not been done. See [RW80] and [HKR]. A consequence of the Nilpotence Theorem (1.1) is that the Morava K–theories, along with ordinary homology with coefficients in a field, are essentially the only homology theories with Künneth isomorphisms.

The Morava K–theories are especially useful for detecting periodic self–maps. This is the subject of the second major result of this paper, the Periodicity Theorem of Hopkins–Smith [HS]. The proof is outlined in [Hop87].

Theorem 1.5 (Periodicity Theorem) *Let X and Y be p–local (noncontractible) finite CW–complexes of type n (1.4).*

(i) There is a self-map $f: \Sigma^d X \to X$ such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is nilpotent for $m > n$. (We will refer to such a map as a v_n–**map**.*) When $n = 0$, $d = 0$ and when $n > 0$ then d is a multiple of $2p^n - 2$.*

(ii) Suppose $h: X \to Y$ is a continuous map where Y is another finite p-local CW-complex of type n. Let $g: \Sigma^e Y \to Y$ be a self-map as in (i). Then there are positive integers i and j with $di = ej$ such that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} \Sigma^{di}X & \xrightarrow{\Sigma^{di}h} & \Sigma^{di}Y \\ f^i\downarrow & & g^j\downarrow \\ X & \xrightarrow{h} & Y. \end{array}$$

There are two special cases of (ii) worth noting. The first is when $X = Y$ and h is the identity map. In that case, (ii) says that f is *assymptotically unique* in the following sense. Suppose g is another such periodic self-map. Then there are positive integers i and j such that f^i is homotopic to g^j.

The second case is when Y is a suspension of X and g is the corresponding suspension of f. Then (ii) shows that f is *assymptotically central* in that any map h commutes with some iterate of it. We will see below in 4.3 that a fixed iterate of f commutes with all h.

2 MU–theory and formal group laws

In this section we will discuss the functor MU_* used in the Nilpotence Theorem. $MU_*(X)$ is defined in terms of maps of manifolds into X as will be explained presently. Unfortunately the geometry in this definition does not appear to be relevant to the applications we have in mind. We will be more concerned with some algebraic properties of the functor which are intimately related to the theory of formal group laws.

Definition 2.1 *Let M_1 and M_2 be smooth closed m-dimensional manifolds and let*

$$f_i: M_i \to X$$

be continuous maps for $i = 1, 2$. These maps are **bordant** *if there is a map*

$$f: W \to X$$

where W is a smooth manifold whose boundary is the disjoint union of M_1 and M_2 such that the restriction of f to M_i is f_i. f is a **bordism** *between f_1 and f_2.*

Bordism is an equivalence relation and the set of bordism classes forms a group under disjoint union, called the m^{th} **bordism group of X.**

A manifold is **stably complex** if it admits a complex linear structure in its stable normal bundle, i.e., the normal bundle obtained by embedding in a large dimensional Euclidean space. A complex analytic manifold (e.g. a complex algebraic variety) is stably complex, but the notion of stably complex is far weaker than that of complex analytic.

Definition 2.2 *$MU_m(X)$, the m^{th}* **complex bordism group of** *X, is the bordism group obtained by requiring that all manifolds in sight be stably complex.*

The fact that these groups are accessible is due to some remarkable work of Thom in the 1950's [Tho54]. A general reference for cobordism theory is Stong's book [Sto68].

The groups $MU_*(X)$ satisfy all but one of the axioms used by Eilenberg–Steenrod to characterize ordinary homology. They fail to satisfy the dimension axiom, which describes the homology of a point. If X is a single point, then the map from the manifold to X is vacuous, and $MU_*(\text{pt.})$ is the group of bordism classes of stably complex manifolds, which we will denote simply by MU_*. It is a graded ring under Cartesian product and its structure was determined independently by Milnor [Mil60] and Novikov ([Nov60] and [Nov62]).

Theorem 2.3 *The complex bordism ring, MU_* is isomorphic to*

$$\mathbf{Z}[x_1, x_2, \ldots]$$

where $\dim x_i = 2i$.

It is possible to describe the generators x_i as complex manifolds, but this is more trouble than it is worth. The complex projective spaces $\mathbf{C}P^i$ serve as polynomial generators of $\mathbf{Q} \otimes MU_*$.

Note that $MU_*(X)$ is an MU_*-module as follows. Given $x \in MU_*(X)$ represented by $f\colon M \to X$ and $\lambda \in MU_*$ represented by a manifold N, λx is represented by the composite map

$$M \times N \longrightarrow M \xrightarrow{f} X.$$

Definition 2.4 *A* **formal group law** *over a commutative ring with unit R is a power series $F(x, y)$ over R that satisfies the following three conditions.*

(i) $F(x, 0) = F(0, x) = x$ *(identity),*

(ii) $F(x, y) = F(y, x)$ *(commutativity) and*

(iii) $F(F(x, y), z) = F(x, F(y, z))$ *(associativity).*

(The existence of an inverse is automatic. It is the power series $i(x)$ determined by the equation $F(x, i(x)) = 0$.)

Example 2.5 *(i) $F(x,y) = x + y$. This is called the* **additive** *formal group law.*

(ii) $F(x,y) = x+y+xy = (1+x)(1+y)-1$. This is called the **multiplicative** *formal group law.*

The theory of formal group laws from the power series point of view is treated comprehensively in [Haz78]. A short account containing all that is relevant for the current discussion can be found in [Rav86, Appendix 2].

The following result is due to Lazard [Laz55a].

Theorem 2.6 (Lazard's Theorem) *(i) There is a universal formal group law defined over a ring L of the form*

$$G(x,y) = \sum_{i,j} a_{i,j}x^i y^j \quad \text{with } a_{i,j} \in L$$

such that for any formal group law F over R there is a unique ring homomorphism θ from L to R such that

$$F(x,y) = \sum_{i,j} \theta(a_{i,j})x^i y^j.$$

(ii) L is a polynomial algebra $\mathbf{Z}[x_1, x_2, \ldots]$. If we put a grading on L such that $a_{i,j}$ has degree $2(1-i-j)$ then x_i has degree $-2i$.

The grading above is chosen so that if x and y have degree 2, then $G(x,y)$ is a homogeneous expression of degree 2. Note that L is isomorphic to MU_* except that the grading is reversed. There is an important connection between the two.

Associated to the covariant functor MU_* there is a contravariant functor MU^*. It bears the same relation to MU_* that ordinary cohomology bears to ordinary homology. The conventions in force in algebraic topology require that $MU^*(\text{pt.})$ (which we will denote by MU^*) be the same as $MU_*(\text{pt.})$ but with the grading reversed. Thus MU^* is isomorphic to the Lazard ring L.

This isomorphism is natural in the following sense. $MU^*(X)$, like $H^*(X)$, comes equipped with cup products, making it a graded algebra over MU^*. Of particular interest is the case when X is the infinite-dimensional complex projective space $\mathbf{C}P^\infty$. We have

$$MU^*(\mathbf{C}P^\infty) \cong MU^*[[x]]$$

where $\dim x = 2$, and

$$MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \cong MU^*[[x \otimes 1, 1 \otimes x]].$$

The space $\mathbf{C}P^\infty$ is an abelian topological group, so there is a map

$$\mathbf{C}P^\infty \times \mathbf{C}P^\infty \xrightarrow{f} \mathbf{C}P^\infty$$

with certain properties. ($\mathbf{C}P^\infty$ is also the classifying space for complex line bundles and the map in question corresponds to the tensor product.) Since MU^* is contravariant we get a map

$$MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \xleftarrow{f^*} MU^*(\mathbf{C}P^\infty)$$

which is determined by its behavior on the generator $x \in MU^2(\mathbf{C}P^\infty)$. The power series

$$f^*(x) = F(x \otimes 1, 1 \otimes x)$$

can easily be shown to be a formal group law over MU^*. Hence by Lazard's Theorem (2.6) it corresponds to a ring homomorphism $\theta: L \to MU^*$. The following was proved by Quillen [Qui69] in 1969.

Theorem 2.7 (Quillen's Theorem) *The homomorphism $\theta: L \to MU^*$ above is an isomorphism. In other words, the formal group law associated with complex cobordism is the universal one.*

Given this isomorphism (and ignoring the reversal of the grading), we can regard $MU_*(X)$ as an L–module.

Now we define a group Γ which acts in an interesting way on L.

Definition 2.8 *Let Γ be the group of power series over $\mathbf{Z}$ having the form*

$$\gamma = x + b_1x^2 + b_2x^3 + \cdots$$

where the group operation is functional composition. Γ acts on the Lazard ring L of 1.5 as follows. Let $G(x,y)$ be the universal formal group law as above and let $\gamma \in \Gamma$. Then $\gamma^{-1}(G(\gamma(x),\gamma(y)))$ is another formal group law over L, and therefore is induced by a homomorphism from L to itself. Since γ is invertible, this homomorphism is an automorphism, giving the desired action of Γ on L.

For reasons too difficult to explain here, Γ also acts naturally on $MU_*(X)$ compatibly with the action on $MU_*(\mathrm{pt.})$ defined above. That is, given $x \in MU_*(X)$, $\gamma \in \Gamma$ and $\lambda \in L$, we have

$$\gamma(\lambda x) = \gamma(\lambda)\gamma(x)$$

and the action of Γ commutes with homomorphisms induced by continuous maps.

For algebraic topologists we can offer some explanation for this action of Γ. It is analogous to the action of the Streenrod algebra in ordinary cohomology. More precisely, it is analogous to the action of the group of multiplicative cohomology operations, such as (in the mod 2 case) the total Streenrod square, $\sum_{i\geq 0}\mathrm{Sq}^i$. Such an operation is determined by its effect on the generator of $H^1(\mathbf{R}P^\infty;\mathbf{Z}/(2))$. Thus the group of multiplicative mod 2 cohomology operations embeds in $\Gamma_{\mathbf{Z}/(2))}$, the group of power series over $\mathbf{Z}/(2)$ analogous to Γ over the integers.

Definition 2.9 *Let* $\mathbf{C\Gamma}$ *denote the category of finitely presented* L*–modules equipped with an action of* Γ *compatible with its action on* L *as above, and let* $\mathbf{FH}$ *denote the category of finite* CW*–complexes and homotopy classes of maps between them.*

Thus we can regard MU_* as a functor from $\mathbf{FH}$ to $\mathbf{C\Gamma}$. The latter category is much more accessible. We will see that it has some structural features which reflect those of $\mathbf{FH}$ very well. The Nilpotence and Periodicity Theorems are examples of this.

In order to study $\mathbf{C\Gamma}$ further we need some more facts about formal group laws. Here are some power series associated with them.

Definition 2.10 *For each integer* n *the* n-**series** $[n](x)$ *is given by*

$$\begin{aligned} [1](x) &= x, \\ [n](x) &= F(x,[n-1](x)) \qquad \textit{for } n>1 \textit{ and} \\ [-n](x) &= i([n](x)). \end{aligned}$$

These satisfy

$$\begin{aligned} [n](x) &\equiv nx \quad \bmod (x^2), \\ [m+n](x) &= F([m](x),[n](x)) \qquad \text{and} \\ [mn](x) &= [m]([n](x)). \end{aligned}$$

For the additive formal group law (2.5), we have $[n](x) = nx$, and for the multiplicative formal group law, $[n](x) = (1+x)^n - 1$.

Of particular interest is the p–series. In characteristic p it always has leading term ax^q where $q = p^h$ for some integer h. This leads to the following.

Definition 2.11 *Let* $F(x,y)$ *be a formal group law over a ring in which the prime* p *is not a unit. If the mod* p *reduction of* $[p](x)$ *has the form*

$$[p](x) = ax^{p^h} + \textit{higher terms}$$

then we say that F *has* **height h at** p*. If* $[p](x) \equiv 0 \bmod p$ *then the height is infinity.*

For the additive formal group law we have $[p](x) = 0$ so the height is ∞. The multiplicative formal group law has height 1 since $[p](x) = x^p$.

The following classification theorem is due to Lazard [Laz55b].

Theorem 2.12 (Lazard's Classification Theorem) *Two formal group laws over the algebraic closure of* $\mathbf{F}_p$ *are isomorphic if and only if they have the same height.*

Let $v_n \in L$ denote the coefficient of x^{p^n} in the p–series for the universal formal group law; the prime p is omitted from the notation. This v_n is closely related to the v_n in the Morava K–theories (1.3). It can be shown that v_n is an indecomposable element in L, i.e., it could serve as a polynomial generator in dimension $2p^n - 2$. Let $I_{p,n} \subset L$ denote the prime ideal $(p, v_1, \ldots v_{n-1})$.

The following result is due to Morava [Mor85] and Landweber [Lan73a].

Theorem 2.13 (Morava–Landweber Theorem) *The only prime ideals in L which are invariant under the action of Γ are the $I_{p,n}$ defined above, where p is a prime integer and n is a nonnegative integer, possibly ∞. ($I_{p,\infty}$ is by definition the ideal $(p, v_1, v_2, \ldots)$ and $I_{p,0}$ is the zero ideal.)*

Moreover in $L/I_{p,n}$ for $n > 0$ the subgroup fixed by Γ is $\mathbf{Z}/(p)[v_n]$. In L itself the invariant subgroup is $\mathbf{Z}$.

This shows that the action of Γ on L is very rigid. L has a bewildering collection of prime ideals, but the only ones we ever have to consider are the ones listed in the theorem. This places severe restriction on the modules in $\mathbf{C}\Gamma$.

Recall that a finitely presented module M over a Noetherian ring R has a finite filtration

$$F_1M \subset F_2M \subset \cdots F_kM = M$$

in which each subquotient $F_iM/F_{i-1}M$ is isomorphic to R/I_i for some prime ideal $I_i \subset R$. Now L is not Noetherian, but it is coherent, which means that finitely presented modules over it admit similar filtrations. For a module in $\mathbf{C}\Gamma$, the filtration can be chosen so that the submodules, and therefore the prime ideals, are all invariant under Γ. The following result is due to Landweber [Lan73b].

Theorem 2.14 (Landweber's Filtration Theorem) *Every module M in $\mathbf{C}\Gamma$ admits a finite filtration by submodules in $\mathbf{C}\Gamma$ as above in which each subquotient is isomorphic to a suspension of $L/I_{p,n}$ for some prime p and some finite n.*

The following are easy consequences of the Landweber Filtration Theorem.

Corollary 2.15 *Suppose M is a p–local module in $\mathbf{C}\Gamma$.*

(i) Then if $v_n^{-1}M = 0$, i.e., if each element in M is annihilated by some power of v_n, then $v_{n-1}^{-1}M = 0$.

(ii) If M is nontrivial, then so is $v_n^{-1}M$ for n sufficiently large.

(iii) If $v_{n-1}^{-1}M = 0$, then there is a positive integer k such that multiplication by v_n^k in M commutes with the action of Γ.

The first two statements should be compared to the last two statements in 1.3. In fact the functor $v_n^{-1}MU_*(X)_{(p)}$ is trivial on a finite p-local CW-complex X if and only if $K(n)_*(X)$ is. One could replace $K(n)_*$ by $v_n^{-1}MU_{(p)*}$ in the statement of the Periodicity Theorem. The third statement above is an algebraic analogue of the Periodicity Theorem.

Now we need to consider certain full subcategories of $\mathbf{C\Gamma}$ and $\mathbf{FH}$.

Definition 2.16 *A full subcategory* $\mathbf{C}$ *of* $\mathbf{C\Gamma}$ *is* **thick** *if it satisfies the following two axioms:*

(i) If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\mathbf{C\Gamma}$ *in which two of the three modules are in* $\mathbf{C}$*, then so is the third.*

(ii) If $M \oplus N$ *is in* $\mathbf{C}$ *then so are* M *and* N*.*

We want to define the corresponding notion for a subcategory of $\mathbf{FH}$, so we need analogs of short exact sequences and direct sums. Given a map $f : X \to Y$, the *cofibre* C_f of f is the quotient of

$$X \times [0,1] \cup Y$$

obtained by collapsing $X \times \{0\}$ to a single point, and identifying $(x,1)$ for $x \in X$ with $f(x) \in Y$. It is easy to show that the cofibre of the map $Y \to C_f$ is homotopy equivalent to ΣX. A *cofibre sequence* is a sequence of maps of the form

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X \longrightarrow \cdots.$$

The composite

$$X \xrightarrow{f} Y \longrightarrow C_f$$

is the homotopy theoretic analog of a short exact sequence. The analog of a direct sum is the *wedge* $X \vee Y$, which is the quotient of $X \cup Y$ obtained by identifying one point in X with one point in Y.

Definition 2.17 *A full subcategory* $\mathbf{F}$ *of* $\mathbf{FH}$ *is* **thick** *if it satisfies the following two axioms:*

(i) If

$$X \xrightarrow{g} Y \longrightarrow C_f$$

is a cofibre sequence in which two of the three spaces are in $\mathbf{F}$*, then so is the third.*

(ii) If $X \vee Y$ *is in* $\mathbf{F}$ *then so are* X *and* Y*.*

Thick subcategories were called *generic subcategories* by Hopkins in [Hop87].

Using the Landweber Filtration Theorem, one can classify the thick subcategories of $\mathbf{C\Gamma}_{(p)}$.

Theorem 2.18 *Let* $\mathbf{C}$ *be a thick subcategory of* $\mathbf{C\Gamma}_{(p)}$ *(the category of all p–local modules* $\mathbf{C\Gamma}$*). Then* $\mathbf{C}$ *is either all of* $\mathbf{C\Gamma}_{(p)}$*, the trivial subcategory (in which the only object is the trivial module), or consists of all p–local modules M in* $\mathbf{C\Gamma}$ *with* $v_{n-1}^{-1}M = 0$. *We denote the latter category by* $\mathbf{C_{p,n}}$.

There is an analogous result about thick subcategories of $\mathbf{FH}_{(p)}$, which is a very useful consequence of the Nilpotence Theorem.

Theorem 2.19 (Thick Subcategory Theorem) *Let* $\mathbf{F} \subset \mathbf{FH}_{(p)}$ *be a thick subcategory of the category of p–local finite CW–complexes. Then* $\mathbf{F}$ *is either all of* $\mathbf{FH}_{(p)}$*, the trivial subcategory (in which the only object is a point) or consists of all p–local finite CW–complexes X with* $v_{n-1}^{-1}MU_*(X)$ *trivial. We denote the latter category by* $\mathbf{F_{p,n}}$.

Thus we have two nested sequences of thick subcategories,

$$\mathbf{FH}_{(p)} = \mathbf{F}_{p,0} \supset \mathbf{F}_{p,1} \supset \mathbf{F}_{p,1} \cdots \{\text{pt.}\}$$

and

$$\mathbf{C\Gamma}_{(p)} = \mathbf{C}_{p,0} \supset \mathbf{C}_{p,1} \supset \mathbf{C}_{p,1} \cdots \{0\}.$$

The functor $MU_*(\cdot)$ sends one to the other. Until 1983 it was not even known that the $\mathbf{F}_{p,n}$ were nontrivial for all but a few small values of n. Mitchell [Mit85] first showed that all of the inclusions of the $\mathbf{F}_{p,n}$ are proper. Now it is a corollary of the Periodicity Theorem.

In Section 3 we will derive the Thick Subcategory Theorem from another form of the Nilpotence Theorem. This is easy since it uses nothing more than elementary tools from homotopy theory.

In Section 4 we will sketch the proof of the Periodicity Theorem. It is not difficult to show that the collection of complexes admitting periodic self maps for given p and n forms a thick subcategory. Given the Thick Subcategory Theorem, it suffices to find just one nontrivial example of a complex of type n with a periodic self–map. This involves some hard homotopy theory. There are two major ingredients in the construction. One is the Adams spectral sequence, a computational tool that one would expect to see used here. The other is a novel application of the modular representation theory of the symmetric group described in as yet unpublished work of Jeff Smith.

3 The proof of the Thick Subcategory Theorem

In this section we will derive the Thick Subcategory Theorem from the Nilpotence Theorem with the use of some standard tools from homotopy theory, which we must introduce before we can give the proof. The proof itself is identical to the one given by Hopkins in [Hop87].

First we have to introduce the category of spectra. Since the category was introduced around 1960 [Lim60], it has taken on a life of its own, as will be seen later in this paper. Most of the theorems in this paper that are stated in terms of spaces are really theorems about spectra that we have done our best to disguise. However we cannot keep up this act any longer.

Definition 3.1 *A* **spectrum** E *is a collection of spaces* $\{E_n : n \gg 0\}$ *and maps* $\Sigma E_n \to E_{n+1}$. *The* **suspension spectrum** *of a space* X *is defined by* $E_n = \Sigma^n X$ *with each map being the identity. The* i^{th} *suspension* $\Sigma^i E$ *of* E *is defined by*

$$(\Sigma^i E)_n = E_{n+i}$$

for any integer i. *Thus any spectrum can be suspended or desuspended any number of times.*

The homotopy groups of E *are given by*

$$\pi_k(E) = \varinjlim \pi_{n+k}(E_n)$$

and other invariants such as homology and cohomology can be similarly defined.

A spectrum X *is* **connective** *if its homotopy groups are bounded below, i.e., if*

$$\pi_{-k}(X) = 0 \text{ for } k \gg 0.$$

It has **finite type** *if* $\pi_k(X)$ *is finitely generated for each* k.

Example 3.2 (Sphere spectrum) *The spectrum* S^k *is defined by setting* $(S^k)_n = S^{k+n}$. *(The abuse of notation here is standard. Hopefully the context will make it clear whether we are talking about a spectrum or a space.)*

Example 3.3 (Mod p Eilenberg–Mac Lane spectrum) *The spectrum* $H/(p)$ *is defined by setting* $H/(p)_n$ *equal to the Eilenberg-Mac Lane space* $K(\mathbf{Z}/(p), n)$.

One of the original motivations for the definition of spectra is the following example.

Example 3.4 (Complex cobordism spectrum) *The spectrum* MU *is defined by setting* MU_{2n} *equal to the Thom space for the universal* $\mathbf{C}^n$*-bundle over* $BU(n)$, *the classifying space for the unitary group* $U(n)$. MU_{2n+1} *defined to be* ΣMU_{2n}. *Then* $\pi_*(MU)$ *is isomorphic to the complex cobordism ring* MU_* *discussed above.* MU *is a ring spectrum (see 4.1 below).*

The homotopy groups of spectra are much more manageable than those of spaces. For example, one has

$$\pi_k(\Sigma^i E) = \pi_{k-i}(E)$$

for all k and i.

Next we need to discuss smash products. For spaces the definition is as follows.

Definition 3.5 *Let X and Y be spaces equipped with base points x_0 and y_0. The* **smash product** *$X \wedge Y$ is the quotient of $X \times Y$ obtained by collapsing $X \times \{y_0\} \cup \{x_0\} \times Y$ to a single point. The k-fold iterated smash product of X with itself is denoted by $X^{(k)}$. For $f: X \to Y$, $f^{(k)}$ denote the evident map from $X^{(k)}$ to $Y^{(k)}$. The map f is* **smash nilpotent** *if $f^{(k)}$ is null homotopic for some k.*

The k-fold suspension $\Sigma^k X$ is the same as $S^k \wedge X$. For CW-complexes X and Y there is an equivalence

$$\Sigma(X \times Y) \simeq (\Sigma X) \vee (\Sigma Y) \vee \Sigma(X \wedge Y).$$

If either of the spectra E or F is a suspension spectrum, then there is an obvious definition of their smash product $E \wedge F$. In particular, smashing E with the sphere spectrum S^0 (3.2) leaves E unchanged. However the general definition of the smash product of two spectra is very difficult; we refer the interested reader to Adams [Ada74].

Definition 3.6 *For a spectrum E the E–***homology** *of X is defined by*

$$E_*(X) = \pi_*(E \wedge X).$$

*The E–***cohomology** *of X, $E^*(X)$, is the graded group of homotopy classes of maps from X to E, i.e.,*

$$E^i(X) = [X, \Sigma^i E]$$

In the case $E = MU$, this is equivalent to the definition given above, 2.2.

The Nilpotence Theorem can be stated in terms of smash products as follows.

Theorem 3.7 (Nilpotence Theorem, Smash Product Form) *Let*

$$F \xrightarrow{f} X$$

be a map of spectra where F is finite. Then f is smash nilpotent if $MU \wedge f$ (i.e., the evident map $MU \wedge F \to MU \wedge X$) is null homotopic.

This can be shown to be equivalent to 1.1. A more useful form of this for our purposes is the following.

Corollary 3.8 *Let W, X and Y be p-local finite spectra with $f\colon X \to Y$. Then $W \wedge f^{(k)}$ is null homotopic for $k \gg 0$ if $K(n)_*(W \wedge f) = 0$ for all $n \geq 0$. (In particular, W could be S^0, in which case $W \wedge f = f$.)*

It is from this result that we will derive the Thick Subcategory Theorem.

Next we need to discuss Spanier–Whitehead duality, which is treated in more detail in [Ada74].

Theorem 3.9 (Spanier–Whitehead Duality) *For a finite spectrum X there is a unique finite spectrum DX (the* **Spanier–Whitehead dual** *of X) with the following properties.*

(i) For any spectrum Y, the graded group $[X,Y]_*$ is isomorphic to $\pi_*(DX \wedge Y)$. We say that the maps $S^n \to DX \wedge Y$ and $\Sigma^n X \to Y$ that correspond under this isomorphism are **adjoint** *to each other. In particular when $Y = X$, the identity map on X is adjoint to a map $e\colon S^0 \to DX \wedge X$.*

(ii) For $Y = X$ this isomorphism is reflected in Morava K-theory, namely

$$K(n)_*(X))$$

is isomorphic to $K(n)_*(DX \wedge X)$. In particular $K(n)_*(e) \neq 0$ when $K(n)_*(X) \neq 0$.

(iii) $DDX \simeq X$.

(iv) For a homology theory E_*, there is a natural isomorphism between $E_k(X)$ and $E^{-k}(DX)$. (These groups are defined in 3.6.)

(v) Spanier–Whitehead duality commutes with smash products, i.e., for finite spectra X and Y, $D(X \wedge Y) = DX \wedge DY$.

The basic geometric idea behind Spanier–Whitehead duality is as follows. A finite spectrum X is the suspension spectrum of a finite CW–complex, which we also denote by X. It can always be embedded in some Euclidean space $\mathbf{R}^N$ and hence in S^N. Then DX is a suitable suspension of the suspension spectrum of the complement $S^N - X$. 3.9(iv) is a generalization of the classical Alexander Duality Theorem, which says that $H_k(X)$ is isomorphic to $H^{N-1-k}(S^N - X)$. A simple example of this is the case where $X = S^k$ and it it linearly embedded in S^N. Then its complement is homotopy equivalent to S^{N-1-k}. The Alexander Duality Theorem says that the complement has the same cohomology as S^{N-1-k} even when the embedding of S^k in S^N is not linear, e.g. when $k = 1$, $n = 3$ and $S^1 \subset S^3$ is knotted.

Before we can proceed with the proof of the Thick Subcategory Theorem we need an elementary lemma about Spanier-Whitehead duality. For a finite spectrum X, let $f\colon W \to S^0$ be the map such that

$$W \xrightarrow{f} S^0 \xrightarrow{e} DX \wedge X$$

is a cofibre sequence. In the category of spectra, such maps always exist. W in this case is finite, and $C_f = DX \wedge X$.

Lemma 3.10 *With notation as above, there is cofibre sequence*

$$C_{f^{(k)}} \longrightarrow C_{f^{(k-1)}} \longrightarrow \Sigma W^{(k-1)} \wedge C_f$$

for each $k > 1$.

Proof. A standard lemma in homotopy theory says that given maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

there is a diagram

$$\begin{tikzcd}
C_f \arrow[r] & \mathrm{pt.} \arrow[r] & \Sigma C_f \\
Y \arrow[u] \arrow[r, "g"] & Z \arrow[u] \arrow[r] & C_g \arrow[u] \\
X \arrow[u, "f"] \arrow[r, "gf"] & Z \arrow[u, equal] \arrow[r] & C_{gf} \arrow[u]
\end{tikzcd}$$

in which each row and column is a cofibre sequence. Setting $X = W^{(k)}$, $Y = W^{(k-1)}$, $Z = S^0$ and $g = f^{(k-1)}$, this diagram becomes

$$\begin{tikzcd}
W^{(k-1)} \wedge C_f \arrow[r] & \mathrm{pt.} \arrow[r] & \Sigma W^{(k-1)} \wedge C_f \\
W^{(k-1)} \arrow[u] \arrow[r, "f^{(k-1)}"] & S^0 \arrow[u] \arrow[r] & C_{f^{(k-1)}} \arrow[u] \\
W^{(k)} \arrow[u, "W^{(k-1)} \wedge f"] \arrow[r, "f^{(k)}"] & S^0 \arrow[u, equal] \arrow[r] & C_{f^{(k)}} \arrow[u]
\end{tikzcd}$$

and the right hand column is the desired cofibre sequence. ∎

Now we are ready to prove the Thick Subcategory Theorem. Let $\mathbf{F} \subset \mathbf{FH}_{(p)}$ be a thick subcategory. Choose n to be the smallest integer such that $\mathbf{F}$ contains a p–local finite spectrum X with $K(n)_*(X) \neq 0$. We want to show that $\mathbf{F} = \mathbf{F}_n$. It is clear from the choice of n that $\mathbf{F} \subset \mathbf{F}_n$, so it suffices to show that $\mathbf{F} \supset \mathbf{F}_n$.

Let Y be a p–local finite CW–spectrum in $\mathbf{F}_n$. From the fact that $\mathbf{F}$ is thick, it follows that $X \wedge F$ is in $\mathbf{F}$ for any finite F, so $X \wedge DX \wedge Y$ (or $C_f \wedge Y$ in the notation of 3.10) is in $\mathbf{F}$. Thus 3.10 implies that $C_{f^{(k)}} \wedge Y$ is in $\mathbf{F}$ for all $k > 0$.

It follows from 3.9(ii) that $K(i)_*(f) = 0$ when $K(i)_*(X) \neq 0$, i.e., for $i \geq n$. Since $K(i)_*(Y) = 0$ for $i < n$, It follows that $K(i)_*(Y \wedge f) = 0$ for all i. Therefore by 3.8 (a corollary to the Nilpotence Theorem), $Y \wedge f^{(k)}$ is null homotopic for some $k > 0$.

Now the cofibre of a null homotopic map is equivalent to the wedge of its target and the suspension of its source, so we have

$$Y \wedge C_{f^{(k)}} \simeq Y \vee \Sigma Y \wedge W^{(k)}.$$

Since $\mathbf{F}$ is thick and contains $Y \wedge C_{f^{(k)}}$, it follows that Y is in $\mathbf{F}$, so $\mathbf{F}$ contains $\mathbf{F}_n$ as desired.

4 The proof of the Periodicity Theorem

In this section we will outline the proof of the Periodicity Theorem, which falls into two parts. The first, which is relatively easy, is to show that the category of spectra admitting self-maps as in the Periodicity Theorem is thick (4.6). Thus by the Thick Subcategory Theorem, this category is either $\mathbf{F}_n$, as asserted in the Periodicity Theorem, or it is trivial. The second and harder step in the proof is to construct a nontrivial example. This relies on some unpublished work of Jeff Smith.

First we need a definition.

Definition 4.1 *A* **ring spectrum** *E is a spectrum equipped with maps $\eta: S^0 \to E$, called the* **unit map**, *and $m: E \wedge E \to E$, called the* **multiplication map**, *such that the composites*

$$\begin{array}{ccccc} E = S^0 \wedge E & \xrightarrow{\eta \wedge E} & E \wedge E & \xrightarrow{m} & E \quad \textit{and} \\ E = E \wedge S^0 & \xrightarrow{E \wedge \eta} & E \wedge E & \xrightarrow{m} & E \end{array}$$

are each the identity on E (this is analogous to the unitary condition on a ring), and the following diagram commutes up to homotopy.

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{m \wedge E} & E \wedge E \\ {\scriptstyle E \wedge m}\downarrow & & {\scriptstyle m}\downarrow \\ E \wedge E & \xrightarrow{m} & E. \end{array}$$

This is an associativity condition on m. The multiplication m need not be commutative up to homotopy, but it is in most cases.

A **module spectrum** *M over E is one equipped with a map*

$$E \wedge M \xrightarrow{\mu} M$$

such that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} E \wedge E \wedge M & \xrightarrow{m \wedge M} & E \wedge M \\ {\scriptstyle E \wedge \mu}\downarrow & & {\scriptstyle \mu}\downarrow \\ E \wedge M & \xrightarrow{\mu} & M. \end{array}$$

We begin by observing that a self-map $f: \Sigma^d X \to X$ is adjoint to $\hat{f}: S^d \to DX \wedge X$. We will abbreviate $DX \wedge X$ by R. Now R is a ring spectrum. The unit is the map $e: S^0 \to DX \wedge X$ adjoint to the identity map on X (3.9). Since $DDX = X$ and Spanier–Whitehead duality commutes with smash products, e is dual to

$$X \wedge DX \xrightarrow{De} S^0.$$

The multiplication on R is the composite

$$DX \wedge X \wedge DX \wedge X \xrightarrow{DX \wedge De \wedge X} DX \wedge S^0 \wedge X = DX \wedge X.$$

We will use $\hat{f}_*$ to denote the element induced by $\hat{f}$ in both $\pi_*(R)$ and $K(n)_*(R)$.

Lemma 4.2 *For a v_n-map f as above, there is an i such that $K(n)_*(f^i)$ is multiplication by some power of v_n.*

Proof. The ring $K(n)_*(R)$ is a finite-dimensional $K(n)_*$-algebra, so the ungraded quotient $K(n)_*(R)/(v_n - 1)$ is a finite ring with a finite group of units. It follows that the group of units in $K(n)_*(R)$ itself is an extension of the group of units of $K(n)_*$ by this finite group. Therefore some power of the unit $\hat{f}_*$ is in $K(n)_*$, and the result follows. ■

Lemma 4.3 *For a v_n-map f as above, there is an $i > 0$ such that $\pi_*(\hat{f}^i)$ is in the center of $\pi_*(R)$.*

Proof. Let A be a noncommutative p-torsion ring, such as $\pi_*(R)$. Given $a \in A$ we define a map

$$\mathrm{ad}(a): A \longrightarrow A$$

by

$$\mathrm{ad}(a)(b) = ab - ba.$$

Thus a is in the center of A if $\mathrm{ad}(a) = 0$.

There is a formula relating $\mathrm{ad}(a^i)$ to $\mathrm{ad}^j(a)$, the j^{th} iterate of $\mathrm{ad}(a)$, namely

$$\mathrm{ad}(a^i)(x) = \sum_{j \le i} \binom{i}{j} \mathrm{ad}^j(a)(x)a^{i-j}.$$

Now suppose $\mathrm{ad}(a)$ is nilpotent and we set $i = p^N$ for some large N. Then the terms on the right for large j are zero because $\mathrm{ad}(a)$ is nilpotent, and the terms for small j vanish because the binomial coefficient is divisible by a large power of p. Hence $\mathrm{ad}(a^i) = 0$ so a^i is in the center of A.

To apply this to the situation at hand, define

$$\Sigma^d R \xrightarrow{\mathrm{ad}(f)} R$$

to be the composite

$$S^d \wedge R \xrightarrow{f \wedge R} R \wedge R \xrightarrow{1-T} R \wedge R \xrightarrow{m} R$$

where T is the map that interchanges the two factors. Then for $x \in \pi_*(R)$,

$$\pi_*(\mathrm{ad}(f))(x) = \mathrm{ad}(\hat{f})(x).$$

By 4.2 (after replacing f by a suitable iterate if necessary), we can assume that $K(n)_*(f)$ is multiplication by a power of v_n, so $K(n)_*(\hat{f})$ is in the center of $K(n)_*(R)$ and $K(n)_*(\mathrm{ad}(f)) = 0$. Hence the Nilpotence Theorem tells us that $\mathrm{ad}(f)$ is nilpotent and the argument above applies to give the desired result. ■

Lemma 4.4 (Uniqueness of v_n–maps) *If X has two v_n–maps f and g then there are integers i and j such that $f^i = g^j$.*

Proof. Replacing f and g by suitable powers if necessary, we may assume that they commute with each other and that $K(m)_*(f) = K(m)_*(g)$ for all m. Hence $K(m)_*(f - g) = 0$ so $f - g$ is nilpotent. For $K \gg 0$ we also have

$$0 = (f - g)^{p^K} \equiv f^{p^K} - g^{p^K}$$

modulo any given power of p, so $f^{p^K} = g^{p^K}$ as claimed. ∎

Lemma 4.5 *If X and Y have v_n–maps f and g and $h: X \to Y$, then there are integers i and j such that the following diagram commutes.*

$$\begin{array}{ccc} \Sigma^? X & \xrightarrow{h} & \Sigma^? Y \\ f^i \downarrow & & g^j \downarrow \\ X & \xrightarrow{h} & Y. \end{array}$$

Note that 4.4 is the special case of this where h is the identity map on X.

Proof. Let $W = DX \wedge Y$, so h is adjoint to an element $\hat{h} \in \pi_*(W)$. W has two v_n–maps, namely $DX \wedge g$ and $Df \wedge Y$, so by 4.4,

$$DX \wedge g^j \simeq Df^i \wedge Y$$

for suitable i and j.

Now hf^i is adjoint to $(Df^i \wedge Y)\hat{h}$ and $g^j h$ is adjoint to $(DX \wedge g^j)\hat{h}$. Since these are homotopic, the diagram commutes. ∎

Theorem 4.6 *The category $\mathbf{C} \subset \mathbf{FH}_{(p)}$ of finite p–local CW–spectra admitting v_n–maps is thick.*

Proof. Suppose $X \vee Y$ is in $\mathbf{C}$ and

$$\Sigma^d(X \vee Y) \xrightarrow{f} X \vee Y$$

is a v_n–map. By 4.3 we can assume that f commutes with the idempotent

$$X \vee Y \longrightarrow X \longrightarrow X \vee Y$$

and it follows that the composite

$$\Sigma^d X \longrightarrow \Sigma^d(X \vee Y) \xrightarrow{f} X \vee Y \longrightarrow X$$

is a v_n–map, so X is in $\mathbf{C}$.

Now suppose $h: X \to Y$ where X and Y have v_n–maps f and g. By 4.5 we can assume that $hf \simeq gh$, so there is a map

$$\Sigma^d C_h \xrightarrow{\ell} C_h$$

making the following diagram commute.

$$\begin{array}{ccccc} X & \xrightarrow{h} & Y & \longrightarrow & C_h \\ {\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & {\scriptstyle \ell}\uparrow \\ \Sigma^d X & \xrightarrow{h} & \Sigma^d Y & \longrightarrow & \Sigma^d C_h \end{array}$$

The 5–lemma implies that $K(n)_*(\ell)$ is an isomorphism.

We also need to show that $K(m)_*(\ell) = 0$ for $m \neq n$. This is *not* implied by the facts that $K(m)_*(f) = 0$ and $K(m)_*(g) = 0$. However, an easy diagram chase shows that they do imply that $K(m)_*(\ell^2) = 0$, so ℓ^2 is the desired v_n–map on C_h. ■

Having proved 4.6, we need only to construct one nontrivial example of a v_n–map in order to complete the proof of the Periodicity Theorem. This requires extensive use of the Adams spectral sequence and the Steenrod algebra. We will not describe these details here because they are quite technical. They will be described in [Rav]. However we will outline a new construction due to Jeff Smith which uses modular (characteristic p) representations of the symmetric group, and which is the most interesting part of the proof.

Suppose X is a finite spectrum, and $X^{(k)}$ is its k–fold smash product. The symmetric group Σ_k acts on $X^{(k)}$ by permuting coordinates. Since we are in the stable category, it is possible to add maps, so we get an action of the group ring $\mathbf{Z}[\Sigma_k]$ on $X^{(k)}$. If X is p–local, we have an action of the p–local group ring $S = \mathbf{Z}_{(p)}[\Sigma_k]$. Now suppose e is an idempotent element ($e^2 = e$) in this group ring. Then $1 - e$ is also idempotent. For any S–module M (such as $\pi_*(X^{(k)})$) we get a splitting

$$M \cong eM \oplus (1-e)M.$$

There is a standard construction in homotopy theory which gives a similar splitting of $X^{(k)}$ or any other spectrum on which S acts, which we write as

$$X^{(k)} \simeq eX^{(k)} \vee (1-e)X^{(k)}.$$

In some cases one of the two summands may be trivial.

Thus each idempotent element $e \in \mathbf{Z}_{(p)}[\Sigma_k]$ leads to a splitting of the smash product $X^{(k)}$ for any X. We will use this to construct a finite spectrum Y of type n that can be shown to admit a v_n–self map, starting with a well known X.

Now suppose V is a finite dimensional vector space over a field of characteristic p. (The example we have in mind is $H^*(X; \mathbf{Z}/(p))$.) Then $W = V^{\otimes k}$ is an S–module so we have a splitting

$$W \cong eW \oplus (1-e)W$$

and the rank of eW is determined by that of V. There are enough idempotents e to give the following.

Theorem 4.7 (J. Smith) *For each positive integer r there is an idempotent*

$$e_r \in \mathbf{Z}_{(p)}[\Sigma_k]$$

(where the number k depends on r) such that the rank of eW above is nonzero if and only if the rank of V is at least r.

Now for $V = H^*(X; \mathbf{Z}/(p))$ there is a technical problem when p is odd and $H^*(X)$ is not concentrated in even dimensions. The action of Σ_k on $H^*(X^{(k)}; \mathbf{Z}/(p)) = V^{\otimes k}$ is not the expected permutation of coordinates, because a minus sign is introduced whenever two odd–dimensional elements are interchanged. Smith has proved a generalization of 4.7 that takes this into account.

For any spectrum Y, $H^*(Y; \mathbf{Z}/(p))$ is a module over the mod p Steenrod algebra A; the best reference for its properties is the classic [SE62]. Using the Adams spectral sequence, it can be shown that if Y is finite and its mod p cohomology is free as a module over a certain subalgebra of A, then Y has type n and admits a v_n–map.

To obtain such a Y, one starts with a finite spectrum X satisfying much milder conditions. An appropriate skeleton of the classifying space of the group with p elements, $B\mathbf{Z}/(p)$, will do. Then one applies a suitable Smith idempotent to a smash power of X. The cohomology of the resulting spectrum Y can be shown to satisfy the required conditions.

This completes our outline of the proof of the Periodicity Theorem.

5 The proof of the Nilpotence Theorem

In this section we will outline the proof of the Nilpotence Theorem. We have previously stated it in two different guises, in terms of self–maps (1.1) and in terms of smash products (3.7). For our purposes here it is convenient to give a third statement, but first we need another definition.

Definition 5.1 *For a ring spectrum E, the* **Hurewicz map** *$h: \pi_*(X) \to E_*(X)$ is the homomorphism induced by*

$$X \simeq S^0 \wedge X \xrightarrow{\eta \wedge X} E \wedge X$$

where $\eta: S^0 \longrightarrow E$ is the unit map for E (4.1).

Theorem 5.2 (Nilpotence Theorem, Ring Spectrum Form) *Let R be a connective ring spectrum of finite type (3.1 and 4.1) and let*

$$\pi_*(R) \xrightarrow{h} MU_*(R)$$

be the Hurewicz map. Then $\alpha \in \pi_*(R)$ is nilpotent if $h(\alpha) = 0$

To see that this implies 1.1, let X be a finite complex and let $R = X \wedge DX$. Recall that a self–map $f\colon \Sigma^d X \to X$ is adjoint to a map $\hat{f}\colon S^d \to R$. Then $h(\hat{f})$ is nilpotent if and only if $MU_*(f)$ is.

In proving this we will make use of certain spectra $X(n)$. They are constructed in terms of vector bundles and Thom spectra. Let SU denote the infinite special unitary group, i.e., the union of all the $SU(n)$'s. The Bott Periodicity Theorem gives us a homotopy equivalence

$$\Omega SU \xrightarrow{\simeq} BU$$

where BU is the classifying space of the infinite unitary group. Composing this with the loops on the inclusion of $SU(n)$ into SU, we get a map

$$\Omega SU(n) \longrightarrow BU.$$

This defines an infinite–dimensional vector bundle over $\Omega SU(n)$. The resulting Thom spectrum is $X(n)$. A routine calculation gives

$$H_*(X(n); \mathbf{Z}) = \mathbf{Z}[b_1, \ldots b_{n-1}]$$

where $|b_i| = 2i$. For $n = \infty$, this is the usual description of $H_*(MU)$.

$X(n)$ is a ring spectrum so we have a Hurewicz map

$$\pi_*(R) \xrightarrow{h(n)} X(n)_*(R).$$

In particular $X(1) = S^0$ so $h(1)$ is the identity map. The map $X(n) \to MU$ is a homotopy equivalence through dimension $2n-1$. It follows that if $h(\alpha) = 0$, then $h(n)(\alpha) = 0$ for large n. Hence, the Nilpotence Theorem will follow from

Theorem 5.3 *With notation as above, if $h(n+1)(\alpha) = 0$ then $h(n)(\alpha)$ is nilpotent.*

The hypothesis that $h(n+1)(\alpha) = 0$ could be replaced by the condition that it is nilpotent, since we could replace α by a suitable power of it.

The element $\alpha \in \pi_*(R)$ is represented by a map

$$S^d \xrightarrow{\alpha} R.$$

Smashing both sides with R and using the multiplication in R, we get a self–map

$$\Sigma^d R = S^d \wedge R \xrightarrow{\alpha \wedge R} R \wedge R \xrightarrow{m} R,$$

which we also designate by α.

Definition 5.4 *Let $\alpha^{-1}R$ denote the homotopy direct limit of*

$$R \xrightarrow{\alpha} \Sigma^{-d}R \xrightarrow{\alpha} \Sigma^{-2d}R \xrightarrow{\alpha} \cdots.$$

This is called the **telescope** *associated with α.*

Lemma 5.5 *If $\alpha^{-1}R \wedge X(n)$ is contractible then $h(n)_*(\alpha)$ is nilpotent.*

Proof. The map $\alpha\colon S^d \to R$ induces a self-map Σ^d. The spectrum $\alpha^{-1}R \wedge X(n)$ is by definition the homotopy direct limit of

$$R \wedge X(n) \longrightarrow \Sigma^{-d}R \wedge X(n) \longrightarrow \Sigma^{-2d}R \wedge X(n) \longrightarrow \cdots$$

It follows that each element of $X(n)_*(R)$, including $h(n)(\alpha)$, is annihilated after a finite number of steps, so $h(n)(\alpha)$ is nilpotent. ∎

In studying questions of this sort, the following notion is convenient.

Definition 5.6 *Two spectra E and F are* **Bousfield equivalent** *if $E \wedge X$ is contractible if and only if $F \wedge X$ is. The resulting equivalence class is denoted by $\langle E \rangle$, called the* **Bousfield class** *of E. We say*

$$\langle E \rangle \geq \langle F \rangle$$

if $E \wedge X$ is contractible implies that $F \wedge X$ is contractible.

Under this partial ordering, the largest Bousfield class is, $\langle S^0 \rangle$ while the smallest is $\langle \text{pt.} \rangle$.

The following is straightforward.

Lemma 5.7 *Let*

$$\Sigma^d X \xrightarrow{f} X \longrightarrow Y$$

be a cofibre sequence, and let $f^{-1}X$ be the telescope associated with f. Then

$$\langle X \rangle = \langle Y \vee f^{-1}X \rangle$$

Now we need to study the spectra $X(n)$ more closely. Consider the diagram

$$\begin{array}{ccccc} \Omega SU(n) & \longrightarrow & \Omega SU(n+1) & \longrightarrow & \Omega S^{2n+1} \\ \| & & \uparrow & & \uparrow \\ \Omega SU(n) & \longrightarrow & B_k & \longrightarrow & J_k S^{2n} \end{array} \tag{5.8}$$

in which each row is a fibration. The top row is obtained by looping the fibration

$$SU(n) \longrightarrow SU(n+1) \xrightarrow{e} S^{2n+1}$$

where e is the evaluation map which sends a matrix $m \in SU(n+1)$ to mu where $u \in \mathbf{C}^{n+1}$ is fixed unit vector.

$J_k S^{2n}$ is the k^{th} space in the James construction [Jam53] on S^{2n}. This is the same thing as the $2nk$–skeleton of ΩS^{2n+1}. It is a CW–complex with one cell in every $2n$ dimensions; it can also be described as a certain quotient of the Cartesian product $(S^{2n})^k$. The space B_k is the pullback, i.e., the $\Omega SU(n)$–bundle over $J_k S^{2n}$ induced by the inclusion map into ΩS^{2n+1}.

Recall that $H_*(\Omega SU(n)) = \mathbf{Z}[b_1, b_2, \ldots b_{n-1}]$. $H_*(B_k) \subset H_*(\Omega SU(n+1))$ is the free module over it generated by b_n^i for $0 \le i \le k$.

Now the composite map

$$B_k \longrightarrow \Omega SU(n+1) \longrightarrow BU \tag{5.9}$$

gives a stable bundle over B_k and we denote the Thom spectrum by F_k. We will be especially interested in F_{p^j-1}, which we will denote by G_j. These spectra interpolate between $X(n)$ and $X(n+1)$. Thus we have $G_0 = X(n)$ and $G_\infty = X(n+1)$.

The following two lemmas clearly imply 5.3 and hence the Nilpotence Theorem. Their proofs will occupy the rest of this section.

Lemma 5.10 *Let $\alpha^{-1}R$ be the telescope associated with $\alpha \in \pi_*(R)$ (5.4). If $h(n+1)_*(\alpha) = 0$ then $G_j \wedge \alpha^{-1}R$ is contractible for large j.*

The following is the harder of the two and is the heart of the Nilpotence Theorem.

Lemma 5.11 *For each $j > 0$, $\langle G_j \rangle = \langle X(n) \rangle$. In particular $\langle G_j \rangle = \langle G_{j+1} \rangle$.*

We will now outline the proof of 5.10. It requires the use of the Adams spectral sequence for a generalized homology theory. It would take too long to define this here. The interested reader can find all the needed definitions in [Rav86]. Fortunately all we require of it here is certain formal properties; we will not have to make any detailed computations.

We need to look at the Adams spectral sequence for $\pi_*(G_j \wedge R)$ based on $X(n+1)$–theory. It has the following properties:

(i) $E_2^{s,t}$ can be identified with a certain Ext group related to $X(n+1)$–theory.

(ii) $E_2^{s,t}$ vanishes unless s and t are both nonnegative.

(iii) α corresponds to an element $x \in E_2^{s,s+d}$ for some $s > 0$.

(iv) $E_2^{s,t}$ vanishes for all (s,t) above a certain line (called a *vanishing line*) of slope

$$\frac{1}{1 + p^{j+1}(n-1)},$$

i.e., $E_2^{s,t} = 0$ whenever

$$s(1 + p^{j+1}(n-1)) > t + c$$

for some constant c.

Hence the slope of the vanishing line can be made arbitrarily small by making j large, and j can be chosen so that some power of x lies above the vanishing line. This means that $G_j \wedge \alpha$ is nilpotent, and 5.10 follows.

We now turn to the more difficult proof of 5.11. We need to show that $\langle G_j \rangle = \langle X(n) \rangle$. Recall that $G_j = F_{p^j-1}$, and $H_*(F_k)$ is the free module over $H_*(X(n))$ generated by b_n^i for $0 \le i \le k$. One has inclusion maps

$$X(n) = F_0 \hookrightarrow F_1 \hookrightarrow F_2 \hookrightarrow \cdots$$

with cofibre sequences

$$F_{k-1} \longrightarrow F_k \longrightarrow \Sigma^{2kn} X(n).$$

From this it follows immediately that

$$\langle F_k \rangle \le \langle X(n) \rangle$$

for all $k \ge 0$.

It can also be shown that (after localizing at p) there is a cofibre sequence

$$F_{kp^j-1} \longrightarrow F_{(k+1)p^j-1} \longrightarrow \Sigma^{2nkp^j} G_j.$$

In particular we have

$$G_j = F_{p^j-1} \hookrightarrow F_{2p^j-1} \hookrightarrow \cdots F_{(p-1)p^j-1} \hookrightarrow F_{p^{j+1}-1} = G_{j+1}$$

where the cofibre of each map is a suspension of G_j. This shows that

$$\langle G_j \rangle \ge \langle G_{j+1} \rangle. \tag{5.12}$$

It is also straightforward to show that there is a cofibre sequence

$$G_j \longrightarrow G_{j+1} \longrightarrow \Sigma^{2np^j} F_{(p-1)p^j-1}$$

which induces a short exact sequence in homology. Thus we can form the composite

$$f_{n,j}\colon G_{j+1} \longrightarrow \Sigma^{2np^j} F_{(p-1)p^j-1} \longrightarrow \Sigma^{2np^j} G_{j+1}$$

in which the first map is surjective in homology while the second is monomorphic. We denote the cofibre of $f_{n,j}$ by $Y_{n,j}$.

In order to understand these spectra, it is useful to consider the simplest case, i.e., $n = 1$ and $j = 0$. Then $G_0 = S^0$, G_1 has p cells and $Y_{1,0}$ is 2–cell complex of the form

$$S^1 \cup e^{2p}$$

so there is a cofibre sequence

$$S^{2p-1} \xrightarrow{b_{1,0}} S^1 \longrightarrow Y_{1,0}. \tag{5.13}$$

(At an odd prime these complexes are actually wedges of spheres, but is is best to ignores this fact for now.)

In the general case one replaces the cells above by copies of G_j, and there is a diagram

$$\begin{array}{ccccc} & & G_j & & \\ & & {\scriptstyle i}\downarrow & & \\ \Sigma^{-1}Y_{n,j} & \longrightarrow & G_{j+1} & \xrightarrow{f_{n,j}} & \Sigma^{2np^j}G_{j+1} \end{array}$$

where the bottom row is a cofibre sequence and i is the inclusion. The composite $f_{n,j}i$ is null, so i lifts to $\Sigma^{-1}Y_{n,j}$ This gives us the following cofibre sequence generalizing (5.13)

$$\Sigma^{2np^{j+1}-2}G_j \xrightarrow{\quad b_{n,j} \quad} G_j \longrightarrow \Sigma^{-1}Y_{n,j}. \tag{5.14}$$

Since $Y_{n,j}$ is the cofibre of a self–map of G_{j+1}, we have

$$\langle G_{j+1}\rangle \geq \langle Y_{n,j}\rangle. \tag{5.15}$$

Using 5.7, we see that *if the telescope $b_{n,j}^{-1}G_j$ is contractible* then we will have

$$\begin{array}{rcll} \langle Y_{n,j}\rangle & = & \langle G_j\rangle & \text{so} \\ \langle G_{j+1}\rangle & = & \langle G_j\rangle & \text{by (5.15) and (5.12)} \end{array}$$

Thus we have reduced the Nilpotence Theorem to the following.

Lemma 5.16 *Let*

$$\Sigma^{2np^{j+1}-2}G_j \xrightarrow{\quad b_{n,j} \quad} G_j$$

be the map of (5.14). It has a contractible telescope for each n and j.

This is equivalent to the statement that for each finite skeleton of G_j, there is an iterate of $b_{n,j}$ whose restriction to the skeleton is null.

Proof. We need to look again at (5.8) for $k = p^j - 1$. The map

$$J_{p^j-1}S^{2n} \longrightarrow \Omega S^{2n+1}$$

is known (after localizing at p) to the be inclusion of the fibre of a map

$$\Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np^j+1}.$$

Thus the diagram (5.8) can be enlarged to

$$\begin{array}{ccccc}
 & & \Omega S^{2np^j+1} & = & \Omega S^{2np^j+1} \\
 & & \uparrow & & H\uparrow \\
\Omega SU(n) & \longrightarrow & \Omega SU(n+1) & \longrightarrow & \Omega S^{2n+1} \\
\| & & \uparrow & & \uparrow \\
\Omega SU(n) & \longrightarrow & B_{p^j-1} & \longrightarrow & J_{p^j-1}S^{2n} \\
 & & \uparrow & & \uparrow \\
 & & \Omega^2 S^{2np^j+1} & = & \Omega^2 S^{2np^j+1}
\end{array} \tag{5.17}$$

in which each row and column is a fibre sequence.

Of particular interest is the map

$$\Omega^2 S^{2np^j+1} \longrightarrow B_{p^j-1}$$

We can think of the double loop space $\Omega^2 S^{2np^j+1}$ as a topological group acting on the space B_{p^j-1}, so there is an action map

$$\Omega^2 S^{2np^j+1} \times B_{p^j-1} \longrightarrow B_{p^j-1}. \tag{5.18}$$

Recall that G_j is the Thom spectrum of a certain stable vector bundle over B_{p^j-1}. This means that (5.18) leads to a stable map

$$\Sigma^\infty \Omega^2 S_+^{2np^j+1} \wedge G_j \xrightarrow{\mu} G_j. \tag{5.19}$$

Here we are skipping over some technical details which can be found in [DHS88].

The space $\Omega^2 S^{2np^j+1}$ was shown by Snaith [Sna74] to have a stable splitting, i.e., the suspension spectrum $\Sigma^\infty \Omega^2 S_+^{2np^j+1}$ is homotopy equivalent to an infinite wedge of finite spectra. After localizing at p, this splitting has the form

$$\Sigma^\infty \Omega^2 S_+^{2np^j+1} \simeq (S^0 \vee S^{2np^j-1}) \wedge \bigvee_{m\geq 0} \Sigma^{m|b_{n,j}|} D_m \tag{5.20}$$

where each D_m is a certain finite complex (independent of n and j) with bottom cell in dimension 0. Moreover there are maps

$$S^0 = D_0 \longrightarrow D_1 \longrightarrow D_2 \longrightarrow \cdots$$

of degree 1 on the bottom cell, and the limit, $\lim_\rightarrow D_m$, is known to be the mod p Eilenberg–Mac Lane spectrum $H/(p)$ (3.3).

Our map $b_{n,j}$ is the composite

$$\Sigma^{|b_{n,j}|}G_j \longrightarrow \Sigma^{|b_{n,j}|}D_1 \wedge G_j \longrightarrow \Sigma^{\infty}\Omega^2 S_+^{2np^j+1} \wedge G_j \xrightarrow{\mu} G_j.$$

and $b_{n,j}^m$ is the composite

$$\Sigma^{m|b_{n,j}|}G_j \longrightarrow \Sigma^{m|b_{n,j}|}D_m \wedge G_j \longrightarrow \Sigma^{\infty}\Omega^2 S_+^{2np^j+1} \wedge G_j \xrightarrow{\mu} G_j.$$

Thus we get a diagram

$$\begin{array}{ccccccc}
G_j & \longrightarrow & D_1 \wedge G_j & \longrightarrow & D_2 \wedge G_j & \longrightarrow & \cdots \\
\| & & \downarrow & & \downarrow & & \\
G_j & \xrightarrow{b_{n,j}} & \Sigma^{-|b_{n,j}|}G_j & \xrightarrow{b_{n,j}} & \Sigma^{-2|b_{n,j}|}G_j & \longrightarrow & \cdots
\end{array}$$

This means that the map

$$G_j \longrightarrow b_{n,j}^{-1}G_j$$

factors through $G_j \wedge H/(p)$.

Now consider the diagram

$$\begin{array}{ccccc}
G_j & \longrightarrow & G_j \wedge H/(p) & \longrightarrow & b_{n,j}^{-1}G_j \\
{\scriptstyle b_{n,j}}\downarrow & & {\scriptstyle b_{n,j}}\downarrow & & \| \\
\Sigma^{-|b_{n,j}|}G_j & \longrightarrow & \Sigma^{-|b_{n,j}|}G_j \wedge H/(p) & \longrightarrow & b_{n,j}^{-1}G_j \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

The middle vertical map is null because $b_{n,j}$ induces the trivial map in homology. Passing to the limit, we get

$$b_{n,j}^{-1}G_j \longrightarrow \text{pt.} \longrightarrow b_{n,j}^{-1}G_j$$

with the composite being the identity map on the telescope $b_{n,j}^{-1}G_j$. This shows that the telescope is contractible as desired. ■

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