Some variations on the telescope conjecture

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Abstract. This paper presents some speculations about alternatives to the recently disproved telescope conjecture in stable homotopy theory. It includes a brief introduction to the parametrized Adams spectral sequence, the main technical tool used to disprove it. An example supporting the new conjectures is described.

1. Introduction

A p-local finite spectrum X is said to have *type* n if $K(n-1)_*(X) = 0$ and $K(n)_*(X) \neq 0$. The periodicity theorem of Hopkins-Smith [**HS**] says that any such complex admits a map

$$\Sigma^d X \xrightarrow{f} X$$

such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(X) = 0$ for all $m > n$. Such a map is called a v_n*-map.* This map is not unique, but the direct limit $\widehat{X}$ of the system

$$X \xrightarrow{f} \Sigma^{-d} X \xrightarrow{f} \Sigma^{-2d} X \xrightarrow{f} \cdots ,$$

called the *telescope associated with* X, is independent of the choice of f. (See [**Rav92a**] for more background.)

This telescope is of interest because its homotopy groups, unlike those of X itself, are computable. By this we mean that it is possible in many interesting cases to give a *complete explicit* description of $\pi_*(\widehat{X})$. We remind the reader that there is not a single example of a noncontractible finite spectrum X for which $\pi_*(X)$ is completely known. The only connected finite complexes X for which the unstable homotopy groups are completely known are the ones (such as surfaces of positive genus) which happen to be $K(\pi, 1)$s.

In the case $n = 0$, i.e., when $H_*(X; \mathbf{Q}) \neq 0$, $\widehat{X}$ is the stable rational homotopy type of X. We have known how to compute its homotopy groups for decades. For $n = 1$, $\pi_*(\widehat{X})$ is called the v_1-periodic homotopy of X. Since 1982 this has been computed in may interesting cases; these results are surveyed by Davis in [**Dav**].

In this paper we will speculate here about this problem for $n \geq 2$, offering some substitutes for the original telescope conjecture [**Rav84**].

1991 *Mathematics Subject Classification.* Primary 55Q10, 55T14; Secondary 55N20, 55P60.
Partially supported by the National Science Foundation and MSRI.

The classical Adams spectral sequence is useless for computing $\pi_*(\widehat{X})$ because $H_*(\widehat{X};\mathbf{Z}/(p))=0$. The Adams-Novikov spectral sequence for $\pi_*(\widehat{X})$ has nice properties and can be completely analyzed in many interesting cases, but is not known to converge for $n>1$.

The telescope conjecture of 1977 has three equivalent formulations:

- The Adams-Novikov spectral sequence for $\pi_*(\widehat{X})$ converges.
- The natural map $\widehat{X}\to L_nX$ is an equivalence, where L_n denotes Bousfield localization with respect to (equivalently) $E(n)$, $v_n^{-1}BP$ or $K(0)\vee K(1)\vee\cdots\vee K(n)$. For X as above, L_nX is the same as $L_{K(n)}X$.
- $\widehat{X}$ has the same Bousfield class as $K(n)$.

These statements are easily proved for $n=0$, known to be true (but not easily proved) for $n=1$ (this is due to Mahowald [**Mah82**] for $p=2$ and to Miller [**Mil81**] for $p>2$), and known to be false ([**Rav92b**] and [**Rava**]) for $n=2$.

We will indicate how far off the telescope conjecture is for $n\geq 2$ by describing our best guesses for the values of $\pi_*(\widehat{X})$ and $\pi_*(L_nX)$ in the simplest cases. What follows is not intended to be a precise statement, but rather an indication of the flavor of the calculations. They have been verified for $n=2$ and $p=2$ in recent joint work with Mahowald and Shick [**MRS**].

With these caveats in mind, $\pi_*(L_nX)$ (for a suitable type n finite ring spectrum X) is a subquotient of an exterior algebra on n^2 generators, while $\pi_*(\widehat{X})$ a subquotient of an exterior algebra on only $\binom{n+1}{2}$generators tensored with $\binom{n}{2}$ factors of the form $\mathbf{Z}/(p)[\mathbf{Q}/\mathbf{Z}_{(p)}]$. (Note that $\binom{n+1}{2}+\binom{n}{2}=n^2$.)

The appearance of this second type of factor, in place of $\binom{n}{2}$ of the exterior factors in $\pi_*(L_nX)$, is a startling development. It implies that in $\pi_*(V(1))$ for $p\geq 5$ (where $V(1)$ stands for Toda's example of a type 2 complex [**Tod71**]) there is a family of elements $x_1,x_2,\cdots$, each having positive Adams-Novikov filtration, such that any product of them (with no repeated factors) is nontrivial. We are not aware of any example of this sort that was known previously.

2. The Adams spectral sequence

The Adams spectral sequence for $\pi_*(X)$ is derived from the following *Adams diagram.*

(1)
$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & X_{-1} & \longleftarrow & X_0 & \longleftarrow & X_1 & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle g_{-1}} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \\
 & & K_{-1} & & K_0 & & K_1, & &
\end{array}$$

Here X_{s+1} is the fibre of g_s, and $X_s=X$ (and $K_{s-1}=\text{pt.}$) for all $s\leq 0$. We get an exact couple of homotopy groups and a spectral sequence with

$$E_1^{s,t}=\pi_{t-s}(K_s)\quad\text{and}\quad d_r: E_r^{s,t}\to E_r^{s+r,t+r-1}.$$

This spectral sequence converges to $\pi_*(X)$ if the homotopy inverse limit $\lim_{\leftarrow} X_s$ is contractible. When X is connective, it is a first quadrant spectral sequence. For more background, see [**Rav86**].

In the classical Adams spectral sequence we set $K_s=X_s\wedge H/p$, where H/p denotes the mod p Eilenberg-Mac Lane spectrum, and in the Adams-Novikov spectral sequence we have $K_s = X_s \wedge BP$. For a connective spectrum X, the classical Adams spectral sequence converges when X is p-adically complete, and the Adams-Novikov spectral sequence converges when X is p-local. In each case E_2 can be identified as an Ext group which can be computed algebraically.

3. The localized Adams spectral sequence

The localized Adams spectral sequence (originally due to Miller [**Mil81**]) is derived from the Adams spectral sequence in the following way. The telescope $\widehat{X}$ is obtained from X by iterating a v_n-map $f : X \to \Sigma^{-d}X$. Suppose that this map has positive Adams filtration (which it always does in the classical case), ie suppose there is a lifting

$$\tilde{f} : X \to \Sigma^{-d} X_{s_0}$$

for some $s_0 > 0$. This will induce maps $\tilde{f} : X_s \to \Sigma^{-d} X_{s+s_0}$ for $s \geq 0$. This enables us to define $\widehat{X}_s$ to be the homotopy direct limit of

$$X_s \xrightarrow{\tilde{f}} \Sigma^{-d} X_{s+s_0} \xrightarrow{\tilde{f}} \Sigma^{-2d} X_{s+2s_0} \xrightarrow{\tilde{f}} \cdots$$

$X_s = X$ for $s < 0$. Thus we get the following diagram, similar to that of (1).

$$\begin{array}{ccccccccc}
\cdots & \leftarrow & \widehat{X}_{-1} & \leftarrow & \widehat{X}_0 & \leftarrow & \widehat{X}_1 & \leftarrow & \cdots \\
 & & \downarrow{\scriptstyle g_{-1}} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \\
 & & \widehat{K}_{-1} & & \widehat{K}_0 & & \widehat{K}_1, & &
\end{array}$$

where the spectra $\widehat{K}_s$ are defined after the fact as the obvious cofibres. This leads to a full plane spectral sequence (the localized Adams spectral sequence) with

$$E_1^{s,t} = \pi_{t-s}(\widehat{K}_s) \quad \text{and} \quad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$

as before. This spectral sequence converges to the homotopy of the homotopy direct limit $\pi_*(\lim_{\to} \widehat{X}_{-s})$ if the homotopy inverse limit $\lim_{\leftarrow} \widehat{X}_s$ is contractible. The following result is proved in [**Ravb**].

Theorem 2 (Convergence of the localized Adams spectral sequence). *For a type n finite complex X, in the localized Adams spectral sequence for $\pi_*(\widehat{X})$ we have*

- *The homotopy direct limit $\lim_{\to} \widehat{X}_{-s}$ is the telescope $\widehat{X}$.*
- *The homotopy inverse limit $\lim_{\leftarrow} \widehat{X}_s$ is contractible if the original (unlocalized) Adams spectral sequence has a vanishing line of slope s_0/d at E_r for some finite r, i.e., if there are constants c and r such that*

$$E_r^{s,t} = 0 \quad \textit{for} \quad s > c + (t-s)(s_0/d).$$

(In this case we say that f has a parallel lifting *$\tilde{f}$.)*

The proof of this result is not deep; it only involves figuring out which diagrams to chase. Here are some informative examples.

- If we start with the Adams-Novikov spectral sequence, then the map f cannot be lifted since $BP_*(f)$ is nontrivial. Thus we have $s_0 = 0$ and the lifting condition requires that X has a horizontal vanishing line in its Adams-Novikov spectral sequence. This is not known (or suspected) to occur for any nontrivial finite X, so we do not get a convergence theorem about the localized Adams-Novikov spectral sequence, which is merely the standard Adams-Novikov spectral sequence applied to $\widehat{X}$.

- If we start with the classical Adams spectral sequence, an unpublished theorem of Hopkins-Smith says that a type n X (with $n > 0$) always has a vanishing line of slope $1/|v_n| = 1/(2p^n - 2)$. (A proof can be found in [**Ravb**].) Thus we have convergence if f has a lifting with $s_0 = d/|v_n|$. This does happen in the few cases where Toda's complex $V(n)$ exists. Then $V(n-1)$ is a type n complex with a v_n-map with $d = |v_n|$ and $s_0 = 1$.
- The lifting described above does not exist in general. For example, let X be the mod 4 Moore spectrum. It has type 1 and $|v_1| = 2$. Adams constructed a v_1-map f with $d = 8$, but its filtration is 3, rather than 4 as required by the convergence theorem. Replacing f with an iterate f^i does not help, because its filtration is only $4i - 1$. This difficulty can be fixed with the localized parametrized Adams spectral sequence (see [**Ravb**] for more details), to be described below.
- In favorable cases (such as Toda's examples) the E_2-term of the localized Adams spectral sequence can be identified as an Ext groups which can be computed explicitly.

4. The parametrized Adams spectral sequence

We will describe a family of Adams spectral sequences parametrized by a rational number ϵ interpolating between the classical Adams spectral sequence (the case $\epsilon = 1$) and the Adams-Novikov spectral sequence ($\epsilon = 0$). The construction is easy to describe, but difficult to carry out in detail. It is the subject of [**Ravb**].

First we need some notation. Let $G = BP$ and $F = H/p$. We have the following homotopy commutative diagram in which the rows are cofibre sequences, and h and h' are the usual unit maps.

$$\begin{array}{ccccc}
\overline{G} & \xrightarrow{r} & S^0 & \xrightarrow{h} & G \\
\big\downarrow{\scriptstyle \bar{\iota}} & & \big\downarrow & & \big\downarrow{\scriptstyle \iota} \\
\overline{F} & \xrightarrow{r'} & S^0 & \xrightarrow{h'} & F
\end{array}$$

Then in the Adams diagram for the classical Adams spectral sequence we can set $X_s = X \wedge \overline{F}^{(s)}$, and in the one for the Adams-Novikov spectral sequence, $X_s = X \wedge \overline{G}^{(s)}$.

We want to smash these two Adams diagrams together to get a 2-dimensional diagram, and we also want to exploit the map $\bar{\iota}$ above. For $i, j \geq 0$ let

$$X_{i,j} = X \wedge \left\{ \begin{array}{ll} S^0 & \text{for } j \leq i \\ \overline{F}^{(j-i)} & \text{for } j > i \end{array} \right\} \wedge \overline{G}^{(i)}$$

We will define maps

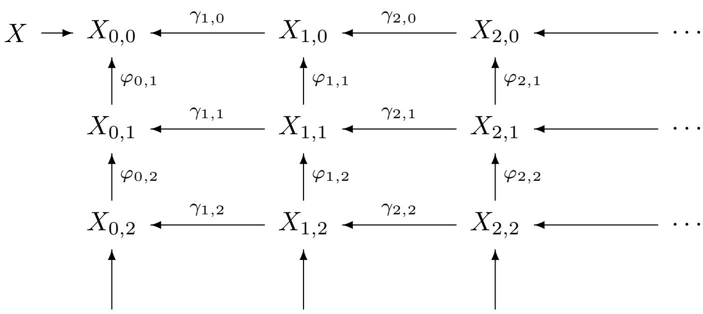
$$X_{i,j} \xrightarrow{\gamma_{i,j}} X_{i-1,j} \text{ for } i > 0, \quad \text{and} \quad X_{i,j} \xrightarrow{\varphi_{i,j}} X_{i,j-1} \text{ for } j > 0$$

by

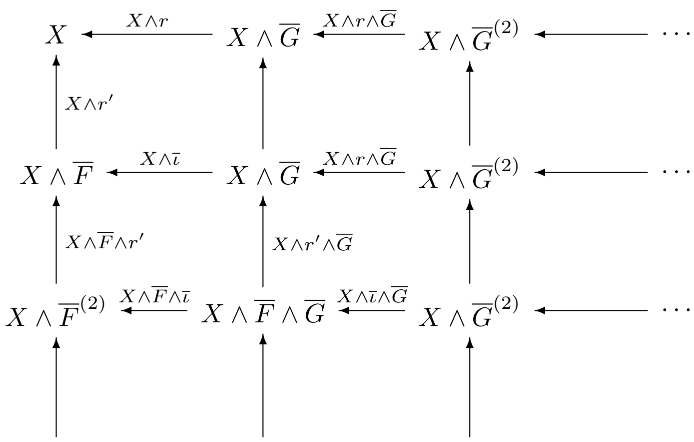
$$\gamma_{i,j} = X \wedge \left\{ \begin{array}{ll} \overline{F}^{(j-i)} \wedge \bar{\iota} & \text{if } j \geq i \\ r & \text{otherwise} \end{array} \right\} \wedge \overline{G}^{(i-1)}$$

$$\varphi_{i,j} = X \wedge \left\{ \begin{array}{ll} \overline{F}^{(j-i-1)} \wedge r' & \text{if } j > i \\ S^0 & \text{otherwise} \end{array} \right\} \wedge \overline{G}^{(i)}$$

It follows that we have a diagram



By our definitions this is the same as



This diagram commutes up to homotopy and is equivalent to one that commutes strictly [**Ravb**]. Hence it makes sense to speak of unions and intersections of the various $X_{i,j}$ as subspectra of X.

Now fix a number $0 \leq \epsilon \leq 1$, and for each $s \geq 0$ let

$$X_s = \bigcup_{(1-\epsilon)i+\epsilon j \geq s} X_{i,j} = \bigcup_{i+\epsilon j \geq s} X \wedge \overline{G}^{(i)} \wedge \overline{F}^{(j)}.$$

DEFINITION 3. *For a rational number $\epsilon = k/m$ (with m and k relatively prime) between 0 and 1, the* **parametrized Adams spectral sequence** *is the homotopy* *spectral sequence based on the exact couple associated with the resolution*

$$\begin{array}{ccccccc} X & \longrightarrow & X_0 & \longleftarrow & X_{1/m} & \longleftarrow & X_{2/m} \longleftarrow \cdots \\ & & \Big\downarrow{\scriptstyle g_0} & & \Big\downarrow{\scriptstyle g_{1/m}} & & \Big\downarrow{\scriptstyle g_{2/m}} \\ & & K_0 & & K_{1/m} & & K_{2/m} \end{array}$$

with X_s as above and K_s the obvious cofibre. This is a reindexed form of the Adams diagram of §2; the index s need not be an integer but will always be a whole multiple of $1/m$. Thus we have

$$E_{1/m}^{s,t} = \pi_{t-s}(K_{s/m}) \quad \text{and} \quad d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

The indices r, s and t need not be integers here, but $t-s$ (the topological dimension) is always a whole number.

In favorable cases, when the classical Adams spectral sequence for $BP_*(X) = \pi_*(BP \wedge X)$ collapses from E_2 (as it does when X is a Toda complex), we can describe $E_{1+\epsilon}$ in terms of Ext groups.

This parametrized Adams spectral sequence can be localized the same way the classical Adams spectral sequence can be, and there is a similar convergence theorem for the localized parametrized Adams spectral sequence. Thus we need to examine the existence of parallel liftings again.

The methods of Hopkins-Smith can be adopted to this situation to show that for any $\epsilon > 0$, the parametrized Adams spectral sequence for a type n complex X has a vanishing line of slope $\epsilon/|v_n|$. We also have the following result.

Theorem 4. *Let $f : X \to \Sigma^{-d}X$ be a v_n-map. Then for $\epsilon < |v_n|/d$, f has a lifting to $\Sigma^{-d}X_{\epsilon d/|v_n|}$.*

Hence for any type n complex X, the localized parametrized Adams spectral sequence converges to $\pi_*(\widehat{X})$ for sufficiently small positive ϵ.

5. Some conjectures

Now we will speculate about the behavior of this localized parametrized Adams spectral sequence converging to $\pi_*(\widehat{X})$. Each statement we will make below has been verified in a special case where $n = 2$ and $p = 2$, in recent joint work with Mahowald and Shick [**MRS**].

Presumably these statements will be proved by showing that the indicated properties are **generic**, i.e., the set of spectra having them is closed under cofibrations and retracts. Then the thick subcategory theorem (originally due to Hopkins-Smith[**HS**] and alos proved in [**Rav92a**, Chapter 5]), which classifies all such sets of finite spectra, will say that if they are true for one type n complex, they are true for all of them. However, at the moment we cannot prove that any of the properties we will discuss are generic.

We have three conjectures. All concern the behavior of the localized parametrized Adams spectral sequence converging to $\pi_*(\widehat{X})$ for sufficiently small positive ϵ.

Conjecture 5. *The localized parametrized Adams spectral sequence collapses from E_r for some finite r.*

Now recall that our spectral sequence has a vanishing line of slope $\epsilon/|v_n|$ at E_∞. Let c be its s-intercept, that is the smallest number such that

$$E_\infty^{s,t} = 0 \quad \text{for} \quad s > c + \epsilon(t-s)/|v_n|.$$

Recall also that the v_n-map f induces isomorphisms

$$E_r^{s,t} \to E_r^{s+\epsilon(d/|v_n|),t+d+\epsilon(d/|v_n|)}$$

commuting with differentials, so E_∞ is determined by $E_\infty^{s,t}$ for $0 \le t-s < d$.

With this in mind, let $\rho(y)$ denote the total rank of the $E_\infty^{s,t}$ with $0 \le t-s < d$ and

$$c + \epsilon(t-s)/|v_n| \ge s > c - y + \epsilon(t-s)/|v_n|.$$

In the usual chart (with horizontal coordinate $t-s$) and vertical coordinate s, this is a parallelogram shaped region bounded by two vertical lines d units apart, the vanishing line, and a line y units below and parallel to it.

CONJECTURE 6. *Let $\rho(y)$ be as above. Then it is finite for all $y \ge 0$ and grows asymptotically with $y^{\binom{n}{2}}$.*

This growth estimate for $\rho(y)$ is related the $\binom{n}{2}$ factors of the form $\mathbf{Z}/(p)[\mathbf{Q}/\mathbf{Z}_{(p)}]$. If the telescope conjecture were true, $\rho(y)$ would be bounded.

Our third conjecture concerns the behavior of the spectral sequence as ϵ approaches 0. This type of analysis was crucial in the disproof of the telescope conjecture.

Given an element $x \in E_r^{s,t}$, we define its **effective filtration** $\phi(x,\epsilon)$ by

$$\phi(x,\epsilon) = s - \frac{|v_n|}{\epsilon}(t-s).$$

On the chart, this is the s-intercept of a line parallel to the vanishing line, through the point corresponding to x. It is invariant under composition with f.

For $x \in \pi_*(\widehat{X})$, for each sufficiently small $\epsilon > 0$, there is a unique nontrivial permanent cycle x_ϵ represented by x, and we define $\phi(x,\epsilon)$ to be $\phi(x_\epsilon,\epsilon)$.

CONJECTURE 7. *For $x \in \pi_*(\widehat{X})$, let $\phi(x,\epsilon)$ be as above. Then the quantity*

$$\lambda_0(x) = \lim_{\epsilon \to 0+} \phi(x,\epsilon)$$

is either $+\infty$ (in which case we say x is **parabolic***), or it is an integer ranging from 0 to $\binom{n+1}{2}$, in which case we say that x is* **linear***.*

A nontrivial element is linear if and only if its image in $\pi_*(L_nX)$ is nontrivial, and $\lambda_0(x)$ is its Adams-Novikov filtration.

For a parabolic element x, the function $\epsilon\phi(x,\epsilon)$ is bounded, and we define

$$\begin{aligned} \lambda_1(x) &= \limsup_{\epsilon \to 0+} \epsilon\phi(x,\epsilon), \\ \lambda_2(x) &= \limsup_{\epsilon \to 0+} \left(\phi(x,\epsilon) - \frac{\lambda_1(x)}{\epsilon}\right), \\ \textit{and} \qquad \mu(x) &= \frac{\limsup_{\epsilon\to 0+} \epsilon\phi(x,\epsilon)}{\liminf_{\epsilon\to 0+} \epsilon\phi(x,\epsilon)}. \end{aligned}$$

(We call these quantities the **focal length***,* **displacement** *and* **magnification** *of x respectively.) These quantities are all nonnegative and finite, and subject to bounds depending only on p and n.*

For $n = 2$, $0 < \lambda_1(x) < \frac{p^2-1}{p}$, $\mu(x) = \frac{(p+1)^2}{4p}$, and $\lambda_2(x) - \lambda_1(x)/(p-1)$ is an integer ranging from 0 to $\binom{n+1}{2}$.

6. An example with $n = 2$

We will now describe an example that illustrates this conjecture and motivates its terminology. The relevant computations were done in [**Rava**]. Let $X = V(1)$ for $p \geq 5$. It has a v_2-map f of degree $|v_2|$. Let $x \in \pi_*(\widehat{X})$ be the image of the composite

$$\begin{array}{ccccc} S^{|v_2|} & \xrightarrow{h} & \Sigma^{|v_2|}V(1) & & & & \\ & & \Big\downarrow{\scriptstyle f} & & & & \\ & & V(1) & \xrightarrow{j} & S^{2p} & \xrightarrow{h} & \Sigma^{2p}V(1) \end{array}$$

where h and j are the obvious inclusion and pinch maps.

In the localized parametrized Adams spectral sequence we have elements

$$b_{i,0} \in E_{1+\epsilon}^{2,2p(p^i-1)} \quad \text{and} \quad h_{i,1} \in E_{1+\epsilon}^{1,2p(p^i-1)}$$

for $i > 0$. For large ϵ, $b_{1,0}$ is the nontrivial permanent cycle represented by x, but for small ϵ, $b_{1,0}$ is killed by a differential and the situation is more complicated.

For $i > 0$, let

$$\epsilon_i = \frac{2p-2}{p^i-1} = \frac{2}{1+p+\cdots+p^{i-1}} \approx \frac{2}{p^{i-1}}$$

Then for $i > 1$ we have differentials

$$d_r(h_{i+1,1}) = \begin{cases} v_2 b_{i,0}^p & \text{for } \epsilon > \epsilon_i \\ v_2 b_{i,0}^p \pm v_2^{p^i} b_{i-1,0} & \text{for } \epsilon = \epsilon_i \\ v_2^{p^i} b_{i-1,0} & \text{for } \epsilon < \epsilon_i \end{cases}$$

for suitable values of r. The element $h_{1,1}$ is always a permanent cycle represented by a linear element, and for suitable values of r we have (ignoring powers of v_2)

$$d_r(h_{2,1}) = \begin{cases} b_{1,0}^p & \text{for } \epsilon > \epsilon_2 \\ b_{i,0}^{p^i} & \text{for } \epsilon_i > \epsilon > \epsilon_{i+1} \\ b_{i,0}^{p^i} \pm b_{i-1,0}^{p^{i-1}} & \text{for } \epsilon = \epsilon_i. \end{cases}$$

From these considerations we can deduce that our element $x \in \pi_*(\widehat{X})$ represents the nontrivial permanent cycle x_ϵ given by

$$x_\epsilon = \begin{cases} b_{1,0} & \text{for } \epsilon > \epsilon_2 \\ b_{i,0}^{p^{i-1}} \pm b_{i+1,0}^{p^i} & \text{for } \epsilon = \epsilon_i \\ b_{i+1,0}^{p^i} & \text{for } \epsilon_i > \epsilon > \epsilon_{i+1} \end{cases}$$

Now since

$$\phi(b_{i,0}, \epsilon) = 2 - \frac{\epsilon}{|v_2|}(2p^{i+1} - 2p - 2) = 2 - \epsilon\left(\frac{p^{i+1} - p - 1}{p^2 - 1}\right),$$

we have

$$\phi(x,\epsilon)=\begin{cases} 2-\epsilon\left(\frac{p^2-p-1}{p^2-1}\right) & \text{for } \epsilon\geq\epsilon_2\\ 2p^{i-1}\;-\epsilon p^{i-1}\left(\frac{p^{i+1}-p-1}{p^2-1}\right) & \text{for } \epsilon_i\geq\epsilon\geq\epsilon_{i+1}.\end{cases} \tag{8}$$

Thus $\phi(x,\epsilon)$ as a function of ϵ is piecewise linear and $\epsilon\phi(x,\epsilon)$ is piecewise quadratic, its graph consisting of a countable collection of parabolic arcs. (This is *not* the reason for the term 'parabolic,' which will be explained below.) Close examination reveals that

$$\frac{(\epsilon+2p-2)^2}{\epsilon p(p^2-1)}\leq\phi(x,\epsilon)\leq\frac{(\epsilon+2p-2)^2(p+1)^2}{4\epsilon p^2(p^2-1)}, \tag{9}$$

i.e., the graph of the $\phi(x,\epsilon)$ lies between two hyperbolas. The lower bound is obtained at the cusp points ϵ_i, and the line segments in the graph are each tangent to the upper hyperbola.

From (9) we get

$$\frac{(\epsilon+2p-2)^2}{p(p^2-1)}\leq\epsilon\phi(x,\epsilon)\leq\frac{(\epsilon+2p-2)^2(p+1)^2}{4p^2(p^2-1)}.$$

The upper and lower bounds are parabolas, but in the relevant range ($0\leq\epsilon\leq 1$) they look very much like straight lines. The graph of $\epsilon\phi(x,\epsilon)$ is a countable collection of parabolic ares, each concave downward. Each arc is tangent to the upper 'line' while the lower 'line' goes through the cusp points, where adjacent arcs meet.

Each arc has its maximum value in the prescribed interval, which is roughly $[2p^{-i},2p^{1-i}]$; the i^{th} arc achieves its maximum value at

$$\epsilon=\frac{p^2-1}{p^{i+1}-p-1}\approx\frac{1}{p^{i-1}}$$

and the maximum value is

$$\frac{p^{i-1}(p^2-1)}{p^{i+1}-p-1}$$

so we get

$$\lambda_1(x)=\frac{p^2-1}{p^2}.$$

This value of $\lambda_1(x)$ differs from the upper bound of the conjecture by a factor of p. Let $x_i\in\pi_*(\widehat{X})$ be an element representing $b_{i,0}$ for $\epsilon>\epsilon_{i+1}$. Then similar computations show that

$$\begin{aligned} \lambda_1(x_i) &= \frac{p^2-1}{p^{i+1}}\\ \text{and}\qquad \lambda_1(x_1^{p-1}x_2^{p-1}\cdots x_k^{p-1}) &= \frac{(p^2-1)(p-1)}{p}\left(\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^k}\right)\\ &= \frac{(p^2-1)(p^k-1)}{p^{k+1}-1}\\ &< \frac{p^2-1}{p}.\end{aligned}$$

Local minima of $\epsilon\phi(x,\epsilon)$ are achieved at the cusp points ϵ_i; we find that

$$\epsilon_i \phi(x, \epsilon_i) = \frac{4(p-1)p^{2i}}{(p+1)(p^i-1)^2},$$

which gives

$$\liminf_{\epsilon \to 0+} \epsilon\phi(x, \epsilon) = \frac{4(p-1)}{p(p+1)}$$

and

$$\mu(x) = \frac{(p+1)^2}{4p}.$$

7. Optical terminology

Finally, we will discuss the terminology used in the last conjecture; it does not refer to the parabolic arcs described above. Consider the points on an Ext chart corresponding to $b_{i,0}^{p^{i-1}}$, for which $s = 2p^{i-1}$ and $t = 2p^i(p^i - 1)$. *These points all lie on the parabola*

$$t - s = \frac{p^2 s^2}{2} - (p+1)s, \tag{10}$$

along which they are exponentially distributed.

In order to find $\epsilon\phi(x, \epsilon)$, look at the lines with slope $\epsilon/|v_2|$ passing through these points, and choose with one with the highest s-intercept. This intercept is $\phi(x, \epsilon)$, and $\epsilon\phi(x, \epsilon)$ is $|v_2|$ times the product of intercept and the slope. *A calculus exercise shows that the limiting value of this product for lines tangent to the parabola is the parabola's focal length, and that $\lambda_1(x)$ is $|v_2|$ times this focal length.*

The number $\lambda_2(x)$ is the s-coordinate (in the $(t-s, s)$-coordinate system) of the vertex of this parabola; hence the term displacement. We see in this case that $\lambda_2(x) = \lambda_1(x)/(p-1)$. Replacing x by its product with a linear element y would translate the parabola raise this quantity by the Adams-Novikov filtration of y.

To find $\liminf_{\epsilon \to 0+} \epsilon\phi(x, \epsilon)$, consider the infinite convex polygon having these points as vertices. Local minima of $\epsilon\phi(x, \epsilon)$ are achieved when $\epsilon/|v_2|$ is the slope of one the edges the polygon, i.e., when $\epsilon = \epsilon_i$ for some $i \geq 2$. These lines are all tangent to the parabola

$$t - s = \frac{p(p+1)^2}{8} s^2 - (p+1)s. \tag{11}$$

The same calculus exercise shows that $\liminf_{\epsilon \to 0+} \epsilon\phi(x, \epsilon)$ is $|v_2|$ times the focal length of this parabola. It follows that $\mu(x)$ is the ratio between the two focal lengths, hence the term 'magnification.'

8. Projective geometry

Here is another approach to the parabola discussed above. The convex polygon in the $(t-s, s)$-plane is dual, in the sense of projective geometry, to the graph of the function $\phi(x, \epsilon)$ (8) in the (ϵ, ϕ)-plane.

This duality is defined as follows. A nonvertical line in one plane has a slope and a 'y-intercept,' i.e., the coordinate of its intersection with the vertical axis. These two numbers are (up to suitable scalar multiplication) the coordinates of the dual point in the other plane. (We assume that the slope of the line is proportional to the horizontal coordinate of the dual point, and the intercept is proportional to the vertical one.) Conversely, given a point in one plane, each nonvertical line through it determines a point in the other plane and these points are collinear, so we get a line in the other plane dual to the original point. If we enlarge both affine planes to projective planes, then it is no longer necessary to exclude vertical lines. They are dual to points at infinity, and the line at infinity is dual to the point at infinity in the vertical direction.

There is also a projective duality between curves. A curve in one plane has a collection of tangent lines, each of which is dual to a point in the other plane. These points all lie on a new curve, which is defined to be the dual of the original curve. The tangent lines of the dual curve are dual to the points of the original curve.

A parabola with horizontal axis, such as the one defined by (10), is dual to a hyperbola having the vertical axis as an asymptote, namely the upper one defined in (9). The lower hyperbola of (9) is dual to the parabola of (11).

References

[Dav] D. M. Davis. Computing v_1-periodic homotopy groups of spheres and some compact lie groups. To appear.

[HS] M. J. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory II. Submitted to *Annals of Mathematics*.

[Mah82] M. E. Mahowald. The image of J in the EHP sequence. *Annals of Mathematics*, 116:65–112, 1982.

[Mil81] H. R. Miller. On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space. *Journal of Pure and Applied Algebra*, 20:287–312, 1981.

[MRS] M. E. Mahowald, D. C. Ravenel, and P. Shick. The v_2-periodic homotopy of a certain Thom complex. To appear.

[Rava] D. C. Ravenel. A counterexample to the telescope conjecture. To appear.

[Ravb] D. C. Ravenel. The parametrized Adams spectral sequence. To appear.

[Rav84] D. C. Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106:351–414, 1984.

[Rav86] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, New York, 1986.

[Rav92a] D. C. Ravenel. *Nilpotence and periodicity in stable homotopy theory.* Volume 128 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, 1992.

[Rav92b] D. C. Ravenel. Progress report on the telescope conjecture. In N. Ray and G. Walker, editors, *Adams Memorial Symposium on Algebraic Topology Volume 2*, pages 1–21, Cambridge University Press, Cambridge, 1992.

[Tod71] H. Toda. On spectra realizing exterior parts of the Steenrod algebra. *Topology*, 10:53–65, 1971.

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