A solution to the Arf-Kervaire invariant problem I

Bilkent University
Ankara

May 17, 2010

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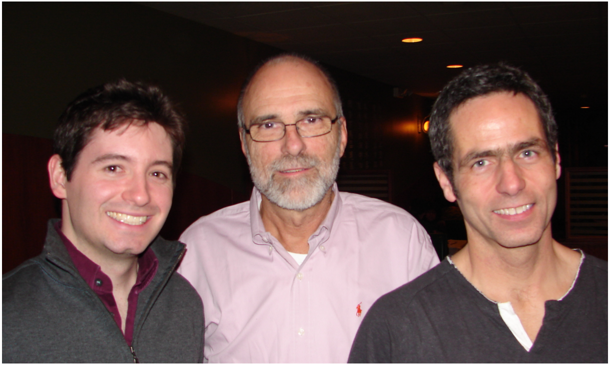

Mike Hill, myself and Mike Hopkins
February 11, 2010
Photo by Bill Browder

A wildly popular dance craze

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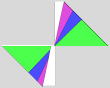
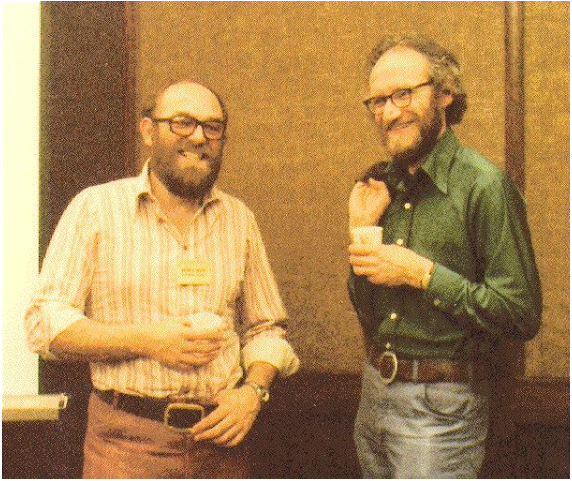
Drawing by Carolyn Snaith 1981
London, Ontario





Vic Snaith and Bill Browder in 1981
Current Trends in Algebraic Topology Conference
University of Western Ontario
Photo by Clarence Wilkerson

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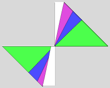

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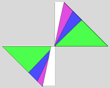

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Snaith's book (continued)

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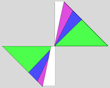

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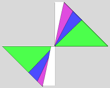

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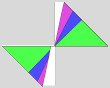
The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It follows from Browder's theorem of 1969 that such things can exist only in dimensions that are 2 less than a power of 2.



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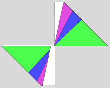

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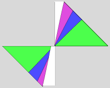

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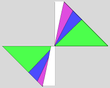

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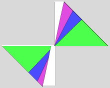

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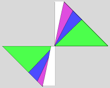

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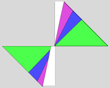

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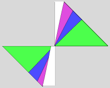
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.



Pontryagin's early work (continued)

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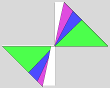

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There is a way to reverse this procedure. A framed manifold $M^k \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^n$.

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Two maps $f_1, f_2 : S^{n+k} \to S^n$ are homotopic if there is a continuous map $h : S^{n+k} \times [0,1] \to S^n$ (called a homotopy between f_1 and f_2) such that

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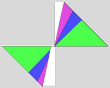

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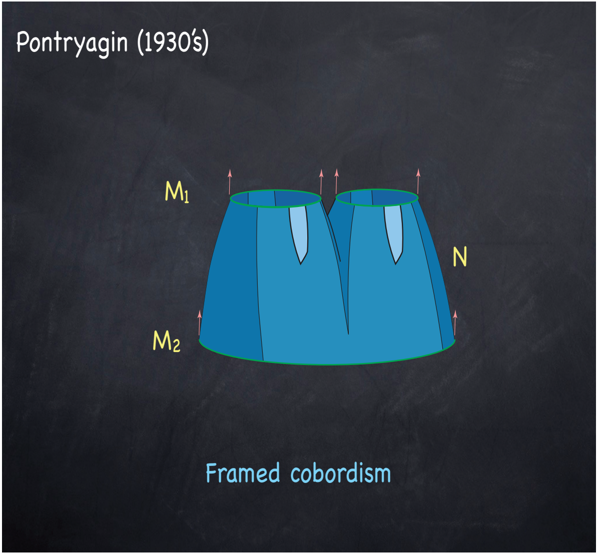
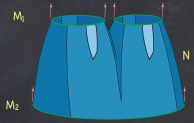
Pontryagin's early work (continued)

Here is an example of a framed cobordism for $n = k = 1$.



Pontryagin's early work (continued)

Pontryagin (1930's)



$$\Omega_k := \{\text{stably framed } k\text{-manifolds}\}/_{\text{cobordism}}$$

Theorem: The above construction gives a bijection

$$\pi_{n+k}(S^n) \approx \Omega_k$$

where

$$\pi_{n+k}(S^n) := \{\text{maps } S^{n+k} \to S^n\}/_{\text{homotopy}}$$

Pontryagin's early work (continued)

Pontryagin (1930's)

k=0

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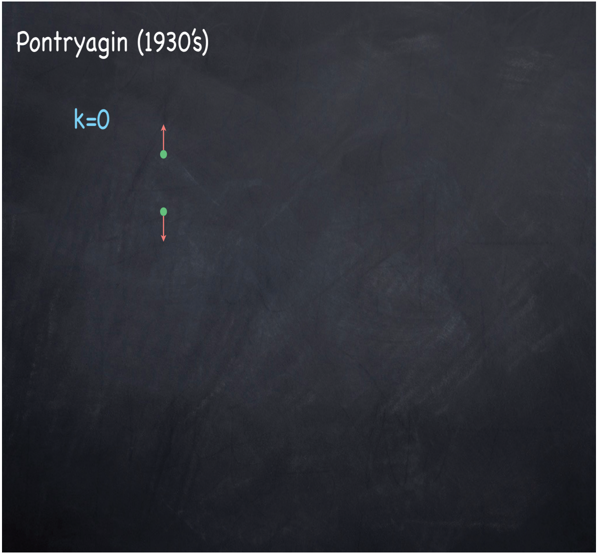
Pontryagin's early work (continued)



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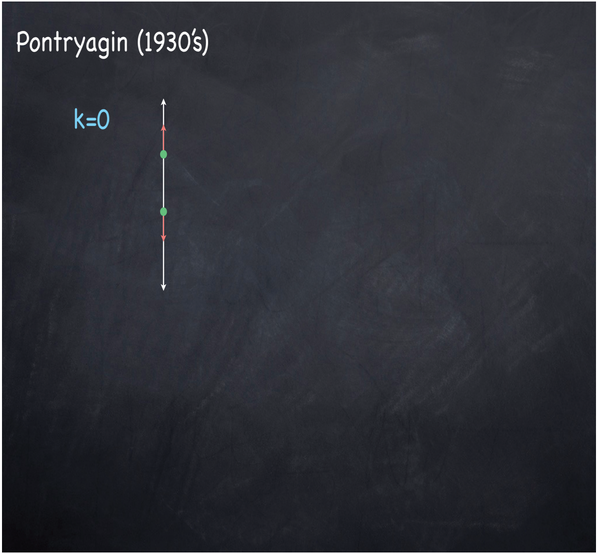

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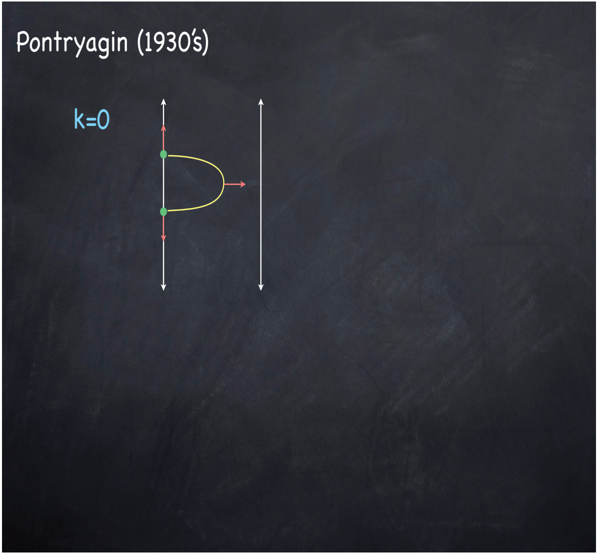

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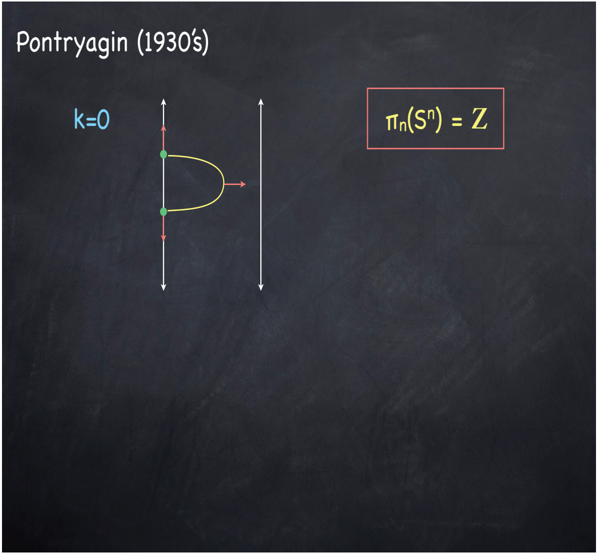

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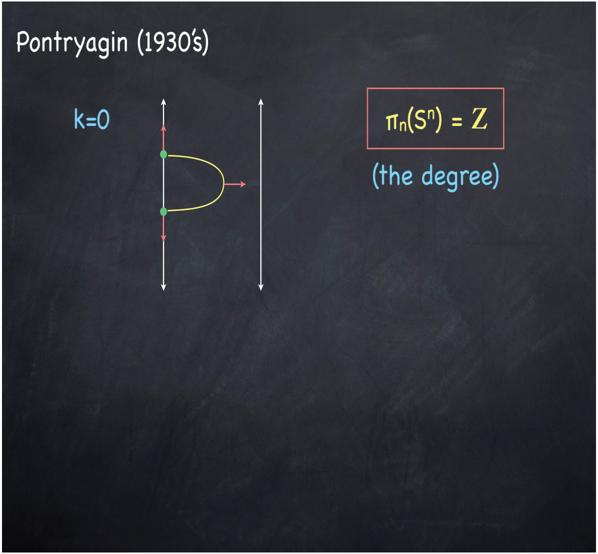

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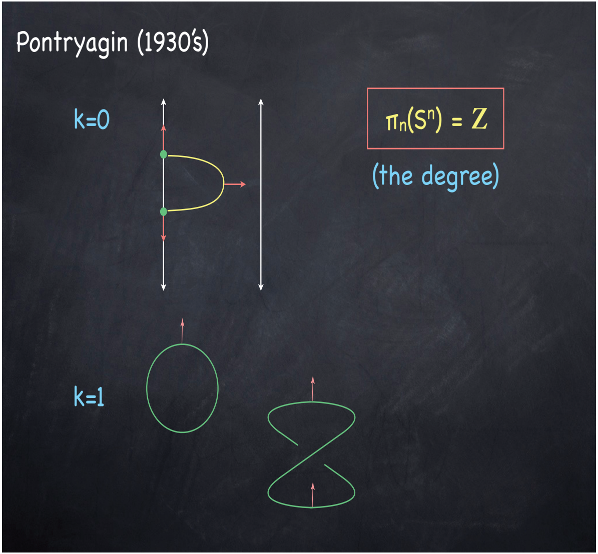

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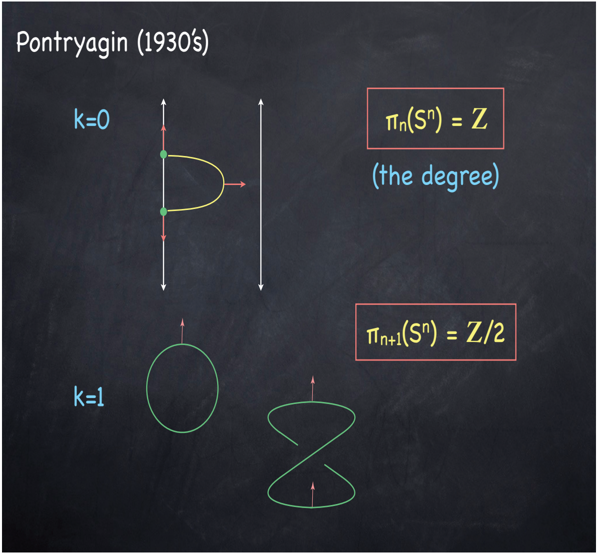

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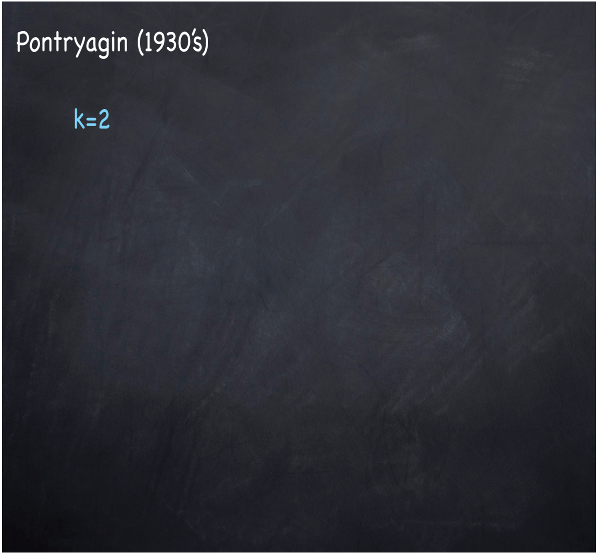

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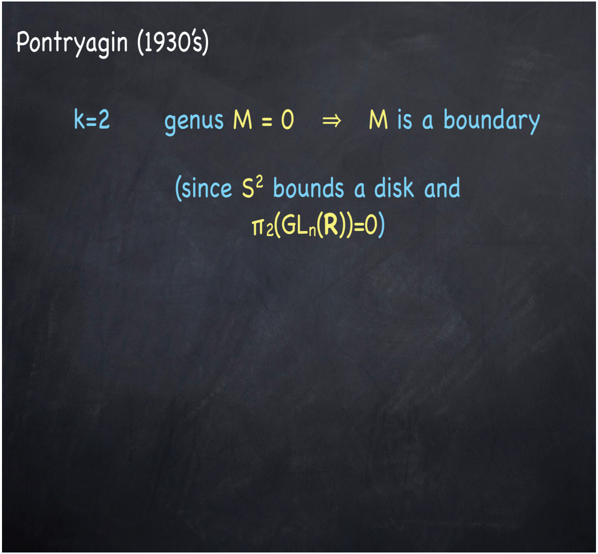

Mike Hill
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Doug Ravenel

Pontryagin's early work (continued)

Pontryagin (1930's)

k=2 genus M = 0 ⇒ M is a boundary

(since S^2 bounds a disk and $\pi_2(GL_n(\mathbf{R}))=0$)

Suppose the genus of M is greater than 0.

Pontryagin’s early work (continued)

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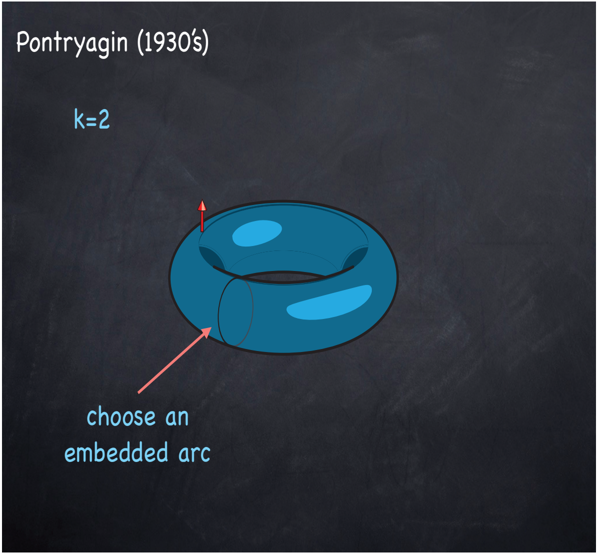

Mike Hill
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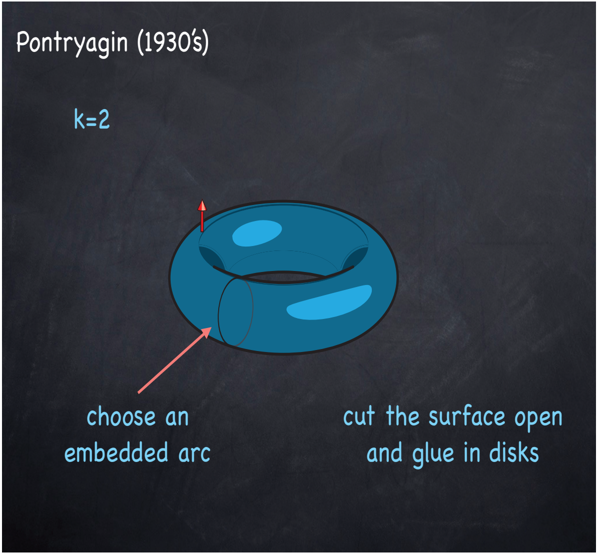

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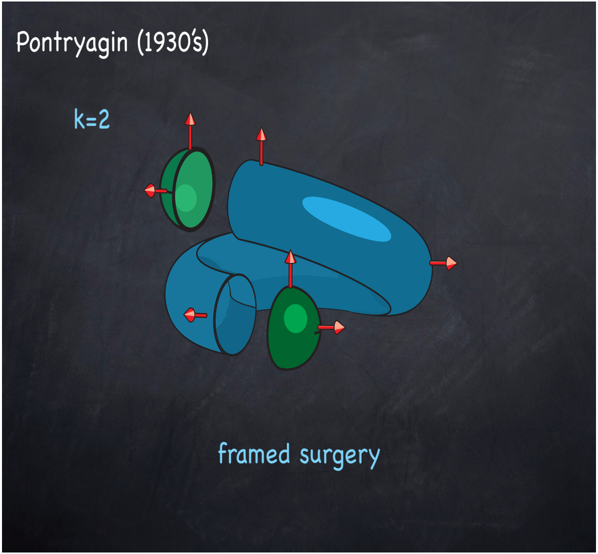

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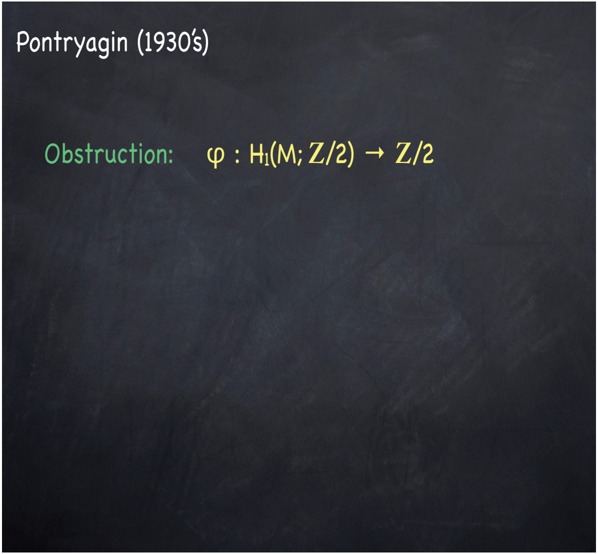

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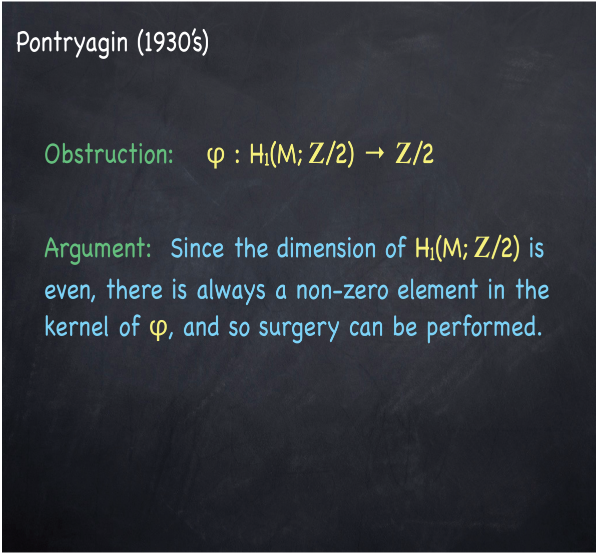

Pontryagin’s early work (continued)

Pontryagin (1930’s)

Obstruction: $\varphi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$

Argument: Since the dimension of $H_1(M; \mathbb{Z}/2)$ is even, there is always a non-zero element in the kernel of φ, and so surgery can be performed.

Conclusion: $\Omega_2 = \pi_{n+2}(S^n) = 0$.

Pontryagin's mistake for $k = 2$

The map $\varphi : H_1(M; \mathbf{Z}/2) \to \mathbf{Z}/2$ is not a homomorphism!

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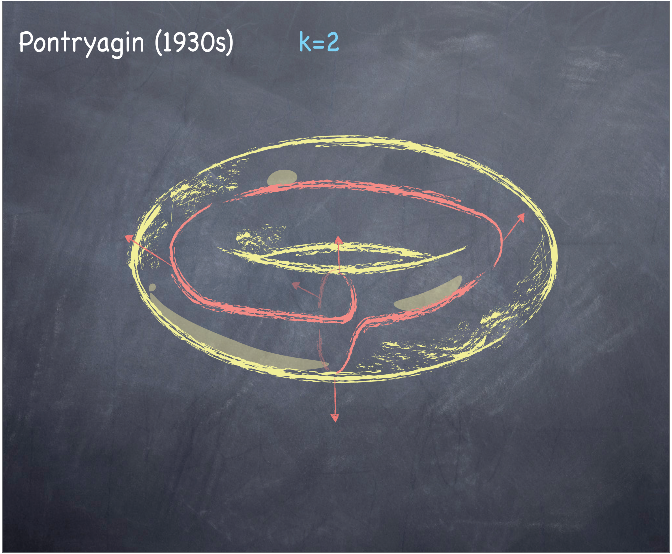
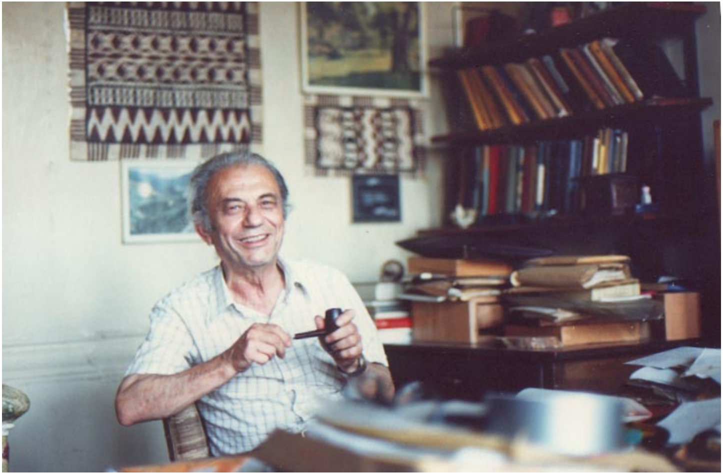
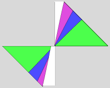


Tuesday, April 21, 2009

The Arf invariant of a quadratic form in characteristic 2



Cahit Arf 1910-1997



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Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$.

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In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix}.$$

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In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q.

Money talks: Arf's definition republished in 2009

Cahit Arf 1910-1997

The Kervaire invariant of a framed $(4m+2)$-manifold

Let M be a $2m$-connected smooth closed framed manifold of dimension $4m+2$.

Mike Hill
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Let M be a $2m$-connected smooth closed framed manifold of dimension $4m + 2$. Let $H = H_{2m+1}(M; \mathbf{Z})$, the homology group in the middle dimension.

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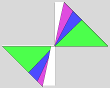

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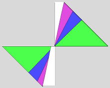

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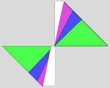

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For $m = 0$, Kervaire's q coincides with Pontryagin's φ.

Mike Hill
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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

What can we say about $\Phi(M)$?

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

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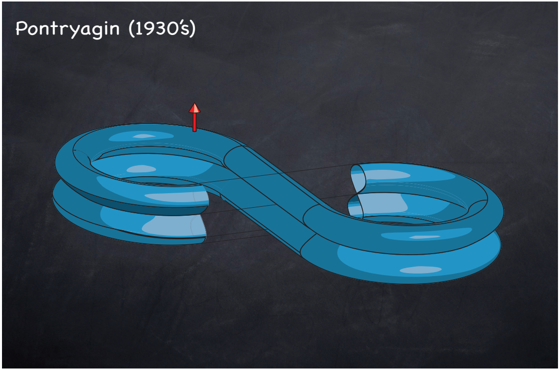

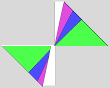

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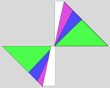

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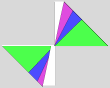

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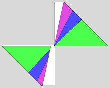

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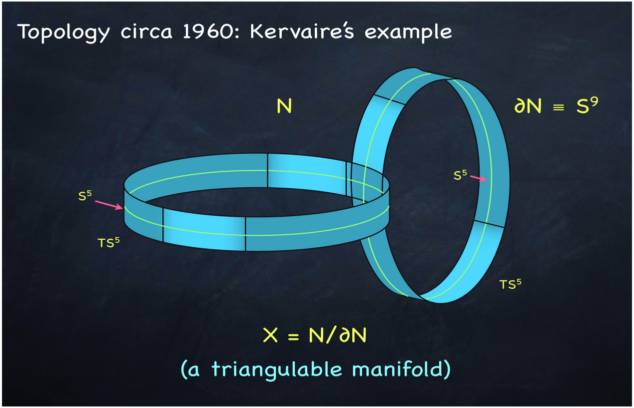

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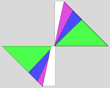

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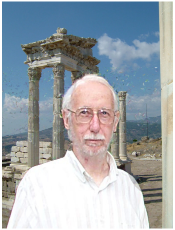
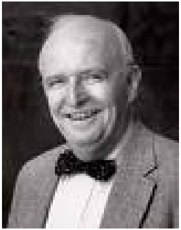
- 

  Ed Brown

  Frank Peterson
  1930-2000

  Brown-Peterson (1966) showed that it vanishes for all positive even m.

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

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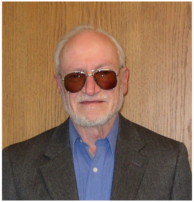
- 

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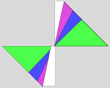

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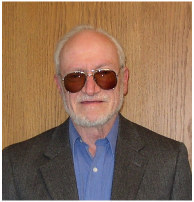
- 

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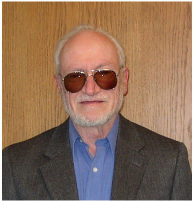
- 

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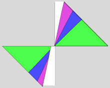

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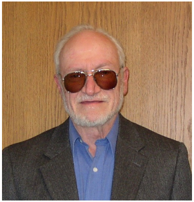
- 

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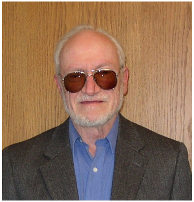
- 

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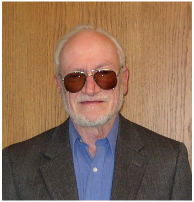
- 

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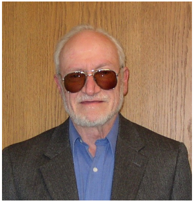
- 

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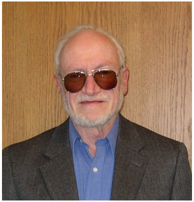
- 

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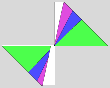
Questions raised by our theorem



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Adams spectral sequence formulation.

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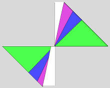

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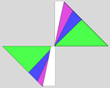

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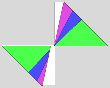
Mike Hill
Mike Hopkins
Doug Ravenel

Ingredients of the proof

Our proof has several ingredients.



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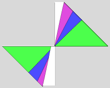

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Mike Hill
Mike Hopkins
Doug Ravenel

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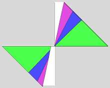

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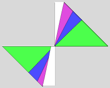

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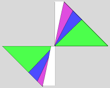

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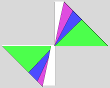
John Milnor


Sergei Novikov


Dan Quillen



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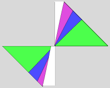

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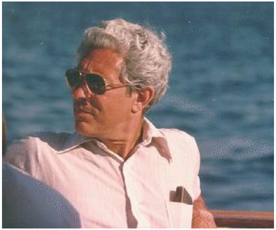
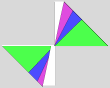


Peter May

John Greenlees

Gaunce Lewis
1949-2006



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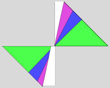

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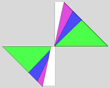
The spectrum Ω (continued)

Here again are the properties of Ω

Mike Hill
Mike Hopkins
Doug Ravenel



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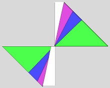

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If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

How we construct Ω

Mike Hill
Mike Hopkins
Doug Ravenel

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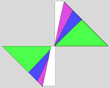
Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.



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How we construct Ω (continued)

Some people who have studied MU as a C_2-spectrum:

Mike Hill
Mike Hopkins
Doug Ravenel

How we construct Ω (continued)

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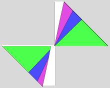
Peter Landweber



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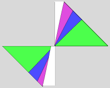
Shoro Araki
1930–2005



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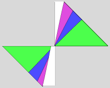

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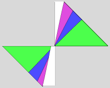

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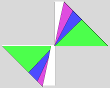

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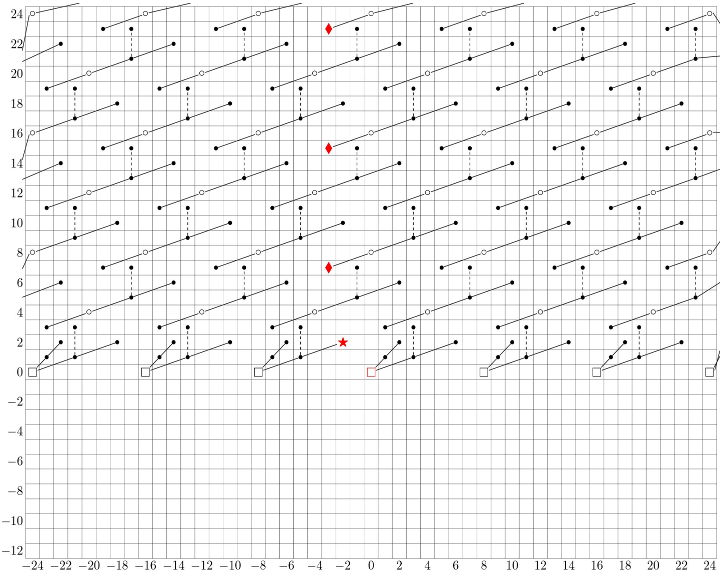
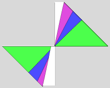
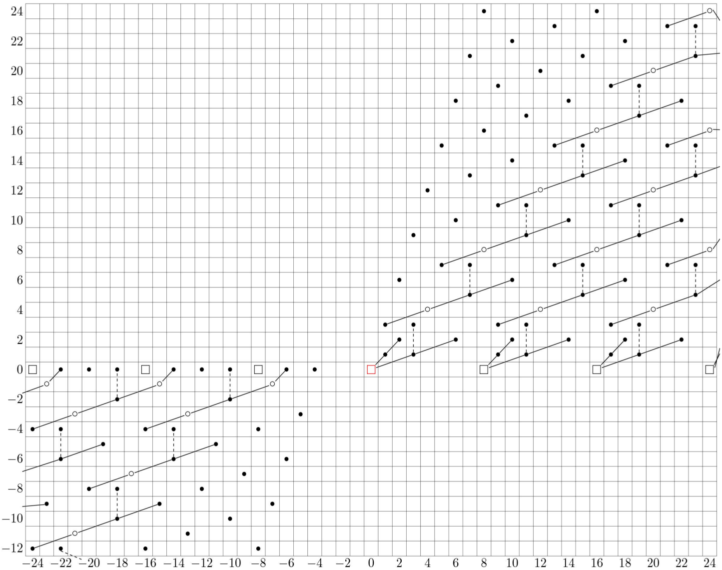
This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.

A homotopy fixed point spectral sequence





The corresponding slice spectral sequence



Mike Hill
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