The 3-primary Arf-Kervaire invariant problem

Mike Hill University of Virginia
Mike Hopkins Harvard University
Doug Ravenel University of Rochester

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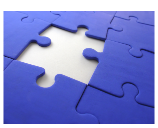
1.1

1 The main point of this talk

The main point of this talk

The 3-primary Arf-Kervaire invariant problem is still open.

We have a program for solving it similar to what we did for $p = 2$.



We are missing a crucial ingredient.

Maybe you can find it!

1.2

2 Introduction

2.1 Defining the problem

Defining the problem

The Arf-Kervaire invariant problem for a prime p is to determine the fate of the elements

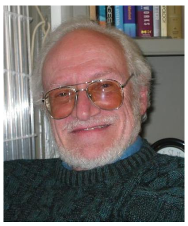
$$\theta_j = \left\{ \begin{array}{ll} h_j^2 & \text{for } p = 2 \\ b_{j-1} & \text{for } p > 2 \end{array} \right\} \in \mathrm{Ext}_A^{2,2p^j(p-1)}(\mathbf{Z}/p, \mathbf{Z}/p) \qquad (1)$$

where A denotes the mod p Steenrod algebra. This Ext group is the E_2-term for the classical Adams spectral sequence converging to the p-component of the stable homotopy groups of spheres.

Frank Adams
1930–1989

In these bidegrees the groups are known to be isomorphic to $\mathbf{Z}/p$ in each case, generated by these elements. 1.3

Introduction (continued)



Bill Browder

Browder's Theorem of 1969 states that for $p = 2$, h_j^2 is a permanent cycle in the Adams spectral sequence if and only if there is a framed manifold with nontrivial Kervaire invariant manifold in dimension $2^{j+1}-2$. Such manifolds are known to exist for $1 \le j \le 5$.

We recently showed that for $p = 2$, θ_j does not exist for $j \ge 7$. The case $j = 6$ remains open. 1.4

Introduction (continued)

Again, we want to know the fate of the elements

$$\theta_j = \left\{ \begin{array}{ll} h_j^2 & \text{for } p = 2 \\ b_{j-1} & \text{for } p > 2 \end{array} \right\} \in \mathrm{Ext}_A^{2,2p^j(p-1)}$$

The corresponding Adams-Novikov group, $\mathrm{Ext}_{BP_*(BP)}^{2,2p^j(p-1)}$, is more complicated. It is an elementary abelian p-group of rank roughly $j/2$. The Thom reduction map

$$\mathrm{Ext}_{BP_*(BP)}^{2,2p^j(p-1)} \xrightarrow{\Phi} \mathrm{Ext}_A^{2,2p^j(p-1)}$$

is onto in all but one case, with

$$\widehat{\theta}_j = \beta_{p^{j-1}/p^{j-1}} \mapsto \left\{ \begin{array}{ll} 0 & \text{for } j = 1 \text{ and } p = 2 \\ \theta_j & \text{otherwise} \end{array} \right.$$

A reformulation of the problem is the following:

Is any element of $\mathrm{Ext}_{BP_*(BP)}^{2,2p^j(p-1)}$ mapping to θ_j a permanent cycle? 1.5

Introduction (continued)

There is no known intrepretation of the problem at odd primes in terms of manifolds. In the late 70s the third author showed for that for $p \ge 5$, the element θ_j for $j > 1$ is not a permanent cycle, while θ_1 is a permanent cycle representing

$$\widehat{\theta}_1 = \beta_1 \in \pi_{2p^2-2p-2}S^0.$$

Modulo some indeterminacy, there are differentials

$$d_{2p-1}(\widehat{\theta}_j) = h_0 \widehat{\theta}_{j-1}^p \qquad (2)$$

where $h_0 \in \mathrm{Ext}_A^{1,2p-1}$ represents $\alpha_1 \in \pi_{2p-3}S^0$. 1.6

2.2 The role of the Morava stabilizer group

The role of the Morava stabilizer group

In order to describe the difficulties at $p = 3$, we need to recall the methods of of [HHR] for $p = 2$ and myself for $p \geq 5$. The starting point for $p \geq 5$ is the following result of Toda:

Hirosi Toda in 2009

In the Adams-Novikov spectral sequence for an odd prime p there is a nontrivial differential

$$d_{2p-1}(\widehat{\theta}_2) = \alpha_1 \widehat{\theta}_1^p. \qquad (3)$$

We also show that there are relations

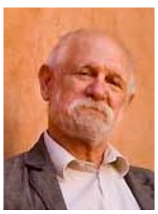
$$\widehat{\theta}_j \widehat{\theta}_2^{p^{j-1}} = \widehat{\theta}_{j+1} \widehat{\theta}_1^{p^{j-1}}. \qquad (4)$$

Using (3-4) one can deduce that

$$d_{2p-1}(\widehat{\theta}_j) = \alpha_1 \widehat{\theta}_{j-1}^p \qquad \text{for all } j \geq 2.$$

The hard part is to use chromatic methods to show that these targets are all nontrivial. 1.7

The role of the Morava stabilizer group (continued)



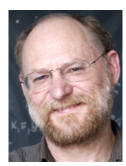
Jack Morava

We now know (but only suspected in the late 70s) that the extended Morava stabilizer group $\mathbf{G}_n$ acts on the Morava spectrum E_n in such a way that the homotopy fixed point set $E_n^{h\mathbf{G}_n}$ is $L_{K(n)}S^0$, the Bousfield localization of the sphere spectrum with respect to the nth Morava K-theory.

This is a corollary of the Hopkins-Miller theorem. For any closed subgroup $H \subset \mathbf{G}_n$ there is a homotopy fixed point spectral sequence

$$H^*(H; \pi_* E_n) \implies \pi_* E_n^{hH}$$

Mike Hopkins



Haynes Miller

which coincides with the Adams-Novikov spectral sequence for E_n^{hH}. One has the expected restriction maps for subgroups. 1.8

The role of the Morava stabilizer group (continued)

The group $\mathbf{G}_n$ is known to have a subgroup of order p (unique up to conjugacy) when $p-1$ divides n. This leads to a composite homomorphism, the detection map

$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_p; \pi_* E_{p-1}) \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}])$$

where the second homomorphism is reduction modulo the maximal ideal in $\pi_* E_{p-1}$ and $|u| = 2$. The action of C_p here is trivial, so the target is a bigraded form of the usual mod p cohomology of C_p. For p odd this cohomology is

$$E(\alpha) \otimes P(\beta) \otimes \mathbf{F}_{p^{p-1}}[u, u^{-1}]$$

where $\alpha \in H^1$ and $\beta \in H^2$ each have topological degree 0. 1.9

The role of the Morava stabilizer group (continued)

Again we have the detection map

$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_p;\pi_* E_{p-1}) \longrightarrow H^*(C_p;\mathbf{F}_{p^{p-1}}[u,u^{-1}]) \tag{5}$$

We showed that under this map we have

$$\begin{array}{ccc} \alpha_1 & \mapsto & u^{p-1}\alpha \\ \widehat{\theta}_j \beta_{p^{j-1}/p^{j-1}} & \mapsto & u^{p^j(p-1)}\beta \end{array} \tag{6}$$

up to unit scalar. Hence all monomials in the $\widehat{\theta}_j$ and their products with α_1 have nontrivial images. This implies that the differentials

$$d_{2p-1}(\widehat{\theta}_j) = \alpha_1 \widehat{\theta}_{j-1}^p$$

are nontrivial as desired. 1.10

The role of the Morava stabilizer group (continued)

To summarize:

- The existence of an element of order p in S_{p-1} leads to the detection map of (5),

$$\begin{array}{ccc} \mathrm{Ext}_{BP_*(BP)} & \longrightarrow & H^*(C_p;\mathbf{F}_{p^{p-1}}[u,u^{-1}]) \\ & & \| \\ & & E(\alpha)\otimes P(\beta)\otimes \mathbf{F}_{p^{p-1}}[u,u^{-1}] \\ \alpha_1 & \longmapsto & u^{p-1}\alpha \\ \widehat{\theta}_j & \longmapsto & u^{p^j(p-1)}\beta \end{array}$$

- The multiplicative relations among the $\widehat{\theta}_j$ and the Toda differential on $\widehat{\theta}_2$ lead to differentials on all higher $\widehat{\theta}_j$. They are nontrivial by the detection data above.

1.11

3 Difficulties at $p = 3$

Difficulties at $p = 3$

Why does this approach fail for $p < 5$?

- For $p = 2$, the target of the Toda "differential",

$$d_3(\widehat{\theta}_2) = \alpha_1 \widehat{\theta}_1^2 = 0,$$

so this method does not show that any $\widehat{\theta}_j$ fails to be a permanent cycle.

- The group $\mathrm{Ext}_{BP_*(BP)}^{2,2p^j(p-1)}$ is known to have $[(j-1)/2]$ other generators besides $\widehat{\theta}_j$. For $p = 3$ these other generators, such as β_7 in the bidegree of $\widehat{\theta}_3$, can have nontrivial images under the detection map. This has to do with the fact that they are v_2-periodic and hence v_{p-1}-periodic. It turns out that $\widehat{\theta}_3 \pm \beta_7$ and hence θ_3 are permanent cycles even though θ_2 is not. The argument above establishes the nonexistence of $\widehat{\theta}_j$ for $j > 1$, but not that of θ_j.

1.12

Difficulties at $p = 3$ (continued)

In order to describe the way out of these difficulties we need to say more about finite subgroups of $\mathbf{G}_n$. It is by definition an extension of the Morava stabilizer group $\mathbf{S}_n$ by $\mathrm{Gal}(\mathbf{F}_{p^n} : \mathbf{F}_p)$. The Galois group (which is cyclic of order n) is there for technical reasons but plays no role on our calculations. $\mathbf{S}_n$ is the group of units in the maximal order of a certain division algebra over the p-adic numbers $\mathbf{Q}_p$. Its finite subgroups have been classified by Hewett.

$\mathbf{S}_n$ has an element of order p iff $p-1$ divides n, a condition that is trivial when $p = 2$. More generally $\mathbf{S}_n$ has an element of order p^{k+1} iff $p^k(p-1)$ divides n. For such n we could replace the detection map (5) by

$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?),$$

for some coefficient ring in the target. The naive choice of $\mathbf{F}_{p^n}[u, u^{-1}]$ for this ring turns out not to detect $\widehat{\theta}_j$ for $n > p-1$. 1.13

Difficulties at $p = 3$ (continued)

Again, for n divisible by $p^k(p-1)$ we have a detection map

$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_{p^{k+1}}; \pi_* E_n) \longrightarrow H^*(C_{p^{k+1}}; ?).$$

Experience has shown two things:

(i) In order to flush out the spurious elements (which are v_2-periodic) having the same bidegree as $\widehat{\theta}_j$, we need to have $n > 2$.

(ii) In order to detect the $\widehat{\theta}_j$ itself, we need to have n be equal to $p^k(p-1)$ for some $k \geq 0$, not just be divisible by it. Then $\widehat{\theta}_j$ will map to an element of order p in a cohomology group isomorphic to $\mathbf{Z}/p^{k+1}$. We cannot detect higher products of these elements for $k > 0$.

For $p = 2$ these considerations suggest using the group C_8 and $n = 4$, which is the approach used in [HHR].

For $p = 3$ we need to use the group C_9 with $n = 6$. 1.14

Difficulties at $p = 3$ (continued)

For the prime 2, our strategy in [HHR] was to construct a ring spectrum Ω with a unit map $S^0 \to \Omega$ satisfying three properties:

(i) Detection Theorem. If θ_j exists, its image in $\pi_*\Omega$ is nontrivial.

(ii) Periodicity Theorem. $\pi_k\Omega$ depends only on the congruence class of k modulo 256.

(iii) Gap Theorem. $\pi_{-2}\Omega = 0$.

The nonexistence of θ_j for $j \geq 7$ follows from the fact that its dimension is congruent to -2 modulo 256.

Ever since the discovery of the Hopkins-Miller theorem, it has been possible to prove that $E_4^{hC_8}$ satisfies the first two of these properties without the use of equivariant stable homotopy theory. 1.15

Difficulties at $p = 3$ (continued)

For $p = 3$, the same goes for $E_6^{hC_9}$ with the periodicity dimension being 972 (2 more than the dimension of θ_5) instead of 256. If all goes well, we would get a theorem saying θ_j does not exist for $j \geq 5$, leaving the status of θ_4 (in the 322-stem) open. We already know that θ_1 (in the 10-stem) and θ_3 (in the 106-stem) exist while θ_2 (in the 34-stem) does not.

For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2. In this case the spectrum also detects the product of α_1 with any monomial in the θ_js. As explained above, this enables us to use Toda's differential to show that none of the θ_j for $j > 1$ exists.

We cannot use Toda's differential for $p < 5$ because

(a) for $p = 2$ its target is trivial, and
(b) since we cannot detect products of the θ_js, we cannot make an inductive argument.

1.16

Difficulties at $p = 3$ (continued)

The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult. We do not know how to do it directly for the case of interest, the Morava E_6 at $p = 3$ as a C_9-spectrum, or for E_4 at $p = 2$ for the group C_8.

We do know how to do it for $MU_{\mathbf{R}}$, which is MU as a C_2-spectrum via complex conjugation, and for $N_2^{2^{n+1}} MU_{\mathbf{R}}$, which is underlain by $MU^{(2^n)}$ with a $C_{2^{n+1}}$-action. A crucial step here is the Reduction Theorem, which says roughly that if we kill all of the underlying homotopy groups in positive dimensions in a certain equivariant way, we get the equivariant Eilenberg-Mac Lane spectrum $H\underline{\mathbf{Z}}$. 1.17

Difficulties at $p = 3$ (continued)

In order to do a similar thing at an $p = 3$ we need an analog MU_{Ξ} of the C_2-spectrum $MU_{\mathbf{R}}$. It should be a C_3-spectrum underlain by $MU^{(2)}$ with two properties:

(i) It has a tractable slice filtration with a certain description.
(ii) Its geometric fixed point spectrum $MU_{\Xi}^{gC_3}$ is a wedge of suspensions of $H/3$, the mod 3 Eilenberg-Mac Lane spectrum. For $p = 2$ we have $MU_{\mathbf{R}}^{gC_2} = MO$, the unoriented cobordism spectrum, which fits this description. This identification is a pivotal step in determining differentials in the slice spectral sequence needed to prove the Periodicity Theorem.

We do not know how to construct this spectrum!

It is our missing piece. Maybe you can find it. 1.18

4 What might happen

4.1 Entering Fantasyland

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Entering Fantasyland

Let's suppose the hypothetical MU_{Ξ} exists as described above. For convenience we will work with its BP analog, BP_{Ξ}.

A useful technical notion. Let E be a connective equivariant spectrum with $\pi_*^u E$ (its underlying homotopy groups) free abelian. A refinement of this group is an equivariant map $W \to E$ where W is underlain by a wedge of spheres mapping to the generators of $\pi_*^u E$. The reduction theorem for E is the statement that the map

$$E \underset{W}{\wedge} S^0 \to H\underline{\mathbf{Z}}$$

is an equivariant equivalence. 1.19

Entering Fantasyland (continued)

If all goes according to plan, $\pi_*^u BP_\Xi$ is refined by a map from

$$W = \bigwedge_{n\geq 1} W_n$$

with

$$W_n = S^0\left[\overline{S}^{2\cdot 3^{n-1}\rho-1}\right]$$

where

- ρ denotes the regular representation of C_3 and
- $\overline{S}^V$ denotes the codimension one skeleton of S^V. 1.20

Entering Fantasyland (continued)

For $n=1$ we have
$$W_1 = S^0\left[\overline{S}^{2\rho-1}\right].$$

Here $\overline{S}^{2\rho-1}$ is underlain by $S^4\vee S^4$, and W_1 is underlain by a wedge of spheres with $k+1$ summands in dimension $4k$ for each $k\geq 0$. There is a C_3-action on the space $X=S^5\times S^5$ such that $W_1=\Sigma^\infty\Omega X$.

Equivariantly we have

$$W_1 = S^0\left[S^{4\rho}\right]\wedge\left(S^0\vee\overline{S}^{2\cdot\rho-1}\vee C_{3+}\wedge\left(\bigvee_{i\geq 2}S^{4i}\right)\right).$$

Free summands here contribute torsion free summands to $\pi_* E^{C_3}$, so they are irrelevant to the Kervaire invariant problem. 1.21

Entering Fantasyland (continued)

Hence will ignore the free summands in W_1 and replace it by

$$W_1' = S^0\left[S^{4\rho}\right]\wedge\left(S^0\vee\overline{S}^{2\rho-1}\right).$$

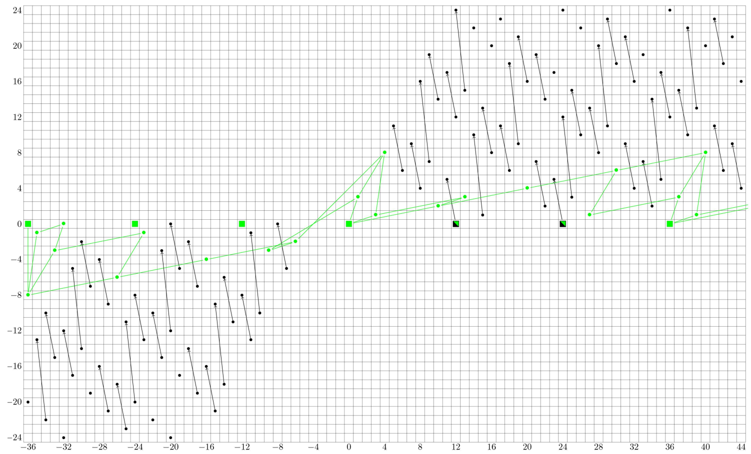
We have a map

$$W_1' = S^0\left[S^{4\rho}\right]\wedge\left(S^0\vee\overline{S}^{2\rho-1}\right)\longrightarrow BP_\Xi$$

Thus there is an element $Nv_1\in\pi_{4\rho}BP_\Xi$. We can invert it and throw away the higher generators. The resulting fixed point spectrum looks a lot like tmf, but with periodicity in dimension 36 instead of 72. 1.22

4.2 Two spectral sequences

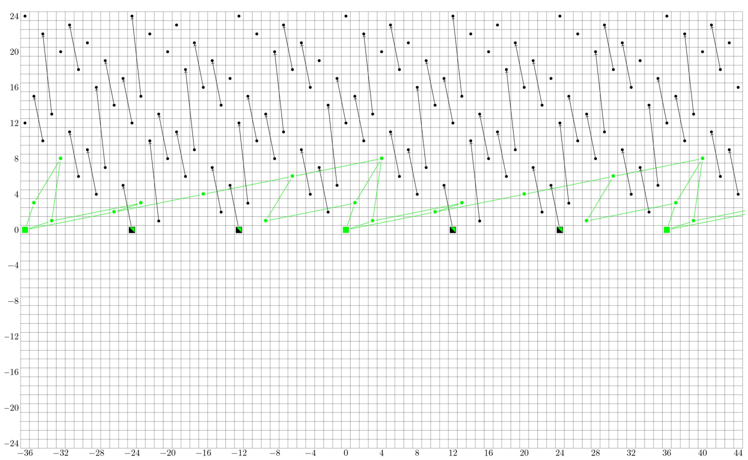
Two spectral sequences



Here is its slice spectral sequence. 1.23

Two spectral sequences (continued)

Here is its homotopy fixed point spectral sequence.



1.24

4.3 Norming up to C_9

Norming up to C_9

Now we need to norm up from C_3 to C_9. Recall that

$$W_1' = \left(S^0 \vee \overline{S}^{2\cdot\rho-1}\right) \wedge S^0\left[S^{4\rho}\right].$$

The norm functor commutes with smash products. For the first factor we have

$$\begin{aligned} N_3^9\left(S^0 \vee \overline{S}^{2\rho-1}\right) &= S^0 \vee \left(C_{9+} \underset{C_3}{\wedge} \left(\overline{S}^{2\rho-1} \vee S^{3\rho-1}\right)\right) \\ &\quad \vee \left(C_{9+} \wedge S^8\right) \vee \overline{S}^{\rho_9+2\lambda} \end{aligned}$$

where λ denotes the 2-dimensional representation of C_9 with a rotation of order 9.

For the second factor of W_1',

$$N_3^9 S^0\left[S^{4\rho}\right] = S^0\left[S^{4\rho_9}\right] \wedge \left(S^0 \vee \left(C_{9+} \underset{C_3}{\wedge} \bigvee_{i,j\geq 0} S^{4(i+j+1)\rho}\right)\right).$$

1.25

4.4 A possible Gap Theorem

A possible Gap Theorem

After inverting the right element in $\pi_{4\rho_9}$, we get a spectrum $\tilde{\Omega}$ whose fixed point set Ω is 972-periodic and detects the θ_j for $j \geq 5$. The key question here is

Do we get a Gap Theorem stating that $\pi_{-2}\Omega$ is torsion free?

To answer this we need to look at the equivariant homotopy groups of

$$X \wedge \Sigma^{4m\rho_9} H\underline{\mathbf{Z}} \quad \text{and} \quad X \wedge C_{9+} \underset{C_3}{\wedge} \Sigma^{4n\rho_3} H\underline{\mathbf{Z}}$$

for $m, n \in \mathbf{Z}$, where X is one of the following:

$$S^0, \quad C_{9+} \underset{C_3}{\wedge} \overline{S}^{2\rho-1}, \quad C_{9+} \underset{C_3}{\wedge} S^{3\rho-1} \quad \text{or} \quad \overline{S}^{\rho_9+2\lambda}.$$

1.26

A possible Gap Theorem (continued)

The following table indicates the dimensions in which

$$\underline{\pi}_i X \wedge \Sigma^{4m\rho_9} H\underline{\mathbf{Z}} \quad \text{and} \quad \underline{\pi}_i X \wedge C_{9+} \underset{C_3}{\wedge} \Sigma^{4n\rho_3} H\underline{\mathbf{Z}}$$

can be nontrivial for $m, n \geq 0$, with one caveat as indicated below.

X	$\underline{\pi}_i X \wedge \Sigma^{4m\rho_9} H\underline{\mathbf{Z}}$	$\underline{\pi}_i X \wedge C_{9+} \underset{C_3}{\wedge} \Sigma^{4n\rho_3} H\underline{\mathbf{Z}}$
S^0	$4m \leq i \leq 36m$	$4n \leq i \leq 12n$
$C_{9+} \underset{C_3}{\wedge} \overline{S}^{2\rho-1}$	$12m+1 \leq i \leq 36m+4$	$4n+1 \leq i \leq 12n+4$
$C_{9+} \underset{C_3}{\wedge} S^{3\rho-1}$	$12m+2 \leq i \leq 36m+8$	$4n+2 \leq i \leq 12n+8$
$\overline{S}^{\rho_9+2\lambda}$	$4m+1 \leq i \leq 36m+12$	$4n+3 \leq i \leq 12n+12$ for $n \geq -1$

1.27

A possible Gap Theorem (continued)

Here is a similar table for $m, n \leq -1$.

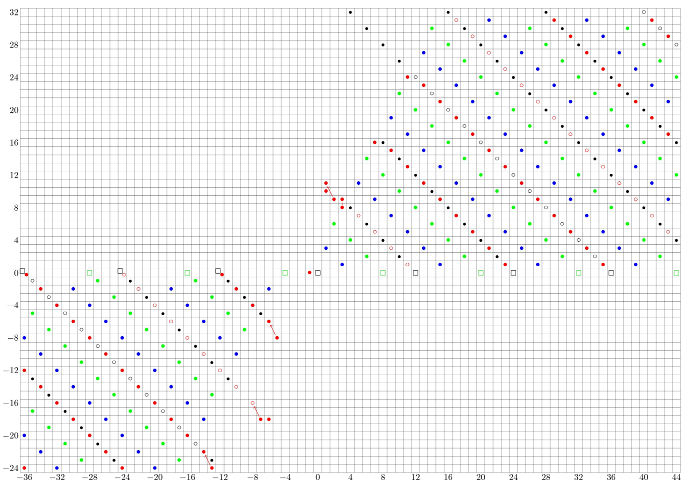
X	$\underline{\pi}_i X \wedge \Sigma^{4m\rho_9} H\underline{\mathbf{Z}}$	$\underline{\pi}_i X \wedge C_{9+} \underset{C_3}{\wedge} \Sigma^{4n\rho_3} H\underline{\mathbf{Z}}$
S^0	$36m \leq i \leq 4m-3$	$12n \leq i \leq 4n-3$
$C_{9+} \underset{C_3}{\wedge} \overline{S}^{2\rho-1}$	$36m+4 \leq i \leq 12m-2$	$12n+4 \leq i \leq 4n-2$
$C_{9+} \underset{C_3}{\wedge} S^{3\rho-1}$	$36m+8 \leq i \leq 12m-1$	$12n+8 \leq i \leq 4n-1$
$\overline{S}^{\rho_9+2\lambda}$	$36m+12 \leq i \leq 4m-2$	$12n+12 \leq i \leq 4n$ for $n \leq -2$

In each case the upper bound here is 3 less than the corresponding lower bound in the previous table. The calculation behind this is the same for $p = 3$ as it was for $p = 2$.

Since $m, n \leq -1$, our upper bound is always ≤ -4, so we have the desired Gap Theorem. 1.28

The C_9 slice spectral sequence

Here is a color coded illustration of these fixed point homotopy groups.



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