The 3-primary Arf-Kervaire invariant problem

Mike Hill   University of Virginia
Mike Hopkins   Harvard University
Doug Ravenel   University of Rochester

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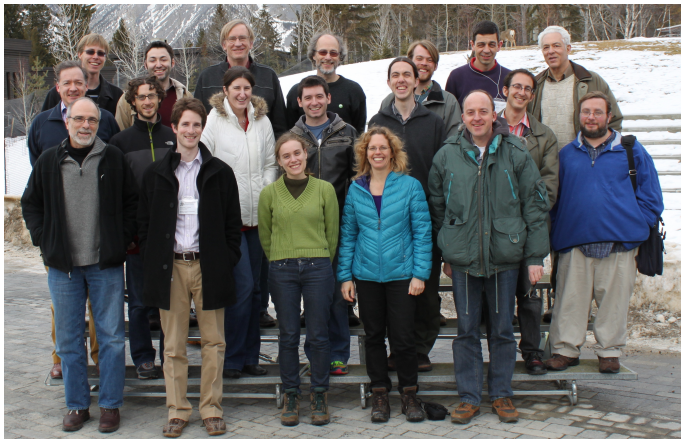
Banff Workshop on Algebraic K-Theory and
Equivariant Homotopy Theory
February 16, 2012

The main point of this talk

The 3-primary
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problem

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We are missing a crucial ingredient.

Maybe you can find it!

Defining the problem

The Arf-Kervaire invariant problem for a prime p is to determine the fate of the elements

$$\theta_j = \left\{ \begin{array}{ll} h_j^2 & \text{for } p = 2 \\ b_{j-1} & \text{for } p > 2 \end{array} \right\} \in \mathrm{Ext}_A^{2,2p^j(p-1)}(\mathbf{Z}/p, \mathbf{Z}/p) \quad (1)$$

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In these bidegrees the groups are known to be isomorphic to $\mathbf{Z}/p$ in each case, generated by these elements.

Frank Adams
1930–1989

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Introduction (continued)

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Entering Fantasyland

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A possible Gap Theorem

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Browder's Theorem of 1969 states that for $p = 2$, h_j^2 is a permanent cycle in the Adams spectral sequence if and only if there is a framed manifold with nontrivial Kervaire invariant manifold in dimension $2^{j+1} - 2$.

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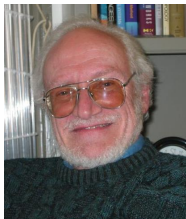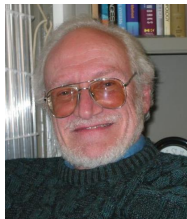
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We recently showed that for $p = 2$, θ_j does not exist for $j \geq 7$. The case $j = 6$ remains open.

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1.5

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A reformulation of the problem is the following:

Is any element of $\mathrm{Ext}_{BP_*(BP)}^{2, 2p^j(p-1)}$ mapping to θ_j a permanent cycle?

1.5

Introduction (continued)

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$$\widehat{\theta}_1 = \beta_1 \in \pi_{2p^2 - 2p - 2} S^0.$$

Introduction (continued)

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Arf-Kervaire invariant
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**Mike Hill
Mike Hopkins
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There is no known intrepretation of the problem at odd primes in terms of manifolds. In the late 70s the third author showed for that for $p \geq 5$, the element θ_j for $j > 1$ is not a permanent cycle, while θ_1 is a permanent cycle representing

$$\widehat{\theta}_1 = \beta_1 \in \pi_{2p^2-2p-2}S^0.$$

Modulo some indeterminacy, there are differentials

$$d_{2p-1}(\widehat{\theta}_j) = h_0\widehat{\theta}_{j-1}^p \tag{2}$$

where $h_0 \in \mathrm{Ext}_A^{1,2p-1}$ represents $\alpha_1 \in \pi_{2p-3}S^0$.

The role of the Morava stabilizer group

In order to describe the difficulties at $p = 3$, we need to recall the methods of of [HHR] for $p = 2$ and myself for $p \geq 5$.

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Hirosi Toda
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The hard part is to use chromatic methods to show that these targets are all nontrivial.

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The role of the Morava stabilizer group (continued)

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The role of the Morava stabilizer group (continued)

Jack Morava

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The role of the Morava stabilizer group (continued)

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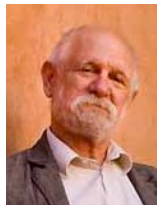Difficulties at $p = 3$

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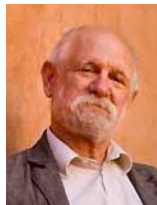Introduction

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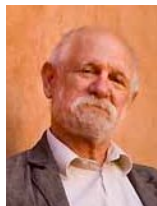
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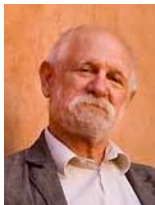Introduction
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The group $\mathbf{G}_n$ is known to have a subgroup of order p (unique up to conjugacy) when $p - 1$ divides n.

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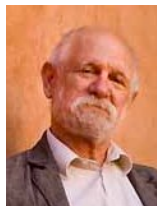
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$$\text{Ext}_{BP_*(BP)} \longrightarrow H^*(C_p; \pi_* E_{p-1}) \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}])$$

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$$E(\alpha) \otimes P(\beta) \otimes \mathbf{F}_{p^{p-1}}[u, u^{-1}]$$

where $\alpha \in H^1$ and $\beta \in H^2$ each have topological degree 0.

The role of the Morava stabilizer group (continued)

Again we have the detection map

$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_p; \pi_* E_{p-1}) \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}]) \quad (5)$$

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We showed that under this map we have

$$\begin{array}{rcl}
\alpha_1 & \mapsto & u^{p-1}\alpha \\
\widehat{\theta}_j \beta_{p^{j-1}/p^{j-1}} & \mapsto & u^{p^j(p-1)}\beta
\end{array} \quad (6)$$

up to unit scalar.

The role of the Morava stabilizer group (continued)

Again we have the detection map

$$\operatorname{Ext}_{BP_*(BP)} \longrightarrow H^*(C_p; \pi_* E_{p-1}) \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}]) \quad (5)$$

We showed that under this map we have

$$\begin{aligned}
\alpha_1 &\mapsto u^{p-1}\alpha \\
\widehat{\theta}_j \beta_{p^{j-1}/p^{j-1}} &\mapsto u^{p^j(p-1)}\beta
\end{aligned} \quad (6)$$

up to unit scalar. Hence all monomials in the $\widehat{\theta}_j$ and their products with α_1 have nontrivial images.

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up to unit scalar. Hence all monomials in the $\widehat{\theta}_j$ and their products with α_1 have nontrivial images. This implies that the differentials

$$d_{2p-1}(\widehat{\theta}_j) = \alpha_1 \widehat{\theta}_{j-1}^p$$

are nontrivial as desired.

The role of the Morava stabilizer group (continued)

To summarize:

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The role of the Morava stabilizer group (continued)

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To summarize:

- The existence of an element of order p in S_{p-1} leads to the detection map of (5),

$$\mathrm{Ext}_{BP_*(BP)} \longrightarrow H^*(C_p; \mathbf{F}_{p^{p-1}}[u, u^{-1}])$$
$$\shortparallel$$
$$E(\alpha) \otimes P(\beta) \otimes \mathbf{F}_{p^{p-1}}[u, u^{-1}]$$

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Difficulties at $p = 3$

Why does this approach fail for $p < 5$?

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Difficulties at $p = 3$ (continued)

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for some coefficient ring in the target. The naive choice of $\mathbf{F}_{p^n}[u, u^{-1}]$ for this ring turns out not to detect $\widehat{\theta}_j$ for $n > p - 1$.

Difficulties at $p = 3$ (continued)

Again, for n divisible by $p^k(p-1)$ we have a detection map

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For $p = 2$ these considerations suggest using the group C_8 and $n = 4$, which is the approach used in [HHR].

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For $p = 3$ we need to use the group C_9 with $n = 6$.

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For the prime 2, our strategy in [HHR] was to construct a ring spectrum Ω with a unit map $S^0 \to \Omega$ satisfying three properties:

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The nonexistence of θ_j for $j \geq 7$ follows from the fact that its dimension is congruent to -2 modulo 256.

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Ever since the discovery of the Hopkins-Miller theorem, it has been possible to prove that $E_4^{hC_8}$ satisfies the first two of these properties without the use of equivariant stable homotopy theory.

Difficulties at $p = 3$ (continued)

For $p = 3$, the same goes for $E_6^{hC_9}$ with the periodicity dimension being 972 (2 more than the dimension of θ_5) instead of 256.

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Difficulties at $p = 3$ (continued)

For $p = 3$, the same goes for $E_6^{hC_9}$ with the periodicity dimension being 972 (2 more than the dimension of θ_5) instead of 256. If all goes well, we would get a theorem saying θ_j does not exist for $j \geq 5$, leaving the status of θ_4 (in the 322-stem) open. We already know that θ_1 (in the 10-stem) and θ_3 (in the 106-stem) exist while θ_2 (in the 34-stem) does not.

For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p - 1)$, which is 2 more than the dimension of θ_2.

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For $p \geq 5$, the same holds for $E_{p-1}^{hC_p}$ with periodicity $2p^2(p-1)$, which is 2 more than the dimension of θ_2. In this case the spectrum also detects the product of α_1 with any monomial in the θ_js.

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Difficulties at $p = 3$ (continued)

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We cannot use Toda's differential for $p < 5$ because

(a) for $p = 2$ its target is trivial, and

(b) since we cannot detect products of the θ_js, we cannot make an inductive argument.

Difficulties at $p = 3$ (continued)

The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration.

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Difficulties at $p = 3$ (continued)

The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration.

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Difficulties at $p = 3$ (continued)

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Difficulties at $p = 3$ (continued)

The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult. We do not know how to do it directly for the case of interest, the Morava E_6 at $p = 3$ as a C_9-spectrum, or for E_4 at $p = 2$ for the group C_8.

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Difficulties at $p = 3$ **(continued)**

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We do know how to do it for $MU_{\mathbf{R}}$, which is MU as a C_2-spectrum via complex conjugation,

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We do know how to do it for $MU_{\mathbf{R}}$, which is MU as a C_2-spectrum via complex conjugation, and for $N_2^{2^{n+1}} MU_{\mathbf{R}}$, which is underlain by $MU^{(2^n)}$ with a $C_{2^{n+1}}$-action.

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Difficulties at $p = 3$ **(continued)**

The proof of the Gap Theorem requires the use of equivariant stable homotopy theory and the slice filtration. The slice filtration is an equivariant analogue of the classical Postnikov filtration. Analyzing it for a general equivariant spectrum is difficult. We do not know how to do it directly for the case of interest, the Morava E_6 at $p = 3$ as a C_9-spectrum, or for E_4 at $p = 2$ for the group C_8.

We do know how to do it for $MU_{\mathbf{R}}$, which is MU as a C_2-spectrum via complex conjugation, and for $N_2^{2^{n+1}} MU_{\mathbf{R}}$, which is underlain by $MU^{(2^n)}$ with a $C_{2^{n+1}}$-action. A crucial step here is the Reduction Theorem, which says roughly that if we kill all of the underlying homotopy groups in positive dimensions in a certain equivariant way, we get the equivariant Eilenberg-Mac Lane spectrum $H\underline{\mathbf{Z}}$.

Difficulties at $p = 3$ (continued)

In order to do a similar thing at an $p = 3$ we need an analog MU_Ξ of the C_2-spectrum $MU_\mathbf{R}$.

Difficulties at $p = 3$ (continued)

In order to do a similar thing at an $p = 3$ we need an analog $MU_{\underline{\equiv}}$ of the C_2-spectrum $MU_{\mathbf{R}}$. It should be a C_3-spectrum underlain by $MU^{(2)}$ with two properties:

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(ii) Its geometric fixed point spectrum $MU_{\Xi}^{gC_3}$ is a wedge of suspensions of $H/3$, the mod 3 Eilenberg-Mac Lane spectrum.

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  unoriented cobordism spectrum, which fits this description.

Difficulties at $p = 3$ (continued)

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Difficulties at $p = 3$ (continued)

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We do not know how to construct this spectrum!

???

Difficulties at $p = 3$ (continued)

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We do not know how to construct this spectrum!

It is our missing piece.

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We do not know how to construct this spectrum!

It is our missing piece. Maybe you can find it.

???

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Let's suppose the hypothetical $MU_{\underline{\underline{\cdot}}}$ exists as described above.

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Let's suppose the hypothetical $MU_{\underline{\underline{\ }}}$ exists as described above. For convenience we will work with its BP analog, $BP_{\underline{\underline{\ }}}$.

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Let's suppose the hypothetical $MU_{\underline{\underline{\hphantom{=}}}}$ exists as described above. For convenience we will work with its BP analog, $BP_{\underline{\underline{\hphantom{=}}}}$.

A useful technical notion.

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Let's suppose the hypothetical $MU_{\underline{\underline{\;}}}$ exists as described above. For convenience we will work with its BP analog, $BP_{\underline{\underline{\;}}}$.

A useful technical notion. Let E be a connective equivariant spectrum with $\pi^u_* E$ (its underlying homotopy groups) free abelian.

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Let's suppose the hypothetical $MU_{\underline{\underline{\Xi}}}$ exists as described above. For convenience we will work with its BP analog, $BP_{\underline{\underline{\Xi}}}$.

A useful technical notion. Let E be a connective equivariant spectrum with $\pi_*^u E$ (its underlying homotopy groups) free abelian. A refinement of this group is an equivariant map $W \to E$ where W is underlain by a wedge of spheres mapping to the generators of $\pi_*^u E$.

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Let's suppose the hypothetical $MU_{\underline{\underline{=}}}$ exists as described above. For convenience we will work with its BP analog, $BP_{\underline{\underline{=}}}$.

A useful technical notion. Let E be a connective equivariant spectrum with $\pi_*^u E$ (its underlying homotopy groups) free abelian. A refinement of this group is an equivariant map $W \to E$ where W is underlain by a wedge of spheres mapping to the generators of $\pi_*^u E$. The reduction theorem for E is the statement that the map

$$E \wedge_W S^0 \to H\underline{\mathbf{Z}}$$

is an equivariant equivalence.

Entering Fantasyland (continued)

If all goes according to plan, $\pi_*^u BP_\Xi$ is refined by a map from

$$W = \bigwedge_{n \geq 1} W_n$$

Entering Fantasyland (continued)

If all goes according to plan, $\pi_*^u BP_{\Xi}$ is refined by a map from

$$W = \bigwedge_{n \geq 1} W_n$$

with

$$W_n = S^0 \left[\overline{S}^{2 \cdot 3^{n-1} \rho - 1} \right]$$

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Entering Fantasyland (continued)

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where

- ρ denotes the regular representation of C_3 and
- $\overline{S}^V$ denotes the codimension one skeleton of S^V.

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For $n = 1$ we have

$$W_1 = S^0 \left[\overline{S}^{2\rho - 1} \right].$$

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Here $\overline{S}^{2\rho - 1}$ is underlain by $S^4 \vee S^4$, and W_1 is underlain by a wedge of spheres with $k + 1$ summands in dimension $4k$ for each $k \geq 0$.

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Equivariantly we have

$$W_1 = S^0 \left[S^{4\rho} \right] \wedge \left(S^0 \vee \overline{S}^{2 \cdot \rho - 1} \vee C_{3+} \wedge \left(\bigvee_{i \geq 2} S^{4i} \right) \right).$$

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Free summands here contribute torsion free summands to $\pi_* E^{C_3}$,

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Free summands here contribute torsion free summands to $\pi_* E^{C_3}$, so they are irrelevant to the Kervaire invariant problem.

Entering Fantasyland (continued)

Hence will ignore the free summands in W_1 and replace it by

$$W_1' = S^0 \left[S^{4\rho} \right] \wedge \left(S^0 \vee \overline{S}^{2\rho - 1} \right).$$

Entering Fantasyland (continued)

Hence will ignore the free summands in W_1 and replace it by

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We have a map

$$W_1' = S^0 \left[S^{4\rho} \right] \wedge \left(S^0 \vee \overline{S}^{2\rho-1} \right) \longrightarrow BP_{\Xi}$$

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Thus there is an element $Nv_1 \in \pi_{4\rho}BP_{\Xi}$.

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Thus there is an element $Nv_1 \in \pi_{4\rho} BP_{\Xi}$. We can invert it and throw away the higher generators.

Entering Fantasyland (continued)

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We have a map

$$W_1' = S^0 \left[S^{4\rho} \right] \wedge \left(S^0 \vee \overline{S}^{2\rho-1} \right) \longrightarrow BP_{\underline{\Xi}}$$

Thus there is an element $Nv_1 \in \pi_{4\rho} BP_{\underline{\Xi}}$. We can invert it and throw away the higher generators. The resulting fixed point spectrum looks a lot like *tmf*, but with periodicity in dimension 36 instead of 72.

Two spectral sequences

Here is its slice spectral sequence.
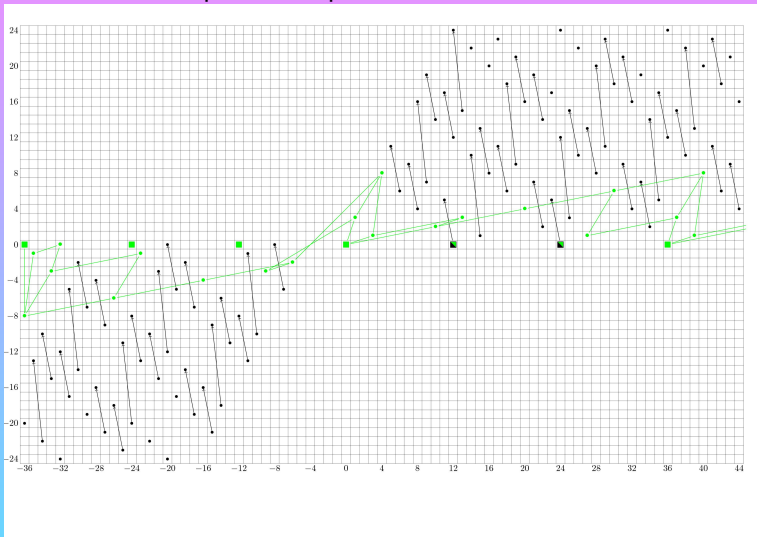
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problem

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Mike Hopkins
Doug Ravenel

Two spectral sequences (continued)

Here is its homotopy fixed point spectral sequence.

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Norming up to C_9

Now we need to norm up from C_3 to C_9.

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The norm functor commutes with smash products.

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The norm functor commutes with smash products. For the first factor we have

$$N_3^9 \left(S^0 \vee \overline{S}^{2\rho - 1} \right) = S^0 \vee \left(C_{9+} \underset{C_3}{\wedge} \left(\overline{S}^{2\rho - 1} \vee S^{3\rho - 1} \right) \right)$$
$$\vee \left(C_{9+} \wedge S^8 \right) \vee \overline{S}^{\rho_9 + 2\lambda}$$

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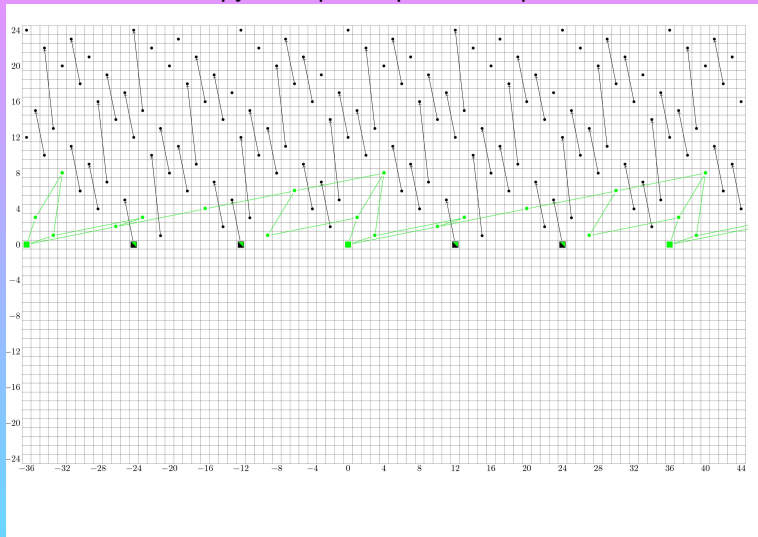The role of the Morava
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Now we need to norm up from C_3 to C_9. Recall that

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where λ denotes the 2-dimensional representation of C_9 with a rotation of order 9.

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For the second factor of W_1',

$$N_3^9 S^0 \left[S^{4\rho} \right] = S^0 \left[S^{4\rho_9} \right] \wedge \left(S^0 \vee \left(C_{9+} \underset{C_3}{\wedge} \bigvee_{i,j \geq 0} S^{4(i+j+1)\rho} \right) \right).$$

A possible Gap Theorem

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After inverting the right element in $\pi_{4\rho_9}$, we get a spectrum $\tilde{\Omega}$ whose fixed point set Ω is 972-periodic and detects the θ_j for $j \geq 5$.

A possible Gap Theorem

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After inverting the right element in $\pi_{4\rho_9}$, we get a spectrum $\tilde{\Omega}$ whose fixed point set Ω is 972-periodic and detects the θ_j for $j \geq 5$. The key question here is

> *Do we get a Gap Theorem stating that $\pi_{-2}\Omega$ is torsion free?*

A possible Gap Theorem

After inverting the right element in $\pi_{4\rho_9}$, we get a spectrum $\tilde{\Omega}$ whose fixed point set Ω is 972-periodic and detects the θ_j for $j \geq 5$. The key question here is

Do we get a Gap Theorem stating that $\pi_{-2}\Omega$ is torsion free?

To answer this we need to look at the equivariant homotopy groups of

$$X \wedge \Sigma^{4m\rho_9} H\underline{\mathbf{Z}} \qquad \text{and} \qquad X \wedge C_{9+} \underset{C_3}{\wedge} \Sigma^{4n\rho_3} H\underline{\mathbf{Z}}$$

for $m, n \in \mathbf{Z}$, where X is one of the following:

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for $m, n \in \mathbf{Z}$, where X is one of the following:

$$S^0, \quad C_{9+} \underset{C_3}{\wedge} \overline{S}^{2\rho-1}, \quad C_{9+} \underset{C_3}{\wedge} S^{3\rho-1} \quad \text{or} \quad \overline{S}^{\rho_9+2\lambda}.$$

A possible Gap Theorem (continued)

The following table indicates the dimensions in which

$$\underline{\pi}_i X \wedge \Sigma^{4m\rho_9} H\underline{\mathbf{Z}} \quad \text{and} \quad \underline{\pi}_i X \wedge C_{9+} \underset{C_3}{\wedge} \Sigma^{4n\rho_3} H\underline{\mathbf{Z}}$$

can be nontrivial for $m, n \geq 0$, with one caveat as indicated below.

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A possible Gap Theorem (continued)

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can be nontrivial for $m, n \geq 0$, with one caveat as indicated below.

X	$\underline{\pi}_i X \wedge \Sigma^{4m\rho_9} H\underline{\mathbb{Z}}$	$\underline{\pi}_i X \wedge C_{9+} \underset{C_3}{\wedge} \Sigma^{4n\rho_3} H\underline{\mathbb{Z}}$
S^0	$4m \leq i \leq 36m$	$4n \leq i \leq 12n$
$C_{9+} \underset{C_3}{\wedge} \overline{S}^{2\rho-1}$	$12m+1 \leq i \leq 36m+4$	$4n+1 \leq i \leq 12n+4$
$C_{9+} \underset{C_3}{\wedge} S^{3\rho-1}$	$12m+2 \leq i \leq 36m+8$	$4n+2 \leq i \leq 12n+8$
$\overline{S}^{\rho_9+2\lambda}$	$4m+1 \leq i \leq 36m+12$	$4n+3 \leq i \leq 12n+12$ for $n \geq -1$

A possible Gap Theorem (continued)

Here is a similar table for $m, n \leq -1$.

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S^0	$36m \leq i \leq 4m - 3$	$12n \leq i \leq 4n - 3$
$C_{9+} \underset{C_3}{\wedge} \overline{S}^{2\rho - 1}$	$36m + 4 \leq i \leq 12m - 2$	$12n + 4 \leq i \leq 4n - 2$
$C_{9+} \underset{C_3}{\wedge} S^{3\rho - 1}$	$36m + 8 \leq i \leq 12m - 1$	$12n + 8 \leq i \leq 4n - 1$
$\overline{S}^{\rho_9 + 2\lambda}$	$36m + 12 \leq i \leq 4m - 2$	$12n + 12 \leq i \leq 4n$ for $n \leq -2$

A possible Gap Theorem (continued)

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In each case the upper bound here is 3 less than the corresponding lower bound in the previous table.

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A possible Gap Theorem (continued)

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1.28

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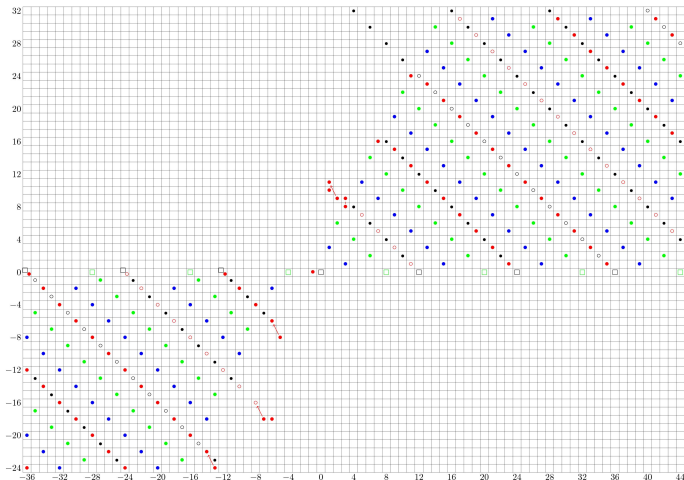 The role of the Morava stabilizer group

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A possible Gap Theorem (continued)

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Since $m, n \leq -1$, our upper bound is always ≤ -4, so we have the desired Gap Theorem.

The C_9 slice spectral sequence

Here is a color coded illustration of these fixed point homotopy groups.

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