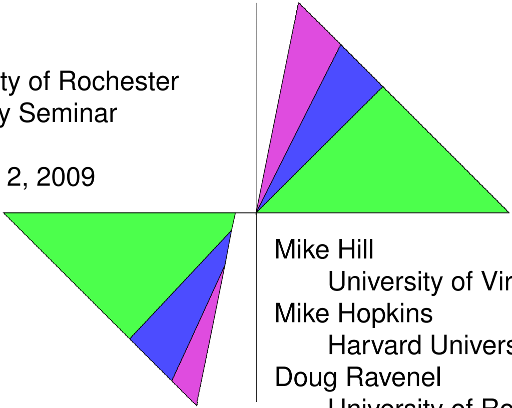
Lecture 2

A solution to the
Arf-Kervaire invariant
problem

University of Rochester
Topology Seminar

October 2, 2009



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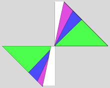

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θ_j is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62.

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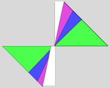

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The case $j = 6$ is still open.

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(iii) Gap Theorem. $\pi_{-2}(\Theta) = 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.

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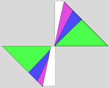
Θ will be the fixed point set associated with a C_8-equivariant spectrum $\tilde{\Theta}$ related to the complex cobordism spectrum. As we will explain below, a G-equivariant spectrum is more than just a spectrum with a G-action.

Ordinary spectra

In order to construct the slice spectral sequence, we need some notions from equivariant stable homotopy theory.



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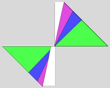
We get a spectrum E from the prespectrum D by defining

$$E_n = \lim_{\substack{\rightarrow \\ k}} \Omega^k D_{n+k}$$

This makes E_n homeomorphic to ΩE_{n+1}.

Ordinary spectra (continued)

Example 1. For a space X, let $D_n = \Sigma^n X$ with the obvious maps.



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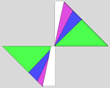
A map of spectra $f : E \to E'$ is a collection of maps of based spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps.



Equivariant spectra

Let G be a finite group. Experience has shown that in order to do equivariant stable homotopy theory, one needs G-spaces E_V indexed by finite dimensional orthogonal representations V sitting in a countably infinite dimensional orthogonal representation U.

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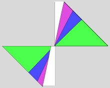
This universe U is said to be complete if it contains infinitely many copies of each irreducible representation of G. A canonical example of a complete universe for finite G is the direct sum of countably many copies of the regular real representation of G.

G-equivariant spectra (continued)

A G-equivariant spectrum (G-spectrum for short) indexed on U consists of a based G-spaces E_V for each finite dimensional subspace $V \subset U$ together with a transitive system of based G-homeomorphisms

$$E_V \xrightarrow[\cong]{\tilde{\sigma}_{V,W}} \Omega^{W-V} E_W$$

for $V \subset W \subset U$.



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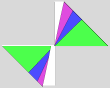

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for $V \subset W \subset U$. Here $\Omega^V X = F(S^V, X)$, the space of equivariant maps to X from the one point compactification of V. $W - V$ is the orthogonal complement of V in W. As in the classical case, the G-homotopy type of E_V depends only on the isomorphism class of V.

G-equivariant spectra (continued)

A map of G-spectra $f : E \to E'$ is a collection of maps of based G-spaces $f_V : E_V \to E'_V$ which commute with the respective structure maps.

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The structure map $\tilde{\sigma}_{V,W}$ is adjoint to a map

$$\sigma_{V,W} : \Sigma^{W-V} E_V \to E_W,$$

where $\Sigma^V X$ is defined to be $S^V \wedge X$.

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A suspension G-prespectrum is a G-prespectrum in which the maps above are G-equivalences for V sufficiently large.

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We define S^{-V} by saying its Wth space for $V \subset W$ is S^{W-V}. This is the analog of formal desuspension in the nonequivariant case.

$RO(G)$-graded homotopy groups (continued)

Given a virtual representation $\nu = V' - V$, we define $S^{\nu} = \Sigma^{V'} S^{-V}$. Hence we have a collection of sphere spectra graded over the orthogonal representation ring $RO(G)$.

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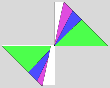

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$$\pi_\nu^G(X) = [S^\nu, X]_G,$$

the group of G-equivariant homotopy classes of maps from S^ν to X.

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For an integer n,

$$\pi_{n}^{G}(X) = [S^{n}, X]_{G} = [S^{n}, X^{G}] = \pi_{n}(X^{G}),$$

the ordinary nth homotopy group of the fixed point spectrum X^{G}.

Inducing and coinducing up to a larger group

Let $H \subset G$ be groups and let X be a H-space. There are two ways to get a G-space from it. The corresponding functors are the left and right adjoints to the forgetful functor from G-spaces to H-spaces.

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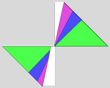
There is the induced G-space

$$G \times_H X = (G \times X)/H$$

where the action of H on $G \times X$ is defined by

$$\eta(\gamma, x) = (\gamma\eta^{-1}, \eta x)$$

for $\eta \in H$, $\gamma \in G$ and $x \in X$.



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for $\eta \in H$, $\gamma \in G$ and $x \in X$. Its underlying space is the disjoint union of $|G/H|$ copies of X.

Inducing and coinducing up to a larger group (continued)

There is the coinduced G-space

$$\mathrm{map}_H(G, X) = \{f \in \mathrm{map}(G, X)\colon f(\gamma\eta^{-1}) = \eta f(\gamma) \\ \forall \eta \in H \text{ and } \gamma \in G\}$$

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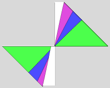

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There is a based analog of the coinduced G-space in which the underlying space is the smash product $X^{(|G/H|)}$.

It extends to H-spectra. For a H-spectrum X we denote the coinduced G-spectrum by $N_H^G X$, the norm of X along the inclusion $H \subset G$.

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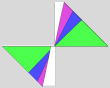
The classical Postnikov tower

The slice spectral sequence is based an equivariant analog of the Postnikov tower.



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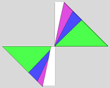

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These two functors have some universal properties. Let $\mathcal{S}$ and $\mathcal{S}_{>n}$ denote the categories of spectra and n-connected spectra.

The classical Postnikov tower (continued)

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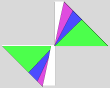
In other words, the map $P_{n+1}X \to X$ is universal among maps from n-connected spectra to X.



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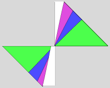
Since $\mathcal{S}_{>n} \subset \mathcal{S}_{>n-1}$, there is a natural transformation $P^n \to P^{n-1}$, whose fiber is denoted by P^n_nX.

An equivariant Postnikov tower

In what follows G will be an arbitrary finite cyclic 2-group, and $g = |G|$.



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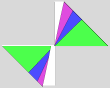

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Recall that $\mathcal{S}_{>n}$ is the category of spectra built up out of spheres of dimension $> n$ using arbitrary wedges and mapping cones.

An equivariant Postnikov tower (continued)

We will replace the set of sphere spectra by

$$\mathcal{A} = \left\{ \widehat{S}(m\rho_h),\ \Sigma^{-1}\widehat{S}(m\rho_h) \colon H \subset G,\ m \in \mathbf{Z} \right\}.$$

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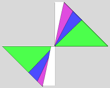
In order to define $\mathcal{S}^G_{>n}$, we need to assign a dimension to each element in $\mathcal{A}$, i.e., to each slcie cell. We do this in terms of the underlying spheres, namely

$$\dim \widehat{S}(m\rho_h) = mh \quad \text{and} \quad \dim \Sigma^{-1}\widehat{S}(m\rho_h) = mh - 1.$$

An equivariant Postnikov tower (continued)

Then $\mathcal{S}^G_{>n}$ is the category built up out of elements in $\mathcal{A}$ of dimension $> n$ using arbitrary wedges, mapping cones and smash products with equivariant suspension spectra.



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$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P^{n+1}_G X & \longrightarrow & P^n_G X & \longrightarrow & P^{n-1}_G X & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & {}^G P^{n+1}_{n+1} X & & {}^G P^n_n X & & {}^G P^{n-1}_{n-1} X & &
\end{array}$$

in which the inverse limit is X and the direct limit is contractible.

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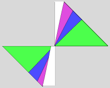

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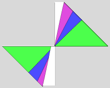

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This is the spectral sequence we will use to study $MU^{(4)}$ and its relatives.

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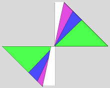

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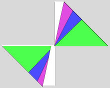

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- $\pi_*(MU) = \mathbf{Z}[x_i : i > 0]$ where $|x_i| = 2i$. This is the complex cobordism ring.

MU as a C_2-spectrum

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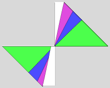

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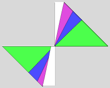

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Since any orthogonal representation V of C_2 is contained in $k\rho$ for $k \gg 0$, we can define the C_2-spectrum *MU* by

$$MU_V = \lim_{\substack{\rightarrow \\ k}} \Omega^{k\rho - V} MU(k).$$

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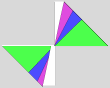

MU as a C_2-spectrum

This spectrum in known as real cobordism theory $MU_{\mathbf{R}}$. It has been studied by Landweber, Araki, Hu-Kriz and Kitchloo-Wilson.

Peter Landweber

Shoro Araki 1930–2005

Nitu Kitchloo

Steve Wilson

Igor Kriz and Po Hu

Norming up from MU

We will now construct a spectrum acted on by a larger cyclic 2-group.

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Norming up from MU

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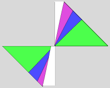

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Let $\gamma \in G$ be a generator and let z_i be a point in MU. Then the action of G on $MU^{(2^n)}$ is given by

$$\gamma(z_1 \wedge \cdots \wedge z_{2^n}) = \overline{z}_{2^n} \wedge z_1 \wedge \cdots \wedge z_{2^n-1},$$

where $\overline{z}_{2^n}$ is the complex conjugate of z_{2^n}.

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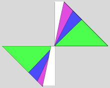

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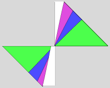

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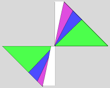

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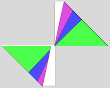

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Recall that the underlying spectrum of $\widehat{W}_1$ is a wedge of 4 copiesof S^2.

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In $\pi_4^u(MU^{(4)})$ there are 14 monomials that fall into 4 orbits under the action of G, each corresponding to a map from a $\widehat{S}(m\rho_h)$.

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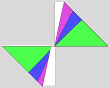

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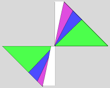

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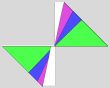
A large portion of our paper is devoted to proving that the slice spectral sequence has the desired properties. From now on we will drop the symbol G from the functors P^n, P_{n+1} and P_n^n.

The slice spectral sequence (continued)

Slice Theorem

In the slice tower for $MU^{(g/2)}$, every odd slice is contractible



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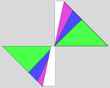

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$$\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z}) = \pi_*^H(S^{m\rho_h} \wedge H\mathbf{Z}).$$

We need this for all integers m because eventually we will obtain the spectrum $\tilde{\Theta}$ by inverting a certain element in $\pi_{32\rho_8}^G(MU^{(4)})$. Here is what we will learn.

Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$

Vanishing Theorem

- *For* $m \geq 0$, $\pi_*^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ *for* $k < m$ *and for* $k > mh$.

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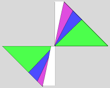

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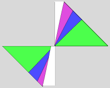

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Gap Corollary

For $h > 1$ *and all integers* m, $\pi_k^H(S^{m\rho_h} \wedge H\mathbf{Z}) = 0$ *for* $-4 < k < 0$.

A solution to the Arf-Kervaire invariant problem

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This means a similar statement must hold for $\pi_*^{C_8}(\tilde{\Theta}) = \pi_*(\Theta)$, which gives the Gap Theorem.

Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$ (continued)

Here is a picture of some slices $S^{m\rho_8} \wedge H\mathbf{Z}$.

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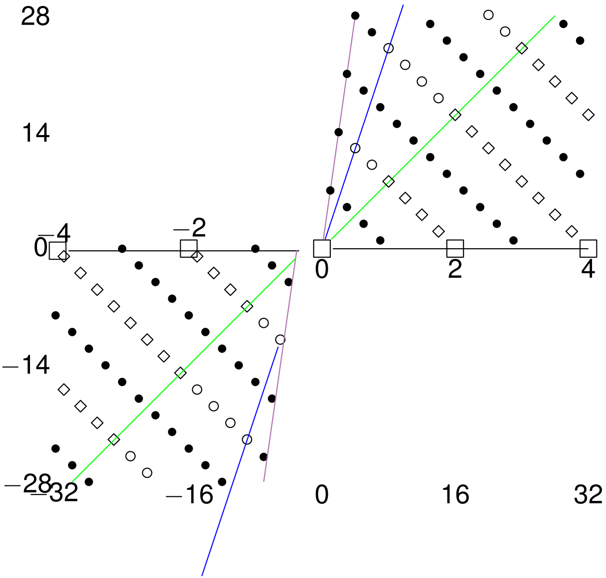

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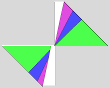
- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7,



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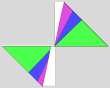

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- Bullets, circles and diamonds indicate cyclic groups of order 2, 4 and 8, and boxes indicate copies of the integers.

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Computing $\pi_*^G(W(m\rho_h) \wedge H\mathbf{Z})$ (continued)

- Note that all elements are in the first and third quadrants between certain black lines with slopes 0 and orchid lines with slope 7, and are concentrated on diagonals where t is divisible by 8.
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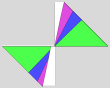

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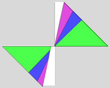
means that the resulting slice spectral sequence is confined to the regions of the first and third quadrants shown in the picture.

The proof of the Vanishing Theorem

Assuming the Slice Theorem, the proofs of the Vanishing Theorem and Gap Corollary are surprisingly easy.



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This means there is a G-CW-complex with one cell in dimension m, two cells in each dimension from $m+1$ to $2m$, four cells in each dimension from $2m+1$ to $4m$, and so on.

The proof of the Vanishing Theorem (continued)

In other words,

$$C(m\rho_g)_k = \begin{cases} 0 & \text{for } k < m \\ \mathbf{Z} & \text{for } k = m \\ \mathbf{Z}[G/H] & \text{for } mg/2h < k \leq mg/h \text{ and } h < g \\ 0 & \text{for } k > gm \end{cases}$$

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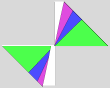
Each of these is a cyclic $\mathbf{Z}[G]$-module.



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Then we have

$$\pi_*^G(S^{m\rho_g} \wedge H\mathbf{Z}) = H_*(\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g))).$$

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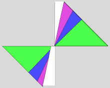
These groups are nontrivial only for $m \leq k \leq gm$, which gives the Vanishing Theorem for $m \geq 0$.



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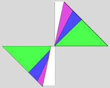
We will look at the bottom three groups in the complex $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, C(m\rho_g)_*)$.



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For $m > 1$ we have

$$\begin{array}{ccccccccc} & & C(m\rho_g)_m & & C(m\rho_g)_{m+1} & & C(m\rho_g)_{m+2} & & \\ & & \| & & \| & & \| & & \\ 0 & \longleftarrow & \mathbf{Z} & \xleftarrow{\epsilon} & \mathbf{Z}[C_2] & \xleftarrow{1-\gamma} & \mathbf{Z}[C_2] & \xleftarrow{1+\gamma} & \cdots \end{array}$$

The proof of the Vanishing Theorem (continued)

Applying $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ to this gives (in dimensions $\leq 2m$)

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \longleftarrow & \cdots \\ m & & m+1 & & m+2 & & m+3 & & m+4 & & \end{array}$$

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$$\pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) \quad = \quad \mathbf{Z}/2$$

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The proof of the Vanishing Theorem (continued)

For the negative multiples of ρ_g, $S^{-m\rho_g}$ is the equivariant Spanier-Whitehead dual of $S^{m\rho_g}$.

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Applying the functor $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$ to our chain complex gives a cochain complex beginning with

$$\begin{array}{ccccccccccc} \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \xrightarrow{0} & \mathbf{Z} & \longrightarrow & \cdots \\ -m & & -m-1 & & -m-2 & & -m-3 & & -m-4 & & \end{array}$$

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The critical fact here is the difference in behavior of the map $\epsilon : \mathbf{Z}[C_2] \to \mathbf{Z}$ under the functors $\mathrm{Hom}_{\mathbf{Z}[G]}(\mathbf{Z}, \cdot)$ and $\mathrm{Hom}_{\mathbf{Z}[G]}(\cdot, \mathbf{Z})$.

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The proof of the Vanishing Theorem (continued)

For $m < 0$ this gives

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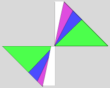

The proof of the Vanishing Theorem (continued)

For $m < 0$ this gives

$$\pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) = 0$$

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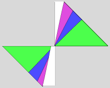

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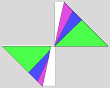
For $m < 0$ this gives

$$\begin{aligned} \pi_m^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= 0 \\ \pi_{m-1}^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= 0 \end{aligned}$$



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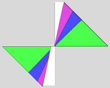

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\pi_{m-3}^G(S^{m\rho_g} \wedge H\mathbf{Z}) &= \begin{cases} 0 & \text{for } (g,m) = (2,1) \text{ or } (2,2) \\ \mathbf{Z}/2 & \text{otherwise} \end{cases}
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This gives both the Vanishing Theorem for $m < 0$ and the Gap Corollary.

Fixed point sets and homotopy fixed point sets

A pointed G-space X (where G acts trivially on the base point) has a fixed point set X^G, which can be thought of as the space of equivariant pointed maps to X, from S^0 (with trivial G-action),

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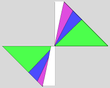
Fixed point sets and homotopy fixed point sets (continued)

There are similar constructions in the category of spectra.



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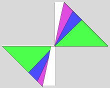

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The slice spectral sequence computes $\pi_*(X^G)$.

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The map $\Theta = \tilde{\Theta}^{C_8} \to \tilde{\Theta}^{hC_8}$ *is an equivalence.*

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Fixed point sets and homotopy fixed point sets (continued)

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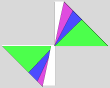

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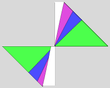
- The two spectral sequences are different, but by the result above, they converge to the same thing.



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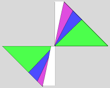

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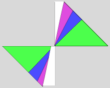

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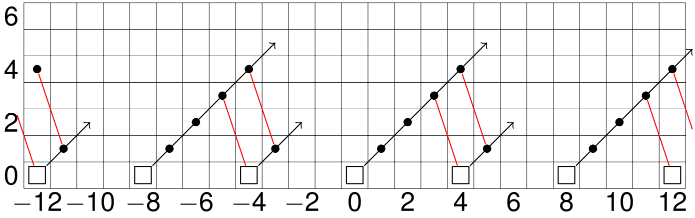
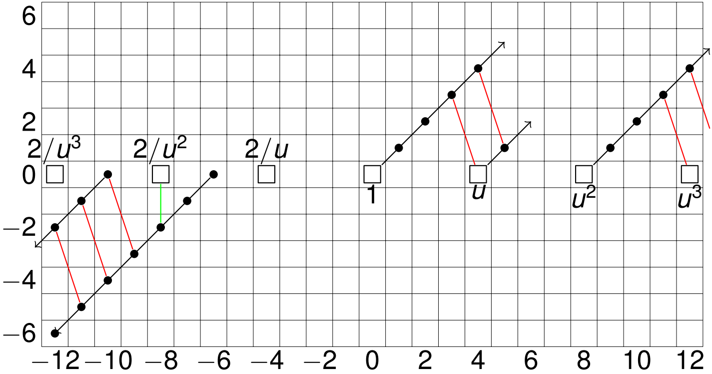


The slice spectral sequence for *KO*

Here is the slice spectral sequence for the actual fixed point set. It was originally studied by Dan Dugger.



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