A solution to the Arf-Kervaire invariant problem

University of Rochester
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The Arf-Kervaire formulation
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The spectrum $M$
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Our main result

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- **Manifold formulation**: It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds $M$ is always zero.
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- **Stable homotopy theoretic formulation**: It says that certain long sought hypothetical maps between high dimensional spheres do not exist.

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A wildly popular dance craze
Here is the stable homotopy theoretic formulation.
Main Theorem

The Arf-Kervaire elements $\theta_j \in \pi_{2j+1-2n}(S^n)$ for large $n$ do not exist for $j \geq 7$. 
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Here $\pi_k(X)$ (for a positive integer $k$) denotes the $k$th homotopy group of the topological space $X$, the set of continuous maps to $X$ from the $k$-sphere $S^k$, up to continuous deformation.
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The $\theta_j$ in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial.
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The \( \theta_j \) in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It has long been known that such things can exist only in dimensions that are 2 less than a power of 2.
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After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.
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Mark Mahowald’s sailboat

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The Arf invariant of a quadratic form in characteristic 2

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In 1941 Arf proved that this invariant (along with the number $n$) determines the isomorphism type of $q$. 
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The Kervaire invariant of a framed \((4k + 2)\)-manifold

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Brown-Peterson (1966) showed that it vanishes for all positive even $k$. 
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- \(\theta_j\) is known to exist for \(1 \leq j \leq 5\), i.e., in dimensions 2, 6, 14, 30 and 62.

- Our theorem says \(\theta_j\) does not exist for \(j \geq 7\). The case \(j = 6\) is still open.
The EHP sequence

Assume all spaces in sight are localized and the prime 2. For each $n > 0$ there is a fiber sequence due to James,

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Here $\Omega X = \Omega^1 X$ where $\Omega^k X$ denotes the space of continuous base point preserving maps to $X$ from the $k$-sphere $S^k$, known as the $k$th loop space of $X$. 
The EHP sequence (continued)

This leads to a long exact sequence of homotopy groups

\[ \cdots \to \pi_m(S^n) \xrightarrow{E} \pi_{m+1}(S^{n+1}) \xrightarrow{H} \pi_{m+1}(S^{2n+1}) \xrightarrow{P} \pi_{m-1}(S^n) \to \cdots \]

Here \( E \) stands for Einhängung, the German word for suspension. \( H \) stands for Hopf invariant. \( P \) stands for Whitehead product.
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For \( m = 2n \) the sequence is

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and we can ask about the image under $P$ of the generator of $\pi_{2n+1}(S^{2n+1})$. We denote it by $w_n \in \pi_{2n-1}(S^n)$, the Whitehead square.
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- When \( n \) is even, \( w_n \) it has infinite order and Hopf invariant two.
- \( w_n \) is trivial for \( n = 1, 3 \) and 7.
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- For such $n$, $w_n$ is divisible by 2 iff $n = 2^{j+1} - 1$ with $j > 2$ and $\theta_j$ exists, in which case $w_n = 2\theta_j$. 
The Hopf-Whitehead $J$ homomorphism

Let $SO(n)$ denote the special orthogonal group acting on $\mathbb{R}^n$. 

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The unstable formulation

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Each Whitehead square $w_{2n+1} \in \pi_{4n+1}(S^{2n+1})$ (except the cases $n = 0, 1$ and $3$) desuspends to a lower sphere until we get an element with a nontrivial Hopf invariant, which is always some $\rho_j$.
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- *The Arf-Kervaire element* \( \theta_j \in \pi_{2j+1-2} \) *exists for all* \( j > 0 \).
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Questions raised by our theorem

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Adams spectral sequence formulation. We now know that the \( h_j \) for \( j \geq 7 \) are not permanent cycles, so they have to support nontrivial differentials. We have no idea what their targets are.

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Spectra are to spaces as integers are to natural numbers.

In particular, recall that a space $X$ has a homotopy group $\pi_k(X)$ for each positive integer $k$. A spectrum $X$ has an abelian homotopy group $\pi_k(X)$ defined for every integer $k$. For the sphere spectrum $S^0$, $\pi_k(S^0)$ is the usual homotopy group $\pi_{n+k}(S^n)$ for $n > k+1$. The hypothetical $\theta_j$ is an element of this group for $k = 2j + 1 - 2$. 

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- It uses complex cobordism theory. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory.
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The spectrum $\Theta$

We will produce a map $S^0 \to \Theta$, where $\Theta$ is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.
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(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k (\Theta)$ depends only on the reduction of $k$ modulo 256.

(iii) Gap Theorem. $\pi_{-2} (\Theta) = 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.
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Background and history

Our main result

The Arf-Kervaire formulation

The unstable formulation

Questions raised by our theorem

Our strategy

Ingredients of the proof

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If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for $\theta_j$ for larger $j$ is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \pmod{256}$ for $j \geq 7$. 
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A solution to the Arf-Kervaire invariant problem

Mike Hill
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How we construct $\varnothing$ (continued)

To get a $C_8$-spectrum, we use the following general construction for getting from a space or spectrum $X$ acted on by a group $H$ to one acted on by a larger group $G$ containing $H$ as a subgroup.

Let $Y = \text{Map}_H(G, X)$, the space (or spectrum) of $H$-equivariant maps from $G$ to $X$. Here the action of $H$ on $G$ is by right multiplication, and the resulting object has an action of $G$ by left multiplication. As a set, $Y = X \mid G/H \mid$, the $|G/H|$-fold Cartesian power of $X$. A general element of $G$ permutes these factors, each of which is left invariant by the subgroup $H$. In particular we get a $C_8$-spectrum $\tilde{\Theta} = \text{Map}_{C_2}(C_8, \text{MU})$. This spectrum is not periodic, but it has a close relative $\Theta$ which is.
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