A solution to the Arf-Kervaire invariant problem

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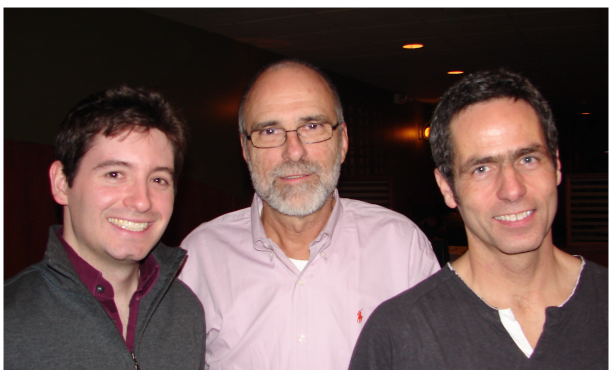


Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010 1.2

1 Background and history

1.1 Our main result

Our main result

Our main theorem can be stated in three different but equivalent ways:

- Manifold formulation: It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds M is always zero.
- Stable homotopy theoretic formulation: It says that certain long sought hypothetical maps between high dimensional spheres do not exist.
- Unstable homotopy theoretic formulation: It says something about the EHP sequence, which has to do with unstable homotopy groups of spheres.

The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved. 1.3

Our main result (continued)

Here is the stable homotopy theoretic formulation.

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2+n}(S^n)$ for large n do not exist for $j \geq 7$.*

Here $\pi_k(X)$ (for a positive integer k) denotes the kth homotopy group of the topological space X, the set of continuous maps to X from the k-sphere S^k, up to continuous deformation. This set has a natural group structure, which is abelian for $k > 1$. 1.4

The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It follows from Browder's theorem of 1969 that such things can exist only in dimensions that are 2 less than a power of 2.

Our main result (continued)

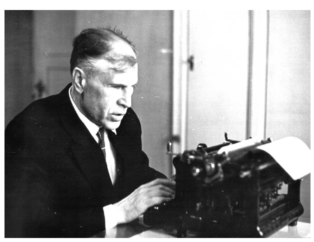
Mark Mahowald

Some homotopy theorists, most notably Mahowald, speculated about what would happen if θ_j existed for all j. They derived numerous consequences about homotopy groups of spheres. The possible nonexistence of the θ_j for large j was known as the Doomsday Hypothesis.

After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then. 1.5

1.2 Pontryagin's early work on homotopy groups of spheres

Pontryagin's early work on homotopy groups of spheres



Lev Pontryagin 1908-1988

Pontryagin's approach to maps $f : S^{n+k} \to S^n$ was

- Assume f is smooth. We know that any such map is can be continuously deformed to a smooth one.
- Pick a regular value $y \in S^n$. Its inverse image will be a smooth k-manifold M in S^{n+k}.
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres. 1.6

Pontryagin's early work (continued)

Let D^n be the closure of an open ball around $y \in S^n$. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^n) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^n$ with boundary homeomorphic to $M \times S^{n-1}$.

A local coordinate system around around y pulls back to one around M called a framing.

There is a way to reverse this procedure. A framed manifold $M^k \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^n$. 1.7

Pontryagin's early work (continued)

To proceed further, we need to be more precise about what we mean by continuous deformation.

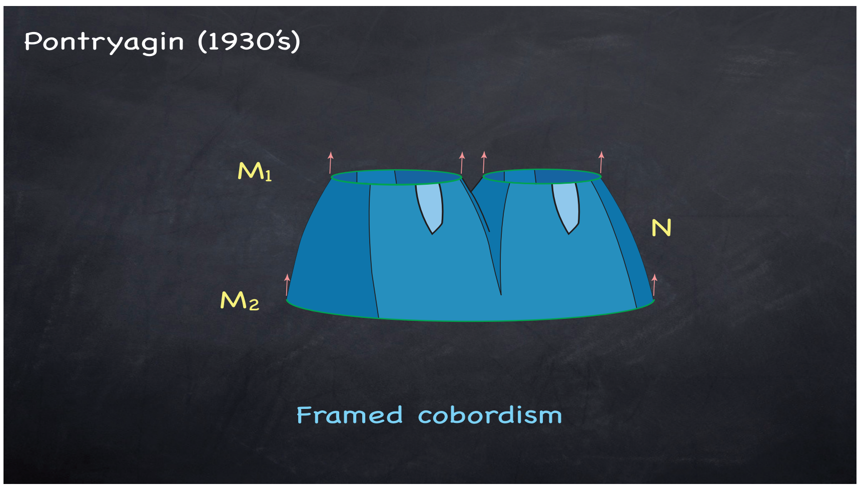
Two maps $f_1, f_2 : S^{n+k} \to S^n$ are homotopic if there is a continuous map $h : S^{n+k} \times [0,1] \to S^n$ (called a homotopy between f_1 and f_2) such that

$$h(x,0) = f_1(x) \qquad \text{and} \qquad h(x,1) = f_2(x).$$

If $y \in S^n$ is a regular value of h, then $h^{-1}(y)$ is a framed $(k+1)$-manifold $N \subset S^{n+k} \times [0,1]$ whose boundary is the disjoint union of $M_1 = f_1^{-1}(y)$ and $M_2 = f_2^{-1}(y)$. This N is called a framed cobordism between M_1 and M_2. When it exists the two closed manifolds are said to be framed cobordant. 1.8

Pontryagin's early work (continued)

Here is an example of a framed cobordism for $n = k = 1$.



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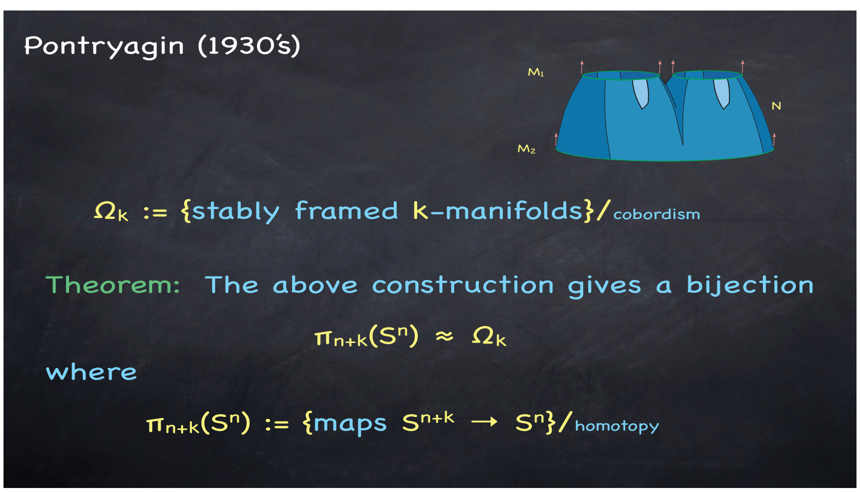
Pontryagin's early work (continued)



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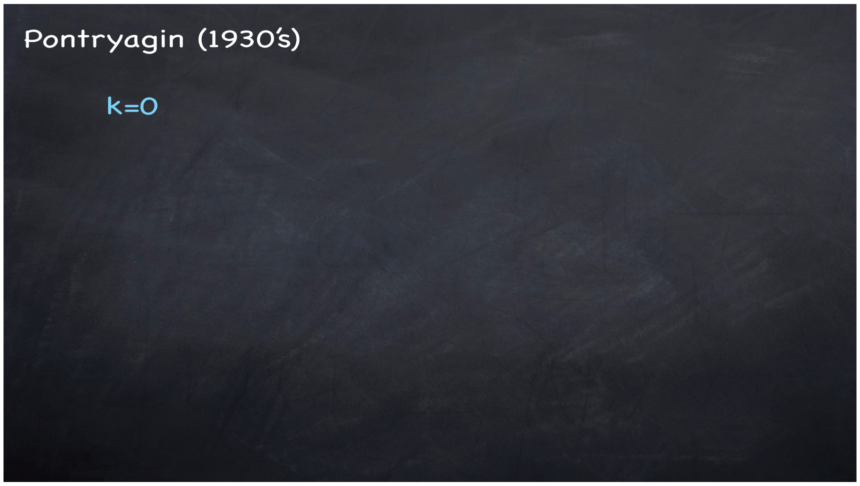
Pontryagin's early work (continued)



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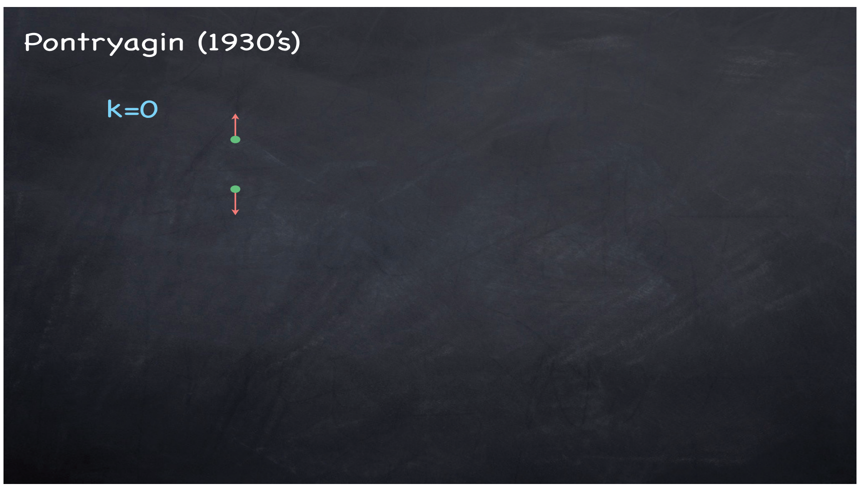
Pontryagin's early work (continued)



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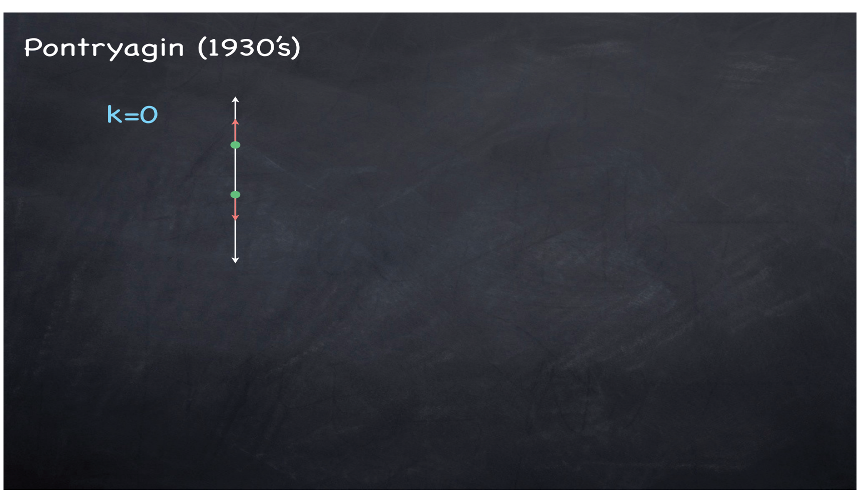
Pontryagin's early work (continued)



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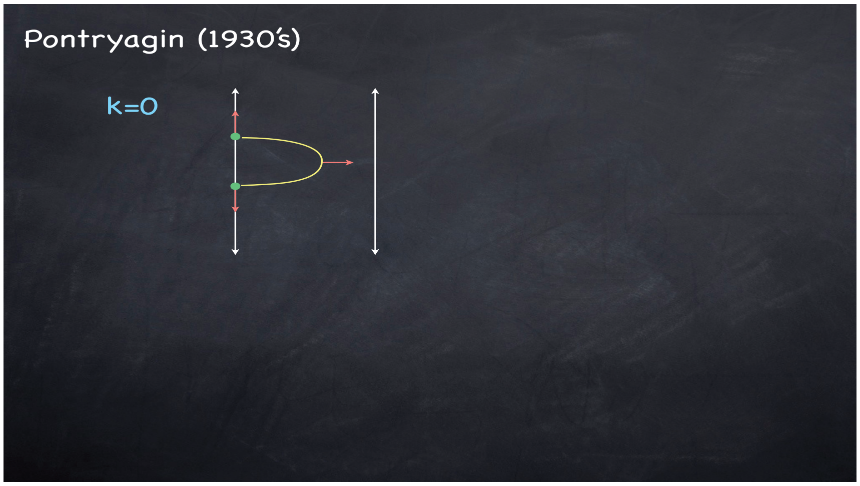
Pontryagin's early work (continued)



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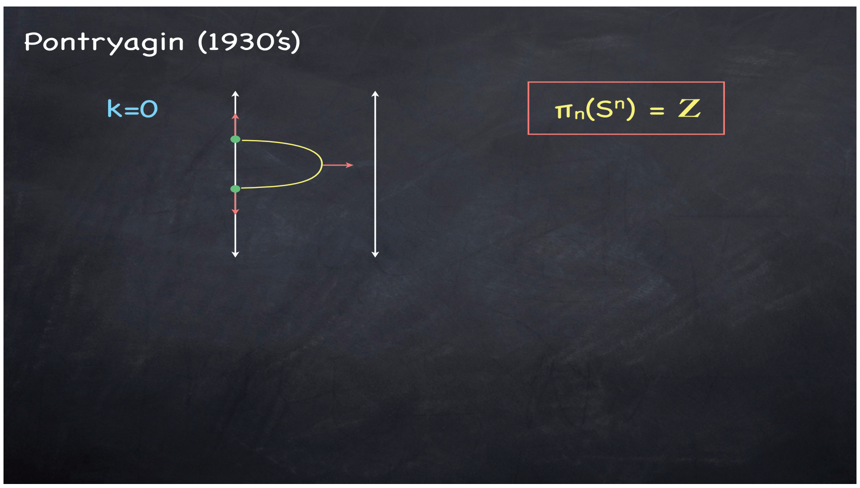
Pontryagin's early work (continued)



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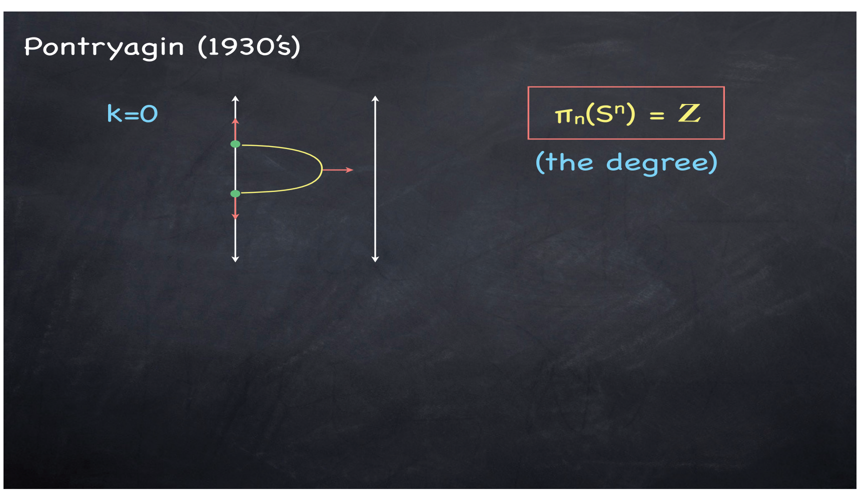
Pontryagin's early work (continued)



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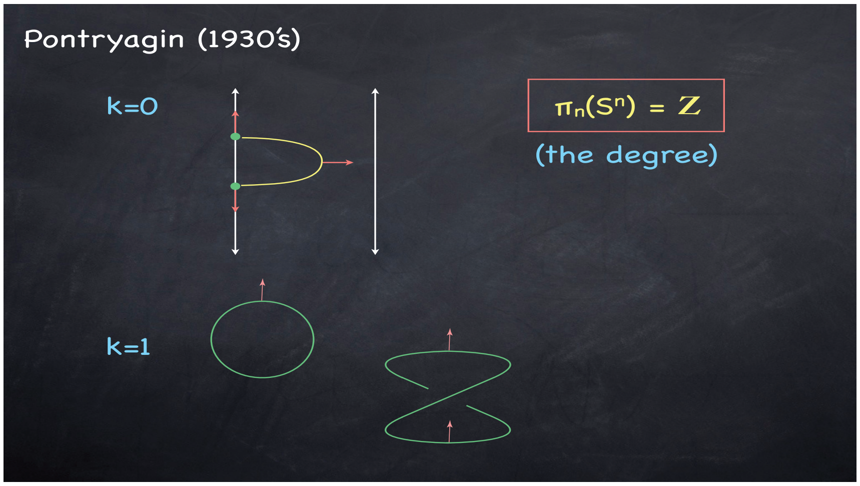
Pontryagin's early work (continued)



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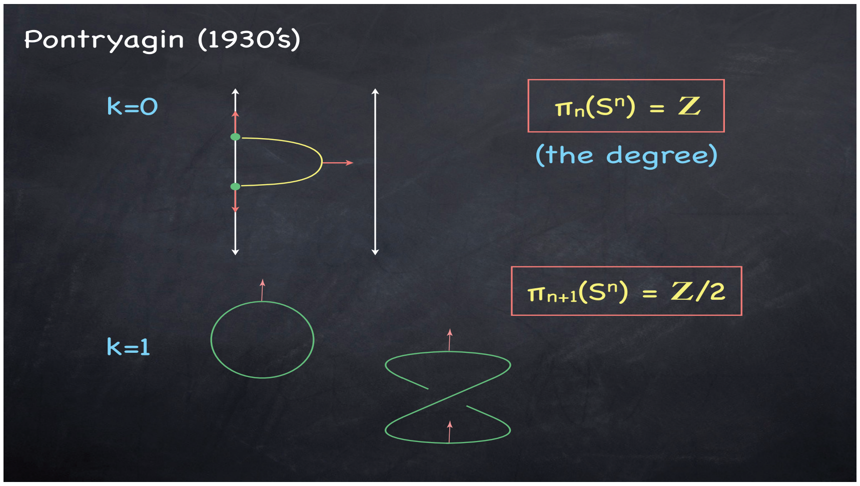
Pontryagin's early work (continued)



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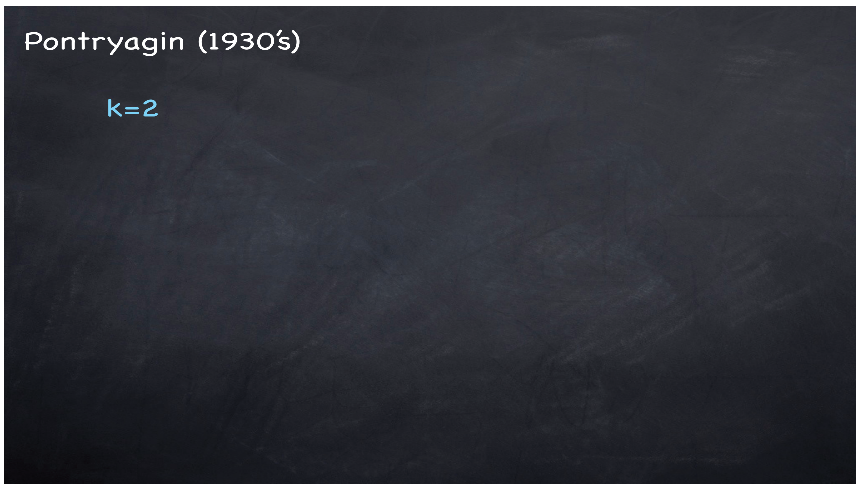
Pontryagin's early work (continued)



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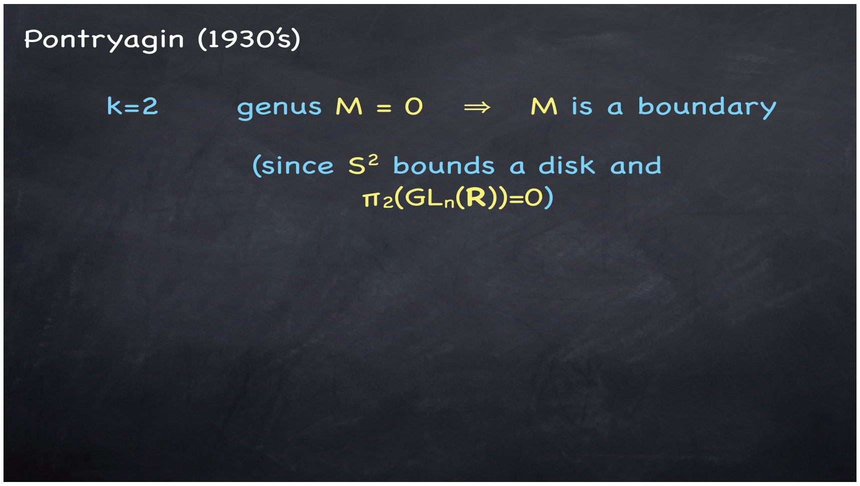
Pontryagin's early work (continued)



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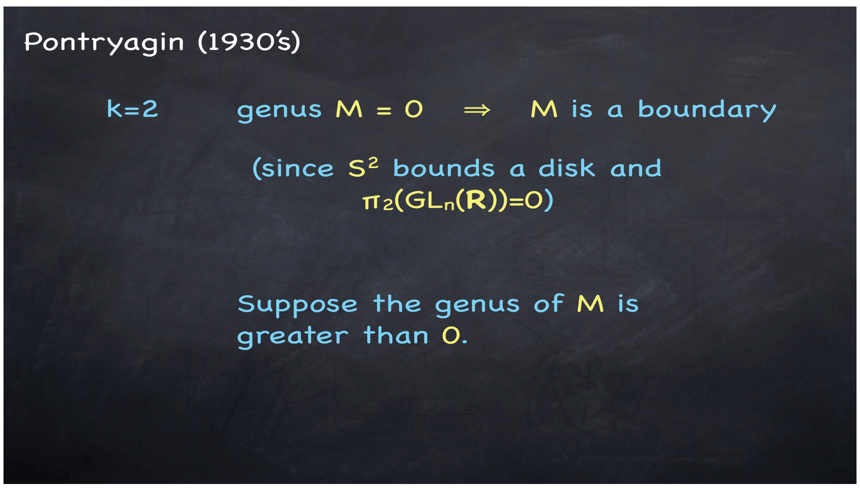
Pontryagin's early work (continued)



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Pontryagin's early work (continued)

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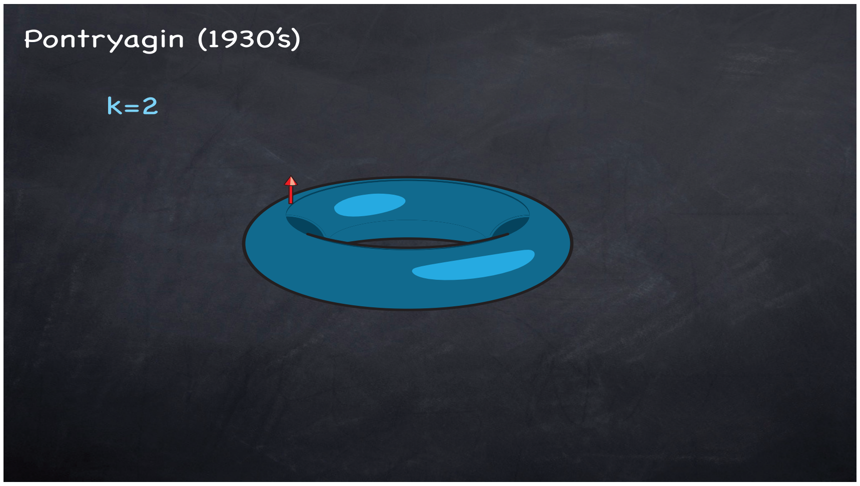
Pontryagin's early work (continued)



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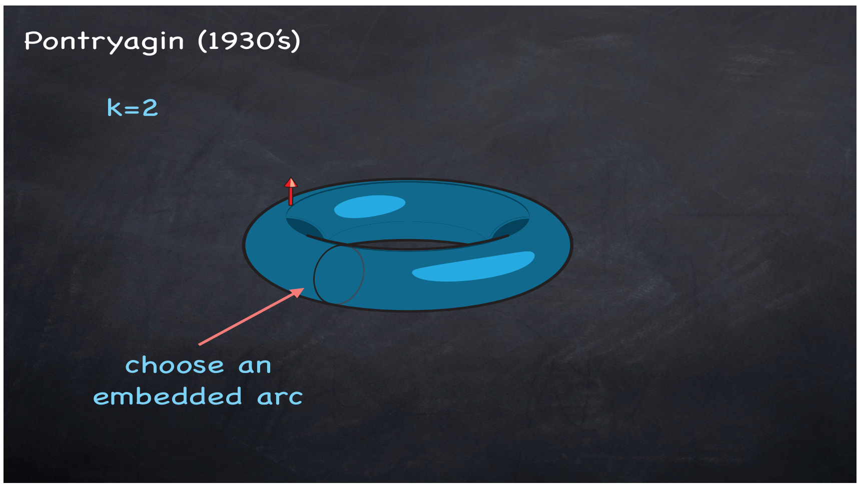
Pontryagin's early work (continued)



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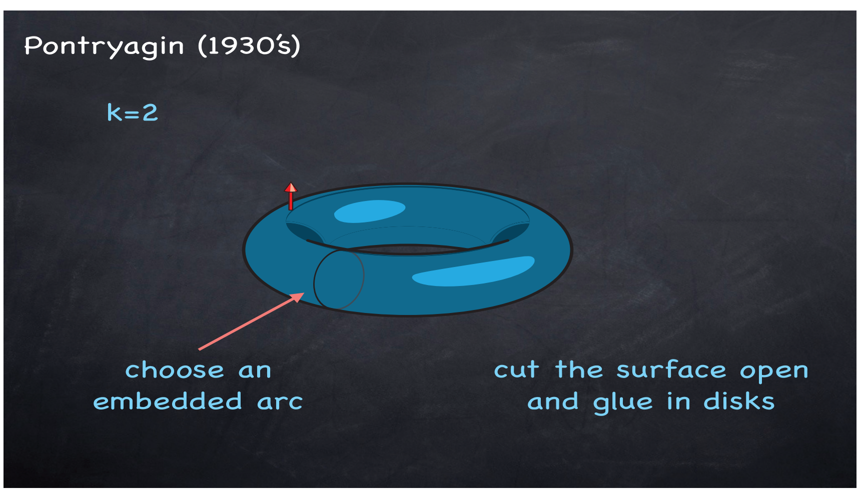
Pontryagin's early work (continued)



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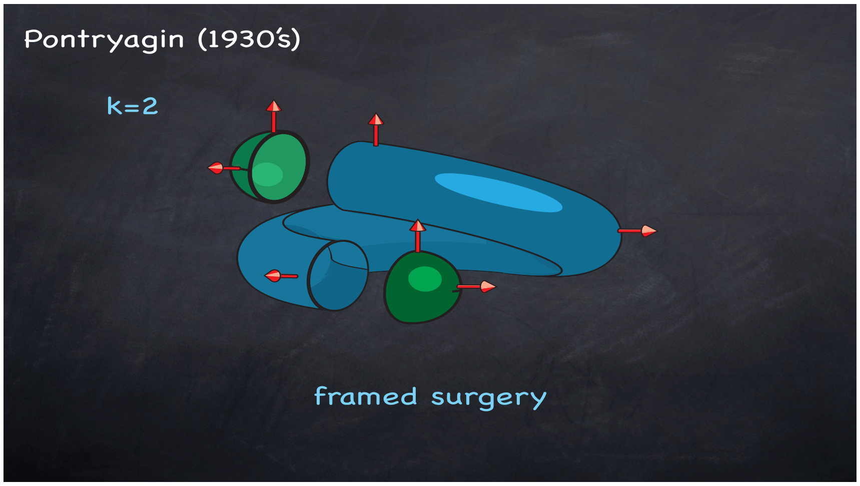
Pontryagin's early work (continued)



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Pontryagin's early work (continued)

Pontryagin (1930's)

Obstruction: $\varphi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$

1.27

Pontryagin's early work (continued)

Pontryagin (1930's)

Obstruction: $\varphi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$

Argument: Since the dimension of $H_1(M; \mathbb{Z}/2)$ is even, there is always a non-zero element in the kernel of φ, and so surgery can be performed.

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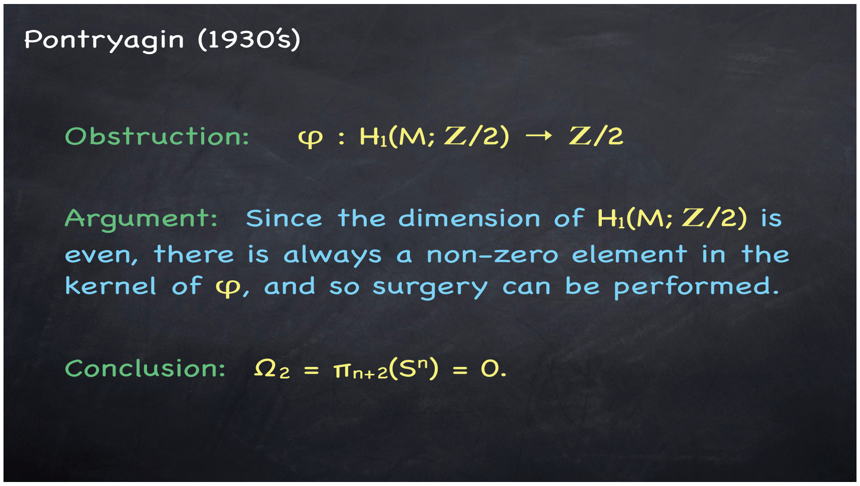
Pontryagin's early work (continued)



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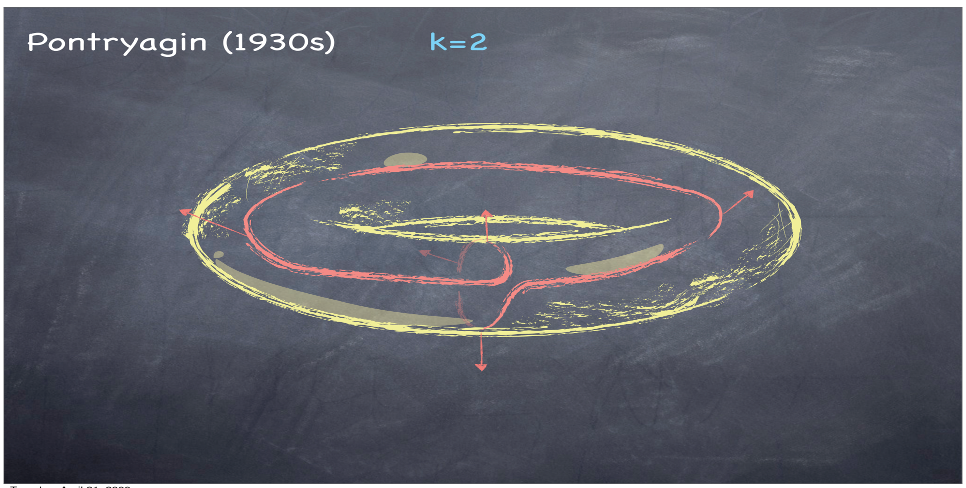
Pontryagin's mistake for $k = 2$

The map $\varphi : H_1(M;\mathbf{Z}/2) \to \mathbf{Z}/2$ is not a homomorphism!



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1.3 The Arf-Kervaire formulation

The Arf invariant of a quadratic form in characteristic 2

Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$. It is known that $\overline{H}$ has a basis of the form $\{a_i, b_i \colon 1 \le i \le n\}$ with

$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

1.31

The Arf invariant of a quadratic form in characteristic 2 (continued)

A quadratic refinement of λ is a map $q : \overline{H} \to \mathbf{Z}/2$ satisfying

$$q(x+y) = q(x) + q(y) + \lambda(x,y)$$

Its Arf invariant is

$$\mathrm{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbf{Z}/2.$$

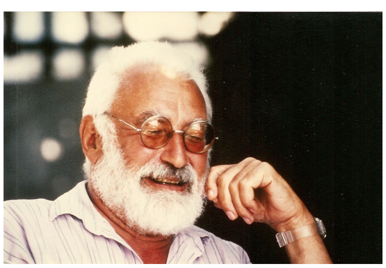
In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q. 1.32

Money talks: Arf's definition republished in 2009

Cahit Arf 1910-1997 1.33

The Kervaire invariant of a framed $(4m+2)$-manifold

Let M be a $2m$-connected smooth closed framed manifold of dimension $4m+2$. Let $H = H_{2m+1}(M;\mathbf{Z})$, the homology group in the middle dimension. Each $x \in H$ is represented by an immersion $i_x : S^{2m+1} \looparrowright M$ with a stably trivialized normal bundle. H has an antisymmetric bilinear form λ defined in terms of intersection numbers.



Michel Kervaire 1927-2007

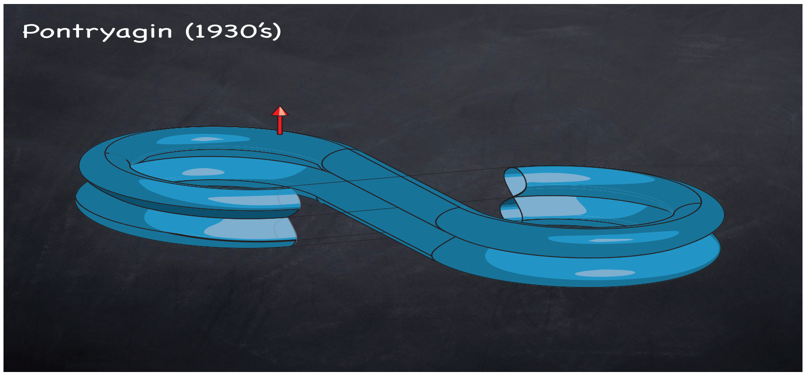
Kervaire defined a quadratic refinement q on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle. The Kervaire invariant $\Phi(M)$ is defined to be the Arf invariant of q.

For $m = 0$, Kervaire's q coincides with Pontryagin's φ. 1.34

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

What can we say about $\Phi(M)$?

- For $m = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant. Pontryagin used it in 1950 (after some false starts in the 30s) to show $\pi_{n+2}(S^n) = \mathbf{Z}/2$ for all $n \geq 2$.



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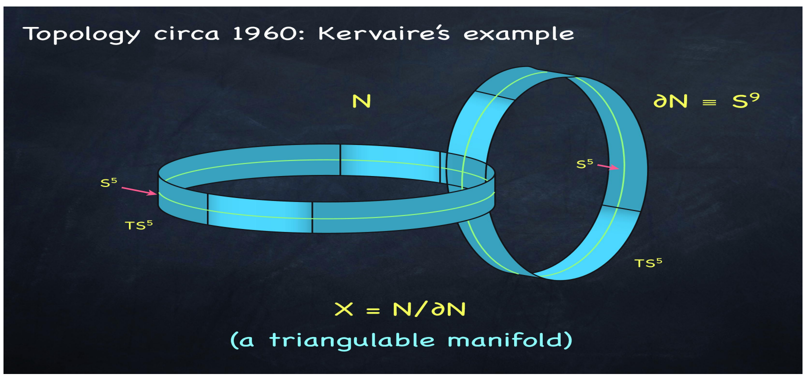
The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

- Kervaire (1960) showed it must vanish when $m = 2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure.



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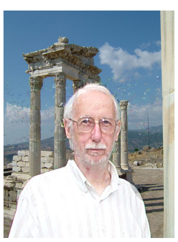
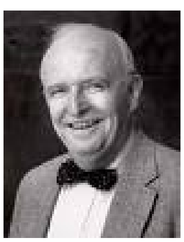
The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

-   Brown-Peterson (1966) showed that it vanishes for all positive even m.

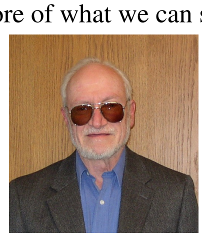
Ed Brown

Frank Peterson 1930-2000

1.37

The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

-  Browder (1969) showed that it can be nontrivial only if $m = 2^{j-1} - 1$ for some positive integer j. This happens iff the element h_j^2 is a permanent cycle in the Adams spectral sequence. The corresponding element in $\pi_{n+2^{j+1}-2}(S^n)$ for large n is θ_j, the subject of our theorem. This is the stable homotopy theoretic formulation of the problem.

Bill Browder

- θ_j is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62.
- In the decade following Browder's theorem, many topologists tried without success to construct framed manifolds with nontrivial Kervaire invariant in all dimensions 2 less than a power of 2.
- Our theorem says θ_j does not exist for $j \geq 7$. The case $j = 6$ is still open.

1.38

1.4 Questions raised by our theorem

Questions raised by our theorem

Adams spectral sequence formulation. We now know that the h_j^2 for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials. We have no idea what their targets are.

Unstable homotopy theoretic formulation. In 1967 Mahowald published an elaborate conjecture about the role of the θ_j (assuming that they all exist) in the unstable homotopy groups of spheres. Since they do not exist, a substitute for his conjecture is needed. We have no idea what it should be.

Our method of proof offers a new tool, the slice spectral sequence, for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future. We will illustrate it at the end of the talk. 1.39

2 Our strategy

2.1 Ingredients of the proof

Ingredients of the proof

Our proof has several ingredients.

- We use methods of stable homotopy theory, which means we use spectra instead of topological spaces. The definition of these would take us too far afield, so instead we offer a slogan:

Spectra are to spaces as integers are to natural numbers.

In particular, recall that a space X has a homotopy group $\pi_k(X)$ for each positive integer k. A spectrum X has an abelian homotopy group $\pi_k(X)$ defined for every integer k.

For the sphere spectrum S^0, $\pi_k(S^0)$ is the usual homotopy group $\pi_{n+k}(S^n)$ for $n > k+1$. The hypothetical θ_j is an element of this group for $k = 2^{j+1} - 2$.

1.40

Ingredients of the proof (continued)

More ingredients of our proof:

- We use complex cobordism theory. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory. It includes some highly developed computational techniques that began with work by Milnor, Novikov and Quillen in the 60s. A pivotal tool in the subject is the theory of formal group laws.


John Milnor


Sergei Novikov


Dan Quillen

1.41

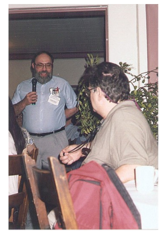
Ingredients of the proof (continued)

More ingredients of our proof:

- We also make use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group G (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbf{Z}$, but by $RO(G)$, the real representation ring of G. Our calculations make use of this richer structure.

Peter May

John Greenlees



Gaunce Lewis
1949-2006

1.42

2.2 The spectrum Ω

The spectrum Ω

We will produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial. This means that if θ_j exists, we will see its image in $\pi_*(\Omega)$.

(ii) Periodicity Theorem. It is 256-periodic, meaning that $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.

(iii) Gap Theorem. $\pi_k(\Omega) = 0$ for $-4 < k < 0$. This property is our zinger. Its proof involves a new tool we call the slice spectral sequence. 1.43

The spectrum Ω (continued)

Here again are the properties of Ω

(i) Detection Theorem. If θ_j exists, it has nontrivial image in $\pi_*(\Omega)$.

(ii) Periodicity Theorem. $\pi_k(\Omega)$ depends only on the reduction of k modulo 256.

(iii) Gap Theorem. $\pi_{-2}(\Omega) = 0$.

(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$. 1.44

2.3 How we construct Ω

How we construct Ω

Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum MU. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The fixed point set of this action is the set of real points, known to topologists as MO, the unoriented cobordism spectrum. In this notation, U and O stand for the unitary and orthogonal groups. 1.45

How we construct Ω (continued)

Some people who have studied MU as a C_2-spectrum:

Peter Landweber

Igor Kriz and Po Hu

Shoro Araki
1930–2005

Nitu Kitchloo

Steve Wilson

1.46

How we construct Ω (continued)

To get a C_8-spectrum, we use the following general construction for getting from a space or spectrum X acted on by a group H to one acted on by a larger group G containing H as a subgroup. Let

$$Y = \mathrm{Map}_H(G, X),$$

the space (or spectrum) of H-equivariant maps from G to X. Here the action of H on G is by right multiplication, and the resulting object has an action of G by left multiplication. As a set, $Y = X^{|G/H|}$, the $|G/H|$-fold Cartesian power of X. A general element of G permutes these factors, each of which is left invariant by the subgroup H.

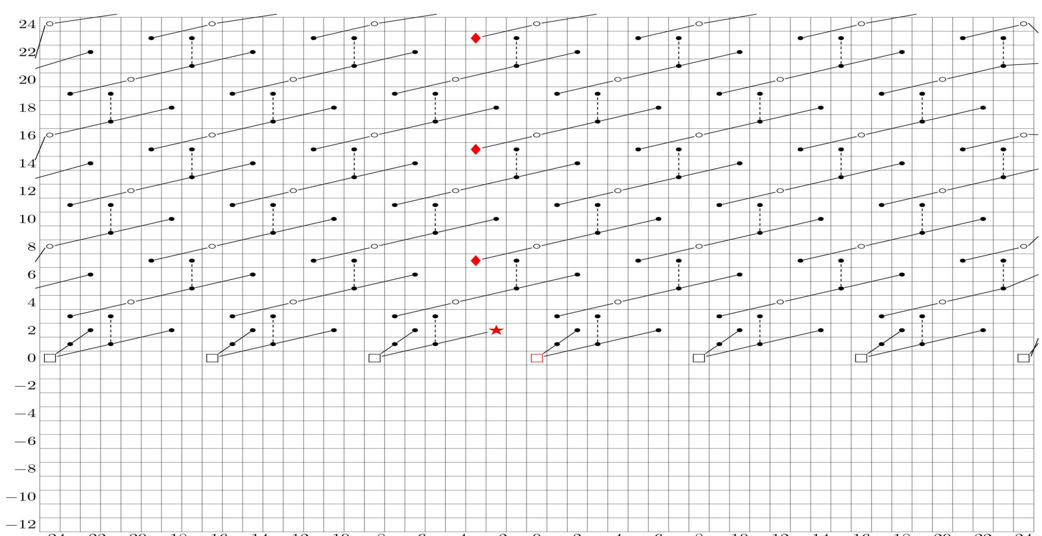
In particular we get a C_8-spectrum

$$MU^{(4)} = \mathrm{Map}_{C_2}(C_8, MU).$$

This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is. 1.47

2.4 The slice spectral sequence

A homotopy fixed point spectral sequence



The corresponding slice spectral sequence