A solution to the Arf-Kervaire invariant problem

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Graduate Student
Topology and Geometry
Conference

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Mike Hopkins
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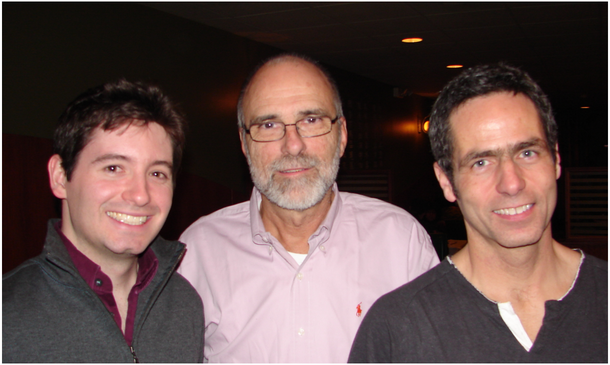
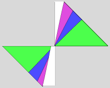


Mike Hill, myself and Mike Hopkins
Photo taken by Bill Browder
February 11, 2010



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Our main result

Our main theorem can be stated in three different but equivalent ways:

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Our main theorem can be stated in three different but equivalent ways:

- Manifold formulation: It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds M is always zero.

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The problem solved by our theorem is nearly 50 years old.

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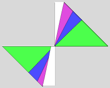

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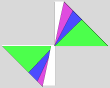
Our main result (continued)

Here is the stable homotopy theoretic formulation.



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Main Theorem

The Arf-Kervaire elements $\theta_j \in \pi_{2^{j+1}-2+n}(S^n)$ for large n do not exist for $j \geq 7$.

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Here $\pi_k(X)$ (for a positive integer k) denotes the kth homotopy group of the topological space X, the set of continuous maps to X from the k-sphere S^k, up to continuous deformation.

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The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial.

Our main result (continued)

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The θ_j in the theorem is the name given to a hypothetical map between spheres for which the Arf-Kervaire invariant is nontrivial. It follows from Browder's theorem of 1969 that such things can exist only in dimensions that are 2 less than a power of 2.

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Our main result (continued)

Mark Mahowald

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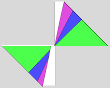

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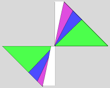

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After 1980, the problem faded into the background because it was thought to be too hard. Our proof is two giant steps away from anything that was attempted in the 70s. We now know that the world of homotopy theory is very different from what they had envisioned then.

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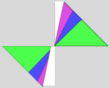

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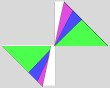
Pontryagin's early work on homotopy groups of spheres

Lev Pontryagin 1908-1988

Pontryagin's approach to maps $f : S^{n+k} \to S^n$ was



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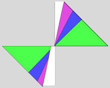

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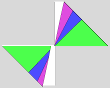
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.



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Pontryagin’s early work (continued)

Let D^n be the closure of an open ball around $y \in S^n$.

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Pontryagin's early work (continued)

Let D^n be the closure of an open ball around $y \in S^n$. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^n) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^n$ with boundary homeomorphic to $M \times S^{n-1}$.

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A local coordinate system around around y pulls back to one around M called a framing.

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There is a way to reverse this procedure. A framed manifold $M^k \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^n$.

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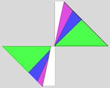
To proceed further, we need to be more precise about what we mean by continuous deformation.



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Two maps $f_1, f_2 : S^{n+k} \to S^n$ are homotopic if there is a continuous map $h : S^{n+k} \times [0, 1] \to S^n$ (called a homotopy between f_1 and f_2) such that

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$$h(x,0) = f_1(x) \qquad \text{and} \qquad h(x,1) = f_2(x).$$

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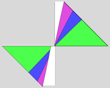
If $y \in S^n$ is a regular value of h, then $h^{-1}(y)$ is a framed $(k + 1)$-manifold $N \subset S^{n+k} \times [0, 1]$



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If $y \in S^n$ is a regular value of h, then $h^{-1}(y)$ is a framed $(k+1)$-manifold $N \subset S^{n+k} \times [0,1]$ whose boundary is the disjoint union of $M_1 = f_1^{-1}(y)$ and $M_2 = f_2^{-1}(y)$.

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Here is an example of a framed cobordism for $n = k = 1$.

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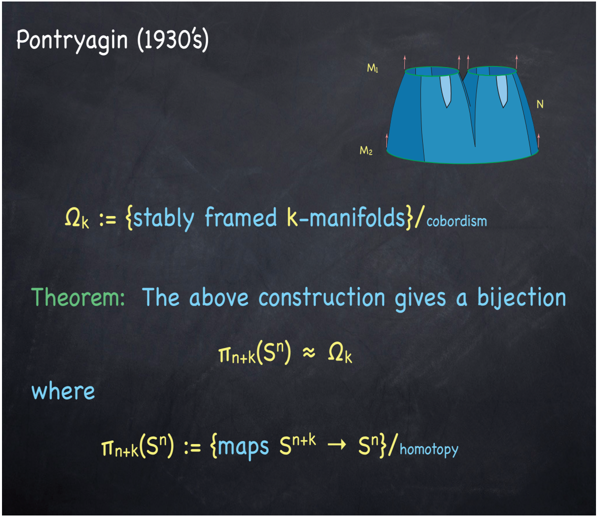

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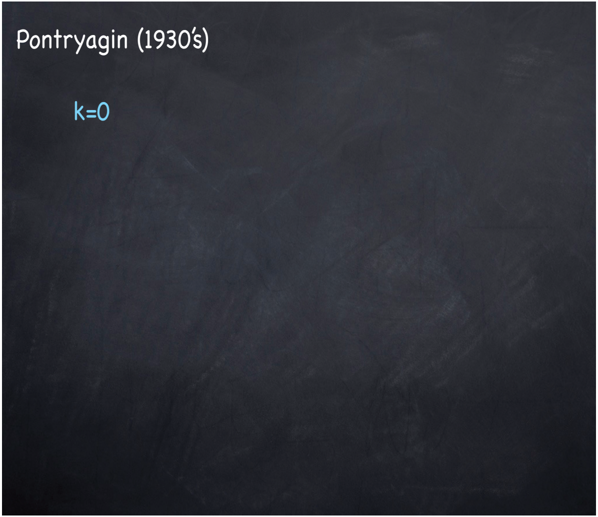

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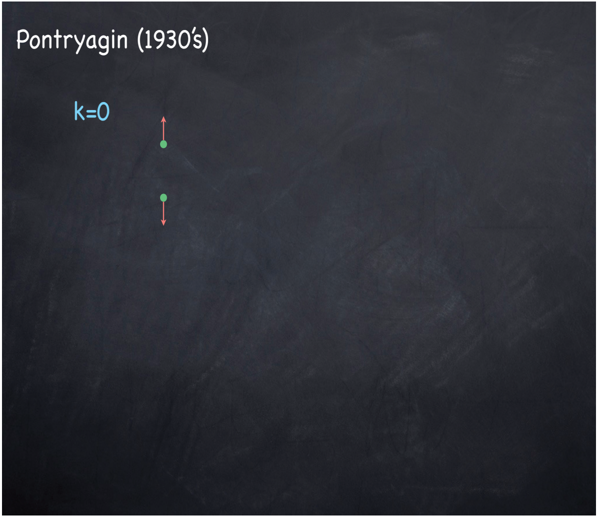

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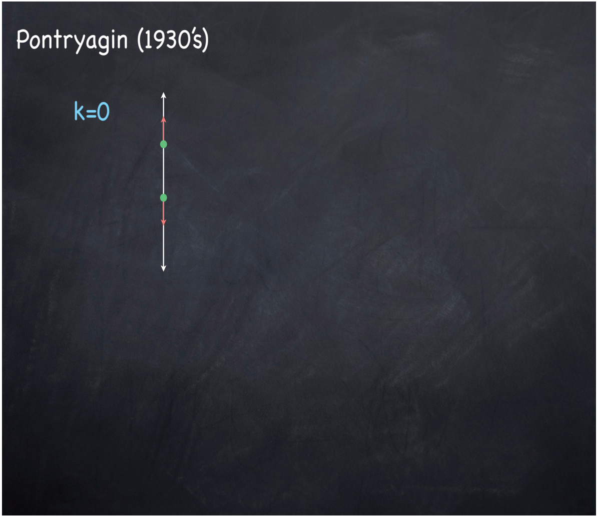

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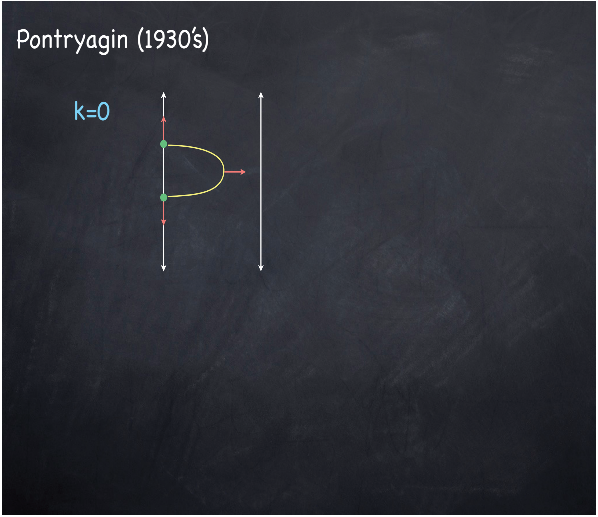

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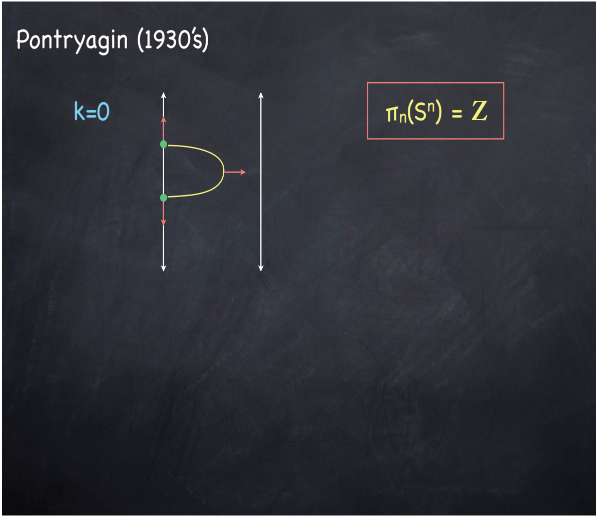

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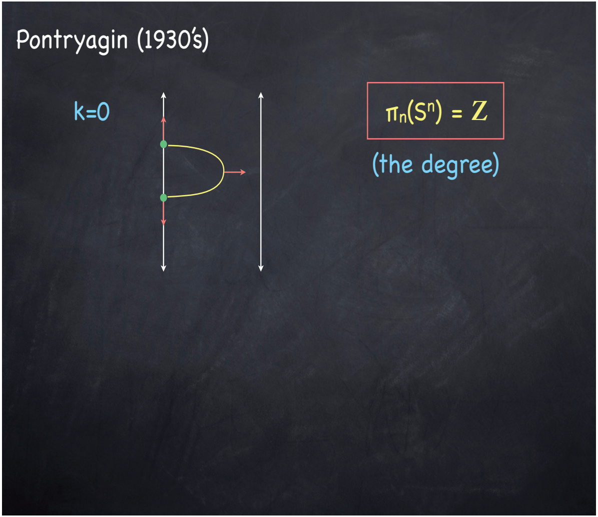

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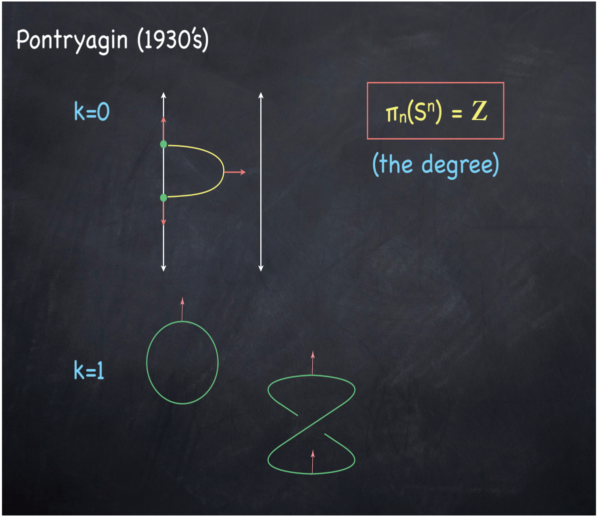

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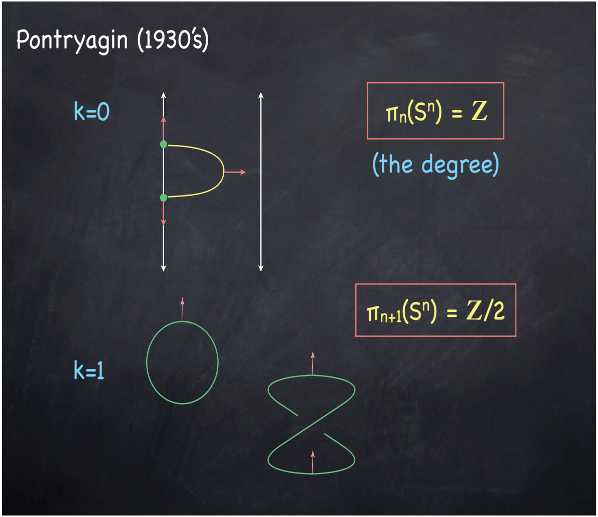

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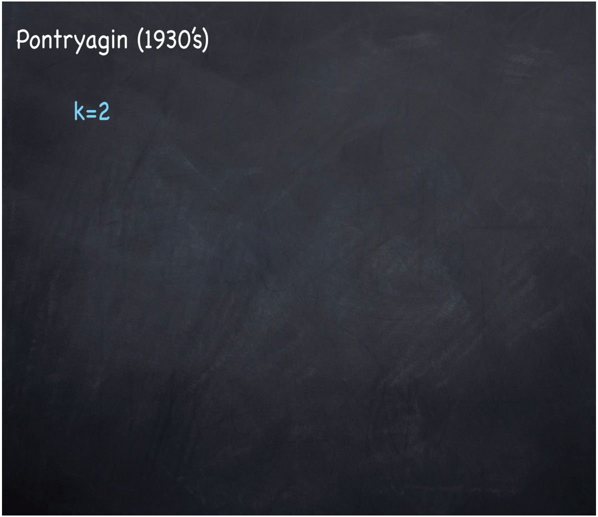

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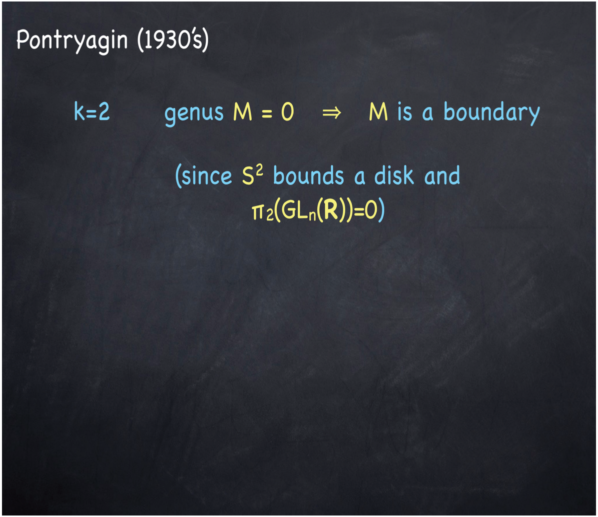

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Pontryagin's early work (continued)

Pontryagin (1930's)

k=2 genus M = 0 ⇒ M is a boundary

(since S^2 bounds a disk and $\pi_2(GL_n(\mathbf{R}))=0$)

Suppose the genus of M is greater than 0.

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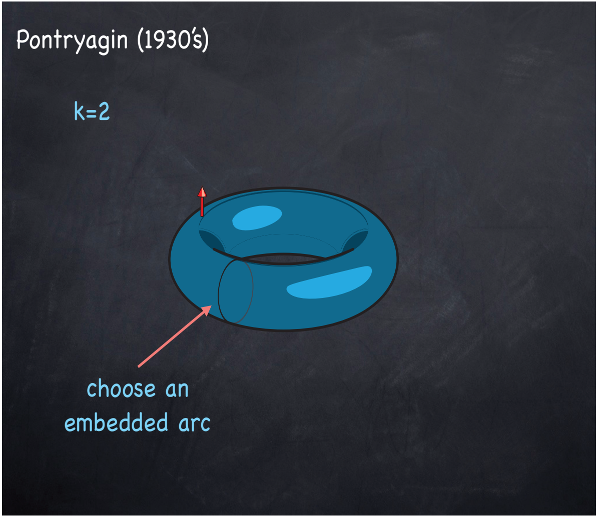

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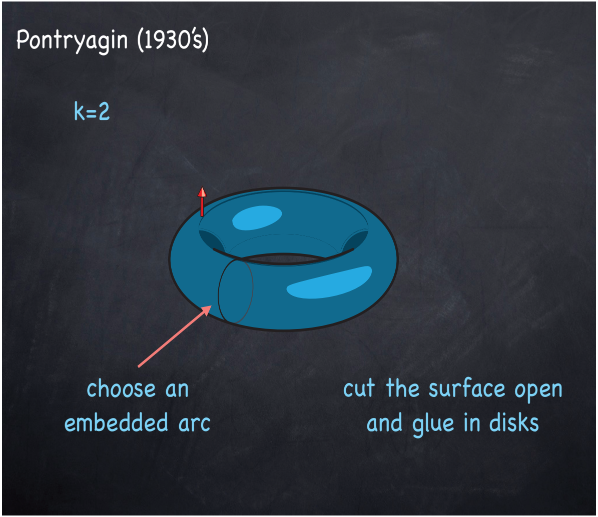

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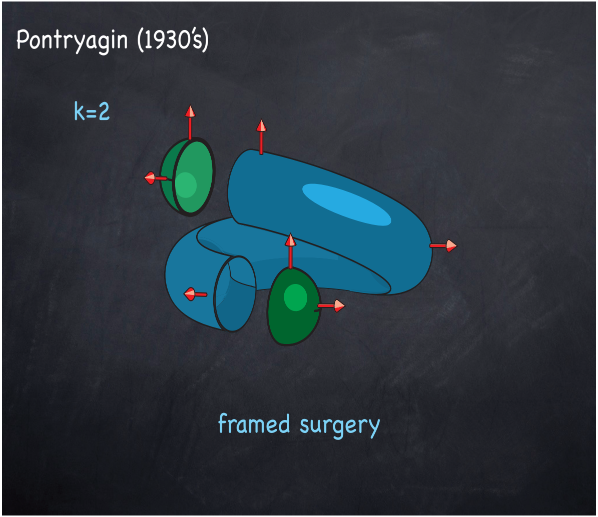

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Pontryagin's early work (continued)

Pontryagin (1930's)

Obstruction: $\varphi : H_1(M; Z/2) \to Z/2$

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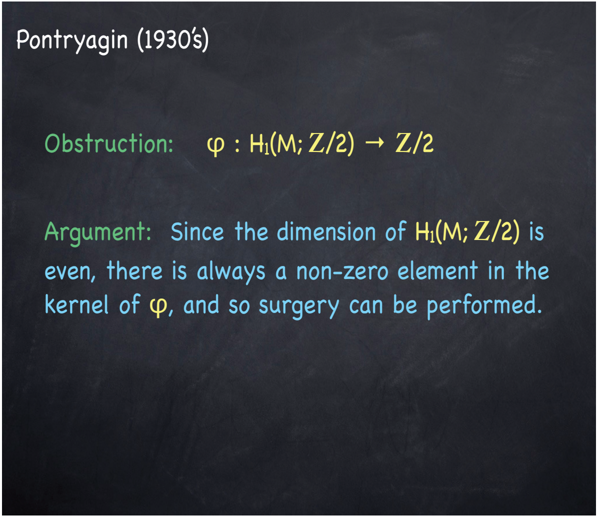

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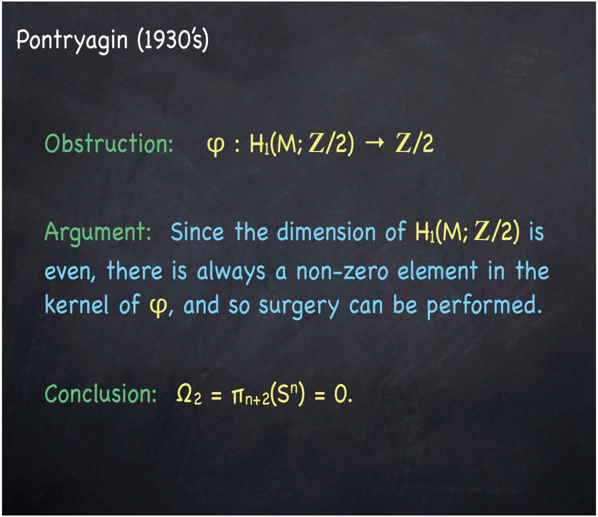

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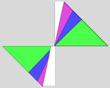
Pontryagin's mistake for $k = 2$

The map $\varphi : H_1(M; \mathbf{Z}/2) \to \mathbf{Z}/2$ is not a homomorphism!



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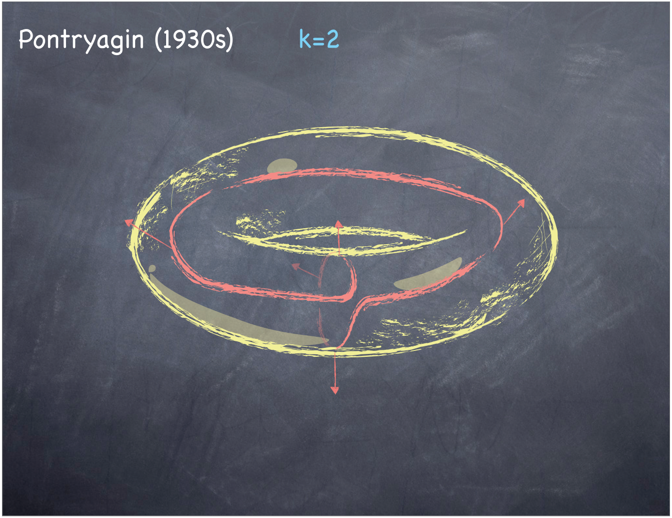

Tuesday, April 21, 2009

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The Arf invariant of a quadratic form in characteristic 2

Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$.

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The Arf invariant of a quadratic form in characteristic 2

Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$. It is known that $\overline{H}$ has a basis of the form $\{a_i, b_i \colon 1 \leq i \leq n\}$ with

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$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

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In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

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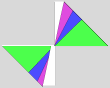
The Arf invariant of a quadratic form in characteristic 2 (continued)

A quadratic refinement of λ is a map $q : \overline{H} \to \mathbf{Z}/2$ satisfying



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The Arf invariant of a quadratic form in characteristic 2 (continued)

A quadratic refinement of λ is a map $q : \overline{H} \to \mathbf{Z}/2$ satisfying

$$q(x+y) = q(x) + q(y) + \lambda(x, y)$$

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Its Arf invariant is

$$\text{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbf{Z}/2.$$

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In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q.

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Money talks: Arf's definition republished in 2009

Cahit Arf 1910-1997

The Kervaire invariant of a framed $(4m+2)$-manifold

Let M be a $2m$-connected smooth closed framed manifold of dimension $4m+2$.

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Let M be a $2m$-connected smooth closed framed manifold of dimension $4m+2$. Let $H = H_{2m+1}(M; \mathbf{Z})$, the homology group in the middle dimension.

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Michel Kervaire 1927-2007

Kervaire defined a quadratic refinement q on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle.

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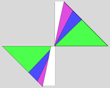

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Kervaire defined a quadratic refinement q on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle. The Kervaire invariant $\Phi(M)$ is defined to be the Arf invariant of q.

For $m = 0$, Kervaire's q coincides with Pontryagin's φ.

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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

What can we say about $\Phi(M)$?

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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

What can we say about $\Phi(M)$?

- For $m = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant.

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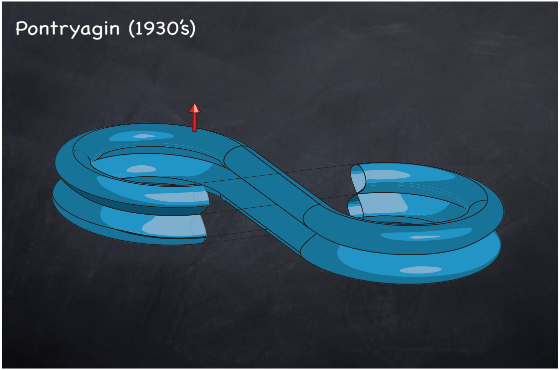

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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

More of what we can say about $\Phi(M)$.

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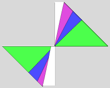

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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

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- Kervaire (1960) showed it must vanish when $m = 2$.

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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

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- Kervaire (1960) showed it must vanish when $m = 2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure.

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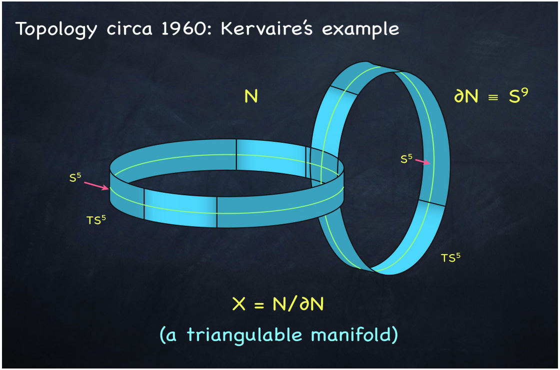

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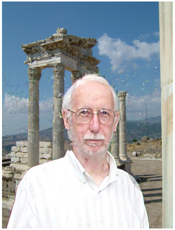
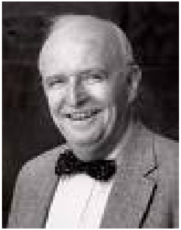
- 

  Ed Brown

  Frank Peterson
  1930-2000

  Brown-Peterson (1966) showed that it vanishes for all positive even m.

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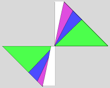

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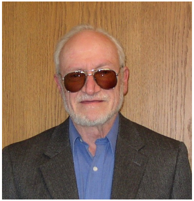
- 

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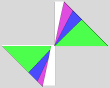

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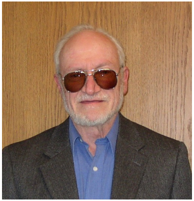
- 

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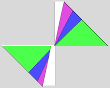

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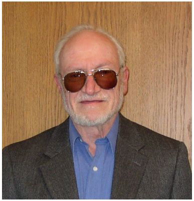
- 

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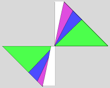

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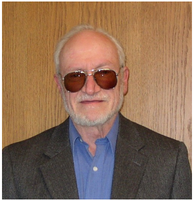
- 

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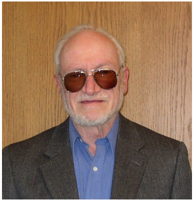
- 

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- θ_j is known to exist for $1 \leq j \leq 5$, i.e., in dimensions 2, 6, 14, 30 and 62.

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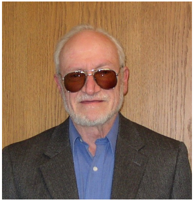
- 

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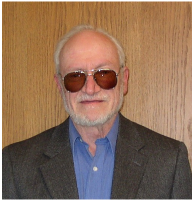
- 

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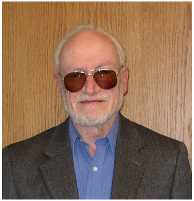
- 

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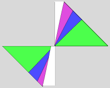
Adams spectral sequence formulation.



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Adams spectral sequence formulation. We now know that the h_j^2 for $j \geq 7$ are not permanent cycles, so they have to support nontrivial differentials.

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Unstable homotopy theoretic formulation.

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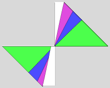
Our method of proof offers a new tool, the slice spectral sequence, for studying the stable homotopy groups of spheres.



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Our proof has several ingredients.

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Our proof has several ingredients.

- We use methods of stable homotopy theory, which means we use spectra instead of topological spaces.

Mike Hill
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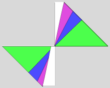

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  In particular, recall that a space X has a homotopy group $\pi_k(X)$ for each positive integer k.

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Ingredients of the proof (continued)

More ingredients of our proof:

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Ingredients of the proof (continued)

More ingredients of our proof:

- We use complex cobordism theory.

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Mike Hopkins
Doug Ravenel

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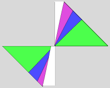
- We use complex cobordism theory. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory.



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Ingredients of the proof (continued)

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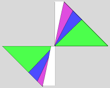
- We use complex cobordism theory. This is a branch of algebraic topology having deep connections with algebraic geometry and number theory. It includes some highly developed computational techniques that began with work by Milnor, Novikov and Quillen in the 60s.



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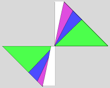
John Milnor

Sergei Novikov

Dan Quillen



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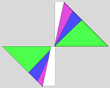
Ingredients of the proof (continued)

More ingredients of our proof:

- We also make use of newer less familiar methods from equivariant stable homotopy theory.



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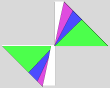
More ingredients of our proof:

- We also make use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group G (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers $\mathbf{Z}$, but by $RO(G)$, the real representation ring of G.



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Ingredients of the proof (continued)

More ingredients of our proof:

- We also make use of newer less familiar methods from equivariant stable homotopy theory. This means there is a finite group G (a cyclic 2-group) acting on all spaces in sight, and all maps are required to commute with these actions. When we pass to spectra, we get homotopy groups indexed not just by the integers **Z**, but by $RO(G)$, the real representation ring of G. Our calculations make use of this richer structure.

Peter May

John Greenlees

Gaunce Lewis 1949-2006

The spectrum Ω

We will produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

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The spectrum Ω

We will produce a map $S^0 \to \Omega$, where Ω is a nonconnective spectrum (meaning that it has nontrivial homotopy groups in arbitrarily large negative dimensions) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial.

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Here again are the properties of Ω

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The spectrum Ω (continued)

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(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

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The spectrum Ω (continued)

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(ii) and (iii) imply that $\pi_{254}(\Omega) = 0$.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist.

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The spectrum Ω (continued)

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If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

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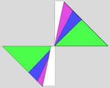
How we construct Ω

Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.



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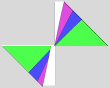
Our spectrum Ω will be the fixed point spectrum for the action of C_8 (the cyclic group of order 8) on an equivariant spectrum $\tilde{\Omega}$.

To construct it we start with the complex cobordism spectrum MU.



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To construct it we start with the complex cobordism spectrum MU. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers.

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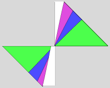

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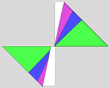
To construct it we start with the complex cobordism spectrum *MU*. It can be thought of as the set of complex points of an algebraic variety defined over the real numbers. This means that it has an action of C_2 defined by complex conjugation. The fixed point set of this action is the set of real points, known to topologists as *MO*, the unoriented cobordism spectrum.



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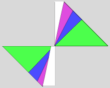
Some people who have studied MU as a C_2-spectrum:

Peter Landweber



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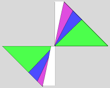
How we construct Ω (continued)

Some people who have studied MU as a C_2-spectrum:

Peter Landweber

Shoro Araki
1930–2005



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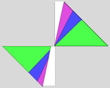

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How we construct Ω (continued)

Some people who have studied MU as a C_2-spectrum:

Peter Landweber

Igor Kriz and Po Hu

Shoro Araki
1930–2005

Nitu Kitchloo

Steve Wilson

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How we construct Ω (continued)

To get a C_8-spectrum, we use the following general construction for getting from a space or spectrum X acted on by a group H to one acted on by a larger group G containing H as a subgroup.

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$$Y = \mathrm{Map}_H(G, X),$$

the space (or spectrum) of H-equivariant maps from G to X.

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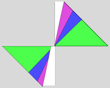

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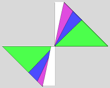

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In particular we get a C_8-spectrum

$$MU^{(4)} = \mathrm{Map}_{C_2}(C_8, MU).$$

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In particular we get a C_8-spectrum

$$MU^{(4)} = \mathrm{Map}_{C_2}(C_8, MU).$$

This spectrum is not periodic, but it has a close relative $\tilde{\Omega}$ which is.

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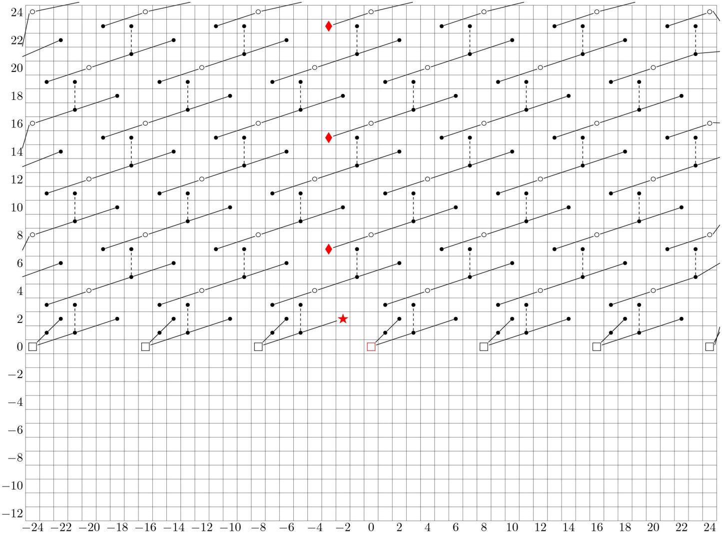
A homotopy fixed point spectral sequence



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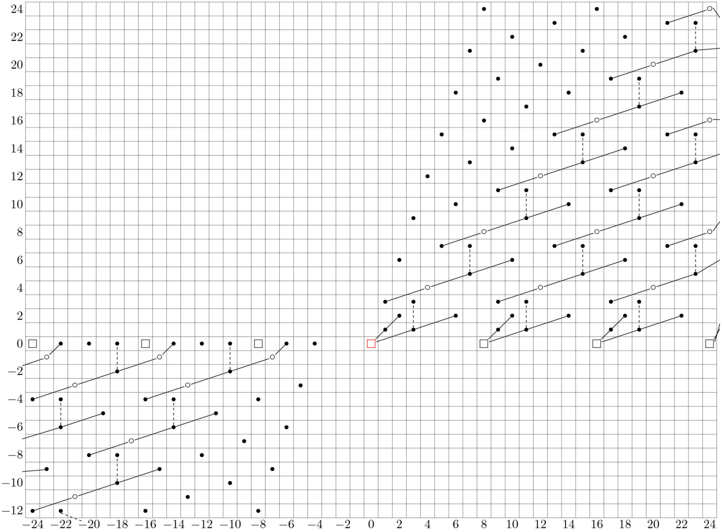

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