A NOVICE'S GUIDE TO THE ADAMS-NOVIKOV SPECTRAL SEQUENCE

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Ever since its introduction by J. F. Adams [8] in 1958, the spectral sequence that bears his name has been a source of fascination to homotopy theorists. By glancing at a table of its structure in low dimensions (such have been published in [7], [10] and [27]; one can also be found in §2) one sees not only the values of but the structural relations among the corresponding stable homotopy groups of spheres. It cannot be denied that the determination of the latter is one of the central problems of algebraic topology. It is equally clear that the Adams spectral sequence and its variants provide us with a very powerful systematic approach to this question.

The Adams spectral sequence in its original form is a device for converting algebraic information coming from the Steenrod algebra into geometric information, namely the structure of the stable homotopy groups of spheres. In 1967 Novikov [44] introduced an analogous spectral sequence (formally known now as the Adams-Novikov spectral sequence, and informally as simply the Novikov spectral sequence) whose input is algebraic information coming from MU^*MU, the algebra of cohomology operations of complex cobordism theory (regarded as a generalized cohomology theory (see [2])). This new spectral sequence is formally similar to the classical one. In both cases, the E_2-term is computable (at least in principle) by purely algebraic methods and the E_∞-term is the bigraded object associated to some filtration of the stable homotopy groups of spheres (the filtrations are <u>not</u> the same for the

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two spectral sequences). However, it became immediately apparent, for odd primes at least, that the Novikov spectral sequence has some striking advantages. Its E_2-term is smaller and there are fewer differentials, i.e. the Novikov E_2-term provides a better approximation to stable homotopy than the Adams E_2-term. Most of the groups in the former are trivial for trivial reasons (the sparseness phenomenon to be described in Corollary 3.17) and this fact places severe restrictions on when nontrivial differentials can occur. It implies for example that $E_2^{**} = E_{2p-1}^{**}$. For $p = 3$, the entire Novikov spectral sequence through dimension 80 can be legibly displayed on a single page (hopefully this will be done in [52]; see [75] for a table through dimension 45), whereas the Adams spectral sequence through a comparable range requires 4 pages (see [36]).

In the Adams spectral sequence for $p > 2$, the first nontrivial differential originates in dimension $pq - 1$ (where $q = 2p - 2$) and is related to the odd primary analogue of the nonexistence of elements of Hopf invariant one (see §2). The latter result is, in the context of the Novikov spectral sequence (even for $p = 2$), a corollary of the structure of the 1-line $E_2^{1,*}$, which is isomorphic to the image of the J-homomorphism (see [1]).

In the Novikov spectral sequence for $p > 2$, the first nontrivial differential does not occur until dimension $p^2q - 2$ and is a consequence of Toda's important relation in stable homotopy $\alpha_1 \beta_1^p = 0$ (see [70], [71] and [56]). An analogous differential occurs in the Adams spectral sequence as well.

The situation at the prime 2 is quite different. At first glance (see Zahler's table in [75]) the Novikov spectral sequence appears to be less efficient than the Adams spectral sequence. The first nontrivial differential in the former originates in dimension 5 whereas the first nontrivial Adams differential does not originate until dimension 15. In looking at Zahler's table one is struck by the abundance of differentials, and also by the nontrivial group extensions occuring in dimensions 3 and 11 (the table stops at dimension 17).

These apparent drawbacks have been responsible for public apathy toward the 2-primary Novikov spectral sequence up until now. An object of this paper, besides providing a general introduction to the subject, is to convince the reader that the Novikov spectral sequence at the prime 2 is a potentially powerful (and almost totally untested) tool for hacking one's way through the jungles of stable homotopy. In particular in §7 we will show how it can be used to detect some interesting new families of elements recently constructed by Mahowald.

The plan of the rest of the paper is as follows:

In §2, we will discuss the classical Adams spectral sequence and some of the questions it raised about the stable homotopy.

In §3, we will set up the Novikov spectral sequence.

In §4, we will discuss the relation between the two spectral sequences and show how comparing the two E_2-terms for $p = 2$ leads to a complete determination of stable homotopy through dimension 17.

In §5, we discuss what we call 'first order' phenomena in the Novikov spectral sequence, i.e. we show how it detects the image of the J-homomorphism and related elements.

In §6 and §7, we discuss second order phenomena, i.e. certain possible new families of homotopy elements which are difficult if not impossible even to conceive of without the Novikov spectral sequence.

In §8, we will discuss some recent theoretical developments which have led to some unexpected insights into the nature of stable homotopy and (most interestingly) the relation between it and algebraic number theory. In other words, we will discuss the theory of Morava stabilizer algebras and the chromatic spectral sequence, in hopes of persuading more people to read (or at least believe) [37], [58], [51] and [39].

I have tried to write this paper in the expository spirit of the talk given at the conference. Naturally, I have expanded the lecture considerably in order to make the paper more comprehensive and useful to someone wishing to begin research in this promising area. At two points however, I have been unable to resist giving some fairly detailed proofs which have not appeared (and probably will not appear) elsewhere. In §5, you will find a new partial proof of Theorem 5.8, which describes the image of the J-homomorphism and related phenomena at the prime 2. The proof uses techniqes which can be generalized to higher order phenomena (such as those described in §6 and §7) and it makes no use of the J-homomorphism itself. In §7 are derivations of some consequences of certain hypotheses concerning the Arf invariant elements and Mahowald's η_j's.

I am painfully aware of the esoteric nature of this subject and of the difficulties faced by anyone in the past who wanted to become familiar with it. I hope that this introduction will make the subject more accessible and that there will be greater activity in what appears to be a very fertile field of research.

The E_2-term can be written either as $\mathrm{Ext}_A^{**}(\mathbb{F}_p, \mathbb{F}_p)$ (Ext in the category of A-modules) or $\mathrm{Ext}_{A_*}^{**}(\mathbb{F}_p, \mathbb{F}_p)$ (Ext in the category A_*-comodules). The distinction here is didactic, but in the case $E = BP$ (the Novikov spectral sequence) the formulation in terms of comodules leads to a substantial simplification.

The identification of the E_2-term can be carried out for general E provided that E is a ring spectrum and $E \wedge E$ is a wedge of suspensions of E. This is the case when $E = MU$, BP or MSp, but not if $E = bo$ or bu. (For the homotopy type of $E \wedge E$ in these two cases, see [35] and [6] §III 17 respectively.)

We now specialize to the case $p = 2$. Table 1, which displays the behavior of the spectral sequence through dimension 19 is provided for the reader's amusement. Before commenting on it, we will discuss $\mathrm{Ext}_A^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$, the Adams "1-line".

Proposition 2.5

$$\mathrm{Ext}_A^{1,t}(\mathbb{F}_2, \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } t = 2^i \\ 0 & \text{otherwise} \end{cases}$$

The generator of $\mathrm{Ext}_A^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ is denoted by h_i and represented by $\xi_1^{2^i}$ in the cobar complex (2.3).

Proof. In (2.3), there are no coboundaries in $\bar{A}_*$, so all cocylces in that group are nontrivial. An element is a cocycle iff its image in A_* is primitive, i.e. if it is dual to a generator of A. A is generated by the elements Sq^{2^i} [66], so the result follows. □

The first 4 of these generators detect well-known elements in stable homotopy: h_o detects 2ι, where ι generates the zero stem, while h_1, h_2, and h_3 detect the suspensions of the 3 Hopf fibrations $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$ respectively.

§2. The Classical Adams Spectral Sequence

In this section, we discuss the outstanding features of the classical mod 2 Adams spectral sequence. Readers who are already knowledgeable in this area will lose very little by skipping this section.

A general formulation of the Adams spectral sequence is the following. We have a diagram of spectra

(2.1)
$$\begin{array}{ccccccccc} X = X_o & \leftarrow & X_1 & \leftarrow & X_2 & \leftarrow & X_3 & \leftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ Y_o & & Y_1 & & Y_2 & & Y_3 & & \end{array}$$

where $X_{s+1} \to X_s \to Y_s$ is a cofibration for each s. Then from the theory of exact couples (see [7]) we have

Theorem 2.2 Associated to the diagram (2.1) there is a spectral sequence $\{E_r^{s,t}\}$ with differentials $d_r: E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that:

(a) $E_1^{s,t} = \pi_{t-s} Y_s$;

(b) $d_1: E_1^{s,t} \to E_1^{s+1,t}$ is induced by the composite
$Y_s \to \Sigma X_{s+1} \to \Sigma Y_{s+1}$;

(c) the spectral sequence converges to $\pi_* \tilde{X}$ where $\tilde{X}$ is the cofibre of $\varprojlim X_i \to X \to \tilde{X}$. □

The diagram (2.1) is called an Adams resolution if $\varprojlim X_i$ is weakly contractible after localizing at some prime p. In this case, the spectral sequence will converge to the p-localization of $\pi_* X$.

Needless to say, the spectral sequence is useful only if one knows $\pi_* Y_s$. This is often the case if we set $Y_s = X_s \wedge E$, where E is the representing spectrum for some familiar homology theory, such as ordinary mod p homology theory. In that case, we have the E_*-homology Adams spectral sequence for $\pi_* X$. For a more detailed discussion, see [6] §III 15. The case $E = MU$ or BP is that of the Novikov spectral sequence.

If X is connective and $E = H\mathbb{F}_p$ (the mod p Eilenberg-MacLane spectrum) or BP (the Brown-Peterson spectrum), then $\tilde{X}$ is the p-adic completion of X or the p-localization of X respectively (see [11] or [12]). If either X or E fail to be connective (e.g. if E is the spectrum representing K-theory) then the relation between X and $\tilde{X}$ (which Bousfield calls the E-nilpotent completion of X) is far from obvious.

Theorem 2.2 yields the classical mod p Adams spectral sequence if we set $X = S^o$, $E = H\mathbb{F}_p$, and $Y_s = X_s \wedge E$. If we denote X_1 by $\bar{E}$, we have $X_s = \bar{E}^{(s)}$ (the s-fold smash product of $\bar{E}$ with itself) and $Y_s = E \wedge \bar{E}^{(s)}$ for $s > 0$. It follows that each Y_s is a wedge of mod p Eilenberg-MacLane spectra and that for $s > 0$, $\pi_* \Sigma^{-s} Y_s = \bar{A}_*^{\otimes s}$ where $\bar{A}_*$ is the agumentation ideal of the dual mod p steenrod algebra A_*. One can show further that the Adams E_1-term in this case is isomorphic to the normalized cobar complex

$$(2.3) \qquad \mathbb{F}_p \xrightarrow{\delta_o} \bar{A}_* \xrightarrow{\delta_1} \bar{A}_* \otimes \bar{A}_* \xrightarrow{\delta_2} \bar{A}_* \otimes \bar{A}_* \otimes \bar{A}_* \xrightarrow{\delta_3} \cdots$$

that one uses to compute the cohomology of the Steenrod algebra. Specifically, we have

$$\delta_s(a_1 \otimes a_2 \cdots \otimes a_s) = \sum_{i=1}^{s} (-1)^i \; a_i \otimes \cdots a_{i-1} \otimes \Delta \, (a_i) \otimes a_{i+1} \cdots a_s$$

where $a_i \in \bar{A}^*$ and $\Delta \colon \bar{A}_* \to \bar{A}_* \otimes \bar{A}_*$ is the coproduct. In this way, we arrive at Adams' celebrated original theorem.

<u>Theorem 2.4</u> (Adams [8]). There is a spectral sequence converging to the p-component of $\pi_* S^o$ with $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p)$, where A is the mod p Steenrod algebra. □

(These elements are customarily denoted by η, ν and σ respectively.)

The question then arises as to whether h_i for $i > 3$ is a permanent cycle in the spectral sequence and therefore detects a homotopy element. This question has some interesting implications.

Theorem 2.6 The following statements are equivalent:

(a) h_i is a permanent cycle in the Adams spectral sequence.

(b) There is a 2-cell complex $X = S^n \cup e^{n+2^i}$ such that Sq^{2^i} is nontrival in $H^*(X; \mathbb{F}_2)$.

(c) $\mathbb{R}^{2^i}$ can be made into a division algebra over $\mathbb{R}$.

(d) S^{2^i-1} is parallelizable. □

A proof can be found in [4].

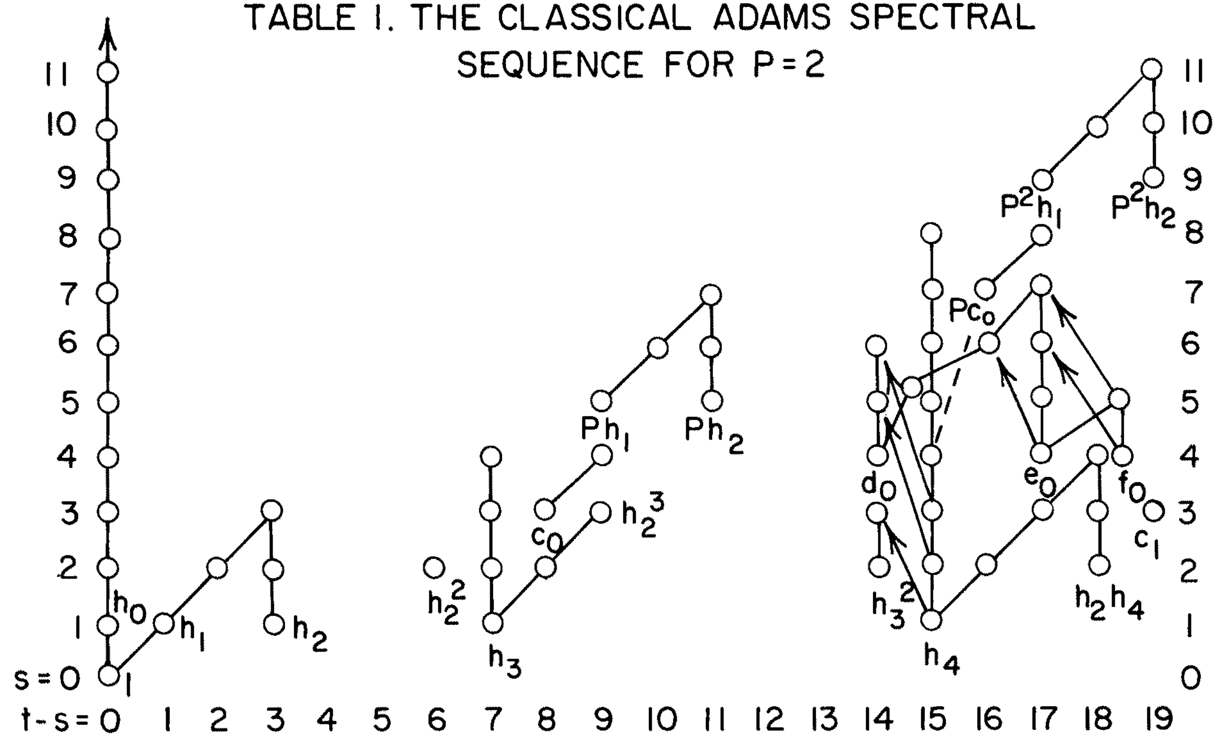
In one of the more glorious moments of algebraic topology, Adams answered the question in the following spectacular way.

Theorem 2.7 (Adams [4]). For $i > 3$, h_i is not a permanent cycle in the Adams spectral sequence. More precisely, $d_2\ h_i = h_0 h_{i-1}^2 \neq 0$. □

We now comment on Table 1. A similar table showing $E_2^{s,t}$ for $t - s \leq 70$ (but not showing any differentials) can be found in [67], where the method for computing it developed by May [32] [33] is discussed. Differentials up to $t - s = 45$ have been computed and published in [10] and [31].

The vertical axis s is filtration or cohomological degree. The horizontal axis is $t - s$, so all elements in the same topological dimension will have the same horizontal co-ordinate. Each small circle represents a basis element of the vector space $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$. When a space is empty, the corresponding vector space is trivial. $\mathrm{Ext}_A^{**}(\mathbb{F}_2, \mathbb{F}_2)$ has a commutative algebra structure, as does $E_r^{s,t}$ for $r > 2$, and the differentials are

TABLE I. THE CLASSICAL ADAMS SPECTRAL SEQUENCE FOR P = 2



HOMOTOPY GROUP $\pi_{t-s}S^{\circ}$

t − s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\pi_{t-s}S^{\circ}$	Z	$\frac{Z}{2}$	$\frac{Z}{2}$	$\frac{Z}{8}$	0	0	$\frac{Z}{2}$	$\frac{Z}{16}$	$\frac{2Z}{2}$	$\frac{3Z}{2}$	$\frac{Z}{2}$	$\frac{Z}{8}$	0	0	$\frac{2Z}{2}$	$\frac{Z}{32}\oplus\frac{Z}{2}$	$\frac{2Z}{2}$	$\frac{4Z}{2}$	$\frac{Z}{8}\oplus\frac{Z}{2}$	$\frac{Z}{8}\oplus\frac{Z}{2}$

derivations, i.e. the spectral sequence is one of commutative algebras. Hence many of the elements of $E_2^{s,t}$ are products of elements in lower filtration (i.e. lower values of s). The vertical lines represent multiplication by h_o (all powers of h_o are nontrivial), and the solid diagonal lines going up and to the right indicate multiplication by h_1. Certain multiplicative relations are built into the table, e.g. $h_1^3 = h_o^2 h_2$, $h_1^3 d_o = h_o^3 e_o$, $h_2^3 = h_1^2 h_3$, etc. Differentials in the spectral sequence are indicated by solid arrows going up and to the left, e.g. $d_2 h_4 = h_o h_3^2$ (by Theorem 2.7) and $d_3 h_o h_4 = h_o d_o$. The broken line going from $h_o^3 h_4$ to Pc_o indicates a nontrivial extension in the multiplicative structure, i.e. if ρ is the element of $\pi_{15} S^o$ detected by $h_o^3 h_4$, then $\eta\rho \in \pi_{16} S^o$ is detected by Pc_o. The elements which are not products of h_i's can be expressed as Massey products (see [34]), e.g. $c_o = \langle h_1, h_o, h_2^2\rangle$, $c_1 = \langle h_2, h_1, h_3^2\rangle$, $d_o = \langle h_o, h_2^2, h_o, h_2^2\rangle$, and $f_o = \langle h_o^2, h_3^2, h_2\rangle$. The letter P denotes a periodicity operator $P\colon E_2^{s,t} \cap \ker h_o^4 \to E_2^{s+4,t+12}$ (see [5]), $Px = \langle x, h_o^4, h_3\rangle$. (In particular, $Ph_3 = h_o^3 h_4$ and $Ph_2^2 = h_o^2 d_o$.) Its analogue in the Novikov spectral sequence will be discussed in some detail in §5.

The corresponding homotopy groups are listed on the lower part of the table. They can be read off from the spectral sequence with the help of

<u>Proposition 2.8</u> If a and $h_o a$ are nonzero permanent cycles in the Adams spectral sequence then the homotopy element detected by the latter is twice that detected by the former, i.e. multiplication by h_o in $E_2^{s,t}$ corresponds to multiplication by 2 in $\pi_* S^o$.

<u>Proof</u>. The statement is certainly true in dimension zero since we know $\pi_o S^o = Z$ and $\mathrm{Ext}_A^{*,*-*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_o]$. The statement in higher dimensions follows from the multiplicative properties of the spectral sequence. □

Note that all differentials in the spectral sequence decrease $t - s$ by 1 and increase s by at least 2. Hence elements that are low enough in the table cannot be targets of nontrivial differentials, while those that are high enough cannot be sources of same. In §4, we will show how all differentials and group extensions through dimension 17 can be determined by comparing the Adams and Novikov E_2-terms.

We conclude this section with a discussion of the Adams 2-line.

<u>Theorem 2.9</u> (Adams [4]) $\mathrm{Ext}_A^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$ has as a basis the elements $h_i h_j$ with $i \le j$ and $i \ne j - 1$. □

As in the case of the 1-line, we can ask what happens to these elements in the spectral sequence. There is no possibility of any of them being the target of a differential, as such a differential would have to originate on the 0-line, which is trivial in positive dimensions. Hence any of these elements which is a permanent cycle will detect a nontrivial homotopy element. In the range of our table, we see that all such elements except $h_o h_4$ are permanent cycles. A big step forward in answering this question is

<u>Theorem 2.10</u> (Mahowald-Tangora [30]) With the exceptions $h_o h_2$, $h_o h_3$, $h_2 h_4$ and possibly $h_2 h_5$ and $h_3 h_6$, the only elements of $\mathrm{Ext}_A^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$ which can possibly be permanent cycles are h_j^2 and $h_1 h_j$. □

The elements h_j^2 are commonly known as the Arf (or Kervaire) invariant elements, due to the following result.

<u>Theorem 2.11</u> (Browder [13]) There is a framed $(2^{j+1} - 2)$-manifold with nontrival Kervaire (or Arf) invariant iff the element h_j^2 is a permanent cycle in the Adams spectral sequence. □

These elements are known to survive (i.e. to be permanent cycles) for $j \leq 5$. They have been the object of intense investigation by Barratt, Mahowald, and others (see [29]). The corresponding homotopy element is commonly known as θ_j. Barratt and Mahowald have privately expressed the belief that if θ_j can be shown to exist for all of stable homotopy will follow with relative ease.

The survival of the elements $h_i h_j$ is closely related to that of h_j^2. If θ_j exists and $2\theta_j = 0$, then the Toda bracket $\eta_{j+1} = \langle \theta_j, 2, \eta \rangle$ is detected by $h_1 h_{j+1}$.

Mahowald has recently devised an extremely ingenious construction to prove

Theorem 2.12 (Mahowald [28]) The element $\eta_j \in \pi_{2^j} S^0$ exists for all $j \geq 3$ i.e. $h_1 h_j$ is a permanent cycle. []

In §7, we will indicate how θ_j and η_j appear in the Novikov spectral sequence and how the latter produces a new family of homotopy elements.

A computation of the Adams 3-line can be found in [74].

§3 Setting up the Novikov Spectral Sequence

The Novikov spectral sequence for the p-localization of the stable homotopy groups of spheres is obtained from Theorem 2.2 by setting $X = S^o$ and $Y_s = X_s \wedge BP$, where BP is the Brown-Peterson spectrum. If we replace BP by MU (the Thom spectrum associated with the unitary group; its homotopy is the complex cobordism ring) we obtain a 'global' Novikov spectral sequence which converges to all of $\pi_* S^o$, not just the p-component. Novikov [44] knew that the p-localizaton of the MU spectral sequence is isomorphic to the BP spectral sequence but he did not know how to compute with the latter. In either case, the identification of the E_2-term is as in the classical case and we have

Theorem 3.1 (Novikov [44], Adams [6] §III 15) There are spectral sequences

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{MU_*MU}(MU_*, MU_*) \Rightarrow \pi_* S^o$$

and

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow (\pi_* S^o)_{(p)}. \quad \square$$

Novikov [44] and Zahler [75] used MU^*MU instead of MU_*MU. Since the former is not of finite type, this approach leads to certain technical difficulties, as one can see by reading [75].

In order to make Theorem 3.1 more explicit (see Proposition 3.16), we will describe BP_*BP. MU_*MU was determined by Novikov [44], and BP_*BP by Quillen [50]. Both are described lucidly by Adams [6], §II 16. An illuminating functorial description of BP_*BP has been given by Landweber [22], and will be discussed briefly in §8.

The structure of MU_*MU is easier to describe than that of BP_*BP. Nevertheless, the latter object, being much smaller, is easier to compute with, even if it takes some time to convince oneself of this. (It took me about four years.)

It would be a disservice to the reader not to begin the description of BP_*BP with a brief discussion of formal group laws. It is safe to say that every major conceptual advance in this subject since Quillen's work [50] has been connected directly or indirectly with the theory of formal group laws.

Definition 3.2 A one dimensional commutative formal group law over a commutative ring with unit R (hereafter and herebefore referred to simply as a formal group law) is a power series $F(x, y) \in R[[x, y]]$ such that

(i) $F(0, x) = F(x, 0) = x$ (identity element)

(ii) $F(x, y) = F(y, x)$ (commutativity)

(iii) $F(F(x, y), z) = F(x, F(y, z))$ (associativity).

Examples 3.3

(i) $F(x, y) = x + y$, the additive formal group law.

(ii) $F(x, y) = x + y + xy$, the multiplicative formal group law so named because $1 + F(x, y) = (1 + x)(1 + y)$.

(iii) $F(x, y) = \dfrac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1 + x^2 y^2}$

This is a formal group law over $Z[1/2]$, originally discovered by Euler in his investigation of elliptic integrals (see [62] pp. 1-9).

There are notions of homomorphisms and isomorphisms of formal groups over R, whose easy definitions we leave to the reader. A comprehensive and down-to-earth account of the theory of formal group laws has recently been provided by Hazewinkel [18]. Fröhlich's book [17] is also useful.

There is also a change of rings homomorphism. If $F(x, y)$ is a formal group law over R and $\theta: R \to S$ is a ring homomorphism, then $\theta(F(x, y))$ is a formal group law over S. With this in mind, we can ask for a universal formal group law $F_U(x, y)$ over a certain ring L (named for its discoverer, Lazard [25]) such that for any formal group law F over any ring R there is a unique ring homomorphism $\theta: L \to R$ such that $F(x, y) = \theta(F_U(x, y))$.

The Lazard ring L and the universal formal group law $F_U(x, y)$ are easy to construct. Simply write $F_U(x, y) = \sum a_{i,j} x^i y^i$ and regard the coefficients $a_{i,j}$ as indeterminates and set $L = Z[a_{i,j}]/(\sim)$, where $\sim$ denotes the relations among the $a_{i,j}$ imposed by Definition 3.2. It is obvious then that L and $F_U(x, y)$ have the desired properties. However, it was not easy to determine L explicitly. After Lazard [25] did so, Quillen made the following remarkable observation. Before stating it, recall that $MU^* = Z[x_1, x_2 \cdots]$ where $\dim x_i = -2i$ and $MU^*CP^\infty \simeq MU^*[[t]]$ where $t \in MU^2CP^\infty$ is canonically defined (see [6] §II 2). Then we have

Theorem 3.4 (Quillen [50]). The complex cobordism ring MU^* is isomorphic to the Lazard ring L, and under this isomorphism, $\Delta(t) = F_U(t\otimes 1, 1\otimes t)$, where $\Delta: MU^*CP^\infty \to MU^*(CP^\infty x CP^\infty)$ is the map induced by the tensor product (of complex line bundles) map $CP^\infty x CP^\infty \to CP^\infty$. □

Proofs can be found in [6] §II 8, and [14]. This result establishes an intimate connection between complex cobordism and formal group laws. Most of the advances in the former since 1969 (the date of Quillen's theorem) have ignored complex manifolds entirely. It would be nice in some sense to have a description of the spectrum MU which is rooted entirely in formal group laws and which makes no mention of Thom spectra or complex manifolds. A recent result of Snaith [65] appears to be a step in this direction. Also the results of [59] imply that MU^*CP^∞ as a Hopf algebra actually characterizes MU.

To proceed with the narrative, we have

Proposition 3.5 Let $F(x, y)$ be a formal group law over a torsion free ring R and define $f(x) \in R\otimes Q[[x]]$ by

$$f(x) = \int_0^x \frac{dt}{f_2(t, 0)}$$

where

$$f_2(x, y) = \frac{\partial F}{\partial y} .$$

Then $f(F(x, y)) = f(x) + f(y)$, i.e. $f(x)$ is an isomorphism over $R\otimes Q$ between F and the additive formal group law. □

Definition 3.6 The power series $f(x)$ above is the logarithm $\log_F(x)$ of the formal group law F.

The word logarithm is used because in the case of the multiplicative formal group law (Example 3.3(ii)),

$$\log_F x = \log(1 + x) = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n}$$

Theorem 3.7 (Mischenko [44]) $\log_{F_U} x = \sum_{n\geq 0} [CP^n] \frac{x^{n+1}}{n+1}$, where $[CP^n]$ denotes the element of MU^* represented by the complex manifold CP^n. □

Definition 3.8 A formal group law F over a torsion free ring is p-typical if $\log_F x = \sum_{i\geq 0} \ell_i x^{p^i}$.

This definition is due to Cartier [16] and can be generalized to rings with torsion (e.g. finite fields). See [18], [23] or [9].

Theorem 3.9 (Cartier [16]) Every formal group law F over a torsion free $Z_{(p)}$-algebra R is canonically isomorphic to a p-typical formal group law F_T, such that if $\log_F x = \sum a_i x^i$, then $\log_{F_T} x = \sum a_{p^i} x^{p^i}$. □

Proofs can be found in [18], [23] and [9]. It is possible to define a universal p-typical formal group law $F_T(x, y)$ over a p-typical Lazard ring L_T and we have

Theorem 3.10 (Quillen [50]). The p-typical Lazard ring L_T is isomorphic to the Brown-Peterson coefficient ring $BP_* = Z_{(p)}[v_1, v_2 \cdots]$ where $\dim v_n = 2(p^n - 1)$ in such a way that in $BP^*(CP^\infty \times CP^\infty)$, $\Delta(t) = F_T(t\otimes 1, 1\otimes t)$, where Δ is as in Theorem 3.4. □

Quillen's proof translates Cartier's canonical isomorphism (theorem 3.9) into a canonical retraction λ of $MU_{(p)}$ (the localization of MU at the prime p) onto BP. Its action on homotopy is determined by

$$\lambda_*[CP^n] = \begin{cases} [CP^n] & \text{if } n = p^i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\ell(x) = \sum_{i\geq 0} [CP^{p^i-1}] \frac{x^{p^i}}{p^i}$ is the logarithm for the universal p-typical formal group law. We let $\ell_i = [CP^{p^i-1}]/p^i \in BP_*\otimes Q$, so $\ell(x) = \sum \ell_i x^{p^i}$, and $BP_*\otimes Q = Q[\ell_i]$. We define the formal sum $\sum^F x_i$ by $\ell(\sum^F x_i) = \sum \ell(x_i)$ or equivalently, $\sum^F x_i = F(x_1, F(x_2, F(x_3, \cdots)\cdots)$.

We are now ready to describe BP_*BP. Since it is $\pi_* BP\wedge BP$, there are two maps $\eta_L, \eta_R: BP_* \to BP_*BP$ (the left and right units) induced by $BP = BP \wedge S^0 \to BP \wedge BP$ and $BP = S^0 \wedge BP \to BP \wedge BP$ respectively. The latter map $\eta_R: \pi_*BP \to BP_*BP$ is the Hurewicz map in BP_*-homology.

Since BP_*BP is the dual of BP^*BP, the algebra of BP^*-cohomology operations, it has a coproduct $\Delta: BP_*BP \to BP_*BP\otimes_{BP_*} BP_*BP$, where the tensor product is with respect to the bimodule structure given by the maps η_R and η_L.

The maps η_R and Δ are related by the commutative diagram

$$(3.11)\quad \begin{array}{ccc} BP_* & \xrightarrow{\eta_R} & BP_*BP = BP_*\otimes_{BP_*} BP_* BP \\ \downarrow{\scriptstyle \eta_R} & & \downarrow{\scriptstyle \eta_R \otimes 1} \\ BP_*BP & \xrightarrow{\Delta} & BP_*BP\otimes_{BP_*} BP_* BP \end{array}$$

Again, we refer the reader to [22] for novel and illuminating interpretation of this structure, which admittedly seems a bit peculiar at first.

Theorem 3.12 (Quillen [50], Adams [6] §II 16). As an algebra, $BP_*BP = BP_*[t_1, t_2, \cdots]$ where $\dim t_i = 2(p^i - 1)$. The structure maps $\Delta\colon BP_*BP \to BP_*BP\otimes_{BP_*} BP_*BP$ and $\eta_L, \eta_R\colon BP \to BP_*BP$ are given by

$$\sum_{i\le 0} \ell(\Delta(t_i)) = \sum_{i,j\ge 0} \ell(t_i \otimes t_j^{p^i})$$

(or equivalently $\sum_{i\ge 0}^F \Delta(t_i) = \sum_{i,j\ge 0}^F t_i\otimes t_j^{p^i})$ where $t_o = 1$, $\eta_L(\ell_n) = \ell_n$, and $\eta_R\, \ell_n = \sum_{0\le i\le n} \ell_i\, t_{n-i}^{p^i}$. $\square$

The reader can easily verify that these formulae satisfy (3.11), and moreover that using (3.11), η_R determines Δ.

Example $\Delta(t_1) = t_1\otimes 1 + 1\otimes t_1$, $\Delta(t_2) = t_2\otimes 1 + t_1\otimes t_1^p + 1\otimes t_2 - \ell_1 \sum_{0<i<p} \binom{p}{i} t_1^i\otimes t_1^{p-i}$, and $\Delta(t_3)$ is too messy to write down in public. Partial simplification of these formulae is achieved in [57] and [54].

We are not finished yet. In order to use Theorem 3.12 in practice, we need to relate the generators ℓ_1 of $BP_*\otimes Q$ to generators v_i of BP_*, which as yet have not been defined.

Theorem 3.13 (Hazewinkel [18], [19]. Generators v_i of BP can be defined recursively by the formula

$$p\ell_n = \sum_{0\le i<n} \ell_i v_{n-i}^{p^i} . \quad \square$$

Example

$$\ell_1 = \frac{v_1}{p}, \qquad \ell_2 = \frac{v_1^{1+p}}{p^2} + \frac{v_2}{p},$$

and

$$\ell_3 = \frac{v_1^{1+p+p^2}}{p^3} + \frac{v_1 v_2^p + v_2 v_1^{p^2}}{p^2} + \frac{v_3}{p}.$$

The following formula for $\eta_R(v_n)$ is useful

Theorem 3.14 [57]

$$\sum_{\substack{i>0\\ j\ge 0}}^{F} v_i t_j^{p^i} \equiv \sum_{\substack{i>0\\ j\ge 0}}^{F} \eta_R(v_i)^{p^j} t_j \mod (p) . \quad \square$$

Example. $\eta_R v_2 \equiv v_2 + v_1 t_1^p - v_1^p t_1 \mod p$.

Araki [9] has defined another set of generators v_i by the slightly different formula $p\ell_n = \sum_{0\le i\le n} \ell_i v_{n-i}^{p^i}$ where $v_o = p$. This gives messier expressions for the ℓ_i, e.g. $\ell_1 = v_1/(p - p^p)$, but the analogue of Theorem 3.14 is true on the nose, not just mod (p). Araki's and Hazewinkel's generators are the same mod (p).

We conclude this section by exhibiting a complex whose cohomology is $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$, i.e. the E_2-term of the Novikov spectral sequence. Let $\Omega^*(BP_*)$ be the complex

$$\Omega^0 BP_* \xrightarrow{d^0} \Omega^1 BP_* \xrightarrow{d^1} \Omega^2 BP_* \xrightarrow{d^2} \Omega^3 BP_* \longrightarrow \cdots$$

where

$$\Omega^s BP_* = BP_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} \cdots \otimes_{BP_*} BP_*BP$$

(s factors of BP_*BP) with

$$\text{(3.15)} \qquad d^s(v \otimes x_1 \otimes \cdots x_s) = \eta_R(v) \otimes x_1 \otimes \cdots x_s$$
$$+ \sum_{i=1}^{s} (-1)^i \, v \otimes x_1 \otimes \cdots x_{i-1} \otimes \Delta(x_i) \otimes x_{i+1} \otimes \cdots x_s$$
$$- (-1)^s v \otimes x_1 \otimes \cdots x_s \otimes 1 .$$

This is the cobar complex for BP_*. Let $\overline{BP_*BP} = (t_1, t_2 \cdots) \subset BP_*BP$ and define the normalized cobar complex $\tilde{\Omega}^s(BP_*)$ by $\tilde{\Omega}^0 BP_* = BP_*$ and $\tilde{\Omega}^s(BP_*) = BP_* \otimes_{BP_*} \overline{BP_*BP} \otimes_{BP_*} \cdots \overline{BP_*BP}$ (with s factors $\overline{BP_*BP}$). Then (3.15) gives $d^s: \tilde{\Omega}^s BP_* \longrightarrow \tilde{\Omega}^{s+1} BP_*$, e.g. $d^0 v = \eta_R(v) - \eta_L(v)$. Then we have

Proposition 3.16

$$H^* \Omega^* BP_* \cong H^* \tilde{\Omega}^* BP_* \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*). \quad \square$$

Corollary 3.17 (Sparseness) In the Novikov spectral sequence $E_r^{s,t} = 0$ if $q \nmid t$, where $q = 2p - 2$, and $E_{qm+2}^{s,t} = E_{qm+q-1}^{s,t}$ for all $m \geq 0$. $\square$

Two systematic methods of computing Ext BP_* through a range of dimensions have been developed [54] [55], and we hope they will be applied soon.

§4 Comparing the Adams and Novikov Spectral Sequences

In this section, we make some general remarks about the relation between the two spectral sequences, and then we make a specific comparison in low dimensions (≤ 17) at $p = 2$.

To begin with, the natural map $BP \to H\mathbb{F}_p$ of spectra induces a map of Adams resolutions (2.1) and hence a spectral sequence homomorphism (i.e. one which commutes with differentials) Φ from the Novikov to the Adams spectral sequence. This implies the following

Proposition 4.1 If a homotopy element is detected in the Adams spectral sequence by an element in $E_\infty^{s,t}$, then it is detected in the Novikov spectral sequence by an element in some $E_\infty^{s',t'}$ with $t' - s' = t - s$ and $s' \leq s$. If a homotopy element is detected in the Novikov spectral sequence by an element in $E_\infty^{s,t}$, then it is detected in the Adams spectral sequence by an element in some $E_\infty^{s',t'}$ with $t' - s' = t - s$ and $s' \geq s$.

This fact, along with knowledge of the behavior of Φ on the 1-line (see §5) leads to an easy proof of the first part of Theorem 2.7 and its odd primary analogue. The latter was first proved by other means by Liulevicius [26] and Shimada-Yamanoshita [61]. In §9 of [39] we calculated the image of Φ on the 2-line for $p > 2$ and thereby proved an odd primary analogue of Theorem 2.10. Our ignorance of the Novikov 2-line (see §6) for $p = 2$ prevented us from giving a similar proof of Theorem 2.10 itself.

Next, we will describe some spectral sequences which indicate a certain relationship between the two E_2-terms. We begin with a Cartan-Eilenberg ([15] p. 349) spectral sequence for the Adams E_2-term. Recall that for $p > 2$ $A_* = \mathbb{F}_p[\xi_1, \xi_2 \cdots] \otimes E(\tau_0, \tau_1 \cdots)$ and for $p = 2$ $A_* = \mathbb{F}_2[\xi_1, \xi_2 \cdots]$ (see [66]). Define extension of Hopf algebras $E \to A \to P$ by $P_* = \mathbb{F}_p[\xi_i]$ for $p > 2$ and $P_* = \mathbb{F}_2[\xi_i^2]$ for $p = 2$.

Theorem 4.2 (a) There is a spectral sequence converging to $\mathrm{Ext}_A(\mathbb{F}_p, \mathbb{F}_p)$ with $E_2^{s,t} = \mathrm{Ext}_P(\mathbb{F}_p, \mathrm{Ext}_E(\mathbb{F}_p, \mathbb{F}_p))$

(b) $\mathrm{Ext}_E(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[a_0, a_1, \cdots]$

where $a_i \in \mathrm{Ext}^{1,2p^i-1}$ is represented by the image of τ_i (or ξ_{i+1} if $p = 2$) in the cobar complex for E.

(c) The spectral sequence collapses from E_2 for $p > 2$.

<u>Proof</u>: (a) is a special case of Theorem XVI 6.1 of [15]. (b) follows from the fact that E is an exterior algebra. For (c) observe that for $p > 2$, we can give A_* a second grading based on the number of τ's which is preserved by both the coproduct and product. (The coproduct does not preserve this grading for $p = 2$). The fact that differentials must respect this grading implies that the spectral sequence collapses. []

Next, we construct the so-called algebraic Novikov spectral sequence ([44], [36]) which converges to the Novikov E_2-term and has itself the same E_2-term (indexed differently) as that of that Cartan-Eilenberg spectral sequence above.

Let $I = (p, v_1, v_2 \cdots) \subset BP_*$. This ideal is independent of the choice of generators v_i. If we filter BP_* by powers of I, the associated bigraded ring $E^0 BP_*$ is isomorphic to $\mathbb{F}_p[a_0, a_1, \cdots]$ where a_i has dimension $2(p^i - 1)$ and filtration 1 and corresponds to the generators v_i (where $v_0 = p$). This filtration can be extended to BP_*BP and to the normalized cobar complex $\tilde{\Omega}^*(BP_*)$. We have $E^0 BP_*BP = E^0 BP_*[t_i]$ and Theorem 3.12 implies that $\Delta t_n = \sum_{0 \le i \le n} t_i \otimes t_{n-i}^{p^i} \in E^0 BP_*BP$. It follows that $BP_*BP/I \cong P_*$ as Hopf algebras. To describe the coboundary operator in $E^0 \tilde{\Omega}^*(BP_*)$, it remains to determine $d^0 = E^0 BP_* \to E^0 \overline{BP_*BP}$. It follows from (3.15) and Theorem 3.14 that $d^0 a_n = \sum_{0 \le i < n} a_i t_{n-i}^{p^i}$. This agrees, via the appropriate isomorphism, with the d_1 in the Cartan-Eilenberg spectral sequence of Theorem 4.2. Combining all these remarks we get

Theorem 4.3 (Novikov [44], Miller [36]) The filtration of $\tilde{\Omega}^{*}BP_{*}$ by powers of I leads to a spectral sequence converging to $\mathrm{Ext}_{BP_{*}BP}(BP_{*}BP_{*})$ whose E_2-term is isomorphic to $\mathrm{Ext}_{P_{*}}(\mathbb{F}_p, \mathbb{F}_p[a_i])$. □

Theorems 4.2 and 4.3 give algebraic spectral sequences having the same E_2-term (up to reindexing) and converging to the Adams and Novikov E_2-terms respectively. For $p > 2$ the former collapses, so in that case the spectral sequence of Theorem 4.3 can be regarded as passing from the Adams E_2-term (reindexed) to the Novikov E_2-term. Presumably (but this has not been proved) differentials in this spectral sequence correspond in some way to differentials in the Adams spectral sequence. For example, one can easily find the Hopf invariant differentials, i.e. those originating on the Adams 1-line, in this manner. Philosophically, Theorems 4.2 and 4.3 imply that for $p > 2$, any information that can be gotten out of the Adams spectral sequence can be obtained more efficiently from the Novikov spectral sequence.

Another way of describing this situation is the following. According to the experts (i.e. M. C. Tangora), all known differentials in the Adams spectral sequence for odd primes are caused by two phenomena. Each is a formal consequence (in some devious way possibly involving Massey products [34]) of either the Hopf invariant differentials or the relations described by Toda in [70] and [71]. In computing the Novikov E_2-term via Theorem 4.3 or any other method one effectively computes all the Hopf invariant differentials in one fell swoop and is left with only the Toda type differentials to contend with. Better yet, for $p = 3$, all known differentials in the Novikov spectral sequence are formal consequences of the first one in dimension 34 (see [52]). One is tempted to conjecture that this is a general phenomenon, i.e. that if one knows the Novikov E_2-term and the first nontrivial differential, then one knows all of the stable homotopy groups of spheres. However, apart from limited empirical evidence, we have no reason to believe in such an optimistic conjecture.

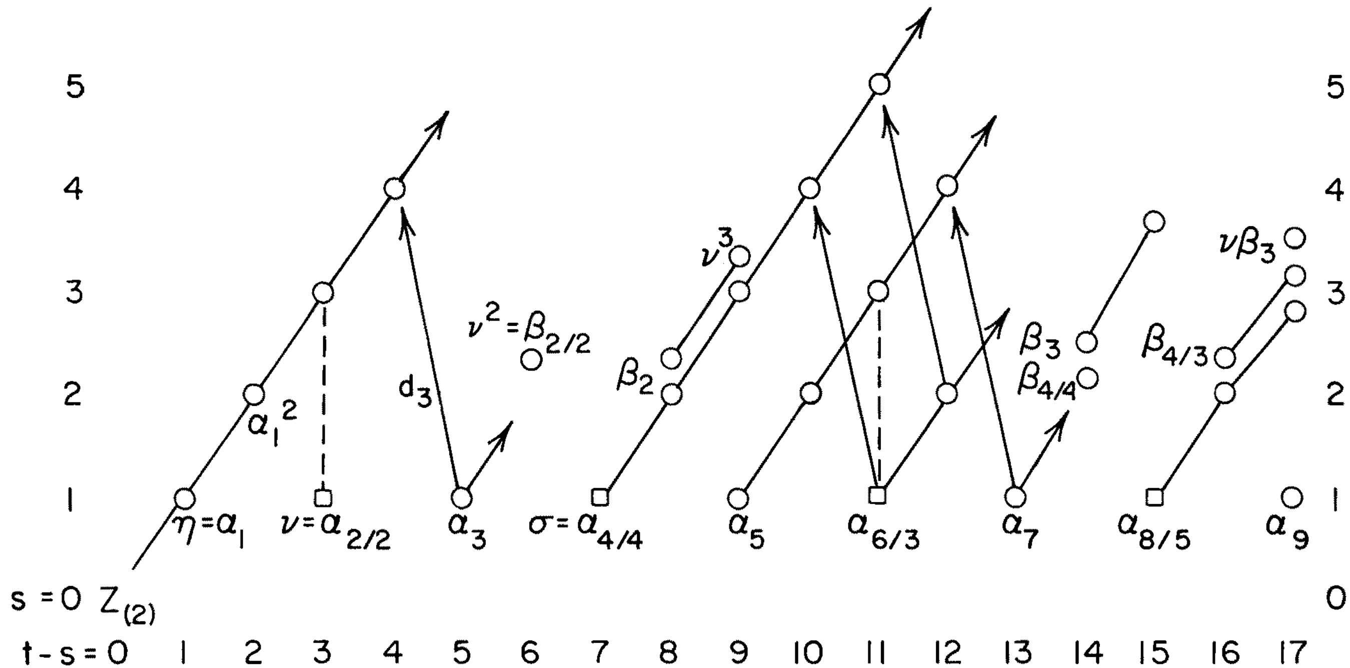
At the prime 2, the relation between the Adams and Novikov E_2-terms is more distant since the spectral sequence of Theorem 4.2 does not collapse. In this case, the Adams spectral sequence does yield some information more readily than the Novikov spectral sequence, and the use of the two spectral sequences in concert provides one with a very powerful tool which has, as yet, no odd primary analogue. We will illustrate by comparing the two through dimension 17, the limit of Zahler's computation [75].

Table 2 is a reproduction of Zahler's table, with the added feature that all elements are named. We will explain this notation in the next two sections. Each $E_2^{s,t}$ is finite except for $E_2^{0,0} = Z_{(2)}$. Each circle in the table represents an element of order 2 and each square represents an element of higher order. Specifically, $\alpha_{2i/j}$ has order 2^j. The diagonal lines going up and to the right indicate multiplication by $\alpha_1 = \eta$, and an arrow pointing in this direction indicates that multiplication by all powers of α_1 is nontrivial. The arrows going up and to the left indicate differentials, and the broken vertical lines indicate nontrivial group extensions.

We now show all the differentials and extensions in the two tables can be deduced by purely algebraic arguments, i.e. without resorting to any geometric considerations.

First, observe that there is no room for any nontrivial differentials in the Adams spectral sequence below dimension 14. (The multiplicative structure precludes nontrivial differentials on h_1 and $h_1 h_3$.) There are also no nontrivial group extensions in this range other than those implied by proposition 2.8. (The fact that $2\eta = 0$ precludes nontrivial extensions in dimensions 8 and 9.) One also knows that $\eta^3\sigma = 0$ because $\eta^3\sigma = \eta\nu^3 = (\eta\nu)\nu^2 = 0$. One can deduce that $\eta^3\sigma = 0$, instead of the element detected by Ph_1^2, by comparing the filtrations of the corresponding elements in the Novikov spectral sequence. The former $\alpha_1{}^3\alpha_{4/4}$ has filtration 4, while the latter, $\alpha_1\alpha_5$, has filtration 2.

TABLE 2. THE NOVIKOV SPECTRAL SEQUENCE FOR P=2



HOMOTOPY GROUP $\pi_{t-s}S^0$

$t-s$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\pi_{t-s}S^0$	Z	$\frac{Z}{2}$	$\frac{Z}{2}$	$\frac{Z}{8}$	0	0	$\frac{Z}{2}$	$\frac{Z}{16}$	$\frac{2Z}{2}$	$\frac{3Z}{2}$	$\frac{Z}{2}$	$\frac{Z}{8}$	0	0	$\frac{2Z}{2}$	$\frac{Z}{32} \oplus \frac{Z}{2}$	$\frac{2Z}{2}$	$\frac{4Z}{2}$

We can use this information to determine the behavior of the Novikov spectral sequence up to dimension 14. The fact that $\pi_3 = Z/8$ implies the nontrivial group extension in dimension 3. The fact that $\pi_4 = \pi_5 = 0$ implies $d_3\alpha_3 = \alpha_1^4$, and that $d_3\alpha_1^t\alpha_3 = \alpha_1^{t+4}$ for all $t \geq 0$. The group extension in dimension 9 is trivial because $2\pi_9 = 0$. The triviality of $\eta^3\sigma$ implies $d_3\alpha_1^t\alpha_{6/3} = \alpha_1^{3+t}\alpha_{4/4}$, and the cyclicity of π_{11} implies a nontrivial group extension in dimension 11. The triviality of π_{12} and π_{13} imply $d_3\alpha_1^t\alpha_7 = \alpha_1^{3+t}\alpha_5$.

In dimensions 14 through 17, the Novikov spectral sequence resolves ambiguities in the Adams spectral sequence as well as _vice versa_. The former now yields $\pi_{14} = 2Z/2$, which forces the Adams differentials $d_2h_4 = h_oh_3^2$ and $d_3h_oh_4 = h_od_o$. The Adams spectral sequence then yields $\pi_{15} = Z/2 \oplus Z/32$, so the group extension 15 of the Novikov spectral sequence is trivial. The latter then shows that η annihilates the elements of order 2 in π_{15}, so $d_2e_o = h_1^2d_o$. On the other hand, η does not annihilate the generator of order 32, so there is the indicated nontrivial multiplicative extension in the Adams spectral sequence. In dimension 17, it can be shown that α_9 and P^2h_1 detect the same element, (see Theorem 5.12) so $2\pi_{17} = 0$ and the Adams elements $h_o^2e_1$ and $h_o^3e_o$ must be hit by differentials. This last fact also follows from the multiplicative structure, i.e. $d_3e_o = h^2d_o$ implies $d_3h_1e_o = h_1^3d_o = h_o^3e_o$, so $d_3f_o = h_oe_o$.

Just how far one can carry this procedure and get away with it is a very tantalizing question. It leads one to the following unsolved, purely algebraic problem: given two Adams type spectral sequences converging to the same thing, find a way to use one of them to get information about the other and _vice versa_. The low dimensional comparison above is based on simplistic, ad hoc arguments which are very unlikely to be strong enough to deal with the more complicated situations which will undoubtedly arise in higher dimensions.

For further discussion of this point, see §7.

§5 First Order Phenomena in the Novikov Spectral Sequence

We will not say exactly what we mean by nth order phenomena until §8. Roughly speaking, first order phenomena consist of Im J and closely related homotopy elements as described by Adams in [1]. The manner in which the Novikov spectral sequence detects these elements was apparently known to Novikov [44] and was sketched by Zahler [75]. Most of the detailed computations necessary were described in §4 of [39] but some of the proofs we present here are new.

We begin by computing the Novikov 1-line. First, we need some notation. For a BP_*BP-comodule M, $\mathrm{Ext}_{BP_*BP}(BP_*, M)$ will be denoted simply by Ext M. If $M = BP_*X$, then Ext M is the E_2-term of the Novikov spectral sequence for π_*X.

Proposition 5.1 If M is a cyclic BP_*-module, Ext M = $H^*(M \otimes_{BP_*} \tilde{\Omega} BP_*)$. □

A proof can be found in §1 of [39].

Now $\mathrm{Ext}^1 BP_*$, the Novikov 1-line, is a torsion group, so we begin by finding the elements of order p. Consider the short exact sequence

$$(5.2) \qquad 0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0.$$

The image of the connecting homomorphism $\delta_0 : \mathrm{Ext}^0 BP_*/(p) \to \mathrm{Ext}^1 BP_*$. is, by elementary arguments, the subgroup of elements of order p. The following result was first published by Landweber [21] and can be derived easily from Theorem 3.14.

Theorem 5.3 Let $I_n = (p, v_1, \cdots v_{n-1}) \subset BP_*$. Then BP_*/I_n is a BP_*BP-comodule and $\mathrm{Ext}^0 BP_*/I_n \cong \mathbb{F}_p[v_n]$. □

Corollary 5.4 $\mathrm{Ext}^0 BP_*/(p) = \mathbb{F}_p[v_1]$ and $\delta_0 v_1^t \equiv \alpha_t \neq 0 \in \mathrm{Ext}^1 BP_*$ for all $t > 0$.

Proof The nontriviality of α_t follows from the long exact sequence in Ext associated with (5.2), in which we have

$$\mathrm{Ext}^0 BP_* \to \mathrm{Ext}^0 BP_*/(p) \xrightarrow{\delta_o} \mathrm{Ext}^1 BP_* \ .$$

In positive dimensions, δ_o is monomorphic because $\mathrm{Ext}^0 BP_*$ is trivial. □

In [75] α_t denotes the generator of $\mathrm{Ext}^{1,2t}\ BP^*$ for $p = 2$, but our α_t is an element of order 2 in that group.

All that remains in computing $\mathrm{Ext}^1 BP_*$, the Novikov 1-line, is determining how many times we can divide α_t by p. From §4 of [39] we have

Theorem 5.5

(a) For $p > 2$, $\alpha_t \in \mathrm{Ext}^{1,qt}\ BP_*$ is divisible by t but not by pt, i.e. $\mathrm{Ext}^{1,qt} BP_* \simeq Z/(p^{1+\nu(t)})$ where $p^{\nu(t)}$ is the largest power of p which divides t.

(b) For $p = 2$, $\alpha_t \in \mathrm{Ext}^{1,2t} BP_*$ is divisible by

$$\begin{cases} t & \text{but not by } 2t \text{ if } t \text{ is odd or } t = 2 \\ 2t & \text{but not by } 4t \text{ if } t \text{ is even and } t > 2; \end{cases}$$

i.e.

$$\mathrm{Ext}^{1,2t} BP_* = \begin{cases} Z/(2) & \text{if } t \text{ is odd} \\ Z/(4) & \text{if } t = 2 \\ Z/(2^{2+\nu(t)}) & \text{if } t \text{ is even and } t > 2. \end{cases}$$ □

It is easy to see that α_t is divisible by $p^{\nu(t)}$. From the fact that $\eta_R v_1 = v_1 + pt_1$ (using Hazewinkel's v_1 (Theorem 3.13) and Theorem 3.12), one computes $\delta_o v_1^t = \frac{1}{p}[(v_1 + pt_1)^t - v_1^t]$ which is easily seen to be divisible by $p^{\nu(t)}$.

We can now explain part of the notation of Table 2. $\alpha_{t/i}$ denotes a certain element (defined precisely in [39]) of order p^i in $\mathrm{Ext}^{1,qt}BP_*$. In particular, $\alpha_{t/1} = \alpha_t$ and $p^{i-1}\alpha_{t/i} = \alpha_t$.

As in §2, one can ask which of these elements are permanent cycles.

Theorem 5.6 (Novikov [44]) For $p > 2$, Im J maps isomorphically to $\mathrm{Ext}^1 BP_*$, i.e. each element of $\mathrm{Ext}^1 BP_*$ is a nontrivial permanent cycle and in homotopy $p\alpha_t = 0$ for all $t > 0$. □

The homotopy elements $\alpha_t \in \pi_{qt-1} S^0$ can also be constructed inductively by Toda brackets, specifically $\alpha_t = \langle \alpha_{t-1}, p\iota, \alpha_1 \rangle$ [72].

As Table 2 indicates, the situation at $p = 2$ is not so simple. Let $x_t \in \mathrm{Ext}^{1,2t}BP_*$ be a generator. Then from [39] §4 we have

Theorem 5.7 For all $s > 0$ and $t \neq 2$, $\alpha_1^s x_t$ generates a nontrivial summand of order 2 in $\mathrm{Ext}^{1+2,2s+2t}BP_*$. □

(This is a consequence of Theorem 5.10 below.) Note that this says that for $t > 2s + 2$ all the groups $E_2^{s,t}$ which are not trivial by sparseness (Corollary 3.17) are in fact nontrivial.

The behavior of these elements in the spectral sequence and in homotopy is as follows.

Theorem 5.8 In the Novikov spectral sequence for $p = 2$

(a) $d_3\, \alpha_1^s\, \alpha_{4t+3} = \alpha_1^{s+3}\, \alpha_{4t+1}$ and $d_3\, \alpha_1^s\, x_{4t+6} = \alpha_1^{3+s}\, x_{4t+4}$ for all $s, t \geq 0$.

(b) For $t > 0$, the elements x_{4t}, $\alpha_1 x_{4t}$, $\alpha_1^2 x_{4t}$, α_{4t+1}, $\alpha_1\alpha_{4t+1}$

$\alpha_1^2\alpha_{4t+1}$ and $2x_{4t+2} = \alpha_{4t+2/2}$ are all nontrivial permanent cycles, as are α_1, α_1^2, α_1^3 and $\alpha_{4/2} = x_2$. In $\pi_* S^0$, we have $2\alpha_{4t} = 2\alpha_{4t+1} = 0$ and $2\alpha_{4t+2} = \alpha_1^2\alpha_{4t+1}$, i.e. there is a nontrivial group extension in dimension $8t + 3$.

(c) The image of the J-homomorphism is the group generated by x_{4t}, $\alpha_1 x_{4t}$, $\alpha_1^2 x_{4t}$ and $\alpha_{4t+2/2}$ (which generates a $Z/8$ summand with $4\alpha_{4t+2/2} = \alpha_1^2\alpha_{4t+1}$).

This result says that the following pattern occurs in the Novikov E_∞-term as a direct summand for all $k > 0$.

$s \uparrow$					
3			$\alpha_1^2 x_{4k}$		$\alpha_1^2\alpha_{4k+1}$
2		$\alpha_1 x_{4k}$		$\alpha_1\alpha_{4k+1}$	⋮
1	x_{4k}		α_{4k+1}		$\alpha_{4k+2/2}$
0					
	$8k-1$	$8k$	$8k+1$	$8k+2$	$8k+3$

$t - s \rightarrow$

where all elements have order 2 except $\alpha_{k+2/2}$ which has order 4 and x_{4k} which has order $2^{\nu(k)+4}$, and the broken vertical line indicates a nontrivial group extension.

In [27] the elements x_{4t}, α_{4t+1} and $\alpha_{4t+2/2}$ are denoted by ρ_t, μ_t and ξ_t respectively, while Adams [1] denotes α_{4t+1} and $\alpha_1\alpha_{4t+1}$ by μ_{8t+1} and μ_{8t+2} respectively.

Parts (a) and (b) seem to have been known to Novikov [44] as was the fact that Im J maps onto the groups indicated in (c). The fact that this map from Im J is an isomorphism requires the Adams Conjecture [1], [49]. We will prove (a) and a weaker form of (b), namely we will only show that the elements said to have order 16 or less are permanent cycles. Another proof of this fact, based on a comparison of the Adams and Novikov spectral sequences can be derived from Theorem 5.12. The J homomorphism can be used to show that x_{4t} is a permanent cycle.

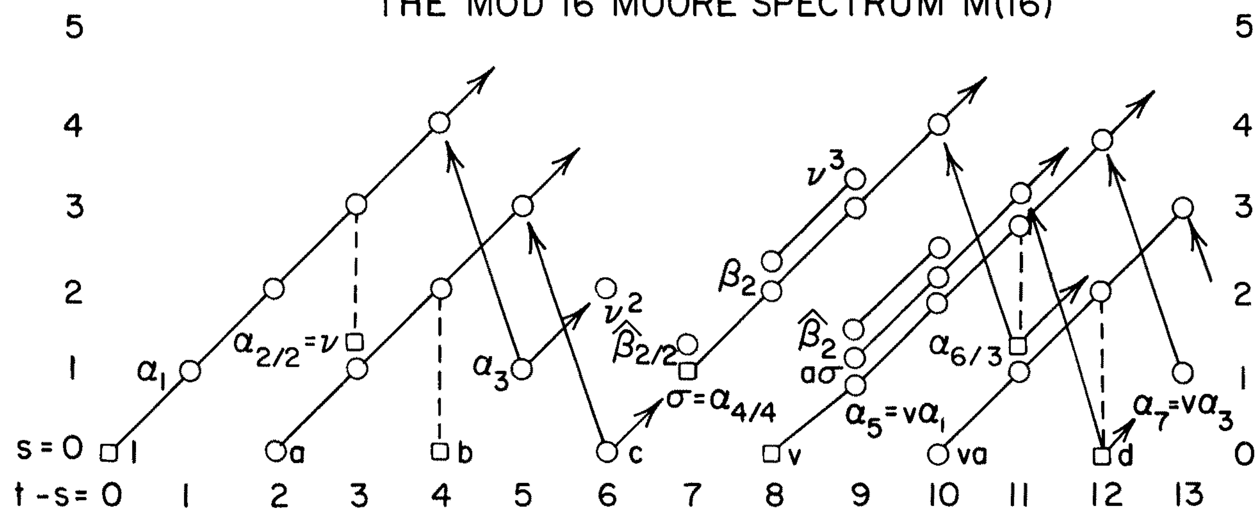
Our proof is based on an analysis of the mod 16 Moore spectrum, which we denote by M(16). As it is somewhat involved, the reader may want to proceed directly to §6.

We begin with Table 3, which displays the Novikov spectral sequence for M(16) through dimension 13. The notation is the same as in Table 2, from which Table 3 can be easily deduced. Circles represent elements or order 2, and squares represent elements of higher order. The orders of 1, ν, b, σ, v, $\alpha_{6/3}$ and d are 16, 4, 4, 16, 16, 8 and 8 respectively. There are various multiplicative relations among these elements, e.g. $2d = vb$, $v\alpha_{2t+1} = \alpha_{2t+5}$, and $v\alpha_{4t+2/3} = \alpha_{4t+6/3}$ which are easy to find.

The element $v \in \mathrm{Ext}^{0,8}BP_*/16$ has the property that $v^t \neq 0$ for all $t > 0$. Since v is a permanent cycle and M(16) is a ring spectrum, nontrivial differentials and group extensions respect multiplication by powers of v. We wish to describe which elements of Ext $BP_*/16$ are not annihilated by any power v, i.e. to describe Ext $BP_*/16$ mod 'v-torsion'. The methods of [39] (also sketched in §8) make this possible. Let $R = (Z/16)[v,\alpha_1]/(2\alpha_1)$. Then we have

Theorem 5.9 In dimensions ≥ 6, Ext $BP_*/16$ mod v-torsion is the R-module generated by c, v, va, d, σ, aσ, $\alpha_{6/3}$ and α_7 with relations $2c = 2va = 8d = 2a\sigma = 8\alpha_{6/3} = 2\alpha_7 = 0$. (More precisely,

TABLE 3. THE NOVIKOV SPECTRAL SEQUENCE FOR THE MOD 16 MOORE SPECTRUM M(16)



$a = 8v_1 \quad b = 4v_1^2 \quad c = 8v_1^3 \quad v = v_1^4 + 8v_1v_2 \quad d = 2v_1^6 + 8v_1^3v_2$

$\hat{\beta}_{2/2} = 8(t_1^4 + v_1^2t_1^2) \quad \hat{\beta}_2 = v_1\hat{\beta}_{2/2}$

this describes the image of Ext $BP_*/16$ in v^{-1} Ext $BP_*/16$ in dimensions ≥ 6. In dimensions 0 through 5 one also has the elements 1, a, ν (note $v\nu = 2\alpha_{6/3}$), b (vb = 2d) and α_3 ($v\alpha_3 = \alpha_7$).) □

This can be deduced from the corresponding statement about Ext $BP_*/2$, namely

Theorem 5.10 Ext $BP_*/2$ modulo v_1-torsion (i.e. the image of Ext $BP_*/2$ in v_1^{-1} Ext $BP_*/2$) is $\mathbb{F}_2[v_1, \alpha_1, \sigma]/(\sigma^2)$. □

The method of proof for this result will be discussed in §8.

In order to relate the behavior of the spectral sequence for M(16) to that for the sphere, we need the Geometric Boundary Theorem.

Theorem 5.11 (Johnson-Miller-Wilson-Zahler [20]) Let $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W$ be a cofibre sequence of finite spectra such that $BP_*(h) = 0$, i.e. such that

$$0 \longrightarrow BP_*W \xrightarrow{f_*} BP_*X \xrightarrow{g_*} BP_*Y \longrightarrow 0$$

is exact, and let $\delta\colon \mathrm{Ext}^*BP_*Y \to \mathrm{Ext}^{*+1}BP_*W$ be the connecting homomorphism. Then if $\bar{x} \in \mathrm{Ext}\, BP_*Y$ is a permanent cycle detecting $x \in \pi_*Y$, then $\delta(\bar{x}) \in \mathrm{Ext}\, BP_*W$ is a permanent cycle detecting $h_*(x) \in \pi_*\Sigma W$. □

Now we can prove Theorem 5.8 a) In $\pi_*M(16)$ we have $\alpha_1^3 a = 0$ so $\delta(x_1^3 v^t a) = \alpha_1^3\alpha_{4t+1} = 0$. Hence by Theorem 5.11, a differential must hit $\alpha_1^3\alpha_{4t+1}$, and by Sparseness (Corollary 3.17) and our knowledge of Ext^1BP_* (Theorem 5.5 (b)), the only possibility $d_3\alpha_{4t+3}$.

For the other family of d_3's, one can show that δ_0 $(a\sigma\alpha_1^2 v^t) = \alpha_1^3 x_{4t+4}$, so by Theorem 5.11, $\alpha_1^3 x_{4t+4}$ must be hit by a differential, and the only possible source is $\alpha_{4t+6/3}$.

For the group extensions in (b), we have $\alpha_1^2 v^t a = 4v^t b$ in $\pi_* M(16)$, so $\delta_o(\alpha_1^2 v^t a) = \alpha_1^2 \alpha_{4t+1}$ detects twice the element by $\delta_o(2v^t b) = \alpha_{4t+2}$. For the permanent cycles of order 16 or less, we have in $\pi_* S^o$, $\alpha_{4t+4/4} = \delta v^t \sigma$, $\alpha_{4t+1} = \delta v^t a$, and $\alpha_{4t+2/2} = \delta v^t \nu$. This concludes the proof of our weakened form of theorem 5.8. □

We draw the reader's attention to the basic idea of the above proof. Theorem 5.9 provides a lever with which we can extrapolate the low dimensional information of Table 3 to the infinite amount of information contained in Theorem 5.8. This kind of extrapolation is typical of applications of the Novikov sepctral sequence to stable homotopy; a finite amount of low dimensional information can often be made to yield an infinite number of nontrivial homotopy elements.

We conclude this section with a discussion of how the phenomena of Theorem 5.8 appear in the Adams spectral sequence. It follows from Corollary 5.4 that any element of order 2 in Ext BP_* can be 'multiplied' (modulo some indeterminacy) by v_1. In other words, the $\mathbb{F}_2[v_1]$-module structure of Ext $BP_*/2$ translates to a Massey product operator which sends an element x of order 2 to $\langle x, 2, \alpha_1 \rangle$. In a similar way, the fact that $\mathrm{Ext}^o\ BP_*/16 \subset (Z/16)\ [v_1^4 + 8v_1v_2]$ leads to an 8-dimensional periodicity operator which sends an element x of order 16 to $\langle x, 16, \alpha_{4/4} \rangle$.

This is readily seen to correspond to the Adams periodicity operator $Px = \langle x, h_o^4, h_3 \rangle$ discussed in §2. The important difference is that the Novikov operator preserves filtration, while the Adams operator raises filtration by 4. With this in mind, one can prove

<u>Theorem 5.12</u> The Novikov elements $\alpha_{4t/4}$, $\alpha_1 x_{4t}$, $\alpha_1^2 x_{4t}$, α_{4t+1}, $\alpha_1 \alpha_{4t+1}$ and $\alpha_{4t+2/2}$ correspond to (i.e. detect the same

homotopy elements as) the Adams elements $P^{t-1} h_3$, $P^{t-1} c_o$, $h_1 P^{t-1} c_o$, $P^t h_1$, $P^t h_1^2$ and $P^t h_2$ respectively for all $t \geq 1$ (the last three elements correspond for $t \geq 0$.) □

Hence the Adams spectral sequence shows that these elements are permanent cycles, since they lie along the vanishing line ([5]) and the Novikov spectral sequence shows that they are nontrivial.

For future reference, we mention the behavior of the elements of Theorem 5.10.

<u>Theorem 5.13</u> In the Novikov spectral sequence for the mod 2 Moore spectrum $M(2)$

(a) $d_3 v_1^{4t+2} = \alpha_1^3 v_1^{4t}$, $d_3 v_1^{4t+3} = \alpha_1^3 v_1^{4t+1}$, $d_3 v_1^{4t+2}\sigma = \alpha_1^3 v_1^{4t}\sigma$ and $d_3 v_1^{4t+3}\sigma = \alpha_1^3 v_1^{4t+1}\sigma$.

(b) The elements $\alpha_1^i \sigma^\varepsilon v_1^{4t+j}$ are nontrivial permanent cycles for $0 \leq i \leq 2$, $\varepsilon = 0, 1$; $j = 0, 1$; $t \geq 0$.

(c) The homotopy element detected by $\alpha_1^2 \sigma^\varepsilon v_1^{4t}$ ($\varepsilon = 0, 1$; $t \geq 0$) is twice that detected by $\sigma^\varepsilon v_1^{4t+1}$.

All homotopy elements implied by (b) except $\sigma^\varepsilon v_1^{4t+1}$ have order 2. □

This can be proved by using Theorem 5.10 and comparing the Adams and Novikov spectral sequences for $M(2)$ through dimension 8.

We will need the following odd primary analogue

<u>Theorem 5.14(a)</u> For $p \geq 3$, Ext BP_*/p modulo v_1-torsion (i.e. the image of Ext BP_*/p in v_1^{-1} Ext $BP_*(p)$ is $\mathbb{F}_p[v_1, \alpha_1]/(\alpha_1^2)$.

(b) Each element v_1^t and $\alpha_1 v_1^t$ a $(t \geq 0)$ is a nontrivial permanent cycle and the corresponding homotopy element has order p. □

§6 Some Second Order Phenomena in the Novikov Spectral Sequence for Odd Primes

As remarked at the beginning of §5, we postpone our definition of nth order phenomena until §8. Unlike the first order phenomena, which was essentially described by Adams in [1], second and higher order phenomena are still largely unexplored. The infinite families of elements discovered in recent years by Larry Smith [63], [64], Oka [45], [46], [47] and Zahler [76] are examples of what we call second order families. The γ-family of Toda [73] is an example of a third order family. We will comment on this family and the unusual publicity received by its first member at the end of the section. The elements η_j recently constructed by Mahowald [28] presumably fit into not one but a series of second order families, as we shall describe in §7.

We will treat the odd primary case first because it is simpler. We begin by considering the computation of $\mathrm{Ext}^2 BP_*$. (See the beginning of §5 for the relevant notation.) As in the case of $\mathrm{Ext}^1 BP_*$, it is a torsion group of finite type, and the subgroup of order p is precisely the image of δ_0: $\mathrm{Ext}^1 BP_*/p \to \mathrm{Ext}^2 BP_*$. By Theorem 5.3, $\mathrm{Ext}^1 BP_*/p$ is a module over $\mathbb{F}_p[v_1]$. Its structure modulo v_1-torsion is given by Theorem 5.14. It can also be shown that for each $t \geq 0$, $v_1^t \alpha_1 \in \mathrm{Ext}^1 BP_*/p$ is the mod p reduction of an element and $\mathrm{Ext}^1 BP_*$, and therefore in $\ker \delta_0$. Hence we are interested in elements of $\mathrm{Ext}^1 BP_*/p$ which are v_1-torsion, i.e. which are annihilated by some power of v_1. To get at the v_1-torsion submodule of $\mathrm{Ext}^1 BP_*/p$, we first study the elements which are killed by v_1 itself.

To this end, consider the short exact sequence

$$(6.1) \qquad 0 \to \Sigma^q BP_*/p \xrightarrow{v_1} BP_*/p \to BP_*/I_2 \to 0 \; ,$$

and let δ_1: $\mathrm{Ext}^* BP_*/I_2 \to \mathrm{Ext}^{1+*} BP_*/I_1$ be the connecting homomorphism. By Theorem 5.3, $\mathrm{Ext}^0 BP_*/I_2 = \mathbb{F}_p[v_2]$, so we get

Proposition 6.2 There are nontrivial elements $\beta_t \equiv \delta_1(v_2^t) \in \mathrm{Ext}^{1,(p+1)tq-q}\, BP_*/p$ for all $t > 0$. □

To finish computing $\mathrm{Ext}^1 BP_*/p$, one needs to determine how many times one can divide β_t by v_1. We let $\beta_{t/i}$ denote an element (if such exists) such that $v_1^{i-1}\beta_{t/i} = \beta_{t/1} \equiv \beta_t$. It is clear then that the v_1-torsion submodule of $\mathrm{Ext}^1 BP_*/p$ is generated over $\mathbb{F}_p$ by such elements.

Theorem 6.3 For all primes p, $0 \neq \beta_{sp^i/j} \in \mathrm{Ext}^{1,(sp^i(p+1)-j)q} BP_*/p$ exists for all $s > 0$, $i \geq 0$ and $0 < j \leq p^i$. (Precise definitions of these elements are given in the proof below.)

Proof: The basic fact that we need is that $\eta_R(v_2) \equiv v_2 + v_1 t_1^p - v_1^p t_1 \mod p$ (Theorem 3.14). From this, we get $\eta_R\, v_2^{p^i} \equiv v_2^{p^i} \mod (p, v_1^{p^i})$, so $v_2^{sp^i} \in \mathrm{Ext}^0 BP_*/(p, v_1^{p^i})$ for $s \geq 0$. Let $\tilde{\delta}$ be the connecting homomorphism for

$$(6.4) \qquad 0 \to \Sigma^{qp^i} BP_*/p \xrightarrow{v_1^{p^i}} BP_*/p \to BP_*/(p, v_1^{p^i}) \to 0 .$$

Then we can define $\beta_{sp^i/p^i} = \tilde{\delta}_1(v_2^{sp^i})$, and for $j < p^i$, $\beta_{sp^i/j} = v_1^{p^i-j}\beta_{sp^i/p^i}$. □

The interested reader can verify that this method of defining elements in $\mathrm{Ext}^1 BP_*/p$ is far easier than writing down explicit cocycles in $\tilde{\Omega} BP_*/p$.

Note that Theorem 6.3 says that $\mathrm{Ext}^1 BP_*/p$ contains $\mathbb{F}_p$-vector spaces of arbitrarily large finite dimension. For example β_{p^2/p^2} and β_{p^2-p+1} are both in $\mathrm{Ext}^{1,p^3 q} BP_*/p$.

Unfortunately, Theorem 6.3 is not the best result possible. Further v_1 divisibility does occur, e.g. one can define β_{2p^2/p^2+p-1}. The complete computation of $\mathrm{Ext}^1 BP_*/p$ for $p > 2$ (and of $\mathrm{Ext}^1 BP_*/I_n$ for all p and $n > 1$) was first done by Miller-Wilson in [40] and redone (including the case $p = 2$, $n = 1$) in §5 of [39]. However, the elements of Theorem 6.4 will suffice for our purposes here.

The next and final step in the computation of $\mathrm{Ext}^2 BP_*$ is to determine how much $\delta_o(\beta_{i/j})$ (which will also be denoted by $\beta_{i/j}$) can be divided by p. This was done for $p > 2$ in §6 of [39] and announced in [38]. The computational difficulties encountered there are formidable. The problem is still open for $p = 2$, but it is certain that the methods of [39], if pushed a little further, will yield the answer.

We denote by $\beta_{i/(j,k)}$ a certain element with $p^{k-1}\beta_{i/(j,k)} = \beta_{i/(j,1)} \equiv \beta_{i/j}$. Then along the lines of Theorem 6.3 we have

<u>Theorem 6.5</u> For all primes p

$0 \neq \beta_{sp^i/(tp^j,1+j)} \in \mathrm{Ext}^{2,(sp^i(p+1)-tp^j)q} BP_*$ exists for all $s > 0$ and $0 < t < p^{i-2j}$ (an will be defined in the proof below), except $\beta_1 = 0$ for $p = 2$.

<u>Proof</u> From $\eta_R v_2 \equiv v_2 + v_1 t_1^p - v_1^p t_1 \mod p$ (Theorem 3.14) we obtain $\eta_R(v_2^{p^{2j+k}}) \equiv v_2^{p^{2j+k}} \mod (p^{1+j}, v_1^{p^{j+k}})$, and since $\eta_R v_1 = v_1 + pt_1$, we have $\eta_R v_1^{p^j} \equiv v_1^{p^j} \mod p^{1+j}$. It follows that $v_2^{p^{2j+k}} \in \mathrm{Ext}^o BP_*/(p^{1+j}, v_1^{p^{j+k}})$ and $v_1^{p^j} \in \mathrm{Ext}^o BP_*(p^{1+j})$. Let $\tilde{\delta}_o$ and $\tilde{\delta}_1$ be the connecting homomorphisms for the short exact sequences

$$0 \to BP_* \xrightarrow{p^{1+j}} BP_* \longrightarrow BP_*/(p^{1+j}) \longrightarrow 0$$

and

$$0 \to \Sigma^{qp^{j+k}} BP_*/(p^{1+j}) \xrightarrow{v_1^{p^{j+k}}} BP_*/(p^{1+j}) \to BP_*/(p^{1+j}, v_1^{p^{j+k}}) \to 0$$

respectively. Then we can define

$$\beta_{sp^{2j+k}/(tp^j,1+j)} = \tilde{\delta}_o(v_1^{p^j(p^k-t)}\tilde{\delta}_1(v_2^{sp^{2j+k}})).$$

The nontriviality of these elements can be seen by looking at the long exact sequences in Ext associated with the short exact sequences above. The one nontrivial fact that is needed is that the image of the mod (p^{1+j}) reduction $\mathrm{Ext}^1 BP_* \to \mathrm{Ext}^1 BP/(p^{1+j})$ consists of elements which are not annihilated by any power of $v_1^{p^j}$. □

Again, this is not the best result possible, but these elements will suffice for our purposes.

Note that Theorem 6.5 says that $\mathrm{Ext}^2 BP_*$ contains elements of arbitrarily high order, but that they occur very infrequently. For example, $\beta_{p^2/(p,2)}$ is the first element of order p^2, and and it is in dimension 130 for $p = 3$, and $\beta_{p^4/(p^2,3)}$, the first element of order p^3, is in dimension 1258 for $p = 3$.

Theorem 6.5 gives most of the additive generators of $\mathrm{Ext}^2 BP_*$ for $p > 2$. This group is much more complicated then $\mathrm{Ext}^1 BP_*$.

As the reader might guess, the question to ask now is which elements in this group are permanent cycles in the Novikov spectral sequence. This problem is far from being solved. Some progress has been made for $p \geq 5$. The current state of the art is

Theorem 6.6 For $p \geq 5$, the following elements in $\mathrm{Ext}^2 BP_*$ are permanent cycles, and the nontrivial homotopy elements they detect have the same order as the corresponding elements in the E_2-term.

(a) (Smith [63]) β_t for $t > 0$.

(b) (Smith [64], Oka [45], Zahler [76]) $\beta_{pt/j}$ for $t > 0$ and $0 < j < p$.

(c) (Oka [46]) $\beta_{tp/p}$ for $t \geq 2$.

(d) (oka [45]) $\beta_{tp^2/j}$ for $t > 0$ and $1 \leq j \leq 2p - 2$

(e) (Oka [47]) $\beta_{tp^2/j}$ for $t \geq 2$. and $1 \leq j \leq 2p$

(f) (Oka [47]) $\beta_{tp^2/(p,2)}$ for $t \geq 2$. []

Some of the elements in (b) - (e) were initially denoted by ε or ρ with various subscripts. On the other hand, we have

Theorem 6.7 [56] For $p \geq 3$ and $i \geq 1$ the element $\beta_{p^i/p^i} \in \mathrm{Ext}^2 BP_*$ is not a permanent cycle; in fact $d_{2p-1}\beta_{p^i/p^i} \equiv \alpha_1 \beta^p_{p^{i-1}/p^{i-1}}$ modulo certain indeterminacy. []

The special case $i = 1$ was first proved by Toda [70], [71] and it gives the first nontrivial differential in the Novikov spectral sequence for $p \geq 3$.

Theorem 6.6 (a) is definitely false for $p = 3$, for we have e.g. $d_5\beta_4 = \pm\alpha_1\beta_1^2\beta_{3/3} \neq 0$. We hope to have more to say about this in [52]. Tentative computations indicate for example that (for $p = 3$) β_t is a permanent cycle iff $t \not\equiv 4, 7$ or $8 \bmod 9$.

We will now sketch the proof of Theorem 6.6 (a), as the proofs of (b) - (f) are all based on the same idea. Let $M(p)$ denote the mod p Moore spectrum. Then applying BP homology to the cofibration

$$(6.8) \qquad S^o \xrightarrow{p} S^o \longrightarrow M(p)$$

yields the short exact sequence (5.2). In this instance, we say 6.8 realizes (5.2). In [63] Smith shows that for $p \geq 3$ there is a map $\alpha: \Sigma^q M(p) \to M(p)$ which in BP homology realizes multiplication by v_1. We denote the cofibre of α by $M(p, v_1)$, so the cofibration

$$\Sigma^q M(p) \xrightarrow{\alpha} M(p) \longrightarrow M(p, v_1)$$

realizes the sequence 6.1. (It is not hard to see that this cannot be done for $p = 2$, but one can construct the spectrum $M(2, v_1^4)$. Our proof of Theorem 5.8 is based on the existences of $M(16, v_1^4 + 8v_1v_2)$.)

Next, Smith shows that for $p \geq 5$ there is a map $\beta: \Sigma^{(p+1)q} M(p, v_1) \to M(p, v_1)$ which realizes multiplication by v_2, so the cofibration

$$\Sigma^{(p+1)q} M(p, v_1) \xrightarrow{\beta} M(p, v_1) \to M(p, v_1, v_2)$$

realizes the short exact sequence

$$(6.9) \quad 0 \to \Sigma^{(p+1)q} BP_*/I_2 \xrightarrow{v_2} BP_*/I_2 \to BP_*/I_3 \to 0 .$$

(The map β does not exist for $p = 3$.)

Then it is not hard to show (with two applications of Theorem 5.11) that the composite

$$S^{t(p+1)q} \to \Sigma^{t(p+1)q} M(p) \to \Sigma^{t(p+1)q} M(p, v_1) \xrightarrow{\beta^t} M(p, v_1) \to \Sigma^{q+1} M(p) \to S^{q+2}$$

(where the first two maps are inclusions of low dimensional skeleta, and the last two maps are projections obtained by pinching low dimensional skeleta) is a homotopy element detected by $\beta_t \in \mathrm{Ext}^2 BP_*$.

In other words, the existence of $\beta_t \in \pi_* S^0$ is based on the existence of the map $\beta: \Sigma^{(p+1)q} M(p, v_1) \to M(p, v_1)$. As in §5, the low dimensional information required to construct this map can be extrapolated by the Novikov spectral sequence into an 'infinite amount' of information, i.e. the existence and nontriviality of β_t for all $t > 0$.

Parts (b) and (d) of Theorem 6.6 are based in a similar manner on the existence of maps

$$\Sigma^{p(p+1)q} M(p, v_1^{p-1}) \to M(p, v_1^{p-1}) \quad \text{and} \quad \Sigma^{p^2(p+1)q} M(p, v_1^{2p-2}) \to M(p, v_1^{2p-2})$$

realizing multiplication by v_2^p and $v_2^{p^2}$ respectively. For (c) the complex $M(p, v_1^p, v_2^p)$ does not exist (its existence would contradict Theorem 6.7 for $i = 1$), but Oka [46] constructs $M(p, v_1^p, v_2^{2p})$ and $M(p, v_1^p, v_2^{3p})$ from self-maps of $M(p, v_1^p)$ which yield the indicated elements. Parts (e) and (f) are proved in a similar manner.

We should point out that the 4-cell and 8-cell complexes M( ) above are not necessarily unique, i.e. a complex whose BP-homology is a cyclic BP_*-module is not in general characterized by that module. What is essential to the argument above is the existence of a self-map of the appropriate 4-cell complex which realizes multiplication by the appropriate power of v_2.

In a similar spirit, Theorem 6.7 implies

<u>Theorem 6.10</u> [56] For $p \geq 3$ and $i \geq 1$ there is no connective spectrum X such that $BP_*X = BP_*/(p, v_1^{p^i}, v_2^{p^i})$. □

In [73] Toda considers the existence of complex $M(p, v_1, v_2 \cdots v_n)$ which he calls $V(n)$ and which he characterizes in terms of their cohomology as modules over the Steenrod algebra. (Such a description of the $M(\ )$'s considered above will not work unless one is willing to resort to (much) higher order cohomology operations. We regard this fact as another advantage of BP-homology.) He proves

Theorem 6.11 (Toda [73]) For $p \geq 7$ the complex $V(3) = M(p, v_1, v_2, v_3)$ exists and is the cofibre of a map $\gamma: \Sigma^{(p^2+p+1)q} M(p, v_1, v_2) \to M(p, v_1, v_2)$. □

Let δ_2 be the connecting homomorphism for the short exact sequence (6.9). (Recall that δ_0 and δ_1 are the connecting homomorphisms of(5.2) and (6.1) respectively.) Then we can define

(6.12) $\gamma_t = \delta_0\delta_1\delta_2(v_3^t) \in \mathrm{Ext}^{3,(t(p^2+p+1)-(p+2))q}\, BP_*$.

From Theorem 6.10, we derive

Corolary 6.13 For $p \geq 7$ the elements $\gamma_t \in \mathrm{Ext}^3 BP_*$ are permanent cycles for all $t > 0$. □

However, the nontriviality of these elements is far from obvious. The status of γ_1 was the subject of a controversy [48], [68], [3] which attracted widespread attention [43], [60]. In order to settle the question for all t one must know $\mathrm{Ext}^2 BP_*$ in all of the appropriate dimensions. Having determined the latter, we proved

Theorem 6.14 [38] [39] For $p \geq 3$, the element $\gamma_t \in \mathrm{Ext}^3 BP_*$ is nontrivial for all $t > 0$. □

The γ's are an example of what we call third order phenomena.

§7. Some Second Order Phenomena in the Novikov Spectral Sequence for the Prime 2.

We must assume that the reader is familiar with the notation introduced in the previous two sections. Our current knowledge of 2-primary second order phenomena is in some sense even sketchier than in the odd primary case. Nevertheless, the situation is quite tantalizing, especially in light of Mahowald's recent result (Theorem 2.12) on the existence of the elements η_j. We will see below that the Novikov spectral sequence provides a very suitable setting for understanding these elements and the families of elements that could possibly derive from them.

As in §6, we begin with a discussion of $\mathrm{Ext}^2 BP_*$, this time for $p = 2$. It is a torsion group, so to get at the elements of order 2, we look at $\mathrm{Ext}^1 BP_*/2$. It is a module over $\mathbb{F}_2[v_1]$ (Theorem 5.3) and its structure modulo v_1-torsion is given by Theorem 5.10. Unlike the odd primary case, not all of the v_1-torsion free part of $\mathrm{Ext}^1 BP_*/2$ is in the kernel of the connecting homomorphism $\delta_0 \colon \mathrm{Ext}^1 BP_*/2 \to \mathrm{Ext}^2 BP_*$. Indeed, the summand of $\mathrm{Ext}^2 BP_*$ indicated in Theorem 5.7 is precisely the image under δ_0 of the summand of $\mathrm{Ext}^1 BP_*/2$ given by Theorem 5.10. We call the former summand the <u>first order part</u> of $\mathrm{Ext}^2 BP_*$. (For $p > 2$, the first order part of $\mathrm{Ext}^2 BP_*$ is trivial.) The <u>second order part</u> of $\mathrm{Ext}^2 BP_*$ is that summand associated (via division by powers of 2) with the image under δ_0 of the v_1-torsion submodule of $\mathrm{Ext}^1 BP_*/2$. This submodule contains all the elements provided by Theorem 6.4 (which is valid for all primes) as well as some more exotic elements which are described in §5 of [39].

Similarly $\mathrm{Ext}^2 BP_*$ itself contains the summand of Theorem 5.7, the subgroup (which is not a summand) provided by Theorem 6.5 (which is also valid for all primes) and some additional elements which have not yet been determined. For emphasis, we repeat that the determination of $\mathrm{Ext}^2 BP_*$ for $p = 2$ is still an open problem, but the methods of [39] are surely adequate for solving it.

We now wish to relate certain elements of Theorem 6.5 to elements in the Adams E_2-term. The manner in which elements of the two E_2-terms correspond to each other is difficult to define precisely, although in many cases it is easy enough to see in practice. Proposition 4.1 gives a correspondance only between nontrivial permanent cycles, and the homomorphism Φ: Ext $BP_* \to \mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ is nontrivial only on a very small number of elements. Most of the elements in $\mathrm{Ext}^2 BP_*$ correspond in some way to elements of higher filtration in the Adams E_2-term.

A working (but not completely precise) procedure for matching elements in the two E_2-terms is the following. Theorem 4.2 and 4.3 give us two spectral sequences having essentially the same E_2-term and converging to the Adams and Novikov E_2-terms respectively. Hence we can take an element in the Novikov E_2-term, represent it (not uniquely in general) by some permanent cycle in the E_2-term of Theorem 4.3, and then see what happens to the corresponding element in the E_2-term of Theorem 4.2. The latter may fail to be a permanent cycle in the spectral sequence of Theorem 4.2. This would probably mean that the element we started with is not a permanent cycle in the Novikov spectral sequence (and that it supports a differential in some way related to a differential of Theorem 4.2), but this assertion has not been proved. It could also happen that the element we get is the target of a differential in 4.2. This might mean either that our original element is the target of some Novikov differential or that it 'corresponds' to an element having higher Adams filtration than originally estimated.

Of course, this procedure could be reversed (i.e. we could start with Adams elements and try to get Novikov elements), and the same remarks would apply. As we tried to indicate at the end of §4, making all of this more precise, especially nailing down the possible method of computing both Adams and Novikov differentials is an important unsolved algebraic problem.

With the above reservations in mind, we make

Assertion 7.1 For $p = 2$

(a) $\Phi\left(\beta_{2^i/2^i}\right) = h_{i+1}^2$ and $\Phi\left(\beta_{2^i/2^i-1}\right) = h_1 h_{i+2}$. (This is a theorem.)

Under the procedure outlined above,

(b) for $j > i \geq 0$, $\beta_{2^j/2^j-2^i}$ corresponds to $h_o^{2^i-1} h_{1+i} \; h_{2+j}$ and $\beta_{2^{1+i}/(2^i,2)}$ corresponds to $h_o^{2^i-2} h_{1+i} \; h_{3+i}$;

(c) for $t \geq 0$, $\beta_{2^j/2^j-2^i-4t}$ corresponds to

$P^t \; h_o^{2^i-1} \; h_{1+i} \; h_{2+j}$ and $\beta_{2^{1+i}/(2^i-4t,2)}$ corresponds to $P^t \; h_o^{2^i-2} \; h_{1+i} \; h_{3+i}$.

Argument: An odd primary analogue of (a) is proved in §9 of [39].

For (b) and (c), recall the definition of $\beta_{2^j/2^j-2^i}$.

Let δ_o and $\tilde{\delta}_1$ be the connecting homomorphisms for the short exact sequences $0 \to BP_* \xrightarrow{2} BP_* \to BP_*/2 \to 0$ and

$$0 \to \Sigma^{2^{j+1}} BP_*/2 \xrightarrow{v_1^{2^j}} BP_*/2 \to BP_*/\left(2, v_1^{2^j}\right) \to 0$$

respectively. Then $v_2^{2^j} \in \mathrm{Ext}^o \; BP_*/\left(2, v_1^{2^j}\right)$ and $\beta_{2^j/2^j-2^i} = \delta_o\left(v_1^{2^i} \; \tilde{\delta}_1\left(v_2^{2^j}\right)\right)$ (see Theorem 6.4).

The spectral sequence of Theorem 4.3 has obvious analogues converging to $\mathrm{Ext} \; BP_*/\left(2, v_1^{2^j}\right)$, and $\mathrm{Ext} \; BP_*/2$, and we can compute $\beta_{2^j/2^j-2^i}$ in the E_2-terms of those spectral sequences.

We have $\tilde{\delta}_1\, v_2^{2^j} = t_1^{2^{j+1}}$ modulo terms with higher I-filtration, and $\delta_o\left(v_1^{2^i}\ \tilde{\delta}_1\left(v_2^{2^j}\right)\right) \equiv \delta_o\ v_1^{2^i}\ t_1^{2^{j+1}} \equiv 2^{2^i-1}\ t_1^{2^i} \mid t_1^{2^{1+j}}$. Since 2 corresponds to h_o and $t_1^{2^i}$ corresponds to h_{1+i}, we get the element $h_o^{2^i-1}\ h_{1+i}\ h_{2+j}$ as desired. The argument for $\beta_{2^{1+i}/(2^i,2)}$ is similar.

For (c) we use the fact (see the discussion preceeding Theorem 5.12) that multiplication by v_1^4 in Ext $BP_*/2$ and Ext $BP_*/4$ corresponds to the Adams periodicity operator P. □

The discussion that follows will be of a more hypothetical nature. We will see how various hypotheses relating to the Arf invariant elements and Mahowald's η_j (Theorem 2.12) imply the existence of new families of homotopy elements. We list our hypothesis in order of decreasing strength.

<u>Hypothesis 7.2</u>$_i$ $(i \geq 2)$ $\beta_{2^i/2^i}$ is a permanent cycle and and the corresponding homotopy element can be factored

$S^{6\cdot 2^i} \to \Sigma^{6\cdot 2^i} M\left(2,\ v_1^{2^i}\right) \xrightarrow{\beta} M\left(2,\ v_1^{2^i}\right) \to S^{2+2\cdot 2^i}$, i.e. the map β realizes multiplication by $v_2^{2^i}$.

<u>Hypothesis 7.3</u>$_i$ $\beta_{2^i/2^i}$ is a nontrivial permanent cycle and the corresponding homotopy element has order 2.

<u>Hypothesis 7.4$_i$</u> $\beta_{2^i/2^i-1}$ is a permanent cycle and the corresponding homotopy element has order 4 and is annihilated by ν.

Theorem 2.12 and 7.1(a) imply that there is a permanent cycle equal to $\beta_{2^i/2^i-1}$ modulo ker Φ. It appears unlikely that the error term in ker Φ would affect any of the arguments that follow, so we assume for simplicity that it is zero.

Similarly, if the Arf invariant element θ_{i+1} exists it is detected by $\beta_{2^i/2^i}$ modulo ker Φ.

Hypothesis 7.2 is known to be false for $i = 2$, and we have included it mainly to illustrate the methodology in as simple a way as possible. The statement that $\beta_{2^i/2^i}$ extends to $M(2, v_1^i)$ or, by duality that it coextends, is equivalent to the Toda bracket [69]. $\langle \beta_{2^i/2^i}, 2\iota, \alpha_{2^i}, 2\iota \rangle$ being defined and trivial. Mahowald has shown the following substitute for it.

<u>Theorem 7.5</u> There is a map $\beta\colon \Sigma^{48} M(4, v_1^4) \to M(4, v_1^4)$ which realizes multiplication by v_2^8. $\square$

<u>Corollary 7.6</u> The elements $\beta_{8t/(4,2)} \in \mathrm{Ext}^{2,48t-8}\, BP_*$ for all $t > 0$ are nontrivial permanent cycles and the corresponding homotopy elements have order 4.

<u>Proof</u>: The argument is similar to that of Theorem 6.6 (which is discussed following Theorem 6.7). We use Theorem 5.11 twice the show that the composite

$$S^{48t} \to \Sigma^{48t} M(4, v_1^4) \xrightarrow{\beta^t} M(4, v_1^4) \to S^{10}$$

is detected by $\beta_{8t/(4,2)}$. □

From 7.1 we see that $\beta_{8/(4,2)} \in \pi_{38}S^o$, $\beta_{16/(4,2)} \in \pi_{86} S^o$ and $\beta_{32/(4,2)} \in \pi_{182}S^o$ are detected in the Adams spectral sequence by $h_o^2 h_3 h_5$, $Ph_o^6 h_4 h_6$, and $P^3 h_o^{14} h_5 h_7$ respectively.

Proposition 7.7

(a) Hypothesis 7.2_i implies 7.3_i.

(b) Hypothesis 7.3_i implies 7.4_i.

Proof: (a) If $\beta_{2^i/2^i}$ extends to $M\left(2, v_1^{2^i}\right)$, it certainly extends to $M(2)$ and so has order 2.

(b) If $\beta_{2^i/2^i}$ has order 2, then it extends to a map $f\colon \Sigma^{4\cdot 2^i-2} M(2) \to S^o$. By Theorem 5.13, $v_1 \in \pi_2 M(2)$ has order 4, $\nu v_1 = 0$ and $f_*(v_1) = \beta_{2^i/2^i-1}$ by Theorem 5.11 and an easy calculation. □

Note that the proof of (b) shows $2\beta_{2^i/2^i-1} = \alpha_1^2\beta_{2^i/2^i} \in \pi_{2^{i+2}} S^o$ if 7.3_i holds.

The Hypothesis 7.2 - 7.4 provide homotopy elements as follows

Theorem 7.8

(a) If 7.2_i holds, then the following elements are (not necessarily nontrivial) permanent cycles: $\beta_{s\cdot 2^i/2^i-4t-a}\sigma^\varepsilon \alpha_1^j$ for $s > 0$; $a = 0, 1$; $\varepsilon = 0, 1$; $j = 0, 1, 2$.

(b) If 7.3_i holds, then the elements of (a) with $s = 1$ are permanent cycles.

(c) If 7.4_i holds, all of the elements of (b) except $\beta_{2^i/2^i}$, and $\alpha_1\beta_{2^i/2^i}$, are permanent cycles.

Proof:

(a) Let $\beta\colon \Sigma^{6\cdot 2^i} M\left(2,\ v_1^{2^i}\right) \to M\left(2,\ v_1^{2^i}\right)$ be the map of 7.2, Then $\beta_{s\cdot 2^i/2^i}$ is the composition

$$S^{6\cdot 2^i} \to \Sigma^{6\cdot 2^i\cdot s} M\left(2,\ v_1^{2^i}\right) \xrightarrow{\beta^s} M\left(2,\ v_1^{2^i}\right) \to S^{2+2^{i+1}} .$$

The other elements are obtained by composing the elements of $\pi_* M(2)$ given by Theorem 5.13 with the map

$$\Sigma^{6s\cdot 2^i} M(2) \to \Sigma^{6s\cdot 2^i} M\left(2,\ v_1^{2^i}\right) \xrightarrow{\beta^s} M\left(2,\ v_1^{2^i}\right) \to S^{2+2^{i+1}} .$$

(b) Compose the elements of Theorem 5.13 with the extension of $\beta_{2^i/2^i}$ to $M(2)$.

(c) The indicated elements with $a = 1$ can be obtained by composing the extension of $\beta_{2^i/2^i-1}$ with the appropriate elements given by the mod 4 analogue of Theorem 5.13. The element $\alpha_1^2\beta_{2^i/2^i}$ is $2\beta_{2^i/2^i-1}$ by the proof of Proposition 7.7(b); $\sigma\beta_{2^i/2^i}$ and $\beta_{2^i/2^i-4}$ can be realized as homotopy elements of order 2 by the Toda brackets [69] $\langle \beta_{2^i/2^i-1},\ \nu,\ \eta\rangle$ and $\langle \beta_{2^i/2^i-1},\ \eta^3,\ 2,\ \eta\rangle$ respectively. (The latter bracket is defined because $\eta^3\beta_{2^i/2^i-1} = 4\nu\beta_{2^i/2^i-1} = 0$.) Then the remaining elements can be obtained by composing the extensions of these two to $M(2)$ and composing with the elements of Theorem 5.13. □

The above theorem does not assert that the indicated elements are nontrivial, and some of them are likly to be trivial, such as $\alpha_1\beta_{s\cdot 2^i/2^i-4t}$, $\alpha_1^2\beta_{s\cdot 2^i/2^i-4t}$ with $2^{i-3} \leq t < 2^{i-2}$ (since $\beta_{s\cdot 2^i/2^i-4t}$ in this is divisible by 2 by Theorem 6.5). The possible nontriviality is the subject of work in progress which will be reported elsewhere. At the moment, we can offer the following.

<u>Theorem 7.9</u> The elements $\alpha_1\beta_{s\cdot 2^i/2^i-4t-1} \in \mathrm{Ext}^3\, BP_*$ and $\alpha_1^2\beta_{s\cdot 2^i/2^i-4t-1} \in \mathrm{Ext}^4\, BP_*$ for $0 \leq t < 2^{i-2}$ (as well as $\beta_{s\cdot 2^i/2^i-4t-1}$, $\beta_{s\cdot 2^i/2^i-4t} \in \mathrm{Ext}^2\, BP_*$) are nontrivial in the Novikov E_2-term. □

<u>Corollary 7.10</u> If the elements of Theorem 7.9 are permanent cycles, then the corresponding homotopy elements are nontrivial.

<u>Proof</u>: By sparseness (Corollary 3.17) a Novikov differential hitting any of these elements would have to originate on the 0-line or the 1-line. The former is trivial in positive dimensions, and all differentials originating on the latter were accounted for in theorem 5.8. □

We cannot resist commenting on how hard it would be to prove similar results using only the Adams spectral sequence. The proof of Theorem 7.9 is based on methods (see §8) which have no counterpart in the Adams spectral sequence. Even if somehow one could prove that the corresponding elements are nontrivial in the Adams E_2-term, they would have such high filtration that it would be extremely difficult to show that they are not hit by nontrivial

Adams differentials. The low filtration of elements in the Novikov spectral sequence makes it a very effective detecting device.

We remind the reader that none of the Hypotheses 7.2 - 7.4 are currently known to be true for all i. This is unfortunate in view of the following

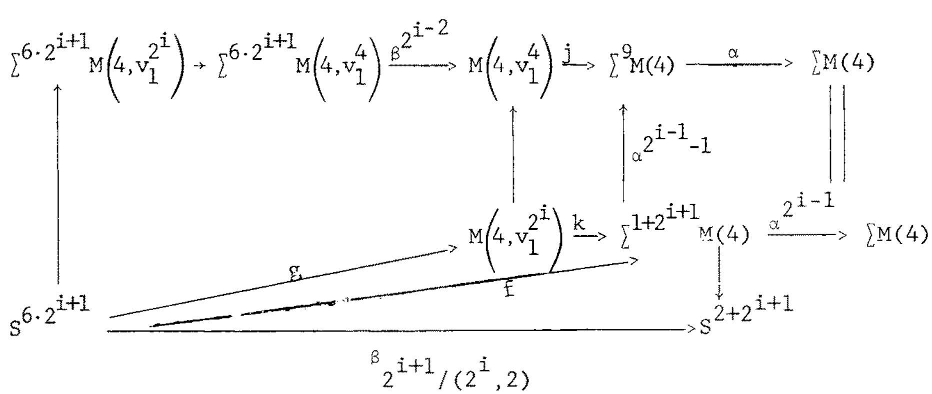
Theorem 7.11 If for some $i \geq 2$

(i) $M\left(4,\ v_1^{2^i}\right)$ is a ring spectrum,

(ii) $\beta_{2^{i+1}/(2^i,2)}$ is a permanent cycle and

(iii) the corresponding homotopy element has order 4, then the elements $\beta_{s\cdot 2^{i+1}/(4j,2)}$ and $\alpha_1^k\beta_{s\cdot 2^{i+1}/4j-1}$ for $s > 0$; $k = 0,\ 1,\ 2;$ $i \geq 2$ and $0 < j \leq 2^{i-2}$ are nontrivial permanent cycles.

Proof: The nontriviality follows from Corollary 7.10. Let $\beta\colon \Sigma^{48} M(4,\ v_1^4) \to M(4,\ v_1^4)$ be the map of Theorem 7.5 and $\alpha\colon \Sigma^8 M(4) \to M(4)$ a map which realizes multiplication by v_1^4. Then consider the following commutative diagram.



All of the maps except f and g are obvious; the last two maps of the two top rows are cofibre sequences, i.e. $\alpha \circ j = \bar{\alpha}^{2^{i-1}} \circ k = 0$. The map f exists because $\beta_{2^{i+1}/(2^i,2)}$ has order 4. The commutativity of the diagram implies that $\alpha^{2^{i-1}} \circ f = 0$, so g exists. The multiplicative structure of $M\left(4,\ v_1^{2^i}\right)$ can be used to extend g to $\Sigma^{6\cdot 2^{i+1}} M\left(4,\ v_1^{2^i}\right)$. Thus we obtain a map

$$\Sigma^{6\cdot 2^{i+1}} M\left(4,\ v_1^{2^i}\right) \xrightarrow{\tilde{\beta}} M\left(4,\ v_1^{2^i}\right)$$

which realizes multiplication by $v_2^{2^{i+1}}$. We can then obtain the desired homotopy elements by composing

$$\Sigma^{6s\cdot 2^{i+1}} M(4) \to \Sigma^{6s\cdot 2^{i+1}} M\left(4, v_1^{2^i}\right) \xrightarrow{\tilde{\beta}^s} M\left(4, v_1^{2^i}\right) \to S^{2+2^{i+1}}$$

with the appropriate elements of $\pi_* M(4)$. □

Hence the hypotheses of Theorem 7.11 imply that a large collection of elements in $\mathrm{Ext}^2 BP_*$ are permanent cycles. Mahowald has an argument for the first hypothesis [77], but the status of the others is less clear. Theorem 7.11 has the following analogue.

<u>Theorem 7.12</u> Let $p \geq 5$. If for some $i \geq 1$

(i) $M\left(p,\ v_1^{p^i-1}\right)$ is a ring spectrum,

(ii) β_{p^i/p^i-1} is a permanent cycle and

(iii) the corresponding homotopy element has order p, then $\beta_{sp^i/j}$ is a permanent cycle (and the corresponding homotopy element of order p) for all $s > 0$, $i \geq 1$ and $0 < j < p^i$.

Proof: We argue as in Theorem 7.11, replacing Theorem 7.5 with the assertion that there is a map

$$\beta: \Sigma^{p(p+1)q} M\left(p, v_1^{p-1}\right) \to M\left(p, v_1^{p-1}\right)$$

realizing multiplication by v_2^p. This map has been constructed by Smith [64] and Oka [45] in the proof of Theorem 6.6(b). ∎

We hope to extend this result to $p = 3$ in [52]. Oka has recently announced [78] a proof of the first hypothesis for all i. The second is likely to follow from an odd primary analogue of Mahowald's Theorem 2.12. The third hypothesis, however, could be quite difficult to prove.

§8 Morava stabilizer Algebras and the Chromatic Spectral Sequence (the inner mysteries of the Novikov E_2-term)

The reader may well wonder how it is possible to prove results such as Theorems 5.7, 5.9, 5.10, 5.14(a) and 7.9, which state that various systematic families of elements in the Novikov E_2-term are nontrivial. The basic technique in each case is to study the map $\mathrm{Ext}\, BP_* \to \mathrm{Ext}\, v_n^{-1} BP_*/I_n$ (where $n = 1$ for the results of §5 and $n = 2$ for Theorem 7.9). The latter group is surprisingly easy to compute due to two startling isomorphisms (Theorems 8.4 and 8.7 below) originally discovered by Jack Morava [42]. It was this computability that motivated us to do the work that led to [39]. Morava's work implies that there is a deep, and previously unsuspected connection between algebraic topology and algebraic number theory. Where it will eventually lead to is anybody's guess.

After describing how to compute $\mathrm{Ext}\, v_n^{-1} BP_*/I_n$, we will set up the chromatic spectral sequence (and explain why it is so named), which is a device for feeding this new found information into the Novikov E_2-term in a most systematic way. We will see that it reveals patterns of periodicity (which may carry over to stable homotopy itself; see [53]) hitherto invisible. In particular, we will define nth order phenomena in the Novikov spectral sequence.

In order to get at $\mathrm{Ext}\, v_n^{-1} BP_*/I_n$, we need to define some auxillary objects. Let $K(n)_* = \mathbb{Q}$ for $n = 0$ and $\mathbb{F}_p[v_n, v_n^{-1}]$ for $n > 0$, and make it a BP_*-module by sending v_i to zero for $i \neq n$ (where $v_0 = p$). For $n > 0$, $K(n)_*$ is a graded field in the sense that every graded module over it is free. Next, define

$$(8.1) \qquad K(n)_* K(n) = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n) ,$$

where the tensor products on the left and right are with respect to the BP_*-module structures on BP_*BP induced by η_L and η_R respectively (see §3). $K(n)_* K(n)$ where a coassociative, non-cocommutative coproduct Δ from BP_*BP. Theorem 3.14 allows us to describe its algebra structure very explicitly.

Theorem 8.2 [58] $K(0)_* K(0) = Q$ and for $n > 0$, $K(n)_* K(n) = K(n)_*[t_1, t_2 \cdots]/\left(v_n t_i^{p^n} - v_n^{p^i} t_n\right)$. The coproduct Δ is given by $\sum_{i\geq 0}^F \Delta(t_i) = \sum_{i,j\geq 0}^F t_i \otimes t_j^{p^i}$, where $t_o = 1$ and F is the formal group law over $K(n)_*$ given by the map $BP_* \to K(n)_*$ (see Theorem 3.10).

Proof: By definition (8.1) $K(n)_*K(n) = v_n^{-1}BP_*BP/(v_i, \eta_R v_i : i \neq n)$. In $K(n)_*K(n)$, $\eta_R v_n = v_n$ and Theorem 3.14 reduces to

$$(8.3) \qquad \sum_{i\geq 0}^F v_n t_i^{p^n} = \sum_{i\geq 0}^F v_n^{p^i} t_i .$$

Each side of (8.3) has at most one formal summand in each dimension, so we can formally cancel and get $v_n t_i^{p^n} = v_n^{p^i} t_i$ by induction on i. The formula for Δ follows from Theorem 3.12. □

Now $K(n)_*K(n)$ is a Hopf algebra over $K(n)_*$, so we can define its cohomology $\mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*)$ in the usual manner. We now come to our first surprise.

Theorem 8.4 [37]

$$\mathrm{Ext}\ v_n^{-1} BP_*/I_n \cong \mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*). \quad \square$$

Since $K(n)_* K(n)$ is much smaller than BP_*BP, this result simplifies the computation of Ext $v_n^{-1} BP_*/I_n$ considerably. In §3 of [58] we filter $K(n)_*K(n)$ in such a way that the associated bigraded object is the dual of the universal enveloping algebra of a restricted Lie algebra. In [51] we use this filtration to construct a May spectral sequence [32] converging to the desired

Ext group. We use this device then to compute Ext $v_n^{-1}BP_*/I_n$ for $n = 0, 1, 2,$ and $\mathrm{Ext}^s\ v_n^{-1}BP_*/I_n$ for all n and $s = 0, 1, 2$.

However, deeper insight into the structure of $K(n)_*K(n)$ is gained as follows. Forgetting the grading, make $\mathbb{F}_p$ into a $K(n)_*$ module by sending v_n to 1, and let

(8.5) $$S(n)_* = K(n)_*K(n) \otimes_{K(n)_*} \mathbb{F}_p .$$

The $S(n)_*$ is a commutative, noncocommuative Hopf algebgra over $\mathbb{F}_p$ with algebra structure

(8.6) $$S(n)_* \cong \mathbb{F}_p[t_1, t_2 \cdots]/(t_i^{p^n} - t_i).$$

Its dual $S(n)$ (defined in the appropriate way in [58]) is called the <u>nth Morava stabilizer algebra</u> $S(n)$. This brings us to our second surprise.

<u>Theorem 8.7</u> [58] $S(n) \otimes \mathbb{F}_{p^n} \cong \mathbb{F}_{p^n}[S_n]$, the group algebra over $\mathbb{F}_{p^n}$ of a certain pro-p group S_n, to be described below. □

<u>Corollary 8.8</u> [58] Ext $v_n^{-1}BP_*(I_n) \otimes_{K(n)_*} \mathbb{F}_p = H_c^*(S_n; \mathbb{F}_p)$, where the latter is the continuous cohomology of S_n with constant mod p coefficients. □

In §2 of [58] we also show how it is possible to recover the bigrading of Ext $v_n^{-1}BP_*/I_n$ from $H_c^*(S_n; \mathbb{F}_p)$. For continuous cohomology of p-adic groups, see Lazard [24].

We will now describe the group S_n. Let Z_p denote the p-adic integers, and $\mathbb{F}_{p^n}$ the field with p^n elements. There is a complete local ring $W(\mathbb{F}_{p^n})$ (called the Witt ring of $\mathbb{F}_{p^n}$) which is a degree n extension of Z_p obtained by adjoining an element ω satisfying $\omega^{p^n-1} = 1$. The residue field of $W(\mathbb{F}_{p^n})$ is $\mathbb{F}_{p^n}$ and the extension $W(\mathbb{F}_{p^n}): Z_p$ is a lifting of the extension $\mathbb{F}_{p^n} : \mathbb{F}_p$. The Frobenius automorphism (which sends x to x^p) of the latter lifts to an automorphism of $W(\mathbb{F}_{p^n})$ over Z_p which sends ω to ω^p.

Let $E_n = W(\mathbb{F}_{p^n}) \langle\langle T \rangle\rangle / (T^n - p)$, i.e. the power series ring over $\mathbb{F}_{p^n}$ on one noncommuting variable T with $T^n = p$ and $T\omega = \omega^p T$. Then E_n is a noncommutative complete local ring with maximal ideal (T) and residue field $\mathbb{F}_{p^n}$. It is a simple algebra over Z_p with rank n^2 with Z_p-basis $\{\omega^i T^j : 0 \le i, j < n\}$. Tensoring it with the p-adic numbers Q_p (the field of fractions of Z_p) gives D_n which is a division algebra with center Q_p and Hasse invariant $\frac{1}{n}$. (The latter is an invariant in Q/Z which classifies such division algebras).

Definition 8.9 The group S_n is the group of units in E_n which are congruent to 1 modulo (T).

Details of the above description can be found in [58].

Examples 8.10 For $p = 2$ $S_1 \approx Z_2^{\times}$, the group of units in Z_2. Hence $S_1 \approx Z/2 \oplus Z_2$ and $H_c^*(S_1; \mathbb{F}_2) = \mathbb{F}_2[x, y]/(y^2)$ with $x, y \in H^1$. Corollary 8.8 leads to Ext $v_1^{-1} BP_*/2 \approx \mathbb{F}_2[v_1, v_1^{-1}, \alpha_1, \sigma]/(\sigma^2)$.

8.11 For $p > 2$, $Z_p^{\times} \approx \mathbb{F}_p^{\times} \oplus Z_p \approx Z/(p-1) \oplus Z_p$ and $H_c^*(S_1; \mathbb{F}_p) = \mathbb{F}_p[X]/(x^2)$ with $x \in H^1$. Corollary 8.8 leads to Ext $v_1^{-1} BP_*/p \approx \mathbb{F}_p[v_1, v_1^{-1}, \alpha_1]/(\alpha_1^2)$.

8.12 For $p \geq 5$,

$$\dim H_c^i(S_2;\ \mathbb{F}_p) = \begin{cases} 1 & \text{for } i = 0, 4 \\ 3 & \text{for } i = 1, 3 \\ 4 & \text{for } i = 2 \\ 0 & \text{for } i > 4 \end{cases}$$

so $\mathrm{Ext}\ BP_*/I_2$ contains a free $\mathbb{F}_p[v_2]$ module on 12 generators.

8.13 For $p = 2$, S_2 contains the quaternion group G of order 8 and the restriction map $H_c^*(S_2;\ \mathbb{F}_2) \to H^*(G;\ S_2)$ is onto.

8.14 For all primes p, S_{p-1} contains a subgroup of order p and the restriction map $H_c^*(S_{p-1};\ \mathbb{F}_p) \to H^*(Z/(p);\ \mathbb{F}_p)$ is onto. In [56] we use this map to show that for $p > 2$ all monomials in the elements β_{p^i/p^i} are nontrivial. This fact is used in the proof of Theorem 6.7.

Details of Examples 8.10 - 8.13 can be found in [51]. A useful reference for the continuous cohomology of p-adic Lie groups is Lazard [24], from which Morava has extracted

Theorem 8.15 [41] If $(p-1) \nmid n$, $H_c^*(S_n;\ \mathbb{F}_p)$ is a Poincare duality algebra of dimension n^2. If $(p-1)|n$, there is an element $b \in H_c^*(S_n^1;\ \mathbb{F}_p)$ such that $H_c^*(S_n;\ \mathbb{F}_p)$ is a finitely generated free module over $\mathbb{F}_p[b]$. □

You may well ask why Theorems 8.4 and 8.7 are true. We will try to give a heuristic explanation. Recall that a groupoid is a small category in which every morphism is an equivalence. The relevant example of such is the category $\underline{\underline{F}}(R)$ of p-typical form group laws (Definition 3.8) over a commutative $Z_{(p)}$-algebra R, and strict isomorphisms between them. (A strict isomorphism f is one with $f(x) \equiv x \bmod x^2$.) Landweber [22] has shown that the set of ring homomorphisms from BP_*BP to R is in one-to-one correspondance with the set of morphisms in $\underline{\underline{F}}(R)$, and the various structure maps of BP_*BP correspond to the various structures of the groupoid. Hence BP_*BP is a cogroupoid object in the algebras, and Haynes Miller [36] has christened such objects <u>Hopf algebroids</u> (a Hopf algebra being a co-group object).

Loosely speaking, Ext BP_* can be thought of as the cohomology of the groupoid of formal group laws over BP_* which are strictly isomorphic to the universal one. The map $BP_* \to v_n^{-1} BP_*/I_n$ induces a formal group law F_n over the latter by Theorem 3.10, and Ext $v_n^{-1} BP_*/I_n$ can be regarded as the cohomology of the groupoid of formal group laws over $v_n^{-1} BP_*/I_n$ which are strictly isomorphic to F_n. It can be shown (e.g. §19.4 of [18]) that any such formal law is canonically strictly isomorphic to the one F_n induced by $BP_* \to K(n)_* \to v_n^{-1} BP_*/I_n$. An argument similar to Landweber's shows that $\mathrm{Hom}_{\mathrm{Rings}}(K(n)_* K(n), R)$ is the groupoid of strict isomorphisms between formal group laws over the Z(p)-algebra R induced from the one over $K(n)_*$. It follows that Ext $v_n^{-1} BP_*/I_n$ and $\mathrm{Ext}_{K(n)_*K(n)}(K(n)_*, K(n)_*)$ are the cohomology groups of equivalent (in the sense of equivalence of categories) groupoids and are therefore isomorphic. This argument is the idea behind Theorem 8.4. The proof given in [37] is less abstract and less enlightening. *

For Theorem 8.7 we set $v_n = 1$ and get the Hopf algebra $S(n)_*$. (Note that $K(n)_*K(n)$ is a Hopf algebra over $K(n)_*$, but a Hopf algebroid over $\mathbb{F}_p$.) We are now dealing with the groupoid of strict isomorphisms of formal group laws over R induced by maps $\mathbb{F}_p \to R$.

* The heuristic proof of Theorem 8.4 described in the second paragraph will be made precise in a forthcoming paper by Jack Morava.

Since there is at most one such map (there are none unless R is an $\mathbb{F}_p$-algebra) our groupoid is actually the strict automorphism group of the induced formal group law over R. In the case $R = \mathbb{F}_{p^n}$, this group is known (§III. 2 of [17] or §20.4 of [18]) to be S_n, whence Theorem 8.7.

We now turn to the chromatic spectral sequence which was first introduced in §3 of [39]. For the reader's amusement, we will try to reconstruct the line of thought which led to its formulation. In §5, we observed that all elements of order p^i in $\mathrm{Ext}^1 BP_*$ are in the image of the connecting homomorphism for the short exact sequence

$$(8.16) \qquad 0 \to BP_* \xrightarrow{p^i} BP_* \longrightarrow BP_*/p^i \longrightarrow 0.$$

We would like to obtain all elements of finite order, and hence all of $\mathrm{Ext}^1 BP_*$ from a single short exact sequence. We have maps of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & BP_* & \xrightarrow{p^i} & BP_* & \longrightarrow & BP_*/p^i & \to & 0 \\ & & \| & & \downarrow p & & \downarrow & & \\ 0 & \to & BP_* & \xrightarrow{p^{i+1}} & BP_* & \longrightarrow & BP_*/p^{i+1} & \to & 0. \end{array}$$

Taking the direct limit over increasing i we get

$$(8.17) \qquad 0 \to BP_* \to p^{-1}BP_* \to BP_*/p^\infty \to 0$$

i.e. the tensor product of BP_* with the short exact sequence $0 \to Z \to Q \to Q/Z \to 0$. Since $\mathrm{Ext}^1 BP_*$ is a torsion group and $\mathrm{Ext}\, p^{-1}BP_* = Q$ concentrated in dimension 0, we see that the connecting homomorphism $\mathrm{Ext}^0 BP_*/p^\infty \to \mathrm{Ext}^1 BP_*$ is an isomorphism in positive dimensions.

In §6, we saw that all of $\mathrm{Ext}^2 BP_*$ (unless $p = 2$, in which case we get all of $\mathrm{Ext}^2 BP_*$ not accounted for in §5) comes from $\mathrm{Ext}^0 BP_*/(p^{1+i}, v_1^{p^{i+j}})$ by composing the connecting homomorphisms of (8.16) and

$$(8.18) \quad 0 \to \Sigma^{qp^{i+j}} BP_*/p^{1+i} \xrightarrow{v_1^{p^{i+j}}} BP_*/p^{1+i} \to BP_*/(p^{1+i}, v_1^{p^{i+j}}) \to 0$$

Moreover, there are maps

$$\begin{array}{ccccccccc}
0 \to & BP_*/p^{1+i} & \xrightarrow{v_1^{p^{i+j}}} & \Sigma^{-qp^{i+j}} BP_*/p^{i+1} & \to & \Sigma^{-qp^{i+j}} BP_*/(p^{1+i}, v_1^{p^{i+j}}) & \to 0 \\
& \| & & \downarrow v_1^{(p-1)p^{i+j}} & & \downarrow & \\
0 \to & BP_*/p^{1+i} & \xrightarrow{v_1^{p^{1+i+j}}} & \Sigma^{-qp^{1+i+j}} BP_*/p^{i+1} & \to & \Sigma^{-qp^{1+i+j}} BP_*/(p^{1+i}, v_1^{p^{1+i+j}}) & \to 0.
\end{array}$$

We can take the direct limit over increasing i and j and get

$$(8.19) \quad 0 \to BP_*/p^\infty \to v_1^{-1} BP_*/p^\infty \to BP_*/(p^\infty, v_1^\infty) \to 0$$

and it can be shown that for $p > 2$ the map (composition of connecting homomorphisms of 8.17 and 8.19 $\mathrm{Ext}^0 BP_*/(p^\infty, v_1^\infty) \to \mathrm{Ext}^2 BP_*$ is also an isomorphism in positive dimensions. Computing an Ext^0 is easier than computing an Ext^2 because there are no coboundaries to worry about.

We can splice 8.17 and 8.19 together to get a 4-term exact sequence

$$(8.20) \quad 0 \to BP_* \to p^{-1} BP_* \to v_1^{-1} BP_*/p^\infty \to BP_*/(p^\infty, v_1^\infty) \to 0.$$

Then $\mathrm{Ext}\, BP_*$ can be computed in terms of the Ext groups of the other comodules by means of a baby spectral sequence. Moreover, Ext $p^{-1}BP_*$ and Ext $v_1^{-1}BP_*/p$ can be computed by the theory of Morava stabilizer algebras discussed above. The latter Ext group is closely related to Ext $v_1^{-1}BP_*/p^\infty$ since there is a short exact sequence

$$(8.21) \qquad 0 \to v_1^{-1}BP_*/p \to v_1^{-1}BP_*/p^\infty \xrightarrow{p} v_1^{-1}BP_*/p^\infty \longrightarrow 0 .$$

Hence the Ext groups for the two middle terms of 8.20 are known, and we are left with computing Ext $BP_*/(p^\infty, v_1^\infty)$. Unfortunately, this seems to be just as difficult as computing Ext BP_* itself, so we have gained very little unless we iterate the procedure as follows.

Define BP_*BP-comodules M^n and N^n inductively as follows. $N^0 = BP_*$, $M^n = v_n^{-1}N^n$ (where $v_0 = p$) and N^{n+1} is the quotient in the short exact sequence

$$(8.22) \qquad 0 \to N^n \to M^n \to N^{n+1} \to 0$$

For $n = 0, 1$ this sequence is 8.17 and 8.19 respectively, and one could write $N^n = BP_*/(p^\infty, v_1^\infty \cdots v_{n-1}^\infty)$ and $M^n = v_n^{-1}BP_*/(p^\infty, v_1^\infty \cdots v_{n-1}^\infty)$. We can splice together the short exact sequences 8.21 to get a long exact sequence

$$(8.23) \qquad 0 \to BP_* \to M^0 \to M^1 \to M^2 \quad \cdots$$

<u>Theorem 8.24</u>. The long exact sequence 8.23 leads to a first quadrant cohomology spectral sequence converging to Ext BP_* with $E_1^{s,t} = \text{Ext}^t M^s$. □

We call this the <u>chromatic spectral sequence</u>. We should warn the reader that it is not at all suited for computing the Novikov E_2-term through a given range of dimensions. We have other devices for that [54], [55]. Its purpose rather is to highlight certain structural patterns in Ext BP_*, as will be explained below.

Very few of the groups $E_1^{s,t}$ have actually been computed. In [39], we compute $E_1^{0,t}$ (which is just one copy of Q in $E_1^{0,0}$ in dimension 0) and $E_1^{1,t}$ for all primes, and $E_1^{2,0}$ for $p > 2$. (We also found the corresponding groups $E_\infty^{s,t}$.) It would be interesting to know all of $E_1^{2,t}$ (especially $E_1^{2,0}$ for $p = 2$) and $E_1^{3,0}$. It is certainly possible (but not easy) to compute these groups with existing techniques. Our knowledge of $\operatorname{Ext} v_2^{-1}BP_*/I_2$, which originally motivated the whole program, has hardly been exploited.

To relate $\operatorname{Ext} M^n$ to $\operatorname{Ext} v_n^{-1}BP_*/I_n$, we need to define some more comodules M_i^{n-i}, which we do by induction on i by setting $M_0^n = M^n$ and M_{i+1}^{n-i-1} is the kernel of the short exact sequence

$$(8.25) \qquad 0 \to M_{i+1}^{n-i-1} \to M_i^{n-i} \xrightarrow{v_i} M_i^{n-i} \to 0 .$$

For $n = 1$ and $i = 0$, this is the sequence 8.21, and one could write

$$M_i^{n-i} = v_n^{-1}BP_*/(p,\ v_1,\ \cdots v_{i-1},\ v_i^\infty,\ v_{i+1}^\infty\ \cdots v_{n-1}^\infty).$$

In particular, $M_n^0 = v_n^{-1}BP_*/I_n$. Each sequence 8.25 gives a long exact sequence of Ext groups and a Bockstein type spectral sequence going from $\operatorname{Ext} M_{i+1}^{n-i-1}$ to $\operatorname{Ext} M_i^{n-i}$. Hence, once can in principle compute $\operatorname{Ext} M^n$ in terms of $\operatorname{Ext} v_n^{-1}BP_*/I_n$, which is accessible through the theory described earlier in this section. In particular, Theorem 8.15 gives a vanishing parabola (instead of a vanishing line), i.e.

<u>Corollary 8.26</u> In the chromatic spectral sequence, $E_1^{s,t} = 0$ if $(p-1)\nmid s$ and $t > s^2$. $\square$

We will now explain how one can use this apparatus to prove Theorems 5.10 and 5.14(a). One can set up chromatic spectral sequences converging to $\mathrm{Ext}\, BP_*/I_n$ by making a long exact sequence

(8.26) $$0 \to BP/I_n \to M_n^0 \to M_n^1 \to M_n^2 \quad \cdots ,$$

where the M_n^i are defined by 8.25. One gets

Theorem 8.27 The long exact sequence 8.26 leads to a first quadrant cohomology spectral sequence converging to $\mathrm{Ext}\, BP_*/I_n$ with $E_1^{s,t} = \mathrm{Ext}^t M_n^s$. []

In the case $n = 1$, we know $E_1^{0,t} = \mathrm{Ext}^t v_1^{-1} BP_*/p$ (Examples 8.10 and 8.11). The image of $\mathrm{Ext}\, BP_*/p$ in this group is simply the subgroup of elements which are permanent cycles in the chromatic spectral sequence. The differentials originating in $E_r^{0,t}$ are easily computed in this case and one finds $E_2^{0,t} = E_\infty^{0,t}$.

Finally, we will explain our use of the word 'chromatic' and define nth order phenomena in the Novikov E_2-term. Both terms refer to various types of periodicity. $\mathrm{Ext}\, v_n^{-1} BP_*/I_n$ is v_n-periodic, i.e. multiplication by v_n induces an isomorphism between $\mathrm{Ext}^{s,k} v_n^{-1} BP_*/I_n$ and $\mathrm{Ext}^{s,k+2(p^n-1)} v_n^{-1} BP_*/I_n$. Moreover, M^n can be shown to be a direct limit of comodules in which increasinly large powers of v_n give similar isomorphisms. Specifically,

Proposition 8.28 Let $M^n(i) = v_n^{-1} BP_*/(p^{1+i}, v_1^{p^i}, v_2^{p^{2i}}, \cdots v_{n-1}^{p^{(n-1)i}})$. Then multiplication by $p(v_1 v_2^2 \cdots v_{n-1}^{n-1})^{(p-1)p^i}$ gives a comodule map $M^n(i) \to M^n(i+1)$ and $M^n = \varinjlim M^n(i)$. Moreover, $v_n^{p^{ni}} \in \mathrm{Ext}^0 M^n(i)$ and multiplication by it gives an isomorphism $\mathrm{Ext}^{s,k} M^n(i) \xrightarrow{\cong} \mathrm{Ext}^{s,k+2p^{ni}(p^n-1)} M^n(i)$. []

Since $\mathrm{Ext}\, M^n = \varinjlim \mathrm{Ext}\, M^n(i)$, the former is a direct limit of periodic groups under periodic maps, or weakly periodic. Each element of $\mathrm{Ext}\, M^n(i)$ can be multiplied nontrivially by $v_n^{p^{ni}}$ and we call this property <u>nth order periodicity</u>.

Hence by <u>nth order phenomena in the Novikov spectal sequence</u> we mean the subquotient of Ext BP_* isomorphic to $E_\infty^{n,*}$ (the nth column) of the chromatic spectral sequence, and related homotopy elements.

We see then that the filtration of Ext BP_* for which the chromatic E_∞-term is the associated trigraded group, is the filtration by order of periodicity. The chromatic spectral sequence is like a spectrum in the astronomical sense that it resolves the Novikov E_2-term Ext BP_* into various 'wavelengths' or orders of periodicity. Hence the adjective 'chromatic'.

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