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Complex Cobordism and its Applications to Homotopy Theory

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In the past few years, the application of complex cobordism to problems in homotopy theory through the medium of the Adams–Novikov spectral sequence has become a lucrative enterprise. We will give a brief survey of some of the foundations and results of this theory, offering nothing new for the experts. See [9] for a more detailed account, including references for some of the statements made here.

The history of the subject begins with Thom's definition [10] of cobordism. Roughly speaking, 2 closed manifolds are *cobordant* if their disjoint union is the boundary of a third manifold. In the complex case, we require that these manifolds possess compatible complex structures on their stable tangent bundles. Cobordism is easily seen to be an equivalence relation and the set of equivalence classes is a ring *(the complex cobordsim ring* MU_**)* under disjoint union and Cartesian product. Thom proved that this ring is canonically isomorphic to the homotopy of the complex Thom spectrum MU. Milnor [5] and Novikov [6] showed that $MU_* = \pi_* MU = Z[x_1, x_2, \ldots]$ where $\dim x_i = 2i$. Brown–Peterson [3] showed that when localized at a prime p, MU splits into an infinite wedge of isomorphic summands known as BP with $\pi_* BP = BP_* = Z_{(p)}[x_{p^i-1}]$.

Since homotopy theory is essentially a local (in the arithmetic sense) subject we shall concern ourselves primarily with the smaller spectrum BP. Once its basic properties have been established, its relation to complex manifolds becomes irrelevant to the applications. Our understanding of these properties rests on a remarkable observation due to Quillen.

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Let $MU^*(\ )$ be the generalized cohomology theory represented by the spectrum MU. Then $MU^*(CP^\infty)=MU^*[[x]]$ where $x\in MU^2(CP^\infty)$ and MU^* is the coefficient ring $\pi_* MU$ negatively graded. We also have $MU^*(CP^\infty\times CP^\infty)=MU^*[[x\otimes 1, 1\otimes x]]$ and the tensor product (of complex line bundles) map $f\colon CP^\infty\times CP^\infty\to CP^\infty$ induces $f^*\colon MU^*(CP^\infty)\to MU^*(CP^\infty\times CP^\infty)$ with $f^*(x)=F(x\otimes 1, 1\otimes x)=\sum a_{ij}x^i\otimes x^j$ with $a_{ij}\in MU^{2(1-i-j)}$. The 2-variable power series F has 3 obvious properties: $F(x,0)=F(0,x)=x$ (identity); $F(x,y)=F(y,x)$ (commutativity); and $F(F(x,y),z)=F(x,F(y,z))$ (associativity). We define a *formal group law* G over a commutative ring R to be a power series $G(x,y)\in R[[x,y]]$ having the three properties of F. Quillen's observation was

THEOREM 1 [8]. *The formal group law F over MU^* is universal in the sense that for any other formal group law G over R, there is a unique ring homomorphism $\theta\colon MU^*\to R$ such that* $G(x,y)=\sum\theta(a_{ij})x^iy^j$. □

THEOREM 2 [8]. *There is a map $\varepsilon\colon MU_*\to BP_*$ such that any formal group law G over a $Z_{(p)}$-algebra R is canonically isomorphic to a formal group law G' induced by $\theta'\varepsilon$ where $\theta'\colon BP_*\to R$ (i.e. there is a power series $f(x)\in R[[x]]$ with leading term x such that $f(G(x,y))=G'(f(x),f(y))$).* □

Quillen was able to use these results to determine the structure of BP^*BP, the algebra of cohomology operations for the theory represented by the spectrum BP. This algebra, the BP analogue of the Steenrod algebra, is difficult to work with because it does not have finite type and cannot be readily described in terms of generators and relations. Instead we will describe its dual $BP_*BP=\pi_*BP\wedge BP$, the analogue of the dual Steenrod algebra.

First, we described the formal group law εF, which we will denote simply by F. Define $\log x\in(Q\otimes BP_*)[[x]]$ by $\log x=\sum_{i\geq 0} l_i x^{p^i}$ where $l_i=\varepsilon[CP^{p^i-1}]/p^i$. Then $F(x,y)$ is defined by

$$\log F(x,y) = F(\log x, \log y). \tag{3}$$

THEOREM 4 ([8], [1]). *As an algebra $BP_*BP=BP_*[t_1,t_2,\ldots]$ with $\dim t_i=2(p^i-1)$. The Hurewicz or right unit map $\eta_R\colon BP_*\to BP_*BP$ (induced by $BP=S^0\wedge BP\to BP\wedge BP$) is given over Q by*

$$\eta_R l_n = \sum l_i t_{n-i}^{p^i}. \tag{5}$$

This map defines a right BP_*-module structure on BP_*BP and the coproduct (dual to composition of cohomology operations) is a map $\Delta\colon BP_*BP\to BP_*BP\otimes_{BP_*}BP_*BP$ defined over Q by

$$\sum_{i\geq 0}\log\Delta(t_i) = \sum_{i,j\geq 0}\log(t_i\otimes t_j^{p^i}) \tag{6}$$

where $t_0=1$. □

The lack of a more explicit formula for $\Delta(t_i)$ was for some time a psychological obstruction to computing with BP. (6) can be rewritten as

$$\sum^F \Delta(t_i) = \sum^F t_i \otimes t_j^{p^i}, \tag{7}$$

(where $\log(\sum^F x_i) = \sum \log x_i$, i.e. $\sum^F x_i$ is the formal sum of the x_i), but this is of little help due to the complexity of F. Another difficulty is that the elements $p^i l_i = \varepsilon[CP^{p^i-1}]$ do not generate BP_*. This problem was surmounted first by Hazewinkel and later by Araki.

THEOREM 8 (ARAKI). *$BP_* = Z_{(p)}[v_1, v_2, \ldots]$ where v_n is defined by $pl_n = \sum_{0 \leq i \leq n} l_i v_{n-i}^{p^i}$ with $v_0 = p$.* □

THEOREM 9. *$\eta_R(v_i)$ is given by*

$$\sum_{i,j\geq 0}^F v_i t_j^{p^i} = \sum_{i,j\geq 0}^F t_i \eta_R(v_j)^{p^i}. \quad \Box$$

This completes our survey of the foundations of the subject. We turn now to some applications in the homotopy groups of spheres. Novikov first formulated an MU analogue of the Adams spectral sequence. His main result can be restated as

THEOREM 10 (NOVIKOV [7]). *Let X be a connective spectrum. There is a spectral sequence converging to $Z_{(p)} \otimes \pi_* X$ with $E_2^{**} = \mathrm{Ext}_{BP_*BP}^{**}(BP_*, BP_*X)$.* □

For the definition of this Ext, see [9]. In it, BP_*X can be replaced by any BP_*BP-comodule M. From now on we will abbreviate this to Ext M.

For $X = S^0$ the E_2-term is $\mathrm{Ext}\, BP_*$ which has the following convenient sparseness property.

PROPOSITION 11. *$\mathrm{Ext}^{s,t} BP_* = 0$ if $t \not\equiv 0 \mod 2(p-1)$. Consequently, in the Adams–Novikov spectral sequence for S^0, $E_{2+2r(p-1)}^{**} = E_{2p-1+2r(p-1)}^{**}$ for $r \geq 0$. In particular, the first nontrivial differential is d_{2p-1} so all nontrivial elements in $E_2^{s,t}$ for $t \leq 2(p-1)$ which are permanent cycles are nontrivial in E_∞^{**}.* □

This spectral sequence has fewer differentials and extensions (at least for p odd) than the classical Adams spectral sequence based on mod p cohomology, i.e. its E_2-term is a closer approximation of stable homotopy. For example, for $p>2$, $\mathrm{Ext}^1 BP_*$ is isomorphic to $\mathrm{Im}\, J$, the image of the Hopf–Whitehead J-homomorphism, and for $p=3$ there are no differentials below dimension 33.

An unstable form of this spectral sequence has recently been constructed and used by Bendersky–Curtis–Miller [2]. It appears to be a very promising device.

In studying the classical Adams spectral sequence one learns that elements in $\mathrm{Ext}^1_{\mathscr{A}}(Z/p, Z/p)$ correspond to generators of the Steenrod algebra $\mathscr{A}$ while elements in $\mathrm{Ext}^2_{\mathscr{A}}(Z/p, Z/p)$ correspond to relations among these generators. However, this point of view appears not to be helpful in understanding $\mathrm{Ext}^1 BP_*$ and $\mathrm{Ext}^2 BP_*$.

We will now describe the Greek letter construction, which is an entirely different method of manufacturing elements in $\mathrm{Ext}\, BP_*$.

An ideal $I \subset BP_*$ is *invariant* if BP_*/I is a BP_*BP-comodule, i.e. if $\eta_R I \subset I BP_*BP$. Invariant ideals are rare as the following result shows.

THEOREM 12 (MORAVA, LANDWEBER). (a) *The only invariant prime ideals in* BP_* *are* $I_n = (p, v_1, \ldots, v_{n-1})$ *for* $0 \leq n \leq \infty$ *(*I_0 *is the zero ideal).*

(b) $\mathrm{Ext}^0\, BP_* = Z_{(p)}$ *and* $\mathrm{Ext}^0\, BP_*/I_n = F_p[v_n]$ *for* $0 < n < \infty$.

(c) *The following is a short exact sequence of* BP_*BP*-comodules.*

$$0 \to \Sigma^{2(p^n-1)} BP_*/I_n \xrightarrow{v_n} BP_*/I_n \to BP_*/I_{n+1} \to 0. \quad \square \tag{13}$$

Now let

$$\delta_n \colon \mathrm{Ext}^{s,t}\, BP_*/I_{n+1} \to \mathrm{Ext}^{s+1,\, t-2(p^n-1)}\, BP_*/I_n$$

be the connecting homomorphism associated with (13). Then we can define the following elements, commonly known as Greek letters, in the Adams–Novikov E_2-term $\mathrm{Ext}\, BP_*$:

$$\begin{aligned} \alpha_t &= \delta_0(v_1^t) \in \mathrm{Ext}^{1,\, 2(p-1)t}\, BP_*, \\ \beta_t &= \delta_0\delta_1(v_2^t) \in \mathrm{Ext}^{2,\, 2(p^2-1)t-2(p-1)}\, BP_*, \\ \gamma_t &= \delta_0\delta_1\delta_2(v_3^t) \in \mathrm{Ext}^{3,\, 2(p^3-1)t-2(p-1)-2(p^2-1)}\, BP_*. \end{aligned} \tag{14}$$

Of course, this definition generalizes to $\eta_t^{(n)}$, where $\eta^{(n)}$ denotes the nth letter of the Greek alphabet.

In order to apply this construction to homotopy theory one must prove two things: that the elements so defined are nontrivial in $\mathrm{Ext}\, BP_*$ and that they are permanent cycles in the Adams–Novikov spectral sequence. It will then follow from Proposition 11 that the resulting elements in E_∞ are nontrivial, so they detect nontrivial homotopy classes.

THEOREM 15 (SEE [9] FOR REFERENCES). (a) *The elements* α_t $(t>0)$ *are nontrivial for* $p \geq 2$ *and are permanent cycles for* $p \geq 3$. *(They detect the elements of order* p *in* $\mathrm{Im}\, J$.*)*

(b) *The elements* β_t $(t>0)$ *are nontrivial for* $p \geq 3$ *and are permanent cycles for* $p \geq 5$.

(c) *The elements* γ_t $(t>0)$ *are nontrivial for* $p \geq 3$ *and are permanent cycles for* $p \geq 7$. $\square$

The nontriviality result is an algebraic computation, while the construction of the corresponding homotopy elements, due to H. Toda and Larry Smith, is as follows. One constructs finite complexes $V(n-1)$ $(n \leq 4)$ with $BP_*V(n-1) = BP_*/I_n$ by means of cofibrations $(n \leq 3)$

$$\Sigma^{2(p^n-1)} V(n-1) \xrightarrow{\varphi_n} V(n-1) \to V(n)$$

realizing the sequence (13), with $V(-1)=S^0$. Then $\eta_t^{(n)}$ is the composition

$$S^{2t(p^n-1)} \xrightarrow{i} \Sigma^{2t(p^n-1)} V(n-1) \xrightarrow{\varphi_n^t} V(n-1) \xrightarrow{j} S^k$$

where i is the inclusion of the bottom cell, j is the collapsing onto the top cell, and $k=\sum_{0\le m<n}(2p^m-1)$.

One can generalize the Greek letter construction by replacing the invariant prime ideals I_n by invariant regular ideals. Regularity is precisely what is needed to get short exact sequences generalizing (13). For $p\geq 3$ it is known that all elements in $\mathrm{Ext}^1 BP_*$ and $\mathrm{Ext}^2 BP_*$ arise in this way.

However, not all elements in $\mathrm{Ext}^3 BP_*$ come from $\mathrm{Ext}^0 BP/I$ for an invariant regular ideal I with 3 generators. For example, the elements $\alpha_1\beta_t$ arise from elements in $\mathrm{Ext}^1 BP_*/I_2$ which are free under multiplication by v_2, so they cannot come from $\mathrm{Ext}^0 BP_*/(p, v_1, v_2^k)$ for any k. What is true is that every element in $\mathrm{Ext}\, BP_*$ is the image of some element in $\mathrm{Ext}\, BP_*/I$ (where I is an invariant regular ideal with n generators) which is free under multiplication by the powers of v_n belonging to $\mathrm{Ext}^0 BP_*/I$.

Hence in some sense every element of the Adams–Novikov E_2-term is a member of an infinite periodic family of the type exemplified most simply by the $\eta_t^{(n)}$ of (14). Whether a similar statement can be made about stable homotopy itself is still an open question. In light of this situation, one would like to classify these periodic families. A machine for doing this known as the chromatic spectral sequence was set up in [4]. One begins by looking at the $F_p[v_n]$-free summand of $\mathrm{Ext}\, BP_*/I_n$, which maps monomorphically to $v_n^{-1}\mathrm{Ext}\, BP_*/I_n=\mathrm{Ext}\, v_n^{-1} BP_*/I_n$. This group is surprisingly easy to compute, due to some farsighted work of Jack Morava. His results indicate a striking connection between homotopy theory and local algebraic number theory. We can only give the barest description here.

$\mathrm{Ext}\, v_n^{-1} BP_*/I_n$ is a free module over $K(n)_*=\mathrm{Ext}^0 v_n^{-1} BP_*/I_n=F_p[v_n, v_n^{-1}]$. We make F_{p^n} a nongraded $K(n)_*$-module by sending v_n to 1. Then we have

THEOREM 16. $F_{p^n}\otimes_{K(n)_*}\mathrm{Ext}\, v_n^{-1} BP_*/I_n=H_c^*(S_n, F_{p^n})$, *the continuous cohomology (with trivial action on* F_{p^n}*) of the compact p-adic Lie group* S_n, *which is the p-Sylow subgroup of the automorphism group of the (height n) formal group law over* F_{p^n} *induced by* $BP_*\to K(n)_*\to F_{p^n}$. □

For example $H_c^* S_n$ has the following Poincaré series $f(t)=\sum(\dim H_c^i S_n)t^i$:

$p \backslash n$	1	2	3
2	$(1+t)/(1-t)$	$(1+t)^3(1-t^6)/(1-t)(1-t^4)$	?
3	$1+t$	$(1+t)^2(1+t^2)/(1-t)$	?
≥ 5	$1+t$	$(1+t)^2(1+t+t^2)$	$(1+t)^3(1+t+6t^2+3t^3+6t^4+t^5+t^6)$

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