DE GRUYTER Forum Math. 2017; 29 (2):383–447

Research Article

Michael A. Hill, Michael J. Hopkins and Douglas C. Ravenel*

The slice spectral sequence for the C_4 analog of real K-theory

DOI: 10.1515/forum-2016-0017
Received January 22, 2016

Abstract: We describe the slice spectral sequence of a 32-periodic C_4-spectrum $K_{[2]}$ related to the C_4 norm $\mathrm{MU}^{((C_4))} = N_{C_2}^{C_4}\mathrm{MU}_{\mathbb{R}}$ of the real cobordism spectrum $\mathrm{MU}_{\mathbb{R}}$. We will give it as a spectral sequence of Mackey functors converging to the graded Mackey functor $\underline{\pi}_* K_{[2]}$, complete with differentials and exotic extensions in the Mackey functor structure. The slice spectral sequence for the 8-periodic real K-theory spectrum $K_{\mathbb{R}}$ was first analyzed by Dugger. The C_8 analog of $K_{[2]}$ is 256-periodic and detects the Kervaire invariant classes θ_j. A partial analysis of its slice spectral sequence led to the solution to the Kervaire invariant problem, namely the theorem that θ_j does not exist for $j \geq 7$.

Keywords: Equivariant stable homotopy theory, Kervaire invariant, Mackey functor, slice spectral sequence

MSC 2010: Primary 55Q10; secondary 55Q91, 55P42, 55R45, 55T99

Communicated by: Frederick R. Cohen

1 Introduction

In [6] we derived the main theorem about the Kervaire invariant elements from some properties of a C_8-equivariant spectrum we called Ω constructed as follows. We started with the C_2-spectrum $\mathrm{MU}_{\mathbb{R}}$, meaning the usual complex cobordism spectrum MU equipped with a C_2 action defined in terms of complex conjugation.

Then we defined a functor $N_{C_2}^{C_8}$, the norm of [6, Section 2.2.3] which we abbreviate here by N_2^8, from the category of C_2-spectra to that of C_8-spectra. Roughly speaking, given a C_2-spectrum X, $N_2^8 X$ is underlain by the fourfold smash power $X^{\wedge 4}$ where a generator γ of C_8 acts by cyclically permuting the four factors, each of which is invariant under the given action of the subgroup C_2. In a similar way one can define a functor N_H^G from H-spectra to G-spectra for any finite groups $H \subseteq G$.

A C_8-spectrum such as $N_2^8\mathrm{MU}_{\mathbb{R}}$, which is a commutative ring spectrum, has equivariant homotopy groups indexed by $RO(C_8)$, the orthogonal representation ring for the group C_8. One element of the latter is ρ_8, the regular representation. In [6, Section 9] we defined a certain element $D \in \pi_{19\rho_8} N_2^8\mathrm{MU}_{\mathbb{R}}$ and then formed the associated mapping telescope, which we denoted by $\Omega_{\mathbb{O}}$. The symbol $\mathbb{O}$ was chosen to suggest a connection with the octonions, but there really is none apart from the fact that the octonions are 8-dimensional like ρ_8.

Note that $\Omega_{\mathbb{O}}$ is also a C_8-equivariant commutative ring spectrum. We then proved that it is equivariantly equivalent to $\Sigma^{256}\Omega_{\mathbb{O}}$; we call this result the Periodicity Theorem. Then our spectrum Ω is $\Omega_{\mathbb{O}}^{C_8}$, the fixed point spectrum of $\Omega_{\mathbb{O}}$.

Michael A. Hill: Department of Mathematics, University of California Los Angeles, Los Angeles, CA 90095, USA,
e-mail: mikehill@math.ucla.edu
Michael J. Hopkins: Department of Mathematics, Harvard University, Cambridge, MA 02138, USA,
e-mail: mjh@math.harvard.edu
***Corresponding author: Douglas C. Ravenel:** Department of Mathematics, University of Rochester, Rochester, NY 14627, USA,
e-mail: doug@math.rochester.edu

It is possible to do this with C_8 replaced by C_{2^n} for any n. The dimension of the periodicity is then $2^{1+n+2^{n-1}}$. For example it is 32 for the group C_4 and 2^{13} for C_{16}. We chose the group C_8 because it is the smallest that suits our purposes, namely it is the smallest one yielding a fixed point spectrum that detects the Kervaire invariant elements θ_j.

We know almost nothing about $\pi_*\Omega$, only that it is periodic with periodic 256, that $\pi_{-2} = 0$ (the Gap Theorem of [6, Section 8]), and that when θ_j exists its image in $\pi_*\Omega$ is nontrivial (the Detection Theorem of [6, Section 11]).

We also know, although we did not say so in [6], that more explicit computations would be much easier if we cut $N_2^8 \mathrm{MU}_{\mathbb{R}}$ down to size in the following way. Its underlying homotopy, meaning that of the spectrum $\mathrm{MU}^{\wedge 4}$, is known classically to be a polynomial algebra over the integers with four generators (cyclically permuted up to sign by the group action) in every positive even dimension. This can be proved with methods described by Adams in [1]. For the cyclic group C_{2^n} one has 2^{n-1} generators in each positive even degree. Specific generators $r_{i,j} \in \pi_{2i}\mathrm{MU}^{\wedge 2^{n-1}}$ for $i > 0$ and $0 \leq j <^{n-1}$ are defined in [6, Section 5.4.2].

There is a way to kill all the generators above dimension $2k$ that was described in [6, Section 2.4]. Roughly speaking, let A be a wedge of suspensions of the sphere spectrum, one for each monomial in the generators one wants to kill. One can define a multiplication and group action on A corresponding to the ones in $\pi_*\mathrm{MU}^{\wedge 4}$. Then one has a map $A \to \mathrm{MU}^{\wedge 4}$ whose restriction to each summand represents the corresponding monomial, and a map $A \to S^0$ (where the target is the sphere spectrum, not the space S^0) sending each positive-dimensional summand to a point. This leads to two maps

$$S^0 \wedge A \wedge \mathrm{MU}^{\wedge 4} \rightrightarrows S^0 \wedge \mathrm{MU}^{\wedge 4}$$

whose coequalizer we denote by $S^0 \wedge_A \mathrm{MU}^{\wedge 4}$. Its homotopy is the quotient of $\pi_*\mathrm{MU}^{\wedge 4}$ obtained by killing the polynomial generators above dimension $2k$. The construction is equivariant, meaning that $S^0 \wedge_A \mathrm{MU}^{\wedge 4}$ underlies a C_8-spectrum.

In [6, Section 7] we showed that for $k = 0$ the spectrum we get is the integer Eilenberg–Mac Lane spectrum $H\mathbf{Z}$; we called this result the Reduction Theorem. In the nonequivariant case this is obvious. We are in effect attaching cells to $\mathrm{MU}^{\wedge 4}$ to kill all of its homotopy groups in positive dimensions, which amounts to constructing the 0th Postnikov section. In the equivariant case the proof is more delicate.

Now consider the case $k = 1$, meaning that we are killing the polynomial generators above dimension 2. Classically we know that doing this to MU (without the C_2-action) produces the connective complex K-theory spectrum, some times denoted by k, bu or (2-locally) $\mathrm{BP}\langle 1\rangle$. Inverting the Bott element via a mapping telescope gives us K itself, which is of course 2-periodic. In the C_2-equivariant case one gets the "real K-theory" spectrum $K_{\mathbb{R}}$ first studied by Atiyah in [3]. It turns out to be 8-periodic and its fixed point spectrum is KO, which is also referred to in other contexts as real K-theory.

The spectrum we get by killing the generators above dimension 2 in the C_8-spectrum $N_2^8 \mathrm{MU}_{\mathbb{R}}$ will be denoted analogously by $k_{[3]}$. We can invert the image of D by forming a mapping telescope, which we will denote by $K_{[3]}$. More generally we denote by $k_{[n]}$ the spectrum obtained from $N_{C_2}^{C_{2^n}} \mathrm{MU}_{\mathbb{R}}$ by killing all generators above dimension 2. In particular, $k_{[1]} = k_{\mathbb{R}}$. Then we denote the mapping telescope (after defining a suitable D) by $K_{[n]}$ and its fixed point set by $\mathrm{KO}_{[n]}$.

For $n \geq 3$, $\mathrm{KO}_{[n]}$ also has a Periodicity, Gap and Detection Theorem, so it could be used to prove the Kervaire Invariant Theorem.

Thus $K_{[3]}$ *is a substitute for* $\Omega_{\mathbb{O}}$ *with much smaller and therefore more tractable homotopy groups.* A detailed study of them might shed some light on the fate of θ_6 in the 126-stem, the one hypothetical Kervaire invariant element whose status is still open. *If we could show that* $\pi_{126}\mathrm{KO}_{[3]} = 0$, *that would mean that* θ_6 *does not exist.*

The computation of the equivariant homotopy $\underline{\pi}_* K_{[3]}$ at this time is daunting. *The purpose of this paper is to do a similar computation for the group* C_4 *as a warmup exercise.* In the process of describing it we will develop some techniques that are likely to be needed in the C_8 case. We start with $N_2^4 \mathrm{MU}_{\mathbb{R}}$, kill its polynomial generators (of which there are two in every positive even dimension) above dimension 2 as described previously, and then invert a certain element in $\pi_{4\rho_4}$. We denote the resulting spectrum by $K_{[2]}$, see Definition 7.3 below. This spectrum is known to be 32-periodic. In an earlier draft of this paper it was denoted by $K_{\mathbf{H}}$.

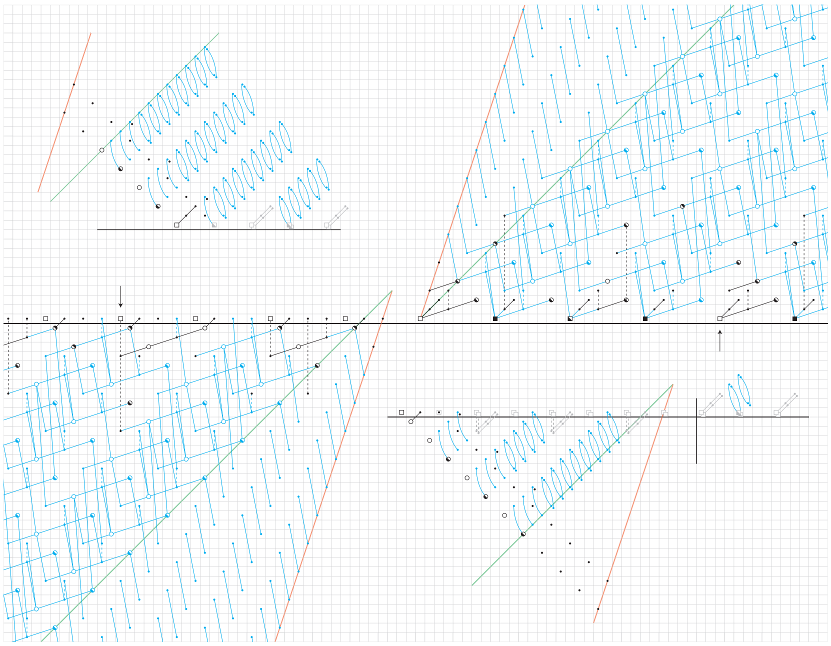


Figure 1. The 2008 poster. The first and third quadrants show $\underline{E}_4(G/G)$ of the slice spectral sequence for $K_{[2]}$ with the elements of Proposition 13.4 excluded. The second quadrant indicates d_3s as in Figures 9 and 10. The fourth quadrant indicates comparable d_3s in the third quadrant of the slice spectral sequence as in Figures 11 and 12.

The computational tool for finding these homotopy groups is the slice spectral sequence introduced in [6, Section 4]. Indeed we do not know of any other way to do it. For $K_{\mathbb{R}}$ it was first analyzed by Dugger [4] and his work is described below in Section 8. In this paper we will study the slice spectral sequence of Mackey functors associated with $K_{[2]}$. We will rely extensively on the results, methods and terminology of [6].

We warn the reader that the computation for $K_{[2]}$ is more intricate than the one for $K_{\mathbf{R}}$. For example, the slice spectral sequence for $K_{\mathbf{R}}$, which is shown in Figure 7, involves five different Mackey functors for the group C_2. We abbreviate them with certain symbols indicated in Table 1. The one for $K_{[2]}$, partly shown in Figure 16, involves over twenty Mackey functors for the group C_4, with symbols indicated in Table 2.

Part of this spectral sequence is also illustrated in an unpublished poster produced in late 2008 and shown in Figure 1. It shows the spectral sequence converging to the homotopy of the fixed point spectrum $K_{[2]}^{C_4}$. The corresponding spectral sequence of Mackey functors converges to the graded Mackey functor $\underline{\pi}_* K_{[2]}$.

In both illustrations some patterns of d_3s and families of elements in low filtration are excluded to avoid clutter. In the poster, representative examples of these are shown in the second and fourth quadrants, the spectral sequence itself being concentrated in the first and third quadrants. In this paper those patterns are spelled out in Section 12 and Section 13.

We now outline the rest of the paper. Briefly, the next five sections introduce various tools we need. Our objects of study, the spectra $k_{[2]}$ and $K_{[2]}$, are formally introduced in Section 7. Dugger's computation for $K_{\mathbf{R}}$ is recalled in Section 8. The final six sections describe the computation for $k_{[2]}$ and $K_{[2]}$.

In more detail, Section 2 collects some notions from equivariant stable homotopy theory with an emphasis on Mackey functors. Definition 2.7 introduces new notation that we will occasionally need.

Section 3 concerns the equivariant analog of the homology of a point namely, the $RO(G)$-graded homotopy of the integer Eilenberg–Mac Lane spectrum $H\mathbf{Z}$. In particular, Lemma 3.6 describes some relations among certain elements in it including the "gold relation" between a_V and u_V.

Section 4 describes some general properties of spectral sequences of Mackey functors. These include Theorem 4.4 about the relation between differential and exotic extensions in the Mackey functor structure and Theorem 4.7 on the norm of a differential.

Section 5 lists some concise symbols for various specific Mackey functors for the groups C_2 and C_4 that we will need. Such functors can be spelled out explicitly by means of Lewis diagrams (5.1), which we usually abbreviate by symbols shown in Tables 1 and 2.

In Section 6 we study some chain complexes of Mackey functors that arise as cellular chain complexes for G-CW complexes of the form S^V.

In Section 7 we formally define (in Definition 7.3) the C_4-spectra of interest in this paper, $k_{[2]}$ and $K_{[2]}$.

In Section 8 we shall describe the slice spectral sequence for an easier case, the C_2-spectrum for real K-theory, $K_{\mathbb{R}}$. This is due to Dugger [4] and serves as a warmup exercise for us. It turns out that everything in the spectral sequence is formally determined by the structure of its $\underline{E}_2$-term and Bott periodicity.

In Section 9 we introduce various elements in the homotopy groups of $k_{[2]}$ and $K_{[2]}$. They are collected in Table 3, which spans several pages. In Section 10 we determine the $\underline{E}_2$-term of the slice spectral sequence for $k_{[2]}$ and $K_{[2]}$.

In Section 11 we use the Slice Differentials Theorem of [6] to determine some differentials in our spectral sequence.

In Section 12 we examine the C_4-spectrum $k_{[2]}$ as a C_2-spectrum. This leads to a calculation only slightly more complicated than Dugger's. It gives a way to remove a lot of clutter from the C_4 calculation.

In Section 13 we determine the $\underline{E}_4$-term of our spectral sequence. It is far smaller than $\underline{E}_2$ and the results of Section 12 enable us to ignore most of it. What is left is small enough to be shown legibly in the spectral sequence charts of Figures 14 and 16. They illustrate integrally graded (as opposed to $RO(C_4)$-graded) spectral sequences of Mackey functors, which are discussed in Section 5. In order to read these charts one needs to refer to Table 2 which defines the "hieroglyphic" symbols we use for the specific Mackey functors that we need.

We finish the calculation in Section 14 by dealing with the remaining differentials and exotic Mackey functor extensions. It turns out that they are all formal consequences of C_2 differentials of the previous section along with the results of Section 4.

The result is a complete description of the *integrally graded* portion of $\underline{\pi}_\star k_{[2]}$. It is best seen in the spectral sequence charts of Figures 14 and 16. Unfortunately, we do not have a clean description, much less an effective way to display the full $RO(C_4)$-graded homotopy groups.

For $G = C_2$, the two irreducible orthogonal representations are the trivial one of degree 1, denoted by the symbol 1, and the sign representation denoted by σ. Thus $RO(G)$ is additively a free abelian group of rank 2, and the spectral sequence of interest is trigraded. In the $RO(C_2)$-graded homotopy of $K_{\mathbf{R}}$, a certain element of degree $1 + \sigma$ (the degree of the regular representation ρ_2) is invertible. This means that each component of $\underline{\pi}_\star K_{\mathbf{R}}$ is canonically isomorphic to a Mackey functor indexed by an ordinary integer. See Theorem 8.6 for a more precise statement. Thus the full (trigraded) $RO(C_2)$-graded slice spectral sequence is determined by bigraded one shown in Figure 7.

For $G = C_4$, the representation ring $RO(G)$ is additively a free abelian group of rank 3, so it leads to a quadrigraded spectral sequence. The three irreducible representations are the trivial and sign representations 1 and σ (each having degree one) and a degree two representation λ given by a rotation of the plane $\mathbb{R}^2$ of order 4. The regular representation ρ_4 is isomorphic to $1 + \sigma + \lambda$. As in the case of $K_{\mathbf{R}}$, there is an invertible element $\bar{\mathfrak{d}}_1$ (see Table 3) in $\underline{\pi}_\star K_{[2]}$ of degree ρ_4. This means we can reduce the quadigraded slice spectral sequence to a trigraded one, but finding a full description of it is a problem for the future.

2 Recollections about equivariant stable homotopy theory

We first discuss some structure on the equivariant homotopy groups of a G-spectrum X. *We will assume throughout that G is a finite cyclic p-group.* This means that its subgroups are well ordered by inclusion and each is uniquely determined by its order. The results of this section hold for any prime p, but the rest of the paper concerns only the case $p = 2$. We will define several maps indexed by pairs of subgroups of G. *We will often replace these indices by the orders of the subgroups, sometimes denoting $|H|$ by h.*

The homotopy groups can be defined in terms of finite G-sets T. Let

$$\underline{\pi}_0^G X(T) = [T_+, X]^G$$

be the set of homotopy classes of equivariant maps from T_+, the suspension spectrum of the union of T with a disjoint base point, to the spectrum X. We will often omit G from the notation when it is clear from the context. For an orthogonal representation V of G, we define

$$\underline{\pi}_V X(T) = [S^V \wedge T_+, X]^G.$$

As an RO(G)-graded contravariant abelian group valued functor of T, this converts disjoint unions to direct sums. This means it is determined by its values on the sets G/H for subgroups $H \subseteq G$.

Since G is abelian, H is normal and $\underline{\pi}_V X(G/H)$ is a $Z[G/H]$-module.

Given subgroups $K \subseteq H \subseteq G$, one has pinch and fold maps between the H-spectra H/H_+ and H/K_+. This leads to a diagram

$$\begin{array}{ccccccc} & & H/H_+ & \underset{\text{fold}}{\overset{\text{pinch}}{\rightleftarrows}} & H/K_+ & & \\ & & & \Big\Downarrow {\scriptstyle G_+\wedge_H(\cdot)} & & & \\ G/H_+ & = & G_+ \wedge_H H/H_+ & \underset{\text{fold}}{\overset{\text{pinch}}{\rightleftarrows}} & G_+ \wedge_H H/K_+ & = G_+ \wedge_K K/K_+ = & G/K_+. \end{array} \tag{2.1}$$

Note that while the fold map is induced by a map of H-sets, the pinch map is not. It only exists in the stable category.

Definition 2.2 (The Mackey functor structure maps in $\underline{\pi}_V^G X$). The fixed point transfer and restriction maps

$$\underline{\pi}_V X(G/H) \underset{\text{res}_K^H}{\overset{\text{tr}_K^H}{\leftrightarrows}} \underline{\pi}_V X(G/K)$$

are the ones induced by the composite maps in the bottom row of (2.1).

These satisfy the formal properties needed to make $\underline{\pi}_V X$ into a Mackey functor; see [6, Definition 3.1]. They are usually referred to simply as the transfer and restriction maps. We use the words "fixed point" to distinguish them from another similar pair of maps specified below in Definition 2.11.

We remind the reader that a Mackey functor $\underline{M}$ for a finite group G assigns an abelian group $\underline{M}(T)$ to every finite G-set T. It converts disjoint unions to direct sums. It is therefore determined by its values on orbits, meaning G-sets for the form G/H for various subgroups H of G. For subgroups $K \subseteq H \subseteq G$, one has a map of G-sets $G/K \to G/H$. In categorical language $\underline{M}$ is actually a pair of functors, one covariant and one contravariant, both behaving the same way on objects. Hence we get maps both ways between $\underline{M}(G/K)$ and $\underline{M}(G/H)$. For the Mackey functor $\underline{\pi}_V X$, these are the two maps of Definition 2.2.

One can generalize the definition of a Mackey functor by replacing the target category of abelian groups by one's favorite abelian category, such as that of R-modules over graded abelian groups.

Definition 2.3. A *graded Green functor* $\underline{R}_*$ *for a group* G is a Mackey functor for G with values in the category of graded abelian groups such that $\underline{R}_*(G/H)$ is a graded commutative ring for each subgroup H and for each pair of subgroups $K \subseteq H \subseteq G$, the restriction map res_K^H is a ring homomorphism and the transfer map tr_K^H satisfies the *Frobenius relation*

$$\text{tr}_K^H(\text{res}_K^H(a)b) = a\,(\text{tr}_K^H(b)) \quad \text{for } a \in \underline{R}_*(G/H) \text{ and } b \in \underline{R}_*(G/K).$$

When X is a ring spectrum, we have the *fixed point Frobenius relation*

$$\text{tr}_K^H(\text{res}_K^H(a)b) = a\,(\text{tr}_K^H(b)) \quad \text{for } a \in \underline{\pi}_\star X(G/H) \text{ and } b \in \underline{\pi}_\star X(G/K). \tag{2.4}$$

In particular, this means that

$$a(\mathrm{tr}_K^H(b)) = 0 \quad \text{when } \mathrm{res}_K^H(a) = 0. \tag{2.5}$$

For a representation V of G, the group

$$\underline{\pi}_V^G X(G/H) = \pi_V^H X = [S^V, X]^H$$

is isomorphic to

$$[S^0, S^{-V} \wedge X]^H = \pi_0(S^{-V} \wedge X)^H.$$

However fixed points do not respect smash products, so we cannot equate this group with

$$\pi_0(S^{-V^H} \wedge X^H) = [S^{V^H}, X^H] = \pi_{|V^H|} X^H = \underline{\pi}_{|V^H|}^G X(G/H).$$

Conversely a G-equivariant map $S^V \to X$ represents an element in

$$[S^V, X]^G = \pi_V^G X = \underline{\pi}_V^G X(G/G).$$

The following notion is useful.

Definition 2.6 (Mackey functor induction and restriction). For s subgroup H of G and an H-Mackey functor $\underline{M}$, the induced G-Mackey functor $\uparrow_H^G \underline{M}$ is given by

$$\uparrow_H^G \underline{M}(T) = \underline{M}(i_H^* T)$$

for each finite G-set T, where i_H^* denotes the forgetful functor from G-sets (or spaces or spectra) to H-sets.

For a G-Mackey functor $\underline{N}$, the restricted H-Mackey functor $\downarrow_H^G \underline{N}$ is given by

$$\downarrow_H^G \underline{N}(S) = \underline{N}(G \times_H S)$$

for each finite H-set S.

This notation is due to Thévenaz–Webb [10]. They put the decorated arrow on the right and denote $G \times_H S$ by $S \uparrow_H^G$ and $i_H^* T$ by $T \downarrow_H^G$.

We also need notation for X as an H-spectrum for subgroups $H \subseteq G$. For this purpose we will enlarge the orthogonal representation ring of G, $\mathrm{RO}(G)$, to the representation ring Mackey functor $\underline{\mathrm{RO}}(G)$ defined by $\underline{\mathrm{RO}}(G)(G/H) = \mathrm{RO}(H)$. This was the motivating example for the definition of a Mackey functor in the first place. In it the transfer map on a representation V of H is the induced representation of a supergroup $K \supseteq H$, and its restriction to a subgroup is defined in the obvious way. In particular, the restriction of the transfer of V is $|K/H|V$.

More generally for a finite G-set T, $\underline{\mathrm{RO}}(G)(T)$ is the ring (under pointwise direct sum and tensor product) of functors to the category of finite-dimensional orthogonal real vector spaces from $B_G T$, the split groupoid (see [9, A1.1.22]) whose objects are the elements of T with morphisms defined by the action of G.

Definition 2.7 ($\underline{\mathrm{RO}}(G)$-graded homotopy groups). For each G-spectrum X and each pair (H, V) consisting of a subgroup $H \subseteq G$ and a virtual orthogonal representation V of H, let the G-Mackey functor $\underline{\pi}_{H,V}(X)$ be defined by

$$\underline{\pi}_{H,V}(X)(T) := \left[(G_+ \wedge_H S^V) \wedge T_+, X\right]^G \cong \left[S^V \wedge i_H^* T_+, i_H^* X\right]^H = \underline{\pi}_V^H(i_H^* X)(i_H^* T),$$

for each finite G-set T. Equivalently, $\underline{\pi}_{H,V}(X) = \uparrow_H^G \underline{\pi}_V^H(i_H^* X)$ (see 2.6) as Mackey functors. We will often denote $\underline{\pi}_{G,V}$ by $\underline{\pi}_V^G$ or $\underline{\pi}_V$.

We will be studying the $\underline{\mathrm{RO}}(G)$-graded slice spectral sequence $\{\underline{E}_r^{s,\star}\}$ of Mackey functors with $r, s \in \mathbf{Z}$ and $\star \in \underline{\mathrm{RO}}(G)$. We will use the notation $\underline{E}_r^{s,(H,V)}$ for such Mackey functors, abbreviating to $\underline{E}_r^{s,V}$ when the subgroup is G. Most of our spectral sequence charts will display the values of $\underline{E}_2^{s,t}$ for integral values of t only.

The following definition should be compared with [2, (2.3)].

Definition 2.8 (An equivariant homeomorphism). Let X be a G-space and Y an H-space for a subgroup $H \subseteq G$. We define the equivariant homeomorphism

$$\tilde{u}_H^G(Y, X) : G \times_H (Y \times i_H^* X) \to (G \times_H Y) \times X$$

by $(g, y, x) \mapsto (g, y, g(x))$ for $g \in G$, $y \in Y$ and $x \in X$. We will use the same notation for a similarly defined homeomorphism

$$\tilde{u}_H^G(Y, X) : G_+ \wedge_H (Y \wedge i_H^* X) \to (G_+ \wedge_H Y) \wedge X$$

for a G-spectrum X and H-spectrum Y. We will abbreviate

$$\tilde{u}_H^G(S^0, X) : G_+ \wedge_H i_H^* X \to G/H_+ \wedge X$$

by $\tilde{u}_H^G(X)$.

For representations V and V' of G both restricting to W on H, but having distinct restrictions to all larger subgroups, we define $\tilde{u}_{V-V'} = \tilde{u}_H^G(S^V)\tilde{u}_H^G(S^{V'})^{-1}$, so the following diagram of equivariant homeomorphisms commutes:

$$\begin{array}{ccc} & \xrightarrow{\tilde{u}_H^G(S^V)} & G/H \wedge S^V \\ G_+ \wedge_H S^W & & \uparrow \tilde{u}_{V-V'} \\ & \xrightarrow[\tilde{u}_H^G(S^{V'})]{} & G/H \wedge S^{V'}. \end{array} \tag{2.9}$$

When $V' = |V|$ (meaning that $H = G_V$ acts trivially on W), then we abbreviate $\tilde{u}_{V-V'}$ by $\tilde{u}_V$.

If V is a representation of H restricting to W on K, we can smash the diagram (2.1) with S^V and get

$$\begin{array}{ccccccc} S^V & \underset{\text{fold}}{\overset{\text{pinch}}{\rightleftarrows}} & H/K_+ \wedge S^V & & & & \\ \Big\Downarrow {\scriptstyle G_+\wedge_H(\cdot)} & & & & & & \\ G_+ \wedge_H S^V & \underset{\text{fold}}{\overset{\text{pinch}}{\rightleftarrows}} & G_+ \wedge_H (H/K_+ \wedge S^V) & \xrightarrow{\approx} & G_+ \wedge_H (H_+ \wedge_K S^W) & = & G_+ \wedge_K S^W, \end{array} \tag{2.10}$$

where the homeomorphism is induced by that of Definition 2.8.

Definition 2.11 (The group action restriction and transfer maps). For subgroups $K \subseteq H \subseteq G$, let $V \in \mathrm{RO}(H)$ be a virtual representation of H restricting to $W \in \mathrm{RO}(K)$. The group action transfer and restriction maps

$$\uparrow_H^G \underline{\pi}_V^H(i_H^* X) = \underline{\pi}_{H,V} X \underset{\underline{r}_K^H}{\overset{\underline{t}_K^{H,V}}{\rightleftarrows}} \underline{\pi}_{K,W} X = \uparrow_K^G \underline{\pi}_W^K(i_K^* X)$$

(see 2.6) are the ones induced by the composite maps in the bottom row of (2.10). The symbols t and r here are underlined because they are maps *of* Mackey functors rather than maps within Mackey functors.

We include V as an index for the group action transfer $\underline{t}_K^{H,V}$ because its target is not determined by its source.

Thus we have abelian groups $\underline{\pi}_{H',V}(X)(G/H'')$ for all subgroups $H', H'' \subseteq G$ and representations V of H'. Most of them are redundant in view of Theorem 2.13 below. In what follows, we will use the notation $H_\cup = H' \cup H''$ and $H_\cap = H' \cap H''$.

Lemma 2.12 (An equivariant module structure). *For a G-spectrum X and H'-spectrum Y,*

$$[G_+ \wedge_{H'} Y, X]^{H''} = \mathbf{Z}[G/H_\cup] \otimes [H_{\cup +} \wedge_{H'} Y, X]^{H''}$$

as $\mathbf{Z}[G/H'']$-modules.

Proof. As abelian groups,

$$[G_+ \wedge_{H'} Y, X]^{H''} = [i_{H''}^*(G_+ \wedge_{H'} Y), X]^{H''} = \Big[\bigvee_{|G/H_\cup|} H_{\cup +} \wedge_{H'} Y, X\Big]^{H''} = \bigoplus_{|G/H_\cup|} [H_{\cup +} \wedge_{H'} Y, X]^{H''}$$

and G/H'' permutes the wedge summands of $\bigvee_{|G/H_\cup|} H_{\cup +} \wedge_{H'} Y$ as it permutes the elements of $G/H_\cup$. □

Theorem 2.13 (The module structure for $\underline{\mathrm{RO}(G)}$-graded homotopy groups). *For subgroups $H', H'' \subseteq G$ with $H_\cup = H' \cup H''$ and $H_\cap = H' \cap H''$, and a virtual representation V of H' restricting to W on $H_\cap$,*

$$\underline{\pi}_{H',V} X(G/H'') \cong \mathbf{Z}[G/H_\cup] \otimes \underline{\pi}_{H_\cap,W} X(G/G) \cong \mathbf{Z}[G/H_\cup] \otimes \underline{\pi}_W^{H_\cap} i_{H_\cap}^* X(H_\cap/H_\cap)$$

as $\mathbf{Z}[G/H'']$-modules.

Suppose that H'' is a proper subgroup of H' and $\gamma \in H'$ is a generator. Then as an element in $\mathbf{Z}[G/H'']$, γ induces multiplication by -1 in $\underline{\pi}_{H',V} X(G/H'')$ if and only if V is nonorientable.

Proof. We start with the definition and use the homeomorphism of Definition 2.8 and the module structure of Lemma 2.12:

$$\begin{aligned}
\underline{\pi}_{H',V} X(G/H'') &= [(G_+ \wedge_{H'} S^V) \wedge G/H''_+, X]^G \\
&= [G_+ \wedge_{H''} (G_+ \wedge_{H'} S^V), X]^G \\
&= [G_+ \wedge_{H'} S^V, X]^{H''} = \mathbf{Z}[G/H_\cup] \otimes [H_{\cup +} \wedge_{H'} S^V, X]^{H''}, \\
[H_{\cup +} \wedge_{H'} S^V, X]^{H''} &= [S^W, X]^{H_\cap} \\
&= [G_+ \wedge_{H_\cap} S^W, X]^G \\
&= \underline{\pi}_W^{H_\cap} (i_{H_\cap}^* X)(H_\cap/H_\cap) = \underline{\pi}_{H_\cap,W} X(G/G).
\end{aligned}$$

For the statement about nonoriented V, we have

$$\underline{\pi}_{H',V} X(G/H'') = \mathbf{Z}[G/H'] \otimes \underline{\pi}_W^{H''} i_{H''}^* X(H''/H'') = \mathbf{Z}[G/H'] \otimes [S^W, X]^{H''}.$$

Then γ induces a map of degree ± 1 on the sphere depending on the orientability of V. □

Theorem 2.13 means that we need only consider the groups

$$\underline{\pi}_{H,V} X(G/G) \cong \underline{\pi}_V^H i_H^* X(H/H).$$

When $H \subset G$ and V is a virtual representation of G, we have

$$\underline{\pi}_V X(G/H) \cong \underline{\pi}_{H,i_H^* V} X(G/G) \cong \underline{\pi}_{i_H^* V}^H i_H^* X(H/H). \tag{2.14}$$

This isomorphism makes the following diagram commute for $K \subseteq H$:

$$\begin{array}{ccccc}
\underline{\pi}_V X(G/H) & \xrightarrow{\cong} & \underline{\pi}_{H,i_H^* V} X(G/G) & \xrightarrow{\cong} & \underline{\pi}_{i_H^* V}^H i_H^* X(H/H) \\
\mathrm{res}_K^H \downarrow \uparrow \mathrm{tr}_K^H & & \underline{r}_K^H \downarrow \uparrow \underline{t}_K^{H,i_H^* V} & & \\
\underline{\pi}_V X(G/K) & \xrightarrow{\cong} & \underline{\pi}_{K,i_K^* V} X(G/G) & \xrightarrow{\cong} & \underline{\pi}_{i_K^* V}^K i_K^* X(K/K).
\end{array}$$

We will use the three groups of (2.14) *interchangeably as convenient and use the same notation for elements in each related by this canonical isomorphism.* Note that the group on the left is indexed by $\mathrm{RO}(G)$ while the two on the right are indexed by $\mathrm{RO}(H)$. This means that if V and V' are representations of G each restricting to W on H, then $\underline{\pi}_V X(G/H)$ and $\underline{\pi}_{V'} X(G/H)$ are canonically isomorphic. The first of these is

$$[G/H_+ \wedge S^V, X]^G \cong [G_+ \wedge_H S^W, X]^G \cong [S^W, i_H^* X]^H,$$

where the first isomorphism is induced by the homeomorphism $\tilde{u}_H^G(X)$ of Definition 2.8 and the second is the fact that $G_+ \wedge_H (\cdot)$ is the left adjoint of the forgetful functor i_H^*.

Remark 2.15 (Factorization via restriction). For a ring spectrum X, such as the one we are studying in this paper, an indecomposable element in $\underline{\pi}_\star X(G/H)$ may map to a product $xy \in \underline{\pi}_{H,\star} X(G/G)$ of elements in groups indexed by representations of H that are not restrictions of representations of G. When this happens we may denote the indecomposable element in $\underline{\pi}_\star X(G/H)$ by $[xy]$. This factorization can make some computations easier.

3 The RO(G)-graded homotopy of $H\mathbf{Z}$

We describe part of the RO(G)-graded Green functor $\underline{\pi}_\star(H\mathbf{Z})$, where $H\mathbf{Z}$ is the integer Eilenberg–Mac Lane spectrum $H\mathbf{Z}$ in the G-equivariant category, for some finite cyclic 2-group G. For each actual (as opposed to virtual) G-representation V we have an equivariant reduced cellular chain complex C_*^V for the space S^V. It is a complex of $\mathbf{Z}[G]$-modules with $H_*(C^V) = H_*(S^{|V|})$.

One can convert such a chain complex C_*^V of $\mathbf{Z}[G]$-modules to one of Mackey functors as follows. Given a $\mathbf{Z}[G]$-module M, we get a Mackey functor $\underline{M}$ defined by

$$\underline{M}(G/H) = M^H \quad \text{for each subgroup } H \subseteq G. \tag{3.1}$$

We call this a *fixed point Mackey functor*. In it each restriction map res_K^H (for $K \subseteq H \subseteq G$) is one-to-one. When M is a permutation module, meaning the free abelian group on a G-set B, we call $\underline{M}$ a *permutation Mackey functor* [6, Section 3.2].

In particular, the $Z[G]$-module $\mathbf{Z}$ with trivial group action (the free abelian group on the G-set G/G) leads to a Mackey functor $\underline{\mathbf{Z}}$ in which each restriction map is an isomorphism and the transfer map tr_K^H is multiplication by $|H/K|$. For each Mackey functor $\underline{M}$ there is an Eilenberg–Mac Lane spectrum $H\underline{M}$ (see [5, Section 5]), and $H\underline{\mathbf{Z}}$ is the same as $H\mathbf{Z}$ with trivial group action.

Given a finite G-CW spectrum X, meaning one built out of cells of the form $G_+ \wedge_H e^n$, we get a reduced cellular chain complex of $\mathbf{Z}[G]$-modules C_*X, leading to a chain complex of fixed point Mackey functors $\underline{C}_*X$. Its homology is a graded Mackey functor $\underline{H}_*X$ with

$$\underline{H}_*X(G/H) = \underline{\pi}_*(X \wedge H\underline{\mathbf{Z}})(G/H) = \pi_*(X \wedge H\underline{\mathbf{Z}})^H.$$

In particular, $\underline{H}_*X(G/\{e\}) = H_*X$, the underlying homology of X. In general $\underline{H}_*X(G/H)$ is not the same as $H_*(X^H)$ because fixed points do not commute with smash products.

For a finite cyclic 2-group $G = C_{2^k}$, the irreducible representations are the 2-dimensional ones $\lambda(m)$ corresponding to rotation through an angle of $2\pi m/2^k$ for $0 < m < 2^{k-1}$, the sign representation σ and the trivial one of degree one, which we denote by 1. The 2-local equivariant homotopy type of $S^{\lambda(m)}$ depends only on the 2-adic valuation of m, so we will only consider $\lambda(2^j)$ for $0 \le j \le k-2$ and denote it by λ_j. The planar rotation λ_{k-1} though angle π is the same representation as 2σ. *We will denote* $\lambda(1) = \lambda_0$ *simply by* λ.

We will describe the chain complex C^V for

$$V = a + b\sigma + \sum_{2 \le j \le k} c_j \lambda_{k-j}$$

for nonnegative integers a, b and c_j. The isotropy group of V (the largest subgroup fixing all of V) is

$$G_V = \begin{cases} C_{2^k} = G & \text{for } b = c_2 = \cdots = c_k = 0, \\ C_{2^{k-1}} =: G' & \text{for } b > 0 \text{ and } c_2 = \cdots = c_k = 0, \\ C_{2^{k-\ell}} & \text{for } c_\ell > 0 \text{ and } c_{1+\ell} = \cdots = c_k = 0. \end{cases}$$

The sphere S^V has a G-CW structure with reduced cellular chain complex C^V of the form

$$C_n^V = \begin{cases} \mathbf{Z} & \text{for } n = d_0, \\ \mathbf{Z}[G/G'] & \text{for } d_0 < n \le d_1, \\ \mathbf{Z}[G/C_{2^{k-j}}] & \text{for } d_{j-1} < n \le d_j \text{ and } 2 \le j \le \ell, \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

where

$$d_j = \begin{cases} a & \text{for } j = 0, \\ a + b & \text{for } j = 1, \\ a + b + 2c_2 + \cdots + 2c_j & \text{for } 2 \le j \le \ell, \end{cases}$$

so $d_\ell = |V|$.

The boundary map $\partial_n : C_n^V \to C_{n-1}^V$ is determined by the fact that $H_*(C^V) = H_*(S^{|V|})$. More explicitly, let γ be a generator of G and

$$\zeta_j = \sum_{0 \le t < 2^j} \gamma^t \quad \text{for } 1 \le j \le k.$$

Then we have

$$\partial_n = \begin{cases} \nabla & \text{for } n = 1 + d_0, \\ (1-\gamma)x_n & \text{for } n - d_0 \text{ even and } 2 + d_0 \le n \le d_n, \\ x_n & \text{for } n - d_0 \text{ odd and } 2 + d_0 \le n \le d_n, \\ 0 & \text{otherwise,} \end{cases}$$

where ∇ is the fold map sending $\gamma \mapsto 1$, and x_n denotes multiplication by an element in $\mathbf{Z}[G]$ to be named below. We will use the same symbol below for the quotient map $\mathbf{Z}[G/H] \to \mathbf{Z}[G/K]$ for $H \subseteq K \subseteq G$. The elements $x_n \in \mathbf{Z}[G]$ for $2 + d_0 \le n \le |V|$ are determined recursively by $x_{2+d_0} = 1$ and

$$x_n x_{n-1} = \zeta_j \quad \text{for } 2 + d_{j-1} < n \le 2 + d_j.$$

It follows that $H_{|V|}C^V = \mathbf{Z}$ generated by either $x_{1+|V|}$ or its product with $1-\gamma$, depending on the parity of b.

This complex is

$$C^V = \Sigma^{|V_0|} C^{V/V_0},$$

where $V_0 = V^G$. This means we can assume without loss of generality that $V_0 = 0$.

An element

$$x \in H_n \underline{C}^V(G/H) = \underline{H}_n S^V(G/H)$$

corresponds to an element $x \in \underline{\pi}_{n-V} H\mathbf{Z}(G/H)$.

We will denote the dual complex $\mathrm{Hom}_{\mathbf{Z}}(C^V, \mathbf{Z})$ by C^{-V}. Its chains lie in dimensions $-n$ for $0 \le n \le |V|$. An element $x \in \underline{H}_{-n}(S^{-V})(G/H)$ corresponds to an element $x \in \underline{\pi}_{V-n} H\mathbf{Z}(G/H)$.

The method we have just described determines only a portion of the $RO(G)$-graded Mackey functor $\underline{\pi}_{(G,\star)} H\underline{\mathbf{Z}}$, namely the groups in which the index differs by an integer from an actual representation V or its negative. For example, it does not give us $\underline{\pi}_{\sigma-\lambda} H\underline{\mathbf{Z}}$ for $|G| \ge 4$.

We leave the proof of the following as an exercise for the reader.

Proposition 3.3 (The top (bottom) homology groups for S^V (S^{-V})). *Let G be a finite cyclic 2-group and V a nontrivial representation of G of degree d with $V^G = 0$ and isotropy group G_V. Then*

$$C_d^V = C_{-d}^{-V} = \mathbf{Z}[G/G_V]$$

and the following hold:

(i) *If V is oriented, then $\underline{H}_d S^V = \underline{\mathbf{Z}}$, the constant $\mathbf{Z}$-valued Mackey functor in which each restriction map is an isomorphism and each transfer tr_H^K is multiplication by $|K/H|$.*

(ii) *$\underline{H}_{-d} S^{-V} = \underline{\mathbf{Z}}(G, G_V)$, the constant $\mathbf{Z}$-valued Mackey functor in which*

$$\mathrm{res}_H^K = \begin{cases} 1 & \text{for } K \subseteq G_V, \\ |K/H| & \text{for } G_V \subseteq H, \end{cases}$$

and

$$\mathrm{tr}_H^K = \begin{cases} |K/H| & \text{for } K \subseteq G_V, \\ 1 & \text{for } G_V \subseteq H. \end{cases}$$

(The above completely describes the cases where $|K/H| = 2$, and they determine all other restrictions and transfers.) The functor $\underline{\mathbf{Z}}(G, e)$ is also known as the dual $\underline{\mathbf{Z}}^*$. These isomorphisms are induced by the maps

$$\begin{array}{ccccc} \underline{H}_d S^V & & & & \underline{H}_{-d} S^{-V} \\ \| & & & & \| \\ \underline{\mathbf{Z}} & \xrightarrow{\Delta} & \underline{\mathbf{Z}[G/G_V]} & \xrightarrow{\nabla} & \underline{\mathbf{Z}}(G, G_V). \end{array}$$

(iii) *If V is not oriented, then $\underline{H}_d S^V = \underline{\mathbf{Z}}_-$, where*

$$\underline{\mathbf{Z}}_-(G/H) = \begin{cases} 0 & \text{for } H = G, \\ \mathbf{Z}_- := \mathbf{Z}[G]/(1+\gamma) & \text{otherwise,} \end{cases}$$

where each restriction map res_H^K *is an isomorphism and each transfer* tr_H^K *is multiplication by* $|K/H|$ *for each proper subgroup K.*

(iv) *We also have $\underline{H}_{-d} S^{-V} = \underline{\mathbf{Z}}(G, G_V)_-$, where*

$$\underline{\mathbf{Z}}(G, G_V)_-(G/H) = \begin{cases} 0 & \text{for } H = G \text{ and } V = \sigma, \\ \mathbf{Z}/2 & \text{for } H = G \text{ and } V \neq \sigma, \\ \mathbf{Z}_- & \text{otherwise,} \end{cases}$$

with the same restrictions and transfers as $\underline{\mathbf{Z}}(G, G_V)$. These isomorphisms are induced by the maps

$$\begin{array}{ccccc} \underline{H}_d S^V & & & & \underline{H}_{-d} S^{-V} \\ \| & & & & \| \\ \underline{\mathbf{Z}}_- & \xrightarrow{\Delta_-} & \underline{\mathbf{Z}[G/G_V]} & \xrightarrow{\nabla_-} & \underline{\mathbf{Z}}(G, G_V)_-. \end{array}$$

The Mackey functor $\underline{\mathbf{Z}(G, G_V)}$ is one of those defined (with different notation) in [7, Definition 2.1].

Definition 3.4 (Three elements in $\underline{\pi}_\star^G(H\mathbf{Z})$). Let V be an actual (as opposed to virtual) representation of the finite cyclic 2-group G with $V^G = 0$ and isotropy group G_V.

(i) The equivariant inclusion $S^0 \to S^V$ defines an element in $\underline{\pi}_{-V} S^0(G/G)$ via the isomorphisms

$$\underline{\pi}_{-V} S^0(G/G) = \underline{\pi}_0 S^V(G/G) = \pi_0 S^{V^G} = \pi_0 S^0 = \mathbf{Z},$$

and we will use the symbol a_V to denote its image in $\underline{\pi}_{-V} H\underline{\mathbf{Z}}(G/G)$.

(ii) The underlying equivalence $S^V \to S^{|V|}$ defines an element in

$$\underline{\pi}_V S^{|V|}(G/G_V) = \underline{\pi}_{V-|V|} S^0(G/G_V)$$

and we will use the symbol e_V to denote its image in $\underline{\pi}_{V-|V|} H\underline{\mathbf{Z}}(G/G_V)$.

(iii) If W is an oriented representation of G (we do not require that $W^G = 0$), there is a map

$$\Delta : \mathbf{Z} \to C_{|W|}^W = \mathbf{Z}[G/G_W]$$

as in Proposition 3.3 giving an element

$$u_W \in \underline{H}_{|W|} S^W(G/G) = \underline{\pi}_{|W|-W} H\mathbf{Z}(G/G).$$

For nonoriented W, Proposition 3.3 gives a map

$$\Delta_- : \mathbf{Z}_- \to C_{|W|}^W$$

and an element

$$u_W \in \underline{H}_{|W|} S^W(G/G') = \underline{\pi}_{|W|-W} H\mathbf{Z}(G/G').$$

The element u_W above is related to the element $\tilde{u}_V$ of (2.9) as follows.

Lemma 3.5 (The restriction of u_W to a unit and permanent cycle). *Let W be a nontrivial representation of G with $H = G_W$. Then the homeomorphism*

$$\Sigma^{-W} \tilde{u}_W : G/H_+ \wedge S^{|W|-W} \to G/H_+$$

of (2.9) *induces an isomorphism $\underline{\pi}_0 H\underline{\mathbf{Z}}(G/H) \to \underline{\pi}_{|W|-W} H\underline{\mathbf{Z}}(G/H)$ sending the unit to $\operatorname{res}_H^K(u_W)$ for u_W as defined in* (iii) *above and $K = G$ or G' depending on the orientability of W.*

The product

$$\operatorname{res}_H^K(u_W) e_W \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/H) = \mathbf{Z}$$

is a generator, so e_W and $\operatorname{res}_H^K(u_W)$ are units in the ring $\underline{\pi}_\star H\underline{\mathbf{Z}}(G/H)$, and $\operatorname{res}_H^K(u_W)$ is in the Hurewicz image of $\underline{\pi}_\star S^0(G/H)$.

Proof. The diagram

$$G/K_+ \wedge S^{|W|-W} \xleftarrow{\text{fold}} G/H_+ \wedge S^{|W|-W} \xrightarrow{\tilde{u}_W} G/H_+$$

induces (via the functor $[\,\cdot\,, H\underline{\mathbf{Z}}]^G$)

$$\begin{array}{ccccc} \underline{\pi}_{|W|-W} H\underline{\mathbf{Z}}(G/K) & \xrightarrow{\operatorname{res}^K_H} & \underline{\pi}_{|W|-W} H\underline{\mathbf{Z}}(G/H) & \xleftarrow{\cong} & \underline{\pi}_0 H\underline{\mathbf{Z}}(G/H) \\ \| & & \| & & \| \\ \underline{H}_{|W|} S^W(G/K) & & \underline{H}_{|W|} S^W(G/H) & & \mathbf{Z}. \end{array}$$

The restriction map is an isomorphism by Proposition 3.3 and the group on the left is generated by u_W.

The product is the composite of H-maps

$$S^W \xrightarrow{e_W} S^{|W|} \xrightarrow{\operatorname{res}^K_H(u_W)} \Sigma^W H\underline{\mathbf{Z}},$$

which is the standard inclusion. □

Note that a_V and e_V are induced by maps to equivariant spheres while u_W is not. This means that in any spectral sequence based on a filtration where the subquotients are equivariant $H\mathbf{Z}$-modules, elements defined in terms of a_V and e_V will be permanent cycles, while multiples and powers of u_W can support nontrivial differentials. Lemma 3.5 says a certain restriction of u_W is a permanent cycle.

Each nonoriented V has the form $W + \sigma$ where σ is the sign representation and W is oriented. It follows that

$$u_V = u_\sigma \operatorname{res}^G_{G'}(u_W) \in \underline{\pi}_{|V|-V} H\mathbf{Z}(G/G').$$

Note also that $a_0 = e_0 = u_0 = 1$. The trivial representations contribute nothing to $\pi_\star(H\mathbf{Z})$. We can limit our attention to representations V with $V^G = 0$. Among such representations of cyclic 2-groups, the oriented ones are precisely the ones of even degree.

Lemma 3.6 (Properties of a_V, e_V and u_W). *The three elements $a_V \in \underline{\pi}_{-V} H\mathbf{Z}(G/G)$, $e_V \in \underline{\pi}_{V-|V|} H\mathbf{Z}(G/G_V)$ and $u_W \in \underline{\pi}_{|W|-W} H\mathbf{Z}(G/G)$ for W oriented of Definition 3.4 satisfy the following:*

(1) $a_{V+W} = a_V a_W$ *and* $u_{V+W} = u_V u_W$.

(2) $|G/G_V| a_V = 0$, *where G_V is the isotropy group of V.*

(3) *For oriented V, $\operatorname{tr}^G_{G_V}(e_V)$ and $\operatorname{tr}^{G'}_{G_V}(e_{V+\sigma})$ have infinite order, while $\operatorname{tr}^G_{G_V}(e_{V+\sigma})$ has order 2 if $|V| > 0$ and $\operatorname{tr}^G_{G_V}(e_\sigma) = \operatorname{tr}^G_{G'}(e_\sigma) = 0$.*

(4) *For oriented V and $G_V \subseteq H \subseteq G$,*

$$\operatorname{tr}^G_{G_V}(e_V) u_V = |G/G_V| \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/G) = \mathbf{Z},$$
$$\operatorname{tr}^{G'}_{G_V}(e_{V+\sigma}) u_{V+\sigma} = |G'/G_V| \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/G') = \mathbf{Z} \quad \text{for } |V| > 0.$$

(5) $a_{V+W} \operatorname{tr}^G_{G_V}(e_{V+U}) = 0$ *if* $|V| > 0$.

(6) *For V and W oriented,* $u_W \operatorname{tr}^G_{G_V}(e_{V+W}) = |G_V/G_{V+W}| \operatorname{tr}^G_{G_V}(e_V)$.

(7) The gold (or au) relation. *For V and W oriented representations of degree 2 with $G_V \subseteq G_W$,*

$$a_W u_V = |G_W/G_V| a_V u_W.$$

For nonoriented W similar statements hold in $\underline{\pi}_\star H\mathbf{Z}(G/G')$. Moreover, $2W$ is oriented and u_{2W} is defined in $\underline{\pi}_{2|W|-2W} H\underline{\mathbf{Z}}(G/G)$ with $\operatorname{res}^G_{G'}(u_{2W}) = u_W^2$.

Proof. (1) This follows from the existence of the pairing $C^V \otimes C^W \to C^{V+W}$. It induces an isomorphism in H_0 and (when both V and W are oriented) in $H_{|V+W|}$.

(2) This holds because $H_0(V)$ is killed by $|G/G_V|$.

(3) This follows from Proposition 3.3.

(4) Using the Frobenius relation we have

$$\begin{aligned} \operatorname{tr}^G_{G_V}(e_V) u_V &= \operatorname{tr}^G_{G_V}(e_V \operatorname{res}^G_{G_V}(u_V)) = \operatorname{tr}^G_{G_V}(1) \quad \text{by Lemma 3.5} \\ &= |G/G_V|, \end{aligned}$$

and

$$\operatorname{tr}_{G_V}^{G'}(e_{V+\sigma})u_{V+\sigma} = \operatorname{tr}_{G_V}^{G'}(e_{V+\sigma}\operatorname{res}_{G_V}^{G'}(u_{V+\sigma})) = \operatorname{tr}_{G_V}^{G'}(1) = |G'/G_V|.$$

(5) We have

$$a_{V+W}\operatorname{tr}_{G_V}^{G}(e_{V+U}) : S^{-|V|-|U|} \to S^{W-U}.$$

It is null because the bottom cell of S^{W-U} is in dimension $-|U|$.

(6) Since V is oriented, we are computing in a torsion free group so we can tensor with the rationals. It follows from (4) that

$$\operatorname{tr}_{G_{V+W}}^{G}(e_{V+W}) = \frac{|G/G_{V+W}|}{u_V u_W} \quad \text{and} \quad \operatorname{tr}_{G_V}^{G}(e_V) = \frac{|G/G_V|}{u_V}$$

so

$$u_W \operatorname{tr}_{G_{V+W}}^{G}(e_{V+W}) = \frac{|G/G_{V+W}|}{u_V} = |G_V/G_{V+W}|\operatorname{tr}_{G_V}^{G}(e_V).$$

(7) For $G = C_{2^n}$, each oriented representation of degree 2 is 2-locally equivalent to a λ_j for $0 \le j < n$. The isotropy group is $G_{\lambda_j} = C_{2^j}$. Hence the assumption that $G_V \subset G_W$ is can be replaced with $V = \lambda_j$ and $W = \lambda_k$ with $0 \le j < k < n$. The statement we wish to prove is

$$a_{\lambda_k} u_{\lambda_j} = 2^{k-j} a_{\lambda_j} u_{\lambda_k}.$$

One has a map $S^{\lambda_j} \to S^{\lambda_k}$ which is the suspension of the 2^{k-j}th power map on the equatorial circle. Hence its underlying degree is 2^{k-j}. We will denote it by $a_{\lambda_k}/a_{\lambda_j}$ since there is a diagram

$$\begin{array}{ccc} & & S^{\lambda_j} \\ & \overset{a_{\lambda_j}}{\nearrow} & \downarrow {\scriptstyle a_{\lambda_k}/a_{\lambda_j}} \\ S^0 & \underset{a_{\lambda_k}}{\searrow} & S^{\lambda_k}. \end{array}$$

We claim there is a similar diagram

$$\begin{array}{ccc} & & S^{\lambda_k} \wedge H\underline{\mathbf{Z}} \\ & \overset{u_{\lambda_k}}{\nearrow} & \downarrow {\scriptstyle u_{\lambda_j}/u_{\lambda_k}} \\ S^2 & \underset{u_{\lambda_j}}{\searrow} & S^{\lambda_j} \wedge H\underline{\mathbf{Z}}, \end{array} \tag{3.7}$$

in which the underlying degree of the vertical map is one.

Smashing $a_{\lambda_k}/a_{\lambda_j}$ with $H\underline{\mathbf{Z}}$ and composing with $u_{\lambda_j}/u_{\lambda_k}$ gives a factorization of the degree 2^{k-j} map on $S^{\lambda_j} \wedge H\underline{\mathbf{Z}}$. Thus we have

$$\frac{u_{\lambda_j}}{u_{\lambda_k}}\frac{a_{\lambda_k}}{a_{\lambda_j}} = 2^{k-j},$$
$$u_{\lambda_j} a_{\lambda_k} = 2^{k-j} u_{\lambda_k} a_{\lambda_j},$$

as desired.

The vertical map in (3.7) would follow from a map

$$S^{\lambda_k - \lambda_j} \to H\underline{\mathbf{Z}}$$

with underlying degree one. Let $G = C_{2^n}$ and $G \supset H = C_{2^j}$. Then $S^{-\lambda_j}$ has a cellular structure of the form

$$G/H_+ \wedge S^{-2} \cup G/H_+ \wedge e^{-1} \cup e^0.$$

We need to smash this with S^{λ_k}. Since λ_k restricts trivially to H,

$$G/H_+ \wedge S^{\lambda_k} = G/H_+ \wedge S^2.$$

This means

$$S^{\lambda_k-\lambda_j} = S^{\lambda_k} \wedge S^{-\lambda_j} = G/H_+ \wedge S^0 \cup G/H_+ \wedge e^1 \cup e^0 \wedge S^{\lambda_k}.$$

Thus its cellular chain complex has the form

$$\begin{array}{cccc}
2 & \mathbf{Z}[G/K] & \xrightarrow{\ \Delta\ } & \\
 & \downarrow{\scriptstyle 1-\gamma} & & \\
1 & \mathbf{Z}[G/K] & \xrightarrow{\ -\Delta\ } & \mathbf{Z}[G/H] \\
 & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle 1-\gamma} \\
0 & \mathbf{Z} & & \mathbf{Z}[G/H],
\end{array}$$

where $K = G/C_{p^k}$ and the left column is the chain complex for S^{λ_k}.

There is a corresponding chain complex of fixed point Mackey functors. Its value on the G-set G/L for an arbitrary subgroup L is

$$\begin{array}{cccc}
2 & \mathbf{Z}[G/\max(K,L)] & \xrightarrow{\ \Delta\ } & \\
 & \downarrow{\scriptstyle 1-\gamma} & & \\
1 & \mathbf{Z}[G/\max(K,L)] & \xrightarrow{\ -\Delta\ } & \mathbf{Z}[G/\max(H,L)] \\
 & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle 1-\gamma} \\
0 & \mathbf{Z} & & \mathbf{Z}[G/\max(H,L)].
\end{array}$$

For each L the map Δ is injective and maps the kernel of the first $1-\gamma$ isomorphically to the kernel of the second one. This means we can replace the above by a diagram of the form

$$\begin{array}{cccc}
1 & \operatorname{coker}(1-\gamma) & \xrightarrow{\ -\Delta\ } & \\
 & \downarrow{\scriptstyle \nabla} & & \\
0 & \mathbf{Z} & & \operatorname{coker}(1-\gamma),
\end{array}$$

where each cokernel is isomorphic to $\mathbf{Z}$ and each map is injective.

This means that $\underline{H}_* S^{\lambda_k-\lambda_j}$ is concentrated in degree 0 where it is the pushout of the diagram above, meaning a Mackey functor whose value on each subgroup is $\mathbf{Z}$. Any such Mackey functor admits a map to $\underline{\mathbf{Z}}$ with underlying degree one. This proves the claim of (3.7). □

The $\mathbf{Z}$-valued Mackey functor $\underline{H}_0 S^{\lambda_k-\lambda_j}$ is discussed in more detail in [7], where it is denoted by $\underline{\mathbf{Z}}(k,j)$.

4 Generalities on differentials and Mackey functor extensions

Before proceeding with a discussion about spectral sequences, we need the following.

Remark 4.1 (Abusive spectral sequence notation). When $d_r(x)$ is a nontrivial element of order 2, the elements $2x$ and x^2 both survive to $\underline{E}_{r+1}$, but in that group they are not the products indicated by these symbols since x itself is no longer present. More generally if $d_r(x) = y$ and $\alpha y = 0$ for some α, then αx is present in $\underline{E}_{r+1}$. This abuse of notation is customary because it would be cumbersome to rename these elements when passing from $\underline{E}_r$ to $\underline{E}_{r+1}$. We will sometimes denote them by $[2x]$, $[x^2]$ and $[\alpha x]$ respectively to emphasize their indecomposability.

Now we make some observations about the relation between exotic transfers and restriction with certain differentials in the slice spectral sequence. By "exotic" we mean in a higher filtration. In a spectral sequence of Mackey functors converging to $\underline{\pi}_* X$, it can happen that an element $x \in \underline{\pi}_V X(G/H)$ has filtration s, but its restriction or transfer has a higher filtration. *In the spectral sequence charts in this paper, exotic transfers and restrictions will be indicated by blue and dashed green lines respectively.*

Lemma 4.2 (Restriction kills a_σ and a_σ kills transfers). *Let G be a finite cyclic 2-group with sign representation σ and index 2 subgroup G', and let X be a G-spectrum. Then in $\underline{\pi}_* X(G/G)$ the image of $\mathrm{tr}_{G'}^G$ is the kernel of multiplication by a_σ, and the kernel of $\mathrm{res}_{G'}^G$ is the image of multiplication by a_σ.*

Suppose further that 4 divides the order of G and let λ be the degree 2 representation sending a generator $\gamma \in G$ to a rotation of order 4. Then restriction kills $2a_\lambda$ and $2a_\lambda$ kills transfers.

Proof. Consider the cofiber sequence obtained by smashing X with

$$S^{-1} \xrightarrow{a_\sigma} S^{\sigma-1} \longrightarrow G_+ \wedge_{G'} S^0 \longrightarrow S^0 \xrightarrow{a_\sigma} S^\sigma. \tag{4.3}$$

Since $(G_+ \wedge_{G'} X)^G$ is equivalent to $X^{G'}$, passage to fixed point spectra gives

$$\Sigma^{-1} X^G \longrightarrow (\Sigma^{\sigma-1} X)^G \longrightarrow X^{G'} \longrightarrow X^G \longrightarrow (\Sigma^\sigma X)^G,$$

so the exact sequence of homotopy groups is

$$\begin{array}{ccccccc}
\underline{\pi}_{k+1} X(G/G) & \xrightarrow{a_\sigma} & \underline{\pi}_{k+1-\sigma} X(G/G) & \longrightarrow & \underline{\pi}_k (G_+ \wedge_{G'} X)(G/G) & \longrightarrow & \underline{\pi}_k X(G/G). \\
 & & & \searrow^{u_\sigma^{-1}\,\mathrm{Res}_{G'}^G} & \| & \nearrow_{\mathrm{Tr}_{G'}^G} & \\
 & & & & \underline{\pi}_k (X)(G/G) & &
\end{array}$$

Note that the isomorphism u_σ is invertible. This gives the exactness required by both statements.

For the statements about a_λ, note that λ restricts to $2\sigma_{G'}$, where $\sigma_{G'}$ is the sign representation for the index 2 subgroup G'. It follows that $\mathrm{res}_{G'}^G(a_\lambda) = a_{\sigma_{G'}}^2$, which has order 2. Using the Frobenius relation, we have for $x \in \underline{\pi}_* X(G/G')$,

$$2a_\lambda \,\mathrm{tr}_{G'}^G(x) = \mathrm{tr}_{G'}^G(\mathrm{res}_{G'}^G(2a_\lambda)x) = \mathrm{tr}_{G'}^G(2a_{\sigma_{G'}}^2 x) = 0. \qquad \square$$

This implies that when $a_\sigma x$ is killed by a differential but $x \in \underline{E}_r(G/G)$ is not, then x represents an element that is $\mathrm{tr}_{G'}^G(y)$ for some y in lower filtration. Similarly if x supports a nontrivial differential but $a_\sigma x$ is a nontrivial permanent cycle, then the latter represents an element with a nontrivial restriction to G' of higher filtration. In both cases the converse also holds.

Theorem 4.4 (Exotic transfers and restrictions in the $\mathrm{RO}(G)$-graded slice spectral sequence). *Let G be a finite cyclic 2-group with index 2 subgroup G' and sign representation σ, and let X be a G-equivariant spectrum with $x \in \underline{E}_r^{s,V} X(G/G)$ (for $V \in \mathrm{RO}(G)$) in the slice spectral sequence for X. Then:*

(i) *Suppose there is a permanent cycle $y' \in \underline{E}_r^{s+r,V+r-1} X(G/G')$. Then there is a nontrivial differential*

$$d_r(x) = \mathrm{tr}_{G'}^G(y')$$

if and only if $[a_\sigma x]$ is a permanent cycle with $\mathrm{res}_{G'}^G(a_\sigma x) = u_\sigma y'$. In this case $[a_\sigma x]$ represents the Toda bracket $\langle a_\sigma, \mathrm{tr}_{G'}^G, y'\rangle$.

(ii) *Suppose there is a permanent cycle $y \in \underline{E}_r^{s+r-1,V+r+\sigma-2} X(G/G)$. Then there is a nontrivial differential*

$$d_r(x) = a_\sigma y$$

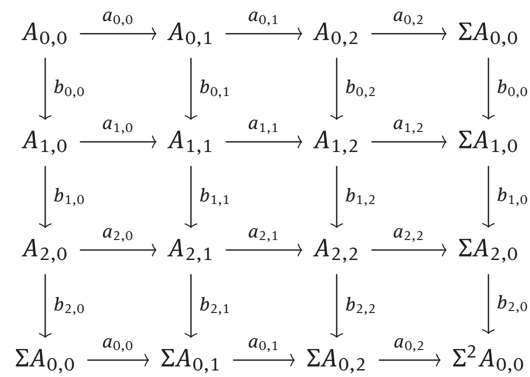
if and only if $\mathrm{res}_{G'}^G(x)$ is a permanent cycle with $\mathrm{tr}_{G'}^G(u_\sigma^{-1}\,\mathrm{res}_{G'}^G(x)) = y$. In this case $\mathrm{res}_{G'}^G(x)$ represents the Toda bracket $\langle \mathrm{res}_{G'}^G, a_\sigma, y\rangle$.

In each case a nontrivial d_r is equivalent to a Mackey functor extension raising filtration by $r-1$. In (i) the permanent cycle $a_\sigma x$ is not divisible in $\underline{\pi}_\star X$ by a_σ and therefore could have a nontrivial restriction in a higher filtration. Similarly in (ii) the element denoted by $\mathrm{res}_{G'}^G(x)$ is not a restriction in $\underline{\pi}_\star X$, so we cannot use the Frobenius relation to equate $\mathrm{tr}_{G'}^G(u_\sigma^{-1}\,\mathrm{res}_{G'}^G(x))$ with $\mathrm{tr}_{G'}^G(u_\sigma^{-1})x$.

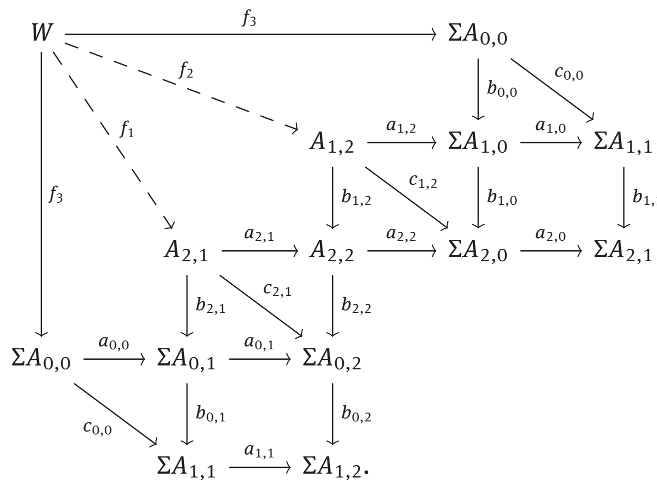
We remark that the proof below makes no use of any properties specific to the slice filtration. The result holds for any equivariant filtration with suitable formal properties.

Before giving the proof we need the following.

Lemma 4.5 (A formal observation). *Suppose we have a commutative diagram up to sign*



in which each row and column is a cofiber sequence. Suppose that from some spectrum W we have a map f_3 and hypothetical maps f_1 and f_2 making the following diagram commute up to sign, where $c_{i,j} = b_{i,j+1}a_{i,j} = a_{i+1,j}b_{i,j}$:

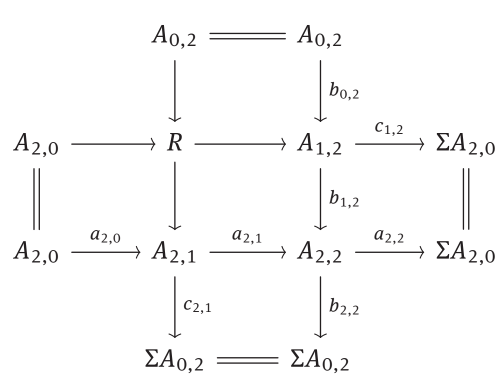


(4.6)

Then f_1 exists if and only if f_2 does. When this happens, $c_{0,0}f_3$ is null and we have Toda brackets

$$\langle a_{1,1}, c_{0,0}, f_3\rangle \ni f_2 \quad \text{and} \quad \langle b_{1,1}, c_{0,0}, f_3\rangle \ni f_1.$$

Proof. Let R be the pullback of $a_{2,1}$ and $b_{1,2}$, so we have a diagram



in which each row and column is a cofiber sequence. Thus we see that R is the fiber of both $c_{1,2}$ and $c_{2,1}$. If f_1 exists, then

$$c_{2,1}f_1 = a_{0,1}b_{2,1}f_1 = a_{0,1}a_{0,0}f_3$$

which is null homotopic, so f_1 lifts to R, which comes equipped with a map to $A_{1,2}$, giving us f_2. Conversely if f_2 exists, it lifts to R, which comes equipped with a map to $A_{2,1}$, giving us f_1.

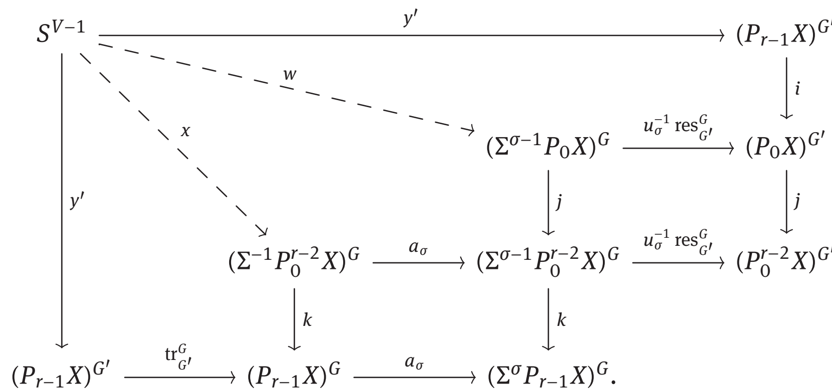
The statement about Toda brackets follows from the way they are defined. □

Proof of Theorem 4.4. For a G-spectrum X and integers $a < b < c \leq \infty$ there is a cofiber sequence

$$P_{b+1}^{c}X \xrightarrow{\ i\ } P_a^c X \xrightarrow{\ j\ } P_a^b X \xrightarrow{\ k\ } \Sigma P_{b+1}^{c} X.$$

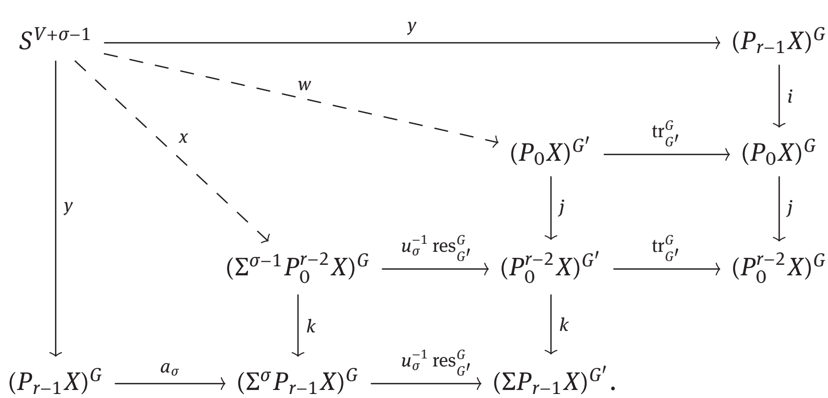
When $c = \infty$, we omit it from the notation. We will combine this and the one of (4.3) to get a diagram similar to (4.6) with $W = S^V$ to prove our two statements.

For (i) note that $x \in \underline{E}_1^{s,t}X(G/G)$ is by definition an element in $\underline{\pi}_{t-s}P_s^s X(G/G)$. We will assume for simplicity that $s = 0$, so x is represented by a map from some S^V to $(P_0^0 X)^G$. Its survival to $\underline{E}_r$ and supporting a nontrivial differential means that it lifts to $(P_0^{r-2}X)^G$ but not to $(P_0^{r-1}X)^G$. The value of $d_r(x)$ is represented by the composite kx in the diagram below, where we can use Lemma 4.5:



The commutativity of the lower left trapezoid is the differential of (i), $d_r(x) = \mathrm{tr}_{G'}^G(y')$. The existence of the map w making the diagram commute follows from that of x and y'. It is the representative of $a_\sigma x$ as a permanent cycle, which represents the indicated Toda bracket. The commutativity of the upper right trapezoid identifies y' as $u_\sigma^{-1}\,\mathrm{res}_{G'}^G(x)$ as claimed. For the converse we have the existence of y' and w and hence that of x.

The second statement follows by a similar argument based on the diagram



Here w represents $u_\sigma^{-1}\,\mathrm{res}_{G'}^G(x)$ as a permanent cycle, so we get a Toda bracket containing $\mathrm{res}_{G'}^G(x)$ as indicated. □

Next we study the way differentials interact with the norm. Suppose we have a subgroup $H \subset G$ and an H-equivariant ring spectrum X with $Y = N_H^G X$. Suppose we have spectral sequences converging to $\underline{\pi}_\star X$ and $\underline{\pi}_\star Y$ based on towers

$$\cdots \to P_n^H X \to P_{n-1}^H X \to \cdots \quad \text{and} \quad \cdots \to P_n^G Y \to P_{n-1}^G Y \to \cdots$$

for functors P_n^H and P_n^G equipped with suitable maps

$$P_m^H X \wedge P_n^H X \to P_{m+n}^H X, \quad P_m^G Y \wedge P_n^G Y \to P_{m+n}^G Y \quad \text{and} \quad N_H^G P_m^H X \to P_{m|G/H|}^G Y.$$

Our slice spectral sequence for each of the spectra studied in this paper fits this description.

Theorem 4.7 (The norm of a differential). *Suppose we have spectral sequences as described above and a differential $d_r(x) = y$ for $x \in \underline{E}_r^{s,\star}X(H/H)$. Let $\rho = \mathrm{Ind}_H^G 1$ and suppose that a_ρ has filtration $|G/H| - 1$. Then in the spectral sequence for $Y = N_H^G X$,*

$$d_{|G/H|(r-1)+1}(a_\rho N_H^G x) = N_H^G y \in \underline{E}_{|G/H|(r-1)+1}^{|G/H|(s+r),\star} Y(G/G).$$

Proof. The differential can be represented by a diagram

$$\begin{array}{ccccccc} S^V & = & S(1+V) & \longrightarrow & D(1+V) & \longrightarrow & S^{1+V} \\ & & \downarrow{\scriptstyle y} & & \downarrow & & \downarrow{\scriptstyle x} \\ & & P^H_{s+r}X & \longrightarrow & P^H_s X & \longrightarrow & P^H_s X/P^H_{s+r}X \end{array}$$

for some orthogonal representation V of H, where each row is a cofiber sequence. We want to apply the norm functor N_H^G to it. Let $W = \mathrm{Ind}_H^G V$. Then we get

$$\begin{array}{ccccccc} S^W & = & N_H^G S(1+V) & \longrightarrow & D(\rho+W) & \longrightarrow & S^{\rho+W} \\ & & \downarrow{\scriptstyle N_H^G y} & & \downarrow & & \downarrow{\scriptstyle N_H^G x} \\ & & N_H^G P^H_{s+r}X & \longrightarrow & N_H^G P^H_s X & \longrightarrow & N_H^G(P^H_s X/P^H_{s+r}X). \end{array}$$

Neither row of this diagram is a cofiber sequence, but we can enlarge it to one where the top and bottom rows are, namely

$$\begin{array}{ccccc} S^W & \longrightarrow & D(1+W) & \longrightarrow & S^{1+W} \\ \| & & \downarrow{\scriptstyle a_\rho} & & \downarrow{\scriptstyle a_\rho} \\ S^W & \longrightarrow & D(\rho+W) & \longrightarrow & S^{\rho+W} \\ \downarrow{\scriptstyle N_H^G y} & & \downarrow & & \downarrow{\scriptstyle N_H^G x} \\ N_H^G P^H_{s+r}X & \longrightarrow & N_H^G P^H_s X & \longrightarrow & N_H^G(P^H_s X/P^H_{s+r}X) \\ \downarrow & & \downarrow & & \downarrow \\ P^G_{(s+r)|G/H|}Y & \longrightarrow & P^G_{s|G/H|}Y & \longrightarrow & P^G_{s|G/H|}Y/P^G_{(s+r)|G/H|}Y. \end{array}$$

Here the first two bottom vertical maps are part of the multiplicative structure the spectral sequence is assumed to have. Composing the maps in the three columns gives us the diagram for the desired differential. □

Given a G-equivariant ring spectrum X, let $X' = i_H^* X$ denote its restriction as an H-spectrum. Then we have $N_H^G X' = X^{(|G/H|)}$ and the multiplication on X gives us a map from this smash product to X. This gives us a map $\pi_\star X' \to \pi_\star X$ called the *internal norm*, which we denote abusively by N_H^G. The argument above yields the following.

Corollary 4.8 (The internal norm of a differential). *With notation as above, suppose we have a differential $d_r(x) = y$ for $x \in \underline{E}_r^{s,\star}X'(H/H)$. Then*

$$d_{|G/H|(r-1)+1}(a_\rho N_H^G x) = N_H^G y \in \underline{E}_{|G/H|(r-1)+1}^{|G/H|(s+r),\star} X(G/G).$$

The following is useful in making such calculations. It is very similar to [6, Lemma 3.13].

Lemma 4.9 (The norm of a_V and u_V). *With notation as above, let V be a representation of H with $V^H = 0$ and let $W = \mathrm{Ind}_H^G V$. Then $N_H^G(a_V) = a_W$. If V is oriented (and hence even-dimensional, making $|V|\rho$ oriented), then*

$$u_{|V|\rho} N(u_V) = u_W.$$

Proof. The element a_V is represented by the map $S^0 \to S^V$, the inclusion of the fixed point set. Applying the norm functor to this map gives

$$S^0 = N_H^G S^0 \to N_H^G S^V = S^W,$$

which is a_W.

When V is oriented, u_V is represented by a map $S^{|V|} \to S^V \wedge H\underline{\mathbf{Z}}$. Applying the norm functor and using the multiplication in $H\underline{\mathbf{Z}}$ leads to a map

$$S^{|V|\rho} = N_H^G S^{|V|} \xrightarrow{N_H^G u_V} S^W \wedge H\underline{\mathbf{Z}}.$$

Now smash both sides with $H\underline{\mathbf{Z}}$, precompose with $u_{|V|\rho}$ and follow with the multiplication on $H\underline{\mathbf{Z}}$, giving

$$S^{|V||\rho|} \xrightarrow{u_{|V|\rho}} S^{|V|\rho} \wedge H\underline{\mathbf{Z}} \xrightarrow{N_H^G u_V \wedge H\underline{\mathbf{Z}}} S^W \wedge H\underline{\mathbf{Z}} \wedge H\underline{\mathbf{Z}} \longrightarrow S^W \wedge H\underline{\mathbf{Z}},$$

which is u_W since $|W| = |V||\rho|$. □

5 Some Mackey functors for C_4 and C_2

We need some notation for Mackey functors to be used in spectral sequence charts. *In this paper, when a cyclic group or subgroup appears as an index, we will often replace it by its order.* We can specify Mackey functors $\underline{M}$ for the group C_2 and $\underline{N}$ for C_4 by means of Lewis diagrams (first introduced in [8]),

$$\begin{array}{c} \underline{M}(C_2/C_2) \\ \mathrm{res}_1^2 \downarrow \uparrow \mathrm{tr}_1^2 \\ \underline{M}(C_2/e) \end{array} \quad \text{and} \quad \begin{array}{c} \underline{N}(C_4/C_4) \\ \mathrm{res}_2^4 \downarrow \uparrow \mathrm{tr}_2^4 \\ \underline{N}(C_4/C_2) \\ \mathrm{res}_1^2 \downarrow \uparrow \mathrm{tr}_1^2 \\ \underline{N}(C_4/e). \end{array} \tag{5.1}$$

We omit Lewis' looped arrow indicating the Weyl group action on $\underline{M}(G/H)$ for proper subgroups H. This notation is prohibitively cumbersome in spectral sequence charts, so we will abbreviate specific examples by more concise symbols. These are shown in Tables 1 and 2. *Admittedly some of these symbols are arbitrary and take some getting used to, but we have to start somewhere.* Lewis denotes the fixed point Mackey functor for a $\mathbf{Z}G$-module M by $R(M)$. He abbreviates $R(\mathbf{Z})$and $R(\mathbf{Z}_-)$ by R and R_-. He also defines (with similar abbreviations) the orbit group Mackey functor $L(M)$ by

$$L(M)(G/H) = M/H.$$

In this case each transfer map is the surjection of the orbit space for a smaller subgroup onto that of a larger one. The functors R and L are the left and right adjoints of the forgetful functor $\underline{M} \mapsto \underline{M}(G/e)$ from Mackey functors to $\mathbf{Z}G$-modules.

Over C_2 we have short exact sequences

$$0 \longrightarrow ◪ \longrightarrow \square \longrightarrow \bullet \longrightarrow 0,$$

$$0 \longrightarrow \bullet \longrightarrow \dot{\square} \longrightarrow \overline{\square} \longrightarrow 0,$$

$$0 \longrightarrow \square \longrightarrow \widehat{\square} \longrightarrow \overline{\square} \longrightarrow 0.$$

We can apply the induction functor to each them to get a short exact sequence of Mackey functors over C_4.

Five of the Mackey functors in Table 2 are fixed point Mackey functors (3.1), meaning they are fixed points of an underlying $Z[G]$-module M, such as $\mathbf{Z}[G]$ or

$$\mathbf{Z} = \mathbf{Z}[G]/(\gamma - 1), \qquad \mathbf{Z}[G/G'] = \mathbf{Z}[G]/(\gamma^2 - 1),$$
$$\mathbf{Z}_- = \mathbf{Z}[G]/(\gamma + 1), \qquad \mathbf{Z}[G/G']_- = \mathbf{Z}[G]/(\gamma^2 + 1).$$

Symbol	$\square$	$\overline{\square}$	$\bullet$	◪	$\dot{\square}$	$\widehat{\square}$
Lewis diagram	$\mathbf{Z}$ $1 \downarrow \uparrow 2$ $\mathbf{Z}$	0 $\downarrow \uparrow$ $\mathbf{Z}_-$	$\mathbf{Z}/2$ $\downarrow \uparrow$ 0	$\mathbf{Z}$ $2 \downarrow \uparrow 1$ $\mathbf{Z}$	$\mathbf{Z}/2$ $0 \downarrow \uparrow 1$ $\mathbf{Z}_-$	$\mathbf{Z}$ $\Delta \downarrow \uparrow \nabla$ $\mathbf{Z}[C_2]$
Lewis symbol	R	R_-	$\langle \mathbf{Z}/2 \rangle$	L	L_-	$R(\mathbf{Z}^2)$

Table 1. Some C_2-Mackey functors.

$\square = \underline{\mathbf{Z}}$	$\widehat{\square} = \underline{\mathbf{Z}[G/G']}$	$\overline{\square} = \underline{\mathbf{Z}}_-$	$\circ$	$\widehat{\widehat{\square}} = \underline{\mathbf{Z}[G]}$	$\widehat{\dot{\square}}$
$\mathbf{Z}$ $1 \downarrow \uparrow 2$ $\mathbf{Z}$ $1 \downarrow \uparrow 2$ $\mathbf{Z}$	$\mathbf{Z}$ $\Delta \downarrow \uparrow \nabla$ $\mathbf{Z}[G/G']$ $1 \downarrow \uparrow 2$ $\mathbf{Z}[G/G']$	0 $\downarrow \uparrow$ $\mathbf{Z}_-$ $1 \downarrow \uparrow 2$ $\mathbf{Z}_-$	$\mathbf{Z}/4$ $1 \downarrow \uparrow 2$ $\mathbf{Z}/2$ $\downarrow \uparrow$ 0	$\mathbf{Z}$ $\Delta \downarrow \uparrow \nabla$ $\mathbf{Z}[G/G']$ $\Delta \downarrow \uparrow \nabla$ $\mathbf{Z}[G]$	$\mathbf{Z}/2$ $\Delta \downarrow \uparrow \nabla$ $\mathbf{Z}/2[G/G']$ $0 \downarrow \uparrow 1$ $\mathbf{Z}[G/G']_-$
$\dot{\square}$	◪ $= \underline{\mathbf{Z}}(G,e)$	▼	▲	$\widehat{\overline{\square}}$	⊡
$\mathbf{Z}/2$ $0 \downarrow \uparrow 1$ $\mathbf{Z}_-$ $1 \downarrow \uparrow 2$ $\mathbf{Z}_-$	$\mathbf{Z}$ $2 \downarrow \uparrow 1$ $\mathbf{Z}$ $2 \downarrow \uparrow 1$ $\mathbf{Z}$	$\mathbf{Z}/2$ $0 \downarrow \uparrow 1$ $\mathbf{Z}/2$ $\downarrow \uparrow$ 0	$\mathbf{Z}/2$ $1 \downarrow \uparrow 0$ $\mathbf{Z}/2$ $\downarrow \uparrow$ 0	0 $\downarrow \uparrow$ 0 $\downarrow \uparrow$ $\mathbf{Z}[G/G']_-$	0 $0 \downarrow \uparrow 0$ $\mathbf{Z}/2$ $0 \downarrow \uparrow \nabla$ $\mathbf{Z}[G/G']_-$
⧩	◩ $= \underline{\mathbf{Z}}(G,G')$	$\bullet$	$\overline{\bullet}$	⧍	$\widehat{\dot{\square}}$
$\mathbf{Z}/2$ $0 \downarrow \uparrow 1$ $\mathbf{Z}/2$ $0 \downarrow \uparrow \nabla$ $\mathbf{Z}[G/G']_-$	$\mathbf{Z}$ $2 \downarrow \uparrow 1$ $\mathbf{Z}$ $1 \downarrow \uparrow 2$ $\mathbf{Z}$	$\mathbf{Z}/2$ $\downarrow \uparrow$ 0 $\downarrow \uparrow$ 0	0 $\downarrow \uparrow$ $\mathbf{Z}/2$ $\downarrow \uparrow$ 0	$\mathbf{Z}/2$ $1 \downarrow \uparrow 0$ $\mathbf{Z}/2$ $0 \downarrow \uparrow \nabla$ $\mathbf{Z}[G/G']_-$	$\mathbf{Z}/2$ $\Delta \downarrow \uparrow \nabla$ $\mathbf{Z}/2[G/G']$ $0 \downarrow \uparrow 1$ $\mathbf{Z}[G/G']_-$
$\overline{◩}$	$\widehat{◩}$	⬒	$\dot{◩}$	⊡	$\widehat{\bullet}$
0 $\downarrow \uparrow$ $\mathbf{Z}_-$ $2 \downarrow \uparrow 1$ $\mathbf{Z}_-$	$\mathbf{Z}$ $\Delta \downarrow \uparrow \nabla$ $\mathbf{Z}[G/G']$ $2 \downarrow \uparrow 1$ $\mathbf{Z}[G/G']$	$\mathbf{Z}$ $1 \downarrow \uparrow 2$ $\mathbf{Z}$ $2 \downarrow \uparrow 1$ $\mathbf{Z}$	$\mathbf{Z}/2$ $0 \downarrow \uparrow 1$ $\mathbf{Z}_-$ $2 \downarrow \uparrow 1$ $\mathbf{Z}_-$	$\mathbf{Z}/4$ $1 \downarrow \uparrow 2$ $\mathbf{Z}/2$ $0 \downarrow \uparrow \nabla$ $\mathbf{Z}[G/G']_-$	$\mathbf{Z}/2$ $\Delta \downarrow \uparrow \nabla$ $\mathbf{Z}/2[G/G']$ $\downarrow \uparrow$ 0
$\overset{\blacktriangledown}{◩}$			$\overset{\circ}{◩}$		
$\mathbf{Z}/2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \downarrow \uparrow \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\mathbf{Z}/2 \oplus \mathbf{Z}_-$ $\begin{bmatrix} 0 & 2 \end{bmatrix} \downarrow \uparrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathbf{Z}_-$			$\mathbf{Z}/4$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \downarrow \uparrow \begin{bmatrix} 2 & 2 \end{bmatrix}$ $\mathbf{Z}/2 \oplus \mathbf{Z}_-$ $\begin{bmatrix} 0 & 2 \end{bmatrix} \downarrow \uparrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\mathbf{Z}_-$		

Table 2. Some C_4-Mackey functors, where $G = C_4$ and G' is its index 2 subgroup. The notation $\underline{\mathbf{Z}}(G, H)$ is defined in Proposition 3.3 (i).

We will use the following notational conventions for C_4-Mackey functors.

(i) Given a C_2-Mackey functor $\underline{M}$ with Lewis diagram

$$\begin{array}{c} A \\ \alpha \big\downarrow \big\uparrow \beta \\ B \end{array}$$

with A and B cyclic, we will use the symbols $\underline{M}$, $\overline{\underline{M}}$ and $\dot{\underline{M}}$ for the C_4-Mackey functors with Lewis diagrams

$$\begin{array}{c} A \\ \alpha \big\downarrow \big\uparrow \beta \\ B \\ 1 \big\downarrow \big\uparrow 2 \\ B, \end{array} \qquad \begin{array}{c} 0 \\ \big\downarrow \big\uparrow \\ A_- \\ \alpha \big\downarrow \big\uparrow \beta \\ B_- \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathbf{Z}/2 \\ 0 \big\downarrow \big\uparrow \tau \\ A_- \\ \alpha \big\downarrow \big\uparrow \beta \\ B_- \end{array}$$

where a generator $\gamma \in C_4$ acts via multiplication by -1 on A and B in the second two, and the transfer τ is nontrivial.

(ii) For a C_2-Mackey functor $\underline{M}$ we will denote $\uparrow_2^4 \underline{M}$ (see Definition 2.6) by $\widehat{\underline{M}}$. For a Mackey functor $\underline{M}$ defined over the trivial group, we will denote $\uparrow_1^2 \underline{M}$ and $\uparrow_1^4 \underline{M}$ by $\widehat{\underline{M}}$ and $\widehat{\widehat{\underline{M}}}$.

Over C_4, in addition to the short exact sequences induced up from C_2, we have

$$\begin{aligned} 0 &\longrightarrow \bullet \longrightarrow \dot{\Box} \longrightarrow \overline{\Box} \longrightarrow 0, \\ 0 &\longrightarrow \blacktriangledown \longrightarrow \circ \longrightarrow \bullet \longrightarrow 0, \\ 0 &\longrightarrow \blacktriangledown \longrightarrow \boxed{\blacktriangledown} \longrightarrow \widehat{\overline{\Box}} \longrightarrow 0, \\ 0 &\longrightarrow \bullet \longrightarrow \circ \longrightarrow \blacktriangle \longrightarrow 0, \\ 0 &\longrightarrow ◪ \longrightarrow \Box \longrightarrow \circ \longrightarrow 0, \\ 0 &\longrightarrow ⧅ \longrightarrow \Box \longrightarrow \bullet \longrightarrow 0. \end{aligned} \tag{5.2}$$

Definition 5.3 (A C_4-enriched C_2-Mackey functor). For a C_2-Mackey functor $\underline{M}$ as above, $\widetilde{\underline{M}}$ will denote the C_2-Mackey functor enriched over $\mathbf{Z}[C_4]$ defined by

$$\widetilde{\underline{M}}(S) = \mathbf{Z}[C_4] \otimes_{\mathbf{Z}[C_2]} \underline{M}(S)$$

for a finite C_2-set S. Equivalently, in the notation of Definition 2.6, $\widetilde{\underline{M}} = \downarrow_2^4 \uparrow_2^4 \underline{M}$.

6 Some chain complexes of Mackey functors

As noted above, a G-CW complex X, meaning one built out of cells of the form $G_+ \wedge_H e^n$, has a reduced cellular chain complex of $\mathbf{Z}[G]$-modules C_*X, leading to a chain complex of fixed point Mackey functors (see (3.1)) $\underline{C}_*X$. When $X = S^V$ for a representation V, we will denote this complex by $\underline{C}_*^V$; see (3.2). Its homology is the graded Mackey functor $\underline{H}_*X$. Here we will apply the methods of Section 3 to three examples.

Example (i). Let $G = C_2$ with generator γ, and $X = S^{n\rho}$ for $n > 0$, where ρ denotes the regular representation. We have seen before [6, Example 3.7] that it has a reduced cellular chain complex C with

$$C_i^{n\rho_2} = \begin{cases} \mathbf{Z}[G]/(\gamma - 1) & \text{for } i = n, \\ \mathbf{Z}[G] & \text{for } n < i \le 2n, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Let $c_i^{(n)}$ denote a generator of $C_i^{n\rho_2}$. The boundary operator d is given by

$$d(c_{i+1}^{(n)}) = \begin{cases} c_i^{(n)} & \text{for } i = n, \\ \gamma_{i+1-n}(c_i^{(n)}) & \text{for } n < i \leq 2n, \\ 0 & \text{otherwise,} \end{cases} \tag{6.2}$$

where $\gamma_i = 1 - (-1)^i \gamma$. For future reference, let

$$\epsilon_i = 1 - (-1)^i = \begin{cases} 0 & \text{for } i \text{ even,} \\ 2 & \text{for } i \text{ odd.} \end{cases}$$

This chain complex has the form

$$\begin{array}{ccccccccc} n & & n+1 & & n+2 & & n+3 & & 2n \\ \square & \xleftarrow{\nabla} & \widehat{\square} & \xleftarrow{\gamma_2} & \widehat{\square} & \xleftarrow{\gamma_3} & \widehat{\square} & \longleftarrow \cdots \xleftarrow{\gamma_n} & \widehat{\square} \end{array}$$

$$\begin{array}{ccccccccc} \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \longleftarrow \cdots \xleftarrow{\epsilon_n} & \mathbf{Z} \\ {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\ \mathbf{Z} & \xleftarrow{\nabla} & \mathbf{Z}[G] & \xleftarrow{\gamma_2} & \mathbf{Z}[G] & \xleftarrow{\gamma_3} & \mathbf{Z}[G] & \longleftarrow \cdots \xleftarrow{\gamma_n} & \mathbf{Z}[G]. \end{array}$$

Passing to homology we get

$$\begin{array}{cccccc} n & n+1 & n+2 & n+3 & & 2n \\ \bullet & 0 & \bullet & 0 & \cdots & \underline{H}_{2n} \\ \\ \mathbf{Z}/2 & 0 & \mathbf{Z}/2 & 0 & \cdots & \underline{H}_{2n}(G/G) \\ \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\ 0 & 0 & 0 & 0 & \cdots & \mathbf{Z}[G]/(\gamma_{n+1}), \end{array}$$

where

$$\underline{H}_{2n}(G/G) = \begin{cases} \mathbf{Z} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd,} \end{cases} \quad \text{and} \quad \underline{H}_{2n} = \begin{cases} \square & \text{for } n \text{ even,} \\ \overline{\square} & \text{for } n \text{ odd.} \end{cases}$$

Here $\square$ and $\overline{\square}$ are fixed point Mackey functors but $\bullet$ is not.

Similar calculations can be made for $S^{n\rho_2}$ for $n < 0$. The results are indicated in Figure 2. This is originally due to unpublished work of Stong and is reported in [8, Theorem 2.1 and Table 2.2]. This information will be used in Section 8.

In other words the RO(G)-graded Mackey functor valued homotopy of $H\underline{\mathbf{Z}}$ is as follows. For $n \geq -1$ we have

$$\underline{\pi}_i \Sigma^{n\rho_2} H\underline{\mathbf{Z}} = \underline{\pi}_{i-n\rho_2} H\underline{\mathbf{Z}} = \begin{cases} \square & \text{for } n \text{ even and } i = 2n, \\ \overline{\square} & \text{for } n \text{ odd and } i = 2n, \\ \bullet & \text{for } n \leq i < 2n \text{ and } i + n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

For $n \leq -2$ we have

$$\underline{\pi}_i \Sigma^{n\rho_2} H\underline{\mathbf{Z}} = \underline{\pi}_{i-n\rho_2} H\underline{\mathbf{Z}} = \begin{cases} ◪ & \text{for } n \text{ even and } i = 2n, \\ \dot{\square} & \text{for } n \text{ odd and } i = 2n, \\ \bullet & \text{for } 2n < i \leq n - 3 \text{ and } i + n \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

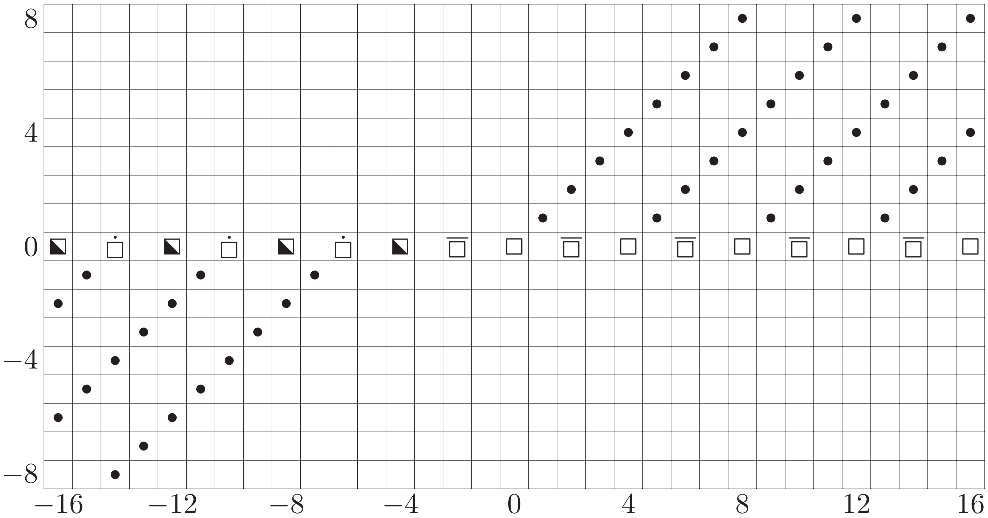


Figure 2. The (collapsing) Mackey functor slice spectral sequence for $\bigvee_{n\in\mathbf{Z}} \Sigma^{n\rho_2} H\underline{\mathbf{Z}}$. The symbols are defined in Table 1. When the Mackey functor $\underline{\pi}_{(2-\rho_2)n-s} H\underline{\mathbf{Z}} = \underline{H}_{2n-s} S^{n\rho_2}$ is nontrivial, it is shown at $(2n-s, s)$ in the chart. Compare with Figure 7.

We can use Definition 3.4 to name some elements of these groups.

Note that $H\underline{\mathbf{Z}}$ is a commutative ring spectrum, so there is a commutative multiplication in $\underline{\pi}_\star H\underline{\mathbf{Z}}$, making it a commutative RO(G)-graded Green functor. For such a functor $\underline{M}$ on a general group G, the restriction maps are a ring homomorphisms while the transfer maps satisfy the Frobenius relations (2.4).

Then the generators of various groups in $\underline{\pi}_\star H\underline{\mathbf{Z}}$ are

- $(4m-2)$-slices for $m > 0$:

$$\begin{aligned} a^{2m-1-2i}u^i &= a_{(2m-1-2i)\sigma}u_{2i\sigma} \in \underline{\pi}_{2m-1+2i}\Sigma^{(2m-1)\rho_2} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{2i-(2m-1)\sigma} H\underline{\mathbf{Z}}(G/G) && \text{for } 0 \le i < m, \\ x^{2m-1} &= u_{(2m-1)\sigma} \in \underline{\pi}_{4m-2}\Sigma^{(2m-1)\rho_2} H\underline{\mathbf{Z}}(G/\{e\}) = \underline{\pi}_{(2m-1)(1-\sigma)} H\underline{\mathbf{Z}}(G/\{e\}) && \text{with } \gamma(x) = -x. \end{aligned}$$

- $4m$-slices for $m > 0$:

$$a^{2m-2i}u^i = a_{(2m-2i)\sigma}u_{2i\sigma} \in \underline{\pi}_{2m-1+2i}\Sigma^{(2m-1)\rho_2} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{2i-(2m-1)\sigma} H\underline{\mathbf{Z}}(G/G) \quad \text{for } 0 \le i \le m$$

  and with $\mathrm{res}(u) = x^2$.
- negative slices:

$$\begin{aligned} z_n &= e_{2n\rho_2} \in \underline{\pi}_{-4n}\Sigma^{-2n\rho_2} H\underline{\mathbf{Z}}(G/\{e\}) = \underline{\pi}_{2n(\sigma-1)} H\underline{\mathbf{Z}}(G/\{e\}) && \text{for } n > 0, \\ a^{-i}\,\mathrm{tr}(x^{-2n-1}) &\in \underline{\pi}_{-4n-2-i}\Sigma^{-(2n+1+i)\rho_2} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{(2n+1)(\sigma-1)+i\sigma} H\underline{\mathbf{Z}}(G/G) && \text{for } n > 0 \text{ and } i \ge 0. \end{aligned}$$

We have relations

$$2a = 0, \qquad \mathrm{res}(a) = 0,$$

$$z_n = x^{-2n}, \quad \mathrm{tr}(x^n) = \begin{cases} 2u^{n/2} & \text{for } n \text{ even and } n \ge 0, \\ \mathrm{tr}(z_{-n/2}) & \text{for } n \text{ even and } n < 0, \\ 0 & \text{for } n \text{ odd and } n > -3. \end{cases}$$

Example (ii). Let $G = C_4$ with generator γ, $G' = C_2 \subseteq G$, the subgroup generated by γ^2, and

$$\widehat{S}(n, G') = G_+ \wedge_{G'} S^{n\rho_2}.$$

Thus we have

$$C_*(\widehat{S}(n, G')) = \mathbf{Z}[G] \otimes_{\mathbf{Z}[G']} C_*^{n\rho_2}$$

with $C_*^{n\rho_2}$ as in (6.1). The calculations of the previous example carry over verbatim by the exactness of Mackey functor induction of Definition 2.6.

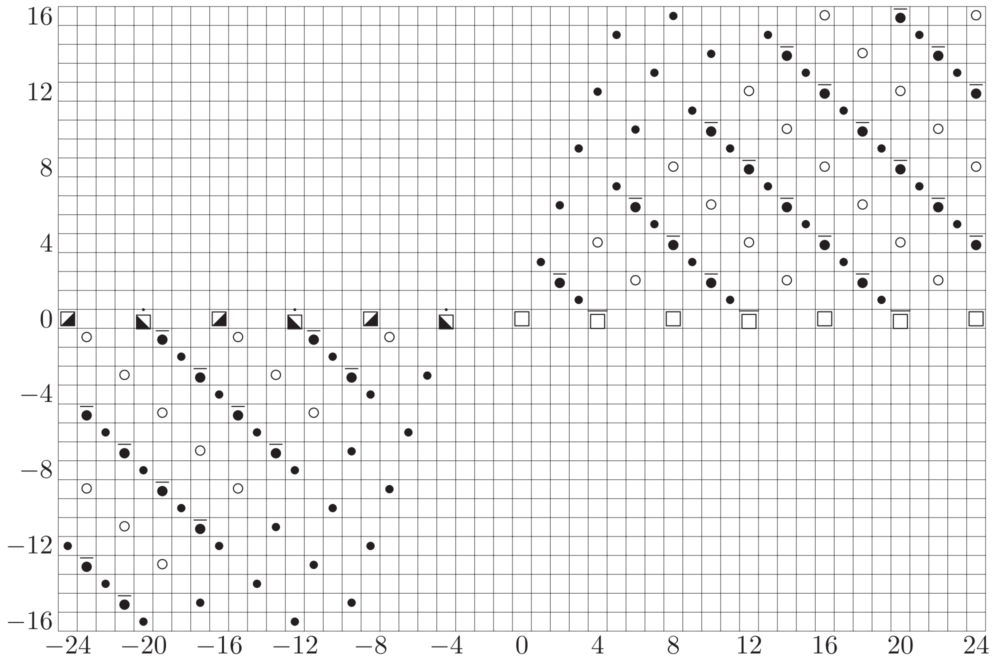


Figure 3. The Mackey functor slice spectral sequence for $\bigvee_{n\in\mathbf{Z}} \Sigma^{n\rho_4} H\underline{\mathbf{Z}}$. The symbols are defined in Table 2. The Mackey functor at position $(4n-s, s)$ is $\underline{\pi}_{n(4-\rho_4)-s} H\underline{\mathbf{Z}} = \underline{H}_{4n-s} S^{n\rho_4}$.

Example (iii). Let $G = C_4$ and $X = S^{n\rho_4}$. Then the reduced cellular chain complex (3.2) has the form

$$C_i^{n\rho_4} = \begin{cases} \mathbf{Z} & \text{for } i = n, \\ \mathbf{Z}[G/G'] & \text{for } n < i \le 2n, \\ \mathbf{Z}[G] & \text{for } 2n < i \le 4n, \\ 0 & \text{otherwise,} \end{cases}$$

in which generators $c_i^{(n)} \in C_i^{n\rho_4}$ satisfy

$$d(c_{i+1}^{(n)}) = \begin{cases} c_i^{(n)} & \text{for } i = n, \\ \gamma_{i+1-n} c_i^{(n)} & \text{for } n < i \le 2n, \\ \mu_{i+1-n} c_i^{(n)} & \text{for } 2n < i < 4n \text{ and } i \text{ even}, \\ \gamma_{i+1-n} c_i^{(n)} & \text{for } 2n < i < 4n \text{ and } i \text{ odd}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mu_i = \gamma_i(1+\gamma^2) = (1-(-1)^i\gamma)(1+\gamma^2).$$

The values of $\underline{H}_* S^{n\rho_4}$ are illustrated in Figure 3. The Mackey functors in filtration 0 (the horizontal axis) are the ones described in Proposition 3.3.

As in (i), we name some of these elements. Let $G = C_4$ and $G' = C_2 \subseteq G$. Recall that the regular representation ρ_4 is $1 + \sigma + \lambda$ where σ is the sign representation and λ is the 2-dimensional representation given by a rotation of order 4.

Note that while Figure 2 shows all of $\underline{\pi}_\star H\underline{\mathbf{Z}}$ for $G = C_2$, Figure 3 shows only a bigraded portion of this trigraded Mackey functor for $G = C_4$, namely the groups for which the index differs by an integer from a multiple of ρ_4. We will need to refer to some elements not shown in the latter chart, namely

$$\left.\begin{aligned} &a_\sigma \in \underline{H}_0 S^\sigma(G/G), && a_\lambda \in \underline{H}_0 S^\lambda(G/G), && \overline{a}_\lambda = \operatorname{res}_2^4(a_\lambda), \\ &u_{2\sigma} \in \underline{H}_2 S^{2\sigma}(G/G)), && u_\sigma \in \underline{H}_1 S^\sigma(G/G'), && \overline{u}_\sigma = \operatorname{res}_1^2(u_\sigma), \\ &u_\lambda \in \underline{H}_2 S^\lambda(G/G), && \overline{u}_\lambda = \operatorname{res}_2^4(u_\lambda), && \overline{\overline{u}}_\lambda = \operatorname{res}_1^4(u_\lambda), \end{aligned}\right\} \tag{6.3}$$

subject to the relations

$$\left.\begin{array}{lll} 2a_\sigma = 0, & \operatorname{res}_2^4(a_\sigma) = 0, & \\ 4a_\lambda = 0, & 2\overline{a}_\lambda = 0, & \operatorname{res}_1^4(a_\lambda) = 0, \\ \operatorname{res}_2^4(u_{2\sigma}) = u_\sigma^2, & a_\sigma^2 u_\lambda = 2a_\lambda u_{2\sigma} & \text{(gold relation);} \end{array}\right\} \tag{6.4}$$

see Definition 3.4 and Lemma 3.6.

We will denote the generator of $\underline{E}_2^{s,t}(G/H)$ (when it is nontrivial) by $x_{t-s,s}$, $y_{t-s,s}$ and $z_{t-s,s}$ for $H = G$, G' and $\{e\}$ respectively. Then the generators for the groups in the 4-slice are

$$\begin{aligned} y_{4,0} &= u_{\rho_4} = u_\sigma \operatorname{res}_2^4(u_\lambda) \in \underline{\pi}_4 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/G') = \underline{\pi}_{3-\sigma-\lambda} H\underline{\mathbf{Z}}(G/G') \quad \text{with } \gamma(x_{4,0}) = -x_{4,0}, \\ x_{3,1} &= a_\sigma u_\lambda \in \underline{\pi}_3 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{2-\sigma-\lambda} H\underline{\mathbf{Z}}(G/G), \\ y_{2,2} &= \operatorname{res}_2^4(a_\lambda) u_\sigma \in \underline{\pi}_2 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/G') = \underline{\pi}_{1-\sigma-\lambda} H\underline{\mathbf{Z}}(G/G'), \\ x_{1,3} &= a_{\rho_4} = a_\sigma a_\lambda \in \underline{\pi}_1 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{-\sigma-\lambda} H\underline{\mathbf{Z}}(G/G) \end{aligned}$$

and the ones for the 8-slice are

$$\begin{aligned} x_{8,0} &= u_{2\lambda+2\sigma} = u_{2\rho_4} \in \underline{\pi}_8 \Sigma^{2\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{6-2\sigma-2\lambda} H\underline{\mathbf{Z}}(G/G) \quad \text{with } y_{4,0}^2 = y_{8,0} = \operatorname{res}_2^4(x_{8,0}), \\ x_{6,2} &= a_\lambda u_{\lambda+2\sigma} \in \underline{\pi}_6 \Sigma^{2\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{4-2\sigma-2\lambda} H\underline{\mathbf{Z}}(G/G) \quad \text{with } x_{3,1}^2 = 2x_{6,2},\ y_{4,0}y_{2,2} = y_{6,2} = \operatorname{res}_2^4(x_{6,2}), \\ x_{4,4} &= a_{2\lambda} u_{2\sigma} \in \underline{\pi}_4 \Sigma^{2\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{2-2\sigma-2\lambda} H\underline{\mathbf{Z}}(G/G) \quad \text{with } y_{2,2}^2 = y_{4,4} = \operatorname{res}_2^4(x_{4,4}),\ x_{1,3}x_{3,1} = 2x_{4,4}, \\ x_{2,6} &= x_{1,3}^2 \in \underline{\pi}_2 \Sigma^{2\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{-2\sigma-2\lambda} H\underline{\mathbf{Z}}(G/G). \end{aligned}$$

These elements and their restrictions generate $\underline{\pi}_* \Sigma^{m\rho_4} H\underline{\mathbf{Z}}$ for $m = 1$ and 2. For $m > 2$ the groups are generated by products of these elements.

The element

$$z_{4,0} = \operatorname{res}_1^2(y_{4,0}) = \operatorname{res}_1^2(u_{\rho_4}) \in \underline{\pi}_4 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/\{e\})$$

is invertible with $\gamma(y_{4,0}) = -y_{4,0}$, $z_{4,0}^2 = z_{8,0} = \operatorname{res}_1^4(x_{8,0})$ and

$$z_{-4m,0} := z_{4,0}^{-m} = e_{m\rho_4} \in \underline{\pi}_{-4m} \Sigma^{-m\rho_4} H\underline{\mathbf{Z}}(G/\{e\}) \quad \text{for } m > 0,$$

where $e_{m\rho_4}$ is as in Definition 3.4. These elements and their transfers generate the groups in

$$\underline{\pi}_{-4m} \Sigma^{-m\rho_4} H\underline{\mathbf{Z}} \quad \text{for } m > 0.$$

Theorem 6.5 (Divisibilities in the negative regular slices for C_4). *There are the following infinite divisibilities in the third quadrant of the spectral sequence in Figure 3.*

(i) $x_{-4,0} = \operatorname{tr}_1^4(z_{-4,0})$ *is divisible by any monomial in* $x_{1,3}$ *and* $x_{4,4}$, *meaning that*

$$x_{1,3}^i x_{4,4}^j x_{-4-4j-i,-4j-3k} = x_{-4,0} \quad \text{for } i, j \geq 0.$$

Moreover, no other basis element killed by $x_{3,1}$ *and* $x_{4,4}$ *has this property.*

(ii) $x_{-4,0}$, *and* $x_{-7,-1}$ *are divisible by any monomial in* $x_{4,4}$, $x_{6,2}$ *and* $x_{8,0}$, *subject to the relation* $x_{6,2}^2 = x_{8,0}x_{4,4}$. *Note here that* $x_{3,1}^2 = 2x_{6,2}$. *Moreover, no other basis element killed by* $x_{4,4}$, $x_{6,2}$ *and* $x_{8,0}$ *has this property.*

(iii) $y_{-7,-1} = \operatorname{res}_2^4(x_{-7,-1})$ *is divisible by any monomial in* $y_{2,2}$ *and* $y_{4,0}$, *meaning that*

$$y_{2,2}^j y_{4,0}^k y_{-7-2j-4k,-1-2j} = y_{-7,-1} \quad \text{for } j, k \geq 0.$$

Moreover, no other basis element killed by $y_{2,2}$,*and* $y_{4,0}$ *has this property.*

We will prove Theorem 6.5 as a corollary of a more general statement (Lemma 6.11 and Corollary 6.13) in which we consider all representations of the form $m\lambda + n\sigma$ for $m, n \geq 0$. Let

$$\underline{R} = \bigoplus_{m,n\geq 0} \underline{H}_* S^{m\lambda+n\sigma}.$$

It is generated by the elements of (6.3) subject to the relations of (6.4).

In the larger ring

$$\underline{\tilde{R}} = \bigoplus_{\substack{m,n\in\mathbf{Z} \\ mn\geq 0}} \underline{H}_* S^{m\lambda+n\sigma},$$

the elements u_σ, $\overline{u}_\sigma$ and $\overline{\overline{u}}_\lambda$ are invertible with

$$e_\sigma = u_\sigma^{-1} \in \underline{H}_{-1} S^{-\sigma}(G/G'), \quad e_\lambda = \overline{\overline{u}}_\lambda^{-1} \in \underline{H}_{-2} S^{-\lambda}(G/e).$$

Define spectra L_m and K_n to be the cofibers of $a_{m\lambda}$ and $a_{n\sigma}$. Thus we have cofiber sequences

$$\Sigma^{-1} L_m \xrightarrow{c_{m\lambda}} S^0 \xrightarrow{a_{m\lambda}} S^{m\lambda} \xrightarrow{b_{m\lambda}} L_m,$$

$$\Sigma^{-1} K_n \xrightarrow{c_{n\sigma}} S^0 \xrightarrow{a_{n\sigma}} S^{n\sigma} \xrightarrow{b_{n\sigma}} K_n.$$

Dualizing gives

$$DL_m \xrightarrow{Db_{m\lambda}} S^{-m\lambda} \xrightarrow{Da_{m\lambda}} S^0 \xrightarrow{Dc_{m\lambda}} \Sigma DL_m,$$

$$DK_n \xrightarrow{Db_{n\sigma}} S^{-n\sigma} \xrightarrow{Da_{n\sigma}} S^0 \xrightarrow{Dc_{n\sigma}} \Sigma DK_n.$$

The maps $Da_{m\lambda}$ and $Da_{n\sigma}$ are the same as desuspensions of $a_{m\lambda}$ and $a_{n\sigma}$, which implies that

$$DL_m = \Sigma^{-1-m\lambda} L_m \quad \text{and} \quad DK_n = \Sigma^{-1-n\sigma} K_n.$$

Inspection of the cellular chain complexes for L_m and K_n and certain of their suspensions reveals that

$$\Sigma^{2-\lambda} L_m \wedge H\underline{\mathbf{Z}} = L_m \wedge H\underline{\mathbf{Z}} = \Sigma^{2-2\sigma} L_m \wedge H\underline{\mathbf{Z}}$$

and

$$\Sigma^{2-2\sigma} K_n \wedge H\underline{\mathbf{Z}} = K_n \wedge H\underline{\mathbf{Z}},$$

while $\Sigma^{1-\sigma}$ alters both $L_m \wedge H\underline{\mathbf{Z}}$ and $K_n \wedge H\underline{\mathbf{Z}}$. We will denote $\Sigma^{k(1-\sigma)} L_m \wedge H\underline{\mathbf{Z}}$ by $L_m^{(-1)^k} \wedge H\underline{\mathbf{Z}}$ and similarly for K_n.

The homology groups of $L_m^\pm$ and $K_m^\pm$ for $m, n > 0$ are indicated in Figures 4 and 5, and those for $S^{m\lambda}$ and $S^{n\sigma}$ are shown in Figure6.

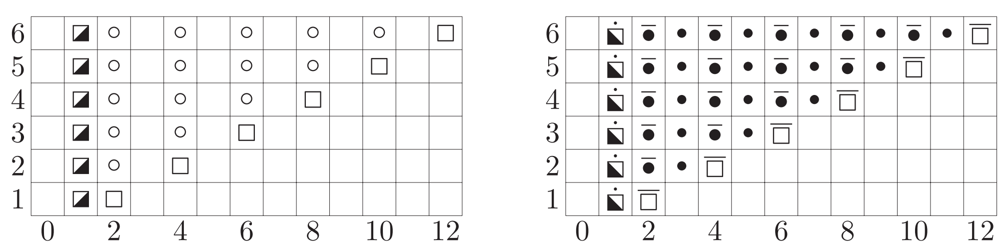


Figure 4. Charts for $\underline{H}_i L_m^\pm$. The horizontal coordinate is i and the vertical one is m; L_m is on the left and L_m^- is on the right.

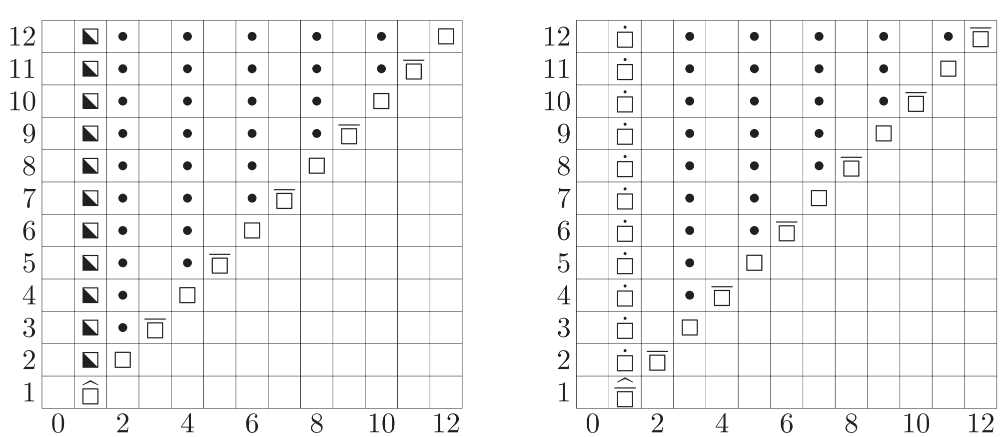


Figure 5. Charts for $\underline{H}_i K_n^\pm$. The horizontal coordinate is i and the vertical one is n; K_n is on the left and K_n^- is on the right.

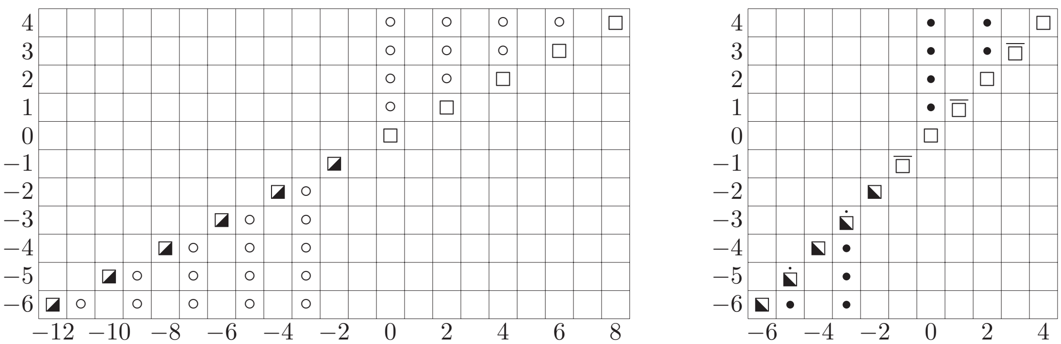


Figure 6. Charts for $\underline{H}_i S^{m\lambda}$ and $\underline{H}_i S^{n\sigma}$. The horizontal coordinates are i and the vertical ones are m and n; $S^{m\lambda}$ is on the left and $S^{n\sigma}$ is on the right.

In the following diagrams we will use the same notation for a map and its smash product with any identity map. Let $V = m\lambda + n\sigma$ with $m, n > 0$, and let R_V denote the fiber of a_V. Since a_V is self-dual up to suspension, we have $DR_V = \Sigma^{-1-V} R_V$. In the following each row and column is a cofiber sequence:

$$\begin{array}{ccccccc} & & & & \Sigma^{n\sigma-1}L_m & = & \Sigma^{n\sigma-1}L_m \\ & & & & \downarrow{\scriptstyle c_{m\lambda}} & & \downarrow \\ \Sigma^{-1}K_n & \xrightarrow{c_{n\sigma}} & S^0 & \xrightarrow{a_{n\sigma}} & S^{n\sigma} & \xrightarrow{b_{n\sigma}} & K_n \\ \downarrow & & \| & & \downarrow{\scriptstyle a_{m\lambda}} & & \downarrow \\ \Sigma^{-1}R_V & \xrightarrow{c_V} & S^0 & \xrightarrow{a_V} & S^V & \xrightarrow{b_V} & R_V \\ & & & & \downarrow{\scriptstyle b_{m\lambda}} & & \downarrow \\ & & & & \Sigma^{n\sigma}L_m & = & \Sigma^{n\sigma}L_m. \end{array} \tag{6.6}$$

The homology sequence for the third column is the easiest way to compute $\underline{H}_* S^V$. That column is

$$\Sigma^{n\sigma-1}L_m \xrightarrow{c_{m\lambda}} S^{n\sigma} \xrightarrow{a_{m\lambda}} S^V \xrightarrow{b_{m\lambda}} \Sigma^{n\sigma}L_m, \tag{6.7}$$

which dualizes to

$$\begin{array}{ccccccc} \Sigma^{1-n\sigma}DL_m & \xleftarrow{c_{m\lambda}} & S^{-n\sigma} & \xleftarrow{a_{m\lambda}} & S^{-V} & \xleftarrow{c_{m\lambda}} & \Sigma^{-n\sigma}DL_m \\ \| & & & & & & \| \\ \Sigma^{-V}L_m & & & & & & \Sigma^{-1-V}L_m \end{array}$$

or

$$\Sigma^{-1-V}L_m \xrightarrow{c_{m\lambda}} S^{-V} \xrightarrow{a_{m\lambda}} S^{-n\sigma} \xrightarrow{b_{m\lambda}} \Sigma^{-V}L_m. \tag{6.8}$$

For (6.7) the long exact sequence in homology includes

$$\underline{H}_{i+1-n}L_m^{(-1)^n} \xrightarrow{c_{m\lambda}} \underline{H}_i S^{n\sigma} \xrightarrow{a_{m\lambda}} \underline{H}_i S^V \xrightarrow{b_{m\lambda}} \underline{H}_{i-n}L_m^{(-1)^n} \xrightarrow{c_{m\lambda}} \underline{H}_{i-1}S^{n\sigma}.$$

Divisibility by a_λ. Multiplication by a_λ leads to

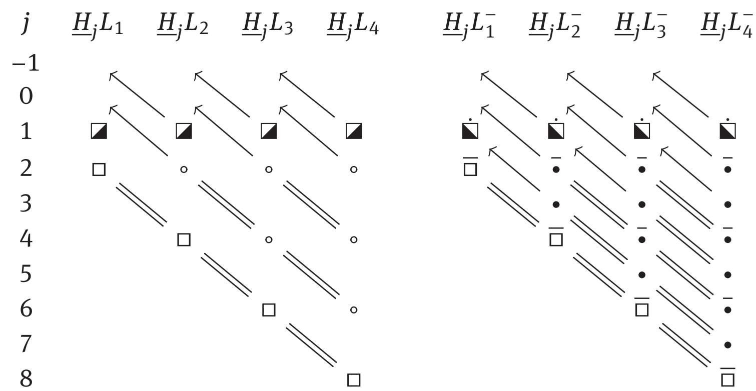
$$\begin{array}{ccccccccc} \underline{H}_{i+1-n}L_m^{(-1)^n} & \xrightarrow{c_{m\lambda}} & \underline{H}_i S^{n\sigma} & \xrightarrow{a_{m\lambda}} & \underline{H}_i S^V & \xrightarrow{b_{m\lambda}} & \underline{H}_{i-n}L_m^{(-1)^n} & \xrightarrow{c_{m\lambda}} & \underline{H}_{i-1}S^{n\sigma} \\ \downarrow{\scriptstyle a'_\lambda} & & \| & & \downarrow{\scriptstyle a_\lambda} & & \downarrow{\scriptstyle a'_\lambda} & & \| \\ \underline{H}_{i+1-n}L_{m'}^{(-1)^n} & \xrightarrow{c_{m'\lambda}} & \underline{H}_i S^{n\sigma} & \xrightarrow{a_{m'\lambda}} & \underline{H}_i S^{V+\lambda} & \xrightarrow{b_{m'\lambda}} & \underline{H}_{i-n}L_{m'}^{(-1)^n} & \xrightarrow{c_{m'\lambda}} & \underline{H}_{i-1}S^{n\sigma}, \end{array}$$

where $m' = m + 1$ and a'_λ is induced by the inclusion $L_m \to L_{m'}$.

In the dual case we get

$$\begin{array}{ccccccccc}
\underline{H}_{i+1}S^{-n\sigma} & \xrightarrow{b} & \underline{H}_{i+1+|V|}L_m^{(-1)^n} & \xrightarrow{c} & \underline{H}_iS^{-V} & \xrightarrow{a} & \underline{H}_iS^{-n\sigma} & \xrightarrow{b} & \underline{H}_{i+|V|}L_m^{(-1)^n} \\
\| & & \uparrow{\scriptstyle Da'_\lambda} & & \uparrow{\scriptstyle a_\lambda} & & \| & & \uparrow{\scriptstyle Da'_\lambda} \\
\underline{H}_{i+1}S^{-n\sigma} & \xrightarrow{b} & \underline{H}_{i+3+|V|}L_{m'}^{(-1)^n} & \xrightarrow{c} & \underline{H}_iS^{-V-\lambda} & \xrightarrow{a} & \underline{H}_iS^{-n\sigma} & \xrightarrow{b} & \underline{H}_{i+2+|V|}L_{m'}^{(-1)^n}.
\end{array} \tag{6.9}$$

Here the subscripts on the horizontal maps ($m\lambda$ in the top row and $m'\lambda$ in the bottom row) have been omitted to save space. The five lemma implies that the middle vertical map is onto when the left hand Da'_λ is onto and the right hand one is one-to-one. The left version of Da'_λ is onto in every case except $i = -|V|$ and the right version of it is one-to-one in all cases except $i = -|V|$ and $i = -1 - |V|$. This is illustrated for small m in the following diagram in which trivial Mackey functors are indicated by blank spaces.

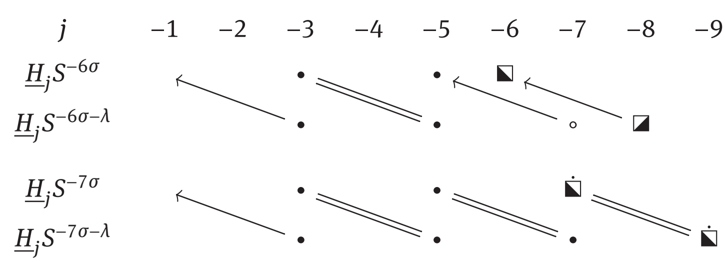


It follows that the map a_λ in (6.9) is onto for all i except $-|V|$. *This is a divisibility result*. Note that a_λ is trivial on $\underline{H}_*X(G/e)$ for any X since $\mathrm{res}_1^4(a_\lambda) = 0$.

Divisibility by u_λ. For u_λ multiplication we use the diagram

$$\begin{array}{ccccccccc}
\underline{H}_{i+1}S^{-n\sigma} & \xrightarrow{b} & \underline{H}_{i+1}L_m^{(-1)^n} & \xrightarrow{c} & \underline{H}_iS^{-V} & \xrightarrow{a} & \underline{H}_iS^{-n\sigma} & \xrightarrow{b} & \underline{H}_iL_m^{(-1)^n} \\
\uparrow{\scriptstyle u_\lambda} & & \| & & \uparrow{\scriptstyle u_\lambda} & & \uparrow{\scriptstyle u_\lambda} & & \| \\
\underline{H}_{i-1}S^{-n\sigma-\lambda} & \xrightarrow{b} & \underline{H}_{i+1}L_m^{(-1)^n} & \xrightarrow{c} & \underline{H}_{i-2}S^{-V-\lambda} & \xrightarrow{a} & \underline{H}_{i-2}S^{-n\sigma-\lambda} & \xrightarrow{b} & \underline{H}_iL_m^{(-1)^n}.
\end{array} \tag{6.10}$$

The rightmost u_λ is onto in all cases except $i = -n$ and n even. This is illustrated for $n = 6$ and 7 in the following diagram.



Thus the central u_λ in (6.10) fails to be onto only in when $i = -n$ and n is even.

Divisibility by a_σ. The corresponding diagram is

$$\begin{array}{ccccccccc}
\underline{H}_{i+1}S^{-n\sigma} & \xrightarrow{b} & \underline{H}_{i+1+|V|}L_m^{(-1)^n} & \xrightarrow{c} & \underline{H}_iS^{-V} & \xrightarrow{a} & \underline{H}_iS^{-n\sigma} & \xrightarrow{b} & \underline{H}_{i+|V|}L_m^{(-1)^n} \\
\uparrow{\scriptstyle a_\sigma} & & \uparrow{\scriptstyle a_\sigma} & & \uparrow{\scriptstyle a_\sigma} & & \uparrow{\scriptstyle a_\sigma} & & \uparrow{\scriptstyle a_\sigma} \\
\underline{H}_{i+1}S^{-n'\sigma} & \xrightarrow{b} & \underline{H}_{i+2+|V|}L_m^{(-1)^{n'}} & \xrightarrow{c} & \underline{H}_iS^{-V-\sigma} & \xrightarrow{a} & \underline{H}_iS^{-n'\sigma} & \xrightarrow{b} & \underline{H}_{i+1+|V|}L_m^{(-1)^{n'}}.
\end{array}$$

Here we have abbreviated $n+1$ by n'. Since $\mathrm{res}_2^4(a_\sigma) = 0$, the map a_σ must vanish on $\underline{H}_* X(G/G')$ and $\underline{H}_* X(G/e)$. It can be nontrivial only on G/G.

By Lemma 4.2, the image of a_σ is the kernel of the restriction map $u_\sigma^{-1}\,\mathrm{res}_2^4$ and the kernel of a_σ is the image of the transfer tr_2^4. From Figure 6 we see that res_2^4 kills $\underline{H}_i S^{-n\sigma}(G/G)$ except the case $i = -n$ for even n. From Figure 4 we see that it kills $\underline{H}_j L_m^-(G/G)$ for all j and $\underline{H}_j L_m(G/G)$ for odd $j > 1$, but not the generators for $j = 1$ nor the ones for even values of j from 2 to $2m$. The transfer has nontrivial image in $\underline{H}_j L_m^-$ only for $j = 1$ and in $\underline{H}_j L_m$ only for $j = 1$ and for even j from 2 to $2m$.

It follows that for odd n, each element of $\underline{H}_i S^{-V}(G/G)$ is divisible by a_σ except when $i = -|V| = -2m - n$. For even n it is onto except when $i = -n$, $i = -n - 2m$, and i odd from $1 - n - 2m$ to $-1 - n$.

Divisibility by $u_{2\sigma}$. For $u_{2\sigma}$ multiplication, the diagram is

$$\begin{array}{ccccccccc}
\underline{H}_{i+1} S^{-n\sigma} & \xrightarrow{b} & \underline{H}_{i+1} L_m^{(-1)^n} & \xrightarrow{c} & \underline{H}_i S^{-V} & \xrightarrow{a} & \underline{H}_i S^{-n\sigma} & \xrightarrow{b} & \underline{H}_i L_m^{(-1)^n} \\
{\scriptstyle u_{2\sigma}}\uparrow & & \| & & {\scriptstyle u_{2\sigma}}\uparrow & & {\scriptstyle u_{2\sigma}}\uparrow & & \| \\
\underline{H}_{i-1} S^{-(n+2)\sigma} & \xrightarrow{b} & \underline{H}_{i+1} L_m^{(-1)^n} & \xrightarrow{c} & \underline{H}_{i-2} S^{-V-2\sigma} & \xrightarrow{a} & \underline{H}_{i-2} S^{-(n+2)\sigma} & \xrightarrow{b} & \underline{H}_i L_m^{(-1)^n}.
\end{array}$$

The rightmost $u_{2\sigma}$ is onto in all cases, so *every element in $\underline{H}_* S^{-V}$ is divisible by $u_{2\sigma}$.*

The arguments above prove the following.

Lemma 6.11 (RO(G)-graded divisibility). *Let $G = C_4$ and $V = m\lambda + n\sigma$ for $m, n \geq 0$.*

(i) *Each element in $\underline{H}_i S^{-V}(G/G)$ or $\underline{H}_i S^{-V}(G/G')$ is divisible by a_λ or $\overline{a}_\lambda$ except when $i = -|V|$.*

(ii) *Each element in $\underline{H}_i S^{-V}(G/H)$ is divisible by a suitable restriction of u_λ except when $i = -n$ for even n.*

(iii) *Each element in $\underline{H}_i S^{-V}(G/G)$ for odd n is divisible by a_σ except when $i = -|V|$. For even n it is divisible by a_σ except when $i = -n$, $i = -|V|$ and i is odd from $i = 1 - |V|$ to $-1 - n$.*

(iv) *Each element in $\underline{H}_i S^{-V}(G/H)$ is divisible by a $u_{2\sigma}$, u_σ or $\overline{u}_\sigma$.*

In Theorem 6.5 we are looking for divisibility by

$$\left.\begin{aligned}
x_{1,3} &= a_\sigma a_\lambda \in \underline{H}_0 S^{\sigma+\lambda}(G/G) = \underline{H}_1 S^{\rho}(G/G), \\
x_{4,4} &= a_\lambda^2 u_{2\sigma} \in \underline{H}_2 S^{2\lambda+2\sigma}(G/G) = \underline{H}_4 S^{2\rho}(G/G), \\
y_{2,2} &= \overline{a}_\lambda u_\sigma \in \underline{H}_1 S^{1\lambda+1\sigma}(G/G') = \underline{H}_2 S^{\rho}(G/G), \\
x_{6,2} &= a_\lambda u_{2\sigma} u_\lambda \in \underline{H}_4 S^{2\lambda+2\sigma}(G/G) = \underline{H}_6 S^{2\rho}(G/G), \\
x_{8,0} &= u_{2\sigma} u_\lambda^2 \in \underline{H}_6 S^{2\lambda+2\sigma}(G/G) = \underline{H}_8 S^{2\rho}(G/G), \\
y_{4,0} &= u_\sigma \overline{u}_\lambda \in \underline{H}_3 S^{\lambda+\sigma}(G/G') = \underline{H}_4 S^{\rho}(G/G').
\end{aligned}\right\} \qquad (6.12)$$

In view of Lemma 6.11 (iv), we can ignore the factors $u_{2\sigma}$ and u_σ when analyzing such divisibility.

Corollary 6.13 (Infinite divisibility by the divisors of (6.12)). *Let*

$$V = m\lambda + n\sigma \quad \text{for } m, n \geq 0.$$

Then the following hold:

- *Each element of $\underline{H}_i S^{-V}(G/G)$ is infinitely divisible by $x_{1,3} = a_\sigma a_\lambda$ for $i > -n$ when n is even and for $i \geq -n$ when n is odd.*
- *Each element of $\underline{H}_i S^{-V}(G/G)$ is infinitely divisible by $x_{4,4} = a_\lambda^2 u_{2\sigma}$ for $i > -|V|$.*
- *Each element of $\underline{H}_i S^{-V}(G/G')$ is infinitely divisible by $y_{2,2} = \overline{a}_\lambda u_\sigma$ for $i > -|V|$.*
- *Each element of $\underline{H}_i S^{-V}(G/G)$ is infinitely divisible by $x_{6,2} = a_\lambda u_{2\sigma} u_\lambda$ for $i > -|V|$ when n is odd and for $-|V| < i < -n$ when n is even.*
- *Each element of $\underline{H}_i S^{-V}(G/G)$ is infinitely divisible by $x_{8,0} = u_{2\sigma} u_\lambda^2$ for $i < -n$ when n is even and for all i when n is odd.*
- *Each element of $\underline{H}_i S^{-V}(G/G')$ is infinitely divisible by $y_{4,0} = u_\sigma \overline{u}_\lambda$ for $i < -n$ when n is even and for all i when n is odd.*

This implies Theorem 6.5.

7 The spectra $k_{\mathbf{R}}$ and $k_{[2]}$

Before defining our spectrum we need to recall some definitions and formulas from [6]. Let $H \subset G$ be finite groups. In [6, Section 2.2.3] we define a norm functor N_H^G from the category of H-spectra to that of G-spectra. Roughly speaking, for an H-spectrum X, $N_H^G X$ is the G-spectrum underlain by the smash power $X^{(|G/H|)}$ with G permuting the factors and H leaving each one invariant. When G is cyclic, we will denote the orders of G and H by g and h, and the norm functor by N_h^g.

There is a C_2-spectrum $\mathrm{MU}_{\mathbb{R}}$ underlain by the complex cobordism spectrum MU with group action given by complex conjugation. Its construction is spelled out in [6, Section B.12]. For a finite cyclic 2-group G we define

$$\mathrm{MU}^{((G))} = N_2^g \mathrm{MU}_{\mathbb{R}}.$$

Choose a generator γ of G. In [6, (5.47)] we defined generators

$$\bar{r}_k = \bar{r}_k^G \in \underline{\pi}_{k\rho_2}^{C_2} i_{C_2}^* \mathrm{MU}^{((G))}(C_2/C_2) \cong \underline{\pi}_{C_2,k\rho_2} \mathrm{MU}^{((G))}(G/G) \tag{7.1}$$

(note that this group is a module over G/C_2) and

$$r_k = \underline{r}_1^2(\bar{r}_k) \in \pi_{\{e\},2k}^u \mathrm{MU}^{((G))}(G/G) \cong \underline{\pi}_{2k}^{\{e\}} \mathrm{MU}^{((G))}(\{e\}/\{e\}) = \pi_{2k}^u \mathrm{MU}^{((G))}.$$

The Hurewicz images of the $\bar{r}_k$ (for which we use the same notation) are defined in terms of the coefficients (see Definition 2.7)

$$\overline{m}_k \in \underline{\pi}_{k\rho_2}^{C_2} H\underline{\mathbf{Z}}_{(2)} \wedge \mathrm{MU}^{((G))}(C_2/C_2) = \underline{\pi}_{C_2,k\rho_2} H\underline{\mathbf{Z}}_{(2)} \wedge \mathrm{MU}^{((G))}(G/G)$$

of the logarithm of the formal group law $\overline{F}$ associated with the left unit map from MU to $\mathrm{MU}^{((G))}$. The formula is

$$\sum_{k\geq 0} \bar{r}_k x^{k+1} = \Big(x + \sum_{\ell>0} \gamma(\overline{m}_{2^\ell-1}) x^{2^\ell}\Big)^{-1} \circ \log_{\overline{F}}(x),$$

where

$$\log_{\overline{F}}(x) = x + \sum_{k>0} \overline{m}_k x^{k+1}.$$

For small k we have

$$\begin{aligned}
\bar{r}_1 &= (1-\gamma)(\overline{m}_1),\\
\bar{r}_2 &= \overline{m}_2 - 2\gamma(\overline{m}_1)(1-\gamma)(\overline{m}_1),\\
\bar{r}_3 &= (1-\gamma)(\overline{m}_3) - \gamma(\overline{m}_1)(5\gamma(\overline{m}_1)^2 - 6\gamma(\overline{m}_1)\overline{m}_1 + \overline{m}_1^2 + 2\overline{m}_2).
\end{aligned}$$

Now let $G = C_2$ or C_4 and, in the latter case $G' = C_2 \subseteq G$. The generators $\bar{r}_k^G$ are the $\bar{r}_k$ defined above. We also have elements $\bar{r}_k^{G'}$ defined by similar formulas with γ replaced by γ^2; recall that $\gamma^2(\overline{m}_k) = (-1)^k \overline{m}_k$. They are the images of similar generators of

$$\underline{\pi}_{k\rho_2}^{C_2} \mathrm{MU}^{((G'))}(C_2/C_2) \cong \underline{\pi}_{C_2,k\rho_2} \mathrm{MU}^{((G'))}(G'/G')$$

under the left unit map

$$\mathrm{MU}^{((G'))} \to \mathrm{MU}^{((G'))} \wedge \mathrm{MU}^{((G'))} \cong i_{G'}^* \mathrm{MU}^{((G))}.$$

Thus we have

$$\begin{aligned}
\bar{r}_1^{G'} &= 2\overline{m}_1,\\
\bar{r}_2^{G'} &= \overline{m}_2 + 4\overline{m}_1^2,\\
\bar{r}_3^{G'} &= 2\overline{m}_3 + 2\overline{m}_1\overline{m}_2 + 12\overline{m}_1^3.
\end{aligned}$$

If we set $\bar{r}_2 = 0$ and $\bar{r}_3 = 0$, we get

$$\left.\begin{aligned} \bar{r}_1^{G'} &= \bar{r}_{1,0} + \bar{r}_{1,1}, \\ \bar{r}_2^{G'} &= 3\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1}^2, \\ \bar{r}_3^{G'} &= 5\bar{r}_{1,0}^2\bar{r}_{1,1} + 5\bar{r}_{1,0}\bar{r}_{1,1}^2 + \bar{r}_{1,1}^3 = \bar{r}_{1,1}(5\bar{r}_{1,0}^2 + 5\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1}^2), \\ \gamma(\bar{r}_3^{G'}) &= -\bar{r}_{1,0}(5\bar{r}_{1,1}^2 - 5\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,0}^2), \\ -\bar{r}_3^{G'}\gamma(\bar{r}_3^{G'})/\bar{r}_{1,0}\bar{r}_{1,1} &= (5\bar{r}_{1,1}^2 - 5\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,0}^2)(\bar{r}_{1,1}^2 + 5\bar{r}_{1,0}\bar{r}_{1,1} + 5\bar{r}_{1,0}^2) \\ &= (5\bar{r}_{1,0}^4 - 20\bar{r}_{1,0}^3\bar{r}_{1,1} + \bar{r}_{1,0}^2\bar{r}_{1,1}^2 + 20\bar{r}_{1,0}\bar{r}_{1,1}^3 + 5\bar{r}_{1,1}^4) \\ &= (5(\bar{r}_{1,0}^2 - \bar{r}_{1,1}^2)^2 - 20\bar{r}_{1,0}\bar{r}_{1,1}(\bar{r}_{1,0}^2 - \bar{r}_{1,1}^2) + 11(\bar{r}_{1,0}\bar{r}_{1,1})^2), \end{aligned}\right\} \tag{7.2}$$

where $\bar{r}_{1,0} = \bar{r}_1$ and $\bar{r}_{1,1} = \gamma(\bar{r}_1)$.

Definition 7.3 ($k_{\mathbf{R}}$, $K_{\mathbf{R}}$, $k_{[2]}$ and $K_{[2]}$). The C_2-spectrum $k_{\mathbb{R}}$ (connective real K-theory), is the spectrum obtained from $MU_{\mathbb{R}}$ by killing the r_ns for $n \geq 2$. Its periodic counterpart $K_{\mathbf{R}}$ is the telescope obtained from $k_{\mathbf{R}}$ by inverting $\bar{r}_1 \in \underline{\pi}_{\rho_2} k_{\mathbf{R}}(C_2/C_2)$.

The C_4-spectrum $k_{[2]}$ is obtained from $MU^{((C_4))}$ by killing the r_ns and their conjugates for $n \geq 2$. Its periodic counterpart $K_{[2]}$ is the telescope obtained from $k_{[2]}$ by inverting a certain element $D \in \underline{\pi}_{4\rho_4} k_{[2]}(C_4/C_4)$ defined below in (9.3) and Table 3.

The image of D in $\underline{\pi}^{C_2}_{8\rho_2} k_{[2]}(C_2/C_2) \cong \underline{\pi}_{C_2, 8\rho_2} k_{[2]}(C_4/C_4)$ is

$$\begin{aligned} \underline{r}_2^4(D) &= \bar{r}_{1,0}\bar{r}_{1,1}\bar{r}_3^{G'}\gamma(\bar{r}_3^{G'}) \\ &= \bar{r}_{1,0}^2\bar{r}_{1,1}^2(-5\bar{r}_{1,0}^4 + 20\bar{r}_{1,0}^3\bar{r}_{1,1} - \bar{r}_{1,0}^2\bar{r}_{1,1}^2 - 20\bar{r}_{1,0}\bar{r}_{1,1}^3 - 5\bar{r}_{1,1}^4) \\ &= -\bar{r}_{1,0}^2\bar{r}_{1,1}^2(5(\bar{r}_{1,0}^2 - \bar{r}_{1,1}^2)^2 - 20\bar{r}_{1,0}\bar{r}_{1,1}(\bar{r}_{1,0}^2 - \bar{r}_{1,1}^2) + 11(\bar{r}_{1,0}\bar{r}_{1,1})^2). \end{aligned} \tag{7.4}$$

It is fixed by the action of G/G', while its factors $\bar{r}_{1,0}\bar{r}_{1,1}$ and $\bar{r}_3^{G'}\gamma(\bar{r}_3^{G'})$ are each negated by the action of the generator γ.

We remark that while $MU^{((C_4))}$ is $MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$ as a C_2-spectrum, $k_{[2]}$ is *not* $k_{\mathbb{R}} \wedge k_{\mathbb{R}}$ as a C_2-spectrum. The former has torsion free underlying homotopy but the latter does not.

8 The slice spectral sequence for $K_{\mathbf{R}}$

In this section we describe the slice spectral sequence for $K_{\mathbf{R}}$. These results are originally due to Dugger [4], to which we refer for many of the proofs. This case is far simpler than that of $K_{[2]}$, but it is very instructive.

Theorem 8.1 (The slice $\underline{E}_2$-terms for $K_{\mathbf{R}}$ and $k_{\mathbf{R}}$). *The slices of $K_{\mathbf{R}}$ are*

$$P_t^t K_{\mathbf{R}} = \begin{cases} \Sigma^{(t/2)\rho_2} H\underline{\mathbf{Z}} & \text{for } t \text{ even}, \\ * & \text{otherwise}. \end{cases}$$

For $k_{\mathbf{R}}$ they are the same in nonnegative dimensions, and contractible below dimension 0.

Hence we know the integrally graded homotopy groups of these slices by the results of Section 6, and they are shown in Figure 2. It shows the $\underline{E}_2$-term for the wedge of all of the slices of $K_{\mathbf{R}}$, and $K_{\mathbf{R}}$ itself has the same $\underline{E}_2$-term. It turns out that the differentials and Mackey functor extensions are determined by the fact that $\underline{\pi}_* K_{\mathbf{R}}$ is 8-periodic, while the $\underline{E}_2$-term is far from it. This explanation is admittedly circular in that the proof of the Periodicity Theorem itself of [6, Section 9] relies on the existence of certain differentials described below in (11.2).

Theorem 8.2 (The slice spectral sequence for $K_{\mathbf{R}}$). *The differentials and extensions in the spectral sequence are as indicated in Figure 7.*

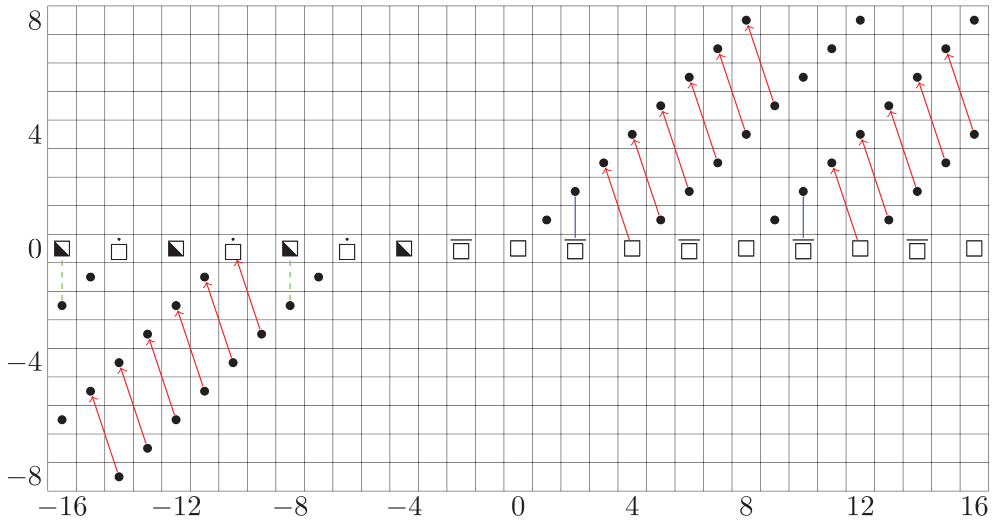


Figure 7. The slice spectral sequence for $K_{\mathbf{R}}$. Compare with Figure 2. Exotic transfers and restrictions are indicated respectively by solid blue and dashed green lines. Differentials are in red.

Proof. There are four phenomena we need to establish:

(i) The differentials in the first quadrant, which are indicated by red lines.
(ii) The differentials in the third quadrant.
(iii) The exotic transfers in the first quadrant, which are indicated by blue lines.
(iv) The exotic restrictions in the third quadrant, which are indicated by dashed green lines.

For (i), note that there is a nontrivial element in $\underline{E}_2^{3,6}(G/G)$, which is part of the 3-stem, but nothing in the (-5)-stem. This means the former element must be killed by a differential, and the only possibility is the one indicated. The other differentials in the first quadrant follow from this one and the multiplicative structure.

For (ii), we know that $\underline{\pi}_7 K_{\mathbf{R}} = 0$, so the same must be true of $\underline{\pi}_{-9}$. Hence the element in $\underline{E}_2^{-3,-12}$ cannot survive, leading to the indicated third quadrant differentials.

For (iii), note that $\underline{\pi}_2$ and $\underline{\pi}_{-6}$ must be the same as Mackey functors. This forces the indicated exotic transfers. For each $m \geq 0$ one has a nonsplit short exact sequence of C_2 Mackey functors

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{E}_2^{2,8m+4} & \longrightarrow & \underline{\pi}_{8m+2}K_{\mathbf{R}} & \longrightarrow & \underline{E}_2^{0,8m+2} & \longrightarrow & 0. \\ & & \| & & \| & & \| & & \\ & & \bullet & & \dot{\Box} & & \overline{\Box} & & \end{array}$$

For (iv), note that $\underline{\pi}_{-8}$ and $\underline{\pi}_0$ must also agree. This forces the indicated exotic restrictions. For each $m < 0$ one has a nonsplit short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{E}_2^{0,8m} & \longrightarrow & \underline{\pi}_{8m}K_{\mathbf{R}} & \longrightarrow & \underline{E}_2^{-2,8m-2} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \boxbslash & & \dot{\Box}, & & \bullet & & \end{array}$$

as desired. □

In order to describe $\underline{\pi}_* K_{\mathbf{R}}$ as a graded Green functor, meaning a graded Mackey functor with multiplication, we recall some notation from Section 6 (i) and Definition 3.4. For $G = C_2$ we have elements

$$\left.\begin{aligned} a = a_\sigma &\in \underline{\pi}_{-\sigma} H\underline{\mathbf{Z}}(G/G), & & \\ u = u_{2\sigma} &\in \underline{\pi}_{2-2\sigma} H\underline{\mathbf{Z}}(G/G), & & \\ x = u_\sigma &\in \underline{\pi}_{1-\sigma} H\underline{\mathbf{Z}}(G/\{e\}) & & \text{with } x^2 = \operatorname{res}(u), \\ z_n = e_{2n\rho_2} &\in \underline{\pi}_{2n(\sigma-1)} H\underline{\mathbf{Z}}(G/\{e\}) & & \text{for } n > 0, \\ a^{-i}\operatorname{tr}(x^{-2n-1}) &\in \underline{\pi}_{(2n+1)(\sigma-1)+i\sigma} H\underline{\mathbf{Z}}(G/G) & & \text{for } n > 0. \end{aligned}\right\} \tag{8.3}$$

We will use the same symbols for the representatives of these elements in the slice E_2-term. The filtrations of u, x and z_n are zero while that of a is one. It follows that $a^{-i}\operatorname{tr}(x^{-2n-1})$ has filtration $-i$. The element x is invertible.

In $\underline{E}_2^{*,*}$ we have relations in

$$\left.\begin{aligned} &2a = 0, && \operatorname{res}(a) = 0, \\ &z_n = x^{-2n}, && \operatorname{tr}(x^n) = \begin{cases} 2u^{n/2} & \text{for } n \text{ even and } n \geq 0, \\ \operatorname{tr}(z_{-n/2}) \neq 0 & \text{for } n \text{ even and } n < 0, \\ 0 & \text{for } n \text{ odd and } n > -3, \\ \neq 0 & \text{for } n \text{ odd and } n \leq -3. \end{cases} \end{aligned}\right\} \tag{8.4}$$

We also have the element $\overline{r}_1 \in \underline{\pi}_{1+\sigma}k_{\mathbf{R}}(G/G)$, the image of the element of the same name in $\overline{r}_1 \in \underline{\pi}_{1+\sigma}\mathrm{MU}_{\mathbb{R}}(G/G)$ of (7.1). We use the same symbol for its representative $\underline{E}_2^{0,1+\sigma}(G/G)$. Then we have integrally graded elements

$$\begin{aligned} \eta = a\,\overline{r}_1 &\in \underline{E}_2^{1,2}(G/G), \\ v_1 = x \cdot \operatorname{res}(\overline{r}_1) &\in \underline{E}_2^{0,2}(G/\{e\}) \quad \text{with } \gamma(v_1) = -v_1, \\ u\overline{r}_1^2 &\in \underline{E}_2^{0,4}(G/G), \\ w = 2u\overline{r}_1^2 &\in \underline{E}_2^{0,4}(G/G), \\ b = u^2\overline{r}_1^4 &\in \underline{E}_2^{0,8}(G/G) \quad \text{with } w^2 = 4b, \end{aligned}$$

where η and v_1 are the images of the elements of the same name in $\pi_1 S^0$ and $\pi_2 k$, and w and b are permanent cycles. The elements x, v_1 and b are invertible. Note that for $n < 0$,

$$\underline{E}_2^{0,2n}(G/G) = \begin{cases} 0 & \text{for } n = 1, \\ \mathbf{Z} \text{ generated by } \operatorname{tr}(v_1^{-n}) & \text{for } n \text{ even}, \\ \mathbf{Z}/2 \text{ generated by } \operatorname{tr}(v_1^{-n}) & \text{for } n \text{ odd and } n < -1, \end{cases}$$

so each group is killed by $\eta = a\overline{r}_1$ by Lemma 4.2.

Then we have

$$\begin{aligned} d_3(u) &= a^3\overline{r}_1 \quad \text{by (11.3) below}, \\ d_3(u\overline{r}_1^2) &= d_3(u)\overline{r}_1^2 = a^3\overline{r}_1^3 = \eta^3, \end{aligned}$$

so

$$\begin{aligned} \operatorname{tr}_1^2(x) &= a^2\overline{r}_1 \quad \text{by (11.4), raising filtration by 2}, \\ \operatorname{tr}_1^2(v_1) &= \eta^2. \end{aligned}$$

Thus we get:

Theorem 8.5 (The homotopy of $K_{\mathbf{R}}$ as an integrally graded Green functor). *With notation as above,*

$$\underline{\pi}_* K_{\mathbf{R}}(G/\{e\}) = \mathbf{Z}[v_1^{\pm 1}], \quad \underline{\pi}_* K_{\mathbf{R}}(G/G) = \mathbf{Z}[b^{\pm 1}, w, \eta]/(2\eta, \eta^3, w\eta, w^2 - 4b)$$

with

$$\operatorname{tr}(v_1^i) = \begin{cases} 2b^j & \text{for } i = 4j, \\ \eta^2 b^j & \text{for } i = 4j+1, \\ wb^j & \text{for } i = 4j+2, \\ 0 & \text{for } i = 4j+3, \end{cases} \quad \operatorname{res}(b) = v_1^4, \quad \operatorname{res}(w) = 2v, \quad \operatorname{res}(\eta) = 0.$$

For each $j < 0$, b^j has filtration -2 and supports an exotic restriction in the slice spectral sequence as indicated in Figure 7. Both $v_1\operatorname{res}(b^j)$ and $\eta^2 b^j$ have filtration zero, so the transfer relating them is does not raise filtration.

Now we will describe the $RO(G)$-graded slice spectral sequence and homotopy of $K_{\mathbf{R}}$. The former is trigraded since $RO(G)$ itself is bigraded, being isomorphic as an abelian group to $\mathbf{Z} \oplus \mathbf{Z}$. For each integer k, one can imagine a chart similar to Figure 7 converging to the graded Mackey functor $\underline{\pi}_{k\sigma+*}K_{\mathbf{R}}$. Figure 7 itself is the one for $k = 0$. The product of elements in the kth and ℓth charts lies in the $(k+\ell)$th chart. We have elements as in (8.3):

$$\begin{aligned} a = a_\sigma &\in \underline{E}_2^{1,1-\sigma}(G/G), \\ u = u_{2\sigma} &\in \underline{E}_2^{0,2-2\sigma}(G/G), \\ x = u_\sigma &\in \underline{E}_2^{0,1-\sigma}(G/\{e\}) && \text{with } \gamma(x) = -1 \text{ and } x^2 = \operatorname{res}(u), \\ z_n = x^{-2n} &\in \underline{E}_2^{0,-2n+2n\sigma}(G/\{e\}) && \text{for } n > 0, \\ a^{-i}\operatorname{tr}(x^{-2n-1}) &\in \underline{E}_2^{-i,-i-2n+2n\sigma}(G/G) && \text{for } i \geq 0 \text{ and } n > 0, \\ \overline{r}_1 &\in \underline{E}_2^{0,1+\sigma}(G/G), \end{aligned}$$

where a, x, z_n and $\overline{r}_1$ are permanent cycles, both x and $\overline{r}_1$ are invertible, and there are relations as in (8.4). We also know that

$$d_3(u) = a^3\overline{r}_1 \quad \text{by (11.3) below,} \qquad \operatorname{tr}_1^2(x) = a^2\overline{r}_1 \quad \text{by (11.4).}$$

Theorem 8.6. *The* $RO(G)$*-graded slice spectral sequence for* $K_{\mathbf{R}}$ *can be obtained by tensoring that of Figure 7 with* $\mathbf{Z}[\overline{r}_1^{\pm 1}]$*, that is for any integer* k*,*

$$\underline{E}_2^{s,t+k\sigma}(G/G) \cong \overline{r}_1^k \underline{E}_2^{s,t-k}(G/G) \quad \textit{and} \quad \underline{E}_2^{s,t+k\sigma}(G/\{e\}) \cong \operatorname{res}(\overline{r}_1^k)\underline{E}_2^{s,t-k}(G/\{e\})$$

and $\underline{\pi}_{t+k\sigma}K_{\mathbf{R}}$ *has a similar description.*

Proof. The element $\overline{r}_1$ and its restriction are invertible permanent cycles, so multiplication by either induces an isomorphism in the spectral sequence. □

Remark 8.7. In *the* $RO(G)$*-graded slice spectral sequence for* $k_{\mathbf{R}}$ one has $d_3(u) = \overline{r}_1 a^3$, but a^3 itself, and indeed all higher powers of a, survive to $\underline{E}_4 = \underline{E}_\infty$. Hence the $\underline{E}_\infty$-term of this spectral sequence does *not* have the horizontal vanishing line that we see in $\underline{E}_4$-term of Figure 7. However when we pass from $k_{\mathbf{R}}$ to $K_{\mathbf{R}}$, $\overline{r}_1$ becomes invertible and we have

$$d_3(\overline{r}_1^{-1}u) = a^3.$$

We can keep track of the groups in this trigraded spectral sequence with the help of four variable Poincaré series $g(\underline{E}_r(G/G)) \in \mathbf{Z}[[x,y,z,t]]$ in which the rank of $\underline{E}_r^{s,i+j\sigma}(G/G)$ is the coefficient in $\mathbf{Z}[[t]]$ of $x^{i-s}y^jz^s$. The variable t keeps track of powers of two. Thus a copy of the integers is represented by $1/(1-t)$ or (when it is the kernel of a differential of the form $\mathbf{Z} \to \mathbf{Z}/2$) $t/(1-t)$. Let

$$\widehat{a} = y^{-1}z, \quad \widehat{u} = x^2y^{-1} \quad \text{and} \quad \widehat{r} = xy. \tag{8.8}$$

Since $\underline{E}_2(G/G) = \mathbf{Z}[a,u,\overline{r}_1]/(2a)$, we have

$$\begin{aligned} g(\underline{E}_2(G/G)) &= \left(\frac{1}{1-t} + \frac{\widehat{a}}{1-\widehat{a}}\right)\frac{1}{(1-\widehat{u})(1-\widehat{r})}, \\ g(\underline{E}_4(G/G)) &= g(\underline{E}_2(G/G)) - \frac{\widehat{u} + \widehat{r}\widehat{a}^3}{(1-\widehat{a})(1-\widehat{u}^2)(1-\widehat{r})}. \end{aligned}$$

We subtract the indicated expression from $g(\underline{E}_2(G/G))$ because we have differentials

$$d_3(a^i\overline{r}_1^ju^{2k+1}) = a^{i+3}\overline{r}_1^{j+1}u^{2k} \quad \text{for all } i,j,k \geq 0.$$

Pursuing this further we get

$$\begin{aligned} g(\underline{E}_4(G/G)) &= \left(\frac{1}{1-t} + \frac{\widehat{a}}{1-\widehat{a}}\right)\frac{1}{(1-\widehat{u})(1-\widehat{r})} - \frac{\widehat{u}}{(1-\widehat{u}^2)(1-\widehat{r})} - \frac{a\widehat{u} + \widehat{r}\widehat{a}^3}{(1-\widehat{a})(1-\widehat{u}^2)(1-\widehat{r})} \\ &= \frac{1+\widehat{u}-\widehat{u}(1-t)}{(1-t)(1-\widehat{u}^2)(1-\widehat{r})} + \frac{\widehat{a}(1+\widehat{u}) - \widehat{a}(\widehat{u}+\widehat{a}^2\widehat{r})}{(1-\widehat{a})(1-\widehat{u}^2)(1-\widehat{r})} \\ &= \frac{1+t\widehat{u}}{(1-t)(1-\widehat{u}^2)(1-\widehat{r})} + \frac{\widehat{a}-\widehat{a}^3+\widehat{a}^3-\widehat{a}^3\widehat{r}}{(1-\widehat{a})(1-\widehat{u}^2)(1-\widehat{r})} \\ &= \frac{1+t\widehat{u}}{(1-t)(1-\widehat{u}^2)(1-\widehat{r})} + \frac{\widehat{a}+\widehat{a}^2}{(1-\widehat{u}^2)(1-\widehat{r})} + \frac{\widehat{a}^3}{(1-\widehat{a})(1-\widehat{u}^2)}. \end{aligned}$$

The third term of this expression represents the elements of filtration above two (referred to in Remark 8.7) which disappear when we pass to $K_{\mathbf{R}}$. The first term represents the elements of filtration zero, which include

$$1, \quad [2u] \in \langle 2, a, a^2\bar{r}_1\rangle \quad \text{and} \quad [u^2] \in \langle a, a^2\bar{r}_1 a, a^2\bar{r}_1\rangle. \tag{8.9}$$

Here we use the notation $[2u]$ and $[u^2]$ to indicate the images in $\underline{E}_4$ of the elements $2u$ and u^2 in $\underline{E}_2$; see Remark 4.1 below. The former *not* divisible by 2 and the latter is not a square since u itself is not present in $\underline{E}_4$, where the Massey products are defined. For an introduction to Massey products, we refer the reader to [9, A1.4].

We now make a similar computation where we enlarge $\underline{E}_2(G/G)$ by adjoining $\bar{r}_1^{-1}u$ and denote the resulting spectral sequence terms by $\underline{E}_2'$ and $\underline{E}_4'$.

Let

$$\widehat{w} = \widehat{r}^{-1}\widehat{u} = xy^{-3}.$$

Then since

$$\underline{E}_2'(G/G) = \mathbf{Z}[a, \bar{r}_1^{-1}u, \bar{r}_1]/(2a),$$

we have

$$\begin{aligned}
g(\underline{E}_2'(G/G)) &= \left(\frac{1}{1-t} + \frac{\widehat{a}}{1-\widehat{a}}\right)\frac{1}{(1-\widehat{w})(1-\widehat{r})},\\
g(\underline{E}_4(G/G)) &= g(\underline{E}_2(G/G)) - \frac{\widehat{w} + \widehat{a}^3}{(1-\widehat{a})(1-\widehat{w}^2)(1-\widehat{r})}\\
&= \left(\frac{1}{1-t} + \frac{\widehat{a}}{1-\widehat{a}}\right)\frac{1}{(1-\widehat{w})(1-\widehat{r})} - \frac{\widehat{w}}{(1-\widehat{w}^2)(1-\widehat{r})} - \frac{a\widehat{w} + \widehat{a}^3}{(1-\widehat{a})(1-\widehat{w}^2)(1-\widehat{r})}\\
&= \frac{1 + \widehat{w} - \widehat{w}(1-t)}{(1-t)(1-\widehat{w}^2)(1-\widehat{r})} + \frac{\widehat{a}(1+\widehat{w}) - \widehat{a}(\widehat{w} + \widehat{a}^2)}{(1-\widehat{a})(1-\widehat{w}^2)(1-\widehat{r})}\\
&= \frac{1 + t\widehat{w}}{(1-t)(1-\widehat{w}^2)(1-\widehat{r})} + \frac{\widehat{a} + \widehat{a}^2}{(1-\widehat{w}^2)(1-\widehat{r})}
\end{aligned}$$

and there is nothing in $\underline{E}_4'$ with filtration above two. As far as we know there is no modification of the spectrum $k_{\mathbf{R}}$ corresponding to this modification of $\underline{E}_r$. However the map $\underline{E}_r k_{\mathbf{R}} \to \underline{E}_r K_{\mathbf{R}}$ clearly factors through $\underline{E}_r'$

9 Some elements in the homotopy groups of $k_{[2]}$ and $K_{[2]}$

For $G = C_4$ we will often use a (second) subscript ϵ on elements such as r_n to indicate the action of a generator γ of $G = C_4$, so $\gamma(x_\epsilon) = x_{1+\epsilon}$ and $x_{2+\epsilon} = \pm x_\epsilon$. Then we have

$$\pi_*^u k_{[2]} = \underline{\pi}_* k_{[2]}(G/\{e\}) = \underline{\pi}_{\{e\},*} k_{[2]}(G/G) = \mathbf{Z}[r_1, \gamma(r_1)] = \mathbf{Z}[r_{1,0}, r_{1,1}], \tag{9.1}$$

where $\gamma^2(r_{1,\epsilon}) = -r_{1,\epsilon}$. Here we use $r_{1,\epsilon}$ and $\bar{r}_{1,\epsilon}$ to denote the images of elements of the same name in the homotopy of $\mathrm{MU}^{((G))}$.

(9.2)

	−2	−1	0	1	2	3	4
2	a_λ						
1		a_σ a_{σ_2}		η η'			
0			u_σ $u_{2\sigma}$ u_{σ_2} u_λ		$\bar{r}_{1,0}$ $\bar{r}_{1,1}$		$\bar{t}_2$ $\bar{t}_2'$ $\bar{\mathfrak{d}}_1$

Here the vertical coordinate is s and the horizontal coordinate is $|t| - s$. More information about these elements can be found in Table 3 below.

We are using the following notational convention. When $x = \mathrm{tr}_2^4(y)$ for some element $y \in \underline{\pi}_\star k_{[2]}(G/G')$, we will write $x' = \mathrm{tr}_2^4(u_\sigma y)$. Examples above include the cases $x = \eta$ and $x = \bar{t}_2$. The primes could be iterated, i.e., we might write $x^{(k)} = \mathrm{tr}_2^4(u_\sigma^k y)$, but this turns out to be unnecessary.

The group action (by G' on $\bar{r}_{1,\epsilon}$, a_{σ_2} and u_{σ_2}, and by G on all the others) fixes each generator but u_σ and u_{σ_2}. For them the action is given by

$$u_\sigma \overset{\gamma}{\longleftrightarrow} -u_\sigma \qquad \text{and} \qquad u_{\sigma_2} \overset{\gamma^2}{\longleftrightarrow} -u_{\sigma_2}$$

by Theorem 2.13. This is compatible with the following G-action:

$$\begin{array}{ccc} r_{1,0} & \xrightarrow{\gamma} & r_{1,1} \\ {\scriptstyle\gamma}\uparrow & & \downarrow{\scriptstyle\gamma} \\ -r_{1,1} & \xleftarrow[\gamma]{} & -r_{1,0}, \end{array}$$

where $r_{1,\epsilon} = \underline{r}_1^2(\bar{r}_{1,\epsilon}) \in \underline{\pi}_{\{e\},2} k_{[2]}(G/G)$.

We will see below (Theorem 11.13) that $d_5(u_{2\sigma}) = a_\sigma^3 a_\lambda \bar{\mathfrak{d}}_1$ and $[u_{2\sigma}^2]$ is a permanent cycle. Since all transfers are killed by a_σ multiplication (Lemma 4.2), this implies that $[u_{2\sigma} x]$ is a permanent cycle representing the Toda bracket

$$[u_{2\sigma} x] = [u_{2\sigma}\, \mathrm{tr}_2^4(y)] = \langle x, a_\sigma, a_\sigma^2 a_\lambda \bar{\mathfrak{d}}_1 \rangle.$$

This element is x'' since in $\underline{E}_2$ we have (using the Frobenius relation (2.4))

$$x'' = \mathrm{tr}_2^4(u_\sigma^2 y) = \mathrm{tr}_2^4(\mathrm{res}_2^4(u_{2\sigma}) y) = u_{2\sigma}\, \mathrm{tr}_2^4(y) = u_{2\sigma} x.$$

Similarly $x''' = u_{2\sigma} x'$. For $k \geq 4$, $x^{(k)} = u_{2\sigma}^2 x^{(k-4)}$ in $\underline{\pi}_\star$ as well as $\underline{E}_2$.

The Periodicity Theorem [6, Theorem 9.19] states that inverting a class in $\underline{\pi}_{4\rho_4} k_{[2]}(G/G)$ whose image under $\underline{r}_2^4\, \mathrm{res}_2^4$ is divisible by $\bar{r}_{3,0}^{G'} \bar{r}_{3,1}^{G'}$ (see (7.2)) and $\bar{r}_{1,0}\bar{r}_{1,1} = \bar{r}_{1,0}^G \bar{r}_{1,1}^G$ makes $u_{8\rho_4}$ a permanent cycle. One such class is

$$D = N_2^4(\bar{\mathfrak{d}}_1^G)\bar{\mathfrak{d}}_2^{G'} = u_{2\sigma}^{-2}(\underline{r}_2^4\, \mathrm{res}_2^4)^{-1}(\bar{r}_{1,0}^G \bar{r}_{1,1}^G \bar{r}_{3,0}^{G'} \bar{r}_{3,1}^{G'}) = \bar{\mathfrak{d}}_1^2(-5\bar{t}_2^2 + 20\bar{t}_2\bar{\mathfrak{d}}_1 + 9\bar{\mathfrak{d}}_1^2) \in \underline{\pi}_{4\rho_4} k_{[2]}(G/G), \tag{9.3}$$

where $\bar{t}_2 = \mathrm{tr}_2^4(u_\sigma^{-1}[\bar{r}_{1,0}^2])$ and $\bar{\mathfrak{d}}_1$ is as in (9.5) below, and $K_{[2]} = D^{-1} k_{[2]}$. Then we know that $\Sigma^{32} K_{[2]}$ is equivalent to $K_{[2]}$.

The Slice and Reduction Theorems [6, Theorems 6.1 and 6.5] imply that the $2k$th slice of $k_{[2]}$ is the $2k$th wedge summand of $H\underline{\mathbf{Z}} \wedge N_2^4(\bigvee_{i\geq 0} S^{i\rho_2})$.

It follows that over G' the $2k$th slice is a wedge of $k+1$ copies of $H\underline{\mathbf{Z}} \wedge S^{k\rho_2}$. Over G we get the wedge of the appropriate number of copies of $G_+ \wedge_{G'} H\underline{\mathbf{Z}} \wedge S^{k\rho_2}$, wedged with a single copy of $H\underline{\mathbf{Z}} \wedge S^{(k/2)\rho_4}$ for even k. This is spelled out in Theorem 10.2 below.

The group $\underline{\pi}_{\rho_2}^{G'} k_{[2]}(G'/\{e\})$ is *not* in the image of the group action restriction $\underline{r}_2^4$ because ρ_2 is not the restriction of a representation of G. However, $\pi_2^u k_{[2]}$ is refined (in the sense of [6, Definition 5.28]) by a map from

$$S_{\rho_2} := G_+ \wedge_{G'} S^{\rho_2} \xrightarrow{\bar{s}_1} k_{[2]}. \tag{9.4}$$

The Reduction Theorem implies that the 2-slice $P_2^2 k_{[2]}$ is $S_{\rho_2} \wedge H\underline{\mathbf{Z}}$. We know that

$$\underline{\pi}_2(S_{\rho_2} \wedge H\underline{\mathbf{Z}}) = \widehat{\overline{\square}}.$$

We use the symbols r_1 and $\gamma(r_1)$ to denote the generators of the underlying abelian group of $\widehat{\overline{\square}}(G/\{e\}) = \mathbf{Z}[G/G']_-$. These elements have trivial fixed point transfers and

$$\underline{\pi}_2(S_{\rho_2} \wedge H\underline{\mathbf{Z}})(G/G') = 0.$$

Table 3 describes some elements in the slice spectral sequence for $k_{[2]}$ in low dimensions, which we now discuss.

Given an element in $\pi_\star \mathrm{MU}^{((G))}$, we will often use the same symbol to denote its image in $\pi_\star k_{[2]}$. For example, in [6, Section 9.1]

$$\bar{\mathfrak{d}}_n \in \pi^G_{(2^n-1)\rho_4}\mathrm{MU}^{((G))} = \underline{\pi}^G_{(2^n-1)\rho_4}\mathrm{MU}^{((G))}(G/G) \tag{9.5}$$

was defined to be the composite

$$S^{(2^n-1)\rho_4} = N_2^4 S^{(2^n-1)\rho_2} \xrightarrow{N_2^4 \bar{r}_{2^n-1}} N_2^4 \mathrm{MU}^{((G))} \longrightarrow \mathrm{MU}^{((G))}.$$

We will use the same symbol to denote its image in the group $\underline{\pi}^G_{(2^n-1)\rho_4} k_{[2]}(G/G)$.

The element $\eta \in \pi_1 S^0$ (coming from the Hopf map $S^3 \to S^2$) has image $a_\sigma \bar{r}_1 \in \underline{\pi}_1^{G'} k_{\mathbf{R}}(G'/G')$. There are t wo corresponding elements

$$\eta_\epsilon \in \underline{\pi}_1^{G'} k_{[2]}(G'/G') \quad \text{for } \epsilon = 0, 1.$$

We use the same symbol for their preimages under $\underline{r}_2^4$ in $\underline{\pi}_1^G k_{[2]}(G/G')$, and there we have

$$\eta_\epsilon = a_{\sigma_2}\bar{r}_{1,\epsilon}.$$

We denote by η again the image of either under the transfer tr_2^4, so

$$\mathrm{res}_2^4(\eta) = \eta_0 + \eta_1.$$

Its cube is killed by a d_3 in the slice spectral sequence, as is the sum of any two monomials of degree 3 in the η_ϵ. It follows that in $\underline{E}_4$ each such monomial is equal to η_0^3. It has a nontrivial transfer, which we denote by x_3.

In [6, Definition 5.51] we defined

$$f_k = a_{\bar\rho}^k N_2^g(\bar{r}_k) \in \underline{\pi}_k \mathrm{MU}^{((G))}(G/G) \tag{9.6}$$

for a finite cyclic 2-group G. In particular, $f_{2^{n-1}} = a_{\bar\rho}^{2^n-1}\bar{\mathfrak{d}}_n$ for $\bar{\mathfrak{d}}_n$ as in (9.5). The slice filtration of f_k is $k(g-1)$ and we will see below (Lemma 4.2 and, for $G = C_4$, Theorem 11.13) that

$$\mathrm{tr}_{G'}^G(u_\sigma) = a_\sigma f_1. \tag{9.7}$$

Note that $u_\sigma \in \underline{E}_2^{0,1-\sigma}(G/G')$ since the maximal subgroup for which the sign representation σ is oriented is G', on which it restricts to the trivial representation of degree 1. This group depends only on the restriction of the $RO(G)$-grading to G', and the isomorphism extends to differentials as well. This means that u_σ is a place holder corresponding to the permanent cycle $1 \in \underline{E}_2^{0,0}(G/G')$.

For $G = C_4$, equation (9.7) implies

$$\mathrm{tr}_2^4(u_\sigma) = a_\sigma f_1 = a_\sigma^2 a_\lambda \bar{\mathfrak{d}}_1.$$

For example,

$$\mathrm{tr}_2^4(\eta_0\eta_1) = \mathrm{tr}_2^4(a_{\sigma_2}^2 \bar{r}_{1,0}\bar{r}_{1,1}) = \mathrm{tr}_2^4(u_\sigma\, \mathrm{res}_2^4(a_\lambda \bar{\mathfrak{d}}_1)) = \mathrm{tr}_2^4(u_\sigma) a_\lambda \bar{\mathfrak{d}}_1 = a_\sigma f_1 a_\lambda \bar{\mathfrak{d}}_1 = f_1^2.$$

The Hopf element $\nu \in \pi_3 S^0$ has image

$$a_\sigma u_\lambda \bar{\mathfrak{d}}_1 \in \underline{\pi}_3 k_{[2]}(G/G),$$

so we also denote the latter by ν. (We will see below in (11.7) that u_λ is not a permanent cycle, but $\bar\nu := a_\sigma u_\lambda$ is (11.8).) It has an exotic restriction η_0^3 (filtration jump two), which implies that

$$2\nu = \mathrm{tr}_2^4(\mathrm{res}_2^4(\nu)) = \mathrm{tr}_2^4(\eta_0^3) = x_3.$$

One way to see this is to use the Periodicity Theorem to equate $\underline{\pi}_3 k_{[2]}$ with $\underline{\pi}_{-29} k_{[2]}$, which can be shown to be the Mackey functor $\circ$ in slice filtration -32. Another argument not relying on periodicity is given below in Theorem 11.13.

The exotic restriction on ν implies

$$\mathrm{res}_2^4(\nu^2) = \eta_0^6,$$

with filtration jump 4.

Theorem 9.8 (The Hurewicz image). *The elements* $\nu \in \underline{\pi}_3 k_{[2]}(G/G)$, $\epsilon \in \underline{\pi}_8 k_{[2]}(G/G)$, $\kappa \in \underline{\pi}_{14} k_{[2]}(G/G)$, *and* $\overline{\kappa} \in \underline{\pi}_{20} k_{[2]}(G/G)$ *are the images of elements of the same names in* $\pi_* S^0$. *The image of the Hopf map* $\eta \in \pi_1 S^0$ *is either* $\eta = \mathrm{tr}_2^4(\eta_\epsilon)$ *or its sum with* f_1.

We refer the reader to [9, Table A3.3] for more information about these elements.

Proof. Suppose we know this for ν and $\overline{\kappa}$. Then $\Delta_1^{-4}\nu$ is represented by an element of filtration −3 whose product with ν^2 is nontrivial. This implies that ν^3 has nontrivial image in $\underline{\pi}_9 k_{[2]}(G/G)$. This is a nontrivial multiplicative extension in the first quadrant, but not in the third. The spectral sequence representative of ν^3 has filtration 11 instead of 3. We will see later that $\nu^3 = 2n$ where n has filtration 1, and ν^3 is the transfer of an element in filtration 1.

Since $\nu^3 = \eta\epsilon$ in $\pi_* S^0$, this implies that η and ϵ are both detected and have the images stated in Table 3. It follows that $\epsilon\overline{\kappa}$ has nontrivial image here. Since $\kappa^2 = \epsilon\overline{\kappa}$ in $\pi_* S^0$, κ must also be detected. Its only possible image is the one indicated.

Both ν and $\overline{\kappa}$ have images of order 8 in $\pi_*\mathrm{TMF}$ and its $K(2)$ localization. The latter is the homotopy fixed point set of an action of the binary tetrahedral group G_{24} acting on E_2. This in turn is a retract of the homotopy fixed point set of the quaternion group Q_8. A restriction and transfer argument shows that both elements have order at least 4 in the homotopy fixed point set of $C_4 \subset Q_8$.

There is an orientation map $\mathrm{MU} \to E_2$, which extends to a C_2-equivariant map $\mathrm{MU}_{\mathbb{R}} \to E_2$. Norming up and multiplying on the right gives us a C_4-equivariant map $N_2^4 \mathrm{MU}_{\mathbb{R}} \to E_2$. This C_4-action on the target is compatible with the G_{24}-action leading to $L_{K(2)}\mathrm{TMF}$.

The image of $\eta \in \pi_1 S^0$ must restrict to $\eta_0 + \eta_1$, so modulo the kernel of res_2^4 it is the element $\mathrm{tr}_2^4(\eta_\epsilon)$, which we are calling η. The kernel of res_2^4 is generated by f_1. □

We now discuss the norm N_2^4, which is a functor from the category of C_2-spectra to that of C_4 spectra. As explained above in connection with Corollary 4.8, for a C_4-ring spectrum X we have an internal norm

$$\underline{\pi}_V^{G'} i_{G'}^* X(G'/G') \cong \underline{\pi}_{G',V}^{G'} X(G/G) \to \underline{\pi}_{\mathrm{Ind}_2^4 V}^{G} X(G/G)$$

and a similar functor on the slice spectral sequence for X. It preserves multiplication but not addition. Its source is a module over G/G', which acts trivially on its target. Consider the diagram

$$\begin{array}{ccccc} \underline{\pi}_{G',V} X(G/G) & \xrightarrow{\cong} & \underline{\pi}_V^{G'} i_{G'}^* X(G'/G') & \xrightarrow{N_2^4} & \underline{\pi}_{\mathrm{Ind}_2^4 V}^{G} X(G/G) \\ & & & & \downarrow{\scriptstyle \mathrm{res}_2^4} \\ \underline{\pi}_{G',2V} X(G/G) & \xrightarrow{\cong} & \underline{\pi}_{2V}^{G'} i_{G'}^* X(G'/G') & \xrightarrow{\cong} & \underline{\pi}_{\mathrm{Ind}_2^4 V}^{G} X(G/G'). \end{array}$$

For $x \in \underline{\pi}_V^{G'} i_{G'}^* X(G'/G')$ we have $x\gamma(x) \in \underline{\pi}_{2V}^{G'} i_{G'}^* X(G'/G')$ and $2V$ is the restriction of some $W \in RO(G)$. The group $\underline{\pi}_W^G X(G/G')$ depends only on the restriction of W to $RO(G')$. If $W' \in RO(G)$ is another virtual representation restricting to $2V$, then $W - W' = k(1-\sigma)$ for some integer k. The canonical isomorphism between $\underline{\pi}_W^G X(G/G')$ and $\underline{\pi}_{W'}^G X(G/G')$ is given by multiplication by u_σ^k.

Definition 9.9 (A second use of square bracket notation). For $0 \le i \le 2d$, let $f(\overline{r}_{1,0}, \overline{r}_{1,1})$ be a homogeneous polynomial of degree $2d - i$, so

$$a_{\sigma_2}^i f(\overline{r}_{1,0}, \overline{r}_{1,1}) \in \underline{\pi}_{(2d-i)+(2d-2i)\sigma_2}^{G'} i_{G'}^* k_{[2]}(G'/G').$$

We will denote by $[a_{\sigma_2}^i f(\overline{r}_{1,0}, \overline{r}_{1,1})]$ its preimage in $\underline{\pi}_{2d-i+(d-i)\lambda} k_{[2]}(G/G')$ under the isomorphism of (2.14).

The first use of square bracket notation is that of Remark 4.1. Note that $\overline{r}_{1,\epsilon} \in \underline{\pi}_{\rho_2}^{G'} i_{G'}^* k_{[2]}$ is not the target of such an isomorphism since $\rho_2 \in RO(G')$ is not the restriction of any element in $RO(G)$, hence the requirement that f has even degree.

We will denote $u_\sigma^{-1}[\overline{r}_{1,\epsilon}^2] \in \underline{\pi}_{\rho_4}^G k_{[2]}(G/G')$ by $\overline{s}_{2,\epsilon}$. Then we have $\gamma(\overline{s}_{2,0}) = -\overline{s}_{2,1}$ and $\gamma(\overline{s}_{2,1}) = -\overline{s}_{2,0}$. We define

$$\overline{t}_2 := (-1)^\epsilon \, \mathrm{tr}_2^4(\overline{s}_{2,\epsilon}),$$

which is independent of ϵ, and we have

$$\operatorname{res}_2^4(\bar{t}_2) = \bar{s}_{2,0} - \bar{s}_{2,1}.$$

Then we have

$$\operatorname{res}_2^4(N_2^4(\bar{r}_{1,0})) = \operatorname{res}_2^4(\bar{\mathfrak{d}}_1) = u_\sigma^{-1}[\bar{r}_{1,0}\bar{r}_{1,1}] \in \underline{\pi}_{\rho_4} k_{[2]}(G/G').$$

More generally, for integers m and n,

$$\begin{aligned}\operatorname{res}_2^4(N_2^4(m\bar{r}_{1,0} + n\bar{r}_{1,1})) &= u_\sigma^{-1}[(m\bar{r}_{1,0} + n\bar{r}_{1,1})(m\bar{r}_{1,1} - n\bar{r}_{1,0})] \\ &= u_\sigma^{-1}((m^2 - n^2)[\bar{r}_{1,0}\bar{r}_{1,1}] + mn([\bar{r}_{1,1}^2] - [\bar{r}_{1,0}^2])) \\ &= (m^2 - n^2)\operatorname{res}_2^4(\bar{\mathfrak{d}}_1) - mn \operatorname{res}_2^4(\bar{t}_2),\end{aligned}$$

so

$$N_2^4(m\bar{r}_{1,0} + n\bar{r}_{1,1}) = (m^2 - n^2)\bar{\mathfrak{d}}_1 - mn\bar{t}_2. \tag{9.10}$$

Similarly, for integers a, b and c,

$$\begin{aligned}u_\sigma^2 \operatorname{res}_2^4(N_2^4(a\bar{r}_{1,0}^2 + b\bar{r}_{1,0}\bar{r}_{1,1} + c\bar{r}_{1,1}^2)) &= [(a\bar{r}_{1,0}^2 + b\bar{r}_{1,0}\bar{r}_{1,1} + c\bar{r}_{1,1}^2)(a\bar{r}_{1,1}^2 - b\bar{r}_{1,0}\bar{r}_{1,1} + c\bar{r}_{1,0}^2)] \\ &= [ac(\bar{r}_{1,0}^4 + \bar{r}_{1,1}^4) + b(c-a)\bar{r}_{1,0}\bar{r}_{1,1}(\bar{r}_{1,0}^2 - \bar{r}_{1,1}^2) + (a^2 - b^2 + c^2)\bar{r}_{1,0}^2\bar{r}_{1,1}^2] \\ &= [ac(\bar{r}_{1,0}^2 - \bar{r}_{1,1}^2)^2 + b(c-a)\bar{r}_{1,0}\bar{r}_{1,1}(\bar{r}_{1,0}^2 - \bar{r}_{1,1}^2) + ((a+c)^2 - b^2)\bar{r}_{1,0}^2\bar{r}_{1,1}^2],\end{aligned}$$

so

$$N_2^4(a\bar{r}_{1,0}^2 + b\bar{r}_{1,0}\bar{r}_{1,1} + c\bar{r}_{1,1}^2) = ac\,\bar{t}_2^2 + b(c-a)\bar{\mathfrak{d}}_1\bar{t}_2 + ((a+c)^2 - b^2)\bar{\mathfrak{d}}_1^2 \tag{9.11}$$

For future reference we need

$$N_2^4(5\bar{r}_{1,0}^2\bar{r}_{1,1} + 5\bar{r}_{1,0}\bar{r}_{1,1}^2 + \bar{r}_{1,1}^3) = N_2^4(\bar{r}_{1,1})N_2^4(5\bar{r}_{1,0}^2 + 5\bar{r}_{1,0}\bar{r}_{1,1} + \bar{r}_{1,1}^2)) = -\bar{\mathfrak{d}}_1(5\bar{t}_2^2 - 20\bar{\mathfrak{d}}_1\bar{t}_2 + 11\bar{\mathfrak{d}}_1^2).$$

Compare with (7.2). We also denote by

$$\eta_\epsilon = [a_{\sigma_2}\bar{r}_{1,\epsilon}] \in \underline{\pi}_1 k_{[2]}(G/G')$$

the preimage of $a_{\sigma_2}\bar{r}_{1,\epsilon} \in \underline{\pi}_1^{G'} i_{G'}^* k_{[2]}(G'/G')$ and by $[a_{\sigma_2}^2] \in \underline{\pi}_{-\lambda} k_{[2]}(G/G')$ the preimage of $a_{\sigma_2}^2$. The latter is $\operatorname{res}_2^4(a_\lambda)$. The values of $N_2^4(a_{\sigma_2})$ and $N_2^4(u_{2\sigma_2})$ are given by Lemma 4.9, namely

$$N_2^4(a_{\sigma_2}) = a_\lambda \quad \text{and} \quad N_2^4(u_{2\sigma_2}) = u_{2\lambda}/u_{2\sigma}.$$

Element	Description
Filtration 0	
$\bar{r}_{1,\epsilon} \in \underline{\pi}_{\rho_2}^{G'} i_{G'}^* k_{[2]}(G'/G') \cong \underline{\pi}_{G',\rho_2} k_{[2]}(G/G)$ with $\bar{r}_{1,2} = -\bar{r}_{1,0}$	Images from (7.1) defined in [6, (5.47)]
$r_{1,\epsilon} \in \underline{\pi}_{\{e\},2} k_{[2]}(G/G) \cong \underline{\pi}_{G,2} k_{[2]}(G/\{e\}) \cong \pi_2^u k_{[2]}$	$\underline{r}_1^2(\bar{r}_{1,\epsilon})$, generating $\underline{\pi}_2^G k_{[2]}$/torsion $= \widehat{\widehat{\Box}}$
$u_{2\sigma} \in \underline{E}_2^{0,2-2\sigma}(G/G)$ with	Element corresponding to $u_{2\sigma} \in \underline{\pi}_{2-2\sigma} H\underline{\mathbf{Z}}(G/G)$
$d_5(u_{2\sigma}) = a_\sigma^3 a_\lambda \bar{\mathfrak{d}}_1$	Slice differential of (11.3)
$[2u_{2\sigma}] = \langle 2, a_\sigma, a_\sigma^2 a_\lambda \bar{\mathfrak{d}}_1 \rangle \in \underline{E}_6^{0,2-2\sigma}(G/G)$	Image of $2u_{2\sigma}$ in $\underline{E}_6^{0,2-2\sigma}(G/G)$, which is a permanent cycle
$[u_{2\sigma}^2] = \langle a_\sigma^3 a_\lambda, \bar{\mathfrak{d}}_1, a_\sigma^3 a_\lambda, \bar{\mathfrak{d}}_1 \rangle \in \underline{E}_6^{0,4-4\sigma}(G/G)$	Image of $u_{2\sigma}^2$ in $\underline{E}_6^{0,2-2\sigma}(G/G)$, which is a permanent cycle
$u_\sigma \in \underline{\pi}_{1-\sigma} k_{[2]}(G/G') \cong \underline{\pi}_{G',0} k_{[2]}(G/G)$ with $\operatorname{res}_2^4(u_{2\sigma}) = u_\sigma^2$, $\gamma(u_\sigma) = -u_\sigma$	Isomorphic image of $1 \in \underline{\pi}_0 k_{[2]}(G/G') \cong \underline{\pi}_{G',0} k_{[2]}(G/G)$
$\operatorname{tr}_2^4(u_\sigma^{4k+1}) = a_\sigma f_1 u_{2\sigma}^{2k}$ (exotic transfer), $\operatorname{tr}_2^4(u_\sigma^{2k}) = 2u_{2\sigma}^k$, $\operatorname{tr}_2^4(u_\sigma^{4k+3}) = 0$	Follows from Theorem 4.4 and $d_5(u_{2\sigma})$ in (11.3)

Continued on next page

Element	Description
$u_\lambda \in \underline{E}_2^{0,2-\lambda}(G/G)$ with	Element corresponding to $u_\lambda \in \underline{\pi}_{2-\lambda} H\underline{\mathbf{Z}}(G/G)$
$[2u_\lambda] \in \underline{\pi}_{2-\lambda} K_{[2]}(G/G)$	$\langle 2, \eta, a_\lambda \rangle$
$a_\sigma^3 u_\lambda = 0$	Follows from the gold relation, Lemma 3.6 (vii)
$d_3(u_\lambda) = \eta a_\lambda = \mathrm{tr}_2^4([a_{\sigma_2}^3 \bar{r}_{1,0}])$	Slice differential of Theorem 11.13
$d_5([u_\lambda^2]) = \bar{\nu} a_\lambda^2 \bar{\mathfrak{d}}_1$	Slice differential of Theorem 11.13
$d_7([2u_\lambda^2]) = \eta' a_\lambda^3 \bar{\mathfrak{d}}_1$	$2\bar{\nu} a_\lambda^2 \bar{\mathfrak{d}}_1$
$[4u_\lambda^2] \in \underline{\pi}_{4-2\lambda} K_{[2]}(G/G)$	$\langle 2, \eta, a_\lambda \rangle^2 = \langle 2, \eta', a_\lambda^3 \bar{\mathfrak{d}}_1 \rangle$
$[2a_\sigma u_\lambda^2] \in \underline{\pi}_{4-\sigma-2\lambda} K_{[2]}(G/G)$	$\langle a_\sigma, \eta', a_\lambda^3 \bar{\mathfrak{d}}_1 \rangle$
$d_7([u_\lambda^4]) = \langle \eta', \bar{\nu}, a_\lambda^2 \bar{\mathfrak{d}}_1 \rangle a_\lambda^3 \bar{\mathfrak{d}}_1$	$[2u_\lambda^2 d(u_\lambda^2)]$
$[2u_\lambda^4] \in \underline{\pi}_{8-4\lambda} K_{[2]}(G/G)$	$\mathrm{tr}_2^4(\overline{u}_\lambda^4)$
$\overline{u}_\lambda \in \underline{E}_2^{0,2-\lambda}(G/G')$ with	$\mathrm{res}_2^4(u_\lambda)$
$d_3(\overline{u}_\lambda) = [a_{\sigma_2}^3(\bar{r}_{1,0} + \bar{r}_{1,1})] = \mathrm{res}_2^4(a_\lambda)(\eta_0 + \eta_1)$	$\mathrm{res}_2^4(d_3(u_\lambda))$
$[2\overline{u}_\lambda] \in \underline{\pi}_{2-\lambda} K_{[2]}(G/G')$	$[\langle 2, a_{\sigma_2}^3, \bar{r}_{1,0} + \bar{r}_{1,1} \rangle] = \langle 2, [a_{\sigma_2}^2], \eta_0 + \eta_1 \rangle$
$d_7([\overline{u}_\lambda^2]) = a_{\sigma_2}^7 \bar{r}_{1,0}^3$	$\mathrm{res}_2^4(d_5(u_\lambda^2))$
$[2\overline{u}_\lambda^2] \in \underline{\pi}_{4-2\lambda} K_{[2]}(G/G')$	$[\langle 2, a_{\sigma_2}^7, \bar{r}_{1,0}^3 \rangle] = \langle 2, [a_{\sigma_2}^2]^2, \eta_0^3 \rangle$
$[\overline{u}_\lambda^4] \in \underline{\pi}_{8-4\lambda} K_{[2]}(G/G')$	$[\langle a_{\sigma_2}^7, \bar{r}_{1,0}^3, a_{\sigma_2}^7, \bar{r}_{1,0}^3 \rangle] = \langle [a_{\sigma_2}^2]^2, \eta_0^3, [a_{\sigma_2}^2]^2, \eta_0^3 \rangle$
$u_{\sigma_2} \in \underline{\pi}_{(G',1-\sigma_2)} k_{[2]}(G/e)$ with $\mathrm{res}_1^2(\overline{u}_\lambda) = u_{\sigma_2}^2$, $\gamma^2(u_{\sigma_2}) = -u_{\sigma_2}$ and $\mathrm{tr}_1^2(u_{\sigma_2}) = a_{\sigma_2}^2(\bar{r}_{1,0} + \bar{r}_{1,1})$ (exotic transfer)	Isomorphic image of $1 \in \underline{\pi}_0 k_{[2]}(G/e)$
$\bar{s}_{2,\epsilon} \in \underline{\pi}_{\rho_4}^G k_{[2]}(G/G')$	$u_\sigma^{-1}[\bar{r}_{1,\epsilon}^2]$
$\bar{\mathfrak{d}}_1 \in \underline{\pi}_{\rho_4}^G k_{[2]}(G/G)$ with $\mathrm{res}_2^4(\bar{\mathfrak{d}}_1) = u_\sigma^{-1}[\bar{r}_{1,0}\bar{r}_{1,1}]$	Image from (9.5) defined in [6, Section 9.1]
$\bar{t}_2 \in \underline{\pi}_{\rho_4}^G k_{[2]}(G/G)$ with $\mathrm{res}_2^4(\bar{t}_2) = \bar{s}_{2,0} - \bar{s}_{2,1}$	$(-1)^\epsilon\, \mathrm{tr}_2^4(\bar{s}_{2,\epsilon})$ for either value of ϵ
$\bar{t}_2' \in \underline{\pi}_{2+\lambda}^G k_{[2]}(G/G)$ with $\mathrm{res}_2^4(\bar{t}_2') = [\bar{r}_{1,0}^2] + [\bar{r}_{1,1}^2]$	$\mathrm{tr}_2^4([\bar{r}_{1,\epsilon}^2])$ for either value of ϵ
$D \in \underline{\pi}_{4\rho_4} k_{[2]}(G/G)$, the periodicity element	$-\bar{\mathfrak{d}}_1^2(5\bar{t}_2^2 - 20\bar{t}_2\bar{\mathfrak{d}}_1 + 11\bar{\mathfrak{d}}_1^2)$
$\Sigma_{2,\epsilon} \in \underline{E}_2^{0,4} k_{[2]}(G/G')$ with $\Sigma_{2,2} = \Sigma_{2,0}$ and $d_3(\Sigma_{2,\epsilon}) = \eta_\epsilon^2(\eta_0 + \eta_1)$	$(-1)^\epsilon u_{\rho_4} \bar{s}_{2,\epsilon} = (-1)^\epsilon \overline{u}_\lambda [\bar{r}_{1,\epsilon}^2]$
$T_2 \in \underline{E}_2^{0,4} k_{[2]}(G/G)$ with $\mathrm{res}_2^4(T_2) = \Sigma_{2,0} - \Sigma_{2,1}$ and $d_3(T_2) = \eta^3$	$\mathrm{tr}_2^4(\Sigma_{2,\epsilon}) = (-1)^\epsilon u_\lambda\, \mathrm{tr}_2^4([\bar{r}_{1,\epsilon}^2])$ for either value of ϵ
$T_4 \in \underline{E}_2^{0,8} k_{[2]}(G/G)$ with $T_4^2 = \Delta_1(T_2^2 - 4\Delta_1)$, $\mathrm{res}_2^4(T_4) = (\Sigma_{2,0} - \Sigma_{2,1})\delta_1$ and $d_3(T_4) = 0$	$(-1)^\epsilon\, \mathrm{tr}_2^4(\Sigma_{2,\epsilon}\delta_1) = u_{2\sigma} u_\lambda^2 \bar{t}_2 \bar{\mathfrak{d}}_1$ for either value of ϵ
$\delta_1 \in \underline{E}_2^{0,4} k_{[2]}(G/G')$ with $\gamma(\delta_1) = -\delta_1$, $\mathrm{tr}_2^4(\delta_1) = 0$ and $d_3(\delta_1) = \eta_0\eta_1(\eta_0 + \eta_1)$	$u_{\rho_4}\, \mathrm{res}_2^4(\bar{\mathfrak{d}}_1) = \overline{u}_\lambda [\bar{r}_{1,0}\bar{r}_{1,1}]$
$\Delta_1 \in \underline{E}_2^{0,8} k_{[2]}(G/G)$ with $\mathrm{res}_2^4(\Delta_1) = \delta_1^2$, $\mathrm{res}_1^4(\Delta_1) = r_{1,0}^2 r_{1,1}^2$ and $d_5(\Delta_1) = \nu x_4$	$u_{2\rho_4} \bar{\mathfrak{d}}_1^2 = u_{2\sigma} u_\lambda^2 \bar{\mathfrak{d}}_1^2$
Filtration 1	
$a_{\sigma_2} \in \underline{\pi}_{G',-\sigma_2} k_{[2]}(G/G) \cong \underline{\pi}_1^{G'} k_{[2]}(G'/G')$ with $2a_{\sigma_2} = 0$	See Definition 3.4
$\eta_\epsilon \in \underline{\pi}_1 k_{[2]}(G/G') \cong \underline{\pi}_1^{G'} k_{[2]}(G'/G')$ with $2\eta_\epsilon = 0$	$[a_{\sigma_2}\bar{r}_{1,\epsilon}]$
$\eta \in \underline{\pi}_1^G k_{[2]}(G/G)$ with $\mathrm{res}_2^4(\eta) = \eta_0 + \eta_1 \in \underline{\pi}_1^G k_{[2]}(G/G')$	$\mathrm{tr}_2^4(\eta_\epsilon) = \mathrm{tr}_2^4([a_{\sigma_2}\bar{r}_{1,0}]) = \mathrm{tr}_2^4([a_{\sigma_2}\bar{r}_{1,1}])$
$\eta' \in \underline{\pi}_{2-\sigma} k_{[2]}(G/G)$ with $\mathrm{res}_2^4(\eta') = u_\sigma(\eta_0 + \eta_1) \in \underline{\pi}_{2-\sigma}^G k_{[2]}(G/G')$	$\mathrm{tr}_2^4(\eta_0 u_\sigma) = \mathrm{tr}_2^4([a_{\sigma_2}\bar{r}_{1,0}]u_\sigma) = \mathrm{tr}_2^4([a_{\sigma_2}\bar{r}_{1,1}]u_\sigma)$
$\bar{\nu} \in \underline{\pi}_{2-\sigma-\lambda} k_{[2]}(G/G)$ with	$[a_\sigma u_\lambda] = \langle a_\sigma, \eta, a_\lambda \rangle$
$\mathrm{res}_2^4(\bar{\nu}) = u_\sigma[a_{\sigma_2}^3 \bar{r}_{1,0}]$ (exotic restriction)	Follows from Theorem 4.4 and $d_3(u_\lambda)$ in Theorem 11.13
$2\bar{\nu} = \eta' a_\lambda$	Transfer of the above
$\eta\bar{\nu} = a_\lambda \langle 2, a_\sigma, f_1^2 \rangle = a_\lambda \langle 2, a_\sigma, \mathrm{tr}_2^4(\eta_0\eta_1) \rangle$	
$\eta'\bar{\nu} = 0$	
$a_\sigma \bar{\nu} = a_\lambda\, \mathrm{tr}_2^4(u_\sigma^2)$	$[a_\sigma^2 u_\lambda] = a_\lambda [2u_{2\sigma}]$ by the gold relation, Lemma 3.6 (vii)
$a_\sigma^2 \bar{\nu} = 0$	$[a_\sigma^3 u_\lambda] = a_\lambda a_\sigma\, \mathrm{tr}_2^4(u_\sigma^2) = 0$

Continued on next page

Element	Description
$\xi \in \underline{\pi}_{4-3\sigma-\lambda} k_{[2]}(G/G)$ with $\mathrm{res}_2^4(\xi) = a_{\sigma_2}^3 u_\sigma^3 \bar{r}_{1,0}$ $2\xi = a_\lambda \langle \eta', a_\sigma^2, f_1 \rangle$ $d_5(u_{2\sigma} u_\lambda^2) = \xi a_\lambda^2 \bar{\mathfrak{d}}_1$ $\eta\xi = 2a_\lambda u_{2\sigma}^2 \bar{\mathfrak{d}}_1$ (exotic multiplication) $\eta'\xi = a_\sigma^2 a_\lambda^3 u_{2\sigma}^2 \bar{\mathfrak{d}}_1^2$ (exotic multiplication)	$[\bar{v} u_{2\sigma}] = \langle \bar{v}, a_\sigma^2, f_1 \rangle$ Follows from value of $\mathrm{res}_2^4(\bar{v})$ Transfer of the above
$v \in \underline{\pi}_3 k_{[2]}(G/G)$ with $\mathrm{res}_2^4(v) = \eta_0^3$ and $2v = x_3$ (exotic restriction and group extension)	$a_\sigma u_\lambda \bar{\mathfrak{d}}_1 = \bar{v}\bar{\mathfrak{d}}_1$, generating $\circ = \underline{\pi}_3 k_{[2]}$ Follows from those on $\bar{v}$
Filtration 2	
$[a_{\sigma_2}^2] \in \underline{\pi}_{-\lambda} k_{[2]}(G/G')$	Preimage of $a_{\sigma_2}^2 \in \underline{\pi}_{-2\sigma_2} i_{G'}^* k_{[2]}(G'/G')$
$a_\lambda \in \underline{\pi}_{-\lambda} k_{[2]}(G/G)$ with $4a_\lambda = 0$ and $\mathrm{res}_2^4(a_\lambda) = [a_{\sigma_2}^2]$	See Definition 3.4
$\eta_\epsilon^2, \eta_0\eta_1 \in \underline{\pi}_2^G k_{[2]}(G/G')$ with $\mathrm{tr}_2^4(\eta_\epsilon^2) = (-1)^\epsilon a_\lambda \bar{t}_2'$ and $\mathrm{tr}_2^4(\eta_0\eta_1) = f_1^2$ (exotic transfer)	$u_\sigma[a_{\sigma_2}^2]\bar{s}_{2,\epsilon}$ and $u_\sigma[a_{\sigma_2}^2]\,\mathrm{res}_2^4(\bar{\mathfrak{d}}_1)$, generating the torsion $\widehat{\bullet} \oplus \blacktriangledown$ in $\underline{\pi}_2^G k_{[2]}$
$\eta^2 = a_\lambda(\bar{t}_2' + a_\sigma^2 a_\lambda \bar{\mathfrak{d}}_1^2) = a_\lambda \bar{t}_2' + f_1^2$ $\eta\eta' = a_\lambda[u_{2\sigma}\bar{t}_2]$, $(\eta')^2 = a_\lambda[u_{2\sigma}\bar{t}_2']$	$a_\lambda \bar{t}_2'$ has order 2 by Lemma 4.2 See (11.5) for the definition of $[u_{2\sigma}\bar{t}_2]$ and $[u_{2\sigma}\bar{t}_2']$
$v^2 \in \underline{\pi}_6 k_{[2]}(G/G)$	$2a_\lambda u_\lambda u_{2\sigma} \bar{\mathfrak{d}}_1^2 = \langle 2, \eta, f_1, f_1^2 \rangle$
$\kappa \in \underline{\pi}_{14} k_{[2]}(G/G)$	$2a_\lambda u_{2\sigma}^2 u_\lambda^3 \bar{\mathfrak{d}}_1^4$
Filtration 3	
$f_1 \in \underline{\pi}_1 k_{[2]}(G/G)$	$a_\sigma a_\lambda \bar{\mathfrak{d}}_1$, generating the summand $\bullet$ of $\underline{\pi}_1 k_{[2]}$
$\eta_0^3 = \eta_0^2\eta_1 = \eta_0\eta_1^2 = \eta_1^3 \in \underline{\pi}_3^G k_{[2]}(G/G')$	$\eta_\epsilon u_\sigma[a_{\sigma_2}^2]\,\mathrm{res}_2^4(\bar{\mathfrak{d}}_1) = \eta_\epsilon u_\sigma[a_{\sigma_2}^2]\bar{s}_{2,\epsilon}$
$x_3 \in \underline{\pi}_3 k_{[2]}(G/G)$ with $\mathrm{res}_2^4(x_3) = 0$	$\mathrm{tr}_2^4(\eta_0^2\eta_1) = a_\lambda \eta' \bar{\mathfrak{d}}_1$
Filtration 4	
$x_4 \in \underline{E}_2^{4,8}(G/G)$ with $d_5(x_4) = f_1^3$, $\mathrm{res}_2^4(x_4) = (\eta_0\eta_1)^2 = \eta_0^4$ and $2x_4 = f_1 v$	$a_\lambda^2 u_{2\sigma} \bar{\mathfrak{d}}_1^2$
$\bar{\kappa} \in \underline{\pi}_{20} k_{[2]}(G/G)$ $2\bar{\kappa} = \mathrm{tr}_2^4(u_\sigma\,\mathrm{res}_2^4(u_{2\sigma}^2 u_\lambda^5 \bar{\mathfrak{d}}_1^5))$ (exotic transfer)	$a_\lambda^2 u_{2\sigma}^3 u_\lambda^4 \bar{\mathfrak{d}}_1^6$
Filtration 8	
$\epsilon \in \underline{\pi}_8 k_{[2]}(G/G)$	$x_4^2 = \langle f_1, f_1^2, f_1, f_1^2 \rangle \in \underline{E}_6^{8,16}(G/G)$
Filtration 11	
$v^3 = \eta\epsilon \in \underline{\pi}_9 k_{[2]}(G/G)$	Represents $f_1 x_4^2 \in \underline{E}_2^{11,20}(G/G)$

Table 3. Some elements in the slice spectral sequence and homotopy groups of $k_{[2]}$, listed in order of ascending filtration.

10 Slices for $k_{[2]}$ and $K_{[2]}$

In this section we will identify the slices for $k_{[2]}$ and $K_{[2]}$ and the generators of their integrally graded homotopy groups. For the latter we will use the notation of Table 3. Let

$$X_{m,n} = \begin{cases} \Sigma^{m\rho_4} H\underline{\mathbf{Z}} & \text{for } m = n \\ G_+ \wedge_{G'} \Sigma^{(m+n)\rho_2} H\underline{\mathbf{Z}} & \text{for } m < n. \end{cases} \tag{10.1}$$

The slices of $k_{[2]}$ are certain finite wedges of these, and those of $K_{[2]}$ are a certain infinite wedges. Fortunately we can analyze these slices by considering just one value of m at a time, this index being preserved by the first differential d_3. These are illustrated below in Figures 9–12. They show both $\underline{E}_2$ and $\underline{E}_4$ in four cases depending on the sign and parity of m.

Theorem 10.2 (The slice $\underline{E}_2$-term for $k_{[2]}$). *The slices of $k_{[2]}$ are*

$$P_t^t k_{[2]} = \begin{cases} \bigvee_{0 \le m \le t/4} X_{m,t/2-m} & \text{for } t \text{ even and } t \ge 0, \\ * & \text{otherwise,} \end{cases}$$

where $X_{m,n}$ is as in (10.1).

The structure of $\pi_*^u k_{[2]}$ as a $\mathbf{Z}[G]$-module (see (9.1)) leads to four types of orbits and slice summands:

(1) $\{(r_{1,0}r_{1,1})^{2\ell}\}$ *leading to $X_{2\ell,2\ell}$ for $\ell \ge 0$; see the leftmost diagonal in Figure 9. On the 0-line we have a copy of $\square$ (defined in Table 2) generated under restrictions by*

$$\Delta_1^\ell = u_{2\ell\rho_4}\bar{\mathfrak{d}}_1^{2\ell} = u_{2\sigma}^\ell u_\lambda^{2\ell}\bar{\mathfrak{d}}_1^{2\ell} \in \underline{E}_2^{0,8\ell}(G/G).$$

In positive filtrations we have

$$\begin{array}{llll} \circ \subseteq \underline{E}_2^{2j,8\ell} & \text{generated by} & a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell-j}\bar{\mathfrak{d}}_1^{2\ell} \in \underline{E}_2^{2j,8\ell}(G/G) & \text{for } 0 < j \le 2\ell, \\ \bullet \subseteq \underline{E}_2^{2k+4\ell,8\ell} & \text{generated by} & a_\sigma^{2k} a_\lambda^{2\ell} u_{2\sigma}^{\ell-k}\bar{\mathfrak{d}}_1^{2\ell} \in \underline{E}_2^{2k+4\ell,8\ell}(G/G) & \text{for } 0 < k \le \ell. \end{array}$$

(2) $\{(r_{1,0}r_{1,1})^{2\ell+1}\}$ *leading to $X_{2\ell+1,2\ell+1}$ for $\ell \ge 0$; see the leftmost diagonal in Figure 10. On the 0-line we have a copy of $\overline{\square}$ generated under restrictions by*

$$\delta_1^{2\ell+1} = u_\sigma^{2\ell+1}\operatorname{res}_2^4(u_\lambda\bar{\mathfrak{d}}_1)^{2\ell+1} \in \underline{E}_2^{0,8\ell+4}(G/G').$$

In positive filtrations we have

$$\begin{array}{llll} \bar{\bullet} \subseteq \underline{E}_2^{2j,8\ell+4} & \text{generated by} & u_\sigma^{2\ell+1}\operatorname{res}_2^4(a_\lambda^j u_\lambda^{2\ell+1-j}\bar{\mathfrak{d}}_1^{2\ell+1}) \in \underline{E}_2^{2j,8\ell+4}(G/G') & \text{for } 0 < j \le 2\ell+1, \\ \bullet \subseteq \underline{E}_2^{2j+1,8\ell+4} & \text{generated by} & a_\sigma a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell+1-j}\bar{\mathfrak{d}}_1^{2\ell+1} \in \underline{E}_2^{2j+1,8\ell+4}(G/G) & \text{for } 0 \le j \le 2\ell+1, \\ \bullet \subseteq \underline{E}_2^{2k+4\ell+3,8\ell+4} & \text{generated by} & a_\sigma^{2k+1} a_\lambda^{2\ell+1} u_{2\sigma}^{\ell-k}\bar{\mathfrak{d}}_1^{2\ell+1} \in \underline{E}_2^{2k+4\ell+3,8\ell+4}(G/G) & \text{for } 0 < k \le \ell. \end{array}$$

(3) $\{r_{1,0}^i r_{1,1}^{2\ell-i}, r_{1,0}^{2\ell-i} r_{1,1}^i\}$ *leading to $X_{i,2\ell-i}$ for $0 \le i < \ell$; see other diagonals in Figure 9. On the 0-line we have a copy of $\widehat{\square}$ generated (under tr_2^4, res_1^2 and the group action) by*

$$u_\sigma^\ell \bar{\mathfrak{s}}_2^{\ell-i}\operatorname{res}_2^4(u_\lambda^\ell\bar{\mathfrak{d}}_1^i) \in \underline{E}_2^{0,4\ell}(G/G').$$

In positive filtrations we have

$$\hat{\bullet} \subseteq \underline{E}_2^{2j,4\ell} \quad \text{generated by } u_\sigma^\ell \bar{\mathfrak{s}}_2^{\ell-i}\operatorname{res}_2^4(a_\lambda^j u_\lambda^{\ell-j}\bar{\mathfrak{d}}_1^i) \begin{cases} \in \underline{E}_2^{2j,4\ell}(G/G') & \text{for } 0 < j \le \ell, \\ = \eta_\epsilon^{2j} u_\sigma^{\ell-j}\bar{\mathfrak{s}}_2^{\ell-i-j}\operatorname{res}_2^4(u_\lambda^{\ell-j}\bar{\mathfrak{d}}_1^i) & \text{for } 0 < j < \ell - i. \end{cases}$$

(4) $\{r_{1,0}^i r_{1,1}^{2\ell+1-i}, r_{1,0}^{2\ell+1-i} r_{1,1}^i\}$ *leading to $X_{i,2\ell+1-i}$ for $0 \le i \le \ell$; see other diagonals in Figure 10. On the 0-line we have a copy of $\widehat{\overline{\square}}$ generated (under transfers and the group action) by*

$$r_{1,0}\operatorname{res}_1^2(u_\sigma^\ell\bar{\mathfrak{s}}_2^{\ell-i})\operatorname{res}_1^4(u_\lambda^\ell\bar{\mathfrak{d}}_1^i) \in \underline{E}_2^{0,4\ell+2}(G/\{e\}).$$

In positive filtrations we have

$$\hat{\bar{\bullet}} \subseteq \underline{E}_2^{2j+1,4\ell+2} \quad \text{generated by } \eta_\epsilon u_\sigma^\ell \bar{\mathfrak{s}}_2^{\ell-i}\operatorname{res}_2^4(a_\lambda^j u_\lambda^{\ell-j}\bar{\mathfrak{d}}_1^i) \begin{cases} \in \underline{E}_2^{2j+1,4\ell+2}(G/G') & \text{for } 0 \le j \le \ell, \\ = \eta_\epsilon^{2j+1} u_\sigma^{\ell-j}\bar{\mathfrak{s}}_2^{\ell-i-j}\operatorname{res}_2^4(u_\lambda^{\ell-j}\bar{\mathfrak{d}}_1^i) & \text{for } 0 \le j \le \ell - i. \end{cases}$$

Corollary 10.3 (A subring of the slice E_2-term). *The ring $\underline{E}_2 k_{[2]}(G/G')$ contains*

$$\mathbf{Z}[\delta_1, \Sigma_{2,\epsilon}, \eta_\epsilon : \epsilon = 0, 1]/(2\eta_\epsilon, \delta_1^2 - \Sigma_{2,0}\Sigma_{2,1}, \eta_\epsilon\Sigma_{2,\epsilon+1} + \eta_{1+\epsilon}\delta_1);$$

see Table 3 for the definitions of its generators. In particular, the elements η_0 and η_1 are algebraically independent modulo 2 with

$$\gamma^\epsilon(\eta_0^m\eta_1^n) \in \underline{\pi}_{m+n}X_{m,n}(G/G') \quad \text{for } m \le n.$$

The element $(\eta_0\eta_1)^2$ is the fixed point restriction of

$$u_{2\sigma}a_\lambda^2\bar{\mathfrak{d}}_1^2 \in \underline{E}_2^{4,8}k_{[2]}(G/G),$$

which has order 4, and the transfer of the former is twice the latter. The element $\eta_0\eta_1$ is not in the image of res_2^4 *and has trivial transfer in $\underline{E}_2$.*

Proof. We detect this subring with the monomorphism

$$\underline{E}_2k_{[2]}(G/G') \xrightarrow{r_2^4} \underline{E}_2k_{[2]}(G'/G'), \quad \eta_t\epsilon \mapsto a_\sigma\bar{r}_{1,\epsilon}, \quad \Sigma_{2,\epsilon} \mapsto u_{2\sigma}\bar{r}_{1,\epsilon}^2, \quad \delta_1 \mapsto u_{2\sigma}\bar{r}_{1,0}\bar{r}_{1,1},$$

in which all the relations are transparent. □

Corollary 10.4 (Slices for $K_{[2]}$). *The slices of $K_{[2]}$ are*

$$P_t^tK_{[2]} = \begin{cases} \bigvee_{m\le t/4} X_{m,t/2-m} & \text{for } t \text{ even,} \\ * & \text{otherwise,} \end{cases}$$

where $X_{m,n}$ is as in Theorem 10.2. Here m can be any integer, and we still require that $m \le n$.

Proof. Recall that $K_{[2]}$ is obtained from $k_{[2]}$ by inverting a certain element

$$D \in \underline{\pi}_{4\rho_4}k_{[2]}(G/G)$$

described in Table 3. Thus $K_{[2]}$ is the homotopy colimit of the diagram

$$k_{[2]} \xrightarrow{D} \Sigma^{-4\rho_4}k_{[2]} \xrightarrow{D} \Sigma^{-8\rho_4}k_{[2]} \xrightarrow{D} \cdots.$$

Desuspending by $4\rho_4$ converts slices to slices, so for even t we have

$$\begin{aligned} P_t^tK_{[2]} &= \lim_{k\to\infty} \Sigma^{-4k\rho_4}P_{t+16k}^{t+16k}k_{[2]} \\ &= \lim_{k\to\infty} \Sigma^{-4k\rho_4} \bigvee_{0\le m\le t/4+4k} X_{m,t/2+8k-m} \\ &= \lim_{k\to\infty} \bigvee_{0\le m\le t/4+4k} X_{m-4k,t/2+4k-m} \\ &= \lim_{k\to\infty} \bigvee_{-4k\le m\le t/4} X_{m,t/2-m} \\ &= \bigvee_{m\le t/4} X_{m,t/2-m}. \end{aligned}$$

□

Corollary 10.5 (A filtration of $k_{[2]}$). *Consider the diagram*

$$\begin{array}{ccccccc} k_{[2]} & \xleftarrow{\bar{\mathfrak{d}}_1} & \Sigma^{\rho_4}k_{[2]} & \xleftarrow{\bar{\mathfrak{d}}_1} & \Sigma^{2\rho_4}k_{[2]} & \xleftarrow{\bar{\mathfrak{d}}_1} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ y_0 & & y_1 = \Sigma^{\rho_4}y_0 & & y_2 = \Sigma^{2\rho_4}y_0, & & \end{array}$$

where y_0 is the cofiber of the map induced by $\bar{\mathfrak{d}}_1$. Then the slices of y_m are

$$P_t^ty_m = \begin{cases} X_{m,t/2-m} & \text{for } t \text{ even and } t \ge 4m, \\ * & \text{otherwise.} \end{cases}$$

Corollary 10.6 (A filtration of $K_{[2]}$). *Let $R = \mathbf{Z}_{(2)}[x]/(11x^2 - 20x + 5)$. After tensoring with R (by smashing with a suitable Moore spectrum M) there is a diagram*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \Sigma^{2\rho_4}k_{[2]} & \xrightarrow{f_2} & \Sigma^{\rho_4}k_{[2]} & \xrightarrow{f_1} & k_{[2]} & \xrightarrow{f_0} & \Sigma^{-\rho_4}k_{[2]} & \xrightarrow{f_{-1}} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & Y_2 & & Y_1 & & Y_0 & & Y_{-1}, & & \end{array}$$

where the homotopy colimit of the top row is $K_{[2]}$ and each Y_m has slices similar to those of y_m as in Corollary 10.5.

Proof. The periodicity element $D = -\bar{\mathfrak{d}}_1^2(5\bar{t}_2^2 - 20\bar{t}_2\bar{\mathfrak{d}}_1 + 11\bar{\mathfrak{d}}_1^2)$ can be factored as

$$D = D_0D_1D_2D_3,$$

where $D_i = a_i\bar{\mathfrak{d}}_1 + b_i\bar{t}_2$ with $a_i \in \mathbf{Z}_{(2)}^{\times}$ and $b_i \in R$. Then let f_{4n+i} be multiplication by D_i. It follows that the composite of any four successive f_ms is D, making the colimit $K_{[2]}$ as desired. The fact that a_i is a unit means that the Y's here have the same slices as the y's in Corollary 10.5. □

Remark 10.7. The 2-adic completion of R is the Witt ring $W(\mathbf{F}_4)$ used in Morava E_2-theory. This follows from the fact that the roots of the quadratic polynomial involve $\sqrt{5}$, which is in $W(\mathbf{F}_4)$ but is not a 2-adic integer.

Moreover, if we assume that $D_0D_1 = 5\bar{t}_2^2 - 20\bar{t}_2\bar{\mathfrak{d}}_1 + 11\bar{\mathfrak{d}}_1^2$, then the composite maps $f_{4n}f_{4n+1}$, as well as f_{4n+2} and f_{4n+3}, can be constructed without adjoining $\sqrt{5}$.

It turns out that $y_m \wedge M$ and Y_m for $m \geq 0$ not only have the same slices, but the same slice spectral sequence, which is shown in Figures 9–12. See Remark 13.2 below. We do not know if they have the same homotopy type.

11 Some differentials in the slice spectral sequence for $k_{[2]}$

Now we turn to differentials. The only generators in (9.2) that are not permanent cycles are the u's. We will see that it is easy to account for the elements in $\underline{E}_2^{0,|V|-V}(G/H)$ for proper subgroups H of $G = C_4$. From (9.2) we see that

$$\underline{E}_2^{s,t} = 0 \quad \text{for } |t| \text{ odd.} \tag{11.1}$$

This sparseness condition implies that d_r can be nontrivial only for odd values of r.

Our starting point is the Slice Differentials Theorem of [6, Theorem 9.9], which is derived from the fact that the geometric fixed point spectrum of $\mathrm{MU}^{((G))}$ is MO. It says that in the slice spectral sequence for $\mathrm{MU}^{((G))}$ for an arbitrary finite cyclic 2-group G of order g, the first nontrivial differential on various powers of $u_{2\sigma}$ is

$$d_r(u_{2\sigma}^{2^{k-1}}) = a_\sigma^{2^k} a_{\bar{\rho}}^{2^k-1} N_2^g(\bar{r}_{2^k-1}^G) \in \underline{E}_r^{r,r+2^k(1-\sigma)-1}\mathrm{MU}^{((G))}(G/G), \tag{11.2}$$

where $r = 1 + (2^k - 1)g$ and $\bar{\rho}$ is the reduced regular representation of G.

In particular,

$$\left.\begin{aligned} d_5(u_{2\sigma}) &= a_\sigma^3 a_\lambda \bar{\mathfrak{d}}_1 \in \underline{E}_5^{5,6-2\sigma}\mathrm{MU}^{((G))}(G/G) && \text{for } G = C_4, \\ d_{13}([u_{2\sigma}^2]) &= a_\sigma^7 a_\lambda^3 \bar{\mathfrak{d}}_2 \in \underline{E}_{13}^{13,16-4\sigma}\mathrm{MU}^{((G))}(G/G) && \text{for } G = C_4, \\ d_3(u_{2\sigma}) &= a_\sigma^3 \bar{r}_1 \in \underline{E}_3^{3,4-2\sigma}\mathrm{MU}_{\mathbb{R}}(G/G) && \text{for } G = C_2, \\ d_7([u_{2\sigma}^2]) &= a_\sigma^7 \bar{r}_3 \in \underline{E}_3^{7,10-4\sigma}\mathrm{MU}_{\mathbb{R}}(G/G) && \text{for } G = C_2. \end{aligned}\right\} \tag{11.3}$$

The first of these leads directly to a similar differential in the slice spectral sequence for $k_{[2]}$. The target of the second one has trivial image in $k_{[2]}$ and we shall see that $[u_{2\sigma}^2]$ turns out to be a permanent cycle.

There are two ways to leverage the third and fourth differentials of (11.3) into information about $k_{[2]}$.

(i) They both lead to differentials in the slice spectral sequence for the C_2 spectrum $i_{G'}^* k_{[2]}$. They are spelled out ind (11.6) and will be studied in detail below in Section 12. They completely determine the slice spectral sequence $\underline{E}_*^{*,\star}(G/G')$ for both $k_{[2]}$ and $K_{[2]}$. Since u_λ restricts to $\bar{u}_\lambda$, which is isomorphic to $u_{2\sigma_2}$, we get some information about differentials on powers of u_λ. The d_3 on $u_{2\sigma_2}$ forces a $d_3(u_\lambda) = \eta a_\lambda$. The target of $d_7([u_{2\sigma_2}^2])$ turns out to be the exotic restriction of an element in filtration 5, leading to $d_5([u_\lambda]^2) = \nu a_\lambda^2$. We will also see that even though $[u_{2\sigma_2}^4]$ is a permanent cycle, $[u_\lambda^4]$ (its preimage under the restriction map res_2^4) is not.

(ii) One can norm up the differentials on $u_{2\sigma_2}$ and its square using Corollary 4.8, converting the d_3 and d_7 to a d_5 and a d_{13}. The source of the latter is $[a_\sigma u_\lambda^4]$, which implies that $[u_\lambda^4]$ is not a permanent cycle.

The differentials of (11.3) lead to Massey products which are permanent cycles,

$$\langle 2, a_\sigma^2, f_1\rangle = [2u_{2\sigma}] = \mathrm{tr}_{G'}^{G}(u_\sigma^2) \in \begin{cases} \underline{E}_6^{0,2-2\sigma}\mathrm{MU}^{((G))}(G/G) & \text{for } G = C_4,\\ \underline{E}_4^{0,2-2\sigma}\mathrm{MU}_{\mathbb{R}}(G/G) & \text{for } G = C_2,\end{cases}$$

$$\langle 2, a_\sigma^4, f_3\rangle = [2u_{2\sigma}^2] = \mathrm{tr}_{G'}^{G}(u_\sigma^4) \in \begin{cases} \underline{E}_{14}^{0,4-4\sigma}\mathrm{MU}^{((G))}(G/G) & \text{for } G = C_4,\\ \underline{E}_8^{0,4-4\sigma}\mathrm{MU}_{\mathbb{R}}(G/G) & \text{for } G = C_4,\end{cases}$$

and (by Theorem 4.4) to exotic transfers

$$\left.\begin{aligned} a_\sigma f_1 &= \begin{cases} \mathrm{tr}_2^4(u_\sigma) \in \underline{E}_\infty^{4,5-\sigma}\mathrm{MU}^{((G))}(G/G) & \text{for } G = C_4 \quad \text{(filtration jump 4)},\\ \mathrm{tr}_1^2(u_\sigma) \in \underline{E}_\infty^{2,3-\sigma}\mathrm{MU}_{\mathbb{R}}(G/G) & \text{for } G = C_2 \quad \text{(filtration jump 2)},\end{cases}\\ a_\sigma^3 f_3 &= \begin{cases} \mathrm{tr}_2^4(u_\sigma^3) \in \underline{E}_\infty^{12,15-3\sigma}\mathrm{MU}_{\mathbb{R}}(G/G) & \text{for } G = C_4 \quad \text{(filtration jump 12)},\\ \mathrm{tr}_1^2(u_\sigma^3) \in \underline{E}_\infty^{6,9-3\sigma}\mathrm{MU}_{\mathbb{R}}(G/G) & \text{for } G = C_2 \quad \text{(filtration jump 6)}.\end{cases}\end{aligned}\right\} \tag{11.4}$$

Since a_σ and $2a_\lambda$ kill transfers by Lemma 4.2, we have Massey products,

$$[u_{2\sigma}\,\mathrm{tr}_2^4(x)] = \mathrm{tr}_2^4(u_\sigma^2 x) = \langle a_\sigma f_1, a_\sigma, \mathrm{tr}_2^4(x)\rangle \quad \text{with } 2a_\lambda[u_{2\sigma}\,\mathrm{tr}_2^4(x)] = 0. \tag{11.5}$$

Now, as before, let $G = C_4$ and $G' = C_2 \subseteq G$. We need to translate the d_3 above in the slice spectral sequence for $\mathrm{MU}_{\mathbb{R}}$ into a statement about the one for $k_{[2]}$ as a G'-spectrum. We have an equivariant multiplication map m of G'-spectra

$$\begin{array}{ccccc} & & \mathrm{MU}^{((G))} & & \\ & & \| & & \\ \mathrm{MU}_{\mathbb{R}} & \xrightarrow{\ \eta_L\ } & \mathrm{MU}_{\mathbb{R}} \wedge \mathrm{MU}_{\mathbb{R}} & \xrightarrow{\ m\ } & \mathrm{MU}_{\mathbb{R}} \\ \bar r_1^{G'} & \longmapsto & \bar r_{1,0}^G + \bar r_{1,1}^G & \longmapsto & \bar r_1^{G'} \\ & & a_\sigma^3(\bar r_{1,0}^G + \bar r_{1,1}^G) & \longmapsto & a_\sigma^3 \bar r_1^{G'} \\ \bar r_3^{G'} & \longmapsto & 5\bar r_{1,0}^G \bar r_{1,1}^G(\bar r_{1,0}^G + \bar r_{1,1}^G) + (\bar r_{1,1}^G)^3 \bmod (\bar r_2^G, \bar r_3^G) & \longmapsto & \bar r_3^{G'}, \end{array}$$

where the elements lie in $\underline{\pi}_{\rho_2}^{G'}(\cdot)(G'/G')$ and $\underline{\pi}_{3\rho_2}^{G'}(\cdot)(G'/G')$. In the slice spectral sequence for $\mathrm{MU}^{((G))}$ as a G'-spectrum, $d_3(u_{2\sigma})$ and $d_7(u_{2\sigma}^2)$ must be G-invariant since $u_{2\sigma}$ is, and they must map respectively to $a_\sigma^3\bar r_1^{G'}$ and $a_\sigma^7\bar r_3^{G'}$, so we have

$$\left.\begin{aligned} d_3(u_{2\sigma_2}) &= d_3(\bar u_\lambda) = a_{\sigma_2}^3(\bar r_{1,0}^G + \bar r_{1,1}^G) = a_{\sigma_2}^2(\eta_0 + \eta_1)\\ d_7([u_{2\sigma_2}^2]) &= d_7([\bar u_\lambda^2]) = a_{\sigma_2}^7(5\bar r_{1,0}^G \bar r_{1,1}^G(\bar r_{1,0}^G + \bar r_{1,1}^G) + (\bar r_{1,1}^G)^3 + \cdots)\\ &= a_{\sigma_2}^7(\bar r_{1,0}^G)^3 + \cdots \quad \text{since } a_{\sigma_2}^3(\bar r_{1,0}^G + \bar r_{1,1}^G) = 0 \text{ in } \underline{E}_4.\end{aligned}\right\} \tag{11.6}$$

We get similar differentials in the slice spectral sequence for $k_{[2]}$ as a C_2-spectrum in which the missing terms in $d_7(\bar u_\lambda^2)$ vanish.

Pulling back along the isomorphism $\underline{r}_2^4$ gives

$$\left.\begin{aligned} d_3(\mathrm{res}_2^4(u_\lambda)) &= d_3(\bar u_\lambda) = [a_{\sigma_2}^2](\eta_0 + \eta_1) = \mathrm{res}_2^4(a_\lambda \eta),\\ d_7(\mathrm{res}_2^4(u_\lambda^2)) &= d_7(\bar u_\lambda^2) = \mathrm{res}_2^4(a_\lambda^2)\eta_0^3 = \mathrm{res}_2^4(a_\lambda^2 \nu).\end{aligned}\right\} \tag{11.7}$$

These imply that

$$d_3(u_\lambda) = a_\lambda \eta \quad \text{and} \quad d_5(u_\lambda^2) = a_\lambda^2 \nu.$$

The differential on u_λ leads to the following Massey products, the second two of which are permanent cycles:

$$\left.\begin{aligned} [u_\lambda^2] &= \langle a_\lambda, \eta, a_\lambda, \eta\rangle \in \underline{E}_4^{0,4-2\lambda}(G/G),\\ [2u_\lambda] &= \langle 2, \eta, a_\lambda\rangle \in \underline{E}_4^{0,2-\lambda}(G/G),\\ \bar\nu := [a_\sigma u_\lambda] &= \langle a_\sigma, \eta, a_\lambda\rangle \in \underline{E}_4^{1,3-\sigma-\lambda}(G/G),\end{aligned}\right\} \tag{11.8}$$

where $\overline{v}$ satisfies

$$\begin{aligned}
a_\sigma^2\overline{v} &= \langle a_\sigma^3, \eta, a_\lambda\rangle = a_\sigma[a_\sigma^2 u_\lambda]\\
&= a_\sigma[2a_\lambda u_{2\sigma}] = [2a_\sigma a_\lambda u_{2\sigma}] = 0,\\
\mathrm{res}_2^4(\overline{v}) &= [a_{\sigma_2}^3\overline{r}_{1,\epsilon}]u_\sigma \in \underline{E}_4^{3,5-\sigma-\lambda}(G/G') && \text{(exotic restriction with filtration jump 2 by Theorem 4.4 (i)),}\\
2\overline{v} &= \mathrm{tr}_2^4(\mathrm{res}_2^4(\overline{v})) = \mathrm{tr}_2^4(u_\sigma[a_{\sigma_2}^3\overline{r}_{1,\epsilon}])\\
&= \eta' a_\lambda \in \underline{E}_4^{3,5-\sigma-\lambda}(G/G) && \text{(exotic group extension with jump 2),}\\
\mathrm{tr}_2^4(x)\overline{v} &= \mathrm{tr}_2^4(x\cdot \mathrm{res}_2^4(\overline{v})) = \mathrm{tr}_2^4(x[a_{\sigma_2}^3\overline{r}_{1,0}]u_\sigma),\\
\eta\overline{v} &= \mathrm{tr}_2^4([a_{\sigma_2}\overline{r}_{1,1}])\overline{v}\\
&= \mathrm{tr}_2^4([a_{\sigma_2}^4\overline{r}_{1,0}\overline{r}_{1,1}]u_\sigma) = a_\lambda^2\bar{\mathfrak{d}}_1\,\mathrm{tr}_2^4(u_\sigma^2)\\
&= a_\lambda\bar{\mathfrak{d}}_1\langle 2, a_\sigma, a_\sigma f_1\rangle = \langle 2, a_\sigma, f_1^2\rangle,\\
\eta'\overline{v} &= a_\lambda^2\bar{\mathfrak{d}}_1\,\mathrm{tr}_2^4(u_\sigma^3) = 0,\\
d_7([\overline{u}_\lambda^2]) &= [a_{\sigma_2}^7\overline{r}_{1,0}^3] && \text{in } \underline{E}_4\\
&= \mathrm{res}_2^4(\overline{v})\,\mathrm{res}_2^4(a_\lambda^2\bar{\mathfrak{d}}_1)\\
&= \mathrm{res}_2^4(\overline{v}a_\lambda^2\bar{\mathfrak{d}}_1)\\
&= \mathrm{res}_2^4(d_5(u_\lambda^2)),\\
d_5([u_\lambda^2]) &= \overline{v}a_\lambda^2\bar{\mathfrak{d}}_1 = a_\lambda^2 v,\\
d_7([2u_\lambda^2]) &= (2\overline{v})a_\lambda^2\bar{\mathfrak{d}}_1 = a_\lambda^3\eta'\bar{\mathfrak{d}}_1.
\end{aligned}$$

Note that $v = \overline{v}\bar{\mathfrak{d}}_1$, with the exotic restriction and group extension on $\overline{v}$ being consistent with those on v.

The differential on $[u_\lambda^2]$ yields Massey products

$$\left.\begin{aligned}
[a_\sigma^2 u_\lambda^2] &= \langle a_\sigma^2, \overline{v}, a_\lambda^2\bar{\mathfrak{d}}_1\rangle,\\
\left[\eta' u_\lambda^2\right] &= \langle \eta', \overline{v}, a_\lambda^2\bar{\mathfrak{d}}_1\rangle.
\end{aligned}\right\} \tag{11.9}$$

Theorem 11.10 (Normed up slice differentials for $k_{[2]}$ and $K_{[2]}$). *In the slice spectral sequences for $k_{[2]}$ and $K_{[2]}$ we have*

$$d_5([a_\sigma u_\lambda^2]) = 0 \quad \textit{and} \quad d_{13}([a_\sigma u_\lambda^4]) = a_\lambda^7[u_{2\sigma}^2]\bar{\mathfrak{d}}_1^3.$$

Proof. The two slice differentials over G' are

$$\begin{aligned}
d_3(u_{2\sigma_2}) &= a_{\sigma_2}^3\overline{r}_1^{G'} = a_{\sigma_2}^3(\overline{r}_{1,0} + \overline{r}_{1,1}),\\
d_7([u_{2\sigma_2}^2]) &= a_{\sigma_2}^7\overline{r}_3^{G'} = a_{\sigma_2}^7(5\overline{r}_{1,0}^2\overline{r}_{1,1} + 5\overline{r}_{1,0}\overline{r}_{1,1}^2 + \overline{r}_{1,1}^3).
\end{aligned}$$

We need to find the norms of both sources and targets. Lemma 4.9 tells us that

$$\begin{aligned}
N_2^4(a_{\sigma_2}^k) &= a_\lambda^k,\\
N_2^4(u_{2\sigma_2}^k) &= u_\lambda^{2k}/u_{2\sigma}^k \quad \text{in } \underline{E}_2(G/G).
\end{aligned}$$

Previous calculations give

$$\begin{aligned}
N_2^4(\overline{r}_{1,0} + \overline{r}_{1,1}) &= -\overline{t}_2 && \text{by (9.10),}\\
N_2^4(5\overline{r}_{1,0}^2\overline{r}_{1,1} + 5\overline{r}_{1,0}\overline{r}_{1,1}^2 + \overline{r}_{1,1}^3) &= -\bar{\mathfrak{d}}_1(5\overline{t}_2^2 - 20\overline{t}_2\bar{\mathfrak{d}}_1 + 11\bar{\mathfrak{d}}_1^2) && \text{by (9.11).}
\end{aligned}$$

For the first differential, Corollary 4.8 tells us that

$$\begin{aligned}
a_\lambda^3\overline{t}_2 &= d_5(a_\sigma u_\lambda^2/u_{2\sigma})\\
&= d_5(a_\sigma u_\lambda^2)/u_{2\sigma} - a_\sigma u_\lambda^2 d_5(u_{2\sigma})/[u_{2\sigma}^2]\\
&= d_5(a_\sigma u_\lambda^2)/u_{2\sigma} - a_\sigma u_\lambda^2 a_\sigma^3 a_\lambda\bar{\mathfrak{d}}_1/[u_{2\sigma}^2].
\end{aligned}$$

Multiplying both sides by the permanent cycle $[u_{2\sigma}^2]$ gives

$$\begin{aligned}[u_{2\sigma}d_5(a_\sigma u_\lambda^2)] &= a_\lambda^3[u_{2\sigma}^2]\bar{t}_2 + a_\sigma u_\lambda^2 a_\sigma^3 a_\lambda \bar{\mathfrak{d}}_1\\ &= a_\lambda^3[u_{2\sigma}^2]\bar{t}_2 + 4a_\lambda^3[u_{2\sigma}^2]\bar{\mathfrak{d}}_1\\ &= a_\lambda^3[u_{2\sigma}^2]\bar{t}_2,\\ d_5(a_\sigma u_\lambda^2) &= a_\lambda^3[u_{2\sigma}\bar{t}_2].\end{aligned}$$

We have seen that

$$\eta\eta' = a_\lambda[u_{2\sigma}\bar{t}_2].$$

This implies that $a_\lambda^2[u_{2\sigma}\bar{t}_2]$ vanishes in $\underline{E}_5$ since $a_\lambda\eta$ is killed by d_3. It follows that $d_5(a_\sigma u_\lambda^2) = a_\lambda^3[u_{2\sigma}\bar{t}_2] = 0$, as claimed.

For the second differential we have

$$\begin{aligned}d_{13}([a_\sigma u_\lambda^4/u_{2\sigma}^2]) &= a_\lambda^7\bar{\mathfrak{d}}_1(-5\bar{t}_2^2 + 20\bar{t}_2\bar{\mathfrak{d}}_1 + 9\bar{\mathfrak{d}}_1^2),\\ d_{13}([a_\sigma u_\lambda^4]) &= a_\lambda^7[u_{2\sigma}^2]\bar{\mathfrak{d}}_1(-5\bar{t}_2^2 + 20\bar{t}_2\bar{\mathfrak{d}}_1 + 9\bar{\mathfrak{d}}_1^2)\\ &= a_\lambda^7[u_{2\sigma}^2]\bar{\mathfrak{d}}_1(-\bar{t}_2^2 + \bar{\mathfrak{d}}_1^2)\end{aligned}$$

since a_λ has order 4. As we saw above, $a_\lambda^2[u_{2\sigma}\bar{t}_2]$ vanishes in $\underline{E}_5$, so $d_{13}([a_\sigma u_\lambda^4])$ is as claimed. □

We can use this to find the differential on $[u_\lambda^4]$. We have

$$\left.\begin{aligned}d([u_\lambda^4]) &= [2u_\lambda^2]d([u_\lambda^2]) = [2u_\lambda^2]\bar{v}a_\lambda^2\bar{\mathfrak{d}}_1 = (2\bar{v})a_\lambda^2[u_\lambda^2]\bar{\mathfrak{d}}_1\\ &= \eta' a_\lambda^3[u_\lambda^2]\bar{\mathfrak{d}}_1 = [\eta' u_\lambda^2]a_\lambda^3\bar{\mathfrak{d}}_1 = \langle \eta', \bar{v}, a_\lambda^2\bar{\mathfrak{d}}_1\rangle a_\lambda^3\bar{\mathfrak{d}}_1.\end{aligned}\right\} \tag{11.11}$$

The differential on $u_{2\sigma}$ yields

$$[xu_{2\sigma}] = \langle x, a_\sigma^2, f_1\rangle$$

for any permanent cycle x killed by a_σ^2. Possible values of x include 2, η, η' (each of which is killed by a_σ as well) and $\bar{v}$. For the last of these we write

$$\xi := [\bar{v}u_{2\sigma}] = \langle \bar{v}, a_\sigma^2, f_1\rangle = \langle [a_\sigma u_\lambda], a_\sigma^2, f_1\rangle \in \underline{E}_6^{1,5-3\sigma-\lambda}(G/G), \tag{11.12}$$

which satisfies

$$\begin{aligned}\mathrm{res}_2^4(\xi) &= a_{\sigma_2}^3 u_\sigma^3 \bar{r}_{1,\epsilon} \in \underline{E}_4^{3,7-3\sigma-\lambda}(G/G') && \text{(exotic restriction with jump 2)},\\ 2\xi &= \mathrm{tr}_2^4(\mathrm{res}_2^4(\xi)) = \eta' a_\lambda u_{2\sigma} \in \underline{E}_4^{3,7-3\sigma-\lambda}(G/G) && \text{(exotic group extension with jump 2)},\\ d_5([u_{2\sigma}u_\lambda^2]) &= a_\sigma^3 a_\lambda u_\lambda^2\bar{\mathfrak{d}}_1 + \bar{v}a_\lambda^2 u_{2\sigma}\bar{\mathfrak{d}}_1 = (a_\sigma^3 u_\lambda^2 + \bar{v}u_{2\sigma})a_\lambda^2\bar{\mathfrak{d}}_1\\ &= (2a_\sigma a_\lambda u_\lambda u_{2\sigma} + \xi)a_\lambda^2\bar{\mathfrak{d}}_1 = \xi a_\lambda^2\bar{\mathfrak{d}}_1,\\ d_7([2u_{2\sigma}u_\lambda^2]) &= 2\xi\cdot a_\lambda^2\bar{\mathfrak{d}}_1 = \eta' a_\lambda^3 u_{2\sigma}\bar{\mathfrak{d}}_1,\\ \mathrm{res}_2^4(d_5([u_{2\sigma}u_\lambda^2])) &= u_\sigma^3 a_{\sigma_2}^3\bar{r}_{1,\epsilon}\,\mathrm{res}_2^4(a_\lambda^2\bar{\mathfrak{d}}_1) = u_\sigma^2 a_{\sigma_2}^7\bar{r}_{1,0}^3 = u_\sigma^2 d_7(\bar{u}_\lambda^2).\end{aligned}$$

Theorem 11.13 (The differentials on powers of u_λ and $u_{2\sigma}$). *The following differentials occur in the slice spectral sequence for $k_{[2]}$; here $\bar{u}_\lambda$ denotes $\mathrm{res}_2^4(u_\lambda)$:*

$$\begin{aligned}d_3(u_\lambda) &= a_\lambda\eta = \mathrm{tr}_2^4(a_{\sigma_2}^3\bar{r}_{1,\epsilon}),\\ d_3(\bar{u}_\lambda) &= \mathrm{res}_2^4(a_\lambda)(\eta_0+\eta_1) = [a_{\sigma_2}^3(\bar{r}_{1,0}+\bar{r}_{1,1})],\\ d_5(u_{2\sigma}) &= a_\sigma^3 a_\lambda\bar{\mathfrak{d}}_1,\\ d_5([u_\lambda^2]) &= a_\lambda^2 a_\sigma u_\lambda\bar{\mathfrak{d}}_1 = a_\lambda^2\bar{v}\bar{\mathfrak{d}}_1 = a_\lambda^2 v && \textit{for } \bar{v} \textit{ as in } (11.8),\\ d_5([u_{2\sigma}u_\lambda^2]) &= a_\sigma^3 a_\lambda u_\lambda^2\bar{\mathfrak{d}}_1 + \bar{v}a_\lambda^2 u_{2\sigma}\bar{\mathfrak{d}}_1 = (a_\sigma^3 u_\lambda^2 + \bar{v}u_{2\sigma})a_\lambda^2\bar{\mathfrak{d}}_1 = \xi a_\lambda^2\bar{\mathfrak{d}}_1 && \textit{for } \xi \textit{ as in } (11.12),\\ d_7([2u_{2\sigma}u_\lambda^2]) &= \eta' a_\lambda^3 u_{2\sigma}\bar{\mathfrak{d}}_1,\\ d_7([2u_\lambda^2]) &= 2a_\lambda^2\bar{v}\bar{\mathfrak{d}}_1 = a_\lambda^3\eta'\bar{\mathfrak{d}}_1,\\ d_7([\bar{u}_\lambda^2]) &= \mathrm{res}_2^4(a_\lambda^2)\eta_0^3 = a_{\sigma_2}^7\bar{r}_{1,0}^3,\\ d_7([u_\lambda^4]) &= [\eta' u_\lambda^2]a_\lambda^3\bar{\mathfrak{d}}_1 = \langle \eta', \bar{v}, a_\lambda^2\bar{\mathfrak{d}}_1\rangle a_\lambda^3\bar{\mathfrak{d}}_1.\end{aligned}$$

The elements

$$\begin{aligned} &u_\sigma, && [2u_\lambda] = \langle 2, \eta, a_\lambda\rangle,\\ [2u_{2\sigma}] &= \langle 2, a_\sigma^2, f_1\rangle = \operatorname{tr}_2^4(u_\sigma^2), && [4u_\lambda^2] = \langle 2, \eta', a_\lambda^3\bar{\mathfrak{d}}_1\rangle = \operatorname{tr}_1^4(u_{\sigma_2}^4),\\ [u_{2\sigma}^2] &= \langle a_\sigma^2, f_1, a_\sigma^2, f_1\rangle && [2\overline{u}_\lambda^2] = \langle 2, a_{\sigma_2}^6, a_{\sigma_2}\bar{r}_{1,0}^3\rangle = \operatorname{tr}_1^2(u_{\sigma_2}^4),\\ [2u_{2\sigma}u_\lambda] &= \langle [2u_{2\sigma}], \eta, a_\lambda\rangle, && [2u_\lambda^4] = \langle 2, \eta', \bar{\nu}, a_\lambda^5\bar{\mathfrak{d}}_1^2\rangle = \operatorname{tr}_2^4(\overline{u}_\lambda^4),\\ [\overline{u}_\lambda^4] &= \langle a_{\sigma_2}^7, \bar{r}_{1,0}^3, a_{\sigma_2}^7, \bar{r}_{1,0}^3\rangle, && [u_\lambda^8] = \langle [\eta' u_\lambda^2], a_\lambda^3\bar{\mathfrak{d}}_1, [\eta' u_\lambda^2], a_\lambda^3\bar{\mathfrak{d}}_1\rangle \end{aligned}$$

are permanent cycles.

We also have the following exotic restriction and transfers:

$$\operatorname{res}_2^4(a_\sigma u_\lambda) = u_\sigma \operatorname{res}_2^4(a_\lambda)\eta_\epsilon = u_\sigma a_{\sigma_2}^3 \bar{r}_{1,\epsilon} \quad (\text{filtration jump } 2),$$

$$\operatorname{tr}_2^4(u_\sigma^k) = \begin{cases} a_\sigma^2 a_\lambda \bar{\mathfrak{d}}_1 u_{2\sigma}^{(k-1)/2} = a_\sigma f_1 u_{2\sigma}^{(k-1)/2} & \text{for } k \equiv 1 \bmod 4 \quad (\text{filtration jump } 4),\\ 2u_{2\sigma}^{k/2} & \text{for } k \text{ even},\\ 0 & \text{for } k \equiv 3 \bmod 4, \end{cases}$$

$$\operatorname{tr}_1^2(u_{\sigma_2}^k) = \begin{cases} a_{\sigma_2}^2(\bar{r}_{1,0} + \bar{r}_{1,1})\overline{u}_\lambda^{(k-1)/2} = a_{\sigma_2}(\eta_0 + \eta_1)\overline{u}_\lambda^{(k-1)/2} & \text{for } k \equiv 1 \bmod 4 \quad (\text{filtration jump } 2),\\ 2\overline{u}_\lambda^{k/2} & \text{for } k \text{ even}\\ a_{\sigma_2}^6 \bar{r}_{1,0}^3 \overline{u}_\lambda^{(k-3)/2} & \text{for } k \equiv 3 \bmod 4 \quad (\text{filtration jump } 6). \end{cases}$$

Proof. All differentials were established above.

The differential on u_λ^2 does *not* lead to an exotic transfer because neither $[\overline{u}_\lambda^2]$ nor $[u_\lambda a_\lambda^2 \bar{\mathfrak{d}}_1]$ is a permanent cycle as required by Theorem 4.4.

We need to discuss the element $[2u_{2\sigma}u_\lambda] = \langle [2u_{2\sigma}], \eta, a_\lambda\rangle$. To see that this Toda bracket is defined, we need to verify that $[2u_{2\sigma}]\eta = 0$. For this we have

$$[2u_{2\sigma}]\eta = [2u_{2\sigma}]\operatorname{tr}_2^4(\eta_0) = \operatorname{tr}_2^4(2u_\sigma^2\eta_0) = \operatorname{tr}_2^4(0) = 0.$$

The exotic restriction and transfers are applications of Theorem 4.4 to the differentials on u_λ and on $[u_{2\sigma}^{(k+1)/2}]$ and $[\overline{u}_\lambda^{(k+1)/2}]$ for odd k. For even k we have

$$\operatorname{tr}_2^4(u_\sigma^k) = \operatorname{tr}_2^4(\operatorname{res}_2^4([u_{2\sigma}^{k/2}])) = [2u_{2\sigma}^{k/2}] \quad \text{since } \operatorname{tr}_2^4(\operatorname{res}_2^4(x)) = (1+\gamma)x,$$

and similarly for even powers of u_{σ_2}.

As remarked above, we lose no information by inverting the class D, which is divisible by $\bar{\mathfrak{d}}_1$. It is shown in [6, Theorem 9.16] that inverting the latter makes $u_{2\sigma}^2$ a permanent cycle. One can also see this from (11.3). Since $d_5(u_{2\sigma}) = a_\sigma^3 a_\lambda \bar{\mathfrak{d}}_1$, we have $d_5(u_{2\sigma}\bar{\mathfrak{d}}_1^{-1}) = a_\sigma^3 a_\lambda$. This means that $d_{13}([u_{2\sigma}^2]) = a_\sigma^7 a_\lambda^3 \bar{\mathfrak{d}}_3$ is trivial in $\underline{E}_6(G/G)$. It turns out that there is no possible target for a higher differential. ☐

12 $k_{[2]}$ as a C_2-spectrum

Before studying the slice spectral sequence for the C_4-spectrum $k_{[2]}$ further, it is helpful to explore its restriction to $G' = C_2$, for which the $\mathbf{Z}$-bigraded portion

$$\underline{E}_2^{*,*} i_{G'}^* k_{[2]}(G'/G') \cong \underline{E}_2^{*,(G',*)} k_{[2]}(G/G) \cong \underline{E}_2^{*,*} k_{[2]}(G/G')$$

(see Theorem 2.13 for these isomorphisms) is the isomorphic image of the subring of Corollary 10.3. In the following we identify $\Sigma_{2,\epsilon}$, δ_1 and $\bar{r}_{1,\epsilon}$ (see Table 3) with their images under $\underline{r}_2^4$. From the differentials of (11.6) we get

$$\left.\begin{aligned} d_3(\Sigma_{2,\epsilon}) &= \eta_\epsilon^2(\eta_0 + \eta_1) = a_\sigma^3 \bar{r}_{1,\epsilon}^2(\bar{r}_{1,0} + \bar{r}_{1,1}),\\ d_3(\delta_1) &= \eta_0^2\eta_1 + \eta_0\eta_1^2 = a_\sigma^3 \bar{r}_{1,0}\bar{r}_{1,1}(\bar{r}_{1,0} + \bar{r}_{1,1}),\\ d_7([\delta_1^2]) &= d_7(u_{2\sigma}^2)\bar{r}_{1,0}^2\bar{r}_{1,1}^2 = a_\sigma^7 \bar{r}_3^{G'}\bar{r}_{1,0}^2\bar{r}_{1,1}^2 = a_\sigma^7(5\bar{r}_{1,0}^2\bar{r}_{1,1} + 5\bar{r}_{1,0}\bar{r}_{1,1}^2 + \bar{r}_{1,1}^3)\bar{r}_{1,0}^2\bar{r}_{1,1}^2. \end{aligned}\right\} \tag{12.1}$$

The differentials d_3 above make all monomials in η_0 and η_1 of any given degree ≥ 3 the same in $\underline{E}_4(G/G')$ and $\underline{E}_4(G'/G')$, so $d_7(\delta_1^2) = \eta_0^7$. Similar calculations show that

$$d_7([\Sigma_{2,\epsilon}^2]) = \eta_0^7 = a_\sigma^7 \bar{r}_{1,0}^7.$$

The image of the periodicity element D here is as in (7.4).

We have the following values of the transfer on powers of u_σ:

$$\operatorname{tr}_1^2(u_\sigma^i) = \begin{cases} [2u_{2\sigma}^{i/2}] & \text{for } i \text{ even,} \\ [a_\sigma^2 u_{2\sigma}^{(i-1)/2}](\bar{r}_{1,0} + \bar{r}_{1,1}) & \text{for } i \equiv 1 \bmod 4, \\ [u_{2\sigma}^4]^{(i-3)/8} a_\sigma^6 \bar{r}_{1,0}^3 = [u_{2\sigma}^4]^{(i-3)/8} a_\sigma^6 \bar{r}_{1,1}^3 & \text{for } i \equiv 3 \bmod 8, \\ 0 & \text{for } i \equiv 7 \bmod 8. \end{cases}$$

This leads to the following, for which Figure 8 is a visual aid.

Theorem 12.2 (The slice spectral sequence for $k_{[2]}$ as a C_2-spectrum). *Using the notation of Table 1 and Definition 5.3, we have*

$$\underline{E}_2^{*,*}(G'/\{e\}) = \mathbf{Z}[r_{1,0}, r_{1,1}] \quad \textit{with } r_{1,\epsilon} \in \underline{E}_2^{0,2}(G'/\{e\}),$$
$$\underline{E}_2^{*,*}(G'/G') = \mathbf{Z}[\delta_1, \Sigma_{2,\epsilon}, \eta_\epsilon : \epsilon = 0, 1]/(2\eta_\epsilon, \delta_1^2 - \Sigma_{2,0}\Sigma_{2,1}, \eta_\epsilon \Sigma_{2,\epsilon+1} + \eta_{1+\epsilon}\delta_1),$$

so

$$\underline{E}_2^{s,t} = \begin{cases} \square \oplus \bigoplus_\ell \widetilde{\square} & \textit{for } (s,t) = (0, 4\ell) \textit{ with } \ell \geq 0, \\ \bigoplus_{\ell+1} \tilde{\tilde{\square}} & \textit{for } (s,t) = (0, 4\ell+2) \textit{ with } \ell \geq 0, \\ \bullet \oplus \bigoplus_{u+\ell} \tilde{\bullet} & \textit{for } (s,t) = (2u, 4\ell+4u) \textit{ with } \ell \geq 0 \textit{ and } u > 0, \\ \bigoplus_{u+\ell} \tilde{\bullet} & \textit{for } (s,t) = (2u-1, 4\ell+4u-2) \textit{ with } \ell \geq 0 \textit{ and } u > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

The first set of differentials and determined by

$$d_3(\Sigma_{2,\epsilon}) = \eta_\epsilon^2(\eta_0 + \eta_1) \quad \textit{and} \quad d_3(\delta_1) = \eta_0\eta_1(\eta_0 + \eta_1)$$

and there is a second set of differentials determined by

$$d_7(\Sigma_{2,\epsilon}^2) = d_7(\delta_1^2) = \eta_0^7.$$

Corollary 12.3 (Some nontrivial permanent cycles). *The elements listed below in $\underline{E}_2^{s,8i+2s} k_{[2]}(G/G')$ are nontrivial permanent cycles. Their transfers in $\underline{E}_2^{s,8i+2s} k_{[2]}(G/G)$ are also permanent cycles.*

- *$\Sigma_{2,\epsilon}^{2i-j}\delta_1^j$ for $0 \leq j \leq 2i$ ($4i+1$ elements of infinite order including δ_1^{2i}), i even and $s = 0$.*
- *$\eta_\epsilon \Sigma_{2,\epsilon}^{2i-j}\delta_1^j$ for $0 \leq j < 2i$ and $\eta_\epsilon \delta_1^{2i}$ ($4i+2$ elements or order 2) for i even and $s = 1$.*
- *$\eta_\epsilon^2 \Sigma_{2,\epsilon}^{2i-j}\delta_1^j$ for $0 \leq j < 2i$ and $\delta_1^{2i}\{\eta_0^2, \eta_0\eta_1, \eta_1^2\}$ ($4i+3$ elements or order 2) for i even and $s = 2$.*
- *$\eta_0^s \delta_1^{2i}$ for $3 \leq s \leq 6$ (four elements or order 2) and i even.*
- *$\Sigma_{2,\epsilon}^{2i-j}\delta_1^j + \delta_1^{2i}$ for $0 \leq j \leq 2i$ ($4i+1$ elements of infinite order including $2\delta_1^{2i}$), i odd and $s = 0$.*
- *$\eta_\epsilon \Sigma_{2,\epsilon}^{2i-j}\delta_1^j + \delta_1^{2i}$ for $0 \leq j \leq 2i-1$ and $\eta_0\delta_1^{2i-1}(\Sigma_{2,1} + \delta_1) = \eta_1\delta_1^{2i-1}(\Sigma_{2,0} + \delta_1)$ ($4i+1$ elements of order 2), i odd and $s = 1$.*
- *$\eta_\epsilon^2 \Sigma_{2,\epsilon}^{2i-j}\delta_1^j + \delta_1^{2i}$ for $0 \leq j \leq 2i-1$, $\eta_0^2\delta_1^{2i-1}(\Sigma_{2,1} + \delta_1) = \eta_0\eta_1\delta_1^{2i-1}(\Sigma_{2,0} + \delta_1)$ and $\eta_0\eta_1\delta_1^{2i-1}(\Sigma_{2,1} + \delta_1) = \eta_1^2\delta_1^{2i-1}(\Sigma_{2,0} + \delta_1)$ ($4i+2$ elements of order 2) for i odd and $s = 2$.*

In $\underline{E}_2^{0,8i+4} k_{[2]}(G/G')$ we have $2\Sigma_{2,\epsilon}^{2i+1-j}\delta_1^j$ for $0 \leq j \leq 2i$ and $2\delta_1^j$, $4i+3$ elements of infinite order, each in the image of the transfer tr_1^2.

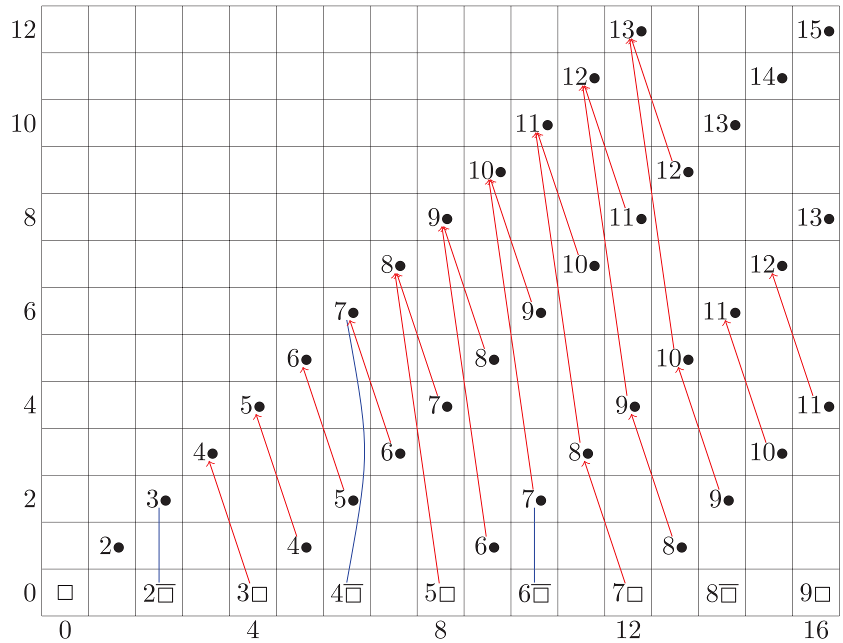
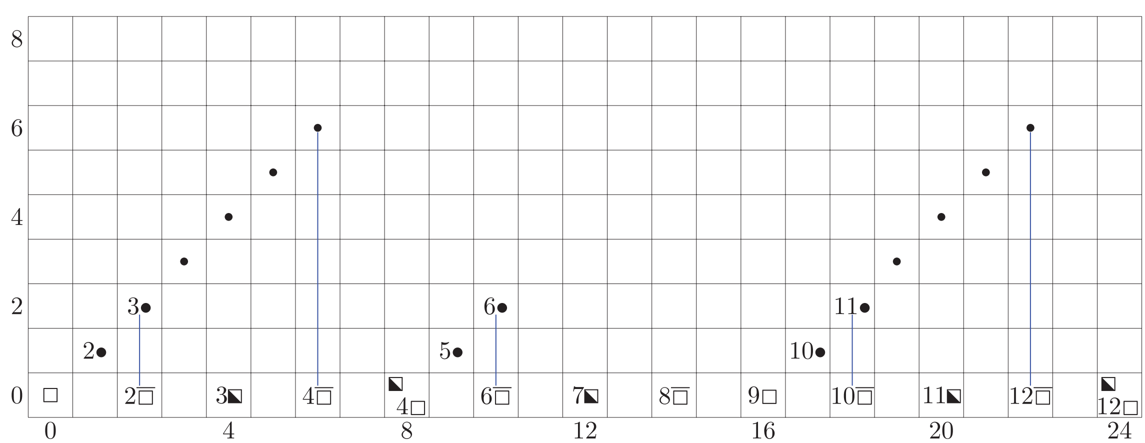


Figure 8. The slice spectral sequence for $k_{[2]}$ as a C_2-spectrum. The Mackey functor symbols are defined in Table 1. The C_4-structure of the Mackey functors is not indicated here. In each bidegree we have a direct sum of the indicated number of copies of the indicated Mackey functor. Each d_3 has maximal rank, leaving a cokernel of rank 1, and each d_7 has rank 1. Blue lines indicate exotic transfers. The ones raising filtration by 2 have maximal rank while the ones raising it by 6 have rank 1. The resulting $\underline{E}_8 = \underline{E}_\infty$-term is shown below.



Remark 12.4. In *the* RO(G)*-graded slice spectral sequence for* $k_{[2]}$ one has

$$d_3(u_{2\sigma}) = a_\sigma^3(\bar{r}_{1,0} + \bar{r}_{1,1}) \quad \text{and} \quad d_7([u_{2\sigma}^2]) = a_\sigma^7 \bar{r}_3^{G'} = a_\sigma^7 \bar{r}_{1,0}^3,$$

but a^7 itself, and indeed all higher powers of a, survive to $\underline{E}_8 = \underline{E}_\infty$. Hence the $\underline{E}_\infty$-term of this spectral sequence does *not* have the horizontal vanishing line that we see in $\underline{E}_8$-term of Figure 7. However when we pass from $k_{[2]}$ to $K_{[2]}$, $\bar{r}_3^{G'} = 5\bar{r}_{1,0}^2\bar{r}_{1,1} + 5\bar{r}_{1,0}\bar{r}_{1,1}^2 + \bar{r}_{1,1}^3$ becomes invertible and we have

$$d_7((\bar{r}_3^{G'})^{-1}[u_{2\sigma}^2]) = d_7(\bar{r}_{1,0}^{-3}[u_{2\sigma}^2]) = a^7.$$

On the other hand, $\bar{r}_{1,0} + \bar{r}_{1,1}$ is not invertible, so we cannot divide $u_{2\sigma}$ by it.

We now give the Poincaré series computation analogous to the one following Remark 8.7, using the notation of (8.8). In RO(G')-graded slice spectral sequence for $k_{[2]}$ we have

$$\underline{E}_2(G'/G') = \mathbf{Z}[a_\sigma, u_{2\sigma}, \bar{r}_{1,0}, \bar{r}_{1,2}]/(2a_\sigma),$$

so

$$g(\underline{E}_2(G'/G')) = \left(\frac{1}{1-t} + \frac{\hat{a}}{1-\hat{a}}\right)\frac{1}{(1-\hat{u})(1-\hat{r})^2},$$

$$g(\underline{E}_4(G'/G')) = g(\underline{E}_2(G'/G')) - \frac{\hat{u}+\hat{r}\hat{a}^3}{(1-\hat{a})(1-\hat{u}^2)(1-\hat{r})^2}$$

$$= \frac{1+t\hat{u}}{(1-t)(1-\hat{u}^2)(1-\hat{r})^2} + \frac{\hat{a}+\hat{a}^2}{(1-\hat{u}^2)(1-\hat{r})^2} + \frac{\hat{a}^3}{(1-\hat{a})(1-\hat{u}^2)(1-\hat{r})}$$

as before. The next differential leads to

$$\begin{aligned}
g(\underline{E}_8(G'/G')) &= g(\underline{E}_4(G'/G')) - \frac{\hat{u}^2+\hat{r}^3\hat{a}^7}{(1-\hat{a})(1-\hat{u}^4)(1-\hat{r})} \\
&= g(\underline{E}_4(G'/G')) - \frac{\hat{u}^2}{(1-\hat{u}^4)(1-\hat{r})} - \frac{\hat{u}^2\hat{a}}{(1-\hat{a})(1-\hat{u}^4)(1-\hat{r})} - \frac{\hat{r}^3\hat{a}^7}{(1-\hat{a})(1-\hat{u}^4)(1-\hat{r})} \\
&= \frac{1+t\hat{u}}{(1-t)(1-\hat{u}^2)(1-\hat{r})^2} + \frac{\hat{a}+\hat{a}^2}{(1-\hat{u}^2)(1-\hat{r})^2} + \frac{\hat{a}^3}{(1-\hat{a})(1-\hat{u}^2)(1-\hat{r})} - \frac{\hat{u}^2}{(1-\hat{u}^4)(1-\hat{r})} \\
&\quad - \frac{\hat{u}^2(\hat{a}+\hat{a}^2)}{(1-\hat{u}^4)(1-\hat{r})} - \frac{\hat{u}^2\hat{a}^3}{(1-\hat{a})(1-\hat{u}^4)(1-\hat{r})} - \frac{\hat{r}^3\hat{a}^7}{(1-\hat{a})(1-\hat{u}^4)(1-\hat{r})} \\
&= \frac{(1+t\hat{u})(1+\hat{u}^2)-(1-t)(1-\hat{r})\hat{u}^2}{(1-t)(1-\hat{u}^4)(1-\hat{r})^2} + \frac{(\hat{a}+\hat{a}^2)(1+\hat{u}^2-\hat{u}^2(1-\hat{r}))}{(1-\hat{u}^4)(1-\hat{r})^2} + \frac{\hat{a}^3(1+\hat{u}^2)-\hat{u}^2\hat{a}^3-\hat{r}^3\hat{a}^7}{(1-\hat{a})(1-\hat{u}^4)(1-\hat{r})} \\
&= \frac{1+t\hat{u}+(t+\hat{r}-t\hat{r})\hat{u}^2+t\hat{u}^3}{(1-t)(1-\hat{u}^4)(1-\hat{r})^2} + \frac{(\hat{a}+\hat{a}^2)(1+\hat{u}^2\hat{r})}{(1-\hat{u}^4)(1-\hat{r})^2} + \frac{\hat{a}-\hat{a}^7+\hat{a}^7-\hat{r}^3\hat{a}^7}{(1-\hat{a})(1-\hat{u}^4)(1-\hat{r})} \\
&= \frac{1+t\hat{u}+(t+\hat{r}-t\hat{r})\hat{u}^2+t\hat{u}^3}{(1-t)(1-\hat{u}^4)(1-\hat{r})^2} + \frac{(\hat{a}+\hat{a}^2)(1+\hat{u}^2\hat{r})}{(1-\hat{u}^4)(1-\hat{r})^2} + \frac{\hat{a}^3+\hat{a}^4+\hat{a}^6+\hat{a}^6}{(1-\hat{u}^4)(1-\hat{r})} + \frac{\hat{a}^7(1+\hat{r}+\hat{r}^2)}{(1-\hat{a})(1-\hat{u}^4)}.
\end{aligned}$$

The fourth term of this expression represents the elements with filtration above six, and the first term represents the elements of filtration 0. The latter include

$$\begin{aligned}
[2u_{2\sigma}] &\in \langle 2, a_\sigma^2, a_\sigma(\bar{r}_{1,0}+\bar{r}_{1,1})\rangle, \\
[2u_{2\sigma}^2] &\in \langle 2, a_\sigma, a_\sigma^6\bar{r}_{1,0}^3\rangle, \\
[(\bar{r}_{1,0}+\bar{r}_{1,1})u_{2\sigma}^2] &\in \langle a_\sigma^4, a_\sigma\bar{r}_{1,0}^3, \bar{r}_{1,0}+\bar{r}_{1,1}\rangle \quad \text{with } (\bar{r}_{1,0}+\bar{r}_{1,1})[2u_{2\sigma}^2] = 2[(\bar{r}_{1,0}+\bar{r}_{1,1})u_{2\sigma}^2], \\
[2u_{2\sigma}^3] &\in \langle 2, a_\sigma^2(\bar{r}_{1,0}+\bar{r}_{1,1}), a_\sigma^2, a_\sigma^6\bar{r}_{1,0}^3\rangle, \\
[u_{2\sigma}^4] &\in \langle a_\sigma^4, a_\sigma^3\bar{r}_{1,0}^3a_\sigma^4, a_\sigma^3\bar{r}_{1,0}^3\rangle
\end{aligned}$$

with notation as in Remark 4.1.

As indicated in Remark 12.4, we can get rid of them by formally adjoining $w := (\bar{r}_3^{G'})^{-1}u_{2\sigma}^2$ to $\underline{E}_2(G'/G')$. As before we denote the enlarged spectral sequence terms by $\underline{E}'_r(G'/G')$ This time let $\widehat{w} = \hat{r}^{-3}\hat{u}^2 = xy^{-7}$. Then we have

$$\underline{E}'_r(G'/G') = \left(\frac{1-\hat{u}^2}{1-\widehat{w}}\right)\underline{E}_r(G'/G') \quad \text{for } r = 2 \text{ and } r = 4$$

and

$$\begin{aligned}
g(\underline{E}'_8(G'/G')) &= g(\underline{E}'_4(G'/G')) - \frac{\widehat{w}+\hat{a}^7}{(1-\hat{a})(1-\widehat{w}^2)(1-\hat{r})} \\
&= g(\underline{E}'_4(G'/G')) - \frac{\widehat{w}}{(1-\widehat{w}^2)(1-\hat{r})} - \frac{(\hat{a}+\hat{a}^2)\widehat{w}}{(1-\widehat{w}^2)(1-\hat{r})} - \frac{\hat{a}^3\widehat{w}+\hat{a}^7}{(1-\hat{a})(1-\widehat{w}^2)(1-\hat{r})} \\
&= \frac{1+t\hat{u}}{(1-t)(1-\widehat{w})(1-\hat{r})^2} + \frac{\hat{a}+\hat{a}^2}{(1-\widehat{w})(1-\hat{r})^2} + \frac{\hat{a}^3}{(1-\hat{a})(1-\widehat{w})(1-\hat{r})} - \frac{\widehat{w}}{(1-\widehat{w}^2)(1-\hat{r})} \\
&\quad - \frac{(\hat{a}+\hat{a}^2)\widehat{w}}{(1-\widehat{w}^2)(1-\hat{r})} - \frac{\hat{a}^3\widehat{w}+\hat{a}^7}{(1-\hat{a})(1-\widehat{w}^2)(1-\hat{r})} \\
&= \frac{(1+t\hat{u})(1+\widehat{w})-(1-t)(1-\hat{r})\widehat{w}}{(1-t)(1-\widehat{w}^2)(1-\hat{r})^2} + \frac{(\hat{a}+\hat{a}^2)(1-(1-\hat{r})\widehat{w})}{(1-\widehat{w}^2)(1-\hat{r})^2} + \frac{\hat{a}^3(1+\widehat{w})-\hat{a}^3\widehat{w}-\hat{a}^7}{(1-\hat{a})(1-\widehat{w}^2)(1-\hat{r})} \\
&= \frac{1+t\hat{u}+(t+\hat{r}-t\hat{r})\widehat{w}+t\widehat{w}\hat{u}}{(1-t)(1-\widehat{w}^2)(1-\hat{r})^2} + \frac{(\hat{a}+\hat{a}^2)(1+\hat{r}\widehat{w})}{(1-\widehat{w}^2)(1-\hat{r})^2} + \frac{\hat{a}^3+\hat{a}^4+\hat{a}^5+\hat{a}^6}{(1-\widehat{w}^2)(1-\hat{r})}.
\end{aligned}$$

Again the first term represents the elements of filtration 0. These include

$$
\begin{aligned}
[2u_{2\sigma}] &\in \langle 2, a_\sigma^2, a_\sigma(\bar{r}_{1,0}+\bar{r}_{1,1})\rangle,\\
[2w] &\in \langle 2, a_\sigma, a_\sigma^6\rangle,\\
[(\bar{r}_{1,0}+\bar{r}_{1,1})w] &\in \langle a_\sigma^4, a_\sigma^3, \bar{r}_{1,0}+\bar{r}_{1,1}\rangle\\
&\qquad \text{with } (\bar{r}_{1,0}+\bar{r}_{1,1})[2w] = 2[(\bar{r}_{1,0}+\bar{r}_{1,1})w],\\
[2u_{2\sigma}w] &\in \langle 2, a_\sigma^2(\bar{r}_{1,0}+\bar{r}_{1,1}), a_\sigma^2, a_\sigma^6\rangle\\
\text{and}\quad [w^2] &\in \langle a_\sigma^4, a_\sigma^3 a_\sigma^4, a_\sigma^3\rangle
\end{aligned}
$$

where, as indicated above, $w = (\bar{r}_3^{G'})^{-1}u_{2\sigma}^2$.

13 The effect of the first differentials over C_4

Theorem 10.2 lists elements in the slice spectral sequence for $k_{[2]}$ over C_4 in terms of

$$r_1,\ \bar{s}_2,\ \bar{\mathfrak{d}}_1;\quad \eta,\ a_\sigma,\ a_\lambda;\quad u_\lambda,\ u_\sigma,\ u_{2\sigma}.$$

All but the u are permanent cycles, and the action of d_3 on u_λ, u_σ, $u_{2\sigma}$ is described above in Theorem 11.13.

Proposition 13.1 (d_3 on elements in Theorem 10.2). *We have the following d_3s, subject to the conditions on i, j, k and ℓ of Theorem 10.2:*

- *On $X_{2\ell,2\ell}$:*

$$d_3(a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell-j}\bar{\mathfrak{d}}_1^{2\ell}) = \begin{cases} a_\lambda^{j+1}\eta u_{2\sigma}^\ell u_\lambda^{2\ell-j-1}\bar{\mathfrak{d}}_1^{2\ell} \in \underline{\pi}_* X_{2\ell,2\ell+1}(G/G) & \text{for } j \text{ odd},\\ 0 & \text{for } j \text{ even,}\end{cases}$$

$$d_3(a_\sigma^{2k}a_\lambda^{2\ell}u_{2\sigma}^{\ell-k}\bar{\mathfrak{d}}_1^{2\ell}) = 0.$$

- *On $X_{2\ell+1,2\ell+1}$:*

$$d_3(\delta_1^{2\ell+1}) = \eta u_\sigma^{2\ell+1}\operatorname{res}_2^4(a_\lambda u_\lambda^{2\ell}\bar{\mathfrak{d}}_1^{2\ell+1}) \in \underline{\pi}_* X_{2\ell+1,2\ell+2}(G/G'),$$

$$d_3(u_\sigma^{2\ell+1}\operatorname{res}_2^4(a_\lambda^j u_\lambda^{2\ell+1-j}\bar{\mathfrak{d}}_1^{2\ell+1})) = \begin{cases} \eta u_\sigma^{2\ell+1}\operatorname{res}_2^4(a_\lambda^{j+1}u_\lambda^{2\ell-j}\bar{\mathfrak{d}}_1^{2\ell+1}) \in \underline{\pi}_* X_{2\ell+1,2\ell+2}(G/G') & \text{for } j \text{ even},\\ 0 & \text{for } j \text{ odd,}\end{cases}$$

$$d_3(a_\sigma a_\lambda^j u_\sigma^{2\ell}u_\lambda^{2\ell+1-j}\bar{\mathfrak{d}}_1^{2\ell+1}) = \begin{cases} \eta a_\sigma a_\lambda^{j+1}u_\sigma^{2\ell}u_\lambda^{2\ell-j}\bar{\mathfrak{d}}_1^{2\ell+1} \in \underline{\pi}_* X_{2\ell+1,2\ell+2}(G/G) & \text{for } j \text{ even},\\ 0 & \text{for } j \text{ odd,}\end{cases}$$

$$d_3(a_\sigma^{2k+1}a_\lambda^{2\ell+1}u_{2\sigma}^{\ell-k}\bar{\mathfrak{d}}_1^{2\ell+1}) = 0.$$

- *On $X_{i,2\ell-i}$:*

$$d_3(u_\sigma^\ell \bar{s}_2^{\ell-i}\operatorname{res}_2^4(u_\lambda^\ell\bar{\mathfrak{d}}_1^i)) = \begin{cases} \eta^3 u_\sigma^{\ell-1}\bar{s}_2^{\ell-i-1}\operatorname{res}_2^4(u_\lambda^{\ell-1}\bar{\mathfrak{d}}_1^i) \in \underline{\pi}_* X_{i,2\ell+1-i}(G/G') & \text{for } \ell \text{ odd},\\ 0 & \text{for } \ell \text{ even,}\end{cases}$$

$$d_3(\eta^{2j}u_\sigma^{\ell-j}\bar{s}_2^{\ell-i-j}\operatorname{res}_2^4(u_\lambda^{\ell-j}\bar{\mathfrak{d}}_1^i)) = \begin{cases} \eta^{2j+1}u_\sigma^{\ell-j}\bar{s}_2^{\ell-i-j}\operatorname{res}_2^4(a_\lambda u_\lambda^{\ell-j-1}\bar{\mathfrak{d}}_1^i) \in \underline{\pi}_* X_{i,2\ell+1-i}(G/G') & \text{for } \ell-j \text{ odd},\\ 0 & \text{for } \ell-j \text{ even.}\end{cases}$$

- *On $X_{i,2\ell+1-i}$:*

$$d_3(r_1\operatorname{res}_1^2(u_\sigma^\ell\bar{s}_2^{\ell-i})\operatorname{res}_1^4(u_\lambda^\ell\bar{\mathfrak{d}}_1^i)) = 0,$$

$$d_3(\eta^{2j+1}u_\sigma^{\ell-j}\bar{s}_2^{\ell-i-j}\operatorname{res}_2^4(u_\lambda^{\ell-j}\bar{\mathfrak{d}}_1^i)) = \begin{cases} \eta^{2j+2}u_\sigma^{\ell-j}\bar{s}_2^{\ell-i-j}\operatorname{res}_2^4(a_\lambda u_\lambda^{\ell-j-1}\bar{\mathfrak{d}}_1^i) \in \underline{\pi}_* X_{i,2\ell+2-i}(G/G') & \text{for } \ell-j \text{ odd},\\ 0 & \text{for } \ell-j \text{ even.}\end{cases}$$

Note that in each case the first index of X is unchanged by the differential, and the second one is increased by one. Since $X_{m,n}$ is a summand of the $2(m+n)$th slice, each d_3 raises the slice degree by 2 as expected.

Remark 13.2 (The spectra y_m and Y_m of Corollaries 10.5 and 10.6). Similar statements can be proved for the case $\ell < 0$. We leave the details to the reader, but illustrate the results in Figures 11 and 12.

The source of each differential in Proposition 13.1 is the product of some element in $\underline{\pi}_\star H\underline{\mathbf{Z}}$ with a power of $\bar{\mathfrak{d}}_1$ or δ_1. The target is the product of a different element in $\underline{\pi}_\star H\underline{\mathbf{Z}}$ with the same power. This means they are differentials in the slice spectral sequence for the spectra y_m of Corollary 10.5.

Similar differentials occur when we replace $\bar{\mathfrak{d}}_1^i$ by any homogeneous polynomial of degree i in $\bar{\mathfrak{d}}_1$ and $\bar{t}_2$ in which the coefficient of $\bar{\mathfrak{d}}_1^i$ is odd. This means they are also differentials in the slice spectral sequence for the spectra Y_m of Corollary 10.6.

These differentials are illustrated in the upper charts in Figures 9–12. In order to pass to $\underline{E}_4$ we need the following exact sequences of Mackey functors:

$$0 \longrightarrow \bullet \longrightarrow \circ \xrightarrow{d_3} \widehat{\bullet} \longrightarrow \overline{\bullet} \longrightarrow 0,$$

$$0 \longrightarrow \widehat{\boxslash} \longrightarrow \widehat{\square} \xrightarrow{d_3} \widehat{\bullet} \longrightarrow 0,$$

$$0 \longrightarrow \overline{\bullet} \xrightarrow{d_3} \widehat{\bullet} \longrightarrow \blacktriangledown \longrightarrow 0,$$

$$0 \longrightarrow \overline{\boxslash} \longrightarrow \overline{\square} \xrightarrow{d_3} \widehat{\bullet} \longrightarrow \blacktriangledown \longrightarrow 0.$$

The resulting subquotients of $\underline{E}_4$ are shown in the lower charts of Figures 9–12 and described below in Theorem 13.3. In the latter the slice summands are organized as shown in the Figures rather than by orbit type as in Theorem 10.2.

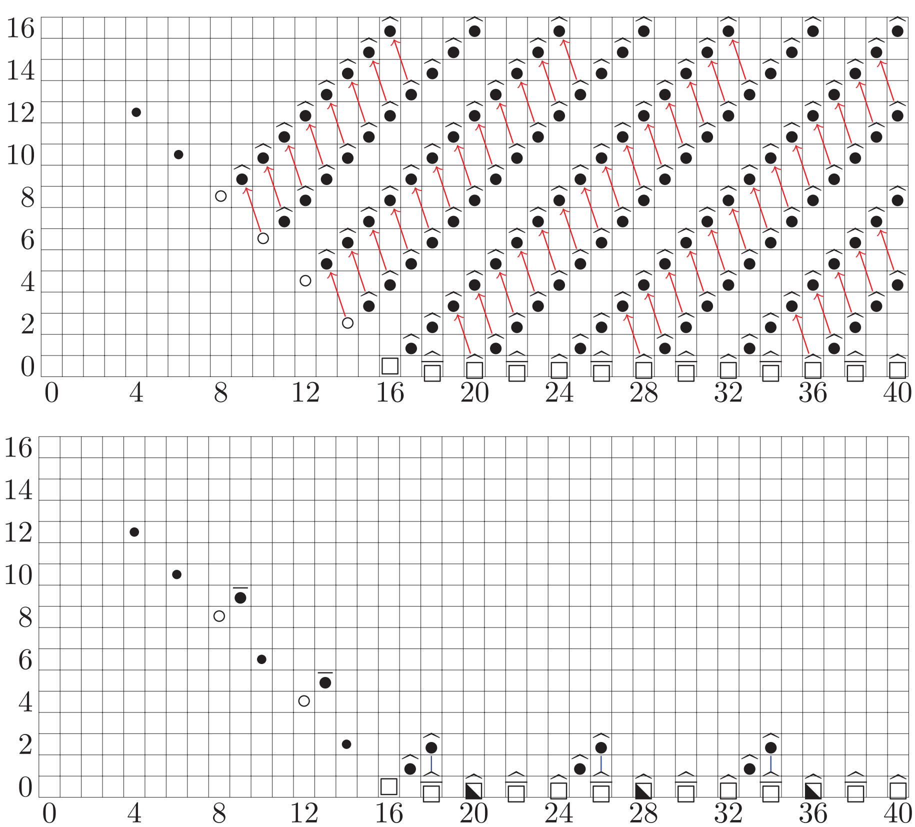


Figure 9. The subquotient of the slice $\underline{E}_2$- and $\underline{E}_4$-terms for $k_{[2]}$ for the slice summands $X_{4,n}$ for $n \geq 4$. Exotic transfers are shown in blue and differentials are in red. The symbols are defined in Table 2. This is also the slice spectral sequence for y_4 as in Corollary 10.5 and Y_4 (after tensoring with R) as in Corollary 10.6.

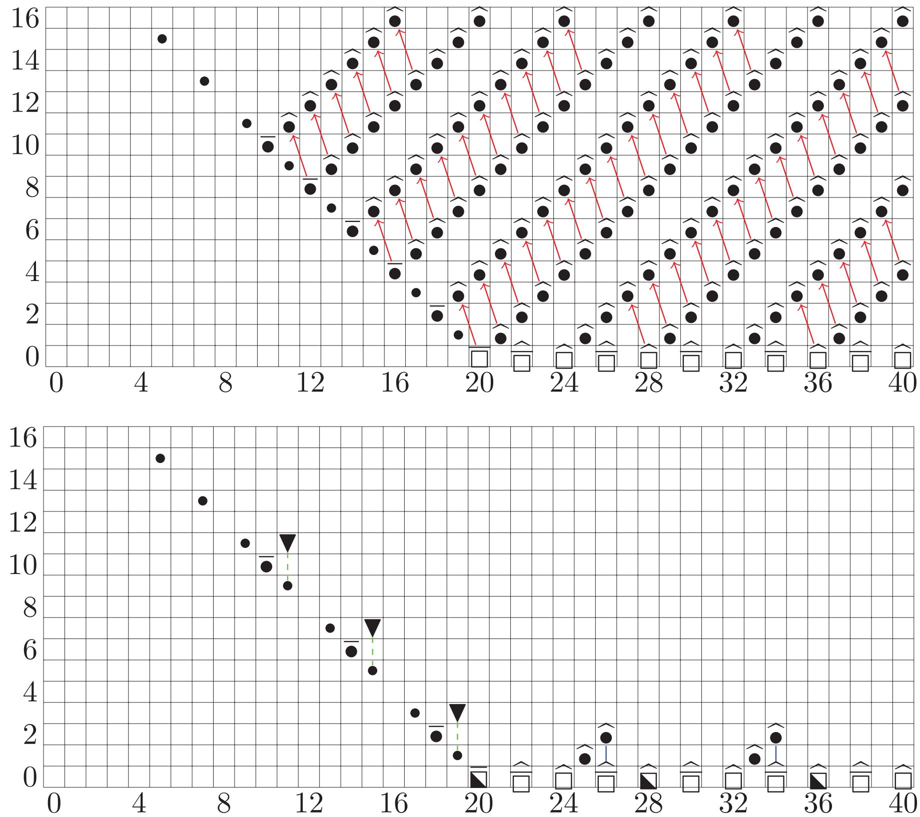


Figure 10. The subquotient of the slice $\underline{E}_2$ and $\underline{E}_4$-terms for $k_{[2]}$ for the slice summands $X_{5,n}$ for $n \geq 5$. Exotic restrictions and transfers are shown in dashed green and solid blue lines respectively. This is also the slice spectral sequence for y_5 as in Corollary 10.5 and for Y_5 (after tensoring with R) as in Corollary 10.6.

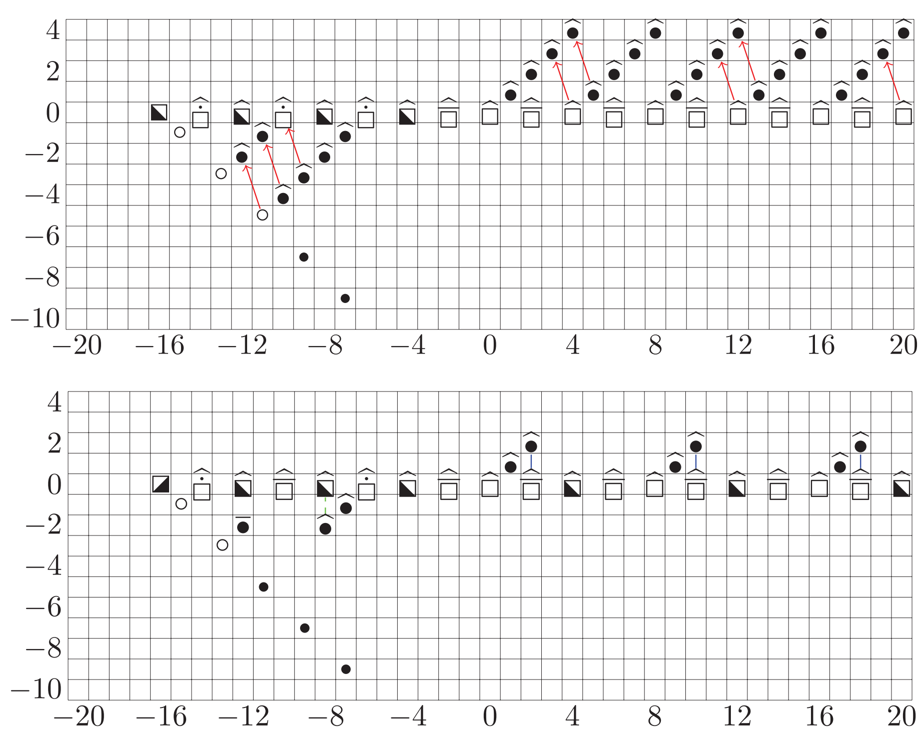


Figure 11. The subquotient of the slice $\underline{E}_2$ and $\underline{E}_4$-terms for $k_{[2]}$ for the slice summands $X_{-4,n}$ for $n \geq -4$. This is also the slice spectral sequence for Y_{-4} (after tensoring with R) as in Corollary 10.6.

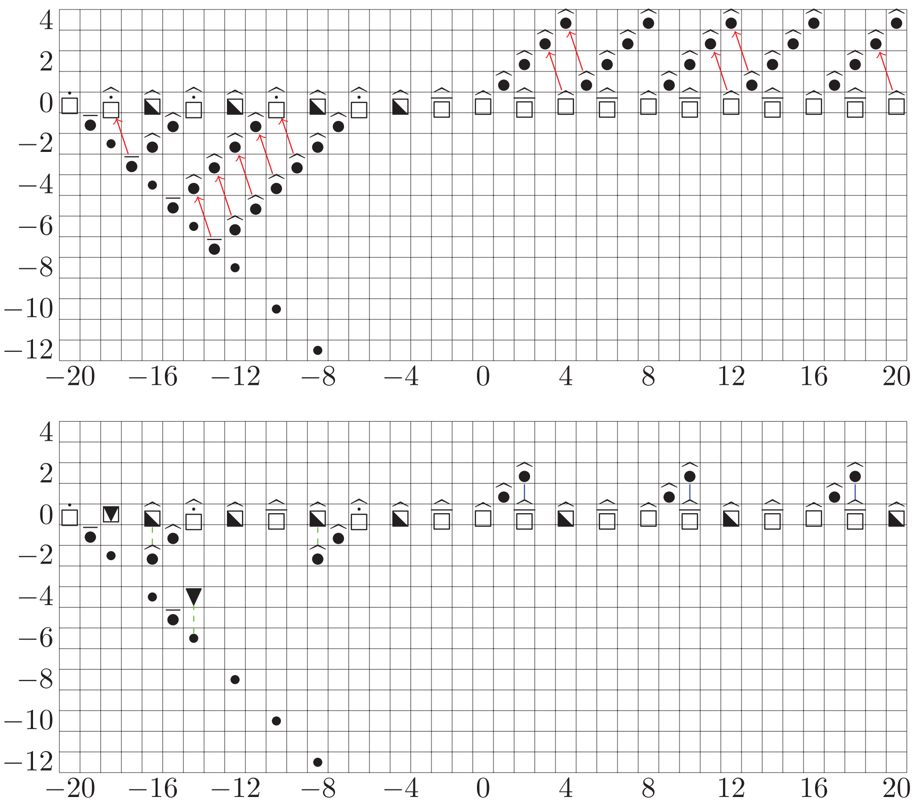


Figure 12. The subquotient of the slice $\underline{E}_2$ and $\underline{E}_4$-terms for $k_{[2]}$ for the slice summands $X_{-5,n}$ for $n \geq -5$. This is also the slice spectral sequence for Y_{-5} (after tensoring with R) as in Corollary 10.6.

Theorem 13.3 (The slice $\underline{E}_4$-term for $k_{[2]}$). *The elements of Theorem 10.2 surviving to $\underline{E}_4$, which live in the appropriate subquotients of $\underline{\pi}_* X_{m,n}$, are as follows:*

(i) *In $\underline{\pi}_* X_{2\ell,2\ell}$ (see the leftmost diagonal in Figure 9), on the 0-line we still have a copy of $\square$ generated under fixed point restrictions by $\Delta_1^\ell \in \underline{E}_4^{0,8\ell}$. In positive filtrations we have*

$$\circ \subseteq \underline{E}_4^{2j,8\ell} \quad \text{generated by} \begin{cases} a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell-j} \bar{\mathfrak{d}}_1^{2\ell} \in \underline{E}_4^{2j,8\ell}(G/G), & j \text{ even}, 0 < j \leq 2\ell, \\ 2a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell-j} \bar{\mathfrak{d}}_1^{2\ell} = a_\sigma^2 a_\lambda^{j-1} u_{2\sigma}^{\ell+1} u_\lambda^{2\ell-j-1} \bar{\mathfrak{d}}_1^{2\ell} \in \underline{E}_4^{2j,8\ell}(G/G), & j \text{ odd}, 0 < j \leq 2\ell, \end{cases}$$

$$\bullet \subseteq \underline{E}_4^{2k+2\ell,8\ell} \quad \text{generated by } a_\sigma^{2k} a_\lambda^{2\ell} u_{2\sigma}^{\ell-k} \bar{\mathfrak{d}}_1^{2\ell} \in \underline{E}_4^{2j+2k,8\ell}(G/G) \quad \text{for } 0 < k \leq \ell.$$

(ii) *In $\underline{\pi}_* X_{2\ell,2\ell+1}$ (see the second leftmost diagonal in Figure 9), in filtration 0 we have $\hat{\hat{\square}}$, generated (under transfers and the group action) by*

$$r_1 \operatorname{res}_1^2(u_\sigma^{2\ell} \operatorname{res}_1^4(u_\lambda^{2\ell} \bar{\mathfrak{d}}_1^{2\ell}) \in \underline{E}_2^{0,8\ell+2}(G/\{e\}).$$

In positive filtrations we have

$\hat{\bullet} \subseteq \underline{E}_4^{1,8\ell+2}$ *generated (under transfers and the group action) by* $\eta u_\sigma^{2\ell} \operatorname{res}_2^4(u_\lambda \bar{\mathfrak{d}}_1)^{2\ell} = \underline{E}_4^{1,8\ell+2}(G/G')$,

$\bar{\bullet} \subseteq \underline{E}_4^{4k+1,8\ell+2}$ *for $0 < k \leq \ell$ generated by* $x = \eta^{4k+1} u_\sigma^{2\ell-2k} \operatorname{res}_2^4(u_\lambda \bar{\mathfrak{d}}_1)^{2\ell-2k} \in \underline{E}_4^{4k+1,8\ell+2}(G/G')$ *with* $(1-\gamma)x = \operatorname{tr}_2^4(x) = 0$.

(iii) *In $\underline{\pi}_* X_{2\ell+1,2\ell+1}$ (see the leftmost diagonal in Figure 10), on the 0-line we have a copy of ◩ generated under fixed point $\Delta_1^{(2\ell+1)/2} \in \underline{E}_4^{0,8\ell+4}$. In positive filtrations we have*

$\bar{\bullet} \subseteq \underline{E}_2^{2j,8\ell+4}$ *generated by* $u_\sigma^{2\ell+1} \operatorname{res}_2^4(a_\lambda^j u_\lambda^{2\ell+1-j} \bar{\mathfrak{d}}_1^{2\ell+1}) \in \underline{E}_2^{2j,8\ell+4}(G/G')$ *for* $0 < j \leq 2\ell+1$,

$\bullet \subseteq \underline{E}_2^{2j+1,8\ell+4}$ *generated by* $a_\sigma a_\lambda^j u_\sigma^{2\ell} u_\lambda^{2\ell+1-j} \bar{\mathfrak{d}}_1^{2\ell+1} \in \underline{E}_2^{2j+2k,8\ell+4}(G/G)$ *for* $0 \leq j \leq 2\ell+1$,

$\bullet \subseteq \underline{E}_2^{2k+4\ell+3,8\ell+4}$ *generated by* $a_\sigma^{2k+1} a_\lambda^{2\ell+1} u_{2\sigma}^{\ell-k} \bar{\mathfrak{d}}_1^{2\ell+1} \in \underline{E}_2^{2k+4\ell+2,8\ell+4}(G/G)$ *for* $0 < k \leq 2\ell+1$.

(iv) *In $\underline{\pi}_* X_{2\ell+1,2\ell+2}$ (see the second leftmost diagonal in Figure 10), in filtration* 0 *we have* $\hat{\hat{\square}}$, *generated (under transfers and the group action) by*

$$r_1 \operatorname{res}_1^2(u_\sigma^{2\ell+1} \operatorname{res}_1^4(u_\lambda^{2\ell+1}\bar{\mathfrak{d}}_1^{2\ell+1}) \in \underline{E}_4^{0,8\ell+6}(G/\{e\}).$$

In positive filtrations we have

$\blacktriangledown \subseteq \underline{E}_4^{4k+3,8\ell+6}$ *for* $0 \le k \le \ell$ *generated under transfer by* $x = \eta^{4k+3}\Delta_1^{\ell-k} \in \underline{E}_4^{4k+3,8\ell+6}(G/G')$ *with* $(1-\gamma)x = 0$.

The generator of $\underline{E}_4^{4k+3,8\ell+6}(G/G')$ *is the exotic restriction of the one in* $\underline{E}_4^{4k+1,8\ell+4}(G/G)$.

(v) *In $\underline{\pi}_* X_{m,m+i}$ for $i \ge 2$ (see the rest of Figures 9 and 10), in filtration* 0 *we have*

$\hat{\hat{\square}} \subseteq \underline{E}_4^{0,4m+4j+2}$ *generated under transfers and group action by* $r_1 \operatorname{res}_1^2(u_\sigma^{m+j}\bar{s}_2^j)\operatorname{res}_1^4(u_\lambda^{m+j}\bar{\mathfrak{d}}_1^m) \in \underline{E}_4^{0,4m+4j+2}(G/\{e\})$ *for* $j \ge 0$,

$\hat{\blacksquare} \subseteq \underline{E}_4^{0,8\ell+4}$ *generated under transfers and group action by* $r_1 \operatorname{res}_1^2(u_\sigma^{m+j}\bar{s}_2^j)\operatorname{res}_1^4(u_\lambda^{m+j}\bar{\mathfrak{d}}_1^m) \in \underline{E}_4^{0,8\ell+4}(G/\{e\})$ *for* $\ell \ge m/2$,

$\hat{\square} \subseteq \underline{E}_4^{0,8\ell}$ *generated under transfers, restriction and group action by* $x_{8\ell,m} = \Sigma_{2,0}^{2\ell-m}\delta_1^m + \ell\delta_1^{2\ell}$,

where

$$\Sigma_{2,\epsilon} = u_{\rho_2}\bar{s}_{2,\epsilon} \quad \textit{and} \quad \delta_1 = u_{\rho_2}\operatorname{res}_2^4(\bar{\mathfrak{d}}_1) \in \underline{E}_4^{0,8\ell}(G/G') \textit{ for } 0 \le m \le 2\ell-1.$$

In positive filtrations we have

$\hat{\bullet} \subseteq \underline{E}_4^{2,8\ell+4}$ *generated under transfers and group action by* $\eta_0^2 \operatorname{res}_2^4(\Delta_1^\ell) = \eta_0^2\delta_1^{2\ell} = \eta_0^2 u_\sigma^{2\ell}\operatorname{res}_2^4(u_\lambda\bar{\mathfrak{d}}_1)^{2\ell} \in \underline{E}_4^{2,8\ell+4}(G/G')$,

$\hat{\bullet} \subseteq \underline{E}_4^{s,8\ell+2s}$ *generated under transfers and group action by* $\eta_\epsilon^s x_{8\ell,m} \in \underline{E}_4^{s,8\ell+2s}(G/G')$ *for* $s = 1, 2$ *and* $0 \le m \le 2\ell-1$.

Each generator of $\underline{E}_4^{2,8\ell+4}(G/G')$ *is an exotic transfer of one in* $\underline{E}_4^{0,8\ell+2}(G/e)$.

Proposition 13.4 (Some nontrivial permanent cycles). *The elements listed in Theorem 13.3* (v) *other than* $\eta_\epsilon^2\delta_1^{2\ell}$ *are all nontrivial permanent cycles.*

Proof. Each such element is either in the image of $\underline{E}_4^{0,*}(G/\{e\})$ under the transfer and therefore a nontrivial permanent cycle, or it is one of the ones listed in Corollary 12.3. ☐

In subsequent discussions and charts, starting with Figure 14, we will omit the elements in Proposition 13.4. These elements all occur in $\underline{E}_4^{s,t}$ *for* $0 \le s \le 2$.

Analogous statements can be made about the slice spectral sequence for $K_{[2]}$. Each of its slices is a certain infinite wedge spelled out in Corollary 10.4. Their homotopy groups are determined by the chain complex calculations of Section 6 and illustrated in Figures 2 (with Mackey functor induction $\uparrow_2^4$ applied) and 3. Analogs of Figures 9–10 are shown in Figures 11–12. In each figure, exotic transfers and restrictions are indicated by blue and dashed green lines respectively. As in the $k_{[2]}$ case, most of the elements shown in this chart can be ignored for the purpose of calculating higher differentials. *In the third quadrant the elements we are ignoring all occur in* $\underline{E}_4^{s,t}$ *for* $-2 \le s \le 0$.

The resulting reduced $\underline{E}_4$ for $K_{[2]}$ is shown in Figure 16. The information shown there is very useful for computing differentials and extensions. The periodicity theorem tells us that $\underline{\pi}_n K_{[2]}$ and $\underline{\pi}_{n-32}K_{[2]}$ are isomorphic. For $0 \le n < 32$ these groups appear in the first and third quadrants respectively, and the information visible in the spectral sequence can be quite different.

For example, we see that $\underline{\pi}_0 K_{[2]}$ has summand of the form $\square$, while $\underline{\pi}_{-32}K_{[2]}$ has a subgroup isomorphic to $\boxslash$. The quotient $\square/\boxslash$ is isomorphic to $\circ$. This leads to the exotic restrictions and transfer in dimension −32 shown in Figure 16. Information that is transparent in dimension 0 implies subtle information in dimension −32. Conversely, we see easily that $\underline{\pi}_{-4}K_{[2]} = \dot{\blacksquare}$ while $\underline{\pi}_{28}K_{[2]}$ has a quotient isomorphic to $\bar{\blacksquare}$. This leads to the “long transfer” (which raises filtration by 12) in dimension 28.

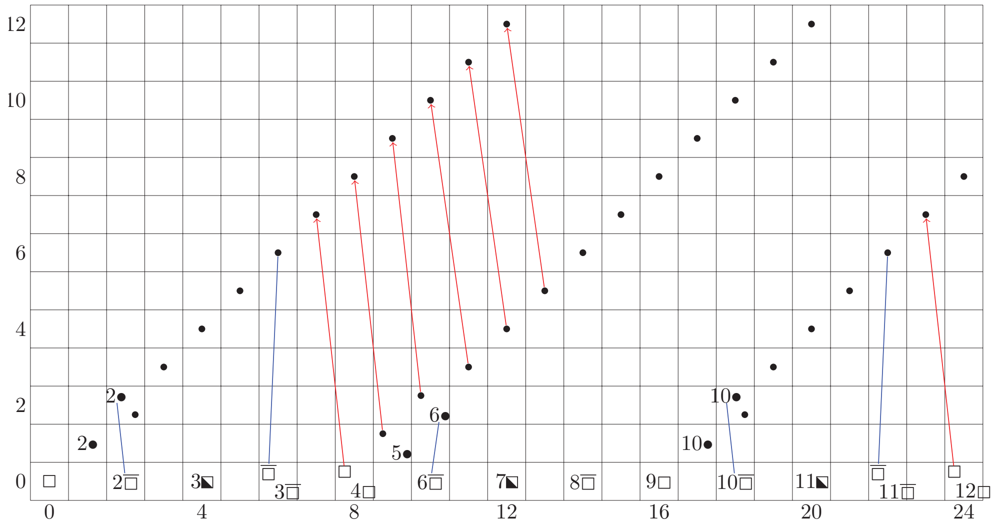


Figure 13. The slice $\underline{E}_7$-term for the C_2-spectrum $i^*_{G'}k_{[2]}$. The Mackey functor symbols are defined in Table 1. A number n in front of a symbol indicates an n-fold direct sum. Blue lines indicate exotic transfers and red lines indicate differentials.

14 Higher differentials and exotic Mackey functor extensions

We can use the results of the Section 12 to study higher differentials and extensions. The $\underline{E}_7$-term implied by them is illustrated in Figure 13. For each $\ell, s \geq 0$ there is a generator

$$y_{8\ell+s,s} := \eta_0^s \delta_1^{2\ell} \in \underline{E}_7^{s,8\ell+2s}(G/G')$$

with

$$d_7(y_{16k+s+8,s}) = y_{16k+s+7,s+7}.$$

Recall that

$$\delta_1 = \bar{u}_\lambda \bar{r}_{1,0}\bar{r}_{1,1} \in \underline{E}_2^{0,4}k_{[2]}(G/G') \cong \underline{E}_2^{0,(G',4)}k_{[2]}(G/G),$$

and in the latter group we denote $\bar{u}_\lambda$ by $u_{2\sigma}$. We have

$$d_3(\delta_1) = d_3(\bar{u}_\lambda)\bar{r}_{1,0}\bar{r}_{1,1} \cong d_3(u_{2\sigma})\bar{r}_{1,0}\bar{r}_{1,1} = a_\sigma^3(\bar{r}_{1,0}+\bar{r}_{1,1})\bar{r}_{1,0}\bar{r}_{1,1}.$$

If the source has the form $\operatorname{res}_2^4(x_{16k+s+8,s})$, then such an x must support a nontrivial d_r for $r \leq 7$. If it has a nontrivial transfer $x'_{16k+s+8,s}$, then such an x' cannot support an earlier differential, and we must have

$$d_r(x'_{16k+s+8,s}) = \operatorname{tr}_2^4(d_7(y_{16k+s+8,s})) = \operatorname{tr}_2^4(y_{16k+s+7,s+7}) \quad \text{for some } r \geq 7.$$

We could get a higher differential (meaning $r > 7$) if $y_{16k+s+7,s+7}$ supports an exotic transfer.

We have seen (Figure 14 and Theorem 13.3) that for $s \geq 3$ and $k \geq 0$,

$$\underline{E}_5^{s,16k+8+2s} = \begin{cases} \circ & \text{for } s \equiv 0 \bmod 4, \\ \bar{\bullet} & \text{for } s \equiv 1,2 \bmod 4, \\ \blacktriangledown & \text{for } s \equiv 3 \bmod 4. \end{cases} \tag{14.1}$$

For $s = 1, 2$, $\underline{E}_5^{s,16k+8+2s}$ has $\bar{\bullet}$ as a direct summand. For $s = 0$ it has $\square$ as a summand, and the differentials on it factor through its quotient $\circ$; see (5.2).

The corresponding statement in the third quadrant is

$$\underline{E}_5^{-s,-16k-2s-24} = \begin{cases} \circ & \text{for } s \equiv 3 \bmod 4, \\ \bar{\bullet} & \text{for } s \equiv 1,2 \bmod 4, \\ \blacktriangledown & \text{for } s \equiv 0 \bmod 4, \end{cases}$$

for $s \geq 3$ and $k \geq 0$. For $s = 1, 2$ the groups have similar summands, and for $s = 0$ there is a summand of the form $\boxed{\blacktriangledown}$, which has $\blacktriangledown$ as a subgroup; again see (5.2). This is illustrated in Figure 16.

Theorem 14.2 (Differentials for C_4 related to the d_7s for C_2). *The differential*

$$d_7(y_{16k+s+8,s}) = y_{16k+s+7,s+7} \quad \textit{with } s \geq 3$$

has the following implications for the congruence classes of s modulo 4.

(i) *For* $s \equiv 0$, $\underline{E}_7^{s,16k+8+2s} = \circ$ *and* $\underline{E}_7^{s+7,16k+14+2s} = \blacktriangledown$. *Hence* $y_{16k+s+8,s}$ *is a restriction with a nontrivial transfer, and*

$$\begin{aligned} d_5(x_{16k+s+8,s}) &= x_{16k+s+7,s+5}, \\ d_7(2x_{16k+s+8,s}) &= d_7(\mathrm{tr}_2^4(y_{16k+s+8,s})) = \mathrm{tr}_2^4(y_{16k+s+7,s+7}) = x_{16k+s+7,s+7}. \end{aligned}$$

(ii) *For* $s \equiv 1$,

$$\begin{aligned} d_7(y_{16k+s+8,s}) &= y_{16k+s+7,s+7}, \\ d_5(x_{16k+s+8,s+2}) &= \mathrm{tr}_2^4(y_{16k+s+7,s+7}) = 2x_{16k+s+7,s+7}. \end{aligned}$$

This leaves the fate of $x_{16k+s+7,s+7}$ *undecided; see below.*

(iii) *For* $s \equiv 2$, $\underline{E}_7^{s,16k+8+2s} = \bar{\bullet}$ *and* $\underline{E}_7^{s+7,16k+14+2s} = \bar{\bullet}$. *Neither the source nor target is a restriction or has a nontrivial transfer, so no additional differentials are implied.*

(iv) *For* $s \equiv 3$, $\underline{E}_7^{s,16k+8+2s} = \blacktriangledown$ *and* $\underline{E}_7^{s+7,16k+14+2s} = \bar{\bullet}$. *In this case the source is an exotic restriction; again see Figure 10). Thus we have*

$$\begin{aligned} d_7(y_{16k+s+8,s}) &= y_{16k+s+7,s+7}, \\ d_5(x_{16k+s+8,s-2}) &= x_{16k+s++7,s+3} \quad \textit{with } \mathrm{res}_2^4(x_{16k+s++7,s+3}) = y_{16k+s+7,s+7}. \end{aligned}$$

Moreover, $\mathrm{tr}_2^4(y_{16k+8+s,s})$ *is nontrivial and it supports a nontrivial* d_{11} *when* $4k + s \equiv 3 \bmod 8$. *The other case,* $4k + s \equiv 7$, *will be discussed below.*

Proof. (i) The target Mackey functor is $\blacktriangledown$ and $y_{16k+s+7,s+7}$ is the exotic restriction of $x_{16k+s+7,s+5}$; see Figure 10 and Theorem 13.3. The indicated d_5 and d_7 follow.

(ii) The differential is nontrivial on the G/G' component of

$$\bar{\bullet} = \underline{E}_7^{s,16k+8+2s} \xrightarrow{d_7} \underline{E}_7^{s+7,16k+14+2s} = \circ.$$

Thus the target has a nontrivial transfer, so the source must have an exotic transfer. The only option is $x_{16k+s+8,s+2}$, and the result follows.

(iv) We prove the statement about d_{11} by showing that

$$y_{16k+s+7,s+7} = \eta_0^{s+7}\delta_1^{4k}$$

supports an exotic transfer that raises filtration by 4. First note that

$$\mathrm{tr}_2^4(\eta_0\eta_1) = \mathrm{tr}_2^4(a_{\sigma_2}^2\bar{r}_{1,0}\bar{r}_{1,0}) = \mathrm{tr}_2^4(u_\sigma\,\mathrm{res}_2^4(a_\lambda\bar{\mathfrak{d}}_1)) = \mathrm{tr}_2^4(u_\sigma)a_\lambda\bar{\mathfrak{d}}_1 = a_\sigma a_\lambda\bar{\mathfrak{d}}_1 a_\lambda\bar{\mathfrak{d}}_1 \quad \text{by (11.4).}$$

Next note that the three elements

$$y_{8,8} = \eta_0^8 = \mathrm{res}_2^4(\epsilon), \quad y_{20,4} = \eta_0^4\delta_1^4 = \mathrm{res}_2^4(\bar{\kappa}) \quad \text{and} \quad y_{32,0} = \delta_1^8 = \mathrm{res}_2^4(\Delta^4)$$

are all permanent cycles, so the same is true of all

$$y_{16m+4\ell,4\ell} = \eta_0^{4\ell}\delta_1^{4m} \quad \text{for } m, \ell \geq 0 \text{ and } m+\ell \text{ even.}$$

It follows that for such ℓ and m,

$$\eta_0\eta_1 y_{16m+4\ell,4\ell} = \eta_0\eta_1\eta_0^{4\ell}\delta_1^{4m} = \eta_0^{4\ell+2}\delta_1^{4m} = y_{16m+4\ell+2,4\ell+2} = \eta_0\eta_1\,\mathrm{res}_2^4(x_{16m+4\ell,4\ell}),$$

so

$$\mathrm{tr}_2^4(y_{16m+4\ell+2,4\ell+2}) = \mathrm{tr}_2^4(\eta_0\eta_1)x_{16m+4\ell,4\ell} = f_1^2 x_{16m+4\ell,4\ell}.$$

This is the desired exotic transfer. □

We now turn to the unsettled part of Theorem 14.2 (iv).

Theorem 14.3 (The fate of $x_{16k+s+8,s}$ for $4k+s \equiv 7 \bmod 8$ and $s \geq 7$). *Each of these elements is the target of a differential d_7 and hence a permanent cycle.*

Proof. Consider the element $\Delta_1^2 \in \underline{E}_2^{0,16}(G/G)$. We will show that

$$d_7(\Delta_1^2) = x_{15,7} = \mathrm{tr}_2^4(y_{15,7}).$$

This is the case $k = 0$ and $s = 7$. The remaining cases will follow via repeated multiplication by ϵ, $\overline{\kappa}$ and Δ_1^4.

We begin by looking at

$$\Delta_1 = u_{2\sigma} u_\lambda^2 \bar{\mathfrak{d}}_1^2.$$

From Theorem 11.13 we have

$$d_5(u_{2\sigma}) = a_\sigma^3 a_\lambda \bar{\mathfrak{d}}_1 \quad \text{and} \quad d_5(u_\lambda^2) = a_\sigma a_\lambda^2 u_\lambda \bar{\mathfrak{d}}_1.$$

Using the gold relation $a_\sigma^2 u_\lambda = 2a_\lambda u_{2\sigma}$, we have

$$\begin{aligned} d_5(\Delta_1) = d_5(u_{2\sigma}u_\lambda^2)\bar{\mathfrak{d}}_1 &= (a_\sigma^3 a_\lambda u_\lambda^2 \bar{\mathfrak{d}}_1 + a_\sigma a_\lambda^2 u_\lambda u_{2\sigma} \bar{\mathfrak{d}}_1)\bar{\mathfrak{d}}_1 \\ &= a_\sigma a_\lambda u_\lambda (a_\sigma^2 u_\lambda + a_\lambda u_{2\sigma})\bar{\mathfrak{d}}_1^2 \\ &= a_\sigma a_\lambda u_\lambda (2a_\lambda u_{2\sigma} + a_\lambda u_{2\sigma})\bar{\mathfrak{d}}_1^2 \\ &= a_\sigma a_\lambda^2 u_\lambda u_{2\sigma} \bar{\mathfrak{d}}_1^2 \qquad \text{since } 2a_\sigma = 0 \\ &= \nu x_4. \end{aligned}$$

Since ν supports an exotic group extension, $2\nu = x_3$, we have

$$2d_5(\Delta_1) = d_7(2\Delta_1) = x_3 x_4.$$

From this it follows that

$$d_7(\Delta_1^2) = \Delta_1 d_7(2\Delta_1) = x_{15,7}$$

as claimed. □

The resulting reduced $\underline{E}_{12}$-term is shown in Figure 15. It is sparse enough that the only possible remaining differentials are the indicated differentials d_{13}. In order to establish them we need the following.

The surviving class in $\underline{E}_{12}^{20,3}(G/G)$ is

$$x_{17,3} = f_1 \Delta_1^2 = a_\sigma a_\lambda \bar{\mathfrak{d}}_1 \cdot [u_{2\sigma}^2] u_\lambda^4 \bar{\mathfrak{d}}_1^4 = (a_\sigma u_\lambda^4)(a_\lambda [u_{2\sigma}^2] \bar{\mathfrak{d}}_1^5).$$

The second factor is a permanent cycle, so Theorem 11.10 gives

$$d_{13}(f_1 \Delta_1^2) = (a_\lambda^7 [u_{2\sigma}^2] \bar{\mathfrak{d}}_1^3)(a_\lambda [u_{2\sigma}^2] \bar{\mathfrak{d}}_1^5) = a_\lambda^8 [u_{2\sigma}^2] \bar{\mathfrak{d}}_1^8 = \epsilon^2 = x_4^4.$$

The surviving class in $\underline{E}_{12}^{32,2}(G/G)$ is

$$x_{30,2} = a_\sigma^2 u_{2\sigma}^3 u_\lambda^8 \bar{\mathfrak{d}}_1^8 \in \underline{E}_{12}^{32,2}(G/G)$$

and satisfies

$$\epsilon x_{30,2} = f_1 \overline{\kappa} x_{17,3} = f_1^2 x_4 \Delta_1^2,$$

so we have proved the following.

Theorem 14.4 (Differentials d_{13} in the slice spectral sequence for $k_{[2]}$). *There are differentials*

$$d_{13}(f_1^\epsilon x_4^m \Delta_1^{2n}) = f_1^{\epsilon - 1} x_4^{m+4} \Delta_1^{2(n-1)}$$

for $\epsilon = 1, 2$, $m + n$ odd, $n \geq 1$ and $m \geq 1 - \epsilon$. The spectral sequence collapses from $\underline{E}_{14}$.

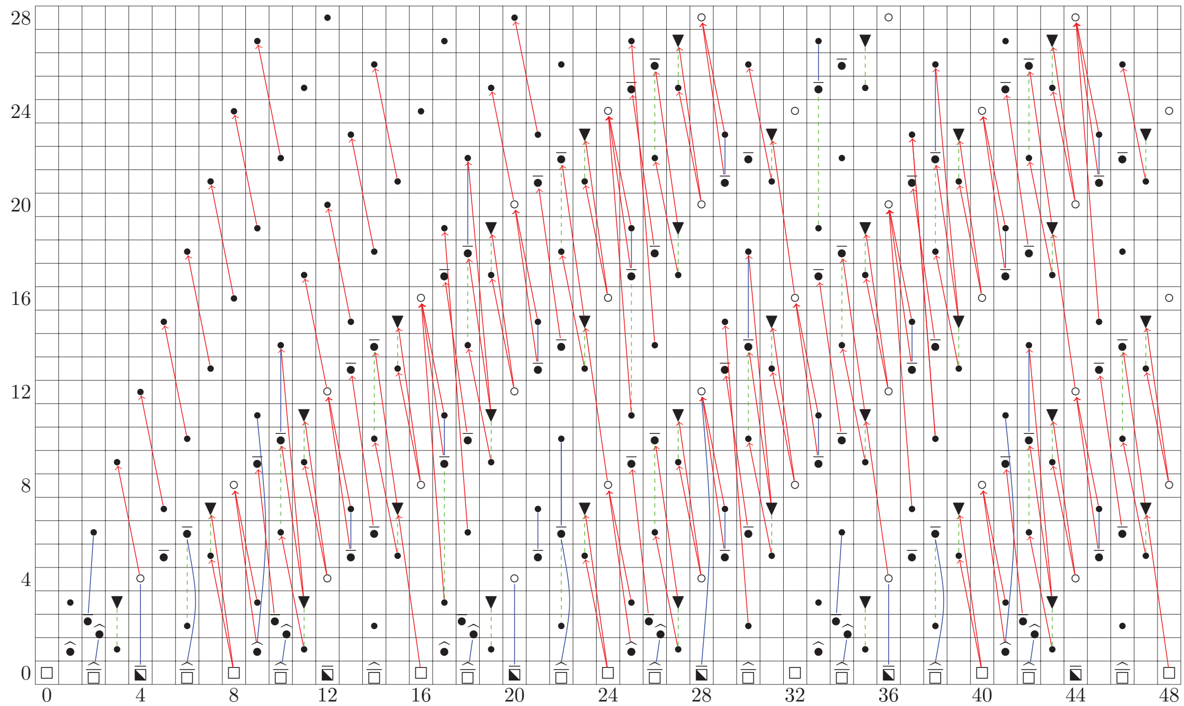


Figure 14. The $\underline{E}_4$-term of the slice spectral sequence for $k_{[2]}$ with elements of Proposition 13.4 removed. Differentials are shown in red. Exotic transfers and restrictions are shown as solid blue and dashed green lines respectively. The Mackey functor symbols are as in Table 2.

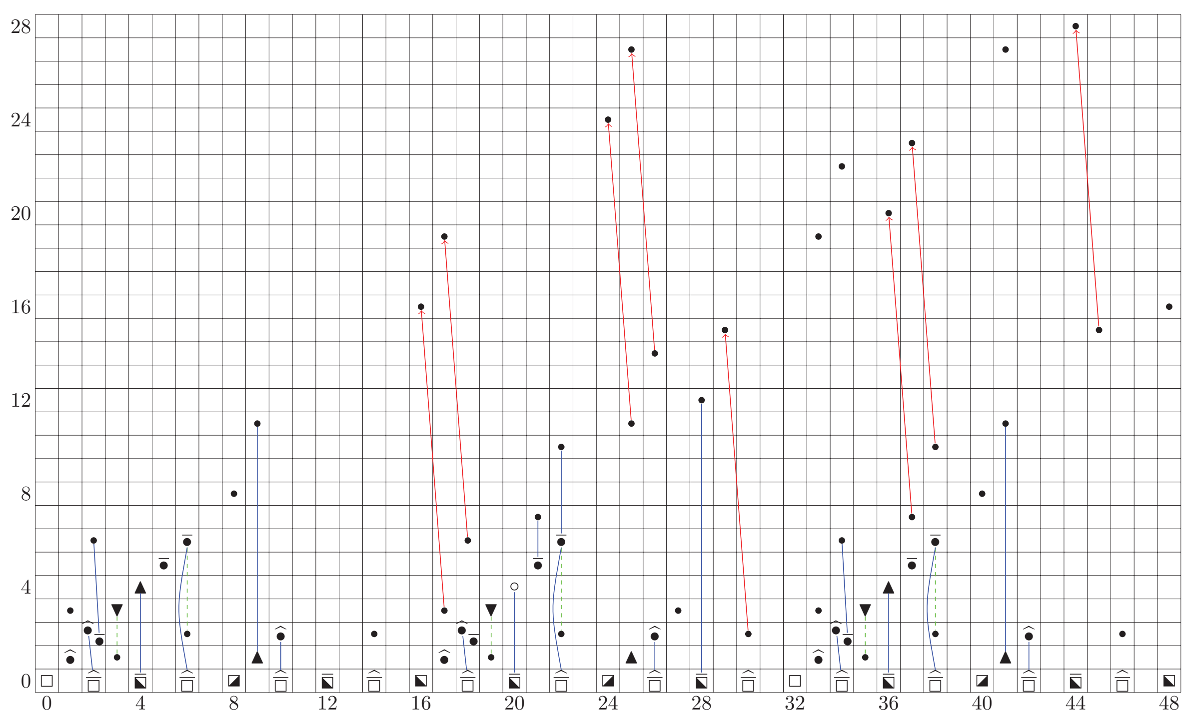


Figure 15. The $\underline{E}_{12}$-term of the slice spectral sequence for $k_{[2]}$ with elements of Proposition 13.4 removed. Differentials are shown in red. Exotic transfers and restrictions are shown as solid blue and dashed green lines respectively. The Mackey functor symbols are as in Table 2.

To finish the calculation we have

Theorem 14.5 (Exotic transfers from and restrictions to the 0-line). *In $\underline{\pi}_* k_{[2]}$, for $i \geq 0$ we have*

$$\begin{aligned}
\operatorname{tr}_1^2(r_{1,\epsilon} r_{1,0}^{4i} r_{1,1}^{4i}) &= \eta_\epsilon^2 \bar{r}_{1,0}^{4i} \bar{r}_{1,1}^{4i} && \in \underline{\pi}_{8i+2} && \text{(filtration jump 2)},\\
\operatorname{tr}_1^4(r_{1,0}^{8i+1} r_{1,1}^{8i+1}) &= 2x_4 \Delta_1^{4i} && \in \underline{\pi}_{32i+4} && \text{(filtration jump 4)},\\
\operatorname{tr}_1^2((r_{1,0}^3 + r_{1,1}^3) r_{1,0}^{8i} r_{1,1}^{8i}) &= \eta_0^3 \eta_1^3 \delta_1^{8i} && \in \underline{\pi}_{32i+6} && \text{(filtration jump 6)},\\
\operatorname{tr}_1^4(r_{1,0}^{8i+5} r_{1,1}^{8i+5}) &= 2x_4 \Delta_1^{4i+2} && \in \underline{\pi}_{32i+20} && \text{(filtration jump 4)},\\
\operatorname{tr}_1^2((r_{1,0}^3 + r_{1,1}^3) r_{1,0}^{8i+4} r_{1,1}^{8i+4}) &= \eta_0^3 \eta_1^3 \delta_1^{8i+4} && \in \underline{\pi}_{32i+22} && \text{(filtration jump 6)},\\
\operatorname{tr}_2^4(2\delta_1^{8i+7}) &= x_4^3 \Delta_1^{4i+2} && \in \underline{\pi}_{32i+28} && \text{(filtration jump 12, the long transfer)}.
\end{aligned}$$

Let $\underline{M}_k$ denote the reduced value of $\underline{\pi}_k k_{[2]}$, meaning the one obtained by removing the elements of Proposition 13.4. Its values are shown in purple in Figure 17, and each has at most two summands. For even k one of them contains torsion free elements, and we denote it by $\underline{M}'_k$. Its values depend on k mod 32 and are as follows, with symbols as in Table 2.

k	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$\underline{M}'_k$	□	$\hat{\dot{\square}}$	$\overset{\blacktriangledown}{⬕}$	$\boxed{\blacktriangle}$	◪	$\hat{\dot{\square}}$	$\overline{⬕}$	$\hat{\bar{\square}}$	⬕	$\hat{\dot{\square}}$	$\overset{\circ}{⬕}$	⧇	◪	$\hat{\dot{\square}}$	$\dot{⬕}$	$\hat{\bar{\square}}$

Proof. We have two tools at our disposal: the periodicity theorem and Theorem 4.4, which relates exotic transfers to differentials.

Figure 16 shows that $\underline{M}'_k$ has the indicated value for $-8 \leq k \leq 0$ because the same is true of $\underline{E}_4^{0,k}$ and there is no room for any exotic extensions. On the other hand $\underline{E}_4^{0,k+32}$ does not have the same value for $k = -8$, $k = -6$ and $k = -4$. This comparison via periodicity forces

- the indicated d_5 and d_7 in dimension 24, which together convert □ to ◪. These were also established in Theorem 14.2.
- the short transfer in dimension 26, which converts $\hat{\bar{\square}}$ to $\hat{\dot{\square}}$. It also follows from the results of Section 12.
- the long transfer in dimension 28, which converts $\overline{⬕}$ to $\dot{⬕}$.

  The differential corresponding to the long transfer is

$$d_{13}([2u_\lambda^7]) = a_\sigma a_\lambda^6 u_{2\sigma} u_\lambda^4 \bar{\mathfrak{d}}_1^3,$$

so

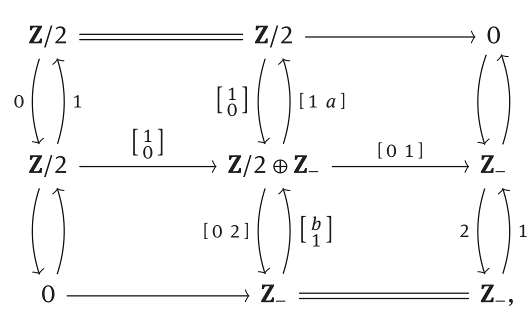
$$d_{13}(a_\sigma [2u_\lambda^7]) = a_\sigma^2 a_\lambda^6 u_{2\sigma} u_\lambda^4 \bar{\mathfrak{d}}_1^3 = 2a_\lambda^7 [u_{2\sigma}^2] u_\lambda^3 \bar{\mathfrak{d}}_1^3.$$

This compares well with the d_{13} of Theorem 11.10, namely

$$d_{13}(a_\sigma [u_\lambda^4]) = a_\lambda^7 [u_{2\sigma}^2] \bar{\mathfrak{d}}_1^3.$$

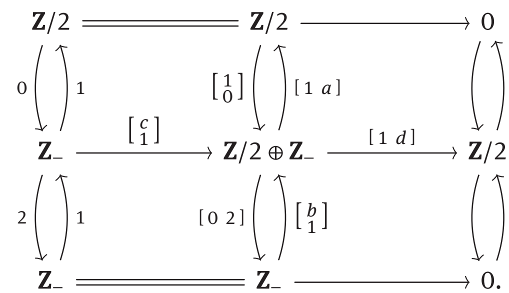
The statements in dimensions 4 and 20 have similar proofs, and we will only give the details for the former. It is based on comparing the $\underline{E}_4$-term for $K_{[2]}$ in dimensions -28 and 4. They must converge to the same thing by periodicity. From the slice $\underline{E}_4$-term in dimension 4 we see there is a short exact sequence

$$0 \longrightarrow \blacktriangledown \longrightarrow \underline{M}'_4 \longrightarrow \overline{⬕} \longrightarrow 0 \qquad (14.6)$$



while the (−28)-stem gives

$$0 \longrightarrow \dot{◪} \longrightarrow \underline{M}_4' \longrightarrow \bar{\bullet} \longrightarrow 0$$



The commutativity of the second diagram requires that

$$a + b = c = 1$$

and

$$b + d = c + d = 0,$$

giving

$$(a, b, c, d) = (0, 1, 1, 1).$$

The diagram for M_4 is that of $\overset{\blacktriangledown}{◪}$ in Table 2.

In dimension 20 the short exact sequence of (14.6) is replaced by

$$0 \to \circ \to \underline{M}_{20}' \to \overline{◪} \to 0$$

and the resulting diagram for $\underline{M}_{20}'$ is that of $\overset{\circ}{◪}$.

Similar arguments can be made in dimensions 6 and 22. □

We could prove a similar statement about exotic restrictions hitting the 0-line in the third quadrant in dimensions congruent to 0, 4, 6, 14, 16, 20 (where there is an exotic transfer) and 22. The problem is naming the elements involved.

In Table 4 we show short or 4-term exact sequences in the sixteen even-dimensional congruence classes. In each case the value of $\underline{M}_k'$ is the symbol appearing in both rows of the diagram. For even k with $0 \le k < 32$, we typically have short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{E}_4^{0,k-32} & \longrightarrow & \underline{M}_k' & \longrightarrow & \text{quotient} & \longrightarrow & 0 \\ & & & & \| & & & & \\ 0 & \longrightarrow & \text{subgroup} & \longrightarrow & \underline{M}_k' & \longrightarrow & \underline{E}_4^{0,k} & \longrightarrow & 0, \end{array}$$

where the quotient or subgroup is finite and may be spread over several filtrations. This happens for the quotient in dimensions −32, −16 and −12, and for the subgroup in dimensions 6 and 22.

This is the situation in dimensions where no differential hits [originates on] the 0-line in the third [first] quadrant. When such a differential occurs, we may need a 4-term sequence, such as the one in dimension −22.

In dimensions 8 and 24 there is more than one such differential, the targets being a quotient and subgroup of the Mackey functor $\circ = \square/◪$.

In dimension −18 we have a d_7 hitting the 0-line. Its source is written as $\circ \subseteq \underline{E}_4^{-7,-24}$ in Figure 16. Its generator supports a d_5, leaving a copy of $\blacktriangledown$ in $\underline{E}_7^{-7,-24}$.

There is no case in which we have such differentials in both the first and third quadrants.

Corollary 14.7 (The $\underline{E}_\infty$-term of the slice spectral sequence for $K_{[2]}$). *The surviving elements in the spectral sequence for $K_{[2]}$ are shown in Figure 17.*

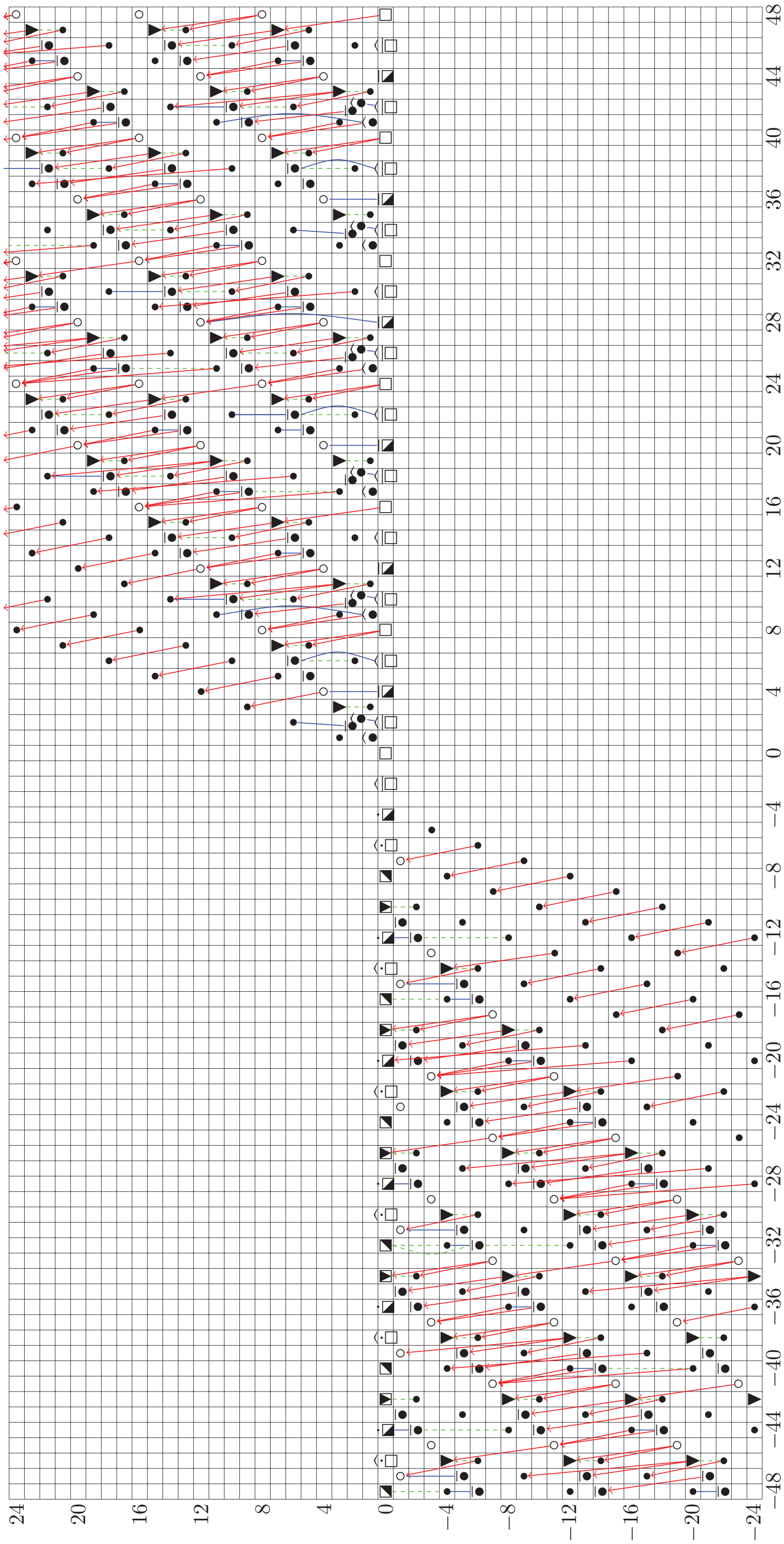


Figure 16. The reduced $\underline{E}_4$-term of the slice spectral sequence for the periodic spectrum $K_{[2]}$. Differentials are shown in red. Exotic transfers and restrictions are shown in solid blue and dashed green vertical lines respectively. The Mackey functor symbols are indicated in the table below Figure 17.

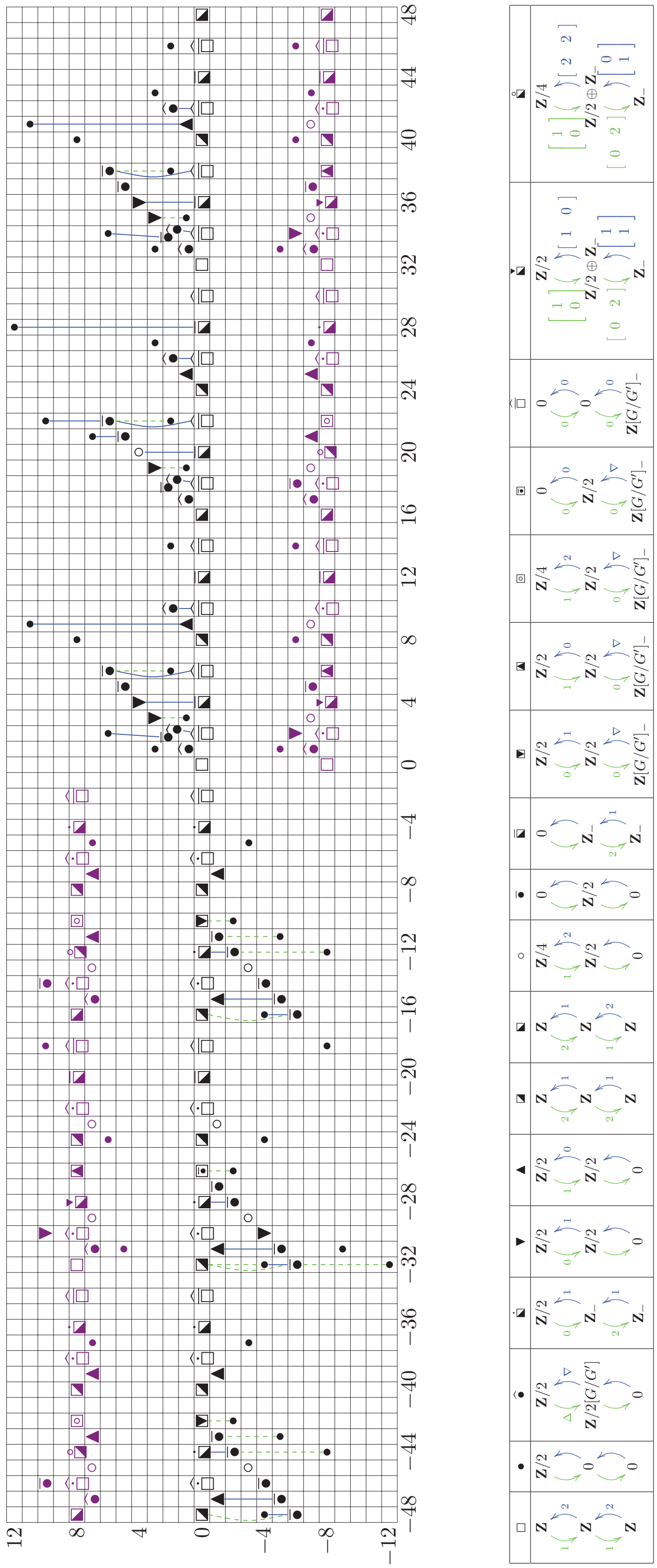


Figure 17. The reduced $\underline{E}_{14} = \underline{E}_{\infty}$-term of the slice spectral sequence for $K_{[2]}$. The exotic Mackey functor extensions lead to the Mackey functors shown in violet in the second and fourth quadrants. The Mackey functor symbols are indicated in the table on the right.

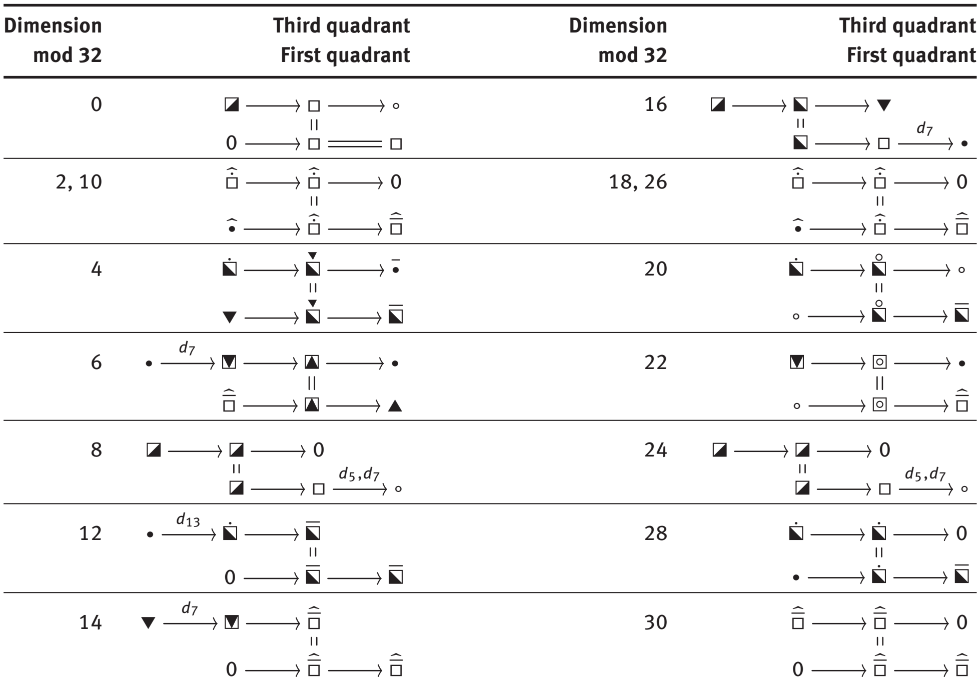


Table 4. Infinite Mackey functors in the reduced $\underline{E}_\infty$-term for $K_{[2]}$. In each even degree there is an infinite Mackey functor on the 0-line that is related to a summand of $\underline{\pi}_{2k}K_{[2]}$ in the manor indicated. The rows in each diagram are short or 4-term exact sequences with the summand appearing in both rows.

Funding: The authors were supported by DARPA Grant FA9550-07-1-0555 and NSF Grants DMS-0905160, DMS-1307896.

References

[1] J. F. Adams, *Stable Homotopy and Generalised Homology*, Chicago Lectures in Math., University of Chicago Press, Chicago, 1974.
[2] J. F. Adams, Prerequisites (on equivariant stable homotopy) for Carlsson's lecture, in: *Algebraic Topology* (Aarhus 1982), Lecture Notes in Math. 1051, Springer, Berlin (1984), 483–532.
[3] M. F. Atiyah, K-theory and reality, *Q. J. Math. Ser. (2)* **17** (1966), 367–386.
[4] D. Dugger, An Atiyah–Hirzebruch spectral sequence for KR-theory, *K-Theory* **35** (2005), no. 3–4, 213–256.
[5] J. P. C. Greenlees and J. P. May, Equivariant stable homotopy theory, in: *Handbook of Algebraic Topology*, North-Holland, Amsterdam (1995), 277–323.
[6] M. A. Hill, M. J. Hopkins and D. C. Ravenel, The non-existence of elements of Kervaire invariant one, preprint (2010), http://arxiv.org/abs/0908.3724v2.
[7] M. A. Hill, M. J. Hopkins and D. C. Ravenel, The slice spectral sequence for RO(C_{p^n})-graded suspensions of $H\underline{\mathbf{Z}}$ I, preprint (2016).
[8] L. G. Lewis, Jr., The $RO(G)$-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbf{Z}/p$ actions, in: *Algebraic Topology and Transformation Groups* (Göttingen 1987), Lecture Notes in Math. 1361, Springer, Berlin (1988), 53–122.
[9] D. C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, Pure App. Math. 121, Academic Press, Orlando, 1986.
[10] J. Thévenaz and P. Webb, The structure of Mackey functors, *Trans. Amer. Math. Soc.* **347** (1995), no. 6, 1865–1961.