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Akhmet'ev's 2008 paper on the Steenrod-Hopf invariant

The is a partial summary of Akhmet'ev's 2008 paper, "Geometric approach towards stable homotopy groups of spheres. The Steenrod-Hopf Invariant."

Let $n = 2^j - 1$ (he says $2^l - 1$). Given a codimension 1 immersion $f : M^{n-1} \looparrowright \mathbf{R}^n$, its stable Hopf invaraint is

$$h(f) = \langle w_1^{n-1}(M),\, [M^{n-1}]\rangle.$$

A codimension k skew framed immersion is a triple (f, κ, Ξ), where $f : M^{n-k} \looparrowright \mathbf{R}^n$ is an immersion, Ξ is an isomorphism between its normal bundle ν_f and $k\kappa$ for a line bundle κ. We will identify κ with its characteristic class in $H^1(M; \mathbf{Z}/2)$. Its Steenrod-Hopf invariant is

$$h(f, \kappa, \Xi) = \langle \kappa^{n-k},\, [M^{n-k}]\rangle.$$

One can get a codimension k skew framed immersion from a codimension 1 immersion (which is necessarily skew framed) by taking the dual of the $(k-1)$th power of κ in M^{n-1}. This operation does not change the Hopf invariant. Either way we get an element in the stable homotopy group $\pi_n(\mathbf{R}P^\infty)$, and the Hopf invariant is its mod 2 Hurewicz image. Using the Kahn-Priddy theorem one can show that the exience of an immersion with nontrivial Hopf invariant is equaivalent to the survival of h_j in the Adams spectral sequence.

The main theorem is that the Hopf invariant vanishes for $j \geq 8$ when

$$k = (n - 15)/2 = 2^{j-1} - 8,$$

which means h_j does not survive for $j \geq 8$.

From f as above one obtains an immersion $g : N^{2-2k} \looparrowright \mathbf{R}^n$ of the manifold of double points. The skew framing Ξ induces an isomorphism Ψ between its normal bundle ν_g and $k\eta$ for a 2-plane bundle η with structure group D_4, the dihedral group of order 8. D_4 has three subgroups of order 2, denoted by I_a, I_b and I_c. I_a is cyclic and I_c is associated with the canonical double covering of $i : \overline{N} \to N$; N by definition isa certain set of unordered pairs of distinct points in M, and $\overline{N}$ is the corresponding set of ordered pairs. The intersection of any two of these subgroups is the center I_d of order 2.

Much of the paper is concerned with conditions that allow one to reduce the structure from D_4 to its cyclic subgroup I_a. (In the Kervaire invariant paper, one wants to reduce to I_b.) Given a skew framed immersion (f, κ, Ξ) with double point manifold N^{n-2k}, a *cyclic structure* is a map $\mu : N^{n-2k} \to K(I_a, 1)$ with

$$\langle \mu^*(t),\, [N^{n-2k}]\rangle = h(f, \kappa, \Xi)$$

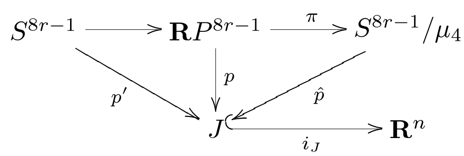
where t generates $H^{n-2k}(K(I_a, 1), \mathbf{Z}/2)$. A lemma attributed to joint work with Eccles in 1998 (but not appearing in a paper they wrote together that year) says that for $j \geq 5$ and $k = (n - 15)/2$, the existence of a cyclic structure implies the vanishing of $h(f, \kappa, \Xi)$. Lemma 4 says that for $j \geq 8$ any such immersion is regularly homotopic to one with a cyclic structure, which implies the main theorem.

To proceed further we need to replace immersions by the more general "generic maps." The defintion is not in the paper, but I learned it from private correspondence with Akhmet'ev. Recall that a map $f : M^{n-k} \to X^n$ of smooth manifolds is an immersion if it si smooth and the induced map of tangent bundles df is one to one at each point of M. More generally, let $\Sigma^i(f)$ (the Σ^i-singular points of f) be the set of points in M where the kernel of *df* is has dimension i. Under favorable circumstances this is a submanifold of M, and we can define

$$\Sigma^{i,j}(f) = \Sigma^j(f|\Sigma^i(f)),$$

the $\Sigma^{i,j}$-singular points of f. In a similar way one can define $\Sigma^{i,j,k}(f)$. In this paper "generic mapping" means that singularities are of types $\Sigma^{1,0}$, $\Sigma^{1,1,0}$ and self-intersections are generic.

Here is a relevant example. Let L^7 be the $\mathbf{Z}/4$-lens space S^7/μ_4, where μ_4 is the group of 4th roots of unity aacting by complex multiplication on the unit sphere in $\mathbf{C}^4$, and let J denote its r-fold join. Let $i_J : J \hookrightarrow \mathbf{R}^n$ be a PL-embedding. We have a diagram



in which the map from each manifold in the top row to $\mathbf{R}^n$ can be approximated by a generic one.

On page 4 he makes a definition that starts with

> Let $g : \mathbf{R}P^{n-k} \to \mathbf{R}^n$ be a generic mapping, $n \geq 3k$, $n = 2^j - 1$ with double point manifold N^{n-2k} and critical points $(n-2k-1)$-dimensional manifold $(\partial N)^{n-2k-1} \subset \mathbf{R}P^{n-k}$.
>
> Let $\eta : (N^{n-2k}, \partial N) \to (K(D_4, 1), K(I_b, 1))$ be the structured mapping corresponding to g. This structured mapping is defined analogously with the case of skew-framed immersions.

The inequality $n \geq 3k$ (which appears two more times) should be $n \leq 3k$. I do not know why the double point manifold should have a boundary or what he means by the map η. In any case he says a *cyclic structure for g* is a lifting of η to a map

$$\mu : (N^{n-2k}, \partial N) \to (K(I_a, 1), K(I_d, 1)),$$

where $I_a \subset D_4$ is the cyclic group of order 4 and $I_d \subset I_b \subset D_4$ is the center, satisfying a certain homological nontriviality condition.

Lemma 5 says that if there is a generic map $g : \mathbf{R}P^{n-k} \to \mathbf{R}^n$ with a cyclic structure with $n \leq 3k$ (here I have corrected his typo), then any skew-framed immersion (f, κ, Ξ) with $f : M^{n-k} \hookrightarrow \mathbf{R}^n$ is regularly homotopic to one with a cyclic structure. He says the idea of the proof is to show that f is regularly homotopic to a generic immersion f' that is a close approximation to $g\kappa$.

This means we need to find a generic $g : \mathbf{R}P^{n-k} \to \mathbf{R}^n$ with a cyclic structure for appropriate n and k. The source of this map is the diagram above with $n = 2^j - 1$ and $r = 1 + 2^{j-4}$, which means $k = 2^{j-1} - 8$. Then g is a generic approximation to $i_J p$. He says that the required PL-embedding of J exists for $j \geq 8$. *This is the source of the lower bound on j in the main theorem.* (He hints that this is proved in a 2004 paper of Melikhov, but the its AMS review makes it seem unlikely to be the source of this result.) He then spends 8 pages proving that this map has the desired cyclic structure. I have not been able to process the argument. According to Don Davis' online table, this real projective space does *not* immerse in $\mathbf{R}^n$ for $j \geq 5$.

After reading this, I asked him the following question. Suppose we replace J with a join J' of r' copies of S^1/μ_u, having dimension $2r'-1$. This space is the sphere $S^{2r'-1}$ (which of course embeds in $\mathbf{R}^{2r'}$) and there is a map to it from $\mathbf{R}P^{2r'-1}$ with the properties needed to get the cyclic structure. Would this work and would it lead to a better lower bound on j? He answered that he had used this construction in a more recent (Russian) version of the paper and that he could get the cyclic structure for any odd n and k divisible by 8. I think this means he can recover Adams' original Hopf invariant one theorem for $j \geq 4$.

Note that no use is made of the desuspension theorem needed in his approach to the Kervaire invariant. In other words, there is no need to take a map to some $\mathbf{R}P^m$ and factor it through a subspace.