Complex Cobordism and Stable Homotopy Groups of Spheres

Douglas C. Ravenel

Department of Mathematics, University of Rochester, Rochester, New York

To my wife, Michelle

Contents

List of Figures

List of Tables

Preface to the second edition

The subject of BP-theory has grown dramatically since the appearance of the first edition 17 years ago. One major development was the proof by Devinatz, Hopkins and Smith (see Devinatz, Hopkins and Smith [**1**] and Hopkins and Smith [**2**]) of nearly all the conjectures made in Ravenel [**8**]. An account of this work can be found in our book Ravenel [**13**]. The only conjecture of Ravenel [**8**] that remains is Telescope Conjecture. An account of our unsuccessful attempt to disprove it is given in Mahowald, Ravenel, and Shick [**1**].

Another big development is the emergence of elliptic cohomology and the theory of topological modular forms. There is still no comprehensive introduction to this topic. Some good papers to start with are Ando, Hopkins and Strickland [**1**], Hopkins and Mahowald [**1**], Landweber, Ravenel and Stong [**8**], and Rezk [**?**], which is an account of the still unpublished Hopkins-Miller theorem.

The seventh and final chapter of the book has been completely rewritten and is nearly twice as long as the original. We did this with an eye to carrying out future research in this area.

I am grateful to the many would be readers who urged me to republish this book and to the AMS for its assistance in getting the original manuscript retypeset. Peter Landweber was kind enough to provide me with a copious list of misprints he found in the first edition. Nori Minami and Igor Kriz helped in correcting some errors in § 4.3. Mike Hill and his fellow MIT students provided me with a timely list of typos in the online version of this edition. Hirofumi Nakai was very helpful in motivationg me to make the revisions of Chapter 7.

Preface to the first edition

My initial inclination was to call this book *The Music of the Spheres*, but I was dissuaded from doing so by my diligent publisher, who is ever mindful of the sensibilities of librarians. The purpose of this book is threefold: (i) to make BP-theory and the Adams–Novikov spectral sequence more accessible to nonexperts, (ii) to provide a convenient reference for workers in the field, and (iii) to demonstrate the computational potential of the indicated machinery for determining stable homotopy groups of spheres. The reader is presumed to have a working knowledge of algebraic topology and to be familiar with the basic concepts of homotopy theory. With this assumption the book is almost entirely self-contained, the major exceptions (e.g., Sections 5.4, 5.4, A1.4, and A1.5) being cases in which the proofs are long, technical, and adequately presented elsewhere.

The subject matter is a difficult one and this book will not change that fact. We hope that it will make it possible to learn the subject other than by the only practical method heretofore available, i.e., by numerous exhausting conversations with one of a handful of experts. Much of the material here has been previously published in journal articles too numerous to keep track of. However, a lot of the foundations of the subject, e.g., Chapter 2 and Appendix 1, have not been previously worked out in sufficient generality and the author found it surprisingly difficult to do so.

The reader (especially if she is a graduate student) should be warned that many portions of this volume contain more than he is likely to want or need to know. In view of (ii), results are given (e.g., in Sections 4.3, 6.3, and A1.4) in greater strengh than needed at present. We hope the newcomer to the field will not be discouraged by abundance of material.

The homotopy groups of spheres is a highly computational topic. The serious reader is strongly encouraged to reproduce and extend as many of the computations presented here as possible. There is no substitute for the insight gained by carrying out such calculations oneself.

Despite the large amount of information and techniques currently available, stable homotopy is still very mysterious. Each new computational breakthrough heightens our appreciation of the difficulty of the problem. The subject has a highly experimental character. One computes as many homotopy groups as possible with existing machinery, and the resulting data form the basis for new conjectures and new theorems, which may lead to better methods of computation. In contrast with physics, in this case the experimentalists who gather data and the theoreticians who interpret them are the same individuals.

The core of this volume is Chapters 2–6 while Chapter 1 is a casual nontechnical introduction to this material. Chapter 7 is a more technical description of actual computations of the Adams–Novikov spectral sequence for the stable homotopy groups of spheres through a large range of dimensions. Although it is likely to be read closely by only a few specialists, it is in some sense the justification for the rest of the book, the computational payoff. The results obtained there, along with some similar calculations of Tangora, are tabulated in Appendix 3.

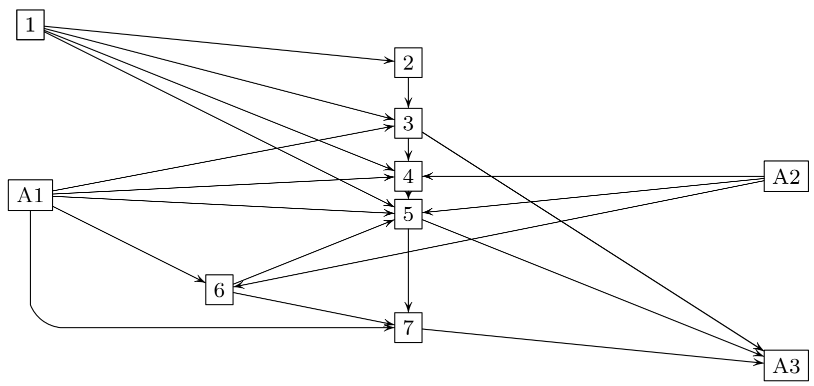
Appendices 1 and 2 are utilitarian in nature and describe technical tools used throughout the book. Appendix 1 develops the theory of Hopf algebroids (of which Hopf algebras are a special case) and useful homological tools such as relative injective resolutions, spectral sequences, Massey products, and algebraic Steenrod operations. It is not entertaining reading; we urge the reader to refer to it only when necessary.

Appendix 2 is a more enjoyable self-contained account of all that is needed from the theory of formal group laws. This material supports a bridge between stable homotopy theory and algebraic number theory. Certain results (e.g., the cohomology of some groups arising in number theory) are carried across this bridge in Chapter 6. The house they inhabit in homotopy theory, the chromatic spectral sequence, is built in Chapter 5.

The logical interdependence of the seven chapters and three appendixes is displayed in the accompanying diagram.

It is a pleasure to acknowledge help received from many sources in preparing this book. The author received invaluable editorial advice from Frank Adams, Peter May, David Pengelley, and Haynes Miller. Steven Mitchell, Austin Pearlman, and Bruce McQuistan made helpful comments on various stages of the manuscript, which owes its very existence to the patient work of innumerable typists at the University of Washington.

Finally, we acknowledge financial help from six sources: the National Science Foundation, the Alfred P. Sloan Foundation, the University of Washington, the Science Research Council of the United Kingdom, the Sonderforschungsbereich of Bonn, West Germany, and the Troisième Cycle of Bern, Switzerland.



Commonly Used Notations

$\mathbf{Z}$	Integers
$\mathbf{Z}_p$	p-agic integers
$\mathbf{Z}_{(p)}$	Integers localized at p
$\mathbf{Z}/(p)$	Integers mod p
$\mathbf{Q}$	Rationals
$\mathbf{Q}_p$	p-adic numbers
$P(x)$	Polynomial algebra on generators x
$E(x)$	Exterior algebra on generators x
$\square$	Cotensor product (Section A1.1)

Given suitable objects A, B, and C and a map $f\colon A \to B$, the evident map $A \otimes C \to B \otimes C$ is denoted by $f \otimes C$.