APPENDIX A3

Tables of Homotopy Groups of Spheres

The Adams spectral sequence for $p = 2$ below dimension 62. The Adams–Novikov spectral sequence for $p = 2$ below dimension 40. Comparison of Toda's, Tangora's and our notation at $p = 2$. 3-Primary stable homotopy excluding in J. 5-Primary stable homotopy excluding in J.

In this appendix we collect most of the known values of the stable homotopy groups of spheres for the primes 2, 3, and 5. Online graphic displays of these are given by Hatcher [**1**]. The results of Toda [**6**] on unstable homotopy groups are shown in Table A3.6. A table of unstable 3-primary homotopy groups up to dimension 80 can be found in Toda [**8**].

Extensive online charts of various Ext groups over the Steenrod algebra have been provided by Nassau [**1**] and Bruner [**3**].

In Figs. A3.1a–c we display the classical Adams E_2-term for $p = 2$,

$$\mathrm{Ext}_A^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

for $t - s \leq 61$, along the differentials and group extensions. The main reference for the calculation of Ext is Tangora [**1**], which includes a table showing the answer for $t - s \leq 70$. We use his notation for the many generators shown in Ext. His table is preceded by a dictionary (not included here) relating this notation to that of the May spectral sequence, which is his main computational tool.

In our table each basis element is indicated by a small circle. Multiplication by the elements h_0, h_1, and h_2 is indicated, respectively, by vertical lines and lines with slopes 1 and $\frac{1}{3}$. Most multiplicative generators are labeled, but there are a few unlabeled generators due to limitations of space. In each case the unlabeled generator is in the image of the periodicity operator P (denoted by Π in Section 3.4), which sends an element $x \in \mathrm{Ext}^{s,t}$ to the Massey product (Section A1.4)

$$\langle x, h_0^4, h_3 \rangle \in \mathrm{Ext}^{s+4,t+12}\,.$$

Differentials are indicated by lines with negative slope. For $t - s \leq 20$ these can be derived by combining the calculation of Ext in this range due to May [**1**] with the calculation of the corresponding homotopy groups by Toda [**6**]. For $21 \leq t - s \leq 45$ the results can be found in various papers by Barratt, Mahowald, Milgram, and Tangora and most recently in Bruner [**2**], where precise references to the earlier work can be found.

Differentials in the range $46 \leq t - s \leq 61$ have been computed (tentatively in some cases) by Mahowald (unpublished) and are included here with his kind permission.

Exotic group extensions and some exotic multiplications by h_1 and h_2 are indicated by broken lines with nonnegative slope.

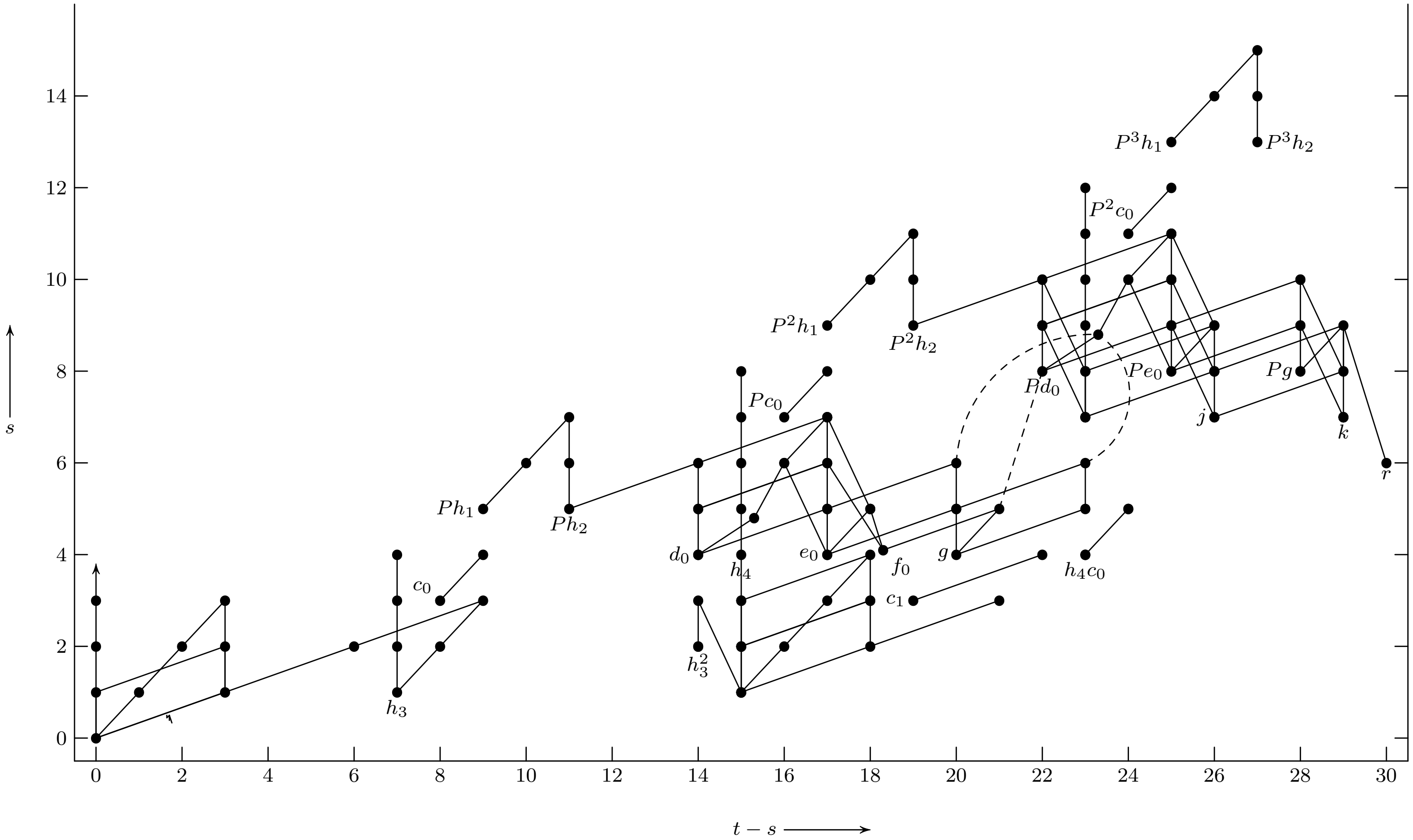
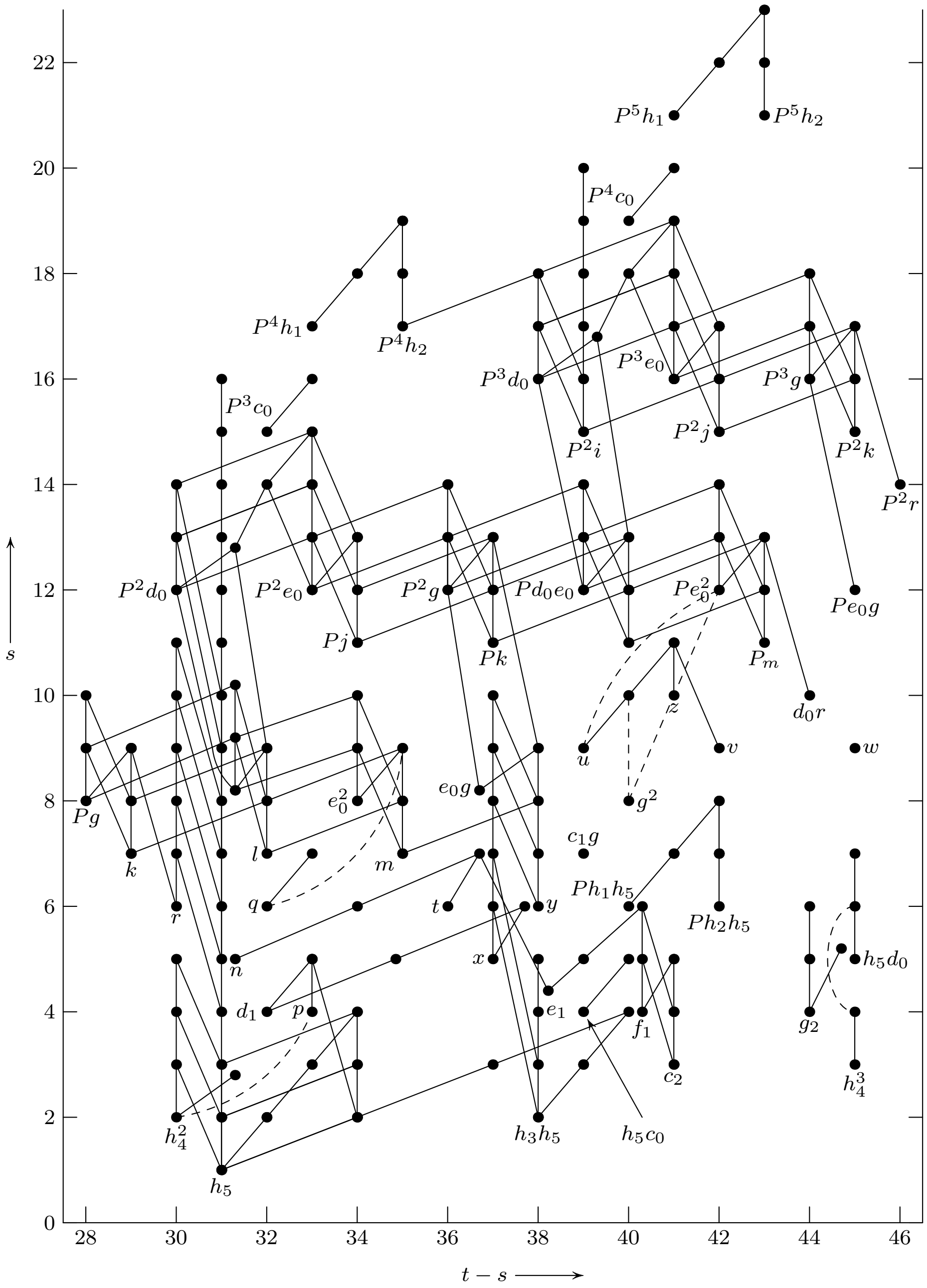


FIGURE A3.1a. The Adams spectral sequence for $p = 2$, $t - s \leq 29$.



$d_4(e_0d_0 + h_0^7h_5) = P^2d_0$ $h_4g = h_2d_1$ $x = \sigma\theta_4$ $h_1x = h_3u = c_1^2$ $h_3x = h_0^2g_2$

$\rho\theta_4 = h_0^2h_5d_0$

$\pi_{45}^S = \mathbf{Z}/(16) \otimes (\mathbf{Z}/(2))^3$

FIGURE A3.1 b. The Adams spectral sequence for $p = 2$, $28 \le t-s \le 45$

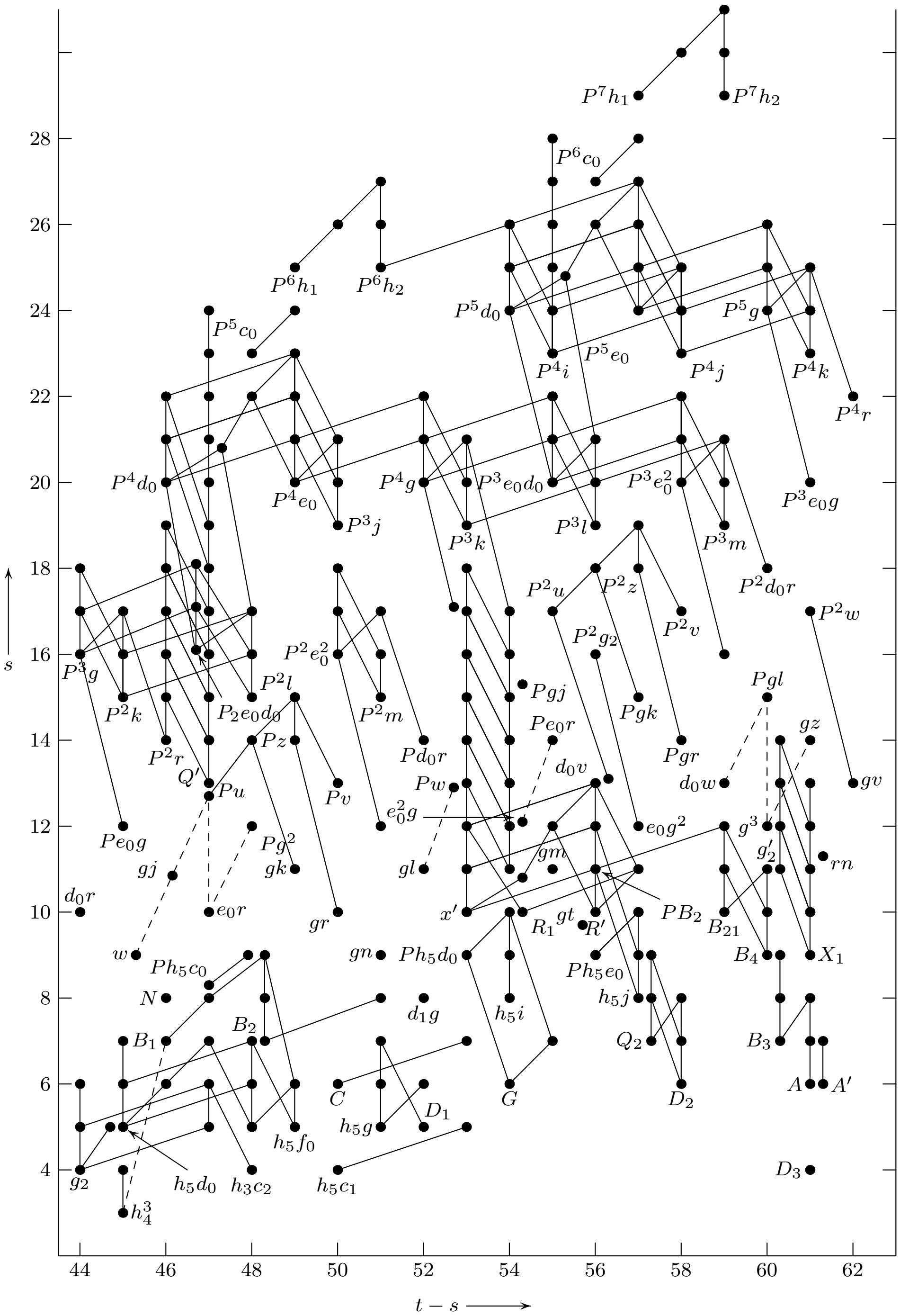


FIGURE A3.1c. The Adams spectral sequence for $p = 2$, $44 \leq t - s \leq 61$. (Differentials tentative)

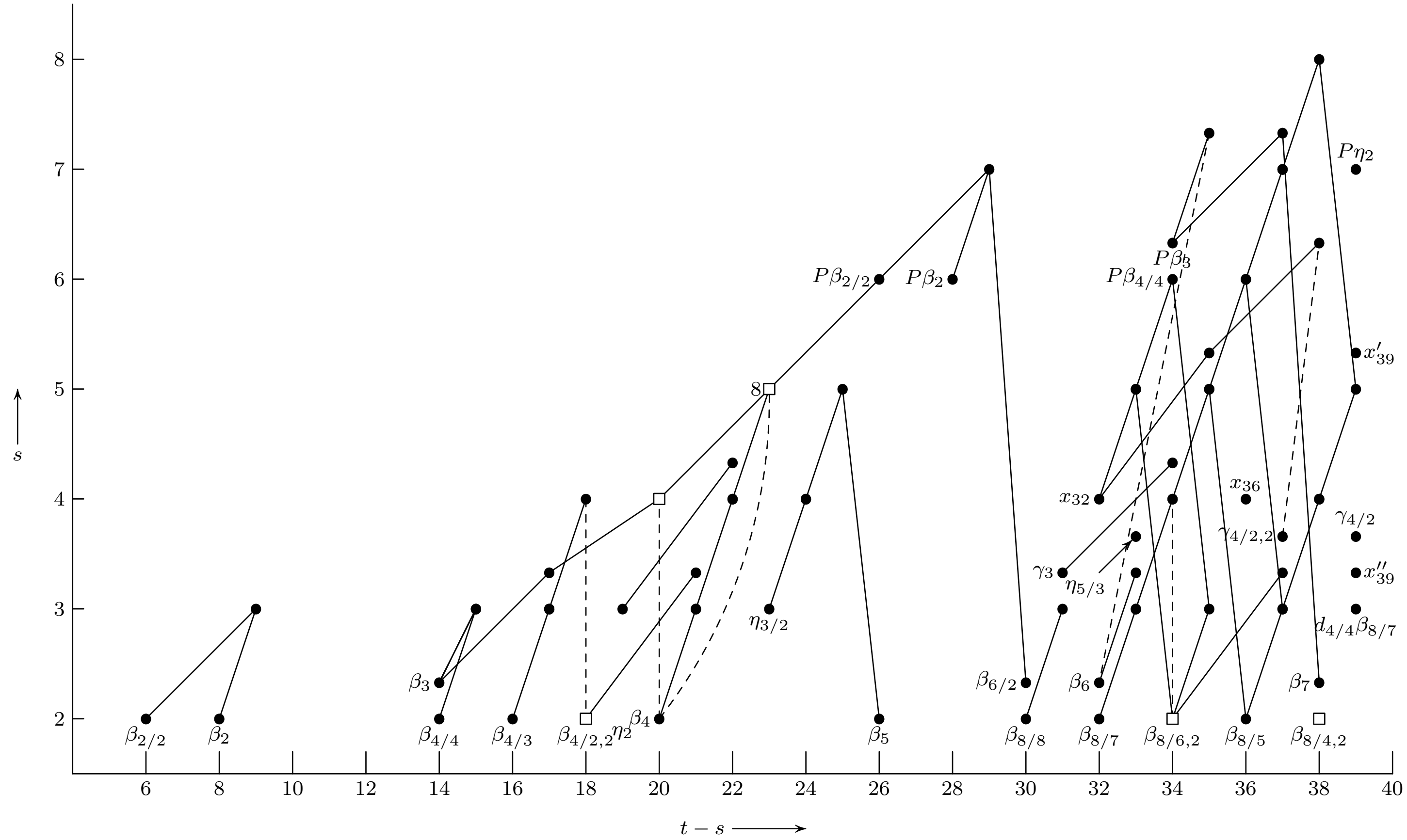


FIGURE A3.2. The Adams–Novikov spectral sequence for $p = 2$, $t - s \leq 39$. (v_1-periodic elements ommited. Computations for $t - s \leq 30$ are tentative.)

In FIG. A3.2 we display the Adams–Novikov E_2-term for $p = 2$ in the range $t-s \leq 39$. The method used is that of Section 4.4, where the calculation is described in detail through dimension 25. The small circles in the chart indicate summands of order 2. Larger cyclic summands are indicated by squares. All such summands in this range have order 4 except the one in $\mathrm{Ext}^{5,28}$, which has order 8. The solid and broken lines in this figure means the same thing as in Figs. A3.1a–c as described above. This figure does *not* include the v_1-periodic elements described in 5.3.7, i.e., the elements in the image of the J-homomorphism and the elements constructed in Adams [**1**].

In TABLE A3.3 we list the values of the 2-component of the stable stems π_k^s for $k \leq 45$, showing the name of each element given by Toda [**7**] (where applicable), by Tangora [**1**] in the Adams spectral sequence, and by us in the Adams–Novikov spectral sequence. Again we omit the v_1-periodic elements described in 5.3.7. These omitted summands are as follows.

$$\begin{array}{ll}
\mathbf{Z} & \text{for } k = 0, \\
\mathbf{Z}/(2) & \text{for } k = 1 \text{ or } 2, \\
\mathbf{Z}/(4) & \text{for } k = 3, \\
\mathbf{Z}/(2^{m+4}) & \text{for } k = 8t-1, \text{ where } t \text{ is an odd multiple of } 2^m, \\
\mathbf{Z}/(2) & \text{for } k \equiv 0 \text{ or } 2 \mod (8) \text{ and } k > 7, \\
(\mathbf{Z}/(2))^2 & \text{for } k \equiv 1 \mod (8) \text{ and } k > 7, \text{ and,} \\
\mathbf{Z}/(8) & \text{for } k \equiv 3 \mod (8) \text{ and } k > 7.
\end{array}$$

In TABLES A3.4 and A3.5 we do the same for the primes 3 and 5, recapitulating the results obtained in Sections 7.4 and 7.5, respectively. Again we omit the v_1-periodic elements described in 5.3.7, which in these cases are (in positive dimensions) precisely $\operatorname{im} J$, i.e.,

$$\begin{array}{ll}
\mathbf{Z} & \text{for } k = 0 \text{ and} \\
\mathbf{Z}/(p^{m+1}) & \text{for } k = (2p-2)t - 1,
\end{array}$$

where $t = sp^m$ and s is prime to p.

In FIG. A3.6 we reproduce the table of unstable homotopy groups of spheres through the 19-stem, given in Toda [**6**].

TABLE A3.3. π_*^S at $p = 2^a$

Stem	Toda's name	Tangora's name	Adams–Novikov name
6	ν^2	h_2^2	$\beta_{2/2}$
8	$\varepsilon = \langle \nu^2, 2, \eta \rangle$	c_0	β_2
9	ν^3	$h_1^2 h_3$	$\alpha_1 \beta_2$
14	σ^2	h_3^2	$\beta_{4/4}$
	κ	d_0	β_3
15	$\eta\kappa$	$h_1 d_0$	$\alpha_1 \beta_3 = \alpha_1 \beta_{4/4}$
16	$\eta^* \langle \sigma, 2\sigma, \eta \rangle$	$h_1 h_4$	$\beta_{4/3}$

TABLE A3.3 (continued)

Stem	Group	Tangora's name	Adams–Novikov name
17	$\eta\eta^*$	$h_1^2h_4$	$\alpha_1\beta_{4/3}$
	$\nu\kappa$	h_2d_0	$\alpha_{2/2}\beta_3 = \alpha_{2/2}\beta_{4/4}$
18	$\nu^*\langle\sigma, 2\sigma, \nu\rangle$	$h_2h_4, h_0h_2h_4$	$\beta_{4/2,2}$
		$h_0^2h_2h_4 = h_1^3h_4$	$4\beta_{4/2,2} = \alpha_1^2\beta_{4/3}$
19	$\bar{\sigma} = \langle\sigma^2 + \kappa, \eta, \nu\rangle$	c_1	η_2
20	$\bar{\kappa}$	g	β_4
	$2\bar{\kappa}$	h_0g	$2\beta_4 = x_{20} = \langle 2, \alpha_1^3, \beta_{4/3}\rangle$
	$4\bar{\kappa}$	h_0^2g	$2x_{20} = \alpha_{2/2}^2\beta_3$
21	σ^3	$h_2^2h_4$	$\alpha_{2/2}\beta_{4/2,2}$
	$\eta\bar{\kappa}$	h_1g	$\alpha_1\beta_4$
22	$\nu\bar{\sigma}$	h_2c_1	$\alpha_{2/2}\eta_2$
	$\eta^2\bar{\kappa}$	Pd_0	$\alpha_1^2\beta_4$
23	$\mathbf{Z}/2 \otimes \mathbf{Z}/8$	h_4c_0	$\eta_{3/2}$
		h_2g	$x_{23} = \langle\alpha_{2/2}, \alpha_1^3, \beta_{4/3}\rangle$
		h_0h_2g	$2x_{23}$
		Ph_1d_0	$4x_{23}$
24	$\mathbf{Z}/2$	$h_1h_4c_0$	$\alpha_1\eta_{3/2}$
26	$\mathbf{Z}/2$	h_2^2g	$\alpha_{2/2}x_{23}$
28	$\mathbf{Z}/2$	$Pg = d_0^2$	$x_{28} = \langle\beta_2, \alpha_1^3, \beta_{4/3}\rangle$
30	$\mathbf{Z}/2$	h_4^2	$\beta_{8/8}$
31	$(\mathbf{Z}/2)^2$	$h_1^2h_4$	$\alpha_1\beta_{8/8}$
		n	γ_3
32	$(\mathbf{Z}/2)^3$	h_1h_5	$\beta_{8/7}$
		d_1	$x_{32} = \langle\alpha_1, \beta_{4/4} + \beta_3,$ $\alpha_1, \beta_{4/4} + \beta_3\rangle$
		q	β_6
33	$(\mathbf{Z}/2)^3$	$h_1^2h_5$	$\alpha_1\beta_{8/7}$
		p	$\eta_{5/6}$
		h_1q	$\alpha_1\beta_3$
34	$\mathbf{Z}/4 \otimes (\mathbf{Z}/2)^2$	$h_0h_2h_5$	$\beta_{8/6}$
		$h_0^2h_2h_5 = h_1^3h_5$	$\alpha_1^2\beta_{8/7}$
		e_0^2	$\alpha_{2/2}\gamma_3$
		C_0^2	$x_{34} = \langle\beta_3, \alpha_1^3, \beta_{4/3}\rangle = P\beta_3$
35	$(\mathbf{Z}/2)^2$	h_2d_1	$\alpha_{2/2}x_{32}$
		$h_1e_0^2$	α_1x_{34}
36	$\mathbf{Z}/2$	t	$x_{36} = ?$
37	$(\mathbf{Z}/2)^2$	$h_2^2h_5$	$\alpha_{2/2}\beta_{8/6,2}$
		x	$\gamma_{4/2,2}$
38	$(\mathbf{Z}/4) \oplus \mathbf{Z}/2$	$h_0^2h_3h_5, h_0^3h_3h_5$	$\beta_{8/4,2}$
		h_1x	$\alpha_1\gamma_{4/2,2}$

TABLE A3.3 (continued)

Stem	Group	Tangora's name	Adams–Novikov name
39	$(\mathbf{Z}/2)^5$	$h_1h_3h_5$	$\alpha_{4/4}\beta_{8/7}$
		h_5c_0	$\gamma_{4/2}$
		h_1c_1	$x_{39} = \langle\alpha_1, \beta_{2/2}, \gamma_3\rangle$
		c_1g	$x'_{39} = \langle\eta_2, \alpha_1^3, \beta_{4/3}\rangle = P\eta_2$
		u	$x''_{39} = \langle\beta_{8/6}, a_1, \alpha_{2/2}\rangle$
40	$\mathbf{Z}/4 + (\mathbf{Z}/2)^4$	$h_1^2h_3h_5$	$\alpha_1\alpha_{4/4}\beta_{8/7}$
		f_1	x_{40}?
		$h_1h_5c_0$	$\beta_2\beta_{8/7} = \gamma_{4/2}\alpha_1$
		Ph_1h_5	$\beta_{8/3}$
		g^2	?
		h_1n	$\alpha_1x''_{39}$
41	$(\mathbf{Z}/2)^3$	h_1f_1	α_1x_{40}
		$Ph_1^2h_5$	α_1x_{40}
		z	?
42	$\mathbf{Z}/8 \oplus \mathbf{Z}/2$	$Ph_2h_5, Ph_0h_2h_5$	$\beta_{\alpha 2,2}$
		$Ph_0^2h_2h_5 = Ph_1^3h_5$	$4\beta_{8/2,2} = \alpha_1^2\beta_{8/3}$
		Pe_0^2	?
44	$\mathbf{Z}/8$	g_2	β_8?
		h_0g_2	
		$h_0^2g_2$	
45	$(\mathbf{Z}/16) \oplus (\mathbf{Z}/2)^3$	h_4^3	γ_4?
		$h_0h_4^3$	
		hg_2	$\alpha_1\beta_8$?
		h_5d_0	
		$h_0h_5d_0$	
		$h_0^2h_5d_0$	
		w	

[a] All element have order 2 unless otherwise indicated. (im J and μ_{8k+1}, μ_{8k+2} omitted.)

TABLE A3.4. 3-Primary Stable Homotopy Excluding im J[a]

Stem	Element
10	β_1
13	$\alpha_1\beta_1$
20	β_1^2
23	$\alpha_1\beta_1^2$
26	β_2
29	$\alpha_1\beta_2$
30	$\beta_1^3 = \langle\beta_2, 3, \alpha_1\rangle$
36	$\beta_1\beta_2$
37	$\langle\alpha_1, \alpha_1, \beta_1^3\rangle = \langle\beta_1, 3, \beta_2\rangle$
38	$\beta_{3/2} = \langle\alpha_1, \beta_1^3, 3, \alpha_1\rangle$
39	$\alpha_1\beta_1\beta_2$
40	β_1^4
42	β_3
45	$x_{45} = \langle\alpha_1, \alpha_1, \beta_{3/2}\rangle$ with $3x_{45} = \alpha_1\beta_3$
46	$\beta_1^2\beta_2$
47	$\langle\alpha_1, \alpha_1, \beta_1^4\rangle$
49	$\alpha_1\beta_1^2\beta_2$
50	β_1^5
52	$\beta_2^2 = \langle\alpha_1, \alpha_1, x_{45}\rangle$
55	$\alpha_1\beta_2^2$
62	$\beta_1\beta_2^2$
65	$\alpha_1\beta_1\beta_2^2$
68	$x_{68} = \langle\alpha_1, \beta_{3/2}, \beta_2\rangle$
72	$\beta_1^2\beta_2^2 = \langle\alpha_1, 3, x_{68}\rangle$
74	β_5
75	$x_{75} = \langle\alpha_1, \alpha_1, x_{68}\rangle = \langle\beta_1, \beta_{3/2,\beta_2\rangle}$ with $3x_{75} = \alpha_1\beta_1^2\beta_2^2$
78	$\beta_2^3 = \beta_1 x_{68}$
81	γ_2
	$x_{81} = \langle\alpha_1, \alpha_1, \beta_5\rangle$
82	$\beta_{6/3}$
84	$\alpha_1\gamma_2$
	$\beta_1\beta_5 = \alpha_1 x_{81}$
85	$\langle\alpha_1, \alpha_1, \beta_2^3\rangle = \beta_1\mu$
	$\alpha_1\beta_{6/3}$
86	$\beta_{6/2}$
90	β_6
91	$\beta_1\gamma_2$
	$\beta_1 x_{81}$
92	$\beta_1\beta_{6/3}$
	$x_{92} = \langle\beta_1, 3, \gamma_2\rangle$
93	$x_{93} = \langle\alpha_1, \alpha_1, \beta_{6/2\rangle}$ with $3x_{93} = \alpha_1\beta_6$
94	$\alpha_1\beta_1\gamma_2$
	$\beta_1^2\beta_5$
95	$\alpha_1\beta_1\beta_{6/3}$
99	$\langle\alpha_1, \alpha_1, x_{92}\rangle$
100	$\beta_2\beta_5$
101	$\beta_1^2\gamma_2$
	$\beta_1^2 x_{81}$
102	$\beta_1^2\beta_{6/3}$
	$\beta_1 x_{92}$
104	$\alpha_1\beta_1^2\gamma_2$
106	$x_{106} = \beta_7 \pm \beta_{9/9}$
107	$\gamma_2\beta_2$
	$\beta_2 x_{81}$
108	$\beta_2\beta_{6/3} = \langle\alpha_1, \alpha_1, \beta_1^2 x_{81}\rangle$

[a] (See 7.5.3 and subsequent discussion.) All elements have order 3 unless otherwise indicated.

TABLE A3.5. 5-Primary Stable Homotopy Excluding im J

Stem	Element	Stem	Element
38	β_1	255	$\alpha_1\beta_1^3\beta_3$
45	$\alpha_1\beta_1$	258	$\beta_1^2\beta_4$
76	β_1^2	265	$\alpha_1\beta_1^2\beta_4$
83	$\alpha_1\beta_1^2$	266	β_1^7
86	β_2	268	$\beta_2\beta_4$ with $\beta_1\beta_5 = 0$
93	$\alpha_1\beta_2$	275	$\alpha_1\beta_2\beta_4$
114	β_1^3	278	β_6
121	$\alpha_1\beta_1^3$	281	$\underline{2}\beta_1^7$
124	$\beta_1\beta_2$	285	$\alpha_1\beta_6$
131	$\alpha_1\beta_1\beta_2$	286	$\beta_1^4\beta_3$
134	β_3	293	$\alpha_1\beta_1^4\beta_3$
141	$\alpha_1\beta_3$	296	$\beta_1^3\beta_4$
152	β_1^4	303	$\alpha_1\beta_1^3\beta_4$
159	$\alpha_1\beta_1^4$	304	β_1^8
162	$\beta_1^2\beta_2$	306	$\beta_1\beta_2\beta_4$
169	$\alpha_1\beta_1^2\beta_2$	313	$\alpha_1\beta_1\beta_2\beta_4$
172	$\beta_1\beta_3$	316	$\beta_1\beta_6$
179	$\alpha_1\beta_1\beta_3$	319	$\underline{2}\beta_1^8$
182	β_4	326	β_7
189	$\alpha_1\beta_4 = \gamma_1$	331	$\underline{2}\beta_1\beta_6$
190	β_1^5	333	$\alpha_1\beta_7$
200	$\beta_1^3\beta_2$	334	$\beta_1^4\beta_4$
205	$\underline{2}\beta_1^5 = \langle\alpha_1, \alpha_1, \beta_1^5\rangle$	341	$\alpha_1\beta_1^4\beta_4$
206	$\beta_{5/4} = \langle\alpha_1, \beta_1^5, 5, \alpha_1\rangle$	342	β_1^9
207	$\alpha_1\beta_1^3\beta_2$	344	$\beta_1^2\beta_2\beta_4$
210	$\beta_1^2\beta_3$	351	$\alpha_1\beta_1^2\beta_2\beta_4$
213	$\alpha_1\beta_{5/4}$	354	$\beta_1^2\beta_6$
214	$\beta_{5/3}$	357	$\underline{2}\beta_1^9$
217	$\alpha_1\beta_1^2\beta_3$	364	$\beta_1\beta_7$
220	$\beta_1\beta_4$	369	$\underline{2}\beta_1^2\beta_6$
221	$\alpha_1\beta_{5/3}$	374	β_8
222	$\beta_{5/2}$	379	$\underline{\beta_1}\beta_7$
227	$\alpha_1\beta_1\beta_4$	380	β_1^{10}
228	β_1^6	381	$\alpha_1\beta_8$
230	β_5	382	$\beta_1^3\beta_2\beta_4$
237	$\underline{2}\beta_{5/2}$ with $5(\underline{2}\beta_{5/2}) = \alpha_1\beta_5$	389	$\alpha_1\beta_1^3\beta_2\beta_4$
238	$\beta_1^4\beta_2$	392	$\beta_1^3\beta_6$
243	$\underline{2}\beta_1^6$	402	$\beta_1^2\beta_7$
245	$\alpha_1\beta_1^4\beta_2$	403	$\underline{3}\beta_1^{10}$
248	$\beta_1^3\beta_3$	404	$x_{404} = \langle\alpha_1\beta_1^4, \beta_1, \beta_{5/4}\rangle$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
407	$\underline{2}\beta_1^3\beta_6$	491	$\underline{2}\,\underline{5}\gamma_2 = \langle\beta_1\alpha_2, \gamma_2\rangle$
411	$\alpha_1 x_{404} = \beta_{5/4}\underline{2}\beta_1^5$		$= \langle\alpha_1\beta_1, \alpha_1, 5, \gamma_2\rangle$
412	$\beta_1\beta_8$	493	$\underline{2}\beta_1^4\beta_7$
	$x_{412} = \beta_1\beta_8 + \beta_{5/4}^2$	494	β_1^{13}
417	$\underline{2}\beta_1^2\beta_7$	498	$\beta_1^2\beta_9$
418	β_1^{11}	503	$\underline{2}\beta_1^3\beta_8$
419	$\alpha_1\beta_1\beta_8$ with $\alpha_1 x_{412} = 0$	508	$\beta_2\beta_9$ with $\beta_1\beta_{10} = 0$
420	$\beta_1^4\beta_2\beta_4 = \langle\alpha_1, 5, x_{412}\rangle$	513	$\beta_1^2\gamma_2$
422	β_9		$\underline{2}\beta_1^2\beta_9$
427	$\underline{2}x_{412}$ with	514	$\beta_1^2\beta_{10/5}$
	$5(\underline{2}x_{412}) = \alpha_1\beta_1^4\beta_2\beta_4$	517	$\underline{3}\beta_1^{13}$
430	$\beta_1^4\beta_6$	518	β_{11}
437	$\underline{2}\beta_9$	520	$\alpha_1\beta_1^2\gamma_2$
	γ_2	523	$\underline{2}\beta_2\beta_9$
438	$\beta_{10/5}$		$\beta_1\gamma_2$
440	$\beta_1^3\beta_7$	524	$\beta_2\beta_{10/5}$
441	$\underline{3}\beta_1^{11}$	525	$\alpha_1\beta_{11}$
444	$\alpha_1\gamma_2$	526	$\beta_1^4\beta_8$
445	$\alpha_1\beta_{10/5}$	529	$\underline{2}\,\underline{5}\beta_1\gamma_2$
	$\underline{2}\beta_1^4\beta_6$	530	$\alpha_1\beta_2\gamma_2$
446	$\beta_{10/4}$	531	$\beta_7\underline{2}\beta_1^5 = \alpha_1\beta_2\beta_{10/5}$
450	$\beta_1^2\beta_8$ with $\beta_1\beta_{5/4}^2 = 0$	532	β_1^{14}
453	$\alpha_1\beta_{10/4}$	536	$\beta_1^3\beta_9$
454	$\beta_{10/3}$	541	$\underline{2}\beta_1^4\beta_8$
455	$\underline{2}\beta_1^3\beta_7$	546	$\beta_1\beta_2\beta_9$
456	β_1^{12}	551	$\beta_1^3\gamma_2$
460	$\beta_1\beta_9$		$\underline{2}\beta_1^3\beta_9$
461	$\alpha_1\beta_{10/3}$	552	$\beta_1^2\underline{5}\gamma_2$
462	$\beta_{10/2}$	555	$\underline{3}\beta_1^{14}$
465	$\underline{2}\beta_1^2\beta_8$	556	$\beta_1\beta_{11}$
470	β_{10}	558	$\alpha_1\beta_1^3\gamma_2$
475	$\beta_1\gamma_2$	561	$\beta_1\underline{2}\beta_2\beta_9$ with $\beta_1\beta_2\gamma_2 =? \beta_1\underline{2}\beta_2\beta_9$
476	$\beta_1\beta_{10/5} = \langle\alpha_1, \beta_1\beta_6, \beta_1^4\rangle$	566	β_{12}
	$\underline{5}\gamma_2 = \langle\beta_1, 5, \gamma_2\rangle$	567	$\beta_1^2\underline{2}\,\underline{5}\gamma_2$
477	$\underline{2}\beta_{10/2}$ with $5(\underline{2}\beta_{10/2}) = \alpha_1\beta_{10}$	570	β_1^{15}
478	$\beta_1^4\beta_7$	571	$\underline{2}\beta_1\beta_{11}$
479	$\underline{3}\beta_1^{12}$	572	$\beta_3\beta_{10/5}$
482	$\alpha_1\beta_1\gamma_2$	573	$\alpha_1\beta_{12}$
483	$\alpha_1\beta_1\beta_{10/5}$	574	$\beta_1^4\beta_9$
488	$\beta_1^3\beta_8$	579	$\beta_9\underline{2}\beta_1^5 = \alpha_1\beta_3\beta_{10/5}$

Table A3.5 (continued)

Stem	Element	Stem	Element
583	$\beta_1^2\beta_2\beta_9$	659	$\alpha_1\beta_1\beta_{13}$
	$\beta_1^4\gamma_2$		with $\alpha_1 x_{652} = 0$
	$\underline{2}\beta_1^4\beta_9$	660	$\beta_1^4\beta_2\beta_9$
590	$\beta_1^3\underline{5}\gamma_2$	662	β_{14}
594	$\beta_1^2\beta_{11}$	665	$\underline{3}\beta_1^2\beta_{12}$
596	$\alpha_1\beta_1^4\gamma_2$		$\beta_1^6\gamma_2$
599	$\beta_1^2\underline{2}\beta_2\beta_9$	666	$\beta_1^5\underline{5}\gamma_2$
601	$\underline{4}\beta_1^{15}$	667	$\underline{2}x_{652}$
602	$x_{602} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{5/4}\rangle$	670	$\beta_1^4\beta_{11}$
604	$\beta_1\beta_{12}$	675	$\underline{2}\beta_1^4\beta_2\beta_9$
605	$\beta_1^3\underline{2}\,\underline{5}\gamma_2$	677	$\underline{2}\beta_{14}$
608	β_1^{16}		$\underline{4}\beta_1^{17}$
609	$\alpha_1 x_{602}$ with $\underline{2}\beta_1^2\beta_{11} =? \alpha_1 x_{602}$	678	$\beta_{15/5}$
610	$\beta_1\beta_3\beta_{10/5} = \langle \alpha_1, 5, x_{602}\rangle$	680	$\beta_1^3\beta_{12}$
614	β_{13}		$\underline{2}\beta_1^6 5\gamma_2$
617	$x_{617} = \langle \alpha_1, (\alpha_1\underline{2}\beta_2\beta_6), \binom{x_{602}}{\beta_6}\rangle$	685	$\alpha_1\beta_{15/5}$
	with	686	$\beta_{15/4}$
	$5x_{617} = \alpha_1\beta_1\beta_3\beta_{10/5}$	689	$\underline{3}\beta_1^5\underline{5}\gamma_2$
620	$\beta_4\beta_{10/5}$	690	$\beta_1^2\beta_{13}$
621	$\alpha_1\beta_{13}$	692	$x_{692} = \langle \alpha_1, \beta_1^5, \beta_1^{13}\rangle$
622	$\beta_1^3\beta_2\beta_9$	693	$\alpha_1\beta_{15/4}$
627	$\underline{3}\beta_1\beta_{12}$		$\underline{3}\beta_1^4\beta_{11}$
	$\beta_1^5\gamma_2$	694	$\beta_{15/3}$
628	$\beta_1^4\underline{5}\gamma_2$	700	$\beta_1\beta_{14}$
632	$\beta_1^3\beta_{11}$	701	$\alpha_1\beta_{15/3}$
635	$\underline{2}\beta_4\beta_{10/5}$	702	$\beta_{15/2}$
636	$x_{636} = \langle \beta_1^3, \alpha_1\beta_1^2, \beta_{10/5}\rangle$	703	$\underline{3}\beta_1^3\beta_{12}$
	$= \langle \beta_{5/4}, \beta_9, \alpha_1\rangle$		$\beta_1^7\gamma_2$
637	$\underline{2}\beta_1^3\beta_2\beta_9$	704	$\beta_1^6\underline{5}\gamma_2$
639	$\underline{4}\beta_1^{16}$	710	β_{15}
642	$\beta_1^2\beta_{12}$	713	$\underline{\beta_1^2}\beta_{13}$
	$\underline{2}\beta_1^5\gamma_2$	714	$x_{714} = \langle \beta_1^4, \beta_1\beta_2, \gamma_2 + \underline{2}\beta_9\rangle$
643	$\underline{2}\,\underline{5}\beta_1^4\gamma_2 = \beta_{5/4}\gamma_2$	715	$\underline{2}\beta_1\beta_{14}$
	$\alpha_1 x_{636}$		$\underline{3}x_{692}$
644	$\beta_{5/4}\beta_{10/5}$	716	$\beta_1\beta_{15/5}$
646	β_1^{17}	717	$\underline{2}\beta_{15/2}$ with
651	$\alpha_1\beta_{5/4}\beta_{10/5}$		$5(\underline{2}\beta_{15/2}) = \alpha_1\beta_{15}$
652	$\beta_1\beta_{13}$	718	$\beta_1^4\beta_{12}$
	$\beta_{5/3}\beta_{10/5} + \beta_1\beta_{13} = x_{652}$		$\underline{2}\beta_1^7\gamma_2$
655	$\underline{3}\beta_1^3\beta_{11}$	721	$\alpha_1 x_{714}$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
723	$\alpha_1\beta_1\beta_{15/5}$	786	$\beta_1\beta_2\beta_{14}$
724	$x_{724} = \langle\beta_1, 5, \beta_1, \beta_1^{17}\rangle$	789	$\underline{3}\beta_1^4\beta_{13}$
	$x'_{724} = \langle\beta_1^2, \beta_1^2\beta_{11}, \alpha_1\alpha_1\rangle$	794	$\underline{2}\beta_1^9\gamma_2$
	with $\beta_1\beta_{15/4} = 0$	796	$\beta_1\beta_{16}$
727	$\underline{3}\beta_1^6\underline{5}\gamma_2$	799	$\underline{3}\beta_1^3\beta_{14}$
728	$\beta_1^3\beta_{13}$		$\beta_1 x_{761}$ with
730	$\beta_1 x_{692}$		$\underline{4}\beta_1^2 x_{692} = ?$
731	$\alpha_1 x'_{724}$ with	800	$\beta_1^2 x_{724}$
	$\alpha_1 x_{724} = 0$		$\beta_1^2 x'_{724}$
738	$\beta_1^2\beta_{14}$	803	$\underline{3}\beta_1^8\underline{5}\gamma_2$
739	$\underline{2}x_{724}$	806	β_{17}
741	$\underline{3}\beta_1^4\beta_{12}$		$\alpha_1\beta_1 x_{761} = \beta_1^3 x_{692}$
	$\beta_1^8\gamma_2$	807	$\alpha_1\beta_1^2 x'_{724}$
742	$\beta_1^7\underline{5}\gamma_2$	809	$\beta_1 x_{771}$
748	$\beta_2\beta_{14}$ with	810	$\beta_2 x'_{724}$
	$\beta_1\beta_{15} = 0$	811	$\underline{2}\beta_1\beta_{16}$
751	$\underline{3}\beta_1^3\beta_{13}$	812	$\beta_3\beta_{15/5}$
753	$\underline{3}\beta_1 x_{692}$	813	$\alpha_1\beta_{17}$
756	$\underline{2}\beta_1^8\gamma_2$	814	$\beta_1^4\beta_{14}$
	with $\beta_1^5\beta_{12} = 0$	815	$\underline{2}\beta_1^2 x_{724}$
758	β_{16}	816	$\alpha_1\beta_1 x_{771}$
761	$x_{761} = \langle\beta_3, \gamma_1, \gamma_2\rangle$	817	$\beta_1^{10}\gamma_2$
	$\underline{3}\beta_1\beta_{14}$		$\underline{4}\beta_1\beta_2\beta_{14}$ with
762	$\beta_1 x_{724}$		$\underline{2}\beta_2\beta_{15/5} = 0$
	$\beta_1 x'_{724}$	818	$\beta_1^{10}\beta_{10/5}$
763	$\underline{2}\beta_2\beta_{14}$	824	$\beta_1^2\beta_1\beta_{14}$
764	$\beta_2\beta_{15/5}$	825	$\underline{2}\beta_2 x'_{724}$
765	$\alpha_1\beta_{16}$	826	$x_{826} = \langle\alpha_1, \beta_1^5, \alpha_1\beta_4, \beta_{10/5}\rangle$
	$\underline{3}\beta_1^7\underline{5}\gamma_2$	827	$\underline{2}\beta_3\beta_{15/5}$
766	$\beta_1^4\beta_{13}$	833	$\alpha_1 x_{826}$
768	$\alpha_1 x_{761} = \gamma_2\underline{2}\beta_1\beta_6$	834	$\beta_1^2\beta_{16}$
	$\beta_1^2 x_{692}$		$x_{834} = \langle\beta_1^4, \underline{2}\beta_1^6, \beta_{10/5}\rangle$
769	$\alpha_1\beta_1 x'_{724}$	837	$\underline{3}\beta_1^4\beta_{14}$
771	$x_{771} = \langle\beta_2, \beta_1^5, \beta_1^{13}\rangle$		$\beta_1^2 x_{761}$
776	$\beta_1^3\beta_{14}$	838	$\beta_1^3 x_{724}$ with
777	$\underline{2}\beta_1 x_{724}$		$\beta_1^3 x'_{724} = 0$
778	$\alpha_1 x_{771}$	840	$\underline{3}\beta_1^{10}\gamma_2$
779	$\beta_1^9\gamma_2$	841	$\alpha_1 x_{834}$
	$\underline{2}\beta_2\beta_{15/5}$		$\underline{3}\beta_1^{10}\beta_{10/5}$
780	$\beta_1^8\underline{5}\gamma_2$	842	$x_{842} = \langle\underline{2}\beta_1^9, \beta_1, \beta_{10/4}\rangle$

Table A3.5 (continued)

Stem	Element	Stem	Element
844	$\alpha_1\beta_1^2 x_{761}$	894	$\beta_1^{12}\beta_{10/9}$
	$\beta_1\beta_{17}$	899	$\alpha_1\beta_1\beta_{18}$ with
847	$\beta_1^2 x_{771}$		$\quad\alpha_1 x_{892}=0$
849	$\alpha_1 x_{842}=\underline{2}\beta_1^2\beta_{16}$	900	$\beta_1^4\beta_2\beta_{14}$
850	$\beta_1\beta_3\beta_{15/5}$	902	β_{19}
853	$\underline{2}\beta_1^3 x_{724}$	903	$\underline{4}\beta_1^3\beta_{16}$
854	β_{18}	905	$\underline{3}\beta_1^2\beta_{17}$
	$\alpha_1\beta_1^2 x_{771}$	906	$\beta_1 x_{868}$
855	$\beta_1^{11}\gamma_2$	907	$\underline{2}x_{892}$
856	$\beta_1^{11}\beta_{10/5}$ with	910	$\beta_1^4\beta_{16}$
	$\quad\underline{4}\,\underline{2}x_{810}=0$	913	$\alpha_1\beta_1 x_{868}$
	$\underline{4}\beta_1^2\beta_1\beta_{14}$		$\beta_1^4 x_{761}$ with
857	$x_{857}=\langle\alpha_1(\alpha_1\underline{2}\beta_1\beta_6),\binom{x_{842}}{\beta_{11}}\rangle$		$\quad\underline{2}\beta_1\beta_4\beta_{15/5}=0$
860	$\beta_4\beta_{15/5}$	914	$\beta_1^5 x_{724}$ with
861	$\alpha_1\beta_{18}$		$\quad\beta_1\beta_{10/5}^2=0$
862	$\beta_1^3\beta_2\beta_{14}$	916	$\underline{\beta}_1^{12}\gamma_2$
865	$\underline{2}\beta_1\beta_3\beta_{15/5}$	917	$\underline{2}\beta_{19}$
867	$\underline{3}\beta_1\beta_{17}$	918	$\beta_{20/5}$
868	x_{868} (see 7.6.5)	920	$\beta_1^3\beta_{17}$
872	$\beta_1^3\beta_{16}$ with $\beta_1 x_{834}=0$	923	$\beta_1^4 x_{771}$
875	$\alpha_1 x_{868}$	925	$\alpha_1\beta_{20/5}$
	$\underline{2}\beta_4\beta_{15/5}$		$\underline{4}\beta_1^{12}\beta_{10/5}$
	$\beta_1^3 x_{761}=\beta_{10/5}\gamma_2$	926	$\beta_{20/4}$
876	$\beta_{10/5}^2$	928	$\underline{2}\beta_1^4 x_{761}$
	$\beta_1^4 x_{724}$	930	$\alpha_1\beta_1^4 x_{771}$
878	$\underline{3}\beta_1^{11}\gamma_2$		$\beta_1^2\beta_{18}$ with
882	$\beta_1^2\beta_{17}$ with		$\quad\beta_1 x_{892}=0$
	$\quad\alpha_1\beta_1^3 x_{761}=0$	931	$\beta_1^{13}\gamma_2$
883	$\alpha_1\beta_{10/5}^2$		$\underline{4}\beta_1^4\beta_2\beta_{14}$
884	$\beta_{5/4}\beta_{15/5}$	932	$\beta_1^{13}\beta_{10/5}$?
885	$\beta_1^3 x_{771}$	933	γ_4?
887	$\underline{4}\beta_1^{11}\beta_{10/5}$		$\alpha_1\beta_{20/4}$
890	$\underline{2}\beta_1^3 x_{761}$	934	$\beta_{20/3}$
891	$\underline{2}\beta_1^4 x_{724}$	937	$\underline{3}\beta_1^5 x_{724}$ with
	$\alpha_1\beta_{5/4}\beta_{15/3}$		$\quad\alpha_1\beta_1\beta_{10/4}^2=0$
892	$\beta_1\beta_{18}$	940	$\alpha_1\gamma_4$
	$\alpha_1\beta_1^3 x_{771}$		$\beta_1\beta_{19}$
	$x_{892}=\beta_{10/4}^2+\beta_1\beta_{18}$	941	$\alpha_1\beta_{20/3}$
893	$\beta_1^{12}\gamma_2$		$\underline{4}\beta_1^4\beta_{16}$
	$\underline{4}\beta_1^3\beta_2\beta_{14}$	942	$\beta_{20/2}$

TABLE A3.5 (continued)

Stem	Element	Stem	Element
950	β_{20}	969	$\beta_1^{14}\gamma_2$
951	$\beta_1^5 x_{761}$		$\underline{2}x_{954}$
	$\underline{4}\beta_1^3\beta_{17}$ with	970	$\beta_1^{14}\beta_{10/5}$?
	$\alpha_1\beta_1^2 x_{868} = 0$	971	$\beta_1\gamma_4$? with
952	$\beta_1^6 x_{724}$		$\alpha_1\beta_1\beta_{20/4} = 0$
	$x_{952} = \langle \beta_1, \beta_1\beta_{10/5} + \underline{5}\gamma_2, \gamma_2 \rangle$	972	$\underline{5}\gamma_4$ with
953	$\underline{3}\beta_1^2\beta_{18}$		$\beta_1\beta_{20/3} = 0$
954	$\underline{3}\beta_1^{13}\gamma_2$	975	$\underline{3}\beta_1^6 x_{724}$
	$x_{954} = \langle \alpha_1, \alpha_1, \beta_1\beta_2\beta_{14}, \beta_1^4 \rangle$	978	$\alpha_1\beta_1\gamma_4$
955	$\underline{2}\beta_1\beta_{19}$		$\beta_1^2\beta_{19}$
956	$\beta_1\beta_{20/5}$	979	$\underline{2}x_{964}$ with
957	$\underline{2}\beta_{20/2}$ with		$\alpha_1\underline{5}\gamma_4 = 0$ and
	$5(\underline{2}\beta_{20/2}) = \alpha_1\beta_{20}$		$\alpha_1/\beta_1\beta_{20/3} = 0$
958	$\beta_1^4\beta_{17}$	987	$\underline{2}\,\underline{5}\gamma_4$
959	$\alpha_1 x_{952}$	988	$\beta_2\beta_{19}$ with $\beta_1\beta_{20} = 0$
963	$\alpha_1\beta_1\beta_{20/5}$	989	$\beta_1^6 x_{761}$
	$\underline{4}\beta_1^{13}\beta_{10/5}$	990	$\beta_1^7 x_{724}$
964	$x_{964} = \underline{3}\,\underline{4}\beta_1^4\beta_{16}$		$\beta_1 x_{952}$
	with $\beta_1\beta_{20/4} = 0$	992	$\underline{3}\beta_1^{14}\gamma_2$ with $\beta_1 x_{954} = 0$
966	$\underline{2}\beta_1^5 x_{761}$	998	β_{21}
968	$\beta_1^3\beta_{18}$	999	$x_{999} = \langle \beta_1\beta_2, \gamma_2, \gamma_2 \rangle$
	with $\alpha_1\beta_1^5 x_{771} = 0$		with $\underline{4}\beta_1^3\beta_{18} = 0$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	∞	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	∞	∞	2	2	12	2	2	3	15	2	2^2	12.2	84.2^2	2^2	6	30	30	6.2	12.2^2	6.2
3		2	2	12	2	2	3	15	2	2^2	12.2	84.2^2	2^2	6	30	30	6.2	12.2^2	12.2^2	132.2
4			2	$\infty.12$	2^2	2^2	24.3	15	2	2^3	120.12.2	84.2^5	2^6	24.6.2	2520.6.2	30	$6^2.2$	$24.12.4.2^2$	$120.12.2^5$	132.2^5
5				24	2	2	2	30	2	2^3	72.2	504.2^2	2^3	6.2	6.2	30.2	2^2	4.2^2	24.2^2	264.2
6					0	∞	2	60	24.2	2^3	72.2	504.4	240	6	12.2	60.6	504.2^2	2^4	24.6.2	1056.8
7						0	2	120	2^3	2^4	24.2	504.2	0	6	24.4	120.2^3	2^4	2^4	24.2	264.2
8							2	$\infty.120$	2^4	2^5	$24^2.2$	504.2	0	6.2	240.24.4	120.2^5	2^7	6.2^4	540.24.2	264.2
9								240	2^3	2^4	24.2	504.2	0	6	16.4	240.2^3	2^4	2^4	24.2	264.2
10									2^2	$\infty.2^3$	12.2	504	12	6	16.2	240.2^2	240.2	2^3	24.2^2	264.6
11										2^3	6.2	504	2	6.2	16.2	240.2	2	2^3	8.4.2	264.2^3
12											6	$\infty.504$	2^2	6.2	48.4.2	240.2	2	2^4	$480.4^2.2$	264.2^5
13												504	2	6	16.2	480.2	2	2^4	$8^2.2$	264.2^3
14													0	$\infty.3$	8.2	480.2	24.2	2^4	$8^2.2$	264.4.2
15														3	4.2	480.2	2^3	2^5	$8^2.2$	264.2^2
16															2^2	$\infty.480.2$	2^4	2^6	$24.8^2.2$	264.2^2
17																480.2	2^3	2^5	$8^2.2$	264.2^2
18																	2^2	$\infty.2^4$	8.4.2	264.2
19																		2^4	8.2^2	264.2
20																			8.2	$\infty.264.2$
21																				264.2

TABLE A3.6. Toda's calculation of unstable homotopy groups $\pi_{n+k}(S^n)$ for $n \leq k+2$ and $k \leq 19$.

For $n > k+1$ the group is isomorphic to the one for $n = k+1$. The notation $a.b.c\dots$ denotes the direct sum of cyclic groups of order a, b, c, etc. The notation a^j denotes the direct sum of j cyclic groups, each having order a. (After Toda [**6**].)