APPENDIX A1

Hopf Algebras and Hopf Algebroids

Commutative, noncocommutative Hopf algebras, such as the dual of the Steenrod algebra A (3.1.1), are familiar objects in algebraic topology and the importance of studying them is obvious. Computations with the Adams spectral sequence require the extensive use of homological algebra in the category of A-modules or, equivalently, in the category of A_*-comodules. In particular there are several change-of-rings theorems (A1.1.18, A1.1.20, and A1.3.13) which are major labor-saving devices. These results are well known, but detailed proofs (which are provided here) are hard to find.

The use of generalized homology theories such as MU- and BP-theory requires a generalization of the definition of a Hopf algebra to that of a *Hopf algebroid*. This term is due to Haynes Miller and its rationale will be explained below. The dual Steenrod algebra A_* is defined over $\mathbf{Z}/(p)$ and has a coproduct $\Delta\colon A_* \to A_* \otimes_{\mathbf{Z}/(p)} A_*$ dual to the product on A. The BP-theoretic analog $BP_*(BP)$ has a coproduct $\Delta\colon BP_*(BP) \to BP_*(BP) \otimes_{\pi_*(BP)} BP_*(BP)$, but the tensor product is defined with respect to a $\pi_*(BP)$-*bimodule* structure on $BP_*(BP)$; i.e., $\pi_*(BP)$ acts differently on the two factors. These actions are defined by two different $\mathbb{Z}_{(p)}$-algebra maps $\eta_L, \eta_R\colon \pi_*(BP) \to BP_*(BP)$, known as the left and right units. In the case of the Steenrod algebra one just has a single unit $\eta\colon \mathbf{Z}/(p) \to A_*$. Hence $BP_*(BP)$ is not a Hopf algebra, but a more general sort of object of which a Hopf algebra is a special case.

The definition of a Hopf algebroid A1.1.1 would seem rather awkward and unnatural were it not for the following category theoretic observation, due to Miller. A Hopf algebra such as A_* is a *cogroup object* in the category of graded $\mathbf{Z}/(p)$-algebras. In other words, given any such algebra R, the coproduct $\Delta\colon A_* \to A_* \otimes A_*$ induces a set map $\mathrm{Hom}(A_*, R) \times \mathrm{Hom}(A_*, R) \to \mathrm{Hom}(A_*, R)$ which makes $\mathrm{Hom}(A_*, R)$ into a group. Now the generalization of Hopf algebras to Hopf algebroids corresponds precisely to that from groups to *groupoids*. Recall that a group can be thought of as a category with a single object in which every morphism is invertible; the elements in the group are identified with the morphisms in the category. A groupoid is a small category in which every morphism is invertible and a Hopf algebroid is a cogroupoid object in the category of commutative algebras over a commutative ground ring K [$\mathbf{Z}_{(p)}$ in the case of $BP_*(BP)$]. The relation between the axioms of a groupoid and the structure of a Hopf algebroid is explained in A1.1.1.

The purpose of this appendix is to generalize the standard tools used in homological computations over a Hopf algebra to the category of comodules over a Hopf algebroid. It also serves as a self-contained (except for Sections 4 and 5) account of the Hopf algebra theory itself. These standard tools include basic definitions (Section 1), some of which are far from obvious; resolutions and homological functors such as Ext and Cotor (Section 2); spectral sequences of various sorts (Section 3), including that of Cartan and Eilenberg [**1**, p. 349]; Massey products (Section 4); and algebraic Steenrod operations (Section 5). We will now describe these five sections in more detail.

In Section 1 we start by defining Hopf algebroids (A1.1.1), comodules and primitives (A1.1.2), cotensor products (A1.1.4), and maps of Hopf algebroids (A1.1.7). The category of comodules is shown to be abelian (A1.1.3), so we can do homological algebra over it in Section 2. Three special types of groupoid give three corresponding types of Hopf algebroid. If the groupoid has a single object (or if all morphisms have the same source and target) we get an ordinary Hopf algebra, as remarked above. The opposite extreme is a groupoid with many objects but at most a single morphism between any pair of them. From such groupoids we get unicursal Hopf algebroids (A1.1.11). A third type of groupoid can be constructed from a group action on a set, and a corresponding Hopf algebroid is said to be split (A1.1.22).

The most difficult definition of Section 1 (which took us quite a while to formulate) is that of an extension of Hopf algebroids (A1.1.15). An extension of Hopf algebras corresponds to an extension of groups, for which one needs to know what a normal subgroup is. We are indebted to Higgins [**1**] for the definition of a normal subgroupoid. A groupoid C_0 is normal in C_1 if

(i) the objects of C_0 are the same as those of C_1,

(ii) the morphisms in C_0 form a subset of those in C_1, and

(iii) if $g\colon X \to Y$ and $h\colon Y \to Y$ are morphisms in C_1 and C_0, respectively, then $g^{-1}hg\colon X \to X$ is a morphism in C_0.

This translates to the definition of a normal map of Hopf algebroids (A1.1.10). The quotient groupoid $C = C_1/C_0$ is the one

(i) whose objects are equivalence classes of objects in C_1, where two objects are equivalent if there is a morphism between them in C_0, and

(ii) whose morphisms are equivalence classes of morphisms in C_1, where two morphisms g and g' are equivalent if $g' = h_1gh_2$ where h_1 and h_2 are morphisms in C_0.

The other major result of Section 1 is the comodule algebra structure theorem (A1.1.17) and its corollaries, which says that a comodule algebra (i.e., a comodule with a multiplication) which maps surjectively to the Hopf algebroid Σ over which it is defined is isomorphic to the tensor product of its primitives with Σ. This applies in particular to a Hopf algebroid Γ mapping onto Σ (A1.1.19). The special case when Σ is a Hopf algebra over a field was first proved by Milnor and Moore [**3**].

In Section 2 we begin our study of homological algebra in the category of comodules over a Hopf algebroid. We show (A1.2.2) that there are enough injectives and define Ext and Cotor (A1.2.3). For our purposes Ext can be regarded as a special case of Cotor (A1.1.6). We find it more convenient here to state and prove our results in terms of Cotor, although no use of it is made in the text. In most cases the translation from Cotor to Ext is obvious and is omitted. After defining these functors we discuss resolutions (A1.2.4, A1.2.10) that can be used to compute them, especially the cobar resolution (A1.2.11). We also define the cup product in Cotor (A1.2.14).

In Section 3 we construct some spectral sequences for computing the Cotor and Ext groups we are interested in. First we have the spectral sequence associated with an LES of comodules (A1.3.2); the example we have in mind is the chromatic spectral sequence of Chapter 5. Next we have the spectral sequence associated with a (decreasing or increasing) filtration of a Hopf algebroid (A1.3.9); examples include the classical May spectral sequence (3.2.9), the spectral sequence of 3.5.2, and the so-called algebraic Novikov spectral sequence (4.4.4).

In A1.3.11 we have a spectral sequence associated with a map of Hopf algebroids which computes Cotor over the target in terms of Cotor over the source. When the map is surjective the spectral sequence collapses and we get a change-of-rings isomorphism (A1.3.12). We also use this spectral sequence to construct a Cartan–Eilenberg spectral sequence (A1.3.14 and A1.3.15) for an extension of Hopf algebroids.

In Section 4 we discuss Massey products, an essential tool in some of the more intricate calculations in the text. The definitive reference is May [**3**] and this section is little more than an introduction to that paper. We refer to it for all the proofs and we describe several examples designed to motivate the more complicated statements therein. The basic definitions of Massey products are given as A1.4.1, A1.4.2, and A1.4.3. The rules for manipulating them are the juggling theorems A1.4.6, A1.4.8, and A1.4.9. Then we discuss the behavior of Massey products in spectral sequences. Theorem A1.4.10 addresses the problem of convergence; A1.4.11 is a Leibnitz formula for differentials on Massey products; and A1.4.12 describes the relation between differentials and extensions.

Section 5 treats algebraic Steenrod operations in suitable Cotor groups. These are defined in the cohomology of any cochain complex having certain additional structure and a general account of them is given by May [**5**]. Our main result (A1.5.1) here (which is also obtained by Bruner *et al.* [**1**]) is that the cobar complex (A1.2.11) has the required structure. Then the theory of May [**5**] gives the operations described in A1.5.2. Our grading of these operations differs from that of other authors including May [**5**] and Bruner *et al.* [**1**]; our $\mathcal{P}^i$ raises cohomological (as opposed to topological) degree by $2i(p-1)$.

1. Basic Definitions

A1.1.1. DEFINITION. *A* Hopf algebroid *over a commutative ring K is a cogroupoid object in the category of (graded or bigraded) commutative K-algebras, i.e., a pair (A, Γ) of commutative K-algebras with structure maps such that for any other commutative K-algebra B, the sets* $\mathrm{Hom}(A, B)$ *and* $\mathrm{Hom}(\Gamma, B)$ *are the objects and morphisms of a groupoid (a small category in which every morphism is an equivalence). The structure maps are*

$\eta_L\colon A \to \Gamma$	*left unit or source,*
$\eta_R\colon A \to \Gamma$	*right unit or target,*
$\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$	*coproduct or composition,*
$\varepsilon\colon \Gamma \to A,$	*counit or identity,*
$c\colon \Gamma \to \Gamma$	*conjugation or inverse.*

Here Γ is a left A-module map via η_L and a right A-module map via η_R, $\Gamma \otimes_A \Gamma$ is the usual tensor product of bimodules, and Δ and ε are A-bimodule maps. The defining properties of a groupoid correspond to the following relations among the structure maps:

(a) $\varepsilon\eta_L = \varepsilon\eta_R = 1_A$, the identity map on A. (The source and target of an identity morphism are the object on which it is defined.)

(b) $(\Gamma\otimes\varepsilon)\Delta = (\varepsilon\otimes\Gamma)\Delta = 1_\Gamma$. (Composition with the identity leaves a morphism unchanged.)

(c) $(\Gamma\otimes\Delta)\Delta = (\Delta\otimes\Gamma)\Delta$. (Composition of morphisms is associative.)

(d) $c\eta_R = \eta_L$ and $c\eta_L = \eta_R$. (Inverting a morphism interchanges source and target.)

(e) $cc = 1_\Gamma$. (The inverse of the inverse is the original morphism.)

(f) Maps exist which make the following commute

$$\begin{tikzcd}
\Gamma & \Gamma\otimes_K\Gamma \arrow[l, "c\cdot\Gamma"'] \arrow[r, "\Gamma\cdot c"] \arrow[d] & \Gamma \\
 & \Gamma\otimes_A\Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\
A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\varepsilon"'] \arrow[r, "\varepsilon"] \arrow[u, "\Delta"] & A \arrow[uu, "\eta_L"']
\end{tikzcd}$$

where $c\cdot\Gamma(\gamma_1\otimes\gamma_2) = c(\gamma_1)\gamma_2$ and $\Gamma\cdot c(\gamma_1\otimes\gamma_2) = \gamma_1 c(\gamma_2)$. (Composition of a morphism with its inverse on either side gives an identity morphism.)

If our algebras are graded the usual sign conventions are assumed; i.e., commutativity means $xy = (-1)^{|x||y|}yx$, where $|x|$ and $|y|$ are the degrees or dimensions of x and y, respectively.

A graded Hopf algebroid is *connected* if the right and left sub-A-modules generated by Γ_0 are both isomorphic to A.

In most cases the algebra A will be understood and the Hopf algebroid will be denoted simply by Γ.

Note that if $\eta_R = \eta_L$, then Γ is a commutative Hopf algebra over A, which is to say a cogroup object in the category of commutative A-algebras. This is the origin of the term Hopf algebroid. More generally if $D \subset A$ is the subalgebra on which $\eta_R = \eta_L$, then Γ is also a Hopf algebroid over D.

The motivating example of a Hopf algebroid is $(\pi_*(E), E_*(E))$ for a suitable spectrum E (see Section 2.2).

A1.1.2. Definition. *A* left Γ-comodule *M is a left A-module M together with a left A-linear map $\psi\colon M \to \Gamma\otimes_A M$ which is counitary and coassociative, i.e., such that $(\varepsilon\otimes M)\psi = M$ (i.e., the identity on M) and $(\Delta\otimes M)\psi = (\Gamma\otimes\psi)\psi$. A* right Γ-comodule *is similarly defined. An element $m\in M$ is* primitive *if $\psi(m) = 1\otimes m$.*

A comodule algebra *M is a comodule which is also a commutative associative A-algebra such that the structure map ψ is an algebra map. If M and N are left Γ-comodules, their* comodule tensor product *is $M\otimes_A N$ with structure map being the composite*

$$M\otimes N \xrightarrow{\psi_M\otimes\psi_N} \Gamma\otimes M\otimes\Gamma\otimes N \to \Gamma\otimes\Gamma\otimes M\otimes N \to \Gamma\otimes M\otimes N,$$

where the second map interchanges the second and third factors and the third map is the multiplication on Γ. All tensor products are over A using only the left A-module structure on A. A differential comodule *C^* is a cochain complex in which each C^s is a comodule and the coboundary operator is a comodule map.*

A1.1.3. THEOREM. *If* Γ *is flat as an* A*-module then the category of left* Γ*-comodules is abelian* (see Hilton and Stammbach [**1**]).

PROOF. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules, then since Γ is flat over A,

$$0 \to \Gamma \otimes_A M' \to \Gamma \otimes_A M \to \Gamma \otimes_A M'' \to 0$$

is also exact. If M is a left Γ-comodule then a comodule structure on either M' or M'' will determine such a structure on the other one. From this fact it follows easily that the kernel or cokemel (as an A-module) of a map of comodules has a unique comodule structure, i.e., that the category has kernels and cokernels. The other defining properties of an abelian category are easily verified. □

In view of the above, *we assume from now on that* Γ *is flat over* A.

A1.1.4. DEFINITION. *Let* M *and* N *be right and left* Γ*-comodules, respectively. Their* cotensor product over Γ *is the* K*-module defined by the exact sequence*

$$0 \to M \square_\Gamma N \to M \otimes_A N \xrightarrow{\psi\otimes N - M\otimes\psi} M \otimes_A \Gamma \otimes_A N,$$

where ψ *denotes the comodule structure maps for both* M *and* N.

Note that $M \square_\Gamma N$ is not a comodule or even an A-module but merely a K-module.

A left comodule M can be given the structure of a right comodule by the composition

$$M \xrightarrow{\psi} \Gamma \otimes M \xrightarrow{T} M \otimes \Gamma \xrightarrow{M\otimes c} M \otimes \Gamma,$$

where T interchanges the two factors and c is the conjugation map (see A1.1.1). A right comodule can be converted to a left comodule by a similar device. With this in mind we have

A1.1.5. PROPOSITION. $M \square_\Gamma N = N \square_\Gamma M$.

The following relates the cotensor product to Hom.

A1.1.6. LEMMA. *Let* M *and* N *be left* Γ*-comodules with* M *projective over* A. *Then*

(a) $\mathrm{Hom}_A(M, A)$ *is a right* Γ*-comodule and*

(b) $\mathrm{Hom}_\Gamma(M, N) = \mathrm{Hom}_A(M, A) \square_\Gamma N$, *e.g.,* $\mathrm{Hom}_\Gamma(A, N) = A \square_\Gamma N$.

PROOF. Let $\psi_M \colon M \to \Gamma \otimes_A M$ and $\psi_N \colon N \to \Gamma \otimes_A N$ be the comodule structure maps. Define

$$\psi_M^*, \psi_N^* \colon \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, \Gamma \otimes_A N)$$

by

$$\psi_M^*(f) = (\Gamma \otimes f)\psi_M \quad \text{and} \quad \psi_N^*(f) = \psi_N f$$

for $f \in \mathrm{Hom}_A(M, N)$. Since M is projective we have a canonical isomorphism,

$$\mathrm{Hom}_A(M, A) \otimes_A N \approx \mathrm{Hom}_A(M, N).$$

Hence for $N = A$ we have

$$\psi_M^* \colon \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(M, A) \otimes_A \Gamma.$$

To show that this is a right Γ-comodule structure we need to show that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_A(M,A) \arrow[r, "\psi_M^*"] \arrow[d, "\psi_M^*"'] & \mathrm{Hom}_A(M,\Gamma) \arrow[d, "\mathrm{Hom}(M,\Delta)"] \\
\mathrm{Hom}_A(M,\Gamma) \arrow[r, "\psi_M^*"] & \mathrm{Hom}_A(M,\Gamma\otimes\Gamma),
\end{tikzcd}$$

i.e., that ψ_M^* is coassociative.

We have a straightforward calculation

$$\begin{aligned}
\psi_M^*\psi_M^*(f) &= (\Gamma\otimes\psi_M^*(f))\psi_M \\
&= (\Gamma\otimes((\Gamma\otimes f)\psi_m))\psi_M \\
&= (\Gamma\otimes\Gamma\otimes f)(\Gamma\otimes\psi_M)\psi_M \\
&= (\Gamma\otimes\Gamma\otimes f)(\Delta\otimes M)\psi_M \\
&= (\Delta\otimes A)(\Gamma\otimes f)\psi_M \\
&= (\Delta\otimes A)\psi_M^* f
\end{aligned}$$

so the diagram commutes and (a) follows.

For (b) note that by definition

$$\mathrm{Hom}(M,N) = \ker(\psi_M^* - \psi_M^*) \subset \mathrm{Hom}_A(M,N)$$

while

$$\begin{aligned}
\mathrm{Hom}_A(M,A)\,\square_\Gamma\, N &= \ker(\psi_M^*\otimes N - \mathrm{Hom}_A(M,A)\otimes\psi_N) \\
&\subset \mathrm{Hom}_A(M,A)\otimes_A N
\end{aligned}$$

and the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(M,A)\otimes N \arrow[r, "\simeq"] \arrow[d, shift right, "\psi_M^*\otimes N"'] \arrow[d, shift left, "\mathrm{Hom}(M,A)\otimes\psi_N"] & \mathrm{Hom}_A(M,N) \arrow[d, shift right, "\psi_M^*"'] \arrow[d, shift left, "\psi_N^*"] \\
\mathrm{Hom}(M,A)\otimes\Gamma\otimes N \arrow[r, "\simeq"] & \mathrm{Hom}_A(M,\Gamma\otimes_A N)
\end{tikzcd}$$ □

The next few definitions and lemmas lead up to that of an extension of Hopf algebroids given in A1.1.15. In A1.3.14 we will derive a corresponding Cartan–Eilenberg spectral sequence.

A1.1.7. DEFINITION. *A map of Hopf algebroids $f\colon (A,\Gamma)\to(B,\Sigma)$ is a pair of K-algebra maps $f_1\colon A\to B$, $f_2\colon \Gamma\to\Sigma$ such that*

$$\begin{gathered}
f_1\varepsilon = \varepsilon f_2, \quad f_2\eta_R = \eta_R f_1, \quad f_2\eta_L = \eta_L f_1, \\
f_2 c = c f_2, \quad \textit{and} \quad \Delta f_2 = (f_2\otimes f_2)\Delta.
\end{gathered}$$

A1.1.8. LEMMA. *Let $f\colon(A,\Gamma)\to(B,\Sigma)$ be a map of Hopf algebroids. Then $\Gamma\otimes_A B$ is a right Σ-comodule and for any left Σ-comodule N, $(\Gamma\otimes_A B)\,\square_\Sigma\, N$ is a sub-left Γ-comodule of $\Gamma\otimes_A N$, where the structure map for the latter is $\Delta\otimes N$.*

PROOF. The map $(\Gamma\otimes f_2)\Delta\colon \Gamma\to\Gamma\otimes_A\Sigma = (\Gamma\otimes_A B)\otimes_B\Sigma$ extends uniquely to $\Gamma\otimes_A B$, making it a right Σ-comodule. By definition $(\Gamma\otimes_A B)\square_\Sigma N$ is the kernel in the exact sequence

$$0\to(\Gamma\otimes_A B)\,\square_\Sigma\, N\to\Gamma\otimes_A N\to\Gamma\otimes_A\Sigma\otimes_B N$$

where the right-hand arrow is the difference between $(\Gamma \otimes f_2)\Delta \otimes N$ and $\Gamma \otimes \psi$. Since $\Gamma \otimes_A N$ and $\Gamma \otimes_A \Sigma \otimes_B N$ are left Γ-comodules it suffices to show that the two maps respect the comodule structure. This is clear for $\Gamma \otimes \psi$, and for $(\Gamma \otimes f)\Delta \otimes N$ we need the commutativity of the following diagram, tensored over B with N.

$$\begin{tikzcd} \Gamma \otimes_A B \arrow[r, "(\Gamma\otimes f_2)\Delta\otimes B"] \arrow[d, "\Delta\otimes B"'] & \Gamma \otimes_A \Sigma \arrow[d, "\Delta\otimes\Sigma"] \\ \Gamma \otimes_A \Gamma \otimes_A B \arrow[r, "\Gamma\otimes(\Gamma\otimes f_2)\Delta\otimes B"] & \Gamma \otimes_A \Gamma \otimes_A \Sigma \end{tikzcd}$$

It follows from the fact that f is a Hopf algebroid map. □

A1.1.9. DEFINITION. *If (A, Γ) is a Hopf algebroid the* associated Hopf algebra (A, Γ') *is defined by* $\Gamma' = \Gamma/(\eta_L(a) - \eta_R(a) \mid a \in A)$. *(The easy verification that a Hopf algebra structure is induced on Γ' is left to the reader.)*

Note that Γ' may not be flat over A even though Γ is.

A1.1.10. DEFINITION. *A map of Hopf algebroids $f\colon (A, \Gamma) \to (A, \Sigma)$ is* normal *if $f_2\colon \Gamma \to \Sigma$ is surjective, $f_1\colon A \to A$ is the identity, and $\Gamma \square_{\Sigma'} A = A \square_{\Sigma'} \Gamma$ in Γ.*

A1.1.11. DEFINITION. *A Hopf algebroid (A, U) is* unicursal *if it is generated as an algebra by the images of η_L and η_R, i.e., if $U = A \otimes_D A$ where $D = A \square_U A$ is a subalgebra of A. (The reader can verify that the Hopf algebroid structure of U is unique.)* □

This term was taken from page 9 of Higgins [**1**].

A1.1.12. LEMMA. *Let M be a right comodule over a unicursal Hopf algebroid (A, U). Then*

(a) *M is isomorphic as a comodule to $M \otimes_A A$ with structure map $M \otimes \eta_R$ and*

(b) *$M = (M \square_U A) \otimes_D A$ as A-modules.*

PROOF. For $m \in M$ let $\psi(m) = m' \otimes m''$. Since U is unicursal we can assume that each m'' is in the image of η_R. It follows that

$$(\psi \otimes U)\psi(m) = (M \otimes \Delta)\psi(m) = m' \otimes 1 \otimes m''$$

so each m' is primitive. Let $\widetilde{m} = m'\varepsilon(m'')$. Then $\psi(\widetilde{m}) = m' \otimes m'' = \psi(m)$, so $m = \widetilde{m}$ since ψ is a monomorphism; Hence M is generated as an A-module by primitive elements and (a) follows. For (b) we have, using (a),

$$(M \square_U A) \otimes_D A = M \otimes_A (A \square_U A) \otimes_D A = M \otimes_A D \otimes_D A = M. \qquad \square$$

A1.1.13. LEMMA. *Let (A, Σ) be a Hopf algebroid, (A, Σ') the associated Hopf algebra* (A1.1.7) *$D = A \square_\Sigma A$, and (A, U) the unicursal Hopf algebroid* (A1.1.9) *with $U = A \otimes_D A$. Then*

(a) *$U = \Sigma \square_{\Sigma'} A$ and*

(b) *for a left Σ-comodule M, $A \square_{\Sigma'} M$ is a left U-comodule and $A \square_\Sigma M = A \square_U (A \square_{\Sigma'} M)$.*

PROOF. By definition, $\Sigma' = A \otimes_U \Sigma$, where the U-module structure on A is given by $\varepsilon\colon U \to A$, so we have

$$\Sigma \otimes_A \Sigma' = \Sigma \otimes_A A \otimes_U \Sigma = \Sigma \otimes_U \Sigma.$$

By A1.1.3, there is a short exact sequence

$$0 \to \Sigma \,\square_{\Sigma'}\, A \to \Sigma \otimes_U \Sigma$$

where the last map is induced by $\Delta - \Sigma \otimes \eta_L$. An element $\sigma \in \Sigma$ has $\Delta(\sigma) = \sigma \otimes 1$ in $\Sigma \otimes_U \Sigma$ iff $\sigma \in U$, so (a) follows.

For (b) we have

$$A \,\square_\Sigma\, M = A \,\square_U\, (U \,\square_\Sigma\, M)$$

and

$$U \,\square_\Sigma\, M = (A \,\square_{\Sigma'}\, \Sigma) \,\square_\Sigma\, M = A \,\square_{\Sigma'}\, M. \qquad \square$$

The following example may be helpful. Let $(A, \Gamma) = (\pi_*(BP), BP_*(BP))$ (4.1.19), i.e., $A = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$ and $\Gamma = A[t_1, t_2, \dots]$ where $\dim v_i = \dim t_i = 2(p^i - 1)$. Let $\Sigma = A[t_{n+1}, t_{n+2}, \dots]$ for some $n \geq 0$. The Hopf algebroid structure on Σ is that of the quotient $\Gamma/(t_1, \dots, t_n)$. The evident map $(A, \Gamma) \to (A, \Sigma)$ is normal (A1.1.10). $D = A \square_\Sigma A$ is $\mathbf{Z}_{(p)}[v_1, \dots, v_n]$ and $\Phi = A \square_\Sigma \Gamma \square_\Sigma A$ is $D[t_1, \dots, t_n]$. (D, Φ) is a sub-Hopf algebroid of (A, Γ) and $(D, \Phi) \to (A, \Gamma) \to (A, \Sigma)$ is an extension (A1.1.15 below).

A1.1.14. THEOREM. *Let $f\colon (A, \Gamma) \to (A, \Sigma)$ be a normal map of Hopf algebroids and let $D = A \,\square_\Sigma\, A$ and $\Phi = A \,\square_\Sigma\, \Gamma \,\square_\Sigma\, A$. Then (D, Φ) is a sub-Hopf algebroid of (A, Γ).*

(Note that by A1.1.8, $A \,\square_\Sigma\, \Gamma$ and $\Gamma \,\square_\Sigma\, A$ are right and left Γ-comodules, respectively, so the expressions $(A \,\square_\Sigma\, \Gamma) \,\square_\Sigma\, A$ and $A \,\square_\Sigma\, (\Gamma \,\square_\Sigma\, A)$ make sense. It is easy to check, without using the normality of f, that they are equal, so Φ is well defined.)

PROOF. By definition an element $a \in A$ is in D iff $f_2\eta_L(a) = f_2\eta_R(a)$ and is in Φ iff $(f_2 \otimes \Gamma \otimes f_2)\,\Delta^2(\gamma) = 1 \otimes \gamma \otimes 1$. To see that η_R sends D to Φ, we have for $d \in D$

$$\begin{aligned}(f_2 \otimes \Gamma \otimes f_2)\Delta^2\eta_R(d) &= 1 \otimes 1 \otimes f_2\eta_R(d)\\ &= 1 \otimes 1 \otimes f_2\eta_L(d) = 1 \otimes \eta_R(d) \otimes 1.\end{aligned}$$

The argument for η_L is similar. It is clear that Φ is invariant under the conjugation c. To show that ε sends Φ to D we need to show $f_2\eta_R\varepsilon(\phi) = f_2\eta_L\varepsilon(\phi)$ for $\phi \in \Phi$. But $f_2\eta_R\varepsilon(\phi) = \eta_R\varepsilon f_2(\phi)$ and since $\Delta^2 f_2(\phi) = 1 \otimes f_2(\phi) \otimes 1$ we have $\Delta f_2(\phi) = 1 \otimes f_2(\phi) = f_2(\phi) \otimes 1$ so $f_2(\phi) \in D$, and $(\eta_R - \eta_L)\varepsilon f_2(\phi) = 0$.

To define a coproduct on Φ we first show that the natural map from $\Phi \otimes_D \Phi$ to $\Gamma \otimes_A \Gamma$ is monomorphic. This amounts to showing that $a\phi \in \Phi$ iff $a \in D$. Now by definition $a\phi \in \Phi$ iff

$$f_2(a\phi') \otimes \phi'' \otimes f_2(\phi''') = 1 \otimes a\phi \otimes 1 = f_2\eta_R(a) \otimes \phi \otimes 1.$$

Since $\phi \in \Phi$ we have

$$f_2(\phi') \otimes \phi'' \otimes f_2(\phi''') = 1 \otimes \phi \otimes 1,$$

so the criterion is

$$f_2(a) \otimes 1 \otimes 1 = f_2\eta_R(a) \otimes 1 \otimes 1,$$

i.e., $a \in D$.

Now consider the commutative diagram

$$\begin{tikzcd}
(D,\Phi) \arrow[r] \arrow[d, equal] & (A,\widetilde{\Phi}) \arrow[r, "g"] \arrow[d] & (A,U) \arrow[d] \\
(D,\Phi) \arrow[r] & (A,\Gamma) \arrow[r, "f"] \arrow[d, "f'"] & (A,\Sigma) \arrow[d] \\
& (A,\Sigma') \arrow[r, equal] & (A,\Sigma')
\end{tikzcd}$$

where Σ' is the Hopf algebra associated to Σ (A1.1.9), f' is the induced map, U is the unicursal Hopf algebroid (A1.1.11) $A\otimes_D A$, $\widetilde{\Phi}=A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A$, and g will be constructed below. We will see that Φ and $\widetilde{\Phi}$ are both Hopf algebroids.

Now the map f' is normal since f is and $A\,\square_{\Sigma'}\,A=A$, so the statement that $\widetilde{\Phi}$ is a Hopf algebroid is a special case of the theorem. Hence we have already shown that it has all of the required structure but the coproduct. Since $\Gamma\,\square_{\Sigma'}\,A=A\,\square_{\Sigma'}\,\Gamma$, we have $\widetilde{\Phi}=A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A=A\,\square_{\Sigma'}\,A\,\square_{\Sigma'}\,\Gamma=A\,\square_{\Sigma'}\,\Gamma$. One easily verifies that the image of $\Delta\colon\Gamma\to\Gamma\otimes_A\Gamma$ is contained in $\Gamma\,\square_\Gamma\,\Gamma$ and hence in $\Gamma\,\square_{\Sigma'}\,\Gamma$. There Δ sends $\widetilde{\Phi}=A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A$ to $A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A=\widetilde{\Phi}\,\square_{\Sigma'}\,\widetilde{\Phi}\subset\widetilde{\Phi}\otimes_A\widetilde{\Phi}$, so $\widetilde{\Phi}$ is a Hopf algebroid.

Since $\widetilde{\Phi}=\Gamma\,\square_{\Sigma'}\,A$ and $U=\Sigma\,\square_{\Sigma'}\,A$ [A1.1.13(a)] we can define g to be $f_2\,\square\,A$. It follows from A1.1.13(b) that

$$\begin{aligned}\Phi&=A\,\square_\Sigma\,\Gamma\,\square_\Sigma\,A=A\,\square_U\,(A\,\square_{\Sigma'}\,\Gamma\,\square_{\Sigma'}\,A)\,\square_U\,A\\&=A\,\square_U\,\widetilde{\Phi}\,\square_U\,A.\end{aligned}$$

By A1.1.12(b) we have $\widetilde{\Phi}=A\otimes_D\Phi\otimes_D A$, so $\widetilde{\Phi}\otimes_A\widetilde{\Phi}=A\otimes_D\widetilde{\Phi}\otimes_D A\otimes_D\Phi\otimes_D A$. The coproduct Δ sends $\widetilde{\Phi}$ to $\widetilde{\Phi}\,\square_U\,\widetilde{\Phi}\subset\Phi\otimes_A\widetilde{\Phi}$ and we have

$$\begin{aligned}\widetilde{\Phi}\,\square_U\,\widetilde{\Phi}&=\widetilde{\Phi}\otimes_A(A\,\square_U\,A)\otimes_A\widetilde{\Phi}\quad\text{by A1.1.12(a)}\\&=A\otimes_D\Phi\otimes_D(A\,\square_U\,A)\otimes_D\Phi\otimes_D A\\&=A\otimes_D\Phi\otimes_D D\otimes_D\Phi\otimes_D A\\&=A\otimes_D\Phi D\otimes\Phi\otimes_D A.\end{aligned}$$

Since Δ is A-bilinear it sends Φ to $\Phi\otimes_D\Phi$ and Φ is a Hopf algebroid. $\square$

A1.1.15. Definition. *An* extension *of Hopf algebroids is a diagram*

$$(D,\Phi)\xrightarrow{i}(A,\Gamma)\xrightarrow{f}(A,\Sigma)$$

where f is normal (A1.1.10) and (D,Φ) is as in A1.1.14.

The extension is cocentral *if the diagram*

$$\begin{tikzcd}
& \Gamma\otimes\Sigma \arrow[dd, "t"] \\
\Gamma \arrow[ur, "(\Gamma\otimes f_2)\Delta"] \arrow[dr, "(f_2\otimes\Gamma)\Delta"'] & \\
& \Sigma\otimes\Gamma
\end{tikzcd}$$

(*where t interchanges factors*) *commutes up to the usual sign. In particular* Σ *must be cocommutative.*

A nice theory of Hopf algebra extensions is developed by Singer [**5**] and in Section II 3 of Singer [**6**].

Note that (as shown in the proof of A1.1.14) if Σ is a Hopf algebra then $\Phi = A \square_\Sigma \Gamma = \Gamma \square_\Sigma A$. More generally we have

A1.1.16. LEMMA. *With notation as above,* $A \square_\Sigma \Gamma = \Phi \otimes_D A$ *as right* Γ-*comodules.*

PROOF. Using A1.1.12 and A1.1.13 we have

$$\begin{aligned}\Phi \otimes_D A &= A \square_\Sigma \Gamma \square_\Sigma A \otimes_D A \\ &= A \square_\Sigma \Gamma \square_{\Sigma'} A \square_I A \otimes_D A \\ &= A \square_\Sigma \Gamma \square_{\Sigma'} A \\ &= A \square_\Sigma A \square_{\Sigma'} \Gamma \\ &= A \square_U A \square_{\Sigma'} A \square_{\Sigma'} \Gamma \\ &= A \square_U A \square_{\Sigma'} \Gamma \\ &= A \square_\Sigma \Gamma. \qquad \square\end{aligned}$$

A1.1.17. COMODULE ALGEBRA STRUCTURE THEOREM. *Let* (B, Σ) *be a graded connected Hopf algebroid,* M *a graded connected right* Σ-*comodule algebra, and* $C = M \square_\Sigma B$. *Suppose*

(i) *there is a surjective comodule algebra map* $f\colon M \to \Sigma$ *and*

(ii) C *is a* B-*module and as such it is a direct summand of* M.

Then M *is isomorphic to* $C \otimes_B \Sigma$ *simultaneously as a left* C-*module and a right* Σ-*comodule.* □

We will prove this after listing some corollaries. If Σ is a Hopf algebra over a field K then the second hypothesis is trivial so we have the following result, first proved as Theorem 4.7 of Milnor and Moore [**3**].

A1.1.18. COROLLARY. *Let* (K, Σ) *be a commutative graded connected Hopf algebra over a field* K. *Let* M *be a* K-*algebra and a right* Σ-*comodule and let* $C = M \square_\Sigma K$. *If there is a surjection* $f\colon M \to \Sigma$ *which is a homomorphism of algebras and* Σ-*comodules, then* M *is isomorphic to* $C \otimes \Sigma$ *simultaneously as a left* C-*module and as a right* Σ-*comodule.* □

A1.1.19. COROLLARY. *Let* $f\colon (A, \Gamma) \to (B, \Sigma)$ *be a map of graded connected Hopf algebroids* (A1.1.7) *and let* $\Gamma' = \Gamma \otimes_A B$ *and* $C = \Gamma' \square_\Sigma B$. *Suppose*

(i) $f_2'\colon \Gamma' \to \Sigma$ *is onto and*

(ii) C *is a* B-*module and there is a* B-*linear map* $g\colon \Gamma' \to C$ *split by the inclusion of* C *in* Γ'.

Then there is a map $\tilde{g}\colon \Gamma' \to C \otimes_B \Sigma$ *defined by* $\tilde{g}(\gamma) = g(\gamma') \otimes f_2'(\gamma'')$ *which is an isomorphism of* C-*modules and* Σ-*comodules.* □

A1.1.20. COROLLARY. *Let* K *be a field and* $f\colon (K, \Gamma) \to (K, \Sigma)$ *a map of graded connected commutative Hopf algebras and let* $C = \Gamma \square_\Sigma K$. *If* f *is surjective then* Γ *is isomorphic to* $C \otimes \Sigma$ *simultaneously as a left* C-*module and as a right* Σ-*comodule.* □

In A1.3.12 and A1.3.13 we will give some change-of-rings isomorphisms of Ext groups relevant to the maps in the previous two corollaries.

PROOF OF A1.1.17. Let $i\colon C \to M$ be the natural inclusion and let $g\colon M \to C$ be a B-linear map such that gi is the identity. Define $\tilde{g}\colon M \to C \otimes_B \Sigma$ to be $(g \otimes \Sigma)\psi$; it is a map of Σ-comodules but not necessarily of C-modules and we will show below that it is an isomorphism.

Next observe that $f \square B\colon C \to B$ is onto. In dimension zero it is simply f, which is onto by assumption, and it is B-linear and therefore surjective. Let $j\colon B \to C$ be a B-linear splitting of $f \square B$. Then $h = \tilde{g}^{-1}(j \otimes \Sigma)\colon \Sigma \to M$ is a comodule splitting of f.

Define $\tilde{h}\colon C \otimes_B \Sigma \to M$ by $\tilde{h}(c \otimes \sigma) = i(c)h(\sigma)$ for $c \in C$ and $\sigma \in \Sigma$. It is clearly a C-linear comodule map and we will show that it is the desired isomorphism. We have

$$\tilde{g}\tilde{h}(c \otimes \sigma) = \tilde{g}(i(c)h(\sigma)) = g(i(c)h(\sigma')) \otimes \sigma'' = c \otimes \sigma$$

where the second equality holds because $i(c)$ is primitive in M and the congruence is modulo elements of lower degree with respect to the following increasing filtration (A1.2.7) on $C \otimes_B \Sigma$. Define $F_n(C \otimes_B \Sigma) \subset C \otimes_B \Sigma$ to be the sub-K-module generated by elements of the form $c \otimes \sigma$ with $\dim \sigma \leq n$. It follows that $\tilde{g}\tilde{h}$ and hence h are isomorphisms.

We still need to show that $\tilde{g}$ is an isomorphism. To show that it is 1-1, let $\widetilde{m} \otimes \sigma$ be the leading term (with respect to the above filtration of $M \otimes \Sigma$) of $\psi(m)$. It follows from coassociativity that $\widetilde{m}$ is primitive, so $g(\widetilde{m}) \neq 0$ if $m \neq 0$ and $\ker \tilde{g} = 0$. To show that $\tilde{g}$ is onto, note that for any $c \otimes \sigma \in C \otimes_B \Sigma$ we can choose $m \in f^{-1}(\sigma)$ and we have

$$\tilde{g}(i(c)m) = g(i(c)m') \otimes m'' = gi(c) \otimes \sigma = c \otimes \sigma$$

so $\operatorname{coker} \tilde{g} = 0$ by standard arguments. $\square$

A1.1.21. DEFINITION. *An ideal $I \subset A$ is* invariant *if it is a sub-Γ-comodule, or equivalently if $\eta_R(I) \subset I\Gamma$.*

A1.1.22. DEFINITION. *A Hopf algebroid (A, Γ) is* split *if there is a Hopf algebroid map $i\colon (K, \Sigma) \to (A, \Gamma)$* (A1.1.19) *such that $i_2'\colon \Sigma \otimes A \to \Gamma$ is an isomorphism of K-algebras.*

Note that composing $\eta_R\colon A \to \Gamma$ with the inverse of i_2' defines a left Σ-comodule structure on A.

2. Homological Algebra

Recall (A1.1.3) that the category of comodules over a Hopf algebroid (A, Γ) is abelian provided Γ is flat over A, which means that we can do homological algebra in it. We want to study the derived functors of Hom and cotensor product (A1.1.4). Derived functors are discussed in most books on homological algebra, e.g., Cartan and Eilenberg [**1**], Hilton and Stammbach [**1**], and Mac Lane [**1**]. In order to define them we must be sure that our category has enough injectives, i.e., that each Γ-comodule can be embedded in an injective one. This can be seen as follows.

A1.2.1. DEFINITION. *Given an A-module N, define a comodule structure on $\Gamma \otimes_A N$ by $\psi = \Delta \otimes N$. Then for any comodule M,*

$$\theta\colon \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_\Gamma(M, \Gamma \otimes_A N)$$

is the isomorphism given by $\theta(f) = (\Gamma \otimes f)\psi_M$ *for* $f \in \mathrm{Hom}_A(M, N)$. *For* $g \in \mathrm{Hom}_\Gamma(M, \Gamma \otimes_A N)$, $\theta^{-1}(g)$ *is given by* $\theta^{-1}(g) = (\varepsilon \otimes N)g$.

A1.2.2. Lemma. *If* I *is an injective* A*-module then* $\Gamma \otimes_A I$ *is an injective* Γ*-comodule. Hence the category of* Γ*-comodules has enough injectives.*

Proof. To show that $\Gamma \otimes_A I$ is injective we must show that if M is a subcomodule of N, then a comodule map from M to $\Gamma \otimes_A I$ extends to N. But $\mathrm{Hom}_\Gamma(M, \Gamma \otimes_A I) = \mathrm{Hom}_A(M, I)$ which is a subgroup of $\mathrm{Hom}_A(N, I) = \mathrm{Hom}_\Gamma(N, \Gamma \otimes_A I)$ since I is injective as an A-module. Hence the existence of enough injectives in the category of A-modules implies the same in the category of Γ-comodules. $\square$

This result allows us to make

A1.2.3. Definition. *For left* Γ*-comodules* M *and* N, $\mathrm{Ext}^i_\Gamma(M, N)$ *is the* i*th right derived functor of* $\mathrm{Hom}_\Gamma(M, N)$*, regarded as a functor of* N*. For* M *a right* Γ*-comodule,* $\mathrm{Cotor}^i_\Gamma(M, N)$*, is the* i*th right derived functor of* $M \square_\Gamma N$ (A1.1.4)*, also regarded as a functor of* N*. The corresponding graded groups will be denoted simply by* $\mathrm{Ext}_\Gamma(M, N)$ *and* $\mathrm{Cotor}_\Gamma(M, N)$*, respectively.*

In practice we shall only be concerned with computing these functors when the first variable is projective over A. In that case the two functors are essentially the same by A1.1.6. We shall therefore make most of our arguments in terms of Cotor and list the corresponding statements about Ext as corollaries without proof.

Recall that the zeroth right derived functor is naturally equivalent to the functor itself if the latter is left exact. The cotensor product is left exact in the second variable if the first variable is flat as an A-comodule.

One knows that right derived functors can be computed using an injective resolution of the second variable. In fact the resolution need only satisfy a weaker condition.

A1.2.4. Lemma. *Let*

$$0 \to N \to R^0 \to R^1 \to \cdots$$

be a long exact sequence of left Γ*-comodules such that* $\mathrm{Cotor}^n_\Gamma(M, R^i) = 0$ *for* $n > 0$*. Then* $\mathrm{Cotor}_\Gamma(M, N)$ *is the cohomology of the complex*

$$\mathrm{Cotor}^0_\Gamma(M, R^0) \xrightarrow{\delta_0} \mathrm{Cotor}^0_\Gamma(M, R^1) \xrightarrow{\delta_1} \cdots . \tag{A1.2.5}$$

Proof. Define comodules N^i inductively by $N^0 = N$ and N^{i+1} is the quotient in the short exact sequence

$$0 \to N^i \to R^i \to N^{i+1} \to 0.$$

These give long exact sequences of Cotor groups which, because of the behavior of $\mathrm{Cotor}_\Gamma(M, R^i)$, reduce to four-term sequences

$$\begin{aligned} 0 \to \mathrm{Cotor}^0_\Gamma(M, N^i) &\to \mathrm{Cotor}^0_\Gamma(M, R^i) \\ &\to \mathrm{Cotor}^0_\Gamma(M, N^{i+1}) \to \mathrm{Cotor}^1_\Gamma(M, N^i) \to 0 \end{aligned}$$

and isomorphisms

$$\mathrm{Cotor}^n_\Gamma(M, N^{i+1}) \approx \mathrm{Cotor}^{n+1}_\Gamma(M.N^i) \quad \text{for} \quad n > 0. \tag{A1.2.6}$$

Hence in A1.2.5, $\ker\delta_i = \mathrm{Cotor}_\Gamma^0(M, N^i)$ while $\mathrm{im}\,\delta_i$ is the image of $\mathrm{Cotor}_\Gamma^0(M, R^i)$ in $\mathrm{Cotor}_\Gamma^0(M, N^{i+1})$ so

$$\ker\delta_i/\,\mathrm{im}\,\delta_{i-1} = \mathrm{Cotor}_\Gamma^1(M, N^{i-1}) = \mathrm{Cotor}_\Gamma^i(M, N)$$

by repeated use of A1.2.6. This quotient by definition is H^i of A1.2.5. □

For another proof see A1.3.2.

We now introduce a class of comodules which satisfy the Ext condition of A1.2.4 when M is projective over A.

A1.2.7. Definition. *An* extended Γ-comodule *is one of the form* $\Gamma\otimes_A N$ *where* N *is an* A*-module. A* relatively injective Γ-comodule *is a direct summand of an extended one.*

This terminology comes from relative homological algebra, for which the standard references are Eilenberg and Moore [**1**] and Chapter IX of Mac Lane [**1**]. Our situation is dual to theirs in the following sense. We have the category $\mathbf{\Gamma}$ of left (or right) Γ-comodules, the category $\mathbf{A}$ of A-modules, the forgetful functor G from $\mathbf{\Gamma}$ to $\mathbf{A}$, and a functor $F\colon \mathbf{A}\to\mathbf{\Gamma}$ given by $F(M) = \Gamma\otimes_A M$ (A1.2.1). Mac Lane [**1**] then defines a resolvent pair to be the above data along with a natural transformation from GF to the identity on $\mathbf{A}$, i.e., natural maps $M\to\Gamma\otimes_A M$ with a certain universal property. We have instead maps $\varepsilon\otimes M\colon \Gamma\otimes_A M\to M$ such that for any A-homomorphism $\mu\colon C\to M$ where C is a Γ-comodule there is a unique comodule map $\alpha\colon C\to\Gamma\otimes_A M$ such that $\mu = (\varepsilon\otimes M)\alpha$. Thus we have what Mac Lane might call a coresolvent pair. Our F produces relative injectives while his produces relative projectives. This duality is to be expected because the example he had in mind was the category of modules over an algebra, while our category $\mathbf{\Gamma}$ is more like that of comodules over a coalgebra. The following lemma is comparable to Theorem IX.6.1 of Mac Lane [**1**].

A1.2.8. Lemma.

(a) *If* $i\colon M\to N$ *is a monomorphism of comodules which is split over* A*, then any map* f *from* M *to a relatively injective comodule* S *extends to* N*.* (*If* i *is not assumed to be split, then this property would make* S *injective.*)

(b) *If* M *is projective as an* A*-module and* S *is a relatively injective comodule, then* $\mathrm{Cotor}_\Gamma^i(M,S) = 0$ *for* $i>0$ *and if* $S = \Gamma\otimes_A N$ *then* $\mathrm{Cotor}_\Gamma^0(M,S) = M\otimes_A N$.

Proof. (a) Let $j\colon N\to M$ be a splitting of i. Then $(\Gamma\otimes f)(\Gamma\otimes j)\psi = g$ is a comodule map from N to $\Gamma\otimes_A S$ such that $gi = \psi f\colon M\to\Gamma\otimes_A S$. It suffices then to show that S is a direct summand of $\Gamma\otimes_A S$, for then g followed by the projection of $\Gamma\otimes_A S$ onto S will be the desired extension of f. By definition S is a direct summand of $\Gamma\otimes_A T$ for some A-module T. Let $k\colon S\to\Gamma\otimes_A T$ and $k^{-1}\colon\Gamma\otimes_A T\to S$ be the splitting maps. Then $k^{-1}(\Gamma\otimes\varepsilon\otimes T)(\Gamma\otimes k)$ is the projection of $\Gamma\otimes_A S$ onto S.

(b) One has an isomorphism $\phi\colon M\otimes_A N\to M\,\square_\Gamma(\Gamma\otimes_A N)$ given by $\phi(m\otimes n) = \psi(m)\otimes n$. Since S is a direct summand of $\Gamma\otimes_A N$, it suffices to replace the former by the latter. Let

$$0\to N\to I^0\to I^1\to\cdots$$

be a resolution of N by injective A-modules. Tensoring over A with Γ gives a resolution of $\Gamma\otimes_A N$ by injective Γ-comodules. $\mathrm{Cotor}_\Gamma(M,\Gamma\otimes_A N)$ is the cohomology of the resolution cotensored with M, which is isomorphic to

$$M \otimes_A I^0 \to M \otimes_A I^1 \to \cdots .$$

This complex is acyclic since M is projective over A. $\square$

Compare the following with Theorem IX.4.3 of Mac Lane [**1**].

A1.2.9. Lemma. (a) *Let*

$$0 \to M \xrightarrow{d_{-1}} P^0 \xrightarrow{d_0} P^1 \xrightarrow{d_1} \cdots$$

and

$$0 \to N \xrightarrow{d_{-1}} R^0 \xrightarrow{d_0} R^1 \xrightarrow{d_1} \cdots$$

be long exact sequences of Γ-comodules in which each P^i and R^i is relatively injective and the image of each map is a direct summand over A. Then a comodule map $f\colon M \to N$ extends to a map of long exact sequences.

(b) *Applying $L \,\square_\Gamma\, (\cdot)$ (where L is a right Γ-comodule projective over A) to the two sequences and taking cohomology gives $\mathrm{Cotor}_\Gamma(L, M)$ and $\mathrm{Cotor}_\Gamma(L, N)$, respectively. The induced map from the former to the latter depends only on f.*

Proof. That the cohomology indicated in (b) is Cotor follows from A1.2.4 and A1.2.8(b). The proof of the other assertions is similar to that of the analogous statements about injective resolutions. Define comodules M^i and N^i inductively by $M^0 = M$, $N^0 = N$, and M^{i+1} and N^{i+1} are the quotients in the short exact sequences

$$0 \to M^i \to P^i \to M^{i+1} \to 0$$

and

$$0 \to N^i \to R^i \to N^{i+1} \to 0.$$

These sequences are split over A. Assume inductively that we have a suitable map from M^i to N^i. Then A1.2.8(a) gives us $f_i\colon P^i \to R^i$, and this induces a map from M^{i+1} to N^{i+1}, thereby proving (a).

For (b) it suffices to show that the map of long exact sequences is unique up to chain homotopy, i.e., given two sets of maps $f_i, f_i'\colon P^i \to R^i$ we need to construct $h_i\colon P^i \to R^{i-1}$ (with $h_0 = 0$) such that $h_{i+1}d_i + d_{i-1}h_i = f_i - f_i'$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M^i \arrow[r, "d_{i-1}"] \arrow[d, "g_{i-1}"'] & P^i \arrow[r, "d_i"] \arrow[d, "g_i"'] & M^{i+1} \arrow[r] \arrow[d, "g_i"'] & 0 \\
0 \arrow[r] & N^i d_{i-1} \arrow[r] & R^i d_i \arrow[r] & N^{i+1} \arrow[r] & 0
\end{tikzcd}$$

where $g_i = f_i - f_i'\colon P^i \to R^i$ and we use the same notation for the map induced from the quotient M^{i+1}. Assume inductively that $h_i\colon P^i \to R^{i-1}$ has been constructed. Projecting it to N^i we get $h_i\colon P^i \to N^i$ with $h_i d_{i-1} = g_{i-1}$. Now we want a map $hath_{i+1}\colon M^{i+1} \to R^i$ such that $hath_{i+1}d_i = g_i - d_{i-1}h_i$. By the exactness of the top row, $hath_{i+1}$ exists iff $(g_i - d_{i-1}h_i)d_{i-1} = 0$. But we have $g_i d_{i-1} - d_{i-1}(h_i d_{i-1}) = g_i d_{i-1} - d_i g_{i-1} = 0$, so $whath_i$ exists. By A1.2.8(a) it extends from M^{i+1} to P^{i+1} giving the desired h_{i+1}. $\square$

Resolution of the above type serve as a substitute for injective resolutions. Hence we have

A1.2.10. Definition. *A* resolution by relative injectives *of a comodule M is a long exact sequence*

$$0 \to M \to R^0 \to R^1 \to \dots$$

in which each R^i is a relatively injective and the image of each map is a direct summand over A. We now give an important example of such a resolution.

A1.2.11. Definition. *Let M be a left Γ-comodule. The* cobar resolution *$D_\Gamma^*(M)$ is defined by $D_\Gamma^s(M) = \Gamma \otimes_A \overline{\Gamma}^{\otimes s} \otimes_A M$, where $\overline{\Gamma} = \ker \varepsilon$, with coboundary $d_s\colon D_\Gamma^s(M) \to D_\Gamma^{s+1}(M)$ given by*

$$\begin{aligned} d_s(\gamma_0 \otimes \gamma_1 \otimes \cdots \gamma_s \otimes m) = \sum_{i=0}^{s} &(-1)^i \gamma_0 \otimes \cdots \gamma_{i-1} \otimes \Delta(\gamma_i) \otimes \gamma_{i+1} \otimes \cdots m \\ &+ (-1)^{s+1} \gamma_0 \otimes \cdots \gamma_s \otimes \psi(m) \end{aligned}$$

for $\gamma_0 \in \Gamma$, $\gamma_1, \dots, \gamma_s \in \overline{\Gamma}$, and $m \in M$. For a right Γ-comodule L which is projective over A, the cobar complex *$C_\Gamma^*(L, M)$ is $L \,\square_\Gamma\, D_\Gamma^*(M)$, so $C_\Gamma^s(L, M) = L \otimes_A \Gamma^{\otimes s} \otimes_A M$, where $\Gamma^{\otimes s}$ denotes the s-fold tensor product of Γ over A. Whenever possible the subscript Γ will be omitted, and $C_\Gamma^*(A, M)$ will be abbreviated to $C_\Gamma^*(M)$. The element $a \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes m \in C_\Gamma(L, M)$, where $a \in L$, will be denoted by $a\gamma_1|\gamma_2|\cdots|\gamma_n m$. If $a = 1$ or $m = 1$, they will be omitted from this notation.*

A1.2.12. Corollary. *$H(C_\Gamma^*(L, M)) = \operatorname{Cotor}_\Gamma(L, M)$ if L is projective over A, and $H(C_\Gamma^*(M)) = \operatorname{Ext}_\Gamma(A, M)$.*

Proof. It suffices by A1.2.9 to show that $D_\Gamma(M) = C_\Gamma(\Gamma, M)$ is a resolution of M by relative injectives. It is clear that $D_\Gamma^s(M)$ is a relative injective and that d^s is a comodule map. To show that $D_\Gamma(M)$ is acyclic we use a contacting homotopy $S\colon D_\Gamma^s(M) \to D_\Gamma^{s-1}(M)$ defined by $S(\gamma\gamma_1|\cdots|\gamma_s m) = \varepsilon(\gamma)\gamma_1\gamma_2|\cdots|\gamma_s m$ for $s > 0$ and $S(\gamma m) = 0$. Then $Sd + dS$ is the identity on $D_\Gamma^s(M)$ for $s > 0$, and $1 - \phi$ on $D_\Gamma^0(M)$, where $\phi(\gamma m) = \varepsilon(\gamma) m' m''$. Hence

$$H^s(D_\Gamma(M)) = \begin{cases} 0 & \text{for } s > 0, \\ \operatorname{im} \phi = M & \text{for } s = 0. \end{cases} \qquad \square$$

Our next job is to define the external cup product in Cotor, which is a map $\operatorname{Cotor}_\Gamma(M_1, N_1) \otimes \operatorname{Cotor}_\Gamma(M_2, N_2) \to \operatorname{Cotor}_\Gamma(M_1 \otimes_A M_2, N_1 \otimes_A N_2)$ (see A1.1.2 for the definition of the comodule tensor product). If $M_1 = M_2 = M$ and $N_1 = N_2 = N$ are comodule algebras (A1.1.2) then composing the above with the map in Cotor induced by $M \otimes_A M \to M$ and $N \otimes_A N \to N$ gives a product on $\operatorname{Cotor}_\Gamma(M, N)$. Let P_1^* and P_2^* denote relative injective resolutions of N_1 and N_2, respectively. Then $P_1^* \otimes_A P_2^*$ is a resolution of $N_1 \otimes_A N_2$. We have canonical maps

$$\operatorname{Cotor}_\Gamma(M_1, N_1) \otimes \operatorname{Cotor}_\Gamma(M_2, N_2) \to H(M_1 \,\square_\Gamma\, P_1^* \otimes M_2 \,\square_\Gamma\, P_2^*)$$

(with tensor products over K) and

$$M_1 \,\square_\Gamma\, P_1^* \otimes M_2 \,\square_\Gamma\, P_2^* \to (M_1 \otimes_A M_2) \,\square_\Gamma\, (P_1^* \otimes_A P_2^*).$$

A1.2.13. Definition. *The external cup product*

$$\operatorname{Cotor}_\Gamma(M_1, N_1) \otimes \operatorname{Cotor}_\Gamma(M_2, N_2) \to \operatorname{Cotor}_\Gamma(M_1 \otimes_A M_2, N_1 \otimes_A N_2)$$

and the internal cup product on $\operatorname{Cotor}_\Gamma(M, N)$ for comodule algebras M and N are induced by the maps described above.

Note that A1.2.9(b) implies that these products are independent of the choices made. Since the internal product is the composition of the external product with the products on M and N and since the latter are commutative and associative we have

A1.2.14. COROLLARY. *If M and N are comodule algebras then* $\mathrm{Cotor}_\Gamma(M, N)$ *is a commutative (in the graded sense) associative algebra.* □

It is useful to have an explicit pairing on cobar complexes

$$C_\Gamma(M_1, N_1) \otimes C_\Gamma(M_2, N_2) \to C_\Gamma(M_1 \otimes M_2, N_1 \otimes N_2).$$

This can be derived from the definitions by tedious straightforward calculation. To express the result we need some notation. For $m_2 \in M_2$ and $n_1 \in N_1$ let

$$m_2^{(0)} \otimes \cdots \otimes m_2^{(s)} \in M_2 \otimes_A \Gamma^{\otimes s}$$

and

$$n_1^{(1)} \otimes \cdots \otimes n_1^{(t+1)} \in \Gamma^{\otimes t} \otimes_A N_1$$

denote the iterated coproducts. Then the pairing is given by

$$\begin{aligned}\text{(A1.2.15)}\quad & m_1\gamma_1|\cdots|\gamma_s n_1 \otimes m_2\gamma_{s+1}|\cdots|\gamma_{s+1}n_2 \\ & \to (-1)^\tau m_1 \otimes m_2^{(0)}\gamma_1 m_2^{(1)}|\cdots|\gamma_s m_2^{(s)}|n_1^{(1)}\gamma_{s+1}|\cdots|n_1^{(t)}\gamma_{s+t}n_1^{(1+t)} \otimes n_2\end{aligned}$$

where

$$\begin{aligned}\tau = \deg m_2 \deg n_1 + \sum_{i=0}^{s} \deg m_2^{(i)}\left(s - i + \sum_{j=i+1}^{s} \deg \gamma_j\right) \\ + \sum_{i=1}^{t+1} \deg n_1^{(i)}\left(i - 1 + \sum_{j=1}^{i-1} \deg \gamma_{j+s}\right).\end{aligned}$$

Note that this is natural in all variables in sight.

Finally, we have two easy miscellaneous results.

A1.2.16. PROPOSITION. (a) *If $I \subset A$ is invariant* (A1.2.12) *then* $(A/I, \Gamma/I\Gamma)$ *is a Hopf algebroid.*

(b) *If M is a left Γ-comodule annihilated by I as above, then*

$$\mathrm{Ext}_\Gamma(A, M) = \mathrm{Ext}_{\Gamma/I\Gamma}(A/I, M).$$

PROOF. Part (a) is straightforward. For (b) observe that the complexes $C_\Gamma(M)$ and $C_{\Gamma/I\Gamma}(M)$ are identical. □

A1.2.17. PROPOSITION. *If (A, Γ) is split* (A1.1.22) *then* $\mathrm{Ext}_\Gamma(A, M) = \mathrm{Ext}_\Sigma(K, M)$ *where the left Σ-comodule structure on the left Γ-comodule M comes from the isomorphism $\Gamma \otimes_A M = \Sigma \otimes M$.*

PROOF. $C_\Gamma(M) = C_\Sigma(M)$. □

3. Some Spectral Sequences

In this section we describe several spectral sequences useful for computing Ext over a Hopf algebroid. The reader is assumed to be familiar with the notion of a spectral sequence; the subject is treated in each of the standard references for homological algebra (Cartan and Eilenberg [**1**], Mac Lane [**1**] and Hilton and Stammbach [**1**]) and in Spanier [**1**]. The reader is warned that most spectral sequences can be indexed in more than one way. With luck the indexing used in this section will be consistent with that used in the text, but it may differ from that appearing elsewhere in the literature and from that used in the next two sections.

Suppose we have a long exact sequence of Γ-comodules

$$0 \to M \to R^0 \xrightarrow{d^0} R^1 \xrightarrow{d^1} R^2 \to \cdots \tag{A1.3.1}$$

Let $S^{i+1} = \operatorname{im} d^i$ and $S^0 = M$ so we have short exact sequences

$$0 \to S^i \xrightarrow{a^i} R^i \xrightarrow{b^i} S^{i+1} \to 0$$

for all $i \geq 0$. Each of these gives us a connecting homomorphism

$$\delta^i\colon \operatorname{Cotor}_\Gamma^{s,t}(L, S^i) \to \operatorname{Cotor}_\Gamma^{s+1,t}(L, S^{i-1}).$$

Let $\delta_{(i)}\colon \operatorname{Cotor}_\Gamma^{s,t}(L, S^i) \to \operatorname{Cotor}_\Gamma^{s+i,t}(L, S^0)$ be the composition $\delta^1\delta^2\cdots\delta^i$. Define a decreasing filtration on $\operatorname{Cotor}_\Gamma^{s,*}(L, M)$ by $F^i = \operatorname{im}\delta_{(i)}$ for $i \leq s$, where $\delta_{(0)}$ is the identity and $F^i = 0$ for $i \leq 0$.

A1.3.2. THEOREM. *Given a long exact sequence of* Γ*-comodules* A1.3.1 *there is a natural trigraded spectral sequence* (E_*^{***}) *(the resolution spectral sequence) such that*

(a) $E_1^{n,s,t} = \operatorname{Cotor}_\Gamma^{s,t}(L, R^n)$;
(b) $d_r\colon E_r^{n,s,t} \to E_r^{n+r,s-r+1,t}$ *and* d_1 *is the map induced by* d^* *in* A1.3.1 *and*
(c) $E_\infty^{n,s,t}$ *is the subquotient* F^n/F^{n+1} *of* $\operatorname{Cotor}_\Gamma^{n+s,t}(L, M)$ *defined above.*

PROOF. We will give two constructions of this spectral sequence. For the first define an exact couple (2.1.6) by

$$\begin{aligned} E_1^{s,t} &= \operatorname{Cotor}_\Gamma^{-t,*}(L, R^s),\\ D_1^{s,t} &= \operatorname{Cotor}_\Gamma^{-t,*}(L, S^s), \end{aligned}$$

$i_1 = \delta^*$, $j_1 = a^*$, and $k_1 = b^*$. Then the associated spectral sequence is the one we want.

The second construction applies when L is projective over A and is more explicit and helpful in practice; we get the spectral sequence from a double complex as described in Cartan and Eilenberg [**1**], Section XV.6 or Mac Lane [**1**], Section XI.6. We will use the terminology of the former. Let

$$\begin{aligned} B^{n,s,*} &= C_\Gamma^s(L, R^n) \qquad \text{(A1.2.11)},\\ \partial_1^{n,s,*} &= (-1)^n C_\Gamma^s(d^n)\colon B^{n,s,*} \to B^{n+1,s,*}, \end{aligned}$$

and

$$\partial_2^{n,s,*} = d^s\colon B^{n,s,*} \to B^{n,s+1,*}.$$

(Our ∂_1, ∂_2 correspond to the d_1, d_2 in Cartan and Eilenberg [**1**], IV.4].) Then $\partial_2^{n+1,s,*}\partial_1^{n,s,*} + \partial_1^{n,s+1,*}\partial_2^{n,s,*} = 0$ since d^s commutes with $C_\Gamma^s(d^n)$. The associated complex $(B^{p,*}, \partial)$ is defined by

$$B^{p,*} = \bigoplus_{n+s=p} B^{n,s,*} = \bigoplus_{n+s=p} C_\Gamma^s(L, R^n) \tag{A1.3.3}$$

with $\partial = \partial_1 + \partial_2 \colon B^{p,*} \to B^{p+1,*}$.

This complex can be filtered in two ways, i.e.,

$$F_I^p B = \bigoplus_{r\geq p} \bigoplus_q B^{r,q,*},$$
$$F_{II}^q B = \bigoplus_{s\geq q} \bigoplus_p B^{p,s,*}$$

and each of these filtrations leads to a spectral sequence. In our case the functor $C_\Gamma^s(L, \cdot)$ is exact since Γ is flat over A, so $H^{s,*}(F_{II}B) = C_\Gamma^s(L, M)$. Hence in the second spectral sequence

$$E_1^{n,s,*} = \begin{cases} C_\Gamma^s(L, M) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E_2^{n,s,*} = E_\infty^{n,s,*} = \begin{cases} \operatorname{Cotor}_\Gamma^{s,*}(L, M) & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The two spectral sequences converge to the same thing, so the first one, which is the one we want, has the desired properties. □

A1.3.4. Corollary. *The cohomology of the complex* B^{**} *of* A1.3.3 *is* $\operatorname{Cotor}_\Gamma^{**}(L, M)$. □

Note that A1.2.4 is a special case of A1.3.3 in which the spectral sequence collapses.

Next we discuss spectral sequences arising from increasing and decreasing filtration of Γ.

A1.3.5. Definition. *An* increasing filtration *on a Hopf algebroid* (A, Γ) *is an increasing sequaence of sub-K-modules*

$$K = F_0\Gamma \subset F_1\Gamma \subset F_2\Gamma \subset \cdots$$

with $\Gamma = \bigcup F_s\Gamma$ *such that*

(a) $F_s\Gamma \cdot F_t\Gamma \subset F_{s+t}\Gamma$,
(b) $c(F_s\Gamma) \subset F_s\Gamma$, *and*
(c) $\Delta F_s\Gamma \subset \bigoplus_{p+q=s} F_p\Gamma \otimes_A F_q\Gamma$.

A decreasing filtration *on* (A, Γ) *is a decreasing sequaence of sub-K-modules*

$$\Gamma = F^0\Gamma \supset F^1\Gamma \supset F^2\Gamma \supset \cdots$$

with $0 = \bigcap F^s\Gamma$ *such that conditions similar to (a), (b), and (c) above (with the inclusion signs reversed) are satisfied. A* filtered Hopf algebroid (A, Γ) *is one equipped with a filtration. Note that a filtration on* Γ *induces one on* A, *e.g.,*

$$F_sA = \eta_L(A) \cap F_s\Gamma = \eta_R(A) \cap F_s\Gamma = \varepsilon(F_s\Gamma).$$

A1.3.6. DEFINITION. *Let (A, Γ) be filtered as above. The associated graded object $E^0\Gamma$ (or $E_0\Gamma$) is defined by*

$$E_s^0\Gamma = F_s\Gamma/F_{s-1}\Gamma$$

or

$$E_0^s\Gamma = F^s\Gamma/F^{s-1}\Gamma.$$

The graded object $E_*^0 A$ (or $E_0^* A$) is defined similarly.

A1.3.7. DEFINITION. *Let M be a Γ-comodule. An increasing filtration on M is an increasing sequence of sub-K-modules*

$$0 = F_1 M \subset F_2 M \subset \cdots$$

such that $M = \bigcup F_s M$, $F_s A \cdot F_t M \subset F_{s+t} M$, and

$$\psi(F_s M) \subset \bigoplus_{p+q=s} F_p\Gamma \otimes F_q M.$$

A decreasing filtration on M is similarly defined, as is the associated graded object $E_*^0 M$ or $E_0^* M$. A filtered comodule *M is a comodule equipped with a filtration.*

A1.3.8. PROPOSITION. *$(E^0 A, E^0\Gamma)$ or $(E_0 A, E_0\Gamma)$ is a graded Hopf algebroid and $E^0 M$ or $E_0 M$ is a comodule over it.* □

Note that if (A, Γ) and M are themselves graded than $(E^0 A, E^0\Gamma)$ and $E^0 M$ are bigraded.

We assume from now on that $E^0\Gamma$ or $E_0\Gamma$ is flat over $E^0 A$ or $E_0 A$.

A1.3.9. THEOREM. *Let L and M be right and left filtered comodules, respectively, over a filtered Hopf algebroid (A, Γ). Then there is a natural spectral sequence converging to $\mathrm{Cotor}_\Gamma(L, M)$ such that*

(a) *in the increasing case*

$$E_1^{s,*} = \mathrm{Cotor}^s_{E^0\Gamma}(E^0 L, E^0 M)$$

where the second grading comes from the filtration and

$$d_r \colon E_r^{s,t} \to E_r^{s+1,t-r};$$

(b) *in the decreasing case*

$$E_1^{s,*} = \mathrm{Cotor}^s_{E_0\Gamma}(E_0 L, E_0 M)$$

and

$$d_r \colon E_r^{s,t} \to E_r^{s+1,t+r}.$$

Note that our indexing differs from that of Cartan and Eilenberg [**1**] and Mac Lane [**1**].

PROOF. The filtrations on Γ and M induce one on the cobar complex (A1.1.14) $C_\Gamma M$ and we have $E_0 C_\Gamma(L, M) = C_{E_0\Gamma}(E_0 L, E_0 M)$ or $E^0 C_\Gamma(L, M) = C_{E^0\Gamma}(E^0 L, E^0 M)$. The associated spectral sequence is the one we want. □

The following is an important example of an increasing filtration.

A1.3.10. Example. Let (K, Γ) be a Hopf algebra. Let $\overline{\Gamma}$ be the unit coideal, i.e., the quotient in the short exact sequence

$$0 \to K \xrightarrow{\eta} \Gamma \to \overline{\Gamma} \to 0,$$

The coproduct map Δ can be iterated by coassociativity to a map $\Delta^s \colon \Gamma \to \Gamma^{\otimes s+1}$. Let $F_s\Gamma$ be the kernel of the composition

$$\Gamma \xrightarrow{\Delta^s} \Gamma^{\otimes s+1} \to \overline{\Gamma}^{\otimes s+1}.$$

This is the *filtration of* $\mathbf{\Gamma}$ *by powers of the unit coideal.*

Next we treat the spectral sequence associated with a map of Hopf algebroids.

A1.3.11. Theorem. *Let $f \colon (A, \Gamma) \to (B, \Sigma)$ be a map of Hopf algebroids* (A1.1.18)*, M a right Γ-comodule and N a left Σ-comodule.*

(a) *$C_\Sigma(\Gamma \otimes_A B, N)$ is a complex of left Γ-comodules, so $\operatorname{Cotor}_\Sigma(\Gamma \otimes_A B, N)$ is a left Γ-comodule.*

(b) *If M is flat over A, there is a natural spectral sequence converging to $\operatorname{Cotor}_\Sigma(M \otimes_A B, N)$ with*

$$E_2^{s,t} = \operatorname{Cotor}_\Gamma^s(M, \operatorname{Cotor}_\Sigma^t(\Gamma \otimes_A B, N))$$

and $d_r \colon E_r^{s,t} \to E_r^{s+r,t-r+1}$.

(c) *If N is a comodule algebra then so is $\operatorname{Cotor}_\Sigma(\Gamma \otimes_A B, N)$. If M is also a comodule algebra, then the spectral sequence is one of algebras.*

Proof. For (a) we have $C_\Sigma^s(\Gamma \otimes_A B, N) = \Gamma \otimes_A \overline{\Sigma}^{\otimes s} \otimes_B N$ with the coboundary d_s as given in A1.2.11. We must show that d_s commutes with the coproduct on Γ. For all terms other than the first in the formula for d_s this commutativity is clear. For the first term consider the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \Gamma \otimes_A \Gamma \arrow[r, "\Gamma \otimes f"] \arrow[d, "\Delta \otimes \Gamma"] & \Gamma \otimes_A \Sigma \arrow[d, "\Delta \otimes \Sigma"] \\
\Gamma \otimes_A \Gamma \arrow[r, "\Gamma \otimes \Delta"] & \Gamma \otimes_A \Gamma \otimes_A \Gamma \arrow[r, "\Gamma \otimes \Gamma \otimes f"] & \Gamma \otimes_A \Gamma \otimes_A \Sigma
\end{tikzcd}$$

The left-hand square commutes by coassociativity and other square commutes trivially. The top composition when tensored over B with $\Sigma^{\otimes s} \otimes_B N$ is the first term in d_s. Hence the commutativity of the diagram shows that d_s is a map of left Γ-comodules.

For (b) consider the double complex

$$C_\Gamma^*(M, C_\Sigma^*(\Gamma \otimes_A B, N)),$$

which is well defined because of (a). We compare the spectral sequences obtained by filtering by the two degrees. Filtering by the first gives

$$E_1 = C_\Gamma^*(M, \operatorname{Cotor}_\Sigma(\Gamma \otimes_A B, N))$$

so

$$E_2 = \operatorname{Cotor}_\Gamma(M, \operatorname{Cotor}_\Sigma(\Gamma \otimes_A B, N))$$

which is the desired spectral sequence. Filtering by the second degree gives a spectral sequence with

$$\begin{aligned} E_1^{s,t} &= \operatorname{Cotor}_\Gamma^s(M, C_\Sigma^t(\Gamma \otimes_A B, N)) \\ &= \operatorname{Cotor}_\Gamma(M, \Gamma \otimes_A \overline{\Sigma}^{\otimes t} \otimes_B N) \\ &= M \otimes_A \overline{\Sigma}^{\otimes t} \otimes_B N \quad \text{by A1.2.8(b)} \\ &= C_\Sigma^t(M \otimes_A B, N) \end{aligned}$$

so $E_2 = E_\infty = \operatorname{Cotor}_\Sigma(M \otimes_A B, N)$.

For (c) note that $\Gamma \otimes_A B$ as well as N is a Σ-comodule algebra. The Γ-coaction on $C_\Sigma(\Gamma \otimes_A B, N)$ is induced by the map

$$\begin{aligned} C(\Delta \otimes B, N) \colon C_\Sigma(\Gamma \otimes_A B, N) &\to C_\Sigma(\Gamma \otimes_A \Gamma \otimes_A B, N) \\ &= \Gamma \otimes_A C_\Sigma(\Gamma \otimes_A B, N). \end{aligned}$$

Since the algebra structure on $C_\Sigma(\ ,\ )$ is functorial, $C(\Delta \otimes B, N)$ induces an algebra map in cohomology and $\operatorname{Cotor}_\Sigma(\Gamma \otimes_A B, N)$ is a Γ-comodule algebra.

To show that we have a spectral sequence of algebras we must define an algebra structure on the double complex used in the proof of (b), which is $M \,\square_\Gamma\, D_\Gamma(\Gamma \otimes_A B \,\square_\Sigma\, D_\Sigma(N))$. Let $\widetilde{N} = \Gamma \otimes_A B \,\square_\Sigma\, D_\Sigma(N)$. We have just seen that it is a Γ-comodule algebra. Then this algebra structure extends to one on $D_\Gamma(\widetilde{N})$ by A1.2.9 since $D_\Gamma(\widetilde{N}) \otimes_A D_\Gamma(\widetilde{N})$ is a relatively injective resolution of $\widetilde{N} \otimes_A \widetilde{N}$. Hence we have maps

$$\begin{aligned} M \,\square_\Gamma\, D_\Gamma(\widetilde{N}) \otimes M \,\square_\Gamma\, D_\Gamma(\widetilde{N}) &\to M \otimes_A M \,\square_\Gamma\, D_\Gamma(\widetilde{N}) \otimes_A D_\Gamma(\widetilde{N}) \\ &\to M \,\square_\Gamma\, D_\Gamma(\widetilde{N}) \otimes_A D_\Gamma(\widetilde{N}) \to M \,\square_\Gamma\, D_\Gamma(\widetilde{N}), \end{aligned}$$

which is the desired algebra structure. □

Our first application of this spectral sequence is a change-of-rings isomorphism that occurs when it collapses.

A1.3.12. Change-of-Rings Isomorphism Theorem. *Let* $f \colon (A, \Gamma) \to (B, \Sigma)$ *be a map of graded connected Hopf algebroids* (A1.1.7) *satisfying the hypotheses of* A1.1.19; *let* M *be a right* Γ*-comodule and let* N *be a left* Σ*-comodule which is flat over* B*. Then*

$$\operatorname{Cotor}_\Gamma(M, (\Gamma \otimes_A B) \,\square_\Sigma\, N) = \operatorname{Cotor}_\Sigma(M \otimes_A B, N).$$

In particular

$$\operatorname{Ext}_\Gamma(A, (\Gamma \otimes_A B) \,\square_\Sigma\, N) = \operatorname{Ext}_\Sigma(B, N),$$

Proof. By A1.1.19 and A1.2.8(b) we have

$$\operatorname{Cotor}_\Sigma^s(\Gamma \otimes_A B, N) = 0 \quad \text{for } s > 0.$$

A1.3.11(b) gives

$$\operatorname{Cotor}_\Gamma(M, \operatorname{Cotor}_\Sigma^0(\Gamma \otimes_A B, N)) = \operatorname{Cotor}_\Sigma(M \otimes_A B, N).$$

Since N is flat over B,

$$\operatorname{Cotor}_\Sigma^0(\Gamma \otimes_A B, N) = (\Gamma \otimes_A B) \,\square_\Sigma\, N$$

and the result follows. □

A1.3.13. COROLLARY. *Let K be a field and $f\colon (K,\Gamma)\to(K,\Sigma)$ be a surjective map of Hopf algebras. If N is a left Σ-comodule then*

$$\mathrm{Ext}_\Gamma(K,\Gamma\,\square_\Sigma\,N)=\mathrm{Ext}_\Sigma(K,N). \qquad \square$$

Next we will construct a change-of-rings spectral sequence for an extension of Hopf algebroids (A1.1.15) similar to that of Cartan and Eilenberg [**1**, XVI 6.1], which we will refer to as the Cartan–Eilenberg spectral sequence.

A1.3.14. CARTAN–EILENBERG SPECTRAL SEQUENCE THEOREM. *Let*

$$(D,\Phi)\xrightarrow{i}(A,\Gamma)\xrightarrow{f}(A,\Sigma)$$

be an extension of graded connected Hopf algebroids (A1.1.15). *Let M be a right Φ-comodule and N a left Γ-comodule.*

(a) $\mathrm{Cotor}_\Sigma(A,N)$ *is a left Φ-comodule. If N is a comodule algebra, then so is this* Cotor.

(b) *There is a natural spectral sequence converging to* $\mathrm{Cotor}_\Gamma(M\otimes_D A,N)$ *with*

$$E_2^{s,t}=\mathrm{Cotor}_\Phi^s(M,\mathrm{Cotor}_\Sigma^t(A,N))$$

and

$$d_r\colon E_r^{s,t}\to E_r^{s+r,t-r+1}.$$

(c) *If M and N are comodule algebras, then the spectral sequence is one of algebras.*

PROOF. Applying A1.3.11 to the map i shows that $\mathrm{Cotor}_\Gamma(\Phi\otimes_D A,N)$ is a left Φ-comodule algebra and there is a spectral sequence converging to $\mathrm{Cotor}_\Gamma(M\otimes_D A,N)$ with

$$E_2=\mathrm{Cotor}_\Phi(M,\mathrm{Cotor}_\Gamma(\Phi\otimes_A D,N)).$$

Hence the theorem will follow if we can show that $\mathrm{Cotor}_\Gamma(\Phi\otimes_D A,N)=\mathrm{Cotor}_\Sigma(A,N)$. Now $\Phi\otimes_D A=A\,\square_\Sigma\,\Gamma$ by A1.1.16. We can apply A1.3.12 to f and get $\mathrm{Cotor}_\Gamma(P\,\square_\Sigma\,\Gamma,R)=\mathrm{Cotor}_\Sigma(P,R)$ for a right Σ-comodule P and left Γ-comodule R. Setting $P=A$ and $R=N$ gives the desired isomorphism

$$\mathrm{Cotor}_\Gamma(\Phi\otimes_D A,N)=\mathrm{Cotor}_\Gamma(A\,\square_\Sigma\,\Gamma,N)=\mathrm{Cotor}_\Sigma(A,N). \qquad \square$$

The case $M=D$ gives

A1.3.15. COROLLARY. *With notation as above, there is a spectral sequence of algebras converging to* $\mathrm{Ext}_\Gamma(A,N)$ *with* $E_2=\mathrm{Ext}_\Phi(D,\mathrm{Ext}_\Sigma(A,N))$. $\square$

Now we will give an alternative formulation of the Cartan–Eilenberg spectral sequence (A1.3.14) suggested by Adams [**12**], 2.3.1 which will be needed to apply the results of the next sections on Massey products and Steenrod operations. Using the notation of A1.2.14, we define a decreasing filtration on $C_\Gamma(M\otimes_D A,N)$ by saying that $m\gamma_1|\dots|\gamma_s n\in F^i$ if i of the γ's are in $\ker f_2$.

A1.3.16. THEOREM. *The spectral sequence associated with the above filtration of $C_\Gamma(M\otimes_D A,N)$ coincides with the Cartan–Eilenberg spectral sequence of* A1.3.14.

PROOF. The Cartan–Eilenberg spectral sequence is obtained by filtering the double complex $C_\Phi^*(M,C_\Gamma^*(\Phi\otimes_D A,N))$ by the first degree. We define a filtration-preserving map θ from this complex to $C_\Gamma(M\otimes_D A,N)$ by

$$\begin{aligned}\theta(m\otimes\phi_1\otimes\cdots\phi_s\otimes\phi\otimes\gamma_{s+1}\otimes\cdots\gamma_{s+t}\otimes n)\\ =m\otimes i_2(\phi_1)\otimes\cdots i_2(\phi_s)i_1\varepsilon(\phi)\otimes\gamma_{s+1}\otimes\cdots\gamma_{s+t}\otimes n.\end{aligned}$$

Let $E_1^{s,t}(M,N) = C_\Phi^s(M, \mathrm{Cotor}_\Gamma^t(\Phi \otimes_D A, N)) = C_\Phi^s(M, \mathrm{Cotor}_\Sigma^t(A, N))$ be the E_1-term of the Cartan–Eilenberg spectral sequence and $\widetilde{E}_1(M,N)$ the E_1-term of the spectral sequence in question. It suffices to show that

$$\theta_*\colon E_1(M,N) \to \widetilde{E}_1(M,N)$$

is an isomorphism.

First consider the case $s = 0$. We have

$$F_0/F^1 = C_\Sigma(M \otimes_D A, N) = M \otimes_D C_\Sigma(A, N)$$

so this is the target of θ for $s = 0$. The source is $M \otimes_D C_\Gamma(\phi \otimes_D A, N)$. The argument in the proof of Theorem A1.3.14 showing that

$$\mathrm{Cotor}_\Gamma(\Phi \otimes_D A, N) = \mathrm{Cotor}_\Sigma(A, N)$$

shows that our two complexes are equivalent so we have the desired isomorphism for $s = 0$.

For $s > 0$ we use the following argument due to E. Ossa.

The differential

$$d_0\colon E_0^{s,t}(M,N) \to E_0^{s,t+1}(M,N)$$

depends only on the Σ-comodule structures of M and N. In fact we may define a complex $\widetilde{D}_\Sigma(N)$ formally by

$$\widetilde{D}_\Sigma^{s,t}(N) = \widetilde{E}_0^{s,t}(\Sigma, N).$$

Then we have

$$\widetilde{E}_0^{s,t}(M,n) = M \,\square_\Sigma\, \widetilde{D}_\Sigma^{s,t}(N).$$

Observe that

$$\widetilde{D}_\Sigma^{0,t}(N) = C_\Sigma^t(\Sigma, N).$$

Now let $G = \ker f$ and

$$C^{s+1} = G^s \,\square_\Sigma\, G = G \,\square_\Sigma\, G \,\square_\Sigma \dots \square_\Sigma\, G$$

with $s+1$ factors.

Note that

$$G = \Sigma \otimes \overline{\Phi} \quad \text{and hence}$$
$$G^s = \Sigma \otimes \overline{\Phi}^{\otimes s}$$

as left Σ-comodules, where the tensor products are over D.

Define

$$\beta_s\colon G^s \,\square_\Sigma\, \widetilde{D}_\Sigma^{0,t}(N) \to \widetilde{D}_\Sigma^{s,t}(N)$$

by

$$\beta_s((g_1 \otimes \dots g_s) \otimes \sigma_1 \otimes \cdots \otimes \sigma_t \otimes n) = \Sigma f(g_1')g_1'' \otimes g_2 \cdots \otimes g_s \otimes \sigma_1 \otimes \cdots \otimes \sigma_t \otimes n.$$

Then β_s is a map of differential Σ-comodules and the diagram

$$\begin{tikzcd}
\widetilde{E}_0^{s,t}(M,N) \arrow[r, equal] \arrow[d, "\theta^{s,t}"] & \widetilde{E}_0^{0,t}(M\otimes\Phi^{\otimes s},N) \arrow[d, "\theta^{0,t}"] \\
\widetilde{E}_0^{s,t}(M,N) \arrow[d, equal] & \widetilde{E}_0^{0,t}(M\otimes\Phi^{\otimes s},N) \arrow[d, equal] \\
M\,\square_\Sigma\,\widetilde{D}_\Sigma^{s,t}(N) & M\,\square_\Sigma\,G^s\,\square_\Sigma\,\widetilde{D}_\Sigma^{s,t}(N) \arrow[l, "\beta_s"']
\end{tikzcd}$$

commutes.

We know that $\theta^{0,t}$ is a chain equivalence so it suffices to show that β_s is one by induction on s. To start this induction note that β_0 is the identity map by definition.

Let

$$F^{s,t}(\Gamma,N)=F^sC_{s+t}(\Gamma,N)$$

and

$$\begin{aligned}F^{s,t}(\Gamma,N)&=F^{s,t}(G,N)+F^{s+1,t-1}(\Gamma,N)\\&=F^{s,t}(\Gamma,N).\end{aligned}$$

Then $F^{s,*}(\Gamma,N)$ is a Σ-comodule subcomplex of $C_\Gamma(\Gamma,N)$ which is invariant under the contraction

$$S(\gamma\otimes\gamma_1\dots\gamma_s\otimes n)=\varepsilon(\gamma)\otimes\gamma_1\dots\gamma_s\otimes n.$$

Since $H_0(F^{s,*}(\Gamma,N))=0$, the complex $F^{s,*}(\Gamma,N)$ is acyclic.

Now look at the short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{F^{s+1}(\Gamma,N)}{\tilde{F}^{s+1}(\Gamma,N)} \arrow[r] \arrow[d, "\phi"', "\cong"] & \dfrac{\tilde{F}^{s}(\Gamma,N)}{\tilde{F}^{s+1}(\Gamma,N)} \arrow[r] & \dfrac{\tilde{F}^{s}(\Gamma,N)}{F^{s+1}(\Gamma,N)} \arrow[r] \arrow[d, "\psi"', "\cong"] & 0 \\
& \tilde{D}_\Sigma^{s+1}(N) & & G\,\square_\Sigma\,\tilde{D}_\Sigma^{s}(N) &
\end{tikzcd}$$

The connecting homomorphism in cohomology is an isomorphism.

We use this for the inductive step. By the inductive hypothesis, the composite

$$G\,\square_\Sigma\,(G^s\,\square_\Sigma\,N)\to G\,\square_\Sigma\,(G^s\,\square_\Sigma\,D_\Sigma^0(N))\to G\,\square_\Sigma\,D_\Sigma^s(N)$$

is an equivalence. If we follow it by $\phi\,\partial\psi$ we get β_{s+1}. This completes the inductive step and the proof. $\square$

A1.3.17. THEOREM. *Let* $\Phi\to\Gamma\to\Sigma$ *be a cocentral extension* (A1.1.15) *of Hopf algebras over a field* K; M *a left* Φ*-comodule and* N *a trivial left* Γ*-comodule. Then* $\mathrm{Ext}_\Sigma(K,N)$ *is trivial as a left* Φ*-comodule, so the Cartan–Eilenberg spectral sequence* (A1.3.14) E_2*-term is* $\mathrm{Ext}_\Phi(M,K)\otimes\mathrm{Ext}_\Sigma(K,K)\otimes N$.

PROOF. We show first that the coaction of Φ on $\mathrm{Ext}_\Sigma(K,N)$ is essentially unique and then give an alternative description of it which is clearly trivial when the extension is cocentral. The coaction is defined for any (not necessarily trivial) left Γ-comodule N. It is natural and determined by its effect when $N=\Gamma$ since we can use an injective resolution of N to reduce to this case. Hence any natural Φ-coaction on $\mathrm{Ext}_\Sigma(K,N)$ giving the standard coaction on $\mathrm{Ext}_\Sigma(K,\Gamma)=\Phi$ must be identical to the one defined above.

Now we need some results of Singer [**5**]. Our Hopf algebra extension is a special case of the type he studies. In Proposition 2.3 he defines a Φ-coaction on Σ, $\rho_\Sigma\colon \Sigma\to\Phi\otimes\Sigma$ via a sort of coconjugation. Its analog for a group extension $N\to G\to H$ is the action of H on N by conjugation. This action is trivial when the extension is central, as is Singer's coaction in the cocentral case.

The following argument is due to Singer.

Since Σ is a Φ-comodule it is a Γ-comodule so for any N as above $\Sigma\otimes_K N$ is a Γ-comodule. It follows that the cobar resolution $D_\Sigma N$ is a differential Γ-comodule and that $\mathrm{Hom}_\Sigma(K,D_\Sigma N)$ is a differential comodule over $\mathrm{Hom}_\Sigma(K,\Gamma)=\Phi$. Hence we have a natural Φ-coaction on $\mathrm{Ext}_\Sigma(K,N)$ which is clearly trivial when N has the trivial Γ-comodule structure and the extension is cocentral.

It remains only to show that this Φ-coaction is identical to the standard one by evaluating it when $N=\Gamma$. In that case we can replace $D_\Sigma N$ by N, since N is an extended Σ-comodule. Hence we have the standard Γ-coaction on Γ inducing the standard Φ-coaction on $\mathrm{Hom}_\Sigma(K,\Gamma)=\Phi$. $\square$

4. Massey Products

In this section we give an informal account of Massey products, a useful structure in the Ext over a Hopf algebroid which will figure in various computations in the text. A parallel structure in the ASS is discussed in Kochman [**4**] and Kochman [**2**, Section 12]. These products were first introduced by Massey [**3**], but the best account of them is May [**3**]. We will give little more than an introduction to May's paper, referring to it for all the proofs and illustrating the more complicated statements with simple examples.

The setting for defining Massey products is a differential graded algebra (DGA) C over a commutative ring K. The relevant example is the cobar complex $C_\Gamma(L,M)$ of A1.2.11, where L and M are Γ-comodule algebras and Γ is a Hopf algebroid (A1.1.1) over K. The product in this complex is given by A1.2.15.

We use the following notation to keep track of signs. For $x\in C$, let $\bar{x}$ denote $(-1)^{1+\deg x}x$, where $\deg x$ is the total degree of x; i.e., if C is a complex of graded objects, $\deg x$ is the sum of the internal and cohomological degrees of x. Hence we have $d(\bar{x})=-\overline{d(x)}$, $(\overline{xy})=-\bar{x}\bar{y}$, and $d(xy)=d(x)y-\bar{x}d(y)$.

Now let $\alpha_i\in H^*(C)$ be represented by cocycles $a_i\in C$ for $i=1,2,3$. If $\alpha_i\alpha_{i+1}=0$ then there are cochains u_i such that $d(u_i)=\bar{a}_i a_{i+1}$, and $\bar{u}_1a_3+\bar{a}_1u_2$ is a cocycle. The corresponding class in $H^*(C)$ is the *Massey product* $\langle\alpha_1,\alpha_2,\alpha_3\rangle$. If $\alpha_i\in H^{s_i}$ the this $\langle\alpha_1,\alpha_2,\alpha_3\rangle\in H^{s-1}$, where $s=\sum s_i$. Unfortunately, this triple product is not well defined because the choices made in its construction are not unique. The choices of a_i do not matter but the u_i could each be altered by adding a cocycle, which means $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ could be altered by any element of the form $x\alpha_3+\alpha_1y$ with $x\in H^{s_1+s_2-1}$ and $y\in H^{s_2+s_3-1}$. The group $\alpha_1H^{s_3+s_2-1}\oplus\alpha_3H^{s_1+s_2-1}$ is called the *indeterminacy*, denoted by $\mathrm{In}\langle\alpha_1,\alpha_2,\alpha_3\rangle$. It may be trivial, in which case $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ is well defined.

A1.4.1. DEFINITION. *With notation as above, $\langle\alpha_1,\alpha_2,\alpha_3\rangle\subset H^s(C)$ is the coset of $\mathrm{In}\langle\alpha_1,\alpha_2,\alpha_2\rangle$ represented by $\bar{a}_1u_2+\bar{u}_1a_3$. Note that $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ is only defined when $\bar{\alpha}_1\alpha_2=\bar{\alpha}_2\alpha_3=0$.*

This construction can be generalized in two ways. First the relations $\alpha_i\bar{\alpha}_{i+1} = 0$ can be replaced by

$$\sum_{j=1}^{m} (\bar{\alpha}_1)_j (\alpha_2)_{j,k} = 0 \quad \text{for } 1 \le k \le n$$

and

$$\sum_{k=1}^{n} (\bar{\alpha}_2)_{j,k} (\alpha_3)_k = 0 \quad \text{for } 1 \le j \le m.$$

Hence the α_i become matrices with entries in $H^*(C)$. We will denote the set of matrices with entries in a ring R by MR. For $x \in MC$ or $MH^*(C)$, define $\bar{x}$ by $(\bar{x})_{j,k} = \bar{x}_{j,k}$.

As before, let $a_i \in MC$ represent $\alpha_i \in MH^*(C)$ and let $u_1 \in MC$ be such that $d(u_i) = \bar{a}_i a_{i+1}$. Then u_1 and u_2 are $(1 \times n)$- and $(m \times 1)$-matrices, respectively, and $\bar{a}_1 u_2 + \bar{u}_1 a_3$ is a cocycle (not a matrix thereof) that represents the coset $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

Note that the matrices α_i need not be homogeneous (i.e., their entries need not all have the same degree) in order to yield a homogeneous triple product. In order to multiply two such matrices we require that, in addition to having compatible sizes, the degrees of their entries be such that the entries of the product are all homogeneous. These conditions are easy to work out and are given in 1.1 of May [**3**]. They hold in all of the applications we will consider and *will be tacitly assumed in subsequent definitions.*

A1.4.2. Definition. *With notation as above, the* matric Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ *is the coset of* $\mathrm{In}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ *represented the cocycle* $\bar{a}_1 u_2 + \bar{u}_1 a_3$, *where* $\mathrm{In}\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ *is the group generated by elements of the form* $x\alpha_3 + \alpha_1 y$ *where* $x, y \in MH^*(C)$ *have the appropriate form.*

The second generalization is to higher (than triple) order products. The Massey product $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ for $\alpha_i \in MH^*(C)$ is defined when all of the lower products $\langle \alpha_i, \alpha_{i+1}, \ldots, \alpha_j \rangle$ for $1 \le i < j \le n$ and $j - i < n - 1$ are defined and contain zero. Here the double product $\langle \alpha_i \alpha_{i+1} \rangle$ is understood to be the ordinary product $\alpha_i \alpha_{i+1}$. Let $a_{i-1,i}$ be a matrix of cocycles representing α_i. Since $\alpha_i \alpha_{i+1} = 0$ there are cochains $a_{i-1,i+1}$ with $d(a_{i-1,i+1}) = \overline{a}_{i-1,i} a_{i,i+1}$. Then the triple product $\langle \alpha_i, \alpha_{i+1}, \alpha_{i+2} \rangle$ is represented by $b_{i-1,i+2} = \bar{a}_{i-1,i+1} a_{i+1,i+2} + \bar{a}_{i-1,i} a_{i,i+2}$. Since this triple product is assumed to contain zero, the above choices can be made so that there is a matrix of cochains $a_{i-1,i+2}$ whose coboundary is $b_{i-1,i+2}$.

Then the fourfold product $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is represented by the cocycle $\bar{a}_{0,3} a_{3,4} + \bar{a}_{0,2} a_{2,4} + \bar{a}_{0,1} a_{1,4}$. More generally, we can choose elements $a_{i,j}$ and $b_{i,j}$ by induction on $j - i$ satisfying $b_{i,j} = \sum_{i<k<j} \bar{a}_{i,k} a_{k,j}$ and $d(a_{i,j}) = b_{i,j}$ for $i - j < n - 1$.

A1.4.3. Definition. *The n-fold Massey product $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ is defined when all of the lowerproducts $\langle \alpha_i, \ldots, \alpha_j \rangle$ contain zero for $i < j$ and $j - i < n - 1$. It is* strictly defined *when these lower products also have trivial indeterminacy, e.g., all triple products are strictly defined. In either case the matrices $a_{i,j}$ chosen above for $0 < i \le j \le n$ and $j - i < n$ constitute a* defining system *for the product in question, which is, modulo indeterminacy (to be described below), the class represented by the cocycle*

$$\sum_{0<i<n} \bar{a}_{0,i} a_{i,n}.$$

Note that if $\alpha_i \in H^{s_i}(C)$*, then* $\langle \alpha_1, \ldots, \alpha_n \rangle \subset H^{s+2-n}(C)$ *where* $s = \sum s_i$*.*

In 1.5 of May [**3**] it is shown that this product is natural with respect to DGA maps f in the sense that $\langle f_*(\alpha_1), \ldots, f_*(\alpha) \rangle$ is defined and contains $f_*(\langle \alpha_1, \ldots, \alpha_n \rangle)$.

The indeterminacy for $n \geq 4$ is problematic in that without additional technical assumptions it need not even be a subgroup. Upper bounds on it are given by the following result, which is part of 2.3, 2.4, and 2.7 of May [**3**]. It expresses the indeterminacy of n-fold products in terms of $(n-1)$-fold products, which is to be expected since that of a triple product is a certain matric double product.

A1.4.4. Indeterminacy Theorem. *Let* $\langle \alpha_1, \ldots, \alpha_n \rangle$ *be defined. For* $1 \leq k \leq n-1$ *let the degree of* x_k *be one less than that of* $\alpha_k \alpha_{k+1}$*.*

(a) *Define matrices* W_k *by*

$$W_1 = (\alpha_1 \ x_1),$$
$$W_k = \begin{pmatrix} \alpha_k & x_k \\ 0 & \alpha_{k+1} \end{pmatrix} \quad \text{for } 2 \leq k \leq n-2$$

and

$$W_{n-1} = \begin{pmatrix} x_{n-1} \\ \alpha_n \end{pmatrix}.$$

Then $\mathrm{In}\langle \alpha_1, \ldots, \alpha_n \rangle \subset \bigcup \langle W_1, \ldots, W_n \rangle$ *where the union is over all* x_k *for which* $\langle W_1, \ldots, W_n \rangle$ *is defined.*

(b) *Let* $\langle \alpha_1, \ldots, \alpha_n \rangle$ *be strictly defined. Then for* $1 \leq k \leq n-1$ $\langle \alpha_1, \ldots, \alpha_{k-1}, x_k, \alpha_{k+2}, \ldots, \alpha_n \rangle$ *is strictly defined and*

$$\mathrm{In}\langle \alpha_1, \ldots, \alpha_n \rangle \subset \bigcup \sum_{k=1}^{n-1} \langle \alpha_1, \ldots, \alpha_{k-1}, x_k, \alpha_{k+2}, \alpha_n \rangle$$

where the union is over all possible x_k*. Equality holds when* $n = 4$*.*

(c) *If* $\alpha_k = \alpha'_k + \alpha''_k$ *and* $\langle \alpha_1, \ldots, \alpha'_k, \ldots, \alpha_n \rangle$ *is strictly defined, then*

$$\langle \alpha_1, \ldots, \alpha_n \rangle \subset \langle \alpha_1, \ldots, \alpha'_k, \ldots, \alpha_n \rangle + \langle \alpha_1, \ldots, \alpha''_k, \ldots, \alpha_n \rangle. \qquad \square$$

There is a more general formula for the sum of two products, which generalizes the equation

$$\alpha_1\beta_1 + \alpha_2\beta_2 = \left\langle (\alpha_1 \alpha_2), \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right\rangle$$

and is part of 2.9 of May [**3**].

A1.4.5. Addition Theorem. *Let* $\langle \alpha_1, \ldots, \alpha_n \rangle$ *and* $\langle \beta_1, \ldots, \beta_n \rangle$ *be defined. Then so is* $\langle \gamma_1, \ldots, \gamma_n \rangle$ *where*

$$\gamma_1 = (\alpha_1, \beta_1), \quad \gamma_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix} \quad \textit{for } 1 < k < n, \quad \textit{and} \quad \gamma_n = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}.$$

Moreover $\langle \alpha_1, \ldots, \alpha_n \rangle + \langle \beta_1, \ldots, \beta_n \rangle \subset \langle \gamma_1, \ldots, \gamma_n \rangle$*.*

In Section 3 of May [**3**] certain associativity formulas are proved, the most useful of which (3.2 and 3.4) relate Massey products and ordinary products and are listed below. The manipulations allowed by this result are commonly known as *juggling*.

A1.4.6. First Juggling Theorem. (a) *If* $\langle \alpha_2, \dots, \alpha_n \rangle$ *is defined, then so is* $\langle \bar{\alpha}_1 \alpha_2, \alpha_3, \dots, \alpha_n \rangle$ *and*

$$\alpha_1 \langle \alpha_2, \dots, \alpha_n \rangle \subset -\langle \bar{\alpha}_1 \alpha_2, \alpha_3, \dots, \alpha_n \rangle.$$

(b) *If* $\langle \alpha_1, \dots, \alpha_{n-1} \rangle$ *is defined, then so is* $\langle \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}\alpha_n \rangle$ *and*

$$\langle \alpha_1, \dots, \alpha_{n-1} \rangle \alpha_n \subset \langle \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}\alpha_n \rangle.$$

(c) *If* $\langle \alpha_1, \dots, \alpha_{n-1} \rangle$ *and* $\langle \alpha_2, \dots, \alpha_n \rangle$ *are strictly defined, then*

$$\alpha_1 \langle \alpha_2, \dots, \alpha_n \rangle = \langle \bar{\alpha}_1, \dots, \bar{\alpha}_{n-1} \rangle \alpha_n.$$

(d) *If* $\langle \alpha_1\alpha_2, \alpha_3, \dots, \alpha_n \rangle$ *is defined, then so is* $\langle \alpha_1, \bar{\alpha}_2\alpha_3, \alpha_4, \dots, \alpha_n \rangle$ *and*

$$\langle \alpha_1\alpha_2, \alpha_3, \dots, \alpha_n \rangle \subset -\langle \alpha_1, \bar{\alpha}_2\alpha_3, \alpha_4, \dots, \alpha_n \rangle.$$

(e) *If* $\langle \alpha_1, \dots, \alpha_{n-2}, \bar{\alpha}_{n-1}\alpha_n \rangle$ *is defined, then so is* $\langle \alpha_1, \dots, \alpha_{n-3}, \alpha_{n-1}, \alpha_n \rangle$ *and*

$$\langle \alpha_1, \dots, \alpha_{n-2}, \bar{\alpha}_{n-1}\alpha_n \rangle \subset -\langle \alpha_1, \dots, \alpha_{n-3}, \alpha_{n-2}\alpha_{n-1}, \alpha_n \rangle.$$

(f) *If* $\langle \alpha_1, \dots, \alpha_{k-1}, \alpha_k\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_n \rangle$ *and* $\langle \alpha_1, \dots, \alpha_k, \bar{\alpha}_{k+1}\alpha_{k+2}, \alpha_{k+3}, \dots, \alpha_n \rangle$ *are strictly defined, then the intersection of the former with minus the latter is nonempty.* □

Now we come to some commutativity formulas. For these the DGA C must satisfy certain conditions (e.g., the cup product must be commutative) which always hold in the cobar complex. We must assume (if $2 \neq 0$ in K) that in each matrix α_i the degrees of the entries all have the same parity ε_i; i.e., ε_i is 0 if the degrees are all even and 1 if they are all odd. Then we define

$$s(i,j) = j - i + \sum_{i \le k \le m \le j} (1+\varepsilon_k)(1+\varepsilon_m) \tag{A1.4.7}$$

and

$$t(k) = (1+\varepsilon_1)\sum_{j=2}^{k}(1+\varepsilon_j).$$

The transpose of a matrix α will be denoted by α'. The following result is 3.7 of May [**3**].

A1.4.8. Second Juggling Theorem. *Let* $\langle \alpha_1, \dots, \alpha_n \rangle$ *be defined and assume that either* $2 = 0$ *in* K *or the degrees of all of the entries of each* α_i *have the same parity* ε_i. *Then* $\langle \alpha_n', \dots, \alpha_1' \rangle$ *is also defined and*

$$\langle \alpha_1, \dots, \alpha_n \rangle' = (-1)^{s(1,n)} \langle \alpha_n', \dots, \alpha_1' \rangle.$$

(*For the sign see* A1.4.7) □

The next result involves more complicated permutations of the factors. In order to ensure that the permuted products make sense we must assume that we have ordinary, as opposed to matric, Massey products. The following result is 3.8 and 3.9 of May [**3**].

A1.4.9. THIRD JUGGLING THEOREM. *Let $\langle\alpha_1,\ldots,\alpha_n\rangle$ be defined as an ordinary Massey product.*

(a) *If $\langle\alpha_{k+1},\ldots,\alpha_n,\alpha_1,\ldots,\alpha_k\rangle$ is strictly defined for $1\le k<n$, then*

$$(-1)^{s(1,n)}\langle\alpha_1,\ldots,\alpha_n\rangle\subset\sum_{k=1}^{n-1}(-1)^{s(1,k)+s(k+1,n)}\langle\alpha_{k+1},\ldots,\alpha_n,\ldots,\alpha_k\rangle.$$

(b) *If $\langle\alpha_2,\ldots,\alpha_k,\alpha_{k+1},\ldots,\alpha_n\rangle$ is strictly defined for $1\le k\le n$ then*

$$\langle\alpha_1,\ldots,\alpha_n\rangle\subset-\sum_{k=2}^{n}(-1)^{t(k)}\langle\alpha_2,\ldots,\alpha_k,\alpha_1,\alpha_{k+1},\ldots,\alpha_n\rangle.$$

(*For the signs see* A1.4.7)

Now we consider the behavior of Massey products in spectral sequences. In the previous section we considered essentially three types: the one associated with a resolution (A1.3.2), the one associated with a filitration (decreasing or increasing) of the Hopf algebroid Γ (A1.3.9), and the Cartan–Eilenberg spectral sequence associated with an extension (A1.3.14). In each case the spectral sequence arises from a filtration of a suitable complex. In the latter two cases this complex is the cobar complex of A1.2.11 (in the case of the Cartan–Eilenberg spectral sequence this result is A1.3.16), which is known to be a DGA (A1.2.14) that satisfies the additional hypotheses (not specified here) needed for the commutativity formulas A1.4.8 and A1.4.9. Hence all of the machinery of this section is applicable to those two spectral sequences; its applicability to the resolution spectral sequence of A1.3.2 will be discussed as needed in specific cases.

To fix notation, suppose that our DGA C is equipped with a decreasing filtration $\{F^pC\}$ which respects the differential and the product. We do not require $F^0C=C$, but only that $\lim_{p\to\infty}F^pC=C$ and $\lim_{p\to\infty}F^pC=0$. Hence we can have an increasing filtration $\{F_pC\}$ by defining $F_pC=F^{-p}C$. Then we get a spectral sequence with

$$\begin{aligned}E_0^{p,q}&=F^pC^{p+q}/F^{p+1}C^{p+q},\\E_1^{p,q}&=H^{p+q}(F^p/F^{p+1}),\\d_r\colon E_r^{p,q}&\to E_r^{p+r,q-r+1},\end{aligned}$$

and

$$E_\infty^{p,q}=F^pH^{p+q}/F^{p+1}H^{p+q}.$$

We let $E_{r,\infty}^{p,q}\subset E_r^{p,q}$ denote the permanent cycles and $i\colon E_{r,\infty}^{p,q}\to E_\infty^{p,q}$ and $\pi\colon F^pC^{p+q}\to E_0^{p,q}$ the natural surjections. If $x\in E_{r,\infty}^{p,q}$ and $y\in F^pH^{p+q}$ projects to $i(x)\in E_\infty^{p,q}$ we say that x *converges to* y. If the entries of a matrix $B\in MC$ are all known to survive to E_r, we indicate this by writing $\pi(B)\in ME_r$. In the following discussions α_i will denote an element in ME_r represented by $a_i\in MC$. If $\alpha_i\in ME_{r,\infty}$, $\beta_i\in MH^*(C)$ will denote an element to which it converges.

Each E_r is a DGA in whose cohomology, E_{r+1}, Massey products can be defined. Suppose $\langle\alpha_1,\ldots,\alpha_n\rangle$ is defined in E_{r+1} and that the total bidegree of the α_i is (s,t), i.e., that the ordinary product $\alpha_1\alpha_2\ldots\alpha_n$ (which is of course zero if $n\ge3$) lies in $E_{r+1}^{s,t}$. Then the indexing of d_r implies that the Massey product is a subset of $E_{r+1}^{s-r(n-2),t+(r-1)(n-2)}$.

May's first spectral sequence result concerns convergence of Massey products. Suppose that the ordinary triple product $\langle\beta_1, \beta_2, \beta_3\rangle \subset H^*(C)$ is defined and that $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ is defined in E_{r+1}. Then one can ask if an element in the latter product is a permanent cycle converging to an element of the former product. Unfortunately, the answer is not always yes. To see how counterexamples can occur, let $\hat{u}_i \in E_r$ be such that $d_r(\hat{u}_i) = \alpha_i\alpha_{i+1}$. Let (p, q) be the bidegree of one of the $\hat{u}_i$. Since $\langle\beta_1, \beta_2, \beta_3\rangle$ is defined we have as before $u_i \in C$ such that $d(u_i) = a_i a_{i+1}$. The difficulty is that $\bar{a}_i a_{i+1}$ need *not* be a coboundary in F^pC; i.e., it may not be possible to find a $u_i \in F^pC$. Equivalently, the best possible representative $\hat{u}_i \in F^pC$ of $\tilde{u}_i$, may have coboundary $\bar{a}_i a_{i+1} - e_i$ with $0 \neq \pi(e_i) \in E_t^{p+t,q-t+1}$ for some $t > r$. Then we have $d(u_i - \tilde{u}_i) = e_i$. and $\pi(u_i - \tilde{u}_i) = \pi(u_i \in E_{m+t}^{p-m,q+n}$ for some $m > 0$, so $d_{m+1}(\pi(u_i)) = \pi(e_i)$. In other words, the failure of the Massey product in E_{r+1} to converge as desired is reflected in the presence of a certain higher differential. Thus we can ensure convergence by hypothesizing that all elements in $E_{m+r+1}^{p-m,q+m}$ for $m \geq 0$ are permanent cycles.

The case $m = 0$ is included for the following reason (we had $m > 0$ in the discussion above). We may be able to find a $u_i \in F^pC$ with $d(u_i) = \bar{a}_i a_{i+1}$ but with $\pi(u_i) \neq u_i$, so $d_t(\pi(u - \tilde{u}_i)) = \pi(e_i) \neq 0$. In this case we can find a convergent element in the Massey product in E_{r+1}, but it would not be the one we started with.

The general convergence result, which is 4.1 and 4.2 of May [**3**], is

A1.4.10. CONVERGENCE THEOREM. (a) *With notation as above let $\langle\alpha_1, \ldots, \alpha_n\rangle$ be defined in E_{r+1}. Assume that $\alpha_i \in ME_{r+1,\infty}$ and α_i converges to β_i, where $\langle\beta_1, \ldots, \beta_n\rangle$ is strictly defined in $H^*(C)$. Assume further that if (p, q) is the bidegree of an entry of some $a_{i,j}$ (for $1 < j-1 < n$) in a defining system for $\langle\alpha_1, \ldots, \alpha_n\rangle$ then each element in $E_{r+m+1}^{p-m,q+m}$ for all $m \geq 0$ is a permanent cycle. Then each element of $\langle\alpha_1, \ldots, \alpha_n\rangle$ is a permanent cycle converging to an element of $\langle\beta_1, \ldots, \beta_n\rangle$.*

(b) *Suppose all of the above conditions are met except that $\langle\alpha_1, \ldots, \alpha_n\rangle$ is not known to be defined in E_{r+1}. If for (p, q) as above every element of $E_{r+m}^{p-m,q+m}$ for $m \geq 1$ is a permanent cycle then $\langle\alpha_1, \ldots, \alpha_n\rangle$ is strictly defined so the conclusion above is valid.* □

The above result does not prevent the product in question from being hit by a higher differential. In this case $\langle\beta_1, \ldots, \beta_n\rangle$ projects to a higher filtration.

May's next result is a generalized Leibnitz formula which computes the differential on a Massey product in terms of differentials on its factors. The statement is complicated so we first describe the simplest nontrivial situation to which it applies. For this discussion we assume that we are in characteristic 2 so we can ignore signs. Suppose $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ is defined in E_{r+1} but that the factors are not necessarily permanent cycles. We wish to compute d_{r+1} of this product. Let α_i have bidegree (p_i, q_i). Then we have $u_i \in F^{p_i+p_{i+1}-2}C$ with $d(u_i) = a_i a_{i+1}$ mod $F^{p_i+p_{i+1}+1}C$. The product is represented by $u_1a_3 + a_1u_2$. Now let $d(a_i) = a_i'$ and $d(u_i) = a_i a_{i+1} + u_i'$. Then we have $d(u_1a_3 + a_1u_2) = u_1'a_3 + u_1a_3' + a_1'u_2 + a_2u_2'$. This expression projects to a permanent cycle which we want to describe as a Massey product in E_{r+1}. Consider

$$\left\langle \begin{pmatrix} d_{r+1}(\alpha_1) & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ d_{r+1}(\alpha_2) & \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_3 \\ d_{r+1}(\alpha_3). \end{pmatrix} \right\rangle.$$

Since $d(u_i) = \alpha_i\alpha_{i+1} + u_i'$ is a cycle, we have $d(u_i')) = d(a_i a_{i+1}) = a_i' a_{i+1} + a_i a_{i+1}'$, so $d_r(\pi(u_i')) = d_{r+1}(\alpha_i)\alpha_{i+1} + \alpha_i d_{r+1}(\alpha_{i+1})$. It follows that the above product contains $\pi(u_1' a_3 + u_1 a_3' + a_1' u_2 + a_2 u_2') \in E_{r+1}$.

Hence we have shown that

$$d_{r+1}(\langle\alpha_1, \alpha_2, \alpha_3\rangle) \subset \left\langle \begin{pmatrix} d_{r+1}(\alpha_1) & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ d_{r+1}(\alpha_2) & \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_3 \\ d_{r+1}(\alpha_3) \end{pmatrix} \right\rangle.$$

We would like to show more generally that for $s > r$ with $d_t(\alpha_i) = 0$ for $r < t < s$, the product is a d_t-cycle and d_s on it is given by a similar formula. As in A1.4.10, there are potential obstacles which must be excluded by appropriate technical hypotheses which are vacuous when $s = r+1$. Let (p, q) be the bidegree of some u_i. By assumption $u_i' \in F^{p+r+1}C$ and $d(u_i') = a_i' a_{i+1} + a_i a_{i+1}'$. Hence $\pi(a_i a_{i+1}' + a_i' a_{i+1}) \in E_*^{p+r+s, q-r-s+2}$ is killed by a d_{r+s-t} for $r < t \le s$. If the new product is to be defined this class must in fact be hit by a d_r and we can ensure this by requiring $E_{r+s-t}^{p+t, q-t+1} = 0$ for $r < t < s$. We also need to know that the original product is a d_t-cycle for $r < t < s$. This may not be the case if $\pi(u_i') \neq 0 \in E_t^{p+t, q-t+1}$ for $r < t < s$, because then we could not get rid of $\pi(u_i')$ by adding to u_i an element in $F^{p+1}C$ with coboundary in $F^{p+r+1}C$ (such a modification of u_i would not alter the original Massey product) and the expression for the Massey product's coboundary could have lower filtration than needed. Hence we also require $E_t^{p+t, q-t+1} = 0$ for $r < t < s$.

We are now ready to state the general result, which is 4.3 and 4.4 of May [**3**].

A1.4.11. THEOREM (LEIBNITZ FORMULA). (a) *With notation as above let $\langle\alpha_1, \alpha_2, \ldots, \alpha_n\rangle$ be defined in E_{r+1} and let $s > r$ be given with $d_t(\alpha_i) = 0$ for all $t < s$ and $1 \le i \le n$. Assume further that for (p, q) as in* A1.4.10 *and for each t with $r < t < s$,*

$$E_t^{p+t, q-t+1} = 0 \quad \text{and} \quad E_{r+s-t}^{p+t, q-1+1} = 0$$

(*for each t one of these implies the other*). *Then each element α of the product is a d_t-cycle for $r < t < s$ and there are permanent cycles $\alpha_i' \in ME_{r+1,\infty}$ which survive to $d_s(\alpha_i)$ such that $\langle\gamma_1, \ldots, \gamma_n\rangle$ is defined in E_{r+1} and contains an element γ which survives to $-d_s(\alpha)$, where*

$$\gamma_1 = (\alpha_1' \bar{\alpha}_1), \quad \gamma_i = \begin{pmatrix} \alpha_i & 0 \\ \alpha_i & \bar{\alpha}_i \end{pmatrix} \quad \text{for } 1 < i < n,$$

and

$$\gamma_n = \begin{pmatrix} \alpha_n \\ \alpha_n' \end{pmatrix}.$$

(b) *Suppose further that each α_i' is unique, that each $\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n\rangle$ is strictly defined, and that all products in sight have zero indeterminacy. Then*

$$d_s(\langle\alpha_1, \ldots, \alpha_s\rangle) = -\sum_{i=1}^{n} \langle\bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1}, \alpha_i', \alpha_{i+1}, \ldots, \alpha_n\rangle. \qquad \square$$

The last result of May [**3**] concerns the case when $\langle\alpha_1, \ldots, \alpha_n\rangle$ is defined in E_{r+1}, the α_i are all permanent cycles, but the corresponding product in $H^*(C)$ is not defined, so the product in E_{r+1} supports some nontrivial higher differential. One could ask for a more general result; one could assume $d_t(\alpha_i) = 0$ for $t < s$ and, without the vanishing hypotheses of the previous theorem, show that the product supports a nontrivial d_t. In many specific cases it may be possible to derive such a result from the one below by passing from the DGA C to a suitable quotient in which the α_i are permanent cycles.

As usual we begin by discussing the situation for ordinary triple products, ignoring signs, and using the notation of the previous discussion. If $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined in E_{r+1} and the a_i are cocycles in C but the corresponding product in $H^*(C)$ is not defined, it is because the $a_i a_{i+1}$ are not both coboundaries; i.e., at least one of the $u_i' = d(u_i) + a_i a_{i+1}$ is nonzero. Suppose $\pi(u_i')$ is nontrivial in $E_{r+1}^{p+r+1,q-r}$. As before, the product is represented by $u_1 a_3 + a_1 u_2$ and its coboundary is $u_1' a_3 + a_1 u_2'$, so $d_{r+1}(\langle \alpha_1, \alpha_2, \alpha_3 \rangle) = \pi(u_1' a_3 + a_1 u_2')$. Here u_i' represents the product $\beta_i \beta_{i+1} \in H^*(C)$, where $\beta_i \in H^*(C)$ is the class represented by a_i. The product $\beta_i \beta_{i+1}$ has filtration greater than the sum of those of β_i and β_{i+1}, and the target of the differential represents the associator $(\beta_1 \beta_2)\beta_3 + \beta_1(\beta_2 \beta_3)$.

Next we generalize by replacing $r+1$ by some $s > r$; i.e., we assume that the filtration of $\beta_i \beta_{i+1}$ exceeds the sum of those of β_i, and β_{i+1} by $s - r$. As in the previous result we need to assume

$$E_t^{p+t,q-t+1} = 0 \quad \text{for } r < t < s;$$

this condition ensures that the triple product is a d_t-cycle.

The general theorem has some hypotheses which are vacuous for triple products, so in order to illustrate them we must discuss quadruple products, again ignoring signs. Recall the notation used in definition A1.4.3. The elements in the defining system for the product in E_{r+1} have cochain representatives corresponding to the defining system the product would have if it were defined in $H^*(C)$. As above, we denote $a_{i-1,i}$ by a_i, $a_{i-1,i+1}$ by u_i, and also $a_{i-1,i+2}$ by v_i. Hence we have $d(a_i) = 0$, $d(u_i) = a_i a_{i+1} + u_i'$, $d(v_i) = a_i u_{i+1} + u_i a_{i+2} + v_i'$, and the product contains an element α represented by $m = a_1 v_2 + u_1 u_3 + v_1 a_4$, so $d(m) = a_1 v_2' + u_1' u_3 + u_1 u_3' + v_1' a_4$. We also have $d(u_i') = 0$ and $d(v_i') = u_i' a_{i+2} + a_i u_{i+1}'$.

We are assuming that $\langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle$ is not defined. There are two possible reasons for this. First, the double products $\beta_i \beta_{i+1}$ may not all vanish. Second, the double products all vanish, in which case $u_i' = 0$, but the two triple products $\langle \beta_i, \beta_{i+1}, \beta_{i+2} \rangle$ must not both contain zero, so $v_i' \neq 0$. More generally there are $n-2$ reasons why an n-fold product may fail to be defined. The theorem will express the differential of the n-fold product in E_{r+1} in terms of the highest order subproducts which are defined in $H^*(C)$. We will treat these two cases separately.

Let (p_i, q_i) be the bidegree of α_i. Then the filtrations of u_i, v_i, and m are, respectively, $p_i + p_{i+1} - r$, $p_i + p_{i+1} + p_{i+2} - 2r$, and $p_1 + p_2 + p_3 + p_4 - 2r$.

Suppose the double products do not all vanish. Let $s > r$ be the largest integer such that each u_i' has filtration $\geq s - r + p_i + p_{i+1}$. We want to give conditions which will ensure that $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is a d_t-cycle for $r < t < s$ and that the triple product

$$\left\langle (\pi(u_1')\ \alpha_1), \begin{pmatrix} \alpha_3 & 0 \\ \pi(u_2') & \alpha_2 \end{pmatrix}, \begin{pmatrix} \alpha_4 \\ \pi(u_3') \end{pmatrix} \right\rangle$$

is defined in E_{r+1} and contains an element which survives to $d_s(\alpha)$; note that if all goes well this triple product contains an element represented by $d(m)$. These conditions will be similar to those of the Leibnitz formula A1.4.11. Let (p, q) be the bidegree of some v_i. As before, we ensure that $d_t(\alpha) = 0$ by requiring $E_t^{p+t,q-t+1}$, and that the triple product is defined in E_{r+1} by requiring $E_{r+s-t}^{p+t,q-t+1} = 0$. The former condition is the same one we made above while discussing the theorem for triple products, but the latter condition is new.

Now we treat the case when the double products vanish but the triple products do not. First consider what would happen if the above discussion were applied here. We would have $s = \infty$ and α would be a permanent cycle provided that $E_t^{p+t,q-t+1} = 0$ for all $t > r$. However, this condition implies that v can be chosen so that $v' = 0$, i.e., that the triple products vanish. Hence the above discussion is not relevant here.

Since $u_i' = 0$, the coboundary of the Massey product m is $a_1 v_2' + v_1' a_4$. Since $d(v_i) = a_i u_{i+1} + u_i a_{i+2} + v_i'$, v_i' is a cocycle representing an element of $\langle \beta_i, \beta_{i+1}, \beta_{i+2}\rangle$. Hence if all goes well we will have $d_s(\alpha) = \alpha_1 \pi(v_2') + \pi(v_1')\alpha_4$, where $s > r$ is the largest integer such that each v_i' has filtration at least $p_i + p_{i+1} + p_{i+2} + s - 2r$. To ensure that $d_t(\alpha) = 0$ for $t < s$, we require $E_t^{p+t,q-t+1} = 0$ for $r < t < s$ as before, where (p, q) is the degree of v_i. We also need to know that $\langle \alpha_i, \alpha_{i+1}, \alpha_{i+2}\rangle$ converges to $\langle \beta_i, \beta_{i+1}, \beta_{i+2}\rangle$; since the former contains zero, this means that the latter has filtration greater than $p_i + p_{i+1} + p_{i+2} - r$. We get this convergence from A1.4.10, so we must require that if (p, q) is the bidegree of $\pi(u_i)$, then each element of $E_{r+m+1}^{p-m,q+m+1}$ for all $m \geq 0$ is a permanent cycle.

Now we state the general result, which is 4.5 and 4.6 of May [**3**].

A1.4.12. Differential and Extension Theorem. (a) *With notation as above, let $\langle \alpha_1, \ldots, \alpha_n\rangle$ be defined in E_{r+1} where each α_i is a permanent cycle converging to $\beta_i \in H^*(C)$. Let k with $1 \leq k \leq n-2$ be such that each $\langle \beta_i, \ldots, \beta_{i+k}\rangle$ is strictly defined in $H^*(C)$ and such that if (p, q) is the bidegree of an entry of some $a_{i,j}$ for $1 < j - i \leq k$ in a defining system for $\langle \alpha_1, \ldots, \alpha_n\rangle$ then each element of $E_{r+m+1}^{p-m,q+m}$ for all $m \geq 0$ is a permanent cycle. Furthermore, let $s > r$ be such that for each (p, q) as above with $k < j - i < n$ and each t with $r < t < s$, $E_t^{p+t,q-t+1} = 0$, and, if $j - i > k + 1$, $E_{r+s-t}^{p+t,q-t+1} = 0$.*

Then for each $\alpha \in \langle \alpha_1, \ldots, \alpha_n\rangle$, $d_t(\alpha) = 0$ for $r < t < s$, and there are permanent cycles $\delta_i \in ME_{r+1,\infty}$ for $1 \leq i \leq n-k$ which converge to elements of $\langle \beta_i, \ldots, \beta_{i+k}\rangle \subset H^*(C)$ such that $\langle \gamma_1, \ldots, \gamma_{n-k}\rangle$ is defined in E_{r+1} and contains an element γ which survives to $-d_s(\alpha)$, where

$$\gamma_1 = (\delta_1\ \bar{\alpha}_1), \quad \gamma_i = \begin{pmatrix} \alpha_{i+k} & 0 \\ \delta_i & \bar{\alpha}_i \end{pmatrix} \quad \textit{for } i < n-k,$$

and

$$\gamma_{n-k} = \begin{pmatrix} \alpha_n \\ \delta_{n-k} \end{pmatrix}.$$

(b) *Suppose in addition to the above that each δ_i is unique, that each $\langle \bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1}, \delta_i, \alpha_{i+k+1}, \ldots, \alpha_n\rangle$ is strictly defined in E_{r+1} and that all Massey products in sight (except possibly $\langle \beta_i, \ldots, \beta_{i+k}\rangle$) have zero indeterminacy. Then*

$$d_s(\langle \alpha_1, \ldots, \alpha_n\rangle) = \sum_{i=1}^{n-k} \langle \bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1}, \delta_i, \alpha_{i+k+1}, \ldots, \alpha_n\rangle. \qquad \square$$

Note that in (b) the uniqueness of δ_i does not make $\langle \beta_i, \ldots, \beta_{i+k}\rangle$ have zero indeterminacy, but merely indeterminacy in a higher filtration. The theorem does not prevent δ_i from being killed by a higher differential. The requirement that $E_{r+m+1}^{p-m,q+m} \subset E_{r+m+1,\infty}$ is vacuous for $k = 1$, e.g., if $n = 3$. The condition

$E_{r+s-t}^{p+t,q-t+1} = 0$ is vacuous when $k = n-2$; both it and $E_t^{p+t,q-t+1} = 0$ are vacuous when $s = r+1$.

A1.4.13. Remark. The above result relates differentials to nontrivial extensions in the multiplicative structure (where this is understood to include Massey product structure) since δ_i represents $\langle \beta_i, \dots, \beta_{i+k} \rangle$ but has filtration greater than that of $\langle \alpha_i, \dots, \alpha_{i+k} \rangle$. The theorem can be used not only to compute differentials given knowledge of multiplicative extensions, but also vice versa. If $d_s(\alpha)$ is known, the hypotheses are met, and there are unique δ_i which fit into the expression for γ, then these δ_i necessarily converge to $\langle \beta_i, \dots, \beta_{i+k} \rangle$.

5. Algebraic Steenrod Operations

In this section we describe operations defined in $\mathrm{Cotor}_\Gamma(M, N)$, where Γ is a Hopf algebroid over $\mathbf{Z}/(p)$ for p prime and M and N are right and left comodule algebras (A1.1.2) over Γ. These operations were first introduced by Liulevicius [**2**], although some of the ideas were implicit in Adams [**12**]. The most thorough account is in May [**5**], to which we will refer for most of the proofs. Much of the material presented here will also be found in Bruner *et al.* [**1**]; we are grateful to its authors for sending us the relevant portion of their manuscript. The construction of these operations is a generalization of Steenrod's original construction (see Steenrod [**1**]) of his operations in the mod (p) cohomology of a topological space X. We recall his method briefly. Let $G = \mathbf{Z}/(p)$ and let E be a contractible space on which G acts freely with orbit space B. X^p denotes the p-fold Cartesian product of X and $X^p \times_G E$ denotes the orbit space of $X^p \times E$ where G acts canonically on E and on X^p by cyclic permutation of coordinates. Choosing a base point in E gives maps $X \to X \times B$ and $X^p \to X^p \times_G E$. Let $\Delta\colon X \to X^p$ be the diagonal embedding. Then there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d] & X^p \arrow[d] \\ X \times B \arrow[r] & X^p \times_G E \end{tikzcd}$$

Given $x \in H^*(X)$ [all H^* groups are understood to have coefficients in $\mathbf{Z}/(p)$ it can be shown that $x \otimes x \otimes \cdots x \in H^*(X^p)$ pulls back canonically to a class $Px \in H^*(X^p \times_G E)$. We have $H^i(B) = \mathbf{Z}/(p)$ generated by e_i for each $i \geq 0$. Hence the image of Px in $H^*(X \times B)$ has the form $\sum_{i \geq 0} x_i \otimes e_i$ with $x_i \in H^*(X)$ and $x_0 = x^p$. These x_i are certain scalar multiples of various Steenrod operations on x.

If C is a suitable DGA whose cohomology is $H^*(X)$ and W is a free R-resolution (where $R = \mathbf{Z}/(p)[G]$) of $\mathbf{Z}/(p)$, then we get a diagram

$$\begin{tikzcd} C \arrow[d] & C_p \arrow[l] \arrow[d] \\ C \otimes_R W & C_p \otimes_R W \arrow[l] \end{tikzcd}$$

where C_p is the p-fold tensor power of C, R acts trivially on C and by cyclic permutation on C_p, and the top map is the iterated product in C. It is this diagram (with suitable properties) that is essential to defining the operations. The fact that C is associated with a space X is not essential. Any DGA C which admits such a diagram has Steenrod operations in its cohomology. The existence of such a diagram is a strong condition on C; it requires the product to be homotopy commutative in a very strong sense. If the product is strictly commutative the diagram exists but gives trivial operations.

In 11.3 of May [**5**] it is shown that the cobar complex (A1.2.11) $C_\Gamma(M,N)$, for M,N as above and Γ a Hopf algebra, has the requisite properties. The generalization to Hopf algebroids is not obvious so we give a partial proof of it here, referring to Bruner *et al.* [**1**] for certain details.

We need some notation to state the result. Let $C = C_\Gamma(M,N)$ for Γ a Hopf algebroid over K (which need not have characteristic p) and M,N comodule algebras. Let C_r denote the r-fold tensor product of C over K. Let π be a subgroup of the r-fold symmetric group Σ_r and let W be a negatively graded $K[\pi]$-free resolution of K. Let π act on C_r by permuting the factors. We will define a map of complexes

$$\theta\colon W \otimes_{K[\pi]} C_r \to C$$

with certain properties.

We define θ by reducing to the case $M=\Gamma$, which is easier to handle because the complex $d = C_\Gamma(\Gamma,N)$ is a Γ-comodule with a contracting homotopy. We have $C = M \,\square_\Gamma\, D$ and an obvious map

$$j\colon W \otimes_{K[\pi]} C_r \to M_r \,\square_\Gamma\, (W \otimes_{K[\pi]} D_r),$$

where the comodule structure on $W \otimes_{K[\pi]} D_r$ is defined by

$$\psi(w \otimes d_1 \cdots \otimes d_r) = d_1' d_2' \ldots d_r' \otimes w \otimes d_1'' \otimes \cdots d_r''$$

for $w \in W$, $d_i \in D$, and $C(d_i) = d_i' \otimes d_i''$, and the comodule structure on M_r is defined similarly. Given a suitable map

$$\tilde\theta\colon W \otimes_{K[\pi]} D_r \to D,$$

we define θ to be the composite $(\mu \,\square\, \tilde\theta)j$, where $\mu\colon M_r \to M$ is the product.

A1.5.1. THEOREM. *With notation as above assume $W_0 = K[\pi]$ with generator e_0. Then there are maps $\theta, \tilde\theta$ as above with the following properties.*

(i) *The restriction of θ to $e_0 \otimes C_r$ is the iterated product* (A1.2.15) $C_r \to C$.

(ii) *θ is natural in M, N, and Γ up to chain homotopy.*

(iii) *The analogs of* (i) *and* (ii) *for $\tilde\theta$ characterize it up to chain homotopy.*

(iv) *Let $\Delta\colon W \to W \otimes W$ be a coassociative differential coproduct on W which is a $K[\pi]$-map (where $K[\pi]$ acts diagonally on $W \otimes W$, i.e., given $\alpha \in \pi$, and $w_1, w_2 \in W$, $\alpha \in (w_1 \otimes w_2) = \alpha(w_1) \otimes \alpha(w_2)$); such coproducts are known to exist. Let $\mu\colon C \otimes C \to C$ be the product of* A1.2.15. *Then the following diagram commutes* *up to natural chain homotopy.*

$$\begin{tikzcd}
W \otimes_{K[\pi]} (C \otimes C)_r \arrow[rr, "W\otimes\mu_r"] \arrow[d, "\Delta\otimes(C\otimes C)_r"'] & & W \otimes_{K[\pi]} C_r \arrow[ddd, "\theta"] \\
W \otimes W \otimes_{K[\pi]} (C \otimes C)_r \arrow[d, "W\otimes T"'] & & \\
W \otimes_{K[\pi]} C_r \otimes W \otimes_{K[\pi]} C_r \arrow[d, "\theta\otimes\theta"'] & & \\
C \otimes C \arrow[rr, "\mu"] & & C
\end{tikzcd}$$

where T is the evident shuffle map.

(v) *Let $\pi = \nu = \mathbf{Z}/(p)$, $\sigma = \Sigma_{p^2}$ and let τ be the split extensions of ν^p by π in which π permutes the factors of ν^p. Let W, V, and Y be resolutions of K over $K[\pi]$, $K[\nu]$, and $K[\sigma]$, respectively. Let $j\colon \tau \to \sigma$ (τ is a p-Sylow subgroup of K) induce a map $j\colon W \otimes V_p \to Y$ ($W \otimes V_p$ is a free $K[\tau]$ resolution of K). Then there is a map $\omega\colon Y \otimes_{K[\sigma]} C_{p^2} \to C$ such that the following diagram commutes up to natural homotopy*

$$\begin{tikzcd}
(W \otimes V_p) \otimes_{K[\tau]} C_{p^2} \arrow[r, "j\otimes C_p^2"] \arrow[dd, "U"] & Y \otimes_{K[\sigma]} C_{p^2} \arrow[dr, "\omega"] & \\
& & C \\
W \otimes_{K[\pi]} (V \otimes_{K[v]} C_p)_p \arrow[r, "W\otimes\theta_p"] & W \otimes_{K[\pi]} C_p \arrow[ur, "\theta"'] &
\end{tikzcd}$$

where U is the evident shuffle.

PROOF. The map $\tilde{\theta}$ satisfying (i), (ii), and (iii) is constructed in Lemma 2.3 of Bruner's chapter in Bruner *et al.* [**1**]. In his notation let $M = N$ and $K = L = C(A, N)$, which is our D. Thus his map Φ is our $\tilde{\theta}$. Since $\tilde{\theta}$ extends the product on N it satisfies (i). For (ii), naturality in M is obvious since cotensor products are natural and everything in sight is natural in Γ. For naturality in N consider the (not necessarily commutative) diagram

$$\begin{tikzcd}
W \otimes_{k[\pi]} C_\Gamma(\Gamma, N)_r \arrow[r] \arrow[d, "\tilde{\theta}"] & W \otimes_{K[\pi]} C_\Gamma(\Gamma, N')_r \arrow[d, "\tilde{\theta}'"] \\
C_\Gamma(\Gamma, N) \arrow[r] & C_\Gamma(\Gamma, N').
\end{tikzcd}$$

Bruner's result gives a map

$$W \otimes_{K[\pi]} C_\Gamma(\Gamma, N)_r \to C_\Gamma(\Gamma, N')$$

extending the map $N_r \to N'$. Both the composites in the diagram have the appropriate properties so they are chain homotopic and $\tilde{\theta}$ is natural in $\widetilde{N}$ up to the chain homotopy.

For (iv) note that π acts on $(C\otimes C)_r = C_{2r}$ by permutation, so π is a subgroup of Σ_{2r}. The two composites in the diagram satisfy (i) and (ii) as maps from $W\otimes_{K[\pi]} C_{2r}$ to C, so they are naturally homotopic by (iii).

To prove (v), construct ω for the group σ in the same way we constructed θ for the group π. Then the compositions $\omega(j\otimes C_{p^2})$ and $(W\otimes\theta_p)U$ both satisfy (i) and (ii) for the group τ, so they are naturally homotopic by (iii). $\square$

With the above result in hand the machinery of May [**5**] applies to $C_\Gamma(M,N)$ and we get Steenrod operations in $\mathrm{Cotor}_\Gamma(M,N)$ when $K=\mathbf{Z}/(p)$. Parts (i), (ii), and (iii) guarantee the existence, naturality, and uniqueness of the operations, while (iv) and (v) give the Cartan formula and Adem relations. These operations have properties similar to those of the topological Steenrod operations with the following three exceptions. First, there is in general no Bockstein operation β. There are operations βP^i, but they need not be decomposable. Recall that in the classical case β was the connecting homomorphism for the short exact sequence

$$0\to C\to \widetilde{C}\otimes \mathbf{Z}/(p^2)\to C\to 0,$$

where $\widetilde{C}$ is a DGA which is free over $\mathbf{Z}$, whose cohomology is the integral cohomology of X and which is such that $\widetilde{C}\otimes\mathbf{Z}/(p)=C$. If C is a cobar complex as above then such a $\widetilde{C}$ may not exist. For example, it does not exist if $C=C_{A_*}(\mathbf{Z}/(p),\mathbf{Z}/(p))$ where A_* is the dual Steenrod algebra, but if $C=C_{BP_*(BP)/(p)}(BP_*/(p))$ we have $\widetilde{C}=C_{BP_*(BP)}(BP_*)$.

Second, when dealing with bigraded complexes there are at least two possible ways to index the operations; these two coincide in the classical singly graded case. In May [**5**] one has $P^i\colon \mathrm{Cotor}^{s,t}\to\mathrm{Cotor}^{s+(2i-t)(p-1),pt}$, which means that $P^i=0$ if either $2i<t$ or $2i>s+t$. (Classically one would always have $t=0$.) We prefer to index our P^i so that they raise cohomological degree by $2i(p-1)$ and are trivial if $i<0$ or $2i>s$ (in May [**5**] such operations are denoted by $\widetilde{P}^i$). This means that we must allow i to be a half-integer with P^i nontrivial only if $2i\equiv t \mod (2)$. (This is not a serious inconvenience because in most of our applications for $p>2$ the complex C^{**} will be trivial for odd t.) The Cartan formula and Adem relations below must be read with this in mind.

Finally, $P^0\colon \mathrm{Cotor}^{s,2t}\to\mathrm{Cotor}^{s,2pt}$ is not the identity as in the classical case. The following is a reindexed form of 11.8 of May [**5**].

A1.5.2. Steenrod Operations Theorem. *Let Γ be a Hopf algebroid over $\mathbf{Z}/(p)$ and M and N right and left Γ-comodule algebras. Denote $\mathrm{Cotor}_\Gamma^{s,t}(M,N)$ by $H^{s,t}$. Then there exist natural homomorphisms*

$$Sq^i\colon H^{s,t}\to H^{s+i,2t}\quad\text{for } p=2,$$
$$P^{i/2}\colon H^{s,t}\to H^{i/2+s,pt}$$

and

$$\beta P^{i/2}\colon H^{s,t}\to H^{i/2+s+1,pt}\quad\text{for } p>2 \text{ and } q=2p-2,$$

all with $i\geq 0$, having the following properties.

(a) *For $p=2$, $Sq^i=0$ if $i>s$. For $p>2$, $P^{i/2}$ and $\beta P^{i/2}=0$ if $i>s$ or $2i\not\equiv t \mod (2)$.*

(b) *For $p=2$, $Sq^i(x)=x^2$ if $i=s$. For $p>2$ and $s+t$ even, $P^i(X)=X^p$ if $2i=s$.*

(c) *If there exists a Hopf algebroid* $\widetilde{\Gamma}$ *and* $\widetilde{\Gamma}$*-comodule algebras* $\widetilde{M}$ *and* $\widetilde{N}$ *all flat over* $\mathbf{Z}_{(p)}$ *with* $\Gamma = \widetilde{\Gamma} \otimes \mathbf{Z}/(p)$, $M = \widetilde{M} \otimes \mathbf{Z}/(p)$, *and* $N = \widetilde{N} \otimes \mathbf{Z}/(p)$, *then* $\beta Sq^i = (i+1)Sq^{i+1}$ *for* $p = 2$ *and for* $p > 2$ βP^i *is the composition of* β *and* P^i, *where* $\beta\colon H^{i,t} \to H^{s+1,t}$ *is the connecting homomorphism for the short exact sequence* $0 \to N \to \widetilde{N} \otimes \mathbf{Z}/(p^2) \to N \to 0$.

(d)

$$Sq^i(xy) = \sum_{0 \le j \le i} Sq^j(x) Sq^{i-j}(y) \quad \text{for } p = 2.$$

For $p > 2$

$$P^{i/2}(xy) = \sum_{0 \le j \le i} P^{j/2}(x) P^{(i-j)/2}(y)$$

and

$$\beta P^{i/2}(xy) = \sum_{0 \le j \le i} \beta P^{j/2}(x) P^{(i-j)/2}(y) + P^{i/2}(x) \beta P^{(i-j)/2}(y).$$

Similar external Cartan formulas hold.

(e) *The following Adem relations hold. For* $p = 2$ *and* $a < 2b$,

$$Sa^q Sq^b = \sum_{i \ge 0} \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i.$$

For $p > 2$, $a < pb$, *and* $\varepsilon = 0$ *or* 1 (*and, by abuse of notation,* $\beta^0 P^i = P^i$ *and* $\beta^1 P^i = \beta P^i$),

$$\beta^\varepsilon P^{a/2} P^{b/2} = \sum_{i \ge 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2} \beta^\varepsilon P^{(a+b-i)/2} P^{i/2}$$

and

$$\begin{aligned}\beta^\varepsilon P^{a/2} \beta P^{b/2} = (1-\varepsilon) &\sum_{i \ge 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2} \beta P^{(a+b-i)/2} P^{i/2} \\ &- \sum_{i \ge 0} (-1)^{(a+i)/2} \binom{(p-1)(b-i)/2 - 1}{(a-pi)/2 - 1} \beta^\varepsilon P^{(a+b-i)/2} P^{i/2},\end{aligned}$$

where, in view of (a), *one only considers terms in which* a, b, *and* i *all have the same parity* (*so the signs and binomial coefficients all make sense*). □

To compute Sq^0 or P^0 we have the following, which is 11.10 of May [**5**].

A1.5.3. PROPOSITION. *With notation as above, let* $x \in H^{s,t}$, *where* t *is even if* $p > 2$, *be represented by a cochain which is a sum of elements of the form* $m\gamma_1|\cdots|\gamma_s n$. *Then* $Sq^0(x)$ *or* $P^0(x)$ *is represented by a similar sum of elements of the form* $m^p\gamma_1^p|\cdots|\gamma_s^p n^p$. □

The operations also satisfy a certain suspension axiom. Consider the category $\mathbf{C}$ of triples (M, Γ, N) with M, Γ, N as above. A morphism in $\mathbf{C}$ consists of maps $M \to M'$, $\Gamma \to \Gamma'$, and $N \to N'$ which respect all the structure in sight. Let C_i, $i = 1, 2, 3$, be the cobar complexes for three objects in $\mathbf{C}$ and suppose there are morphisms which induce maps

$$C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$$

such that the composite gf is trivial in positive cohomological degree. Let H^{**}, $i = 1, 2, 3$, denote the corresponding Cotor groups. Define a homomorphism σ (the *suspension*) from $\ker f^* \subset H_1^{s+t,t}$ to $H_3^{s,t}/\operatorname{im} g^*$ as follows. Given $x \in \ker f^*$, choose a cocycle $a \in C_1$ representing x and a cochain $b \in C_2$ such that $d(b) = f(a)$. Then $g(b)$ is a cocycle representing $\sigma(x)$. It is routine to verify that $\sigma(x)$ is well defined.

A1.5.4. SUSPENSION LEMMA. *Let σ be as above. Then for $p > 2$, $\sigma(P^i(x)) = P^i(\sigma(x))$ and $\sigma(\beta P^i(x) = \beta P^i(\sigma(x))$ and similarly for $p = 2$.*

PROOF. We show how this statement can be derived from ones proved in May [**5**]. Let $\overline{C}_1 \subset C_1$ be the subcomplex of elements of positive cohomological degree. It has the structure necessary for defining Steenrod operations in its cohomology since C_1 does. Then May's theorem 3.3 applies to

$$\overline{C}_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$$

and shows that suspension commutes with the operations in $\ker f^* \subset H^*(C_1)$. We have $H^s(\overline{C}_1) = H^s(C_1)$ for $s > 1$ and a four-term exact sequence

$$0 \to M_1 \,\square_{\Gamma_1}\, N_1 \to M_1 \otimes_{A_1} M_1 \to H^1(\overline{C}_1) \to H^1(C_1) \to 0$$

so the result follows. $\square$

A1.5.5. COROLLARY. *Let δ be the connecting homomorphism associated with an short exact sequence of commutative associative Γ-comodule algebras. Then $P^i\delta = \delta P^i$ and $\beta P^i\delta = -\delta\beta P^i$ for $p > 2$ and similarly for $p = 2$. (In this situation the subcomodule algebra must fail to have a unit.)*

PROOF. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be such a short exact sequence. Then set $N_i = N$ and $\Gamma_i = \Gamma$ in the previous lemma. Then δ is the inverse of σ so the result follows. $\square$

We need a transgression theorem.

A1.5.6. COROLLARY. *Let $(D, \Phi) \xrightarrow{i} (A, \Gamma) \xrightarrow{f} (A, \Sigma)$ be an extension of Hopf algebroids over $\mathbf{Z}/(p)$ (A1.1.15); let M be a right Φ-comodule algebra and N a left Γ-comodule algebra, both commutative and associative. Then there is a suspension map σ from $\ker i^* \subset \operatorname{Cotor}_\Phi^{s+1,t}(M, A \,\square_\Sigma\, N)$ to $\operatorname{Cotor}_\Sigma^{s,t}(M \otimes_D A, N)/\operatorname{im} f^*$ which commutes with Steenrod operations as in* A1.5.4.

PROOF. $A \,\square_\Sigma\, N$ is a left Φ-comodule algebra by A1.3.14(a). We claim the composite $\overline{\Phi} \xrightarrow{i} \overline{\Gamma} \xrightarrow{f} \overline{\Sigma}$ is zero; since $\Phi = A\square_\Sigma\Gamma\square_\Sigma A$, $fi(\Phi) = A\square_\Sigma\Sigma\square_\Sigma A = A\square_\Sigma A = D$, so $fi(\overline{\Phi}) = 0$. Hence $C_\Phi(M, A\square_\Sigma N) \to C_\Gamma(M\otimes_D A, N) \to C_\Sigma(M\otimes_D A, N)$ is zero in positive cohomological degree. Hence the result follows from A1.5.4. $\square$

The following is a reformulation of theorem 3.4 of May[**5**].

A1.5.7. KUDO TRANSGRESSION THEOREM. *Let $\Phi \to \Gamma \to \Sigma$ be a cocentral extension (A1.1.15) of Hopf algebras over a field K of characteristic p. In the Cartan–Eilenberg spectral sequence (A1.3.14) for $\operatorname{Ext}_\Gamma(K, K)$ we have $E_2^{s,t} = \operatorname{Ext}_\Phi^s(K, K) \otimes \operatorname{Ext}_\Sigma^t(K, K)$ with $d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$. Then the transgression $d_r\colon E_r^{0,r-1} \to E_r^{r,0}$ commutes with Steenrod operations up to sign as in A1.5.4; e.g., if $d_r(x) = y$ then $d_{r+2s(p+1)}(P^s(x)) = P^s(y)$. Moreover for $p > 2$ and $r - 1$ even we have $d_{(p-1)(r-1)+1}(x^{p-1}y) = -\beta P^{(r-1)/2}(y)$.* $\square$