CHAPTER 7

Computing Stable Homotopy Groups with the Adams–Novikov Spectral Sequence

In this chapter we apply the Adams–Novikov spectral sequence to the motivating problem of this book, the stable homotopy groups of spheres. Our main accomplishment is to find the first thousand stems for $p = 5$, the previous record being 760 by Aubry [**1**]. In Section 1 we describe the method of infinite descent for computing the Adams–Novikov spectral sequence E_2-term in a range of dimensions, namely to find it for the spectra $T(m)$ of Section 6.5 by downward induction on m. Recall $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$ as a comodule, so $T(m)$ is equivalent to BP in dimensions less than $|v_{m+1}|$. This starts our downward induction since we always restrict our attention to a finite range of dimensions.

In Section 2 we construct a resolution enabling us in theory to extract the Adams–Novikov E_2-term for S^0 from that for $T(1)$. In practice we must proceed more slowly, computing for skeleta $T(1)^{(p^i-1)q}$ by downward induction on i. In Section 3 we do this down to $i = 1$; see 7.5.1. $T(1)^{(p-1)q}$ is a complex with p cells, its Adams–Novikov spectral sequence collapses in our range, and its homotopy is surprisingly regular.

In Section 4 we take the final step from $T(1)^{(p-1)q}$ to S^0. We have a spectral sequence (7.1.16) for this calculation and a practical procedure (7.1.18) for the required bookkeeping. We illustrate this method for $p = 3$, but here our range of dimensions is not new; see Tangora [**6**] and Nakamura [**3**].

In Section 5 we describe the calculations for $p = 5$, giving a running account of the more difficult differentials in the spectral sequence of 7.1.16 for that case. The results are tabulated in Appendix 3 and range up to the 1000-stem.

In more detail, the method in question involves the connective p-local ring spectra $T(m)$ of 6.5, which satisfy

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m] \subset BP_*(BP).$$

$T(0)$ is the p-local sphere spectrum, and there are maps

$$S^0 = T(0) \to T(1) \to T(2) \to \cdots \to BP.$$

The map $T(m) \to BP$ is an equivalence below dimension $|v_{m+1}| - 1 = 2p^{m+1} - 3$.

To descend from $\pi_*(T(m))$ to $\pi_*(T(m-1))$ we need some spectra interpolating between $T(m-1)$ and $T(m)$. Note that $BP_*(T(m))$ is a free module over $BP_*(T(m-1))$ on the generators $\{t_m^j : j \geq 0\}$. In Lemma 7.1.11 we show that for each h there is a $T(m-1)$-module spectrum $T(m-1)_h$ with

$$BP_*(T(m-1)_h) = BP_*(T(m-1))\{t_m^j : 0 \leq j \leq h\}.$$

We will be most interested in the case where h is one less than a power of p, and we will denote $T(m)_{p^i-1}$ by $T(m)_{(i)}$.

We have inclusions

$$T(m-1) = T(m-1)_{(0)} \to T(m-1)_{(1)} \to T(m-1)_{(2)} \to \cdots T(m)$$

and the map $T(m-1)_{(i)} \to T(m)$ is an equivalence below dimension $p^i|t_m| - 1 = 2(h+1)(p^m-1)-1$.

For example when $m = i = 0$, the spectrum $T(m)_{(i)}$ is S^0 while $T(m)_{p^{i+1}-1}$ is the p-cell complex

$$Y = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cdots \cup_{\alpha_1} e^{(p-1)q},$$

where $q = 2p-2$.

In Theorem 7.1.16 we give a spectral sequence for computing $\pi_*(T(m-1)_{(i)})$ in terms of $\pi_*(T(m-1)_{(i+1)}$. Its E_1-term is

$$E(h_{m,i}) \otimes P(b_{m,i}) \otimes \pi_*(T(m-1)_{(i+1)}$$

where the elements

$$\begin{array}{rrcl} & h_{m,i} & \in & E_1^{1,2p^i(p^m-1)} \\ \text{and} & b_{m,i} & \in & E_1^{2,2p^{i+1}(p^m-1)} \end{array}$$

are permanent cycles.

In the case $m = i = 0$ cited above, the E_1-term of this spectral sequence is

$$E(h_{1,0}) \otimes P(b_{1,0}) \otimes \pi_*(Y).$$

where $h_{1,0}$ and $b_{1,0}$ represent the homotopy elements α_1 and β_1 (α_1^2 for $p = 2$) respectively.

Thus to compute $\pi_*(S^0)$ below dimension $p^3(2p-2)$ we could proceed as follows. In this range we have

$$BP \cong T(3) \cong T(2)_{(1)}.$$

We then use the spectral sequence of 7.1.16 to get down to $T(2)$, which is equivalent in this range to $T(1)_{(2)}$, then use it twice to get down to $T(1) \cong T(0)_{(3)}$, and so on. This would make for a total of six applications of 7.1.16. Fortunately we have some shortcuts that make this process easier.

The Adams–Novikov E_2-term for $T(m)$ is

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))).$$

From now on we will drop the first variable when writing such Ext groups, since we will never consider any value for it other than BP_*. There is a change-of-rings isomorphism that equates this group with

$$\mathrm{Ext}_{\Gamma(m+1)}(BP_*)$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \dots, t_m) = BP_*[t_{m+1}, t_{m+2}, \dots].$$

Using our knowledge of $\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*)$ (Proposition 7.1.24) and $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$ (Theorem 7.1.31) in all dimensions, we will construct a 4-term exact sequence

$$0 \to BP_* \to D^0_{m+1} \to D^1_{m+1} \to E^2_{m+1} \to 0$$

of $\Gamma(m+1)$-comodules. The two D^i_{m+1} are weak injective, meaning that all of their higher Ext groups (above Ext^0) vanish (we study such comodules systematically at the end of Section 1), and below dimension $p^2|v_{m+1}|$

$$\mathrm{Ext}^0_{\Gamma(m+1)}(D^i_{m+1}) \cong \mathrm{Ext}^i_{\Gamma(m+1)}(BP_*).$$

It follows that in that range

$$\mathrm{Ext}^s_{\Gamma(m+1)}(E^2_{m+1}) \cong \mathrm{Ext}^{s+2}_{\Gamma(m+1)}(BP_*) \qquad \text{for all } s \geq 0.$$

The comodule E^2_{m+1} is $(2p^{m+2}-2p-1)$-connected. In Theorem 7.2.6 we determine its Ext groups (and hence those of BP_*) up to dimension $p^2|v_{m+1}|$. There are no Adams–Novikov differentials or nontrivial group extensions in this range (except in the case $m = 0$ and $p = 2$), so this also determines $\pi_*(T(m))$ in the same range.

Thus Theorem 7.2.6 gives us the homotopy of $T(0)_{(3)}$ in our range directly without any use of 7.1.16. In a future paper with Hirofume Nakai we will study the homotopy of $T(m)_{(2)}$ and the spectral sequence of 7.1.16 for the homotopy of $T(m)_{(1)}$ below dimension $p^3|v_{m+1}|$. There are still no room for Adams–Novikov differentials, so the homotopy and Ext calculations coincide. For $m = 0$ this computation was the subject of Ravenel [**11**].

It is only when we pass from $T(m)_{(1)}$ to $T(m)_{(0)} = T(m)$ that we encounter Adams–Novikov differentials below dimension $p^3|v_{m+1}|$. For $m = 0$ the first of these is the Toda differential

$$d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$$

of Toda [**3**] and Toda [**2**].

1. The method of infinite descent

First we define some Hopf algebroids that we will need.

7.1.1. Definition. *$\Gamma(m+1)$ is the quotient $BP_*(BP)/(t_1, t_2, \ldots, t_m)$,*

$$A(m) = BP_*\Box_{\Gamma(m+1)}BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_m]$$

and

$$G(m+1, k-1) = \Gamma(m+1)\Box_{\Gamma(m+k+1)}BP_* = A(m+k)[t_{m+1}, t_{m+1} \ldots, t_{m+k}]$$

We abbreviate $G(m+1, 0)$ by $G(m+1)$, and is understood that $G(m+1, \infty) = \Gamma(m+1)$.

In particular, $\Gamma(1) = BP_*(BP)$.

7.1.2. Proposition. *$G(m+1, k-1) \to \Gamma(m+1) \to \Gamma(m+k+1)$ is a Hopf algebroid extension (A1.1.15). Given a left $\Gamma(m+1)$-comodule M there is a Cartan–Eilenberg spectral sequence (A1.3.14) converging to $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ with*

$$\tilde{E}_2^{s,t} = \mathrm{Ext}^s_{G(m+1,k-1)}(A(m+k), \mathrm{Ext}^t_{\Gamma(m+k+1)}(BP_*, M))$$

and $d_r : E_r^{s,t} \to \tilde{E}_r^{s+r,t-r+1}$. (We use the notation $\tilde{E}_r^{s,t}$ to distinguish the Cartan–Eilenberg spectral sequence from the resolution spectral sequence.)

7.1.3. Corollary. *Let M be a $\Gamma(m+1)$-comodule concentrated in nonnegative dimensions. Then*

$$\mathrm{Ext}_{\Gamma(m+k+1)}(BP_*, M) = \mathrm{Ext}_{\Gamma(m+1)}(BP_*, G(m+1, k-1) \otimes_{A(m+k)} M).$$

In particular, $\mathrm{Ext}^{s,t}_{\Gamma(m+1)}(BP_*, M)$ *for* $t < 2(p^{m+1}-1)$ *is isomorphic to* M *for* $s = 0$ *and vanishes for* $s > 0$. *Moreover for the spectrum* $T(m)$ *constructed in 6.5 and having* $BP_*(T(m)) = BP_*[t_1, \dots, t_m]$,

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m))) = \mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*).$$

The following characterization of the Cartan–Eilenberg spectral sequence is a special case of (A1.3.16).

7.1.4. Lemma. *The Cartan–Eilenberg spectral sequence of 7.1.2 is the one associated with the decreasing filtration of the cobar complex* $C_{\Gamma(m+1)}(BP_*, M)$ *(see below) defined by saying that*

$$\gamma_1 \otimes \cdots \otimes \gamma_s \otimes m \in C^s_{\Gamma(m+1)}(BP_*, M)$$

is in F^i *if* i *of the* γ*'s project trivially to* $\Gamma(m+k+1)$.

The *method of infinite descent* for computing $\mathrm{Ext}_{BP_*(BP)}(BP_*, M)$ for a connective comodule M (e.g. the BP-homology of a connective spectrum) is to compute over Ext over $\Gamma(m+1)$ by downward induction on m. To calculate through a fixed range of dimensions k, we choose m so that $k \le 2(p^{m+1}-1)$ and use 7.1.3 to start the induction. For the inductive step we could use the Cartan–Eilenberg spectral sequence of 7.1.2, but it is more efficient to use a different spectral sequence, which we now describe.

7.1.5. Definition. *A comodule* M *over a Hopf algebroid* (A, Γ) *is* **weak injective** *(through a range of dimensions) if* $\mathrm{Ext}^s(M) = 0$ *for* $s > 0$ *(through the same range).*

We will study such comodules in the at the end of this section.

7.1.6. Definition. *For a left* $G(m+1, k-1)$*-comodule* M *let*

$$\widehat{r}_j : M \to \Sigma^{j|t_{m+1}|} M$$

be the group homomorphism defined by

$$M \xrightarrow{\psi_M} G(m+1, k-1) \otimes M \xrightarrow{\rho_j \otimes M} \Sigma^{j|t_{m+1}|} M$$

where $\rho_j : G(m+1, k-1) \to A(m+k)$ *is the* $A(m+k)$*-linear map sending* t^j_{m+1} *to 1 and all other monomials in the* t_{m+i} *to 0.*

We will refer to this map as a **Quillen operation**. *When* $m = 0$ *we denote it simply by* r_j.

It follows that

$$\psi(x) = \sum_j t^j_{m+1} \otimes \widehat{r}_j(x) + \dots,$$

where the missing terms involve t_ℓ for $\ell > m+1$.

The following is proved in Ravenel [**12**] as Lemma 1.10.

7.1.7. Lemma. *The Quillen operation* $\widehat{r}_j$ *of 7.1.6 is a comodule map and for* $j > 0$ *it induces the trivial endomorphism in Ext.*

7.1.8. Definition. *Let* $T^h_m \subset G(m+1, k-1)$ *denote the sub-*$A(m+k)$*-module generated by* $\{t^j_{m+1} \colon 0 \le j \le h\}$. *We will denote* $T^{p^i-1}_m$ *by* $T^{(i)}_m$. *A* $G(m+1, k-1)$*-comodule* M *is* i**-free** *if the comodule tensor product* $T^{(i)}_m \otimes_{A(m+k)} M$ *is weak injective.*

We have suppressed the index k from the notation T_m^h because it will usually be clear from the context. In the case $k = \infty$ the Ext group has the topological interpretation given in Lemma 7.1.11 below. The following lemma is useful in dealing with such comodules. It is proved in Ravenel [**12**] as Lemma 1.12.

7.1.9. Lemma. *For a left $G(m+1)$-comodule M, the group*

$$\mathrm{Ext}^0_{G(m+1)}(A(m+1), T_m^{(i)} \otimes_{A(m+k)} M)$$

is isomorphic as an $A(m)$-module to

$$L = \bigcap_{j \geq p^i} \ker \widehat{r}_j \subset M.$$

The following is proved in Ravenel [**12**] as Lemma 1.14.

7.1.10. Lemma. *Let D be a weak injective comodule over $\Gamma(m+1)$. Then $T_m^{(i)} \otimes D$ is also weak injective with*

$$\mathrm{Ext}^0_{\Gamma(m+1)}(T_m^{(i)} \otimes D) \cong A(m)\left\{t_{m+1}^j \colon 0 \leq j \leq p^i - 1\right\} \otimes \mathrm{Ext}^0_{\Gamma(m+1)}(D).$$

Given $x_0 \in \mathrm{Ext}^0_{\Gamma(m+1)}(D)$, the element isomorphic to $t_{m+1}^j \otimes x_0$ is

$$\sum_{0 \leq k \leq j} (-1)^k \binom{j}{k} t_{m+1}^k \otimes x_{j-k} \in T_m^{(i)} \otimes D$$

where $x_j \in D$ satsifies

$$\psi(x_j) = \sum_{0 \leq k \leq j} \binom{j}{k} t_{m+1}^{j-k} \otimes x_k.$$

The following is proved in Ravenel [**12**] as Lemma 1.15. The only case of it that we will need here is for $m = 0$, where $T(0)_h$ is the $2(p-1)h$-skeleton of $T(1)$.

7.1.11. Lemma. *For each nonnegative m and h there is a spectrum $T(m)_h$ where $BP_*(T(m)_h) \subset BP_*(BP)$ is a free module over*

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$$

on generators $\{t_{m+1}^j \colon 0 \leq j \leq h\}$. Its Adams–Novikov E_2-term is

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m)_h)) \cong \mathrm{Ext}_{\Gamma(m+1)}(BP_*, T_m^h).$$

We will denote $T(m)_{p^i-1}$ by $T(m)_{(i)}$.

To pass from $\mathrm{Ext}_{G(m+1,k-1)}(T_m^{(i+1)} \otimes M)$ to $\mathrm{Ext}_{G(m+1,k-1)}(T_m^{(i)} \otimes M)$ we can make use of the tensor product (over $A(m+k)$) of M with the long exact sequence

$$0 \longrightarrow T_m^{(i)} \xrightarrow{\ i\ } R^0 \xrightarrow{\ d^0\ } R^1 \xrightarrow{\ d^1\ } R^2 \xrightarrow{\ d^2\ } \cdots, \tag{7.1.12}$$

where

$$\begin{aligned} R^{2s+e} &= \Sigma^{(ps+e)2p^i(p^{m+1}-1)} T_m^{(i)} \qquad \text{for } e = 0, 1 \\ \text{and} \qquad d^s &= \begin{cases} \widehat{r}_{p^i} & \text{for } s \text{ even} \\ \widehat{r}_{(p-1)p^i} & \text{for } s \text{ odd,} \end{cases} \end{aligned}$$

which leads to a spectral sequence as in (A1.3.2).

7.1.13. THEOREM. *For a $G(m+1,k-1)$-comodule M there is a spectral sequence converging to* $\mathrm{Ext}_{G(m+1,k-1)}(M\otimes T_m^{(i)})$ *with*

$$E_1^{*,t} = E(h_{m+1,i})\otimes P(b_{m+1,i})\otimes \mathrm{Ext}^t_{G(m+1,k-1)}(T_m^{(i)}\otimes M)$$

with $h_{m+1,i}\in E_1^{1,0}$, $b_{m+1,i}\in E_1^{2,0}$, and $d_r : E_r^{s,t}\to E_r^{s+r,t-r+1}$. If M is $(i+1)$-free in a range of dimensions, then the spectral sequence collapses from E_2 in the same range.

Moreover d_1 is induced by the action on M of $\widehat{r}_{p^i\Delta_{m+1}}$ for s even and $\widehat{r}_{(p-1)p^i}$ for s odd.

The action of d_1 is as follows. Let

$$x = \sum_{0\le j<p^{i+1}} t^j_{m+1}\otimes m_j \in T_m^{(i)}\otimes M$$

Then d_1 is induced by the endomorphism

$$x\mapsto \begin{cases} -\displaystyle\sum_{0\le k<p^i}\ \sum_{k\le j<p^{i+1}} \binom{j}{k} t^{j-k}_{m+1}\otimes \widehat{r}_{(p^i-k)}(m_j) & \textit{for } s \textit{ even}\\ -\displaystyle\sum_{0\le k<(p-1)p^i}\ \sum_{k\le j<p^{i+1}} \binom{j}{k} t^{j-k}_{m+1}\otimes \widehat{r}_{((p-1)p^i-k)}(m_j) & \textit{for } s \textit{ odd.}\end{cases}$$

We will refer to this as the **small descent spectral sequence**.

PROOF. Additively this spectral sequence is a special case of the one in (A1.3.2) associated with M tensored with the long exact sequence (7.1.12), and the collapsing for $(i+1)$-free M follows from the fact that the spectral sequence is in that case concentrated on the horizontal axis.

For the identification of d_1, note that by (7.1.12) it is induced by the endomorphism

$$\begin{aligned} x \quad &\mapsto \quad \begin{cases} \displaystyle\sum_{0\le j<p^{i+1}} \widehat{r}_{p^i}(t^j_{m+1})\otimes m_j & \text{for } s \text{ even}\\ \displaystyle\sum_{0\le j<p^{i+1}} \widehat{r}_{(p-1)p^i}(t^j_{m+1})\otimes m_j & \text{for } s \text{ odd}\end{cases}\\ &= \quad \begin{cases} \displaystyle\sum_{p^i\le j<p^{i+1}} \binom{j}{p^i} t^{j-p^i}_{m+1}\otimes m_j & \text{for } s \text{ even}\\ \displaystyle\sum_{(p-1)p^i\le j<p^{i+1}} \binom{j}{(p-1)p^i} t^{j-(p-1)p^i}_{m+1}\otimes m_j & \text{for } s \text{ odd.}\end{cases}\end{aligned}$$

It follows from Lemma 7.1.7 that $\widehat{r}_{p^i\Delta_{m+1}}$ and $\widehat{r}_{(p-1)p^i\Delta_{m+1}}$ each induce trivial endomorphisms in Ext, so d_1 is also induced by

$$x \quad\mapsto \quad \begin{cases} -\widehat{r}_{p^i}(x) + \displaystyle\sum_{0\le j<p^{i+1}} \widehat{r}_{p^i}(t^j_{m+1})\otimes m_j & \text{for } s \text{ even}\\ -\widehat{r}_{(p_1)p^i}(x) + \displaystyle\sum_{0\le j<p^{i+1}} \widehat{r}_{(p-1)p^i}(t^j_{m+1})\otimes m_j & \text{for } s \text{ odd,}\end{cases}$$

which leads to the stated formula.

The multiplicative structure requires some explanation. The elements $h_{m+1,i}$ and $b_{m+1,i}$ correspond under Yoneda's isomorphism Hilton and Stammbach [**1**, page

155] to the tensor product of M with the exact sequences

$$0 \to T_m^{(i)} \to T_m^{2p^i-1} \to \Sigma^{p^i|t_{m+1}|}T_m^{(i)} \to 0$$
and
$$0 \to T_m^{(i)} \to T_m^{(i+1)} \to \Sigma^{p^i|t_{m+1}|}T_m^{(i+1)} \to \Sigma^{p^{i+1}|t_{m+1}|}T_m^{(i)} \to 0$$

respectively. Products of these elements correspond to the splices of the these. It follows that these two elements are permanent cycles and that the spectral sequence is one of modules over the algebra $E(h_{m+1,i}) \otimes P(b_{m+1,i})$. □

In practice we will find higher differentials in this spectral sequence by computing in the cobar complex $C_{G(m+1,k-1)}(M \otimes T_m^{(i)})$ or its subcomplex $C_{G(m+1,k-1)}(M)$. As explained in the proof of (A1.3.2), it can be embedded by a quasi-isomorphism (i.e., a map inducing an isomorphism in cohomology) into the double complex $B = \oplus_{s,t\geq 0} B^{s,t}$ defined by

$$B^{s,t} = C^t_{G(m+1,k-1)}(M \otimes R^s)$$

with coboundary

$$\partial = d + (-1)^s d^s,$$

where d is the coboundary operator in the cobar complex. Our spectral sequence is obtained from the filtration of B by horizontal degree, i.e., the one defined by

$$F^r B = \bigoplus_{s\geq r, t\geq 0} B^{s,t}.$$

Theorem 7.1.13 also has a topological counterpart in the case $M = BP_*$. Before stating it we need to define topological analogs of the operations $\widehat{r}_{p^i}$ and $\widehat{r}_{(p-1)p^i}$. One can show that there are cofiber sequences

$$T(m)_{(i)} \to T(m)_{(i+1)} \to \Sigma^{p^i|t_{m+1}|}T(m)_{p^i(p-1)-1} \tag{7.1.14}$$

and

$$T(m)_{p^i(p-1)-1} \to T(m)_{(i+1)} \to \Sigma^{(p-1)p^i|t_{m+1}|}T(m)_{(i)}. \tag{7.1.15}$$

We define

$$T(m)_{(i+1)} \xrightarrow{\rho_{p^i}} \Sigma^{p^i|t_{m+1}|}T(m)_{(i+1)}$$

and

$$T(m)_{(i+1)} \xrightarrow{\rho_{p^i(p-1)}} \Sigma^{(p-1)p^i|t_{m+1}|}T(m)_{(i+1)}$$

to be the composites

$$T(m)_{(i+1)} \to \Sigma^{p^i|t_{m+1}|}T(m)_{p^i(p-1)-1} \to \Sigma^{p^i|t_{m+1}|}T(m)_{(i+1)}$$

and

$$T(m)_{(i+1)} \to \Sigma^{(p-1)p^i|t_{m+1}|}T(m)_{(i)} \to \Sigma^{(p-1)p^i|t_{m+1}|}T(m)_{(i+1)}.$$

7.1.16. Theorem. *Let $T(m)_{(i)}$ be the spectrum of Lemma 7.1.11. There is a spectral sequence converging to $\pi_*(T(m)_{(i)})$ with*

$$E_1^{s,t} = E(h_{m+1,i}) \otimes P(b_{m+1,i}) \otimes \pi_*(T(m)_{(i+1)}) \qquad \textit{and} \qquad d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$$

with $h_{m+1,i} \in E_1^{1,2p^i(p^{m+1}-1)}$ *and* $b_{m+1,i} \in E_1^{2,2p^{i+1}(p^{m+1}-1)}$. *Moreover* d_1 *is* ρ_{p^i} *for* s *even and* $\rho_{(p-1)p^i}$ *for* s *odd. The elements* $h_{m+1,i}$ *and* $b_{m+1,i}$ *are permanent cycles, and the spectral sequence is one of modules over the ring*

$$R = E(h_{m+1,i}) \otimes P(b_{m+1,i})$$

We will refer to this as the **topological small descent spectral sequence**.

Proof. This the spectral sequence based on the Adams diagram

$$\begin{array}{ccccccc} X & \longleftarrow & \Sigma^a X' & \longleftarrow & \Sigma^b X & \longleftarrow & \Sigma^{a+b} X' \longleftarrow \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & & \Sigma^a Y & & \Sigma^b Y & & \Sigma^{a+b} Y \end{array}$$

where

$$\begin{aligned} a &= 2p^i(p^{m+1}-1)-1, \\ b &= 2p^{i+1}(p^{m+1}-1)-2, \\ X &= T(m)_{(i)}, \\ X' &= T(m)_{p^i(p-1)-1}, \\ \text{and} \quad Y &= T(m)_{(i+1)}. \end{aligned}$$

We will show that the elements $h_{m+1,i}$ and $b_{m+1,i}$ can each be realized by maps of the form

$$S^0 \longrightarrow X \xrightarrow{f} \Sigma^{-?} X$$

For $h_{m+1,i}$, f is the boundary map for the cofiber sequence

$$T(m)^{(i)} \to T(m)^{2p^i-1} \to \Sigma^{h+1} T(m)^{(i)},$$

and for $b_{m+1,i}$ it is the composite (in either order) of the ones for (7.1.14) and (7.1.15). □

7.1.17. Example. When $m = i = 0$, the spectrum $T(0)_{(0)}$ is S^0 while $T(0)_{(1)}$ is the p-cell complex

$$Y = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cdots \cup_{\alpha_1} e^{(p-1)q},$$

where $q = 2p-2$. The E_1-term of the spectral sequence of Theorem 7.1.16 is

$$E(h_{1,0}) \otimes P(b_{1,0}) \otimes \pi_*(Y).$$

where $h_{1,0}$ and $b_{1,0}$ represent the homotopy elements α_1 and β_1 (α_1^2 for $p = 2$) respectively.

We will use this spectral sequence through a range of dimensions in the following way. For each spectrum $T(m)_{(i+1)}$ the elements of Adams–Novikov filtration 0 and 1 are all permanent cycles, so we ignore them, replacing $\pi_*(T(m)_{(i+1)})$ by an appropriate subquotient of $\mathrm{Ext}_{\Gamma(m+1)}(T_m^{(i)} \otimes E_{m+1}^2)$. Let $\mathbf{N}$ be a list of generators of this group arranged by dimension. When an element x has order greater than p, we also list its nontrivial multiples by powers of p. Thus

$$\mathbf{N} \otimes E(h_{m+1,i}) \otimes P(b_{m+1,i})$$

contains a list of generators of the E_1-term in our range. Rather than construct similar lists for each E_r term we use the following method.

7.1.18. Input/output procedure. *We make two lists* **I** *(input) and* **O** *(output).* **I** *is the subset of* $\mathbf{N} \otimes E(h_{m+1,i})$ *that includes all elements in our range. Then* **O** *is constructed by dimensional induction starting with the empty list as follows. Assuming* **O** *has been constructed through dimensions* $k-1$*, add to it the* k*-dimensional elements of* **I***. If any of them supports a nontrivial differential in the spectral sequence, remove both the source and target from* **O***. (It may be necessary to alter the list of* $(k-1)$*-dimensional elements by a linear transformation so that each nontrivial target is a "basis" element.) Then if* $k > |b_{m+1,i}|$*, we append the product of* $b_{m+1,i}$ *with each element of* **O** *in dimension* $k - |b_{m+1,i}|$*. This completes the inductive step.*

Note that each element in E_1 of filtration greater than 1 is divisible by $b_{m+1,i}$. Since the spectral sequence is one of R-modules, that same is true of each E_r. In 7.1.18 we compute the differentials originating in filtrations 0 and 1. If $d_r(x) = y$ is one of them, there is no chance that for some minimal $t > 0$

$$d_{r'}(x') = b^t_{m+1,i} y \qquad \text{with } r' < r$$

because such an x' would have to be divisible by $b_{m+1,i}$. This justifies the removal of $b^t_{m+1,i}x$ and $b^t_{m+1,i}y$ for all $t \geq 0$ from consideration.

We will consider various $\Gamma(m+1)$-comodules M and will abbreviate $\mathrm{Ext}_{\Gamma(m+1)}(BP_*, M)$ by $\mathrm{Ext}_{\Gamma(m+1)}(M)$ or simply $\mathrm{Ext}(M)$.

Excluding the case $m = 0$ and $p = 2$, we will construct a diagram of 4-term exact sequences of $\Gamma(m+1)$-comodules

$$\begin{tikzcd}
0 \arrow[r] & BP_* \arrow[r] \arrow[d, equal] & D^0_{m+1} \arrow[r] \arrow[d, equal] & D^1_{m+1} \arrow[r] \arrow[d] & E^2_{m+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BP_* \arrow[r] \arrow[d, equal] & D^0_{m+1} \arrow[r] \arrow[d] & v_1^{-1}E^1_{m+1} \arrow[r] \arrow[d] & E^1_{m+1}/(v_1^\infty) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BP_* \arrow[r] & M^0 \arrow[r] & M^1 \arrow[r] & N^2 \arrow[r] & 0
\end{tikzcd} \tag{7.1.19}$$

where each vertical map is a monomorphism, M^i and N^2 are as in 5.1.5, the D^i_{m+1} are weak injectives with $\mathrm{Ext}^0(D^0_{m+1}) = \mathrm{Ext}^0(BP_*)$, $\mathrm{Ext}^0(D^1_{m+1})$ contains $\mathrm{Ext}^1(BP_*)$ (with equality holding for $m = 0$ and p odd), and $E^1_{m+1} = D^0_{m+1}/BP_*$. $\mathrm{Ext}^0(BP_*)$ and $\mathrm{Ext}^1(BP_*)$ are given in 7.1.24 and 7.1.31 respectively.

It follows that for $m = 0$ and p odd, there is an isomorphism

$$\mathrm{Ext}^s_{\Gamma(m+1)}(E^2_{m+1}) \cong \mathrm{Ext}^{s+2}_{\Gamma(m+1)}(BP_*),$$

and for $m > 0$ there is a similar isomorphism below dimension $p^2|v_{m+1}|$ for all primes. E^2_{m+1} is locally finite and $(p|v_{m+1}| - 1)$-connected, which means that $\mathrm{Ext}^s_{\Gamma(m+1)}$ for $s > 1$ vanishes below dimension $p|v_{m+1}|$.

We will construct the map from BP_* to the weak injective D^0_{m+1}, inducing an isomorphism in Ext^0 , explicitly in Theorem 7.1.28. For $m > 0$ we cannot construct a similar map out of $E^1_{m+1} = D^0_{m+1}/BP_*$. Instead we will construct a map to a weak injective D^1_{m+1} which enlarges Ext^0 by as little as possible. We will do this

by producing a comodule $E^2_{m+1} \subset E^1_{m+1}/v_1^\infty$ and using the induced extension

$$\begin{tikzcd} 0 \arrow[r] & E^1_{m+1} \arrow[r] \arrow[d, equal] & v_1^{-1}E^1_{m+1} \arrow[r] & E^1_{m+1}/(v_1^\infty) \arrow[r] & 0 \\ 0 \arrow[r] & E^1_{m+1} \arrow[r] & D^1_{m+1} \arrow[r] \arrow[u] & E^2_{m+1} \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{7.1.20}$$

The comodule E^2_{m+1} for $m > 0$ will be described in the next section. For $m = 0$ and p odd, a map from E^1_1 to a weak injective D^1_1 inducing an isomorphism in Ext^0 will be constructed in below in Lemma 7.2.1.

We will use the following notations for $m > 0$. We put hats over the symbols in order to distinguish this notation from the usual one for elements in $\mathrm{Ext}_{BP_*(BP)}$. For $m = 0$ we will use similar notation without the hats.

$$\left\{ \begin{array}{lllllllll} \widehat{v}_i & = & v_{m+i}, & & \widehat{t}_i & = & t_{m+i}, & \omega & = & p^m, \\ \widehat{h}_{i,j} & = & h_{m+i,j}, & \text{and} & \widehat{b}_{i,j} & = & b_{m+i,j}. & & & \end{array} \right. \tag{7.1.21}$$

We will show that in dimensions below $p^2|\widehat{v}_1|$, E^2_{m+1} is the $A(m+1)$-module generated by the set of chromatic fractions

$$\left\{ \frac{\widehat{v}_2^{e_2}}{p^{e_0} v_1^{e_1}} \colon e_0, e_1 > 0,\ e_2 \geq e_0 + e_1 - 1 \right\}, \tag{7.1.22}$$

and its Ext group in this range is

$$A(m+1)/I_2 \otimes E(\widehat{h}_{1,0}) \otimes P(\widehat{b}_{1,0}) \otimes \left\{ \frac{\widehat{v}_2^{e_2}}{p v_1} \colon e_2 \geq 1 \right\}, \tag{7.1.23}$$

where $\widehat{h}_{1,0} \in \mathrm{Ext}^{1,2(p\omega-1)}$ corresponds to the primitive $\widehat{t}_1 \in \Gamma(m+1)$, and $\widehat{b}_{1,0} \in \mathrm{Ext}^{1,2p(p\omega-1)}$ is its transpotent. In both cases there is no power of v_1 in the numerator when $m = 0$. These statements will be proved below as Theorem 7.2.6.

An Adams–Novikov differential for $T(m)$ originating in the 2-line would have to land in filtration $2p+1$, which is trivial in the is range of dimensions, so by proving 7.2.6 we have determined $\pi_*(T(m))$ in this range.

Our first goal here is to compute Ext^0 and Ext^1. The following generalization of the Morava-Landweber theorem (4.3.2) is straightforward.

7.1.24. Proposition.

$$\mathrm{Ext}^0_{\Gamma(m+1)}(BP_*/I_n) = A(n+m)/I_n.$$

For $n = 0$ each of the generators is a permanent cycle.

Proof. The indicated elements are easily seen to be invariant in $\Gamma(m+1)$. In dimensions less that $|\widehat{v}_1|-1$, $T(m)$ is homotopy equivalent to BP, so the generators v_i for $i \leq m$ are permanent cycles as claimed. □

Now we will describe a map from BP_* to a weak injective D^0_{m+1} inducing an isomorphism in Ext^0. D^0_{m+1} is the sub-$A(m)$-algebra of $p^{-1}BP_*$ generated by certain elements $\widehat{\lambda}_i$ for $i > 0$ congruent to $\widehat{v}_i/p$ modulo decomposables.

To describe them we need to recall the formula of Hazewinkel [**4**] (see A2.2.1) relating polynomial generators $v_i \in BP_*$ to the coefficients ℓ_i of the formal group law, namely

$$p\ell_i = \sum_{0\le j<i} \ell_j v_{i-j}^{p^j} \qquad \text{for } i>0. \tag{7.1.25}$$

This recursive formula expands to

$$\begin{aligned}
\ell_1 &= \frac{v_1}{p} \\
\ell_2 &= \frac{v_2}{p} + \frac{v_1^{p+1}}{p^2} \\
\ell_3 &= \frac{v_3}{p} + \frac{v_1 v_2^p}{p^2} + \frac{v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3} \\
&\vdots
\end{aligned}$$

We need to define reduced log coefficients $\widehat{\ell}_i$ for $i>0$ obtained from the ℓ_{m+i} by subtracting the terms which are monomials in the v_j for $j \le m$. Thus for $m>0$ we have

$$\begin{aligned}
\widehat{\ell}_1 &= \frac{\widehat{v}_1}{p} \\
\widehat{\ell}_2 &= \frac{\widehat{v}_2}{p} + \frac{v_1 \widehat{v}_1^p}{p^2} + \frac{\widehat{v}_1 v_1^{p\omega}}{p^2} \\
&\vdots
\end{aligned}$$

The analog of Hazewinkel's formula for these elements is

$$p\widehat{\ell}_i = \sum_{0\le j<i} \ell_j \widehat{v}_{i-j}^{p^j} + \sum_{0<j<\min(i,m+1)} \widehat{\ell}_{i-j} v_j^{p^{i-j}\omega}. \tag{7.1.26}$$

We use these to define our generators $\widehat{\lambda}_i$ recursively for $i>0$ by

$$\widehat{\lambda}_i = \widehat{\ell}_i - \sum_{0<j<i} \ell_j \widehat{\lambda}_{i-j}^{p^j}. \tag{7.1.27}$$

For $m=0$ we will denote these by λ_i.

The following is proved as Theorem 3.12 and equation (3.15) in Ravenel [**12**].

7.1.28. Theorem. *The BP_*-module $D^0_{m+1} \subset p^{-1}BP_*$ described above is a subcomodule over $\Gamma(m+1)$ that is weak injective (7.1.5) with $\mathrm{Ext}^0 = A(m)$. In it we have*

$$\eta_R(\widehat{\lambda}_i) \equiv \widehat{\lambda}_i + \widehat{t}_i \bmod \textit{decomposables.}$$

Before proceeding further we need the following technical tool.

7.1.29. Definition. *Let H be a graded connected torsion abelian p-group of finite type, and let H_i have order p^{h_i}. Then the* **Poincaré series** *for H is*

$$g(H) = \Sigma h_i t^i.$$

7.1.30. Example. Let $I \subset BP_*$ be the maximal ideal so that $BP_*/I = \mathbf{Z}/(p)$. Then the Poincaré series for $\Gamma(m+1)/I$ is

$$G_m(t) = \prod_{i>0}(1 - t^{|v_{m+i}|})^{-1}.$$

We will abbreviate $t^{|v_{m+i}|}$ by x_i and denote x_1 simply by x. When $m > 0$ we will denote $t^{|v_i|}$ for $i \le m$ by y_i and $t^{|v_1|}$ simply by y.

For Ext^1 we have

7.1.31. Theorem. *Unless $m = 0$ and $p = 2$ (which is handled in (5.2.6)), $\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*, BP_*)$ is the $A(m)$-module generated by the set*

$$\left\{\delta_0\left(\frac{\widehat{v}_1^j}{jp}\right) : j > 0\right\},$$

where δ_0 is the connecting homomorphism for the short exact sequence

$$0 \to BP_* \to M^0 \to N^1 \to 0$$

as in (5.1.5). Its Poincaré series is

$$g_m(t)\sum_{i>0}\frac{x^{p^{i-1}}}{1 - x^{p^{i-1}}},$$

where $x = t^{|v_{m+1}|}$. Moroever each of these elements is a permanent cycle.

Proof. The Ext calculation follows from (6.5.11) and (7.1.3). For the Poincaré series, note that the set of $A(m)$-module generators of order p^i is

$$\left\{\delta_0\left(\frac{\widehat{v}_1^{jp^{i-1}}}{p^i}\right) : j > 0\right\},$$

and its Poincaré series is

$$\frac{x^{p^{i-1}}}{1 - x^{p^{i-1}}}.$$

To show that each of these elements is a permanent cycle, we will study the Bockstein spectral sequence converging to $\pi_*(T(m))$ with

$$E_1 = \mathbf{Z}/(p)[v_0] \otimes \pi_*(V(0) \wedge T(m)).$$

$V(0) \wedge T(m)$ is a ring spectrum in all cases except $m = 0$ and $p = 2$. We know that $T(m)$ is a ring spectrum for all m and p and that $V(0)$ is one for p odd. The case $p = 2$ and $m > 0$ is dealt with in Lemma 3.18 of Ravenel [**12**].

Low dimensional calculations reveal that $\widehat{v}_1 \in \mathrm{Ext}^0(BP_*/p)$ is a homotopy element. The elements $\widehat{\alpha}_j = \frac{\widehat{v}_1^j}{p}$ can then be constructed in the usual way using the self-map of $V(0) \wedge T(m)$ inducing multiplictation by $\widehat{v}_1^j$ followed by the pinch map

$$V(0) \wedge T(m) \to \Sigma T(m).$$

In the Bockstein spectral sequence it follows that $\widehat{v}_1^{sp^i}$ survives to E_{i+1}, so $\widehat{\alpha}_{sp^i}$ is divisible (as a homotopy element) by p^i. □

Now we will recall some results about weak injective comodules M over a general Hopf algebroid (A, Γ) over $\mathbf{Z}_{(p)}$. In most cases we will refer to Ravenel [**12**] for the proofs. We will abbreviate $\mathrm{Ext}_\Gamma(A, M)$ by $\mathrm{Ext}(M)$.

The definition 7.1.5 of a weak injective should be compared with other notions of injectivity. A comodule I (or more generally an object in an abelian category) is *injective* if any homomorphism to it extends over monomorphisms, i.e., if one can always fill in the following diagram.

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & M \arrow[u] \arrow[r, "i"] & N \arrow[ul, dashed] \end{tikzcd}$$

This definition is rather limiting. For example if A is a free $\mathbf{Z}_{(p)}$-module, then an injective I must be p-divisible since a homomorphism $A \to I$ must extend over $A \otimes \mathbf{Q}$.

There is also a notion of relative injectivity (A1.2.7) requiring I to be a summand of $\Gamma \otimes_A I$, which implies that the diagram above can always be completed when i is split over A. This implies weak injectivity as we have defined it here (see (A1.2.8)(b)), but we do not know if the converse is true. In any case the requirements of our definition can be said to hold only through a range of dimensions. The following is proved in Ravenel [**12**] as Lemma 2.1.

7.1.32. Lemma. *A connective comodule M over (A, Γ) is weak injective in a range of dimensions iff $\operatorname{Ext}^1(M) = 0$ in the same range.*

The following is proved in Ravenel [**12**] as Lemma 2.2.

7.1.33. Lemma. *Let*

$$(D, \Phi) \to (A, \Gamma) \to (A, \Sigma)$$

be an extension (A1.1.15) of graded connected Hopf algebroids of finite type, and suppose that M is a weak injective comodule over Γ. Then M is also weak injective over Σ, and $\operatorname{Ext}^0_\Sigma(A, M)$ is weak injective over Φ with

$$\operatorname{Ext}^0_\Phi(D, \operatorname{Ext}^0_\Sigma(A, M)) \cong \operatorname{Ext}^0_\Gamma(A, M).$$

Here is a method of recognizing weak injectives without computing any higher Ext groups. The following is proved in Ravenel [**12**] as Theorem 2.6.

7.1.34. Theorem. *Let (A, Γ) be a graded connected Hopf algebroid over $\mathbf{Z}_{(p)}$, and let M be a connected torsion Γ-comodule of finite type. Let $I \subset A$ be the maximal ideal (so that $A/I = \mathbf{Z}/(p)$). Then*

$$g(M) \le g(\operatorname{Ext}^0(M)) g(\Gamma/I),$$

meaning that each coefficient of the power series on the left is dominated by the corresponding one on the right, with equality holding if and only if M is a weak injective (7.1.5).

It would be nice if for any comodule M one could find a map $M \to W$ to a weak injective inducing an isomorphism in Ext^0, but this is not always possible. In Ravenel [**12**, Example 2.8] we showed that it fails when $(A, \Gamma) = (A(1), G(1))$ and $M = A/(p^2)$.

For future reference will need the Poincaré series of $E^1_{m+1} = D^0_{m+1}/BP_*$. The following is proved as Lemma 3.16 in Ravenel [**12**].

7.1.35. Lemma. *Let*

$$\begin{aligned} g_m(t) &= \prod_{1\le i\le m} \frac{1}{1-y_i} \\ \text{and} \qquad G_m(t) &= \prod_{i>0} \frac{1}{1-x_i}, \end{aligned}$$

(with x_i and y_i as in 7.1.30) the series for $A(m)/(p)$ and $\Gamma(m+1)/I$ respectively. Then the Poincaré series for $E^1_{m+1} = D^0_{m+1}/BP_*$ is

$$g_m(t)G_m(t)\sum_{i>0}\frac{x_i}{1-x_i}.$$

2. The comodule E^2_{m+1}

In this section we will describe the comodule E^2_{m+1} needed above in (7.1.20) below dimension $p^2|\widehat{v}_1|$. This will determine $\pi_*(T(m))$ below dimension $p^2|\widehat{v}_1|-3$. For $m=0$ and p odd we can construct D^1_1 in all dimensions directly as follows.

7.2.1. Lemma. *For p odd there is a map $E^1_1 \to D^1_1$ to a weak injective inducing an isomorphism in* Ext^0.

Proof. $M^1 = v_1^{-1}E^1_1$ is not a weak injective for $m=0$ since $\mathrm{Ext}^1_{\Gamma(1)}(M^1) = \mathbf{Q}/\mathbf{Z}$ concentrated in degree 0.

We will construct D^1_1 as a union of submodules of M^1 as follows. Let $K_0 = E^1_1 \subset M^1$. For each $i \ge 0$ we will construct a diagram

$$\begin{tikzcd} & L_{i+1} \arrow[r, equal] & L_{i+1} \\ K_i \arrow[r] \arrow[d, equal] & M^1 \arrow[r] \arrow[u] & L_i \arrow[u] \\ K_i \arrow[r] & K_{i+1} \arrow[r] \arrow[u] & L'_i \arrow[u] \end{tikzcd}$$

in which each row and column is exact. L'_i will be the sub-BP_*-module of $L_i = M^1/K_i$ generated by the positive dimensional part of $\mathrm{Ext}^0(L_i)$. It is a subcomodule of L_i, K_{i+1} is defined to be the induced extension by K_i, and $L_{i+1} = M^1/K_{i+1}$. Hence K_i, K_{i+1}, and L'_i are connective while L_i and L_{i+1} are not.

We know that in positive dimensions $K_0 = E^1_1$ has the same Ext^0 as M^1. We will show by induction that the same is true for each K_i. In the long exact sequence of Ext groups associated with the right column, the map $\mathrm{Ext}^0(L'_i) \to \mathrm{Ext}^0(L_i)$ is an isomorphism in positive dimensions, so the positive dimensional part of $\mathrm{Ext}^0(L_{i+1})$ is contained in $\mathrm{Ext}^1(L'_i)$, which has higher connectivity than $\mathrm{Ext}^0(L_i)$.

It follows that the connectivity of L'_i increases with i, and therefore the limit K_∞ has finite type. The connectivity of the positive dimensional part of $\mathrm{Ext}^0(L_i)$ also increases with i, so $\mathrm{Ext}^0(L_\infty)$ is trivial in positive dimensions. From the long exact sequence of Ext groups for the short exact sequence

$$0 \to K_\infty \to M^1 \to L_\infty \to 0$$

we deduce that $\mathrm{Ext}^1(K_\infty) = 0$, so K_∞ is a weak injective by Lemma 7.1.32. It has the same Ext^0 as E^1_1, so it is our D^1_1. □

Now we are ready to study the hypothetical comodule E^2_{m+1} of (7.1.19) for $m > 0$.

7.2.2. Lemma. *The Poincaré series for E^2_{m+1} is at least*

$$g_m(t)G_m(t)\sum_{i>0}\frac{x^{p^i}(1-y_i)}{(1-x^{p^i})(1-x_{i+1})}$$

(where $g_m(t)$ and $G_m(t)$ are as in Lemma 7.1.35), with equality holding for $m = 0$ and $p > 2$. In dimensions less than $p^2|\widehat{v}_1|$ it simplifies to

$$g_{m+2}(t)\left(\frac{x^p(1-y)}{(1-x_2)(1-x^p)}\right),$$

where x, y, x_i and y_i are as in 7.1.30.

We will see in Theorem 7.2.6 below that equality also holds in dimensions less than $p^2|\widehat{v}_1|$.

Proof of 7.2.2. The relevant Poincaré series (excluding the case $m = 0$ and $p = 2$) are

$$\begin{aligned} g(E^1_{m+1}) &= g_m(t)G_m(t)\sum_{i>0}\frac{x_i}{(1-x_i)} \qquad \text{by 7.1.35} \\ &= g_m(t)G_m(t)\left(\frac{x}{1-x}+\sum_{i>0}\frac{x_{i+1}}{1-x_{i+1}}\right), \\ \text{and}\quad g(\mathrm{Ext}^0(E^1_{m+1})) &= g(\mathrm{Ext}^1(BP_*)) \\ &= g_m(t)\sum_{i>0}\frac{x^{p^{i-1}}}{1-x^{p^{i-1}}} \qquad \text{by 7.1.31} \\ &= g_m(t)\left(\frac{x}{1-x}+\sum_{i>0}\frac{x^{p^i}}{1-x^{p^i}}\right). \end{aligned}$$

If there were a map $E^1_{m+1} \to D^1_{m+1}$ to a weak injective inducing an isomorphism in Ext^0 (which there is for $m = 0$ and p odd by 7.2.1), we would have

$$\begin{aligned} g(D^1_{m+1}) &= G_m(t)g(\mathrm{Ext}^0(E^1_{m+1})) \qquad \text{by 7.1.34} \\ &= G_m(t)g(\mathrm{Ext}^1(BP_*)) \\ &= g_m(t)G_m(t)\left(\frac{x}{1-x}+\sum_{i>0}\frac{x^{p^i}}{1-x^{p^i}}\right). \end{aligned}$$

It follows that

$$\begin{aligned} g(E^2_{m+1}) &\geq g_m(t)G_m(t)\left(\frac{x}{1-x}+\sum_{i>0}\frac{x^{p^i}}{1-x^{p^i}}\right) - g(E^1_{m+1}) \\ &= g_m(t)G_m(t)\sum_{i>0}\left(\frac{x^{p^i}}{1-x^{p^i}}-\frac{x_{i+1}}{1-x_{i+1}}\right) \\ &= g_m(t)G_m(t)\sum_{i>0}\frac{x^{p^i}(1-y_i)}{(1-x^{p^i})(1-x_{i+1})}. \end{aligned}$$

In our range of dimensions we can replace $g_m(t)G_m(t)$ by $g_{m+2}(t)$, and only the first term of the last sum is relevant. Hence we have

$$g(E^2_{m+1}) \quad \equiv \quad g_{m+2}(t)\left(\frac{x^p(1-y)}{(1-x_2)(1-x^p)}\right) \qquad \bmod (t^{p^2|\widehat{v}_1|}). \quad \square$$

7.2.3. Corollary. *For a locally finite bounded below subcomodule*

$$E \subset E^1_{m+1}/(v_1^\infty),$$

let D denote the induced (as in (7.1.20)) extension by E^1_{m+1} shown in the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & E^1_{m+1} \arrow[r] \arrow[d, equal] & v_1^{-1}E^1_{m+1} \arrow[r] & E^1_{m+1}/(v_1^\infty) \arrow[r] & 0 \\
0 \arrow[r] & E^1_{m+1} \arrow[r] & D \arrow[r] \arrow[u] & E \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Let K denote the kernel of the connecting homomorphism

$$\delta : \mathrm{Ext}^0(E) \to \mathrm{Ext}^1(E^1_{m+1}) = \mathrm{Ext}^2(BP_*).$$

Then D is weak injective if and only if the Poincaré series $g(E)$ is $g(K)G_m(t)$ plus the series specified in Lemma 7.2.2. In particular it is weak injective if δ is a monomorphism and $g(E)$ is the specifed series.

Proof. The specified series is $G_m(t)g(\mathrm{Ext}^0(E^1_{m+1})) - g(E^1_{m+1})$, and

$$g(\mathrm{Ext}^0(D)) = g(\mathrm{Ext}^0(E^1_{m+1})) + g(K).$$

Hence our hypothesis implies

$$\begin{aligned}
g(D) &= g(E^1_{m+1}) + g(E) \\
&= g(E^1_{m+1}) + G_m(t)g(\mathrm{Ext}^0(E^1_{m+1})) - g(E^1_{m+1}) + g(K)G_m(t) \\
&= G_m(t)(g(\mathrm{Ext}^0(E^1_{m+1}) + g(K)) \\
&= G_m(t)g(\mathrm{Ext}^0(D)),
\end{aligned}$$

which is equivalent to the weak injectivity of D by Theorem 7.1.34. $\square$

Now we need to identify some elements in $E^1_{m+1}/(v_1^\infty)$.

7.2.4. Lemma. *The comodule $E^1_{m+1}/(v_1^\infty)$ contains the sets*

(a)

$$\begin{cases}
\left\{\dfrac{1}{p^{1+e_0}v_1^{e_1-e_0}} : e_1 > e_0 \geq 0\right\} & \textit{for } m = 0 \\
\left\{\dfrac{\widehat{v}_1^{1+e_0}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0\right\} & \textit{for } m > 0
\end{cases}$$

(b)

$$\left\{\frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0\right\}.$$

These generators will be discussed further in Theorem 7.2.6 below.

PROOF. (i) The element in question is the image of $v_1^{-1-e_1}\widehat{\lambda}_1^{1+e_0}$.
(ii) In D^0_{m+1} we have

$$\begin{aligned}
\widehat{\lambda}_2 &= \widehat{\ell}_2 - \ell_1\widehat{\lambda}_1^p \\
&= \frac{\widehat{v}_2}{p} - \frac{v_1\widehat{\lambda}_1^p}{p} + \begin{cases} \dfrac{v_1\widehat{v}_1^p}{p^2} + \dfrac{v_1^{p\omega}\widehat{v}_1}{p^2} & \text{for } m=0 \\ \dfrac{v_1^{p+1}}{p^2} & \text{for } m>0 \end{cases} \\
&= \frac{\widehat{v}_2}{p} + \frac{v_1}{p}(1-p^{p-1})\widehat{\lambda}_1^p + \begin{cases} 0 & \text{for } m=0 \\ v_1^{p\omega}\widehat{\lambda}_1 & \text{for } m>0 \end{cases}
\end{aligned}$$

so

$$\frac{\widehat{v}_2}{p} = \widehat{\lambda}_2 + \frac{v_1}{p}\mu \tag{7.2.5}$$

where

$$\mu = (1-p^{p-1})\widehat{\lambda}_1^p + \begin{cases} 0 & \text{for } m=0 \\ v_1^{\omega-1}\widehat{\lambda}_1 & \text{for } m>0. \end{cases}$$

Hence in $p^{-1}v_1^{-1}BP_*$ we have

$$\begin{aligned}
\frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} &= \frac{p^{e_1}}{v_1^{1+e_1}}\left(\frac{\widehat{v}_2}{p}\right)^{1+e_0+e_1} \\
&= \frac{p^{e_1}}{v_1^{1+e_1}}\left(\widehat{\lambda}_2 + \frac{v_1}{p}\mu\right)^{1+e_0+e_1} \\
&= \frac{p^{e_1}}{v_1^{1+e_1}}\sum_{k\ge 0}\binom{1+e_0+e_1}{k}\widehat{\lambda}_2^{1+e_0+e_1-k}\frac{v_1^k}{p^k}\mu^k \\
&= \sum_{k\ge 0}\binom{1+e_0+e_1}{k}\frac{p^{e_1-k}}{v_1^{1+e_1-k}}\widehat{\lambda}_2^{1+e_0+e_1-k}\mu^k.
\end{aligned}$$

The image of this element in $p^{-1}BP_*/(v_1^\infty)$ is

$$\sum_{0\le k\le e_1}\binom{1+e_0+e_1}{k}\frac{p^{e_1-k}}{v_1^{1+e_1-k}}\widehat{\lambda}_2^{1+e_0+e_1-k}\mu^k.$$

The coefficient of each term is an integer, so the expression lies in $D^0_{m+1}/(v_1^\infty)$, and its image in $E^1_{m+1}/(v_1^\infty)$ is the desired element.

□

We will now construct a comodule $E^2_{m+1} \subset E^1_{m+1}/(v_1^\infty)$ satisfying the conditions of Corollary 7.2.3 with δ monomorphic below dimension $p^2|\widehat{v}_1|$.

7.2.6. THEOREM. *Let $E^2_{m+1} \subset E^1_{m+1}/(v_1^\infty)$ be the $A(m+2)$-module generated by the set*

$$\left\{\frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \ge 0\right\}.$$

Below dimension $p^2|\widehat{v}_1|$ it has the Poincaré series specified in Lemma 7.2.2, it is a comodule, it is 1-free, and its Ext group is

$$A(m+1)/I_2 \otimes E(\widehat{h}_{1,0}) \otimes P(\widehat{b}_{1,0}) \otimes \left\{\frac{\widehat{v}_2^{e_2}}{pv_1} : e_2 \ge 1\right\}.$$

In particular Ext^0 *maps monomorphically to* $\mathrm{Ext}^2(BP_*)$ *in that range.*

PROOF. Recall that the Poincaré series specified in Lemma 7.2.2 in this range is

$$g_{m+2}(t)\left(\frac{x^p(1-y)}{(1-x_2)(1-x^p)}\right) = g(BP_*/I_2)\frac{x^p}{(1-x_2)(1-x^p)}.$$

Each generator of E^2_{m+1} can be written as

$$x_{e_0,e_1} = \frac{\widehat{v}_2^{1+e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}} = \frac{\widehat{v}_2}{pv_1}\left(\frac{\widehat{v}_2}{p}\right)^{e_0}\left(\frac{\widehat{v}_2}{v_1}\right)^{e_1}$$

with $e_0, e_1 \geq 0$. Since $|\frac{\widehat{v}_2}{pv_1}| = p|\widehat{v}_1|$, the Poincaré series for this set of generators is

$$\frac{x^p}{(1-x_2)(1-x^p)}.$$

We can filter E^2_{m+1} by defining F_i to be the submodule generated by the x_{e_0,e_1} with $e_0+e_1 \leq i$. Then each subquotient is a direct sum of suspensions of BP_*/I_2, so the Poincaré series is as claimed.

To see that E^2_{m+1} is a comodule, we will use the I-adic valuation as defined in the proof of Lemma 7.1.35. In our our range the set of elements with valuation at least -1 is the $A(m)$-submodule M generated by

$$\left\{\frac{\widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}} : e_0, e_1 \geq 0,\ i+j \geq 1+e_0+e_1\right\},$$

while E^2_{m+1} is generated by a similar set with $j \geq 1+e_0+e_1$. Thus it suffices to show that the decreasing filtration on M defined by letting F^kM be the submodule generated by all such generators with $j-e_0-e_1 \geq k$ is a comodule filtration. For this observe that modulo $\Gamma(m+1)\otimes F^{1+j-e_0-e_1}M$ we have

$$\frac{\eta_R(\widehat{v}_1^i\widehat{v}_2^j)}{p^{1+e_0}v_1^{1+e_1}} \equiv \frac{\widehat{v}_1^i(\widehat{v}_2+v_1\widehat{t}_1^p+p\widehat{t}_2)^j}{p^{1+e_0}v_1^{1+e_1}} \in \Gamma(m+1)\otimes F^{j-e_0-e_1}M,$$

so $E^2_{m+1} = F^1M$ is a subcomodule.

We use the same filtration for the Ext computation. Assuming that $j \geq 1+e_0+e_1 > 1$ we have

$$\begin{aligned}\frac{\eta_R(\widehat{v}_1^i\widehat{v}_2^j)-\widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}} &\equiv \frac{\widehat{v}_1^i(\widehat{v}_2+v_1\widehat{t}_1^p+p\widehat{t}_2)^j-\widehat{v}_1^i\widehat{v}_2^j}{p^{1+e_0}v_1^{1+e_1}}\\ &\equiv \binom{j}{e_0+e_1}\frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}(v_1\widehat{t}_1^p+p\widehat{t}_2)^{e_0+e_1}}{p^{1+e_0}v_1^{1+e_1}}+\dots\\ &\equiv (e_0,e_1,j-e_0-e_1)\frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}\widehat{t}_1^{pe_1}\widehat{t}_2^{e_0}}{pv_1}+\dots\end{aligned}$$

where the missing terms involve higher powers of $\widehat{v}_2$. The multinomial coefficient $(e_0,e_1,j-e_0-e_1)$ is always nonzero since $j<p$. This means no linear combination of such elements is invariant, and the only invariant generators are the ones with $e_0=e_1=0$, so Ext^0 is as claimed.

We will use this to show that E^2_{m+1} is 1-free (as defined in 7.1.8), i.e., that $T_m^{p-1}\otimes_{BP_*}E^2_{m+1}$ is weak injective in this range. For $0 \leq k \leq p-1$ we have

$$\frac{\psi(\widehat{v}_1^i\widehat{v}_2^j\widehat{t}_1^k)-\widehat{v}_1^i\widehat{v}_2^j\widehat{t}_1^k}{p^{1+e_0}v_1^{1+e_1}}=(e_0,e_1,j-e_0-e_1)\widehat{t}_1^{pe_1+k}\widehat{t}_2^{e_0}\otimes\frac{\widehat{v}_1^i\widehat{v}_2^{j-e_0-e_1}}{pv_1}+\dots.$$

This means that

$$\mathrm{Ext}^0(T_m^{p-1}\otimes_{BP_*}E^2_{m+1})=\mathrm{Ext}^0(E^2_{m+1}).$$

It follows that

$$\begin{aligned} g(\mathrm{Ext}^0) &= g_{m+1}(t)(1-y)\frac{x^p}{1-x_2}\\ \text{so}\qquad g(E^2_{m+1}) &= g(\mathrm{Ext}^0)\frac{1}{(1-x^p)(1-x_2)},\\ \text{and}\qquad g(T_m^{p-1}\otimes_{BP_*}E^2_{m+1}) &= g(\mathrm{Ext}^0)\frac{1}{(1-x)(1-x_2)}\\ &= g(\mathrm{Ext}^0)G_m(t)\end{aligned}$$

This makes $T_m^{p-1}\otimes_{BP_*}E^2_{m+1}$ weak injective in this range by Theorem 7.1.34.

We can use the small descent spectral sequence of Theorem 7.1.13 to pass from $\mathrm{Ext}(T_m^{p-1}\otimes_{BP_*}E^2_{m+1})$ to $\mathrm{Ext}(E^2_{m+1})$. It collapses from E_1 since the two comodules have the same Ext^0. This means that $\mathrm{Ext}(E^2_{m+1})$ is as claimed.

To show that $\mathrm{Ext}^0(E^2_{m+1})$ maps monomorphically to $\mathrm{Ext}^2(BP_*)$, the chromatic method tells us that $\mathrm{Ext}^2(BP_*)$ is a certain subquotient of $\mathrm{Ext}^0(M^2)$, namely the kernel of the map to $\mathrm{Ext}^0(M^3)$ modulo the image of the map from $\mathrm{Ext}^0(M^1)$. We know that the latter is the $A(m)$-module generated by the elements $\dfrac{\widehat{v}_1^i}{pi}$, and the elements in question, the $A(m+1)$ multiples of $\dfrac{\widehat{v}_2^i}{pv_1}$ are not in the image. The latter map trivially to $\mathrm{Ext}^0(M^3)$ because they involve no negative powers of v_2. □

7.2.7. Corollary. *Excluding the case $(p,m)=(2,0)$, below dimension $p^2|\widehat{v}_1|$,*

$$\mathrm{Ext}^s_{\Gamma(m+1)}=\begin{cases} A(m) & \text{for } s=0\\ A(m)\left\{\dfrac{\widehat{v}_1^j}{pj}:j>0\right\} & \text{for } s=1\\ \mathrm{Ext}^{s-2}_{\Gamma(m+1)}(E^2_{m+1}) & \text{for } s\ge 2.\end{cases}$$

The Adams–Novikov spectral sequence collapses with no nontrivial extensions in this range, so $\pi_*(T(m))$ has a similar description below dimension $p^2|\widehat{v}_1|-3$.

The group $\mathrm{Ext}_{\Gamma(m+1)}(E^2_{m+1})$ was described in Theorem 7.2.6.

We now specialize to the case $m=0$ and p odd. Using Lemma 7.2.1 we get the 4-term exact sequence

$$0\to BP_*\to D_1^0\to D_1^1\to E_1^2\to 0, \tag{7.2.8}$$

for which the resolution spectral sequence (A1.3.2) collapses from E_1.

We could get at $\mathrm{Ext}_{\Gamma(1)}(E_1^2)$ via the Cartan–Eilenberg spectral sequence for the extension

$$(A(1),G(1))\to(BP_*,\Gamma(1))\to(BP_*,\Gamma(2))$$

if we knew the value of $\mathrm{Ext}_{\Gamma(2)}(E_1^2)$ as a $G(1)$-comodule. For this we need to consider (7.2.8) as an exact sequence of $\Gamma(2)$-comodules and study the resulting

resolution spectral sequence. By Lemma 7.1.33 we know that D_1^0 and D_1^1 are weak injectives over $\Gamma(2)$. It follows that the resolution spectral sequence collapses from E_2 and that the connecting homomorphism

$$\delta : \mathrm{Ext}^s_{\Gamma(2)}(E_1^2) \to \mathrm{Ext}^{s+1}_{\Gamma(2)}(E_1^1) = \mathrm{Ext}^{s+2}_{\Gamma(2)}$$

is an isomorphism for $s > 0$. This implies that

$$\mathrm{Ext}^s_{\Gamma(2)}(E_1^2) \cong \mathrm{Ext}^{s+2}_{\Gamma(2)},$$

which is described in our range by Theorem 7.2.6.

For $s = 0$, the situation is only slightly more complicated. Recall that the 4-term exact sequence (7.2.8) is the splice of two short exact sequences,

$$0 \to BP_* \to D_1^0 \to E_1^1 \to 0$$

(where $E_1^1 = D_1^0/BP_*$) and

$$0 \to E_1^1 \to D_1^1 \to E_1^2 \to 0,$$

Thus we have a short exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma(2)}(E_1^1) \to \mathrm{Ext}^0_{\Gamma(2)}(D_1^1) \to L \to 0 \tag{7.2.9}$$

and this L is the kernel of δ for $s = 0$. Thus there is a short exact sequence

$$0 \to L \to \mathrm{Ext}^0_{\Gamma(2)}(E_1^2) \to U \to 0, \tag{7.2.10}$$

where $U = \mathrm{Ext}^2_{\Gamma(2)}$, which is described in our range by Theorem 7.2.6.

7.2.11. Theorem. *The comodule L of (7.2.9) is the $A(1)$-submodule $B \subset N^2$ generated by the set*

$$\left\{ \frac{v_2^i}{ipv_1^i} : i > 0 \right\}.$$

We will denote the element $\frac{v_2^i}{ipv_1^i}$ by $\beta'_{i/i}$. Theorem 7.2.11 implies

7.2.12. Theorem. *In the resolution spectral sequence for (7.2.8) we have*

$$\begin{aligned} E_1^{0,s} = E_\infty^{0,s} &= \begin{cases} \mathbf{Z}_{(p)} & \text{for } s = 0 \\ 0 & \text{for } s > 0, \end{cases} \\ E_2^{1,s} = E_\infty^{1,s} &= \begin{cases} \mathrm{Ext}^1_{\Gamma(1)} & \text{for } s = 0 \\ 0 & \text{for } s > 0, \end{cases} \end{aligned}$$

and for

$$E_1^{2,*} = \mathrm{Ext}_{\Gamma(1)}(E_1^2).$$

In the Cartan–Eilenberg spectral sequence (A1.3.14) for this group we have

$$\tilde{E}_2^{s,t} = \mathrm{Ext}^s_{G(1)}(\mathrm{Ext}^t_{\Gamma(2)}(E_2^1)).$$

For $t > 0$,

$$\mathrm{Ext}^s_{G(1)}(\mathrm{Ext}^t_{\Gamma(2)}(E_2^1)) = \mathrm{Ext}^s_{G(1)}(\mathrm{Ext}^{t+2}_{\Gamma(2)})$$

and for $t = 0$ there is long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^0_{G(1)}(B) \arrow[r] & \tilde{E}_2^{0,0} \arrow[r] & \mathrm{Ext}^0_{G(1)}(U) \arrow[dll] \\
& \mathrm{Ext}^1_{G(1)}(B) \arrow[r] & \tilde{E}_2^{1,0} \arrow[r] & \mathrm{Ext}^1_{G(1)}(U) \arrow[dll] \\
& \mathrm{Ext}^2_{G(1)}(B) \arrow[r] & \cdots &
\end{tikzcd}$$

associated with the short exact sequence (7.2.10).

We will also need to consider the tensor product of (7.2.8) with $T_0^{(j)}$, and we will denote the resulting resolution spectral sequence by $\left\{E_r^{s,t}(T_0^{(j)})\right\}$. Let $\left\{\tilde{E}_r^{s,t}(T_0^{(j)})\right\}$ denote the Cartan–Eilenberg spectral sequence for $\mathrm{Ext}_{\Gamma(1)}(T_0^{(j)} \otimes E_1^2)$. For a $\Gamma(1)$-comodule M, we have

$$\mathrm{Ext}_{\Gamma(2)}(T_0^{(j)} \otimes_{BP_*} M) \cong \overline{T}_0^{(j)} \otimes_{A(1)} \mathrm{Ext}_{\Gamma(2)}(M),$$

where $T_0^{(j)} \subset \Gamma(1)$ and $\overline{T}_0^{(j)} \subset G(1)$, since $T_0^{(j)}$ is isomorphic over $\Gamma(2)$ to a direct sum of p^j suspensions of BP_*. It follows that we have a short exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma(2)}(T_0^{(j)} \otimes E_1^1) \to \mathrm{Ext}^0_{\Gamma(2)}(T_0^{(j)} \otimes D_1^1) \to \overline{T}_0^{(j)} \otimes B \to 0$$

and the long exact sequence of Theorem 7.2.12 generalizes to

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^0_{G(1)}(\overline{T}_0^{(j)} \otimes B) \arrow[r] & \tilde{E}_2^{0,0}(T_0^{(j)}) \arrow[r] & \mathrm{Ext}^0_{G(1)}(\overline{T}_0^{(j)} \otimes U) \arrow[dll] \\
& \mathrm{Ext}^1_{G(1)}(\overline{T}_0^{(j)} \otimes B) \arrow[r] & \tilde{E}_2^{1,0}(T_0^{(j)}) \arrow[r] & \mathrm{Ext}^1_{G(1)}(\overline{T}_0^{(j)} \otimes U) \arrow[dll] \\
& \mathrm{Ext}^2_{G(1)}(\overline{T}_0^{(j)} \otimes B) \arrow[r] & \cdots &
\end{tikzcd} \tag{7.2.13}$$

The following is helpful in proving Theorem 7.2.11.

7.2.14. Lemma. *Let $M \subset \mathrm{Ext}^0_{\Gamma(2)}(E_1^1/(v_1^\infty))$ be a $G(1)$-subcomodule with trivial image (under the connecting homomorphism) in*

$$\mathrm{Ext}^1_{\Gamma(2)}(E_1^1) = \mathrm{Ext}^2_{\Gamma(2)};$$

equivalently let

$$M \subset E/(v_1^\infty).$$

where $E = \mathrm{Ext}^0_{\Gamma(2)}(E_1^1)$. Then it is a subcomodule of $\mathrm{Ext}^0_{\Gamma(2)}(E_1^2)$ if it has a preimage

$$\tilde{M} \subset \mathrm{Ext}^0_{\Gamma(2)}(v_1^{-1}E_1^1) \subset v_1^{-1}E_1^1$$

that is obtained from E by adjoining elements divisible by the ideal $J = (\widehat{\lambda}_2, \widehat{\lambda}_3, \dots)$.

PROOF. We have a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & \tilde{M} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E \arrow[r] & v_1^{-1}E \arrow[r] & E/v_1^\infty \arrow[r] & 0
\end{tikzcd}$$

We need to verify that the monomorphism

$$\mathrm{Ext}^0_{\Gamma(1)}(E^1_1) = \mathrm{Ext}^0_{G(1)}(E) \to \mathrm{Ext}^0_{G(1)}(\tilde{M})$$

is an isomorphism. If an element $x \in \tilde{M}$ is invariant, then some v_1-multiple of it must lie in $\mathrm{Ext}^0_{\Gamma(1)}(E^1_1)$, which has no elements divisible by J. Hence x has trivial image in M and therefore lies in in E, and we have our isomorphism.

Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] \arrow[dd, bend right] & \tilde{M} \arrow[r] \arrow[d, dotted] \arrow[dd, bend right] & M \arrow[r] \arrow[d, dotted] \arrow[dd, bend right] & 0 \\
0 \arrow[r] & E^1_1 \arrow[r] \arrow[d, equal] \arrow[dd, bend right] & D^1_1 \arrow[r] \arrow[d] \arrow[dd, bend right] & E^2_1 \arrow[r] \arrow[d] \arrow[dd, bend right] & 0 \\
0 \arrow[r] & E^1_1 \arrow[r] & v_1^{-1}E^1_1 \arrow[r] \arrow[d] & E^1_1/(v_1^\infty) \arrow[r] \arrow[d] & 0 \\
 & & v_1^{-1}E^1_1/D^1_1 \arrow[r, equal] & v_1^{-1}E^1_1/D^1_1 &
\end{tikzcd}$$

We have shown that the map $\tilde{M} \to v_1^{-1}E^1_1/D^1_1$ is trivial in Ext^0, so it is trivial. It follows that $\tilde{M}$ maps to D^1_1, so M maps to E^2_1. □

7.2.15. LEMMA. *Let L be as in (7.2.9). Then*

$$g(L) = \frac{1}{1-x}\sum_{i\geq 0}\frac{x^{p^{i+1}}(1-x^{p^i})}{(1-x^{p^{i+1}})(1-x_2^{p^i})},$$

where $x = t^{|v_1|}$ and $x_2 = t^{|v_2|}$.

PROOF. Since D^0_1 is weak injective, applying the functor $\mathrm{Ext}_{\Gamma(2)}$ to the short exact sequence

$$0 \to BP_* \to D^0_1 \to E^1_1 \to 0$$

yields a 4-term exact sequence

$$0 \to A(1) \to \mathrm{Ext}^0_{\Gamma(2)}(D^0_1) \to \mathrm{Ext}^0_{\Gamma(2)}(E^1_1) \to \mathrm{Ext}^1_{\Gamma(2)} \to 0$$

and hence a short exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma(2)}(D^0_1)/A(1) \to \mathrm{Ext}^0_{\Gamma(2)}(E^1_1) \to \mathrm{Ext}^1_{\Gamma(2)} \to 0,$$

where

$$\mathrm{Ext}^0_{\Gamma(2)}(D^0_1) = A(1)[p^{-1}v_1].$$

A calculation similar to that of Lemma 7.1.35 shows that

$$g(\mathrm{Ext}^0_{\Gamma(2)}(D^0_1)/A(1)) = \frac{x}{(1-x)^2}$$

so

$$\mathrm{Ext}^0_{\Gamma(2)}(E^1_1) = \frac{x}{1-x}\left(\frac{x}{1-x} + \sum_{i\geq 0}\frac{x_2^{p^i}}{1-x_2^{p^i}}\right). \tag{7.2.16}$$

Now consider the short exact sequence

$$0 \to \mathrm{Ext}^0_{\Gamma(2)}(E^1_1) \to \mathrm{Ext}^0_{\Gamma(2)}(D^1_1) \to L \to 0. \tag{7.2.17}$$

Since D^1_1 is weak injective over $\Gamma(1)$, Lemma 7.1.33 tells us that $\mathrm{Ext}^0_{\Gamma(2)}(D^1_1)$ is weak injective over $G(1,0)$ with

$$\mathrm{Ext}^0_{G(1,0)}(\mathrm{Ext}^0_{\Gamma(2)}(D^1_1)) = \mathrm{Ext}^0_{\Gamma(1)}(D^1_1) = \mathrm{Ext}^1_{\Gamma(1)}$$

so

$$g(\mathrm{Ext}^0_{\Gamma(2)}(D^1_1)) = \frac{x}{1-x}\sum_{i\geq 0}\frac{x^{p^i}}{1-x^{p^i}}. \tag{7.2.18}$$

Combining (7.2.16), (7.2.17), and (7.2.18) gives

$$\begin{aligned} g(L) &= g(\mathrm{Ext}^0_{\Gamma(2)}(D^1_1)) - g(\mathrm{Ext}^0_{\Gamma(2)}(E^1_1)) \\ &= \frac{x}{1-x}\left(\sum_{i\geq 0}\frac{x^{p^i}}{1-x^{p^i}}\frac{x}{1-x} - \sum_{i\geq 0}\frac{x_2^{p^i}}{1-x_2^{p^i}}\right) \\ &= \frac{x}{1-x}\sum_{i\geq 0}\left(\frac{x^{p^{i+1}}}{1-x^{p^{i+1}}} - \frac{x_2^{p^i}}{1-x_2^{p^i}}\right) \\ &= \frac{x}{1-x}\sum_{i\geq 0}\frac{x^{p^{i+1}}(1-x_1^{p^i})}{(1-x^{p^{i+1}})(1-x_2^{p^i})}. \end{aligned}$$

□

7.2.19. Lemma. *Let B be as in Theorem 7.2.11. Its Poincaré series is the same as the one for L, as given in Lemma 7.2.15.*

Proof. Let $F_kB \subset B$ denote the submodule of exponent p^k, with $B_0 = \phi$. Then we find that

$$F_kB = F_{k-1}B + A(1)\left\{\beta_{ip^{k-1}/ip^{k-1},k} \colon i > 0\right\}$$

so

$$F_kB/F_{k-1}B = A(1)/I_1\left\{\beta_{ip^{k-1}/ip^{k-1},k} \colon i > 0\right\},$$

and

$$\begin{aligned}
F_kB &= F_{k-1}B + g(F_kB/F_{k-1}B) \\
&= g(A(1)/I_2)\sum_{i>0}\frac{x^{ip^k}-x_2^{ip^{k-1}}}{1-x} \\
&= \frac{x}{1-x}\sum_{i>0}\left(x^{ip^k}-x_2^{ip^{k-1}}\right) \\
&= \frac{x}{1-x}\left(\frac{x^{p^k}}{1-x^{p^k}}-\frac{x_2^{p^{k-1}}}{1-x_2^{p^{k-1}}}\right) \\
&= \frac{x}{1-x}\frac{x^{p^k}(1-x^{p^{k-1}})}{(1-x^{p^k})(1-x_2^{p^{k-1}})}.
\end{aligned}$$

Summing this for $k \geq 1$ gives the desired Poincaré series of B. □

Proof of Theorem 7.2.11. We first show that B is a $G(1)$-comodule by showing that it is invariant over $\Gamma(2)$. In $\Gamma(2)$ we have

$$\eta_R(v_2) = v_2 + pt_2,$$

so for each $i > 0$, the elements

$$\frac{v_2^i}{ip} \in N^1 \qquad \text{and hence} \qquad \frac{v_2^i}{ipv_1^i} \in N^2$$

are invariant.

Next we show that $B \subset E_1^1/(v_1^\infty)$. Note that

$$\begin{aligned}
v_1^{-1}v_2 &= pv_1^{-1}\widehat{\lambda}_2 + (1-p^{p-1})\widehat{\lambda}_1^p \\
&= pv_1^{-1}(\widehat{\lambda}_2 + \widehat{\lambda}_1 w) \\
&\qquad \text{where } w = (1-p^{p-1})\widehat{\lambda}_1^p \qquad (7.2.20) \\
\text{so} \qquad \beta'_{i/i} &= \frac{p^i(\widehat{\lambda}_2+\widehat{\lambda}_1 w)^i}{ipv_1^i} = \frac{p^{i-1}(\widehat{\lambda}_2+\widehat{\lambda}_1 w)^i}{iv_1^i}.
\end{aligned}$$

The coefficient p^{i-1}/i in this expression is always a p-local integer, so

$$\beta'_{i/i} \in E_1^1/(v_1^\infty).$$

Let

$$\tilde{\beta}'_{i/i} = \frac{v_1^{-i}v_2^i - w^i}{pi}.$$

Then we have

$$\begin{aligned}
\tilde{\beta}'_{i/i} &= \frac{v_1^{-i}(p\widehat{\lambda}_2 + v_1 w)^i - w^i}{ip} \\
&= \sum_{j>0}\binom{i}{j}\frac{(pv_1^{-1}\widehat{\lambda}_2)^j w^{i-j}}{ip} \\
&\in v_1^{-1}E_1^1,
\end{aligned}$$

so $\tilde{\beta}'_{i/i}$ is a lifting of $\beta'_{i/i}$ to $v_1^{-1}E_1^1$. Let $\tilde{B} \subset \mathrm{Ext}^0_{\Gamma(2)}(v_1^{-1}E_1^1)$ be the $A(1)$-submodule obtained by ajoining the elements $\tilde{\beta}'_{i/i}$ to $\mathrm{Ext}^0_{\Gamma(2)}(E_1^1)$; it projects to $B \subset E_1^1/(v_1^\infty)$. Since each $\tilde{\beta}'_{i/i}$ is divisible by $\widehat{\lambda}_2$, it follows from Lemma 7.2.14 that $B \subset E_1^2$.

B and L have the same Poincaré series by 7.2.15 and 7.2.19, so they are equal. □

3. The homotopy of $T(0)_{(2)}$ and $T(0)_{(1)}$

In this section we will determine the Adams–Novikov E_2-term

$$\mathrm{Ext}_{\Gamma(1)}(BP_*(T(0)_{(2)}))$$

and $\pi_*(T(0)_{(2)})$ in dimensions less than $(p^3+p)|v_1|-3$. This is lower than the range of the previous section for reasons that will be explained below in Lemma 7.3.5. All assertions about Ext groups and related objects will apply only in that range.

We will then state a theorem (7.3.15) about differentials in the spectral sequence of (7.2.13) for $j=1$, which we will prove in the next section.

Our starting point is the determination in Corollary 7.2.7 of $\pi_*(T(1))$ and its Adams–Novikov E_2-term through a larger range, roughly $p^2|v_2|$. There is an equivalence

$$T(1) \cong T(0)_{p^3+p^2-1},$$

so we could use the small descent spectral sequence of Theorem 7.1.13 and the topological small descent spectral sequence 7.1.16 (which turn out to be the same up to regrading) to get what we want. It turns out that we can finesse this by standard algebra.

Theorem 7.2.12 gives a Cartan–Eilenberg spectral sequence converging to $\mathrm{Ext}_{\Gamma(1)}$ whose E_2-term is expressed in terms of $\mathrm{Ext}_{G(1)}(B)$ and $\mathrm{Ext}_{G(1)}(\mathrm{Ext}^s_{\Gamma(2)})$ for $s \geq 2$.

First we have the following partial result about $\mathrm{Ext}_{G(1)}(B)$.

7.3.1. Lemma. *For each $j>0$, the $G(1)$-comodule B of Theorem 7.2.11 is j-free below dimension $p^j|v_2|$, and $\mathrm{Ext}^0_{G(1)}(T_0^{(j)} \otimes B)$ is additively isomorphic in this range to the $A(1)$-submodule of $E_1^1/(v_1^\infty)$ generated by the set*

$$\left\{\beta'_{i/\min(i,p^{j-1})} : i>0\right\} \cup \left\{\beta_{i/p^j} : p^j \leq i < p^j + p^{j-1}\right\}.$$

In particular it is 2-free in our range of dimensions.

Proof. We will denote the indicated group by $H^0(B)$. Given a $G(1)$-comodule M, let $M' = T_0^{(j)} \otimes_{A(1)} M$. According to Theorem 7.1.34, M is j-free (i.e. M' is weak injective) if

$$g(M') = \frac{g(\mathrm{Ext}^0(M'))}{1-x},$$

where as before $x = t^{|v_1|}$. We also know that

$$g(M') = g(M)\frac{1-x^{p^j}}{1-x},$$

so the condition for weak injectivity can be rewritten as

$$g(M) = \frac{g(\mathrm{Ext}^0(M'))}{1 - x^{p^j}}.$$

Now in B we have

$$\left\{\begin{array}{rcl} r_{kp^j}(\beta'_{i/i}) & = & \binom{i}{kp^{j-1}} \frac{v_2^{i-kp^{j-1}}}{ipv_1^{i-kp^{j-1}}} = \binom{i-1}{kp^{j-1}} \beta'_{i-kp^{j-1}/i-kp^{j-1}} \\ r_{kp^j}(\beta_{i/i}) & = & \binom{i}{kp^{j-1}} \beta_{i-kp^{j-1}/i-kp^{j-1}}. \end{array}\right. \tag{7.3.2}$$

For $p^{j-1} < i < p^j + p^{j-1}$, choose k so that $0 < i - kp^{j-1} \leq p^{j-1}$. Then the coefficients of β and β' above are units in every case except for $r_{kp^j}(\beta_{p^j/p^j})$. It follows that for each element in B, applying r_{kp^j} for some k will yield an element in $H^0(B)$. This means that in our range we have

$$g(B) = \frac{g(H^0(B))}{1 - x^{p^j}},$$

so B is j-free if $H^0(B)$ is additively isomorphic to $\mathrm{Ext}^0(B')$.

Each element in $H^0(B)$ is killed by r_i for $i \geq p^j$, so there is a corresponding invariant element in $T_0^{(j)} \otimes B'$ by Lemma 7.1.9. On the other hand, (7.3.2) implies that no element in B' outside of $T_0^{(j)} \otimes H^0(B)$ is invariant, so $\mathrm{Ext}^0(B')$ is as desired. □

The groups $\mathrm{Ext}^s_{\Gamma(2)}$ for $s \geq 2$ in our range were determined in Theorem 7.2.6. Translated to the present context, it reads as follows.

7.3.3. Theorem. *Below dimension $p^2|v_2|$, the group $\mathrm{Ext}^{2+*}_{\Gamma(2)}$ is*

$$E(h_{2,0}) \otimes P(b_{2,0}) \otimes U$$

(where $U = \mathrm{Ext}^2_{\Gamma(2)}$), or more explicitly

$$A(1)/I_2 \otimes E(h_{2,0}) \otimes P(b_{2,0}) \otimes \left\{\delta_0\delta_1\left(\frac{v_3^i v_2^j}{pv_1}\right) : i > 0,\ j \geq 0\right\},$$

where δ_0 and δ_1 are the connecting homomorphisms for the short exact sequences

$$0 \to BP_* \to M^0 \to N^1 \to 0$$

and

$$0 \to N^1 \to M^1 \to N^2 \to 0$$

respectively.

7.3.4. Theorem. *For $i, j \geq 0$, let*

$$u_{i,j} = v_2^j\left(\frac{v_3^i}{i!pv_1} - \frac{v_2^{i+ip}}{c_{i,j}pv_1^{1+ip}}\right) \in N^2$$

where

$$c_{i,j} = \prod_{1 \leq k \leq i} \binom{i+j+kp}{p}.$$

Then $u_{i,j}$ has the following properties.

(i) *$u_{i,j}$ lies in $E_1^1/(v_1^\infty)$ and is invariant over $\Gamma(2)$, i.e., it lies in*
$$\mathrm{Ext}^0_{\Gamma(2)}(E_1^1/(v_1^\infty)).$$
(ii) *Its image in U is that of*
$$\frac{v_3^i v_2^j}{i! p v_1}.$$
(iii) *For $i > 0$*
$$r_{p^2}(u_{i,j}) = u_{i-1,j+1},$$
where $u_{0,j} = 0$.

(iv) *For $j \geq 0$,*
$$r_p(u_{1,j}) = -\frac{j+1}{(p,j)}\beta_{j+p/p}.$$

We will denote $u_{1,j}$ by u_j. The coefficients $i!$, $c_{i,j}$ and (p,j) are always nonzero modulo p in our range except in the case
$$u_{p^2-p-1} = \frac{v_2^{p^2-p-1}v_3}{pv_1} - \frac{v_2^{p^2}}{p^2 v_1^{p+1}}.$$

Proof of Theorem 7.3.4. (i) Recall (7.2.5) that
$$\frac{v_2}{p} = \lambda_2 + (1-p^{p-1})\lambda_1^{p+1},$$
while the definition of λ_3 implies that
$$\frac{v_3}{p} \equiv \lambda_3 \bmod (v_1).$$
Hence
$$\begin{aligned} \frac{v_2^j v_3^i}{pv_1} &= \frac{p^{i+j-1}\lambda_2^j\lambda_3^i}{v_1} \\ &\in E_1^1/(v_1^\infty), \\ \text{and} \qquad \frac{v_2^{j+i+ip}}{pv_1^{1+ip}} &= p^{j+i+ip-1}\frac{(\lambda_2 + (1-p^{p-1})\lambda_1^{p+1})^{j+i+ip}}{v_1^{1+ip}} \\ &\in E_1^1/(v_1^\infty), \end{aligned}$$
so $u_{i,j} \in E_1^1/(v_1^\infty)$.

The invariance of $u_{i,j}$ over $\Gamma(2)$ follows from the fact (Proposition 7.1.24) that v_2 is invariant modulo (p) and v_3 is invariant modulo I_2.

(ii) We will show that the difference between the two elements maps trivially to U. It is a scalar multiple of
$$e = \frac{v_2^{i+ip}}{pv_1^{1+ip}},$$
which is the image of
$$\frac{v_1^{-1-ip}v_2^{i+ip}}{p} \in M^1.$$
This is invariant over $\Gamma(2)$, so our element $e \in \mathrm{Ext}^0_{\Gamma(2)}(N^2)$ is in the image of $\mathrm{Ext}^0_{\Gamma(2)}(M^1)$, so it maps trivally to $\mathrm{Ext}^2_{\Gamma(2)} = U$.

(iii) Since

$$\begin{aligned} \eta_R(v_3) &\equiv v_3 + v_2 t_1^{p^2} - v_2^p t_1 \mod I_2 \\ \text{and} \quad \eta_R(v_2) &\equiv v_2 + v_1 t_1^p - v_1^p t_1 \mod I_1, \end{aligned}$$

we have

$$\begin{aligned} \eta_R\left(\frac{v_2^j v_3^i}{i! p v_1}\right) &= \frac{v_2^j (v_3^i + i v_2 v_3^{i-1} t_1^{p^2} + \dots)}{i! p v_1}, \\ \text{so} \quad r_{p^2}\left(\frac{v_2^j v_3^i}{i! p v_1}\right) &= \frac{v_2^{j+1} v_3^{i-1}}{(i-1)! p v_1}. \end{aligned}$$

For the second term we have

$$\begin{aligned} \eta_R\left(\frac{v_2^{j+i+ip}}{c_{i,j} p v_1^{1+ip}}\right) &= \frac{(v_2 + v_1 t_1^p - v_1^p t_1)^{j+i+ip}}{c_{i,j} p v_1^{1+ip}} \\ &= \sum_{0 \le k \le ip} \binom{j+i+ip}{k} \frac{v_2^{j+i+ip-k} (t_1^p - v_1^{p-1} t_1)^k}{c_{i,j} p v_1^{1+ip-k}} \\ &= \sum_{0 \le k \le ip} \binom{j+i+ip}{k} \sum_{0 \le \ell \le k} (-1)^\ell \binom{k}{\ell} \frac{v_2^{j+i+ip-k} t_1^{p(k-\ell)+\ell}}{c_{i,j} p v_1^{1+ip-k-(p-1)\ell}}. \end{aligned}$$

We need to collect the terms in which the exponent of t_1 is p^2, i.e. for which $(p-1)\ell = p(k-p)$. Hence $k-p$ must be divisible by $p-1$, so we can write $k = p + (p-1)k'$ and $\ell = pk'$. This gives

$$\begin{aligned} & r_{p^2}\left(\frac{v_2^{j+i+ip}}{c_{i,j} p v_1^{1+ip}}\right) \\ &= \sum_{0 \le k' \le p} (-1)^{pk'} \binom{j+i+ip}{p+(p-1)k'} \binom{p+(p-1)k'}{pk'} \frac{v_2^{j+i+ip-p-(p-1)k'}}{c_{i,j} p v_1^{1+ip-p-(p^2-1)k'}} \\ &= \binom{j+i+ip}{p} \frac{v_2^{j+i+ip-p}}{c_{i,j} p v_1^{1+ip-p}} \\ &= \frac{v_2^{j+i+(i-1)p}}{c_{i-1,j+1} p v_1^{1+(i-1)p}} \end{aligned}$$

and the result follows.

(iv) Using the methods of (iii), we find that

$$\begin{aligned} r_p(u_{1,j}) &= r_p\left(\frac{v_2^j v_3}{p v_1}\right) - r_p\left(\frac{v_2^{j+p+1}}{c_{1,j} p v_1^{p+1}}\right) \\ &= -\binom{j+p+1}{1} \frac{v_2^{j+p}}{c_{1,j} p v_1^p} \\ &= -\frac{j+1}{(p,j)} \frac{v_2^{j+p}}{c_{1,j} p v_1^p}. \end{aligned}$$

□

In order to use the Cartan–Eilenberg spectral sequence of (7.2.13) we need to know $\mathrm{Ext}_{G(1)}(T_0^{(j)} \otimes U)$. We will compute it by downward induction on j using the small descent spectral sequence of Theorem 7.1.13. Recall (Theorem 7.3.3) that U in our range is generated as an $A(1)$-module by the elements

$$\delta_0\delta_1(u_{i,j}) = \delta_0\delta_1\left(\frac{v_3^i v_2^j}{pv_1}\right).$$

We start with the following.

7.3.5. Lemma. *Let $U = \mathrm{Ext}^2_{\Gamma(2)}$ as before. In dimensions less than $(p^3+p)|v_1|$, there is a short exact sequence of $G(1)$-comodules*

$$0 \to U \to U_0 \to U_1 \to 0 \tag{7.3.6}$$

where $U_0 \subset v_2^{-1}U$ is the $A(2)$-submodule generated by

$$\left\{\delta_0\delta_1\left(\frac{v_2^{1-i}v_3^i}{pv_1}\right) : i > 0\right\}.$$

U_0 and U_1 are each 2-free (7.1.8) as $G(1)$-comodules, and we have

$$\begin{aligned} \mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes U_0) &= A(1)\left\{\delta_0\delta_1(u_{1,j}) \colon j \geq 0\right\} \\ \textit{and} \qquad \mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes U_1) &= A(1)\left\{\delta_0\delta_1\delta_2\left(\frac{u_{i,j}}{v_2}\right) : i \geq 2,\ j \geq 0\right\} \end{aligned}$$

(where δ_2 is the connecting homomorphism for 7.3.6) so

$$\mathrm{Ext}^s_{G(1)}(T_0^{(2)} \otimes U) = \begin{cases} A(1)\left\{\delta_0\delta_1(u_{1,j}) \colon j \geq 0\right\} & \textit{for } s = 0 \\ A(1)\left\{\gamma_i \colon i \geq 2\right\} & \textit{for } s = 1 \\ 0 & \textit{for } s > 1, \end{cases}$$

where

$$\gamma_i = \delta_0\delta_1\delta_2\left(\frac{u_{i,0}}{v_2}\right).$$

Note that we have reduced our range of dimensions from $(p^3+p^2)|v_1|$ to $(p^3+p)|v_1|$. A 2-free subcomodule of M^2 containing U must contain the element

$$x = \frac{v_2^{1-p}v_3^p}{p^2v_1},$$

and $|x| = (p^3+p)|v_1|$. $v_2^{p-1}x$ is in $\mathrm{Ext}^2_{\Gamma(2)}$, but is out of the range of Theorems 7.2.6 and 7.3.3.

Proof. We will construct the desired extension of $\mathrm{Ext}^2_{\Gamma(2)}$ by inducing from one of $\mathrm{Ext}^0_{\Gamma(2)}(E_1^2)$ as in the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^0_{\Gamma(2)}(E_1^2) \arrow[r] \arrow[d, "\delta_0\delta_1"] & U_0' \arrow[r] \arrow[d] & U_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U \arrow[r] & U_0 \arrow[r] & U_1 \arrow[r] & 0 \end{tikzcd}$$

We can extend the defintion of $u_{i,j}$ to negative j and we have $u_{i,j-i} = v_2^{j-1} u_{i,0}$ for $1 \le j \le i$. U_0' is the $A(1)$-submodule of $v_2^{-1} \operatorname{Ext}^0_{\Gamma(2)}(E_1^2)$ obtained by adjoining the elements

$$\left\{ u_{i,j-i} \colon i > 0, 1 \le j \le i \right\}$$

Theorem 7.3.4(v) implies that U_0' and hence U_1 and U_0 are comodules.

It follows that $U_0 \subset v_2^{-1} \operatorname{Ext}^2_{\Gamma(2)}$ is as claimed. The 2-freeness of U_0 and U_1 follows from Theorem 7.3.4(iii).

For the computation of $\operatorname{Ext}^0_{G(1)}(T_0^{(2)} \otimes U_k)$ for $k = 0$ and 1, the following pictures for $p = 3$ may be helpful. We denote $\delta_0\delta_1(u_{i,j})$ by $u'_{i,j}$ and each diagonal arrow represents the action of r_{p^2}. For U_0 (which is v_2-torsion free) we have

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
u'_{1,0} & u'_{2,0} \arrow[ul] & u'_{3,0} \arrow[ul] \\
 & u'_{2,-1} \arrow[ul] & u'_{3,-1} \arrow[ul] \\
 & & u'_{3,-2} \arrow[ul]
\end{tikzcd}$$

where the missing elements have higher second subscripts. For U_1 (which is all v_2-torsion) we denote $\delta_0\delta_1\delta_2(v_2^{-k} u_{i,j})$ by $\frac{u'_{i,j}}{v_2^k}$, and the picture is

$$\begin{tikzcd}
\frac{u'_{1,0}}{v_2} & \frac{u'_{2,0}}{v_2} & \frac{u'_{3,0}}{v_2} \\
 & \frac{u'_{2,0}}{v_2^2} \arrow[ul] & \frac{u'_{3,0}}{v_2^2} \arrow[ul] \\
 & & \frac{u'_{3,0}}{v_2^3} \arrow[ul]
\end{tikzcd}$$

In each case Ext^0 is generated by the elements not supporting an arrow, i.e., the ones in the left column of the first picture and the top row of the second. $\square$

Now consider the Cartan–Eilenberg spectral sequence of (7.2.13) for $j = 2$. For $t > 2$, $\operatorname{Ext}^t_{\Gamma(2)}$ is a suspension of $U = \operatorname{Ext}^2_{\Gamma(2)}$, so $\tilde{E}_r^{s,t}(T_0^{p^{2-1}}) = 0$ for $s > 1$. More precisely for $t = \varepsilon + 2t'$ with $\varepsilon = 0$ or 1,

$$\operatorname{Ext}^{t+2}_{\Gamma(2)} = h_{2,0}^{\varepsilon} b_{2,0}^{t'} U,$$

which we abbreviate by $\Sigma^{(t)} U$. Then we have

7.3.7. Corollary. *In the resolution spectral sequence we have the following short exact sequences for the groups $E_2^{2,t}(T_0^{(2)})$: for $t = 0$*

$$0 \longrightarrow \mathrm{Ext}^0(T_0^{(2)} \otimes B) \longrightarrow E_2^{2,0}(T_0^{(2)}) \longrightarrow \mathrm{Ext}^0(T_0^{(2)} \otimes U) \longrightarrow 0,$$

and for $t > 0$

$$0 \longrightarrow \mathrm{Ext}^1(T_0^{(2)} \otimes \Sigma^{(t-1)}U) \longrightarrow E_2^{2,t}(T_0^{(2)}) \longrightarrow \mathrm{Ext}^0(T_0^{(2)} \otimes \Sigma^{(t)}U) \longrightarrow 0,$$

where $\Sigma^{(t)}U$ is as above and the Ext groups are over $G(1)$.

The groups $\mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes B)$ and $\mathrm{Ext}_{G(1)}(T_0^{(2)} \otimes U)$ are described in Lemmas 7.3.1 and 7.3.5 respectively.

Proof. The long exact sequence of (7.2.13) and Lemma 7.3.1 imply that

$$\tilde{E}_2^{s,t}(T_0^{(2)}) = \mathrm{Ext}^s_{G(1)}(T_0^{(2)} \otimes \Sigma^{(t)}U) \qquad \text{for } t > 0.$$

For $t = 0$ there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes B) \longrightarrow \tilde{E}_2^{0,0}(T_0^{p^2-1}) \longrightarrow \mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes U) \longrightarrow 0$$

and

$$\tilde{E}_2^{s,0}(T_0^{p^2-1}) = \begin{cases} \mathrm{Ext}^1_{G(1)}(T_0^{(2)} \otimes \Sigma^{(t)}U) & \text{for } s = 1 \\ 0 & \text{for } s > 1. \end{cases}$$

Since$\tilde{E}_2^{s,t}$ vanishes for $s > 1$, this spectral sequence collapses from E_2 and reduces to the indicated collection of short exact sequences for the groups $E_2^{2,*}(T_0^{p^2-1})$ in the resolution spectral sequence. □

7.3.8. Corollary. *The Adams–Novikov spectral sequence for $\pi_*(T(0)_{(2)})$ collapses in our range of dimensions, i.e., below dimension $(p^2)|v_2| - 3$.*

Proof. This will follow by a spareness argument if we can show that in this range $E_2^{s,*}$ (for the Adams–Novikov spectral sequence) vanishes for $s < 2p+1$. We can rull out differentials originating in filtrations 0 or 1 by the usual arguments, and by sparesness the each nontrivial differential d_r has $r \equiv 1 \bmod 2p-2$. Thus the shortest possible one is d_{2p-1}, for which the filtration of the target would be too high.

For the vanishing statement the first element in filtration $2p+1$ is $u_1 b_{2,0}^{p-1} h_{2,0}$, and we have

$$\begin{aligned} |u_1| = |b_{2,0}| &= p|v_2| - 2 \\ &= p(2p^2-2) - 2 = 2p^3 - 2p - 2 \\ \text{and} \qquad |h_{2,0}| &= |v_2| - 1 = 2p^2 - 3 \\ \text{so} \qquad |u_1 b_{2,0}^{p-1} h_{2,0}| &= p(2p^3 - 2p - 2) + 2p^2 - 3 \\ &= 2p^4 - 2p - 3 \\ &> p^2|v_2| - 3. \end{aligned}$$

□

Now we will analyze the Cartan–Eilenberg spectral sequence of (7.2.13) for $j = 1$. It has a rich pattern of differentials. This (in slightly different language)

was the subject of Ravenel [**11**]. In order to use this spectral sequence we need to know $\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes B)$ and $\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes U)$. We will derive these from the corresponding Ext groups for $T_0^{(2)}$ given in Lemmas 7.3.1 and 7.3.5 using the the small descent spectral sequence of Theorem 7.1.13.

The former collapses from E_2 since $\mathrm{Ext}_{G(1)}(T_0^{(2)} \otimes B)$ is concentrated in degree 0. The action of r_p on $\mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes B)$ is given by

$$\begin{aligned} r_p\left(\beta'_{i/e_1}\right) &= \beta_{i-1/e_1-1} \\ \text{and} \qquad r_p\left(\beta_{pi/e_1}\right) &= 0. \end{aligned}$$

In order to understand this, the following picture for $p = 3$ may be helpful.

$$\begin{tikzcd}
\beta_1 & \beta_2 & \beta'_3 & \beta_3 & & \\
 & \beta_{2/2} \arrow[ul] & \beta'_{3/2} \arrow[ul] & \beta_{3/2} & \beta_{4/2} \arrow[ul] & \\
 & & \beta'_{3/3} \arrow[ul] & \beta_{3/3} & \beta_{4/3} \arrow[ul] & \beta_{5/3} \arrow[ul]
\end{tikzcd} \tag{7.3.9}$$

Each arrow represents the action of r_p up to unit scalar. Thought of as a graph, this picture has $2p$ components, two of which have maximal size. Each component corresponds to an $A(m)$-summand of our E_2-term, with the caveat that $p\beta'_{p/e_1} = \beta_{p/e_1}$ and $v_1\beta'_{i/e} = \beta'_{i/e-1}$.

In the summand containing β_1, the subset of E_1

$$\left\{\beta_1,\, \beta_{2/2},\, \beta'_{3/3}\right\} \otimes E(h_{1,1}) \otimes P(b_{1,1})$$

reduces on passage to E_2 to simply $\{\beta_1\}$. Similarly

$$\left\{\beta_2,\, \beta'_{3/2}\right\} \otimes E(h_{1,1}) \otimes P(b_{1,1})$$

reduces to

$$\left\{\beta_2,\, \beta'_{3/2}h_{1,1}\right\} \otimes P(b_{1,1}),$$

where

$$\begin{aligned} \beta'_{3/2}h_{1,1} &= \langle h_{1,1},\, h_{1,1},\, \beta_2\rangle \\ \text{and} \qquad h_{1,1}(\beta'_{3/2}h_{1,1}) &= h_{1,1}\langle h_{1,1},\, h_{1,1},\, \beta_2\rangle \\ &= \langle h_{1,1},\, h_{1,1},\, h_{1,1}\rangle\beta_2 \\ &= b_{1,1}\beta_2. \end{aligned}$$

The entire configuration is v_2^p-periodic. This leads to the following.

7.3.10. Proposition. *In dimensions less than $p^2|v_2|$, $\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes B)$ has $\mathbf{Z}/(p)$ basis*

$$P(b_{1,1}) \otimes \left\{ \begin{array}{c} \left\{\beta_{1+pi}, \beta_{p+pi};\ \beta_{p^2/p^2-p+1}, \ldots, \beta_{p^2/p+1}\right\} \\ \bigoplus \\ \left\{\beta'_{2+pi}, \ldots, \beta'_{p+pi};\ \beta_{p+pi/p}, \ldots, \beta_{p+pi/2};\ \beta_{p^2/p^2}, \ldots, \beta_{p^2/p^2-p+2}\right\} \\ \oplus \\ h_{1,1}\left\{\beta'_{p+pi/p-1}, \ldots, \beta'_{p+pi};\ \beta_{p+pi/p}, \ldots, \beta_{2p-1+pi/p};\right. \\ \left. \qquad\qquad \beta_{p^2/p^2}, \ldots, \beta_{p^2+p-2/p^2}\right\}, \end{array} \right.$$

where $0 \le i < p$, subject to the caveat that $v_1\beta_{p/e} = \beta_{p/e-1}$ and $p\beta'_{p/e} = \beta_{p/e}$. In particular $\mathrm{Ext}^0_{G(1)}(T_0^{(1)} \otimes B)$ has basis

$$\left\{\beta'_{1+pi}, \ldots, \beta'_{p+pi};\ \beta_{p+pi/p}, \ldots, \beta_{p+pi/1};\ \beta_{p^2/p^2}, \ldots, \beta_{p^2/p+1}\right\}.$$

The action of r_p on U is trivial, so $E_1 = E_2$ in the small descent spectral sequence for $\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes U)$. In theory there could be a nontrivial differential

$$d_2 : E_2^{s,1} \to E_2^{s+2,0},$$

but this cannot happen since $E_2^{*,1}$ is v_2-torsion while $E_2^{*,0}$ is v_2-torsion free. Hence the spectral sequence collapses and we have

$$\mathrm{Ext}_{G(1)}(T_0^{(1)} \otimes U) = E(h_{1,1}) \otimes P(b_{1,1}) \otimes \mathrm{Ext}_{G(1)}(T_0^{(2)} \otimes U), \tag{7.3.11}$$

where $\mathrm{Ext}_{G(1)}(T_0^{(2)} \otimes U)$ is as in Lemma 7.3.5.

We now have the ingredients needed to study the Cartan–Eilenberg spectral sequence $\left\{\tilde{E}_r^{s,t}(T_0^{(1)})\right\}$ of (7.2.13). We first need to analyze the connecting homomorphism δ in the long exact sequence for $t = 0$. Since the target groups of it are v_2-torsion free, δ is trivial on the v_2-torsion module

$$E(h_{1,1}) \otimes P(b_{1,1}) \otimes \mathrm{Ext}^1_{G(1)}(T_0^{(2)} \otimes U).$$

For its behavior on

$$E(h_{1,1}) \otimes P(b_{1,1}) \otimes \mathrm{Ext}^0_{G(1)}(T_0^{(2)} \otimes U)$$

we have

7.3.12. Lemma. *In the long exact sequence of (7.2.13) for $j = 1$ we have (up to unit scalar)*

$$\begin{aligned} \delta^{2k}(b_{1,1}^k u_i) &= (i+1)h_{1,1}b_{1,1}^k\beta_{i+p/p}, \\ \textit{and} \qquad \delta^{2k+1}(h_{1,1}b_{1,1}^k u_i) &= \binom{i+1}{p-1} b_{1,1}^{k+1}\beta_{i+2/2} \end{aligned}$$

for all $i, k \ge 0$.

This means that $\tilde{E}_2^{*,0}(T_0^{(1)})$ looks like the Ext group one would would have if the picture of (7.3.9) were replaced by

(7.3.13)

$$\begin{tikzcd}
\beta_1 & \beta_2 & \beta_3' & \beta_3 & & \\
 & \beta_{2/2} \arrow[ul] & \beta_{3/2}' \arrow[ul] & \beta_{3/2} & \beta_{4/2} \arrow[ul] & \\
 & & \beta_{3/3}' \arrow[ul] & \beta_{3/3} & \beta_{4/3} \arrow[ul] & \beta_{5/3} \arrow[ul] \\
 & & & u_0 & u_1 \arrow[ul] & u_2.
\end{tikzcd}$$

The graph now has $2p+1$ instead of $2p$ components, three of which are maximal.

Proof of Lemma 7.3.12. It suffices to show that $r_p(u_i)$ is as indicated in the picture above. We have (using Theorem 7.3.4)

$$u_i = u_{1,i} = v_2^i \left(\frac{v_3}{pv_1} - \frac{v_2^{p+1}}{pc_{1,i}v_1^{p+1}} \right)$$

so

$$r_p(u_i) = -(i+p+1)\frac{v_2^{i+p}}{pc_{1,i}v_1^p} = -\frac{i+1}{(p,i)}\beta_{i+p/p}.$$

□

7.3.14. Corollary. *In the Cartan–Eilenberg spectral sequence of (7.2.13), $\tilde{E}_2(T_0^{(1)})$ has $\mathbf{Z}/(p)$-basis*

$$\left\{\beta_{1+pi}, \beta_{p+pi}, \beta_{p+pi/2};\ \beta_{p^2/p^2-p+1}, \dots, \beta_{p^2/p+1}\right\}$$
$$\bigoplus$$
$$P(b_{1,1}) \otimes \left\{ \begin{array}{c} \left\{\beta'_{2+pi}, \dots, \beta'_{p+pi};\ \beta_{p+pi/p}, \dots, \beta_{p+pi/2};\right. \\ \left. u_{pi+p-1};\ \beta_{p^2/p^2}, \dots, \beta_{p^2/p^2-p+2}\right\} \\ \oplus \\ h_{1,1}\left\{\beta'_{p+pi/p-1}, \dots, \beta'_{p+pi};\ u_{pi}, \dots, u_{pi+p-2};\right. \\ \left. \beta_{p^2/p^2}, \dots, \beta_{p^2+p-2/p^2}\right\} \end{array} \right.$$
$$\bigoplus$$
$$E(h_{1,1}) \otimes P(b_{1,1}, b_{2,0}) \otimes \{h_{2,0}u_j,\ b_{2,0}u_j : j \geq 0\}$$
$$\bigoplus$$
$$E(h_{1,1},\ h_{2,0}) \otimes P(b_{1,1}, b_{2,0}) \otimes \{\gamma_2, \gamma_3, \dots\},$$

where $0 \leq i < p$, (omitting unnecessary subscripts)

$$u,\ v,\ \beta, \in \tilde{E}_2^{0,0} \qquad \textit{and} \qquad \gamma \in \tilde{E}_2^{0,1},$$

and the operators $h_{i,j}$, $b_{i,j}$, etc. behave as if they had the following bidegrees.

$$h_{2,0} \in \tilde{E}_2^{0,1}, \qquad h_{1,1} \in \tilde{E}_2^{1,0},$$
$$b_{2,0} \in \tilde{E}_2^{0,2}, \quad \textit{and} \quad b_{1,1} \in \tilde{E}_2^{2,0}.$$

Now we need to study higher differentials.

7.3.15. THEOREM. *The Cartan–Eilenberg spectral sequence of (7.2.13) for $j = 1$ has the following differentials and no others in dimensions less than $(p^3 + p)|v_1|$.*

(i)

$$d_2(h_{2,0}u_i) \quad = \quad b_{1,1}\beta'_{i+2}.$$

(ii)

$$d_3(h_{2,0}^{\varepsilon}b_{2,0}^k u_i) \quad = \quad (k+i+1)h_{1,1}h_{2,0}^{\varepsilon}b_{1,1}b_{2,0}^{k-1}u_i \qquad \textit{for } k > 0 \textit{ and } \varepsilon = 0 \textit{ or } 1.$$

(iii)

$$d_{2k+2}(h_{1,1}h_{2,0}b_{2,0}^k u_{pi'-2-k}) \quad = \quad h_{1,1}b_{1,1}^{k+1}\beta'_{pi'/k+1} \qquad \textit{for } k < p-1.$$

(iv)

$$d_{2k+1}(h_{1,1}b_{2,0}^k u_{pi'-2-k}) \quad = \quad b_{1,1}^{k+1}\beta_{pi'/k+2} \qquad \textit{for } k > 0.$$

(v)

$$d_3(h_{2,0}^{\varepsilon}b_{2,0}^k\gamma_j) = kh_{1,1}h_{2,0}^{\varepsilon}b_{1,1}b_{2,0}^{k-1}\gamma_j.$$

We will prove Theorem 7.3.15 in the next section. For a more explicit description of the resulting Ext group, see Theorem 7.5.1. An illustration of it for $p = 5$ can be found in Figure 7.3.17. There are no Adams–Novikov differentials in this range. In the figure

- Ext^0 and Ext^1 are not shown.
- Short vertical and horizontal lines indicate multiplication by p and v_1.
- Diagonal lines indicate multiplication by $h_{1,1}$, $h_{2,0}$ and the Massey product operations $\underline{5i}$ of 7.4.12.

Now that we have computed $\mathrm{Ext}_{\Gamma(1)}(T_0^{(1)} \otimes E_1^2)$, it is a simple matter to get to $\mathrm{Ext}_{\Gamma(1)}(T_0^{(1)})$ itself. We have the 4-term exact sequence

$$0 \to T_0^{(1)} \to T_0^{(1)} \otimes D_1^0 \to T_0^{(1)} \otimes D_1^1 \to T_0^{(1)} \otimes E_1^2 \to 0 \tag{7.3.16}$$

in which the two middle terms are weak injectives by Lemma 7.1.10 with

$$\begin{aligned}\mathrm{Ext}^0_{\Gamma(1)}(T_0^{(1)} \otimes D_1^i) \quad &\cong \quad \mathbf{Z}_{(p)}\left\{t_1^j \colon 0 \le j < p\right\} \otimes \mathrm{Ext}^0_{\Gamma(1)}(D_1^i) \\ &\cong \quad \mathbf{Z}_{(p)}\left\{t_1^j \colon 0 \le j < p\right\} \otimes \mathrm{Ext}^i_{\Gamma(1)}(BP_*).\end{aligned}$$

We will compute Ext^0 of the middle map of (7.3.16) using the description of the groups given in 7.1.10. Recall that D_1^0 contains all powers of $\lambda_1 = p^{-1}v_1$. Then $\mathrm{Ext}^0_{\Gamma(1)}(T_0^{(1)} \otimes D_1^0)$ is the free $\mathbf{Z}_{(p)}$-module on the set $\{z_j \colon 0 \le j < p\}$ where

$$z_j = \sum_{0 \le k \le j} (-1)^k \binom{j}{k} t_1^k \otimes \lambda_1^{j-k} = t_1^j \otimes 1 + \dots.$$

The image of

$$\begin{aligned} p^t z_j \quad &= \quad \sum_{0 \le k \le j} (-1)^k p^t \binom{j}{k} t_1^k \otimes \lambda_1^{j-k} \\ &= \quad \sum_{0 \le k \le j} (-1)^k \binom{j}{k} t_1^k \otimes p^{t+k-j} v_1^{j-k}\end{aligned}$$

in $\mathrm{Ext}^0_{\Gamma(1)}(T_0^{(1)} \otimes D_1^1)$ is

$$\begin{aligned}\sum_{0\le k\le j}(-1)^k\binom{j}{k}t_1^k\otimes\frac{v_1^{j-k}}{p^{j-k-t}} &= (-1)^{j-1-t}\binom{j}{t+1}t_1^{j-1-t}\otimes\alpha_{t+1}+\dots\\ &= 0 \qquad \text{if } t\ge j.\end{aligned}$$

From this we deduce that

$$\mathrm{Ext}^0(T_0^{(1)}) = \mathbf{Z}_{(p)}\left\{p^j z_j\colon 0\le j<p\right\},$$

and Ext^0 of the third map of (7.3.16) sends

$$t_1^{(1)}\otimes\alpha_1+\dots\mapsto 1\otimes\beta_1.$$

Thus the map

$$\mathrm{Ext}^0(T_0^{(1)}\otimes E_1^2)\to\mathrm{Ext}^2(T_0^{(1)})$$

has a kernel, namely the $\mathbf{Z}_{(p)}$-summand generated by β_1, and for $s>2$,

$$\mathrm{Ext}^s_{\Gamma(1)}(T_0^{(1)})\cong\mathrm{Ext}^{s-2}_{\Gamma(1)}(T_0^{(1)}\otimes E_1^2),$$

which can be read off from Theorem 7.3.15.

4. The proof of Theorem 7.3.15

Recall that our range of dimensions is now $(p^3+p)|v_1|$.

It is easy to see that all of the elements in Corollary 7.3.14 save those involving u_j or $b_{2,0}$ are permanent cycles. Establishing the indicated differentials will ultimately be reduced to computing Ext groups for certain comodules over the Hopf algebra

$$P(1)_* = \mathbf{Z}/(p)[c(t_1),c(t_2)]/(c(t_1^{p^2}),c(t_2)^p)$$

with coproduct inherited from that of $BP_*(BP)$, i.e., with

$$\begin{aligned}\Delta(c(t_1)) &= c(t_1)\otimes 1+1\otimes c(t_1)\\ \text{and}\qquad \Delta(c(t_2)) &= c(t_2)\otimes 1+c(t_1)^p\otimes c(t_1)+1\otimes c(t_2).\end{aligned}$$

It is dual to the subalgebra $P(1)$ of the Steenrod algebra generated by the reduced power operations P^1 and P^p. For a $P(1)_*$-comodule M, we will abbreviate $\mathrm{Ext}_{P(1)_*}(\mathbf{Z}/(p),M)$ by $\mathrm{Ext}_{P(1)_*}(M)$, or, when $M=\mathbf{Z}/(p)$, by simply $\mathrm{Ext}_{P(1)_*}$

In principle one could get at $\mathrm{Ext}_{\Gamma(1)}(T_0^{(1)}\otimes E_1^2)$ in our range of dimensions (i.e., below dimension $p^3|v_1|$) by finding $\mathrm{Ext}_{\Gamma(3)}(T_0^{(1)}\otimes E_1^2)$ and using the Cartan–Eilenberg spectral sequence for the extension

$$G(1,1)\to\Gamma(1)\to\Gamma(3).$$

(Recall that $G(1,1)=A(2)[t_1,t_2]$.)

Consider our 4-term exact sequence

$$0\to BP_*\to D_1^0\to D_1^1\to E_1^2\to 0$$

The two middle terms are weak injective over $\Gamma(1)$ and hence over $\Gamma(3)$. For the last term we have,

$$\mathrm{Ext}^s_{\Gamma(3)}(E_1^2)=\mathrm{Ext}^{s+2}_{\Gamma(3)} \qquad \text{for } s>0.$$

The first generator for $s=1$ is $\frac{v_4t_3}{pv_1}$, which is out of our range. This means that the fourth term is also weak injective over $\Gamma(3)$ in our range.

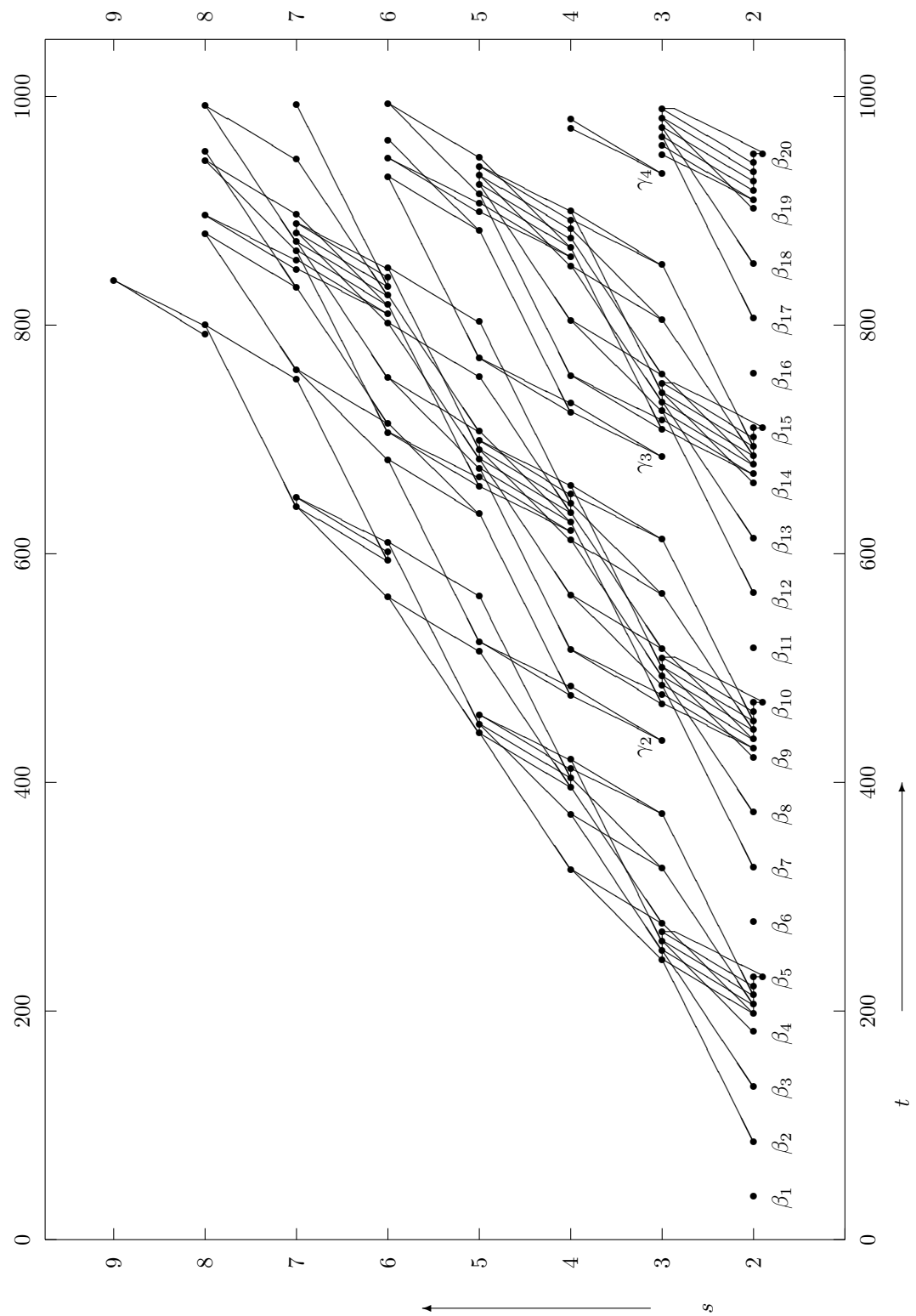

FIGURE 7.3.17. $\mathrm{Ext}(T_0^{(1)})$ for $p = 5$ in dimensions below 998.

For a $\Gamma(1)$-comodule M, we will denote the $G(1,1)$-comodule $\mathrm{Ext}^0_{\Gamma(3)}(M)$ by $\tilde{M}$. Applying $\mathrm{Ext}^0_{\Gamma(3)}(\cdot)$ to our 4-term exact sequence yields a 4-term exact (in our range) sequence of $G(1,1)$-comodules

$$0 \to A(2) \to \tilde{D}^0_1 \to \tilde{D}^1_1 \to \tilde{E}^2_1 \to 0.$$

Let $\tilde{D}^2_1$ be the $A(2)$-submodule of $\tilde{M}^2$ (where M^2 is the chromatic comodule) obtained by adjoining the elements

$$\left\{ \frac{v_2^{-j} v_3^k}{p v_1^i} \colon i,j>0,\ k \geq i+j \right\}$$

to $\tilde{E}^2_1$, so we have a short exact sequence of $G(1,1)$-comodules

$$0 \to \tilde{E}^2_1 \to \tilde{D}^2_1 \to \tilde{E}^3_1 \to 0, \tag{7.4.1}$$

where $\tilde{E}^3_1$ is the $A(2)$-submodule of $\tilde{N}^3$ generated by

$$\left\{ \frac{v_3^{2+e_1+e_2+e_3}}{p v_1^{1+e_1} v_2^{1+e}} \colon e_1, e_2, e_3 \geq 0 \right\}.$$

Its Poincaré series is

$$g(\tilde{E}^3_1) = \frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)}. \tag{7.4.2}$$

7.4.3. Definition. *Let P be the left $G(1,1)$-comodule*

$$\begin{aligned} P &= A(2)\left\{ c(t_1^i t_2^j) \colon 0 \leq i, pj < p^2 \right\} \\ &= A(2)\left\{ t_1^i (t_2 - t_1^{p+1})^j \colon 0 \leq i, pj < p^2 \right\} \subset G(1,1). \end{aligned}$$

*A $G(1,1)$-comodule M is P-**free** (in a range of dimensions) if $P \otimes_{A(2)} M$ is weak injective (in the same range).*

7.4.4. Lemma. *$\tilde{D}^2_1$ and $\tilde{E}^3_1$ are P-free in our range, i.e. below dimension $p^2|v_2|$.*

Proof. For $\tilde{E}^3_1$ we can show this by direct calculation. Up to unit scalar we have

$$r_{(j-1)p^2\Delta_1+(i-1)p\Delta_2}\left(\frac{v_3^k}{p v_1^i v_2^j} \right) = \frac{v_3^{k+2-i-j}}{p v_1 v_2} = \gamma_{k+2-i-j},$$

so these elements form a basis for $\mathrm{Ext}^0_{G(1,1)}(\tilde{E}^3_1)$ and for $\mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{E}^3_1)$. (Here $r_{a,b}$ denotes the Quillen operation dual to $t_1^a t_2^b$.) The Poincaré series for this Ext^0 is

$$\frac{x^{2p^2+p}}{1-x_3}.$$

Meanwhile we have

$$\begin{aligned} g(P \otimes_{A(2)} \tilde{E}^3_1) &= g(P(1)) g(\tilde{E}^3_1) \\ &= \frac{(1-x^{p^2})(1-x_2^p)}{(1-x)(1-x_2)} \frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)} \\ &= \frac{x^{2p^2+p}}{(1-x)(1-x_2)(1-x_3)} \\ &= g_2(t) g(\mathrm{Ext}^0_{G(1,1)}(\tilde{E}^3_1)) \end{aligned}$$

so $\tilde{E}_1^3$ is P-free as claimed.

For $\tilde{D}_1^2$ we will first show that $P \otimes \tilde{D}_1^2$ is weak injective over $G(2)$. Then it will suffice to show that

$$\mathrm{Ext}^0_{G(2)}(P \otimes \tilde{D}_1^2)$$

is weak injective over $G(1)$, i.e. that $\mathrm{Ext}^0_{G(2)}(\tilde{D}_1^2)$ is 2-free.

As a $G(2)$-comodule, P is isomorphic to a direct sum of certain suspensions of $T_1^{(1)}$. We know by Theorem 7.2.6 that $T_1^{(1)} \otimes E_2^2$ is weak injective over $\Gamma(2)$ in our range. The same is true of $T_1^{(1)} \otimes E_1^2$ since it has the same positively graded Ext groups over $\Gamma(2)$. Thus the same goes for $T_1^{(1)} \otimes \tilde{E}_1^2$ and $P \otimes \tilde{E}_1^2$ over $G(2)$. Since we already know that $P \otimes \tilde{E}_1^3$ is weak injective over $G(1,1)$ and hence over $G(2)$, this implies that $P \otimes \tilde{D}_1^2$ is weak injective over $G(2)$.

This means that it suffices to show that $\mathrm{Ext}^0_{G(2)}(\tilde{D}_1^2)$ is 2-free. For this we have the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \mathrm{Ext}^0_{G(2)}(\tilde{E}_1^2) \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] & \mathrm{Ext}^0_{G(2)}(\tilde{D}_1^2) \arrow[r] \arrow[d] & U_0 \arrow[r] \arrow[d] & 0 \\
 & & \mathrm{Ext}^0_{G(2)}(\tilde{E}_1^3) \arrow[r, equal] \arrow[d] & U_1 \arrow[d] & \\
 & & 0 & 0, &
\end{tikzcd}$$

where B is as in Theorem 7.2.11 and the column on the right is as in Lemma 7.3.5. Since B and U_0 are both 2-free in our range, so is $\mathrm{Ext}^0_{G(2)}(\tilde{E}_1^2)$. $\square$

We will show that $\mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{D}_1^2)$ and $\mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{E}_1^3)$ each admit filtrations whose associated bigraded objects are comodules over $P(1)_*$, and analyzing them will lead to a proof of Theorem 7.3.15.

As in the above lemma, $\tilde{E}_1^3$ is easier to handle. We have

$$\mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{E}_1^3) = \mathrm{Ext}^0_{G(1,1)}(\tilde{E}_1^3) = \mathbf{Z}/(p)\Big\{\gamma_k \colon k \geq 2\Big\}. \tag{7.4.5}$$

No filtration is necessary here since it is annihilated by I_2, and we have

$$\mathrm{Ext}_{G(1,1)}(\tilde{E}_1^3) = \mathbf{Z}/(p)\Big\{\gamma_k \colon k \geq 2\Big\} \otimes \mathrm{Ext}_{P(1)_*}(\mathbf{Z}/(p)).$$

The case of $\tilde{D}_1^2$ is more complicated.

7.4.6. Lemma. *Let*

$$M = \mathrm{Ext}^0_{G(1,1)}(P \otimes \tilde{D}_1^2).$$

In our range it is generated by the following set.

$$\begin{aligned}&\left\{\beta_{i/j,k}\colon 1\le j,k\le p,\ i\ge j+k-1\right\}\\ \cup&\left\{\tilde\beta_{i/j,\min(p+1,i+2-j)}\colon 1\le j\le p,\ i\ge p\right\}\\ \cup&\left\{\tilde\beta_{i/p+1}\colon i\ge p+1\right\}\cup\left\{\beta_{p^2/p^2-j}\colon 0\le j<p\right\}\end{aligned}$$

Here

$$\begin{aligned}\tilde\beta_{i/j,k} &= \frac{v_2^i}{p^kv_1^j}(1+x)\left(1-\binom{i}{p}y+\binom{i}{2p}y^2\right)\\ &\qquad \textit{where } x=p^pv_1^{-1-p}v_2 \textit{ and } y=v_1^pv_2^{-1-p}v_3\\ &= \frac{v_2^i}{p^kv_1^j}+\begin{cases}0 & \textit{for } j,k<p+1\\ -\binom{i}{p}\dfrac{v_2^{i-p-1}v_3}{pv_1} & \textit{for } (j,k)=(p+1,1)\\ \dfrac{v_2^{i+1}}{pv_1^{j+p+1}}-\binom{i}{p}\dfrac{v_2^{i-p}v_3}{pv_1^{j+1}}+\binom{i}{2p}\dfrac{v_2^{i-2p-1}v_3^2}{pv_1^{j+1-p}} & \\ & \textit{for } k=p+1.\end{cases}\end{aligned}$$

It has a decreasing filtration defined by

$$||\tilde\beta_{i/j,k}||=i+[i/p]-j-k.$$

The above set is a $\mathbf{Z}/(p)$-basis for the associated bigraded object, which is a $P(1)_*$-comodule. Its structure as a $P(1)$-module is given by

$$\begin{aligned}r_1(\tilde\beta_{i/j,k}) &= j\tilde\beta_{i/j+1,k-1}\\ r_p(\tilde\beta_{i/j,k}) &= \begin{cases}\left(\dfrac{i}{p}\right)\tilde\beta_{i-1/j-1,k-1} & \textit{for } p|i\\ \begin{cases} i\tilde\beta_{i-1/j-1,k} & \textit{for } j>1\\ i\tilde\beta_{i/j+p,1} & \textit{for } (j,k)=(1,p+1)\\ 0 & \textit{for } j=1 \textit{ and } k<p+1.\end{cases} & \textit{for } p\nmid i.\end{cases}\end{aligned}$$

Note that $\tilde\beta_{i/p+1}$ is a unit multiple of the element u_{i-p-1} of Theorem 7.3.4.

Proof. Recall that $g(E_1^2)$ was determined in Lemma 7.2.2, which implies that in our range,

$$\begin{aligned}g(\tilde E_1^2) &= g_2(t)\sum_{i\ge1}\frac{x^{p^i}(1-x_i)}{(1-x^{p^i})(1-x_{i+1})}\\ &= \frac{1}{(1-x)(1-x_2)}\left(\frac{x^p(1-x)}{(1-x^p)(1-x_2)}+\frac{x^{p^2}(1-x_2)}{(1-x^{p^2})(1-x_3)}\right.\\ &\qquad\left.+\frac{x^{p^3}(1-x_3)}{(1-x^{p^3})(1-x_4)}\right)\\ &= \frac{x^p}{(1-x^p)(1-x_2)^2}+\frac{x^{p^2}}{(1-x)(1-x^{p^2})(1-x_3)}+\frac{x^{p^3}}{1-x},\end{aligned}$$

so we have

$$\begin{aligned}
g(M) &= \frac{g(P(1))}{g_2(t)}\left(g(\tilde{E}_1^2)+g(\tilde{E}_1^3)\right)\\
&= (1-x^{p^2})(1-x_2^p)\left(\frac{x^p}{(1-x^p)(1-x_2)^2}+\frac{x^{p^2}}{(1-x)(1-x^{p^2})(1-x_3)}\right.\\
&\qquad\qquad \left.+\frac{x^{2p^2+p}}{(1-x^{p^2})(1-x_2^p)(1-x_3)}+\frac{x^{p^3}}{1-x}\right)\\
&= \frac{x^p(1-x^{p^2})(1-x_2^p)}{(1-x^p)(1-x_2)^2}+\frac{x^{p^2}}{1-x}+\frac{x^{p^3}}{1-x}\\
&= \sum_{1\le j\le p} x^{jp}\frac{1-x_2^p}{(1-x_2)^2}+\sum_{1\le j\le p}\frac{x^{p^2+p-j}}{1-x_2}+\frac{x_2^p}{1-x_2}+\frac{x^{p^3}}{1-x}.
\end{aligned}$$

The four indicated subsets correspond to these four terms.

In order to show that we have the right elements, we need to show that for each indicated generator z, the invariant element

$$\tilde{z}=\sum_{a,b\ge 0}\bar{t}_1^a\bar{t}_2^b\otimes r_{a,b}(z)\in G(1,1)\otimes M$$

actually lies in $P\otimes M$. For dimensional reasons we need only consider the cases where $a<p^3$ and $b<p^2$. Then if $a\ge p^2$ or $b\ge p$, $r_{a,b}(z)$ vanishes if both $r_{p^2}(z)$ and $r_{0,p}(z)$ do. But for each of our generators, the correcting terms (i.e. $\tilde{\beta}_{i/j,k}-\beta_{i/j,k}$) are chosen to insure that r_{p^2} and $r_{0,p}$ act trivially.

Our putative filtration is similar to the I-adic one, which is given by

$$||\tilde{\beta}_{i/j,k}||=i-j-k.$$

Note that we are not assigning a filtration to each chromatic monomial, but to each of the generators listed in Lemma 7.4.6.

Roughly speaking, it suffices to show that an operation $r_{a,b}$ raises this filtration by the amount by which it lowers the value of $[i/p]$. Since r_{p^2} and $r_{0,p}$ act trivially, it suffices to consider the action of r_1 and r_p. The actions of r_1 on v_2 and v_3, and the action of r_p on v_3 raise I-adic filtration by at least $p-1$ and can therefore be

ignored. It follows that modulo such terms, we have

$$\begin{aligned}
r_1\left(\tilde\beta_{i/j,k}\right) &= -j\frac{v_2^i}{p^{k-1}v_1^{j+1}} = j\tilde\beta_{i/j+1,k-1} \\
&= \begin{cases} 0 & \text{for } p|j \\ -j\tilde\beta_{i/j+1,k-1} & \text{for } p\nmid j \end{cases} \\
r_p\left(\tilde\beta_{i/j,k}\right) &= i\frac{v_2^{i-1}}{p^k v_1^{j-1}} + \binom{-j}{p}\frac{v_2^i}{p^{k-p}v_1^{j+p}} \\
&\qquad + \begin{cases} 0 & \text{for } k<p+1 \\ (i+1)\dfrac{v_2^i}{pv_1^{j+p}} - i\dbinom{i}{p}\dfrac{v_2^{i-p-1}v_3}{pv_1^j} & \text{for } k=p+1 \end{cases} \\
&= i\left(\frac{v_2^{i-1}}{p^k v_1^{j-1}} + \left\{\begin{array}{ll} 0 & \text{for } k<p+1 \\ \dfrac{v_2^i}{pv_1^{j+p}} - \dbinom{i}{p}\dfrac{v_2^{i-p-1}v_3}{pv_1^j} & \text{for } k=p+1 \end{array}\right\}\right) \\
&= \begin{cases} \dbinom{i}{p}\tilde\beta_{i-1/j-1,k-1} & \text{for } p|i \\ \begin{cases} i\tilde\beta_{i-1/j-1,k} & \text{for } j>1 \\ i\tilde\beta_{i/j+p,1} & \text{for } (j,k)=(1,p+1) \\ 0 & \text{for } j=1 \text{ and } k<p+1. \end{cases} & \text{for } p\nmid i. \end{cases}
\end{aligned}$$

Note that r_1 never changes the value of i or the I-adic filtration, while r_p raises the latter by 1 precisely when lowers the value of $[i/p]$ by 1. It follows that the indicated filtration is preserved by r_1 and r_p.

The associated bigraded is killed by I_2 because multiplication by it always raises filtration. □

In what follows we will ignore the elements

$$\left\{\beta_{p^2/p^2-j} : 0 \le j < p\right\}.$$

They are clearly permanent cycles and will thus have no bearing on the proof of Theorem 7.3.15. *From now on, M will denote the quotient of M (as defined previously) by the subspace spanned by these elements.*

To explore the structure of E_0M further, we need to introduce some auxiliary $P(1)_*$-comodules. For $0 \le i < p$ let

$$C_i = \mathbf{Z}/(p)\left\{t_1^{pj} : 0 \le j \le i\right\},$$

and let $C_{-1} = 0$. Let

$$H = P(1)_* \Box_{P(0)_*} \mathbf{Z}/(p).$$

7.4.7. Lemma. (i) *For $i \ge 0$, let $c(i) = p\left[\frac{i+p}{p}\right] - i - 1$. There is a 4-term exact sequence*

$$0 \longrightarrow \Sigma^e C_{c(i)-1} \longrightarrow \Sigma^e H \longrightarrow E_0^{||u_i||}M \longrightarrow \Sigma^{|u_i|}C_{c(i)} \longrightarrow 0,$$

where

$$e = \begin{cases} |\beta_{i+2}| & \textit{for } i \equiv -1 \bmod (p) \\ |\beta_{i+1}| = |u_i| - |b_{1,1}| & \textit{otherwise.} \end{cases}$$

When i is congruent to -1 modulo p, then $c(i) = 0$ so the first term is trivial. The sequence splits in that case, i.e. for $j > 0$

$$E_0^{(p+1)j-1}M = \Sigma^{|\beta_{pj+1}|}H \oplus \Sigma^{|u_{pj-1}|}\mathbf{Z}/(p).$$

The value of $||u_i||$ is never congruent to -2 modulo $p+1$, and for $j > 0$

$$E_0^{(p+1)j-2}M = \Sigma^{|\beta_{pj}|}H.$$

(ii) *For i not congruent to -1 modulo p, there is are maps of 4-term sequences*

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^e C_{c(i)-1} \arrow[r] \arrow[d, equal] & \Sigma^e H \arrow[r] & E_0^{||u_i||}M \arrow[r] & \Sigma^{|u_i|}C_{c(i)} \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^e C_{c(i)-1} \arrow[r] & \Sigma^e C_{p-1} \arrow[u] \arrow[r, "r_{pc(i)}"] \arrow[d, equal] & \Sigma^{e+|t_1^{pc(i)}|}C_{p-1} \arrow[u] \arrow[r] & \Sigma^{|u_i|}C_{c(i)-1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^e \mathbf{Z}/(p) \arrow[r] \arrow[u] & \Sigma^e C_{p-1} \arrow[r, "r_p"] & \Sigma^{e+|t_1^p|}C_{p-1} \arrow[u, "r_{p(c(i)-1)}"] \arrow[r] & \Sigma^{|u_i|}\mathbf{Z}/(p) \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

in which each vertical map is a monomorphism. The bottom sequence is a Yoneda representative for the class $b_{1,1} \in \mathrm{Ext}^2_{P(1)_*}$.

Proof. (i) Let $r_{0,1} = r_p r_1 - r_1 r_p \in P(1)$. It generates a truncated polynomial algebra of height p which we denote by $T(r_{0,1})$. It follows from 7.4.6 that

$$r_{0,1}(\tilde{\beta}_{i/j,k}) = i\tilde{\beta}_{i-1/j,k-1}.$$

The element on the right is nonzero modulo higher filtration when i/p^{k-1} is not a $p$local integer, i.e. when k is not too small. Thus up to unit scalar we get

$$\begin{aligned}
r_{0,1}^{p-1}(\tilde{\beta}_{i+p-1/j,p+1}) &= \binom{i+p-1}{p-1}\tilde{\beta}_{i/j,2} \\
&= \begin{cases} \tilde{\beta}_{i/j} & \text{for } p \nmid i \\ \tilde{\beta}_{i/j,2} & \text{for } p|i \end{cases} \\
r_{0,1}^{p-1}(\tilde{\beta}_{pi+p-1/j,p}) &= \tilde{\beta}_{pi/j}
\end{aligned}$$

This means that each element in the first two subsets in Lemma 7.4.6 is part of free module over $T(r_{0,1})$.

In $P(1)$, r_p commmutes with $r_{0,1}$, and H is free as a module over $T(r_p, r_{0,1})$ on its top element x. It is characterized as a cyclic $P(1)$-module by $r_1(x) = 0$ and $r_{p(p-1),p}(x) \neq 0$.

In $E_0^{(p+1)j-2}M$, the top element is $\tilde{\beta}_{pj+2p-2/p,p+1}$. It is killed by r_1, and up to unit scalar,

$$r_{p(p-1),p}(\tilde{\beta}_{pj+2p-2/p,p+1}) = \beta_{pj},$$

so $E_0^{(p+1)j-2}M$ has the indicated structure.

In $E_0^{(p+1)j-2}M$ for $j > 0$, u_{pj-1} is killed by both r_1 and r_p and generates a $P(1)$-summand. It is not present for $j = 0$. For $j \geq 0$, the class $\tilde{\beta}_{pj+2p-1/p,p+1}$ generates a summand isomorphic to a suspension of H as claimed.

In $E_0^{(p+1)j}M$ consider the sub-$P(1)$-module generated by the element $x = \beta_{pj+2p-1/p,p}$. Up to unit scalar we have

$$\begin{aligned} r_1(x) &= 0 \\ r_{0,p-1}(x) &= \beta_{pj+p/p} \\ r_{(p-1)p,p-2}(x) &= \beta_{pj+2} \\ r_{p,p-1}(x) &= 0. \end{aligned}$$

Thus there is a homomorphism from the indicated suspension of H to $E_0^{(p+1)j}M$ sending the top element to x with kernel isomorphic to C_{p-2}. Its cokernel is a copy of C_{p-1} in which top element is the image of $\tilde{\beta}_{pj+2p-1/p-1,p+1}$ and the bottom element is the image of $\tilde{\beta}_{pj+p+1/p+1}$.

The remaining cases, $E_0^{(p+1)j+k}M$ for $1 \le k \le p-2$, are similar. The top element in the image of H is $\tilde{\beta}_{pj+k+2p-1/p,p+1}$, and the top and bottom elements in the cokernel are the images of $\tilde{\beta}_{pj+k+2p-1/p-1,p+1}$ and $\tilde{\beta}_{pj+p+1+k/p+1}$ respectively.

(ii) The existence of the map of follows by inspection. Consider the case $p = 3$ and $i = 0$. Then the diagram is

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{12}C_1 \arrow[r] & \Sigma^{12}H \arrow[r] & E_0^0M \arrow[r] & \Sigma^{48}C \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^{12}C_0 \arrow[r] \arrow[u] & \Sigma^{12}C_2 \arrow[r] \arrow[u] & \Sigma^{24}C_2 \arrow[r] \arrow[u] & \Sigma^{48}C_0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The following diagram may be helpful in understanding the vertical maps.

$$\begin{tikzcd}
* & * \arrow[l] & \beta_{3/3} \arrow[l] & \tilde{\beta}_{4/4} \arrow[l] \\
 & \beta_2 \arrow[u] & \beta_{3/2,2} \arrow[l] \arrow[u] & \beta_{4/3,2} \arrow[l] \arrow[u] \\
 & & \beta_{3/1,3} \arrow[u] & \beta_{4/2,3} \arrow[l] \arrow[u] & \beta_{5/3,3} \arrow[l] \\
 & & & \tilde{\beta}_{4/1,4} \arrow[u] \arrow[uu, bend right] & \tilde{\beta}_{5/2,4} \arrow[l] \arrow[u]
\end{tikzcd}$$

Here the short vertical arrows represent the action of r_1, and the longer arrows represent r_3. The named elements form a basis of E_0^0M and the asterisks are elements in $\Sigma^{12}H$ which map trivially to E_0^0M. H consists of all elements in the first three rows except $\tilde{\beta}_{4/4}$. $\square$

We will use Lemma 7.4.7 to determine $\mathrm{Ext}_{P(1)_*}(E_0M)$ in the following way. We regard the 4-term sequence of 7.4.7(i) as a resolution of 0, apply the functor $\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)} \otimes \cdot)$, and get a 4-column spectral sequence converging to 0. It turns out to have a d_3 that is determined by 7.4.7(ii), and this information will determine our Ext group.

In order to proceed further we need to know

$$\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)} \otimes H) \qquad \text{and} \qquad \mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)} \otimes C_i),$$

where

$$\overline{T}_0^h = T_0^h \otimes_{BP_*} \mathbf{Z}/(p) \qquad \text{with} \qquad \overline{T}_0^{(i)} = \overline{T}_0^{p^i-1}.$$

This is a comodule over $P(1)_*$.

We will abbreviate $\mathrm{Ext}^s_{P(1)_*}(\overline{T}_0^{(1)} \otimes N)$ by $F^s(N)$.

Since $\overline{T}_0^{(1)} \otimes H = P(1)_*$, we have

$$F^{s,t}(H) = \begin{cases} \mathbf{Z}/(p) & \text{for } (s,t) = (0,0) \\ 0 & \text{otherwise.} \end{cases} \tag{7.4.8}$$

Next we compute $F^*(\mathbf{Z}/(p))$. There is a Hopf algebra extension

$$\mathbf{Z}/(p)[t_1]/(t_1^{p^2}) \to P(1)_* \to \mathbf{Z}/(p)[t_2]/(t_2^p) \tag{7.4.9}$$

and we have

$$\mathrm{Ext}_{\mathbf{Z}/(p)[t_2]/(t_2^p)} = E(h_{2,0}) \otimes P(b_{2,0})$$

where

$$h_{2,0} \in \mathrm{Ext}^{1,2(p^2-1)} \qquad \text{and} \qquad b_{2,0} \in \mathrm{Ext}^{2,2p(p^2-1)}.$$

In particular $\overline{T}_0^{(2)} = \mathbf{Z}/(p)[t_1]/(t_1^{p^2})$, so

$$\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(2)}) = \mathrm{Ext}_{\mathbf{Z}/(p)[t_2]/(t_2^p)} = E(h_{2,0}) \otimes P(b_{2,0}) \tag{7.4.10}$$

where $h_{2,0} \in \mathrm{Ext}^{1,2p^2-2}$ and $b_{2,0} \in \mathrm{Ext}^{2,2p^3-2p}$.

To compute $F^*(\mathbf{Z}/(p))$, we will use the long exact sequence

$$0 \longrightarrow \overline{T}_0^{(1)} \longrightarrow \overline{T}_0^{(2)} \xrightarrow{r_p} \Sigma^{p|v_1|}\overline{T}_0^{(2)} \xrightarrow{r_{p^2-p}} \Sigma^{p^2|v_1|}\overline{T}_0^{(2)} \xrightarrow{r_p} \cdots. \tag{7.4.11}$$

This leads to a resolution spectral sequence converging to $\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)})$ with

$$E_1^{s,t} = E(h_{1,1}, h_{2,0}) \otimes P(b_{1,1}, b_{2,0}),$$

where

$$\begin{aligned} h_{1,1} &\in E_1^{1,0}, & h_{2,0} &\in E_1^{0,1}, \\ b_{1,1} &\in E_1^{2,0}, \quad \text{and} & b_{2,0} &\in E_1^{0,2}. \end{aligned}$$

Alternatively, one could use the same resolution to show that

$$\mathrm{Ext}_{\mathbf{Z}/(p)[t_1]/(t_1^{p^2})}(\overline{T}_0^{(1)}) = E(h_{1,1}) \otimes P(b_{1,1})$$

and then use the Cartan–Eilenberg spectral sequence for (7.4.9). It is isomorphic to the resolution spectral sequence above.

Before describing this spectral sequence we need some notation for certain Massey products.

7.4.12. Definition. *Let i be an integer with $0 < i < p$. Then $\underline{i}x$ denotes the Massey product (when it is defined)*

$$\langle h_{1,0}, \ldots, h_{1,0}, x\rangle$$

with i factors $h_{1,0}$, and $\underline{pi}x$ denotes the Massey product (when it is defined)

$$\langle h_{1,1}, \ldots, h_{1,1}, x\rangle$$

with i factors $h_{1,1}$.

Under suitable hypotheses we have $b_{1,0}x \in \underline{p-i} \cdot \underline{i}x$ and $b_{1,1}x \in \underline{p(p-i)} \cdot \underline{pi}x$.

7.4.13. THEOREM. *The differentials in the above spectral sequence are as follows*:

(a) $d_3(h_{2,0}^{\varepsilon} b_{2,0}^i) = i h_{1,1} h_{2,0}^{\varepsilon} b_{11} b_{2,0}^{i-1}$;

(b) $d_{2p-1}(h_{2,0}^{\varepsilon} h_{1,1} b_{2,0}^{pi+p-1}) = h_{2,0}^{\varepsilon} b_{1,1}^p b_{2,0}^{pi}$,

where $\varepsilon = 0$ or 1. These differentials commute with multiplication by $h_{2,0}$, $h_{1,1}$, and $b_{1,1}$, and all other differentials are trivial. Consequently $\mathrm{Ext}_{P(1)_*}(\overline{T}_0^{(1)})$ *is a free module over*

$$P(b_{2,0}^p) \otimes E(h_{2,0})$$

on the set

$$\{b_{1,1}^i \colon 0 \le i \le p-1\} \cup \{h_{1,1} b_{2,0}^i \colon 0 \le i \le p-2\}.$$

There are Massey product relations

$$h_{1,1} b_{2,0}^i \in \underline{p(i+1)} b_{1,1}^i \qquad \textit{and} \qquad b_{1,1}^{i+1} \in \underline{p(p-i-1)} h_{1,1} b_{2,0}^i$$

for $0 \le i \le p-2$. We will denote this object by R.

PROOF. In the Cartan–Eilenberg spectral sequence for (7.4.9) one has

$$d_2(h_{2,0}) = \pm h_{1,0} h_{1,1}$$

since the reduced diagonal on t_2 is $t_1 \otimes t_1^p$. Now we use the theory of algebraic Steenrod operations of A1.5 and the Kudo transgression theorem A1.5.7. Up to sign we have $\beta P^0(h_{2,0}) = b_{2,0}$, so

$$d_3(b_{2,0}) = \beta P^0(h_{1,0} h_{1,1}) = \beta(h_{1,1} h_{1,2}) = h_{1,1} b_{1,1}$$

as claimed in (a). Then A1.5.7 implies that

$$d_{2p-1}(h_{1,1} b_{1,1} b_{2,0}^{p-1}) = \beta(h_{1,2} b_{1,1}^p) = b_{1,1}^{p+1},$$

so $d_{2p-1}(h_{1,1} b_{2,0}^{p-1}) = b_{1,1}^p$ as claimed in (b). The stated Massey product relations follow easily from (a) and (b). □

To compute $F^*(C_i)$ for $0 < i < p$, we use the spectral sequence associated with the skeletal filtration of C_i. In it we have

$$E_1^{j,k} = F^k(\Sigma^{pj|t_1|}\mathbf{Z}/(p)) \quad \text{for } 0 \le j \le i \qquad \text{and} \qquad d_r : E_r^{j,k} \to E_r^{j-r,k+1}.$$

We will denote the generator of $E_1^{j,0}$ by x_{pj} and write x_0 as 1. Since

$$\overline{T}_0^{(1)} \otimes C_{p-1} = \overline{T}_0^{(2)},$$

its Ext group is given by (7.4.10). There is a pattern of differentials implied by the Massey product relations of Theorem 7.4.13.

7.4.14. PROPOSITION. *In the skeletal filtration spectral sequence for*

$$F^*(C_i) = \mathrm{Ext}^*_{P(1)_*}(\overline{T}_0^{(1)} \otimes C_i)$$

we have the following differentials and no others.

$$\begin{aligned}
&d_{k+1}(x_{pj}h_{2,0}^{\varepsilon}b_{1,1}^{k}) \\
&\qquad = \quad x_{p(j-k-1)}h_{2,0}^{\varepsilon}h_{1,1}b_{2,0}^{k} \quad && \textit{for } 0 \le k < j \le i \\
&d_{p-1-k}(x_{pj}h_{2,0}^{\varepsilon}h_{1,1}b_{2,0}^{k}) \\
&\qquad = \quad x_{p(j+k+1-p)}h_{2,0}^{\varepsilon}b_{1,1}^{k+1} \quad && \textit{for } p-1-j \le k \le p-1-j+i \\
& && \textit{and } 0 \le j \le i,
\end{aligned}$$

where $\varepsilon = 0$ or 1.

The following diagram illustrates this for $p = 5$.

(7.4.15)

$$\begin{tikzcd}
1 & x_5 \arrow[l, "d_1"'] & x_{10} \arrow[l, "d_1"'] & x_{15} \arrow[l, "d_1"'] & x_{20} \arrow[l, "d_1"'] \\
h_{1,1} & x_5h_{1,1} & x_{10}h_{1,1} & x_{15}h_{1,1} & x_{20}h_{1,1} \arrow[lllll, "d_4"'] \\
b_{1,1} & x_5b_{1,1} & x_{10}b_{1,1} & x_{15}b_{1,1} & x_{20}b_{1,1} \\
h_{1,1}b_{2,0} & x_5h_{1,1}b_{2,0} & x_{10}h_{1,1}b_{2,0} & x_{15}h_{1,1}b_{2,0} \arrow[lll, "d_2"'] & x_{20}h_{1,1}b_{2,0} \arrow[lll, "d_2"'] \\
b_{1,1}^2 & x_5b_{1,1}^2 & x_{10}b_{1,1}^2 & x_{15}b_{1,1}^2 \arrow[lll, "d_3"'] & x_{20}b_{1,1}^2 \arrow[lll, "d_3"'] \\
h_{1,1}b_{2,0}^2 & x_5h_{1,1}b_{2,0}^2 & x_{10}h_{1,1}b_{2,0}^2 & x_{15}h_{1,1}b_{2,0}^2 \arrow[lll, "d_3"'] & x_{20}h_{1,1}b_{2,0}^2 \arrow[lll, "d_3"'] \\
b_{1,1}^3 & x_5b_{1,1}^3 & x_{10}b_{1,1}^3 \arrow[lll, "d_2"'] & x_{15}b_{1,1}^3 \arrow[lll, "d_2"'] & x_{20}b_{1,1}^3 \arrow[lll, "d_2"'] \\
h_{1,1}b_{2,0}^3 & x_5h_{1,1}b_{2,0}^3 & x_{10}h_{1,1}b_{2,0}^3 & x_{15}h_{1,1}b_{2,0}^3 & x_{20}h_{1,1}b_{2,0}^3 \arrow[lllll, "d_4"'] \\
b_{1,1}^4 & x_5b_{1,1}^4 \arrow[l, "d_1"'] & x_{10}b_{1,1}^4 \arrow[l, "d_1"'] & x_{15}b_{1,1}^4 \arrow[l, "d_1"'] & x_{20}b_{1,1}^4 \arrow[l, "d_1"']
\end{tikzcd}$$

Each row and column corresponds to a different value of k and j respectively. The skeletal filtration spectral sequence for C_{p-1} is obtained by tensoring the pattern indicated above with $E(h_{2,0}) \otimes P(b_{2,0}^p)$. Note that the only element in the jth column not on either end of a differential is $x_{pj}b_{1,1}^j$, which represents $b_{2,0}^j$.

The skeletal filtration spectral sequence for C_i is obtained from that for C_{p-1} by looking only at the first $i+1$ columns.

Now we consider the resolution spectral sequence converging to 0 associated the 4-term exact sequence of Lemma 7.4.7(i). In it we have

$$\begin{aligned} E_1^{0,s} &= F^s(\Sigma^e C_{c(i)-1}) \\ E_1^{1,s} &= F^s(\Sigma^e H) \\ E_1^{2,s} &= F^s(\Sigma^e E_0^{b(i)} M) \\ E_1^{3,s} &= F^s(\Sigma^{|u_i|} C_{c(i)}); \end{aligned}$$

each of these groups is graded by dimension. The last differential here is

$$d_3 : E_3^{0,s} \to E_3^{3,s-2}.$$

It is an isomorphism and hence has an inverse since the spectral sequence converges to 0. The bottom dimension is $e = |\beta_{b(i)+1}|$. By (7.4.8) we have

$$E_1^{1,s} = \begin{cases} \mathbf{Z}/(p) & \text{concentrated in dimension } e \text{ for } s = 0 \\ 0 & \text{for } s > 0. \end{cases}$$

The bottom class here is killed by a d_1 coming from the one in $E_1^{1,s}$. Above the bottom dimension, the only differentials in addition to the d_3 above are

$$d_2 : E_2^{0,s} \to E_2^{2,s-1} \qquad \text{and} \qquad d_1 : E_1^{2,s} \to E_1^{3,s}.$$

It follows that above dimension e there is a short exact sequence

$$0 \longrightarrow \operatorname{coker} \tilde{d}_3^{-1} \xrightarrow{d_2} F^s(\Sigma^e E_0^{b(i)} M) \xrightarrow{d_1} \ker \tilde{d}_3^{-1} \longrightarrow 0, \tag{7.4.16}$$

where $\tilde{d}_3^{-1}$ denotes the composite

$$F^s(\Sigma^{b_{1,1}} C_{c(i)}) \longrightarrow E_3^{3,s} \xrightarrow[\cong]{d_3^{-1}} E_3^{0,s+2} \longrightarrow F^{s+2}(C_{c(i)-1})$$

Here $\operatorname{coker} \tilde{d}_3^{-1}$ is a quotient of $F^{s+1}(\Sigma^e C_{c(i)-1})$ and $\ker \tilde{d}_3^{-1}$ is a subgroup of $F^s(\Sigma^{|u_i|} C_{c(i)})$. Note that $|u_i| - e = |b_{1,1}|$ in all cases. Lemma 7.4.7(ii) implies that $\tilde{d}_3^{-1}$, roughly speaking, multiplication by $b_{1,1}$.

We illustrate this for the case $p = 5$ and $i = 0$. The 4-term sequence is

$$0 \longrightarrow \Sigma^{40} C_3 \longrightarrow \Sigma^{40} H \longrightarrow E_0^0 M \longrightarrow \Sigma^{240} C_4 \longrightarrow 0.$$

Referring to (7.4.15) we see that the product of $b_{1,1}$ with any element in $F^*(C_4)$ (except $x_{20} b_{1,1}^4$, which is out of our range) is killed by a differential originating in the last column, which means that it is alive in $F^*(C_3)$. Thus $\tilde{d}_3^{-1}$ is a monomorphism in our range, so its kernel is trivial and the d_2 in (7.4.16) is an isomorphism. The cokernel of $\tilde{d}_3^{-1}$ is the quotient of

$$\Sigma^{40} E(h_{2,0}) \otimes \left\{ 1,\, x_5 h_{1,1},\, x_5 b_{1,1},\, x_{10} h_{1,1} b_{2,0},\, x_{10} b_{1,1}^2,\, x_5 h_{1,1} b_{2,0}^2,\, x_{15} b_{1,1}^3,\, h_{1,1} b_{2,0}^3 \right\}$$

obtained by killing the bottom class. The classes $h_{2,0}$ and $x_{15} h_{11}$ map to β_2 and $\beta_{5/5}$. By inspection this leads to the desired value of $F^*(E_0^0 M)$.

For $i = 1$, the 4-term sequence is

$$0 \longrightarrow \Sigma^{88} C_2 \longrightarrow \Sigma^{88} H \longrightarrow E_0^1 M \longrightarrow \Sigma^{288} C_3 \longrightarrow 0.$$

Again $\tilde{d}_3^{-1}$ is a monomorphism in our range. The cokernel of $\tilde{d}_3^{-1}$ is the quotient of

$$\Sigma^{88}E(h_{2,0}) \otimes \left\{1,\, x_{10}h_{1,1},\, x_5b_{1,1},\, x_5h_{1,1}b_{2,0},\, x_{10}b_{1,1}^2,\, h_{1,1}b_{2,0}^2\right\}$$

obtained by killing the bottom class. The classes $h_{2,0}$ and $x_{10}h_{11}$ map to β_3 and $\beta_{5/4}$. By inspection this leads to the desired value of $F^*(E_0^1M)$.

In order to see that this works in general it is useful to compare the comodules $\overline{T}_0^{(1)} \otimes E_0^iM$ with certain others with known Ext groups. Let

$$0 \longrightarrow \overline{T}_0^{(1)} \longrightarrow F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \cdots \tag{7.4.17}$$

be a minimal free resolution of $\overline{T}_0^{(1)}$. Its structure is as follows.

7.4.18. Proposition. *The free $P(1)_*$-comodule F_i above is*

$$F_i = \begin{cases} P(1)_* & \text{for } i = 0 \\ \Sigma^{i'|b_{1,1}|}P(1)_* \oplus \Sigma^{(i'-1)|b_{2,0}|+|t_1^pt_2|}P(1)_* & \text{for } i = 2i' \text{ and } 0 < i' < p \\ \Sigma^{i'|b_{2,0}|+|t_1^p|}P(1)_* \oplus \Sigma^{i'|b_{1,1}|+|t_2|}P(1)_* & \text{for } i = 2i'+1 \text{ and } \\ & \qquad 0 < i' < p-1 \\ \Sigma^{|t_2|+(p-1)|b_{1,1}|}P(1)_* & \text{for } i = 2p-1 \\ \Sigma^{p|b_{2,0}|}F_{i-2p} & \text{for } i \geq 2p. \end{cases}$$

In $P(1)$ let $x = P^1$, $y = P^p$, $z = yx - xy$. Then there are relations

$$x^p = 0,\ [x,z] = 0,\ [y,z] = 0, \quad \text{and } y^p = xz^{p-1},$$

(corresponding to the four generators of $\mathrm{Ext}^2_{P(1)_*}$) which imply that $z^p = 0$. Then d_i is represented (via left multiplication) by a matrix M_i over $P(1)$ as follows.

$$M_i = \begin{cases} \begin{bmatrix} y \\ z \end{bmatrix} & \text{for } i = 0 \\ \begin{bmatrix} y^{p-i'} & -xz^{p-2} \\ z & -y^{i'} \end{bmatrix} & \text{for } i = 2i'-1 \text{ with } 0 < i' < p \\ \begin{bmatrix} y^{i'+1} & -(xy+(i'+1)z)z^{p-2} \\ z & -y^{p-i'} \end{bmatrix} & \text{for } i = 2i' \text{ with } 0 < i' < p-1 \\ \begin{bmatrix} z & -y \end{bmatrix} & \text{for } i = 2p-2 \\ \begin{bmatrix} y^{p-1}z^{p-1} \end{bmatrix} & \text{for } i = 2p-1 \\ M_{i-2p} & \text{for } i \geq 2p. \end{cases}$$

Let K_i denote the kernel of d_i, and consider the following diagram with exact rows and columns for $0 < i < p$.

(7.4.19)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & X_i \arrow[r] \arrow[d] & \Sigma^{a_i} P(1)_* \arrow[r] \arrow[d] & K_{2i} \arrow[r] \arrow[d, equal] & Y_i \arrow[r] & 0 \\
0 \arrow[r] & K_{2i-1} \arrow[r, "d_{2i-2}"] \arrow[d] & F_{2i-1} \arrow[r, "d_{2i-1}"] \arrow[d] & K_{2i} \arrow[r] & 0 & \\
 & \Sigma^{b_i} P(1)_* \arrow[r, equal] \arrow[d] & \Sigma^{b_i} P(1)_* \arrow[d] & & & \\
 & Y_i \arrow[d] & 0 & & & \\
 & 0 & & & &
\end{tikzcd}$$

where $a_i = (i-1)|b_{2,0}| + |t_1^p|$, $b_i = (i-1)|b_{1,1}| + |t_2|$, and the middle column is split. We will see that the top row (up to reindexing and suspension) is the 4-term sequence of Lemma 7.4.7(i) tensored with $\overline{T}_0^{(1)}$. For this we need to identify X_i and Y_i.

X_i is the kernel of the map represented by the first column of M_{2i-1}, namely

$$\begin{bmatrix} y^{p-i} \\ z \end{bmatrix}.$$

This kernel is the ideal generated by $y^i z^{p-1}$, which is

$$\Sigma^{a_i} \overline{T}_0^{p(p-i)-1} = \Sigma^{i|b_{2,0}|-|b_{1,1}|} \overline{T}_0^{(1)} \otimes C_{p-1-i}.$$

Y_i is the cokernel of the map to $\Sigma^{b_i} P(1)_*$ represented by the bottom row of M_{2i-2}, namely

$$\begin{cases} \begin{bmatrix} z \end{bmatrix} & \text{for } i = 1 \\ \begin{bmatrix} z & -y^{p+1-i} \end{bmatrix} & \text{for } 1 < i < p. \end{cases}$$

This cokernel is

$$\Sigma^{b_i + |y^{i-1} z^{p-1}|} \overline{T}_0^{(p+1-i)p-1} = \Sigma^{i|b_{2,0}|} \overline{T}_0^{(1)} \otimes C_{p-i}.$$

This enables us to prove the following analog of Lemma 7.4.7.

7.4.20. Lemma. *For $0 < i < p$ there are maps of 4-term exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{a_i}\overline{T}_0^k \arrow[r] \arrow[d, equal] & \Sigma^{a_i}P(1)_* \arrow[r] & K_{2i} \arrow[r] & \Sigma^{a_i+|b_{1,1}|}\overline{T}_0^{k+p} \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^{a_i}\overline{T}_0^k \arrow[r] & \Sigma^{a_i}\overline{T}_0^{(2)} \arrow[r, "r_{k+1}"] \arrow[u] \arrow[d, equal] & \Sigma^{i|b_{1,1}|}\overline{T}_0^{(2)} \arrow[r] \arrow[u] & \Sigma^{a_i+|b_{1,1}|}\overline{T}_0^k \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Sigma^{a_i}\overline{T}_0^{(1)} \arrow[r] \arrow[u] & \Sigma^{a_i}\overline{T}_0^{(2)} \arrow[r, "r_p"] & \Sigma^{a_i+|t_1^p|}\overline{T}_0^{(2)} \arrow[r] \arrow[u, "r_{k+1-p}"'] & \Sigma^{a_i+|b_{1,1}|}\overline{T}_0^{(1)} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where $k = p(p-i)-1$, the top row is the same as that in (7.4.19), and each vertical map is a monomorphism.

Proof. The statement about the top row is a reformulation of our determination of X_i and Y_i above. Each vertical map is obvious except the one to K_{2i}. K_{2i} is the kernel of the map d_{2i} from

$$F_{2i} = \Sigma^{i|b_{1,1}|}P(1)_* \oplus \Sigma^{(i-1)|b_{2,0}|+|t_1^p t_2|}P(1)_*$$

(note that $i|b_{1,1}| = a_i + |t_1^{p(p-i)}|$) to

$$F_{2i+1} = \Sigma^{i|b_{2,0}|+|t_1^p|}P(1)_* \oplus \Sigma^{i|b_{1,1}|+|t_2|}P(1)_*$$

represented by the matrix

$$M_{2i} = \begin{bmatrix} y^{i+1} & -(xy+(i+1)z)z^{p-2} \\ z & -y^{p-i} \end{bmatrix}.$$

The map $\Sigma^{a_i}P(1)_*$ is the restriction of d_{2i-1}, under which we have

$$t_1^{p(p-i)} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so this is the image of the bottom element in $\Sigma^{i|b_{1,1}|}\overline{T}_0^{(2)}$ in K_{2i}. This means that the top element in $\Sigma^{i|b_{1,1}|}\overline{T}_0^{(2)}$ must map to an element of the form

$$\begin{bmatrix} t_1^{p(p-1)} + \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

where ε_1 and ε_2 are each killed by y^{p-1}. We also need this element to be in K_{2i}, so it must satisfy

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= M_{2i}\begin{bmatrix} t_1^{p(p-1)} + \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \\ &= \begin{bmatrix} ct_1^{p(p-2-i)} + y^{i+1}(\varepsilon_1) - (xy+(i+1)z)z^{p-2}(\varepsilon_2) \\ z(\varepsilon_1) - y^{p-i}(\varepsilon_2) \end{bmatrix} \end{aligned}$$

for a certain unit scalar c. We can get this by setting $\varepsilon_1 = 0$ and making ε_2 a linear combination of $t_1^{p(p-2-i)}t_2^{p-1}$ and $t_1^{1+p(p-1-i)}t_2^{p-2}$ chosen to make the element in the top row vanish. Such an ε_2 will be killed by y^{p-i}, so the element in the bottom row will vanish as well. $\square$

This means that the Ext computation for the K_{2i} is essentially identical to that $\overline{T}_0^{(1)} \otimes E_0M$ described above. It follows from the way the K_i were constructed that for all i and s,

$$Ext^s_{P(1)_*}(K_i) = \mathrm{Ext}^{s+i}_{P(1)_*}(K_0) = \mathrm{Ext}^{s+i}_{P(1)_*}(\overline{T}_0^{(1)}). \tag{7.4.21}$$

These groups are known by Theorem 7.4.13.

We need the following analog of Theorem 7.4.13 for these comodules, whose proof we leave as an exercise for the reader.

7.4.22. THEOREM. *In the Cartan–Eilenberg spectral sequence converging to* $\mathrm{Ext}_{P(1)_*}(K_{2i})$ *based on the extension (7.4.9) for* $0 < i < p$, $\tilde{E}_2$ *is a subquotient (determined by the* d_1 *indicated below) of*

$$P(b_{1,1}) \otimes \begin{cases} \{b^i_{2,0}\} \otimes E(h_{2,0}, h_{1,1}) \otimes P(b_{2,0}) \\ \oplus \\ \left\{h_{1,1}h_{2,0}b^{i-1}_{2,0},\ \underline{p(p-i)}(h_{1,1}h_{2,0}b^{i-1}_{2,0}),\ b^i_{1,1},\ \underline{pi}b^i_{1,1}\right\}. \end{cases}$$

Here we are using the isomorphism of (7.4.21) to name the generators in the two indicated sets. Thus we have

$$\begin{array}{rcl} b^i_{2,0},\ h_{1,1}h_{2,0}b^{i-1}_{2,0},\ b^i_{1,1} & \in & \tilde{E}_2^{0,0} \\ \underline{p(p-i)}h_{1,1}h_{2,0}b^{i-1}_{2,0},\ \underline{pi}b^i_{1,1} & \in & \tilde{E}_2^{1,0} \\ h_{1,1} \in E_2^{1,0} \qquad b_{1,1} & \in & E_2^{2,0} \\ h_{2,0} \in E_2^{0,1} \qquad b_{2,0} & \in & E_2^{0,2}, \end{array}$$

and the differentials are (up to unit scalar)

$$\begin{array}{rcl} d_1(b^i_{2,0}) & = & \underline{pi}b^i_{1,1} \\ d_2(h_{2,0}b^i_{2,0}) & = & b_{1,1} \cdot h_{1,1}h_{2,0}b^{i-1}_{2,0} \\ d_3(h^\varepsilon_{2,0}b^k_{2,0} \cdot b^i_{2,0}) & = & (i+k)h_{1,1}h^\varepsilon_{2,0}b_{1,1}b^{k-1}_{2,0} \cdot b^i_{2,0} \\ & & \qquad \textit{for } k > 0 \textit{ and } \varepsilon = 0 \textit{ or } 1 \\ d_{2p-2i-1}(h_{1,1}b^{p-1-i}_{2,0} \cdot b^i_{2,0}) & = & b^{p-i}_{1,1} \cdot b^i_{1,1} \\ d_{2p-2i-2}(h_{1,1}h_{2,0}b^{p-1-i}_{2,0} \cdot b^i_{2,0}) & = & b^{p-i}_{1,1} \cdot \underline{p(p-i)}h_{1,1}h_{2,0}b^{i-1}_{2,0}. \end{array}$$

The last four differentials listed above should be compared with the first four listed in Theorem 7.3.15. The first differential of Theorem 7.4.13 corresponds to the last one of 7.3.15, while the second differential of 7.4.13 would correspond to one in 7.3.15 that is out of our range.

Thus Theorem 7.3.15 is a consequence of the relation between the K_{2i} and E_0M.

5. Computing $\pi_*(S^0)$ for $p = 3$

We begin by recalling the results of the previous sections. We are considering groups $\mathrm{Ext}^{s,t} = 0$ for $t < p^3|v_1|$ (where $|v_1| = 2p-2$) with $p > 2$. For each odd prime p, we have the 4-term exact sequence (7.1.19) of comdules over $BP_*(BP)$

$$0 \longrightarrow BP_* \longrightarrow D^0_1 \longrightarrow D^1_1 \longrightarrow E^2_1 \longrightarrow 0$$

in which D_1^0 and D_1^1 are weak injective (meaning that their higher Ext groups vanish, see 7.1.5) and the maps

$$\mathrm{Ext}^0(D_1^0) \longrightarrow \mathrm{Ext}^0(D_1^1) \longrightarrow \mathrm{Ext}^0(E_1^2)$$

are trivial. This means that the resolution spectral sequence collapses from E_1 and we have isomorphisms

$$\mathrm{Ext}^s = \begin{cases} \mathrm{Ext}^0(D_1^0) & \text{for } s = 0 \\ \mathrm{Ext}^0(D_1^1) & \text{for } s = 1 \\ \mathrm{Ext}^{s-2}(E_1^2) & \text{for } s \geq 2 \end{cases}$$

We have determined $\mathrm{Ext}(T_0^{(1)} \otimes E_1^2)$ in Theorem 7.3.15, which can be reformulated as follows.

7.5.1. ABC Theorem. *For $p > 2$ and $t < (p^3 + p)|v_1|$*

$$\mathrm{Ext}(T_0^{(1)} \otimes E_1^2) = A \oplus B \oplus C$$

where A is the $\mathbf{Z}/(p)$-vector space spanned by

$$\left\{\beta_i = \frac{v_2^i}{pv_1} : i > 0 \text{ and } i \equiv 0, 1 \mod (p)\right\} \cup \left\{\beta_{p^2/p^2-j} : 0 \leq j < p\right\},$$

$$B = R \otimes \left\{\gamma_k \in \mathrm{Ext}^{1,2k(p^3-1)-2(p^2+p-2)} : k \geq 2\right\}$$

and

$$C^{s,t} = \bigoplus_{i \geq 0} R^{2+s+2i, t+i(p^2-1)q}.$$

Here $R = \mathrm{Ext}_{P(1)_*}(\mathbf{Z}/(p), \overline{T}_0^{(1)})$ as described in Theorem 7.4.13. □

This result is illustrated for $p = 5$ in Figure 7.3.17. Each dot represents a basis element. Vertical lines represent multiplication by 5 and horizontal lines represent the Massey product operation $\langle -, 5, \alpha_1 \rangle$, corresponding to multiplication by v_1. The diagonal lines correspond either to mulitplication by $h_{2,0}$ or to Massey product operations $\langle -, h_{11}, h_{11}, \ldots, h_{11} \rangle$.

The next step is to pass from this group to $\mathrm{Ext}(E_1^2)$ using the small descent spectral sequence of Theorem 7.1.13. Alternatively one could observe that $E_1^2 = BP_*(\mathrm{coker}\, J)$ and that the Adams–Novikov spectral sequence for $\pi_*(T(0)_{(1)} \wedge \mathrm{coker}\, J)$ collapses for dimensional reasons. We can then use the topological small descent spectral sequence of Theorem 7.1.16 to pass from this group to $\pi_*(\mathrm{coker}\, J)$. We will do this using the input/output procedure of 7.1.18.

We give a basis for $\mathbf{N}$. Recall the the input $\mathbf{I}$ in this case is $\mathbf{N} \otimes E(h_{1,0})$.

7.5.2. Proposition. *For $p = 3$, $\mathbf{N}$ as in 7.1.18 has basis elements in dimensions indicated below.*

10 β_1	42 $\beta_3, \beta_{3/1,2}$	58 β_4
26 β_2	49 $h_{2,0}b_{1,1}$	68 $b_{1,1}^2$
34 $\beta_{3/3} = b_{1,1}$	53 $h_{11}\beta_{3/1,2}$	72 $b_{2,0}\beta_2$
38 $\beta_{3/2}$	57 $\eta_1 = h_{11}u_0 = \underline{6}\beta_{3/3}$	74 β_5

78 u_2
81 γ_2
82 $\beta_{6/3}$
83 $\beta_2\eta_1 = h_{2,0}b_{1,1}^2$
86 $\beta_{6/2}$
89 $\eta_3 = h_{11}u_2$
90 $\beta_6, \beta_{6/1,2}$
92 $h_{11}\gamma_2$
93 $h_{2,0}u_2$
96 $h_{2,0}\gamma_2$
97 $h_{2,0}\beta_{6/3}$
97 $h_{2,0}\beta_{6/3}$
101 $h_{11}\beta_{6/1,2}$
104 $\beta_2 u_2$
105 $\eta_4 = h_{11}u_3 = \underline{6}\beta_{6/3}$
106 $\beta_7, \beta_{9/9}$
107 $\gamma_2\beta_2$ □

The notation $\underline{n}x$ for an integer n denotes a certain Massey product involving x as in 7.4.12. $\underline{3}x$ and $\underline{6}x$ denote $h_{11}x$ and $\langle h_{11}, h_{11}, x\rangle$, respectively.

Now we turn to the list **O** of 7.1.18, shown in 7.5.3. Elements from **N** are underlined. A differential is indicated by enclosing the target in square brackets and indicating the source on the right. Hence such pairs are to be ommited from the final output. The computation of the differentials will be described below.

7.5.3. Theorem. *With notation as above the list* **O** *of* 7.1.18 *for* $p = 3$ *is as follows.*

10 $\underline{\beta_1}$
13 $\alpha_1\beta_1$
20 β_1^2
23 $\alpha_1\beta_1^2$
26 $\underline{\beta_2}$
29 $\alpha_1\beta_2$
30 β_1^3
33 $[\alpha_1\beta_1^3]\underline{\beta_{3/3}}$
36 $\beta_1\beta_2$
37 $\underline{2}\beta_1^3$
38 $\underline{\beta_{3/2}}$
39 $\alpha_1\beta_1\beta_2$
40 β_1^4
41 $[\alpha_1\beta_{3/2}]\underline{\beta_{3/1,2}}$
42 $\underline{\beta_3}$
45 $\underline{2}\beta_{3/2}$
$\alpha_1\beta_3$
46 $\beta_1^2\beta_2$
47 $\underline{2}\beta_1^4$
48 $[\beta_1\beta_{3/2}]\underline{h_{2,0}b_{1,1}}$
49 $\alpha_1\beta_1^2\beta_2$
50 β_1^5
52 β_2^2
$[\beta_1\beta_3]\underline{h_{11}\beta_{3/1,2}}$
55 $\alpha_1\beta_2^2$
$[\alpha_1\beta_1\beta_3]\alpha_1 h_{11}\beta_{3/1,2}$
56 $[\beta_1^3\beta_2]\eta_1$
57 $[\underline{2}\beta_1^5]\underline{\beta_4}$
59 $[\alpha_1\beta_1^3\beta_2]\alpha_1\eta_1$
60 $[\beta_1^6]\alpha_1\beta_4$
62 $\beta_1\beta_2^2$
65 $\alpha_1\beta_1\beta_2^2$
68 $\underline{\beta_{3/3}^2} \pm \beta_4\beta_1 = x_{68}$
71 $[\alpha_1 x_{68}]\underline{b_{2,0}\beta_2}$
72 $\beta_1^2\beta_2^2$
74 $\underline{\beta_5}$
75 $\underline{2}x_{68}$
$\alpha_1\beta_1^2\beta_2^2$
77 $[\alpha_1\beta_5]\underline{u_2}$
78 $\beta_2^3 = \beta_1 x_{68}$
81 $\underline{\gamma_2}$
$\underline{2}\beta_5$
82 $\underline{\beta_{6/3}}$
$[\beta_1^3\beta_2^2]\underline{\beta_2\eta_1}$
84 $\alpha_1\gamma_2$
$\beta_1\beta_5$
85 $\underline{2}\beta_2^3$
$\alpha_1\beta_{6/3}$
$[\alpha_1\beta_1^3\beta_2^2]\alpha_1\beta_2\eta_1$
86 $\underline{\beta_{6/2}}$
88 $[\beta_1\beta_2^3]\underline{\eta_3}$
89 $[\alpha_1\beta_{6/2}]\underline{\beta_{6/1,2}}$
90 $\underline{\beta_6}$
91 $\beta_1\gamma_2$
$\underline{2}\beta_1\beta_5$
92 $\beta_1\beta_{6/3}$
$\underline{h_{11}\gamma_2}$
$[\alpha_1 u_3 + \beta_1\beta_{6/3}]\underline{h_{2,0}u_2}$
93 $\underline{2}\beta_{6/2}$
$\alpha_1\beta_6$
94 $\alpha_1\beta_1\gamma_2$
$\beta_1^2\beta_5$

95 $[\alpha_1 \underline{3}\gamma_2]\underline{h_{2,0}\gamma_2}$

$\alpha_1\beta_1\beta_{6/3}$

$[\underline{2}\beta_1\beta_2^3]\alpha_1 h_{2,0}u_2$

96 $[\beta_1\beta_{6/2}]\underline{h_{2,0}\beta_{6/3}}$

99 $\underline{2}\cdot\underline{3}\gamma_2$

100 $\beta_2\beta_5 = \underline{2}\cdot\underline{2}\beta_{6/2}$

$[\beta_1\beta_6]\underline{h_{11}\beta_{6/1,2}}$

$\underline{2}\beta_1^2\beta_5$

101 $\underline{2}\beta_1^2\beta_5$

102 $\beta_1^2\beta_{6/3} = \pm\beta_{3/3}^3$

$\underline{3}\beta_1\gamma_2$

103 $[\alpha_1\beta_2\beta_5]\underline{\beta_2 u_2}$

$[\alpha_1\beta_1\beta_6]\alpha_1 h_{11}\beta_{6/1,2}$

104 $\alpha_1\beta_1^2\gamma_2$

$[\beta_1^3\beta_5]\underline{\eta_4}$

105 $[\alpha_1\beta_{3/3}^3]\underline{\beta_{9/9}}$

106 $\underline{\beta_7} \pm \beta_{9/9}$

7.5.4. Remark. In the calculations below we shall make use of *Toda brackets* (first defined by Toda [**6**]) and their relation to Massey products. Suppose we have spaces (or spectra) and maps $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ with gf and hg null-homotopic. Let $\bar{f}\colon CW \to Y$ and $\bar{g}\colon CX \to Z$ be null homotopies. Define a map $k\colon \Sigma W \to Z$ by regarding ΣW as the union of two copies of CW, and letting the restrictions of k be $h\bar{f}$ and $\bar{g}(Cf)$. k is not unique up to homotopy as it depends on the choice of the null homotopies $\bar{f}$ and $\bar{g}$. Two choices of $\bar{f}$ differ by a map $\Sigma W \to Y$ and similarly for $\bar{g}$. Hence we get a certain coset of $[\Sigma W, Z]$ denoted in Toda [**6**] by $\{f, g, h\}$, but here by $\langle f, g, h\rangle$. Alternatively, let C_g be the cofiber of g, $\bar{h}\colon C_g \to Z$ an extension of h and $\bar{f}\colon \Sigma W \to C_g$ a lifting of Σf. Then k is the composite $\overline{hf}$.

Recall (A1.4.1) that for a differential algebra C with $a, b, c \in H^*C$ satisfying $ab = bc = 0$ the Massey product $\langle a, b, c\rangle$ is defined in a similar way. The interested reader can formulate the definition of higher matric Toda brackets, but any such map can be given as the composite of two maps to and from a suitable auxiliary spectrum (such as C_g). For example, given

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

satisfying suitable conditions with each X_i a sphere, the resulting n-fold Toda bracket is a composite $\Sigma^{n-2}X_0 \to Y \to X_n$, where Y is a complex with $(n-1)$ cells.

The relation between Toda brackets and Massey products and their behavior in the Adams spectral sequence is studied by Kochman [**2, 4, 5**]. The basic idea of Kochman [**4**] is to show that the Adams spectral sequence arises from a filtered complex, so the spectral sequence results of A1.4 apply. Given Kochman's work we will use Toda brackets and Massey products interchangeably.

7.5.5. Remark. In the following discussions we will not attempt to keep track of nonzero scalars mod (p). For $p = 3$ this means that a $\pm$ should appear in front of every symbol in an equation. The reader does not have the right to sue for improper coefficients.

Now we provide a running commentary on this list. The notation $\underline{2}x$ denotes the Massey product $\langle\alpha_1, \alpha_1, x\rangle$. If $d_r(y) = \alpha_1 x$ then $\alpha_1 y$ represents $\underline{2}x$. Also note that $\alpha_1\underline{2}x = \pm\beta_1 x$.

In the 33-stem we have the Toda differential of 4.4.22. The element $\alpha_1\beta_{3/2}$ is a permanent cycle giving $\underline{2}\beta_1^3$. The coboundary of $\frac{v_1 v_2^3}{9v_1^3}$ gives

$$\langle\alpha_1, \alpha_1, \beta_1^3\rangle\langle\beta_2, 3, \beta_1\rangle. \tag{7.5.6}$$

The differentials shown in the 41-, 48-, 52-, and 55-stems can be computed algebraically; i.e., they correspond to relations in Ext. The elements $\alpha_1\beta_{3/2}$, $\beta_1\beta_{3/2}$, and $\beta_1\beta_3$ are the coboundaries of

$$\frac{v_2^3}{9v_1}, \quad \frac{v_2^3t_1^2}{9v_1}+\frac{v_2^3t_2}{3v_1^3}, \quad \text{and} \quad \frac{v_1^2v_2^3t_1^3}{9v_1^3}-\frac{v_2^3t_2}{3v_1^2},$$

respectively. We also have $3(\underline{2}\beta_{3/2}) = \alpha_1\beta_3$, i.e., $\pi_{45}(S^0) = \mathbf{Z}/(9)$.

For the differential in the 56-stem we claim $\alpha_1\eta_1 = \pm\beta_2\beta_{3/3}$, forcing $d_5(\eta_1) = \pm\beta_1^3\beta_2$ in the Adams–Novikov spectral sequence. The claim could be verified by direct calculation, but the following indirect argument is easier. $\beta_2\beta_{3/3}$ must be nonzero and hence a multiple of $\alpha_1\eta_1$ because $\alpha_1\beta_1^3\beta_2 \neq 0$ in Ext and must be killed by a differential.

However, we will need the direct calculation in the future, so we record it now for general p. Consider the element

$$\frac{v_2^{i-1}v_3(t_2-t_1^{1+p})-v_2^i(t_3-t_1t_2^p-t_2t_1^{p^2}+t_1^{1+p+p^2})+v_2^{i+p-1}(t_1^{p+2}-t_1t_2)}{pv_1}$$
$$+\frac{2v_2^{i+p}}{p^2v_1^p(i+p)}\sum_{0<j<p}\frac{(-1)^j}{j}p^{j-1}v_1^{p-j}t_1^j \quad \text{with} \quad i>0.$$

Straightforward calculation shows the coboundary is

$$\frac{v_2^{i+1}b_{1,1}}{pv_1}+\frac{2p^{p-3}v_2^{i+p}b_{1,0}}{(i+p)v_1^p}-\frac{v_2^{i-1}(v_2t_2^p+v_2^pt_2-v_2t_1^{p+p^2}-v_3t_1^p)|t_1}{pv_1}, \tag{7.5.7}$$

which gives the desired result since the third term represents η_i. The second term is nonzero in our range only in the case $i = p = 3$, where we have $\pm\alpha_1\eta_3 = \beta_4b_{1,1}+\beta_{6/3}b_{1,0}$. This element is also the coboundary of $\frac{v_2^6t_1^3}{9v_1^3}+\frac{v_2^5t_1^6}{3v_1^2}$, so $\alpha_3\eta_3 = 0$.

For the differentials in the 57- and 60-stems we claim $\beta_2^3 = \pm\beta_1^2\beta_4 \pm \beta_{3/3}^2\beta_1$ in Ext. This must be a permanent cycle since β_2 is. It is straightforward that $d_5(\beta_{3/3}^2) = \pm\underline{2}\beta_1^6$ in the Adams–Novikov spectral sequence, so we get $d_5(\beta_4) = \pm\underline{2}\beta_1^5$. Then $\beta_1^6 = \alpha_1\underline{2}\beta_1^5 = 0$ in $\pi_*(S^0)$, so $d_9(\alpha_1\beta_4) = \beta_1^6$.

To verify our claim that $\beta_2^3 = \pm\beta_1^2\beta_4 \pm \beta_1\beta_{3/3}^2$, it suffices to compute in $\mathrm{Ext}(BP_*/I_2)$. The mod I_2 reductions of β_2, β_4, and $\beta_{3/3}$ are $v_2b_{1,0} \pm k_0$, $v_2^3b_{1,0}$, and $b_{1,1}$, respectively, where $k_0 = \langle h_{10}, h_{11}, h_{11}\rangle$. A Massey product manipulation shows $k_0^3 = b_{1,0}b_{1,1}^2$ and the result follows.

Now we will show

$$x_{68} = \langle \alpha_1, \beta_{3/2}, \beta_2\rangle. \tag{7.5.8}$$

We can do this calculation in Ext and work mod I, i.e., in Ext_P, and it suffices to show that the indicate product is nonzero. We have

$$\begin{aligned}\langle h_{10}, h_{10}h_{12}, \langle h_{11}, h_{11}, h_{10}\rangle\rangle &= \langle h_{10}, h_{10}, h_{12}\langle h_{11}, h_{11}, h_{10}\rangle\rangle\\ &= \langle h_{10}, h_{10}, \langle h_{12}, h_{11}, h_{11}\rangle h_{10}\rangle\\ &= b_{1,0}\langle h_{12}, h_{11}, h_{11}\rangle\\ &= \langle b_{1,0}h_{12}, h_{11}, h_{11}\rangle = \langle b_{1,1}h_{11}, h_{11}, h_{11}\rangle\\ &= b_{1,1}^2 \neq 0.\end{aligned}$$

This element satisfies $\beta_1 x_{68} = \beta_2^3$. To show $\alpha_1 x_{68} = 0$, consider the coboundary of

$$\frac{v_2^2 b_{2,0} \pm v_2 v_3 b_{1,0}}{3v_1} \pm \frac{v_1 v_2^3 b_{1,0}}{9v_1^3}.$$

Next we show that there is a nontrivial extension in the 75-stem. We have

$$\begin{aligned}\beta_1^3\langle\alpha_1,\alpha_1,\beta_{3/2}\rangle &= \langle\beta_1^3,\alpha_1,\alpha_1\rangle\beta_{3/2}\\ &= \langle\beta_2,3,\beta_1\rangle\beta_{3/2} \quad \text{by } 7.5.6\\ &= 3\langle\beta_1,\beta_{3/2},\beta_2\rangle = 3\langle\langle\alpha_1,\alpha_1,\alpha_1\rangle,\beta_{3/2},\beta_2\rangle\\ &= 3\underline{2}\langle\alpha_1,\beta_{3/2},\beta_2\rangle = 3\underline{2}x_{68}.\end{aligned}$$

For the differential in the 77-stem note that $\alpha_1\beta_5$ is the coboundary of $u_2 = \frac{v_2^6}{3v_1^4} + \frac{v_2^2 v_3}{3v_1}$.

This brings us to the 88-stem, where we need to show $\beta_1^2 x_{68} = 0$. Since $x_{68} = \langle\alpha_1,\alpha_1,\beta_1^5,\beta_1\rangle$ we can show $\beta_1^3 x_{68} = 0$. There is no element in the 99-stem other than $\beta_1\eta_3$ to kill it, so the differential follows.

The differential in the 89-stem is similar to that in the 41-stem. The one in the 92-stem follows from 7.5.7.

In the 95-stem $\alpha_1 h_{11}\gamma_2$ the coboundary of $\frac{v_3^2 t_2}{3v_1 v_2}$. The differential in the 105-stem is a special case of 6.4.1. The others are straightforward. The resulting homotopy groups are shown in Table A3.4.

6. Computations for $p = 5$

We will apply the results and techniques of Section 9 to compute up to the 1000-stem for $p = 5$. Naturally the lists **I** and **O** are quite long. The length of **O**, i.e., the number of additive generators in coker J through dimension k, appears to be roughly a quadratic function of k in our range. The conventions of 7.5.4 and 7.5.5 are still in effect.

The highlight of the 5-primary calculation is the following result

7.6.1. Theorem. *For $p=5$, $\beta_1^{17} \neq 0$ and there are Adams–Novikov differentials $d_{33}(\gamma_3) = \beta_1^{18}$. Consequently the Smith–Toda complex $V(3)$ does not exist, and $V(2)$ is not a ring spectrum.* □

7.6.2. Conjecture. *For $p \geq 7$, $\beta_1^{p^2-p} \neq 0$ and $\beta_1^{p^2-p+1} = 0$. Moreover $\langle\gamma_3,\gamma_2,\ldots,\gamma_2\rangle = \beta_1^{(2p-1)(p-1)/2}$ where γ_2 appears in the bracket $(p-5)/2$ times.* □

We will prove 7.6.1 modulo certain calculations to be carried out below. First we give a classical argument due to Toda for $\beta_1^{p^2-p+1} = 0$. We know $\alpha_1\beta_1^p = 0$ from Toda [**2, 3**]. It follows by bracket manipulations that $w_i = \langle\alpha_1,\alpha_1,\ldots,\alpha_1,\beta_1^{ip}\rangle$ is defined with $(i+1)$ factors α_1 and $1 \leq i \leq p-2$. The corresponding ANSS element is $\alpha_1\beta_{p/p}^i$. Now since $\beta_1 = \langle\alpha_1,\ldots,\alpha_1\rangle$ with p factors we have [using A1.4.6(c)]

$$\alpha_1 w_{p-2} = \langle\alpha_1,\ldots,\alpha_1\rangle\beta_1^{p^2-2p} = \beta_1^{p^2-2p+1}.$$

Hence $\beta_1^{p^2-p+1}$ is divisible by $\alpha_1\beta_1^p$ and is therefore zero. The corresponding Adams–Novikov differential is $d_r(\alpha_1\beta_{p/p}^{p-1}) = \beta_1^{p^2-p+1}$ with $r = 2p^2 - 4p + 3$.

We will give a more geometric translation of this argument for $p = 5$. Let $X_i = T(1)^{iq} = S^0 \bigcup_{\alpha_1} e^q \bigcup_{\alpha_1} \cdots \bigcup_{\alpha_1} e^{iq}$. The Toda bracket definition of β_1 means

there is a diagram

(7.6.3)
$$\begin{tikzcd} S^{31} \arrow[r] & X_3 \\ S^{38} \arrow[u, "\alpha_1"] \arrow[r, "\beta_1"] & S^0 \arrow[u] \end{tikzcd}$$

where the cofiber of the top map is X_4. From $\alpha_1\beta_1^5 = 0$ we get a diagram

$$\begin{tikzcd} \Sigma^{190}X_1 \arrow[dr, "f"] & \\ S^{190} \arrow[u] \arrow[r, "\beta_1^5"] & S^0. \end{tikzcd}$$

We smash this with itself three times and use the fact that X_3 is a retract of X_1^3 to get

$$\begin{tikzcd} \Sigma^{570}X_3 \arrow[dr, "g"] & \\ S^{570} \arrow[u] \arrow[r, "\beta_1^{15}"] & S^0. \end{tikzcd}$$

Combining this with 7.6.3 we get

$$\begin{tikzcd} & S^{601} \arrow[r] & \Sigma^{570}X_3 \arrow[dr, "g"] & \\ S^{79q} \arrow[ur, "0"] \arrow[r, "\beta_1^5"] & S^{608} \arrow[u, "\alpha_1"] \arrow[r, "\beta_1"] & S^{570} \arrow[u] \arrow[r, "\beta_1^{15}"] & S^0 \end{tikzcd}$$

so $\beta_1^{21} = 0$.

The calculation below shows that $\alpha_1\beta_{5/5}^4$ is a linear combination of $\beta_1^3\gamma_3$, $\underline{3}\beta_1^3\beta_{14}$, and $\beta_1 x_{761}$, where

$$x_{761} = \langle \alpha_1\beta_3, \beta_4, \gamma_2 \rangle \in \mathrm{Ext}^{7,768}.$$

Each factor of x_{761} is a permanent cycle, so x_{761} can fail to be one only if one of the products $\alpha_1\beta_3\beta_4$ and $\beta_4\gamma_2$ is nonzero in homotopy. But these products lie in stems 323 and 619 which are trivial, so x_{761} is a permanent cycle, as is $\underline{3}\beta_1^3\beta_{14}$. Since $d_{33}(\alpha_1\beta_{5/5}^4) = \beta_1^{21}$, we must have $d_{33}(\gamma_3) = \beta_1^{18}$ as claimed.

The nonexistence of γ_3 as a homotopy element shows the Smith–Toda complex $V(3)$ [satisfying $BP_*(V(3)) = BP_*/I_4$] cannot exist for $p = 5$. If one computes the Adams–Novikov spectral sequence for $V(2)$ through dimension 248, one finds that $v_3 \in \mathrm{Ext}^0$ is a permanent cycle; i.e., v_3 is realized by a map $S^{248} \xrightarrow{f} V(2)$. If $V(2)$ were a ring spectrum we could use the multiplication to extend f to a self-map with cofiber $V(3)$, giving a contradiction.

Now we proceed with the calculation for $p = 5$.

7.6.4. Theorem. *For $p = 5$ $\mathbf{N}$ as in 7.1.18 has basis elements in dimensions indicated below, with notation as in 7.5.2. η_i denotes $h_{1,1}u_{i-1}$.*

38 β_1	182 β_4	214 $\beta_{5/3}$
86 β_2	198 $\beta_{5/5} = b_{1,1}$	222 $\beta_{5/2}$
134 β_3	206 $\beta_{5/4}$	

230 β_5
$\beta_{5/1,2}$
245 $h_{2,0}b_{1,1}$
253 $h_{2,0}\beta_{5/4}$
261 $h_{2,0}\beta_{5/3}$
269 $h_{11}\beta_{5/1,2}$
277 η_1
278 β_6
324 $b_{2,0}\beta_2$
325 η_2
326 β_7
372 $b_{2,0}\beta_3$
373 η_3
374 β_8
396 $\beta_{5/5}^2$
404 $\beta_{5/5}\beta_{5/4}$
412 $\beta_{5/4}^2$
420 $b_{2,0}\beta_4$
422 β_9
430 u_4
437 γ_2
438 $\beta_{10/5}$
443 $h_{2,0}b_{1,1}^2$
446 $\beta_{10/4}$
451 $h_{2,0}b_{1,1}\beta_{5/4}$
454 $\beta_{10/3}$
459 $h_{2,0}b_{1,1}\beta_{5/3}$
462 $\beta_{10/2}$
469 η_5
470 β_{10}
$\beta_{10/1,2}$
477 $h_{2,0}u_4$
476 $h_{11}\gamma_2$
484 $h_{2,0}\gamma_2$
485 $h_{2,0}\beta_{10/5}$
493 $h_{2,0}\beta_{10/4}$
501 $h_{2,0}\beta_{10/3}$
509 $h_{11}\beta_{10/1,2}$
515 $b_{2,0}\eta_1$
516 $h_{2,0}\eta_5$
517 η_6
518 β_{11}
523 $\beta_2\gamma_2$
562 $b_{2,0}^2\beta_2$
563 $b_{2,0}\eta_2$
564 $b_{2,0}\beta_7$
565 η_7
566 β_{12}
594 $\beta_{5/5}^3$
602 $\beta_{5/5}^2\beta_{5/4}^2$
610 $b_{2,0}^2\beta_3$
612 $b_{2,0}\beta_8$
613 η_8
614 β_{13}
620 $b_{2,0}u_3$
628 $b_{1,1}u_4$
635 $b_{1,1}\gamma_2$
636 $b_{1,1}\beta_{10/5}$
641 $h_{2,0}b_{1,1}^3$
644 $\beta_{5/4}\beta_{10/5}$
649 $h_{2,0}b_{1,1}^2\beta_{5/4}$
652 $\beta_{10/5}\beta_{5/3}$
659 $h_{2,0}b_{2,0}\beta_8$
660 $b_{2,0}\beta_9$
662 β_{14}
667 $h_{2,0}b_{2,0}u_3$
670 u_9
675 $h_{2,0}b_{1,1}u_4$
678 $\beta_{15/5}$
682 $h_{2,0}b_{1,1}\gamma_2$
683 $h_{2,0}b_{1,1}\beta_{10/5}$
685 γ_3
686 $\beta_{15/4}$
691 $h_{2,0}b_{1,1}\beta_{10/4}$
694 $\beta_{15/3}$
699 $\eta_1\beta_9$
702 $\beta_{15/2}$
706 $h_{2,0}b_{1,1}b_{2,0}u_3$
707 $b_{2,0}\eta_5$
709 η_{10}
710 β_{15}
$\beta_{15/1,2}$
714 $h_{11}b_{2,0}\gamma_2$
717 $h_{2,0}u_9$
724 $h_{11}\gamma_3$
725 $h_{2,0}\beta_{15/5}$
732 $h_{2,0}\gamma_3$
733 $h_{2,0}\beta_{15/4}$
741 $h_{2,0}\beta_{15/3}$
749 $h_{11}\beta_{15/1,2}$
753 $b_{2,0}^2\eta_1$
754 $h_{2,0}b_{2,0}\eta_5$
755 $b_{2,0}\eta_6$
756 $h_{2,0}\eta_{10}$
757 η_{11}
758 β_{16}
761 $b_{2,0}\beta_2\gamma_2$
771 $\beta_2\gamma_3$
792 $\beta_{5/5}^4$
800 $b_{2,0}^3\beta_2$
802 $b_{2,0}^2\beta_7$
803 $b_{2,0}\eta_7$
804 $b_{2,0}\beta_{12}$
805 η_{12}
806 β_{17}

810 $b_{2,0}^2u_2$	892 $\beta_{10/4}^2$	949 η_{15}
818 $b_{1,1}b_{2,0}u_3$	896 $h_{2,0}h_{11}b_{2,0}^2u_2$	950 β_{20}
826 $b_{1,1}^2u_4$	897 $h_{11}b_{2,0}^2u_3$	$\beta_{20/1,2}$
833 $b_{1,1}^2\gamma_2$	899 $h_{2,0}b_{2,0}\beta_{13}$	952 $h_{11}b_{2,0}^2\gamma_2$
834 $b_{1,1}^2\beta_{10/5}$	900 $b_{2,0}\beta_{14}$	957 $h_{2,0}u_{14}$
839 $h_{2,0}b_{1,1}^4$	902 β_{19}	962 $h_{11}b_{2,0}\gamma_3$
842 $b_{1,1}^2\beta_{10/4}$	907 $h_{2,0}b_{2,0}u_8$	965 $h_{2,0}\beta_{20/5}$
849 $h_{2,0}b_{2,0}^2\beta_7$	910 u_{14}	972 $h_{11}\gamma_4$
850 $b_{2,0}^2\beta_8$	915 $h_{2,0}b_{1,1}u_9$	980 $h_{2,0}\gamma_4$
852 $b_{2,0}\beta_{13}$	918 $\beta_{20/5}$	981 $h_{2,0}\beta_{20/3}$
873 η_{13}	923 $h_{2,0}b_{1,1}\beta_{15/5}$	989 $h_{11}\beta_{10/1,2}$
854 β_{18}	926 $\beta_{20/4}$	992 $h_{2,0}b_{2,0}^2\eta_5$
857 $h_{2,0}b_{2,0}^2u_2$	930 $h_{2,0}b_{1,1}\gamma_3$	993 $b_{2,0}^2\eta_6$
860 $b_{2,0}u_8$	931 $h_{2,0}b_{1,1}\beta_{15/4}$	994 $h_{2,0}b_{2,0}\eta_{10}$
865 $h_{2,0}b_{1,1}b_{2,0}u_3$	933 γ_4	995 $b_{2,0}\eta_{11}$
868 $b_{1,1}u_9$	934 $\beta_{20/3}$	996 $h_{2,0}\eta_{15}$
873 $h_{2,0}b_{1,1}^2u_4$	939 $\eta_1\beta_{14}$	997 η_{16}
876 $\beta_{10/5}^2$	942 $\beta_{20/2}$	998 β_{21}
880 $h_{2,0}b_{1,1}^2\gamma_2$	944 $h_{2,0}h_{11}b_{2,0}^2u_3$	$\beta_{25/25}$
881 $h_{2,0}b_{1,1}^2\beta_{10/5}$	945 $b_{2,0}^2\eta_5$	999 $b_{2,0}^2\beta_2\gamma_2$
884 $\beta_{5/4}\beta_{15/5}$	946 $h_{2,0}h_{11}b_{2,0}u_8$	1000 $b_{2,0}^2u_1$ $\square$
889 $h_{2,0}b_{1,1}^2\beta_{10/4}$	947 $b_{2,0}\eta_{10}$	

Now we will describe the list **O**, i.e., the analog of 7.5.3. The notation of that result is still in force, and we assume the reader is familiar with techniques used there. We will not comment on differentials with an obvious 3-primary analog, in particular on those following from 7.5.7. Many differentials we encounter are periodic under v_2 or v_2^p.

Since the list **O** is quite long, we will give it in six installments, pausing for comments and proofs when appropriate.

7.6.5. Theorem. *For $p = 5$ the list* **O** (7.1.18) *is as follows.* (*First installment*)

38 $\underline{\beta_1}$	93 $\alpha_1\beta_2$	134 $\underline{\beta_3}$
45 $\alpha_1\beta_1$	114 β_1^3	141 $\alpha_1\beta_3$
76 β_1^2	121 $\alpha_1\beta_1^3$	152 β_1^4
83 $\alpha_1\beta_1^2$	124 $\beta_1\beta_2$	159 $\alpha_1\beta_1^4$
86 $\underline{\beta_2}$	131 $\alpha_1\beta_1\beta_2$	162 $\beta_1^2\beta_2$

169 $\alpha_1\beta_1^2\beta_2$
172 $\beta_1\beta_3$
179 $\alpha_1\beta_1\beta_3$
182 $\underline{\beta_4}$
189 $\alpha_1\beta_4$
190 β_1^5
197 $[\alpha_1\beta_1^5]\underline{\beta_{5/5}}$
200 $\beta_1^3\beta_2$
205 $\underline{2}\beta_1^5$
206 $\underline{\beta_{5/4}}$
207 $\alpha_1\beta_1^3\beta_2$
210 $\beta_1^2\beta_3$
213 $\alpha_1\beta_{5/4}$
214 $\underline{\beta_{5/3}}$
217 $\alpha_1\beta_1^2\beta_3$
220 $\beta_1\beta_4$
221 $\alpha_1\beta_{5/3}$
222 $\underline{\beta_{5/2}}$
227 $\alpha_1\beta_1\beta_4$
228 $\underline{\beta_1^6}$
229 $[\alpha_1\beta_{5/2}]\underline{\beta_{5/1,2}}$
230 $\underline{\beta_5}$
237 $\underline{2}\beta_{5/2}$
$\alpha_1\beta_5$
238 $\beta_1^4\beta_2$
243 $\underline{2}\beta_1^6$
244 $[\beta_1\beta_{5/4}]\underline{h_{2,0}b_{1,1}}$
245 $\alpha_1\beta_1^4\beta_2$
248 $\beta_1^3\beta_3$
251 $[\alpha_1\beta_1\beta_{5/4}]\alpha_1 h_{2,0}b_{1,1}$
252 $[\beta_1\beta_{5/3}]\underline{h_{2,0}\beta_{5/4}}$
255 $\alpha_1\beta_1^3\beta_3$
258 $\beta_1^2\beta_4$
259 $[\alpha_1\beta_1\beta_{5/3}]\alpha_1 h_{2,0}\beta_{5/4}$
260 $[\beta_1\beta_{5/2}]\underline{h_{2,0}\beta_{5/3}}$
265 $\alpha_1\beta_1^2\beta_4$
266 β_1^7
268 $[\beta_1\beta_5]\underline{h_{11}\beta_{5/1,2}}$
$\beta_2\beta_4$
275 $[\alpha_1\beta_1\beta_5]\alpha_1 h_{11}\beta_{5/1,2}$
276 $[\beta_1^5\beta_2]\underline{\eta_1}$
278 $\underline{\beta_6}$
281 $\underline{2}\beta_1^7$
283 $[\alpha_1\beta_1^4\beta_2]\alpha_1\eta_1$
285 $\alpha_1\beta_6$
286 $\beta_1^4\beta_3$
293 $\alpha_1\beta_1^4\beta_3$
296 $\beta_1^3\beta_4$
303 $\alpha_1\beta_1^3\beta_4$
304 β_1^8
306 $\beta_1\beta_2\beta_4$
313 $\alpha_1\beta_1\beta_2\beta_4$
316 $\beta_1\beta_6$
319 $\underline{2}\beta_1^8$
323 $[\alpha_1\beta_1\beta_6]\underline{b_{2,0}\beta_2}$
324 $[\beta_1^5\beta_3]\underline{\underline{\eta_2}}$
326 β_7
331 $[\alpha_1\beta_1^5\beta_3]\alpha_1\eta_2$
$\underline{2}\beta_1\beta_6$
333 $\alpha_1\beta_7$
334 $\beta_1^4\beta_4$
341 $\alpha_1\beta_1^4\beta_4$
342 β_1^9
344 $\beta_1^2\beta_2\beta_4$
351 $\alpha_1\beta_1^2\beta_2\beta_4$
354 $\beta_1^2\beta_6$
357 $\underline{2}\beta_1^9$
364 $\beta_1\beta_7$
369 $\underline{2}\beta_1^2\beta_6$
371 $[\alpha_1\beta_1\beta_7]\underline{b_{2,0}\beta_3}$
372 $[\beta_1^5\beta_4]\underline{\eta_3}$
374 $\underline{\beta_8}$
379 $[\alpha_1\beta_1^5\beta_4]\alpha_1\eta_3$
$\underline{2}\beta_1\beta_7$
380 β_1^{10}
381 $\alpha_1\beta_8$
382 $\beta_1^3\beta_2\beta_4$
389 $\alpha_1\beta_1^3\beta_2\beta_4$
392 $\beta_1^3\beta_6$
395 $[\underline{2}\beta_1^{10}]\underline{\beta_{5/5}^2}$
402 $\beta_1^2\beta_7$
403 $\underline{3}\beta_1^{10}$
404 $\underline{\beta_{5/5}\beta_{5/4}}$
$= \langle\alpha_1\beta_1^4, \beta_1, \beta_{5/4}\rangle$
$= x_{404}$
407 $\underline{2}\beta_1^3\beta_6$
411 $\alpha_1 x_{404} = \beta_{5/4}\underline{2}\beta_1^5$
412 $\beta_1\beta_8$
$\beta_1\beta_8 + \underline{\beta_{5/4}^2} = x_{412}$
417 $\underline{2}\beta_1^2\beta_7$
418 β_1^{11}
419 $[\alpha_1 x_{412}]\underline{b_{2,0}\beta_4}$
$\alpha_1\beta_1\beta_8$
420 $\beta_1^4\beta_2\beta_4$
422 β_9
427 $\underline{2}x_{412}$
$\alpha_1\beta_1^4\beta_2\beta_4$
429 $[\alpha_1\beta_9]\underline{u_4}$ □

7.6.6. Remark. The small descent spectral sequences of 7.1.13 and 7.1.16 have some useful multiplicative structure even though $T(0)_{(1)}$ (the complex with p cells)

is not a ring spectrum and its BP-homology is not a comodule algebra. Recall that $T(0)_i$ is the iq-skeleton of $T(1)$. Then $\pi_*(T(0)_{(1)})$ is filtered by the images of $\pi_*(T(0)_i)$ for $i \le p-1$. One has maps $T(0)_i \wedge T(0)_j \to T(0)_{i+j}$ inducing pairings $F_i \otimes F_j \to F_{i+j}$ for $i+j \le p-1$. spectral sequence differentials always lower this filtration degree and respect this pairing. The filtration can be dualized as follows. A map $S^m \to T(0)_i$ is dual to a map $\Sigma^{m-iq}T(0)_i \to S^0$ since $DT(0)_i = \Sigma^{-iq}T(0)_i$ for $i \le p-1$. An element in $\pi_m(T(0)_{p-1})$ is in F_i iff the diagram

$$\begin{tikzcd} S^m \arrow[r] & T(0)_{p-1} \\ \Sigma^{m-qi}T(0)_i \arrow[u] \arrow[r, dashed] & S^0 \arrow[u] \end{tikzcd}$$

can be completed. The pairing $T(0)_i \wedge T(0)_j \to T(0)_{i+j}$ dualizes to $DT(0)_{i+j} \to DT(0)_i \wedge DT(0)_j$. If $\alpha \in \pi_m(T(0)_{p-1})$ is in F_i and $\beta \in \pi_n(T(0)_{p-1})$ is in F_j with $i+j=p$, then we get a map $\Sigma^{m+n}DT(0)_p \to S^0$. If this map is trivial on the bottom cell then it factors through $\Sigma^{m+n}DT(0)_{p-1} = \Sigma^{m+n-q(p-1)}T(0)_{p-1}$. This factorization will often lead to a differential in our spectral sequence.

For the differentials in dimensions 323, 371, and 419 recall (4.3.22) that there is an element $b_{2,0} \in C(BP_*/I_2)$ with $d(b_{2,0}) = (b_{1,0}|t_1^{p^2}) - (t_1^9|b_{1,1})$. Since $b_{1,0}$ and $t_1^{p^2}$ are both cycles there is a $y \in C(BP_*/I_2)$ such that $d(y) = (b_{1,0}|t_1^{p^2}) - (t_1^{p^2}|b_{1,0})$. Hence the coboundary of

$$\frac{v_2^{i-p}v_1b_{1,0} + v_2^{i+p-1}(y-b_{2,0})}{pv_1} - \frac{v_2^{i+2-p}b_{1,1}}{(1+2-p)pv_1^2} \quad \text{for } i \ge p$$

is

$$\frac{v_2^i t_1|b_{1,0}}{pv_1} + \frac{2v_2^{i+2-p}|b_{1,1}}{(i+2-p)v_1^3},$$

where the second term is nonzero only if $i \equiv -2 \mod (p)$. This gives

$$\alpha_1\beta_1\beta_i = \begin{cases} 0 & \text{for } i \ge p,\ i \not\equiv -2 \mod (p) \\ \alpha_1\beta_{(i+2-p)/4}\beta_{p/p-1} & \text{for } i \equiv -2. \end{cases} \tag{7.6.7}$$

Remember (7.5.5) we are not keeping track of nonzero scalar coefficients. The differentials in question follow.

Next we show that there is a nontrivial group extension in the 427-stem, similar to that for $p=3$ in the 75-stem. We want to prove $\alpha_1\beta_1^4\beta_2\beta_4 = 5\underline{2}x_{412}$. Since $\alpha_1\beta_2\beta_4 = \beta_1\underline{2}\beta_{5/2}$ we need to look at $\beta_1^5\underline{2}\beta_{5/2}$. We have

$$\begin{aligned} \beta_1^5\underline{2}\beta_{5/2} &= \beta_1^5\langle\alpha_1,\alpha_1,\beta_{5/2}\rangle = \beta_1^5\langle\alpha_1,\alpha_3,\beta_{5/4}\rangle = \langle\beta_1^5,\alpha_1,\alpha_3\rangle\beta_{5/4} \\ &= \alpha_1\beta_{5/3}\beta_{5/4} = \alpha_1\langle\alpha_1,5,\beta_{5/4}\rangle\beta_{5/4} = \alpha_1\langle\alpha_1,5,\beta_{5/4}^2\rangle \\ &= \alpha_1\langle\alpha_1,5,x_{412}\rangle = x_{412}\langle\alpha_1,\alpha_1,5\rangle = 5\underline{2}x_{412}. \end{aligned}$$

More generally one has

$$\beta_1^p\underline{2}\beta_{p/2} = p\underline{2}(\beta_{p/4}\beta_{p/p-1} + \beta_1\beta_{2p-2}). \tag{7.6.8}$$

Since

$$\alpha_1\langle\alpha_1,5,x_{412}\rangle = \beta_1^5\underline{2}\beta_{5/2} = \alpha_1\beta_1^4\beta_2\beta_4$$

we have

$$\langle\alpha_1,5,x_{412}\rangle = \beta_1^4\beta_2\beta_4$$

7.6.5 (*Second installment*)

430 $\beta_1^4\beta_6$

437 $\underline{2}\beta_9$

$\underline{\gamma_2}$

438 $\underline{\beta_{10/5}}$

440 $\underline{3}\beta_1^4$

442 $[\beta_1 x_{404}]h_{2,0}b_{1,1}^2$

444 $\alpha_1\gamma_2$

445 $\alpha_1\beta_{10/5}$

$\underline{2}\beta_1^4\beta_6$

446 $\underline{\beta_{10/4}}$

449 $[\alpha_1\beta_1 x_{404}]\alpha_1 h_{2,0}b_{1,1}^2$

450 $\beta_1^2\beta_8$

$[\beta_1\beta_{5/4}^2]\underline{h_{2,0}b_{1,1}\beta_{5/4}}$

453 $\alpha_1\beta_{10/4}$

454 $\underline{\beta_{10/3}}$

455 $\underline{2}\beta_1^3\beta_7$

456 β_1^{12}

457 $[\alpha_1\beta_1^2\beta_8]\alpha_1 h_{2,0}b_{1,1}\beta_{5/4}$

458 $[\beta_1^5\beta_2\beta_4]\underline{h_{2,0}b_{1,1}\beta_{5/3}}$

460 $\beta_1\beta_9$

461 $\alpha_1\beta_{10/3}$

462 $\underline{\beta_{10/2}}$

465 $\underline{2}\beta_1^2\beta_8$

$[\alpha_1\beta_1^5\beta_2\beta_4]\alpha_1 h_{2,0}b_{1,1}\beta_{5/3}$

468 $[\beta_1^5\beta_6]\underline{\eta_5}$

469 $[\alpha_1\beta_{10/2}]\underline{\beta_{10/1,2}}$

470 $\underline{\beta_{10}}$

475 $\beta_1\gamma_2$

$\underline{2}\beta_1\beta_9$

476 $[\alpha_1\eta_5 + \beta_1\beta_{10/5}]h_{2,0}u_4$

$\beta_1\beta_{10/5}$

$\underline{5}\gamma_2$

477 $\underline{2}\beta_{10/2}$

$\alpha_1\beta_{10}$

478 $\beta_1^4\beta_7$

479 $\underline{3}\beta_1^{12}$

482 $\alpha_1\beta_1\gamma_2$

483 $\alpha_1\beta_1\beta_{10/5}$

$[\underline{2}\beta_1^5\beta_6]\alpha_1 h_{2,0}u_4$

483 $[\alpha_1\underline{5}\gamma_2]\underline{h_{2,0}\gamma_2}$

484 $[\beta_1\beta_{10/4}]\underline{h_{2,0}\beta_{10/5}}$

488 $\beta_1^3\beta_8$

491 $\underline{2}\,\underline{5}\gamma_2$

$[\alpha_1\beta_1\beta_{10/4}]\alpha_1 h_{2,0}\beta_{10/5}$

492 $[\beta_1\beta_{10/3}]\underline{h_{2,0}\beta_{5/4}}$

493 $\underline{2}\beta_1^4\beta_7$

494 β_1^{13}

498 $\beta_1^2\beta_9$

499 $[\alpha_1\beta_1\beta_{10/3}]\alpha_1 h_{2,0}\beta_{5/4}$

500 $[\beta_1\beta_{10/2}]\underline{h_{2,0}\beta_{10/3}}$

503 $\underline{2}\beta_1^3\beta_8$

508 $\alpha_1 h_{2,0}\beta_{10/3} = \beta_2\beta_9$

$[\beta_1\beta_{10}]\underline{h_{11}\beta_{10/1,2}}$

513 $\beta_1^2\gamma_2$

$\underline{2}\beta_1^2\beta_9$

514 $\beta_1^2\beta_{10/5}$

515 $[\alpha_1\beta_2\beta_9]\underline{h_{2,0}\eta_5}$

$[\beta_1^2\beta_{10/5} + \beta_1\underline{5}\gamma_2]\underline{b_{2,0}\eta_1}$

$[\alpha_1\beta_1\beta_{10}]\alpha_1 h_{11}\beta_{10/1,2}$

516 $[\beta_1^5\beta_7]\eta_6$

517 $\underline{3}\beta_1^{13}$

518 $\underline{\beta_{11}}$

520 $\alpha_1\beta_1^2\gamma_2$

521 $[\alpha_1\beta_1^2\beta_{10/5}]\alpha_1 b_{2,0}\eta_1$

523 $\underline{2}\beta_2\beta_9$

$\underline{\beta_2\gamma_2}$

524 $\alpha_1\eta_6 = \beta_2\beta_{10/5}$

525 $\alpha_1\beta_{11}$

526 $\beta_1^4\beta_8$

529 $\underline{2}\,\underline{5}\beta_1\gamma_2$

530 $\alpha_1\beta_2\gamma_2$

531 $\underline{2}\beta_1^5\beta_7 = \alpha_1\beta_2\beta_{10/5}$

532 β_1^{14}

536 $\beta_1^3\beta_9$

541 $\underline{2}\beta_1^4\beta_8$

546 $\beta_1\beta_2\beta_9$

551 $\beta_1^3\gamma_2$

$\underline{2}\beta_1^3\beta_9$

552 $\underline{5}\beta_1^2\gamma_2$

555 $\underline{3}\beta_1^{14}$

558 $\alpha_1\beta_1^3\gamma_2$

556 $\beta_1\beta_{11}$

561 $\underline{2}\beta_1\beta_2\beta_9$

$[\beta_1\beta_2\gamma_2 + \underline{2}\beta_1\beta_2\beta_9]\underline{b_{2,0}^2\beta_2}$

562 $[\beta_1\beta_2\beta_{10/5}]\underline{b_{2,0}\eta_2}$

563 $[\alpha_1\beta_1\beta_{11}]\underline{b_{2,0}\beta_7}$

564 $[\beta_1^5\beta_8]\underline{\eta_7}$

566 $\underline{\beta}_{12}$

567 $\underline{2}\,\underline{5}\beta_1^2\gamma_2$

568 $[\alpha_1\beta_1\beta_2\gamma_2]\alpha_1 b_{2,0}^2\beta_2$

569 $[\alpha_1\beta_1\beta_2\beta_{10/5}]\alpha_1 b_{2,0}\eta_2$ □

For the differential in the 514-stem, note that in the corresponding spectral sequence for $\mathrm{Ext}_P(\mathbf{Z}/(p), \mathbf{Z}/(p))$ the image of $b_{2,0}\eta_1$ kills that of $\beta_1\underline{5}\gamma_2$, so the target in our spectral sequence of $b_{2,0}\eta_1$ is $\beta_1\underline{5}\gamma_2$ plus some multiple of $\beta_1^2\beta_{10/5}$. On the other hand, we have

$$\begin{aligned}\alpha_1\beta_1^2\beta_{10/5} &= \alpha_1\beta_1\beta_6\beta_{5/5}\\ &= \alpha_1\beta_1\langle\beta_6, \alpha_1\beta_1, \beta_1^4\rangle\\ &= \langle\alpha_1\beta_1\beta_6, \alpha_1\beta_1, \beta_1^4\rangle\\ &= 0\end{aligned}$$

and the result follows.

The relation in the 524-stem follows from 7.5.7. The differential in dimension 561 is $h_{2,0}$ times that in the 514-stem. The one in dimension 562 comes from a relation in Ext, i.e., $\beta_1^2\beta_2\beta_{10/5} = \beta_1^2\beta_7\beta_{5/5} = \beta_1\beta_6\alpha_1\eta_1 = 0$ since $\alpha_1\beta_1\beta_6 = 0$. Theorem 7.6.4 shows that there is no element in dimension 601 to give this relation, so we must have $\beta_1\beta_2\beta_{10/5}$ as indicated.

More generally, we have in Ext for $1 < i < p$ and $j > 1$

$$\begin{aligned}\beta_1^2\beta_i\beta_{pi/p} &= \beta_1\beta_{i+p}\beta_{i+pj-2p}\beta_{p/p}\\ &= \beta_1\beta_{1+p}\alpha_1\eta_{i+pj-1-2p} && \text{by 7.5.7}\\ &= 0 && \text{by 7.6.7.}\end{aligned} \tag{7.6.9}$$

In some cases this result along with inspection of $\underline{I}$ implies $\beta_1\beta_i\beta_{pj/p} = 0$.

7.6.5 (*Third installment*)

570 β_1^{15}

571 $\underline{2}\beta_1\beta_{11}$

572 $\alpha_1\eta_7 = \beta_3\beta_{10/5}$

573 $\alpha_1\beta_{12}$

574 $\beta_1^4\beta_9$

579 $\underline{2}\beta_1^5\beta_8 = \alpha_1\beta_3\beta_{10/5}$

584 $\beta_1^2\beta_2\beta_9$

589 $\beta_1^4\gamma_2$

$\underline{2}\beta_1^4\beta_9$

590 $\underline{5}\beta_1^3\gamma_2$

593 $[\underline{3}\beta_1^{15}]\underline{\beta_{5/5}^3}$

594 $\beta_1^2\beta_{11}$

596 $\alpha_1\beta_1^4\gamma_2$

599 $\underline{2}\beta_1^2\beta_2\beta_9$

601 $\underline{4}\beta_1^{15}$

602 $\underline{\beta_{5/5}^2\beta_{5/4}} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{5/4}\rangle = x_{602}$

604 $\beta_1\beta_{12}$

605 $\underline{2}\,\underline{5}\beta_1^3\gamma_2$

608 β_1^{16}

609 $\alpha_1 x_{602}$

$[\underline{2}\beta_1^2\beta_{11} + \alpha_1 x_{602}]b_{2,0}^2\beta_3$

610 $\beta_1\beta_3\beta_{10/5} = \langle \alpha_1, 5, x_{602}\rangle$

611 $[\alpha_1\beta_1\beta_{12}]\underline{b_{2,0}\beta_8}$

612 $[\beta_1^5\beta_9]\underline{\eta_8}$

614 $\underline{\beta}_{13}$

617 $x_{617} = \alpha_1\beta_3 b_{2,0}^2$

$\alpha_1\beta_1\beta_3\beta_{10/5}$

619 $[2\beta_1\beta_{12}]\underline{b_{2,0}u_3}$

620 $\alpha_1\eta_8 = \beta_4\beta_{10/5}$

621 $\alpha_1\beta_{13}$

622 $\beta_1^3\beta_2\beta_9$

627 $\underline{3}\beta_1\beta_{12}$

$\beta_1^5\gamma_2$

$[\alpha_1\beta_4\beta_{10/5}]\underline{b_{1,1}u_4}$

628 $\underline{5}\beta_1^4\gamma_2$

632 $\beta_1^3\beta_{11}$

634 $[\alpha_1\beta_1^5\gamma_2]\underline{b_{1,1}\gamma_2}$

□

The differential in the 609-stem is an Ext relation derived as follows. Since x_{602} is divisible in Ext by $\beta_{5/4}$ we have $d(h_{2,0}b_{1,1}^3) = \beta_1 x_{602}$, so $d(\alpha_1 h_{2,0}b_{1,1}^3) = \alpha_1\beta_1 x_{602}$. On the other hand, whenever $\alpha_1 x = \alpha_1 y = 0$, $\underline{2}xy = 0$, e.g., $\underline{2}(\beta_1\beta_6)^2 = \underline{2}\beta_1^3\beta_{11} = 0$, forcing the image of $b_{2,0}^2\beta_3$ to contain a nonzero multiple of $\underline{2}\beta_1^2\beta_{11}$. Similary

$$\underline{2}\beta_1^3\beta_k = 0 \quad \text{for all} \quad k \geq 2p+1. \tag{7.6.10}$$

In many cases (such as $k = 12$) inspection of $\underline{N}$ (7.6.4) shows $\underline{2}\beta_1^2\beta_k = 0$. To get the other term we compute modulo filtration 2 in our spectral sequence (7.1.16), i.e., mod β_{10}. Then we get $\langle h_{11}, h_{11}, b_{1,1}^2\rangle$ is killed by $b_{2,0}^2$ in $\mathrm{Ext}(BP_*/I_2)$, so $\beta_3 b_{2,0}^2$ kills $\langle \beta_3, h_{11}, h_{11}\rangle b_{1,1}^2$ and the coboundary of $\frac{v_2^5 v_1^2}{25 v_1^5}$ shows $\langle \beta_3, h_{11}, h_{11}\rangle = \beta_{5/4}\alpha_1$.

There is a nontrivial group extension in dimension 617 similar to the one in the 427-stem. We have

$$x_{617} = \left\langle \alpha_1, (\alpha_1\underline{2}\beta_1\beta_6), \begin{pmatrix} x_{602} \\ \beta_6 \end{pmatrix} \right\rangle$$

so

$$\begin{aligned} 5x_{617} &= \langle 5, \alpha_1, (\alpha_1\underline{2}\beta_1\beta_6)\rangle \begin{pmatrix} x_{602} \\ \beta_6 \end{pmatrix} \\ &= \langle 5, \alpha_1, \alpha_1\rangle x_{602} = \alpha_1 x_{602} \\ &= \alpha_2\langle \alpha_1, \beta_1^5, x_{404}\rangle \\ &= \langle \alpha_2, \alpha_1, \beta_1^5\rangle x_{404}. \end{aligned}$$

On the other hand

$$\begin{aligned}\beta_1^5\langle\alpha_1,\alpha_1,x_{412}\rangle &= \beta_1^5\left\langle \alpha_1,(\alpha_2\alpha_1\beta_2),\begin{pmatrix}x_{404}\\ \beta_7\end{pmatrix}\right\rangle\\ &= \langle\beta_1^5,\alpha_1,(\alpha_2\alpha_1\beta_2)\rangle\begin{pmatrix}x_{404}\\ \beta_7\end{pmatrix}\\ &= \langle\beta_1^5,\alpha_1,\alpha_2\rangle x_{404}\end{aligned}$$

so the result follows. We also have $\alpha_2 x_{602} = \alpha_1\langle\alpha_1,5,x_{602}\rangle$ so $\langle\alpha_1,5,x_{602}\rangle = \beta_1\beta_3\beta_{10/5}$.

In the 627-stem we have an Ext relation

$$\alpha_1\beta_4\beta_{10/5} = \alpha_1\beta_9\beta_{5/5} = 0.$$

7.6.5 (*Fourth installment*)

635 $\underline{2}\beta_4\beta_{10/5} = \langle\beta_9,\alpha_1,\underline{2}\beta_1^5\rangle$

636 $\underline{\beta_{5/5}\beta_{10/5}} = \langle\beta_1^3,\alpha_1\beta_1^2,\beta_{10/5}\rangle = x_{636}$

637 $\underline{2}\beta_1^3\beta_2\beta_9$

639 $\underline{4}\beta_1^{16}$

640 $[\beta_1 x_{602}]\underline{h_{2,0}b_{1,1}^3}$

642 $\beta_1^2\beta_{12}$

643 $\underline{2}\,\underline{5}\beta_1^4\gamma_2 = \beta_{5/4}\gamma_2$

$\alpha_1 x_{636}$

644 $\underline{\beta_{5/4}\beta_{10/5}}$

646 β_1^{17}

647 $[\underline{2}\beta_1^3\beta_{11}]\alpha_1 h_{2,0}b_{1,1}^3$

648 $[\beta_1^2\beta_3\beta_{10/5}]\underline{h_{2,0}b_{1,1}^2\beta_{5/4}}$

651 $\alpha_1\beta_{5/4}\beta_{10/5}$

652 $\beta_1\beta_{13}$

$\underline{\beta_{5/3}\beta_{10/5}} + \beta_1\beta_{13} = x_{652}$

655 $[\alpha_1\beta_1^2\beta_3\beta_{10/5}]\alpha_1 h_{2,0}b_{1,1}^2\beta_{5/4}$

$\beta_1 x_{617}$

658 $[\beta_1\beta_4\beta_{10/5}]\underline{h_{2,0}b_{2,0}\beta_8}$

659 $[\alpha_1 x_{652}]b_{2,0}\beta_9$

660 $\beta_1^4\beta_2\beta_9$

662 $\underline{\beta}_{14}$

665 $\underline{3}\beta_1^2\beta_{12}$

$\beta_1^6\gamma_2$

666 $\underline{5}\beta_1^5\gamma_2$

$[\alpha_1 h_{2,0}b_{2,0}\beta_8]\underline{h_{2,0}b_{2,0}u_3}$

$\underline{2}\beta_1^5\gamma_2$

667 $\underline{2}x_{652}$

669 $[\alpha_1\beta_{14}]\underline{u_9}$

670 $\beta_1^4\beta_{11}$

673 $[\underline{2}\beta_1\beta_4\beta_{10/5}]\alpha_1 h_{2,0}b_{2,0}u_3$

674 $[\beta_1 x_{636}]\underline{h_{2,0}b_{1,1}u_4}$

675 $\underline{2}\beta_1^4\beta_2\beta_9$

677 $\underline{2}\beta_{14}$

$\underline{4}\beta_1^{17}$

678 $\underline{\beta}_{15/5}$

680 $\beta_1^3\beta_{12}$

$\underline{2}\beta_1^6\gamma_2$

681 $[\alpha_1\beta_1 x_{636}]\alpha_1 h_{2,0}b_{1,1}u_4$

$[\underline{2}\,\underline{5}\beta_1^5\gamma_2]\underline{h_{2,0}b_{1,1}\gamma_2}$

682 $[\beta_1\beta_{5/4}\beta_{10/5}]\underline{h_{2,0}b_{1,1}\beta_{10/5}}$

684 $[\beta_1^{18}]\underline{\gamma_3}$

$\alpha_1\beta_1\beta_{13}$

685 $\alpha_1\beta_{15/5}$

686 $\underline{\beta}_{15/4}$

689 $\underline{3}\,\underline{5}\beta_1^6\gamma_2$

$[\alpha_1\beta_1\beta_{5/4}\beta_{10/5}]\alpha_1 h_{2,0}b_{1,1}\beta_{10/5}$

690 $\beta_1^2\beta_{13}$

$[\beta_1\beta_{5/3}\beta_{10/5}]\underline{h_{2,0}b_{1,1}\beta_{10/4}}$

692 $\alpha_1\gamma_3 = \langle \alpha_1, \beta_1^5, \beta_1^{13}\rangle = x_{692}$

693 $\alpha_1\beta_{15/4}$

$\underline{3}\beta_1^4\beta_{11}$

694 $\underline{\beta}_{15/3}$

697 $[\alpha_1\beta_1\beta_{5/3}\beta_{10/5}]\alpha_1 h_{2,0}b_{1,1}\beta_{10/4}$

698 $[\beta_1^5\beta_2\beta_9]\eta_1\beta_9$

700 $\beta_1\beta_{14}$

701 $\alpha_1\beta_{15/3}$

702 $\underline{\beta}_{15/2}$

703 $\underline{3}\beta_1^3\beta_{12}$

$\beta_1^7\gamma_2$

704 $\underline{5}\beta_1^6\gamma_2$

705 $[\underline{2}\beta_1^2\beta_{13}]\underline{h_{2,0}h_{11}b_{2,0}u_3}$

706 $[\alpha_1\eta_1\beta_9]\underline{b_{2,0}\eta_5}$

708 $[\beta_1^5\beta_{11}]\underline{\eta}_{10}$

709 $[\alpha_1\beta_{15/2}]\underline{\beta}_{15/1,2}$

710 $\underline{\beta_{15}}$

713 $\underline{3}\beta_1^2\beta_{13}$

$[\underline{2}\beta_1^5\beta_2\beta_9]\alpha_1 b_{2,0}\beta_5$

714 $\underline{\eta_1\gamma_2} = \langle \beta_1^4, \beta_1\beta_2, \gamma_2 + \underline{2}\beta_9\rangle = x_{714}$

715 $\underline{3}x_{692}$

$\underline{2}\beta_1\beta_{14}$

716 $\beta_1\beta_{15/5}$

$[\alpha_1\eta_{10} + \beta_1\beta_{15/5}]\underline{h_{2,0}u_9}$

717 $\underline{2}\beta_{15/2}$

$\alpha_1\beta_{15}$

718 $\beta_1^4\beta_{12}$

$\underline{2}\beta_1^7\gamma_2$

721 $\alpha_1 x_{714}$

723 $\alpha_1\beta_1\beta_{15/5}$

724 $\underline{5\gamma_3} = \langle \beta_1, 5, \beta_1, \beta_1^{17}\rangle = x_{724}$

$\alpha_1 h_{2,0}u_9 = \langle \beta_1^2, \beta_1^3\beta_{11}, \alpha_1, \alpha_1\rangle$

$= x'_{724}$

$[\beta_1\beta_{15/4}]\underline{h_{2,0}\beta_{15/5}}$

727 $\underline{3}\,\underline{5}\beta_1^6\gamma_2$

728 $\beta_1^3\beta_{13}$

730 $\beta_1 x_{692}$

731 $\underline{3}\beta_1^5\beta_{11}$

$[\alpha_1\beta_1\beta_{15/4}]\alpha_1 h_{2,0}\beta_{15/5}$

$[\alpha_1 x_{724}]\underline{h_{2,0}\gamma_3}$

732 $[\beta_1\beta_{15/3}]\underline{h_{2,0}\beta_{15/4}}$

738 $\beta_1^2\beta_{14}$

739 $\underline{2}x_{724}$

$[\alpha_1\beta_1\beta_{15/3}]\alpha_1 h_{2,0}\beta_{15/4}$

740 $[\beta_1\beta_{15/2}]\underline{h_{2,0}\beta_{15/3}}$

741 $\underline{3}\beta_1^4\beta_{12}$

$\beta_1^8\gamma_2$

742 $\underline{5}\beta_1^7\gamma_2$

748 $\beta_2\beta_{14}$

$[\beta_1\beta_{15}]\underline{h_{11}\beta_{15/1,2}}$

751 $\underline{3}\beta_1^3\beta_{13}$

752 $[\beta_1 x_{714}]\underline{b_{2,0}^2\eta_1}$

753 $[\underline{2}\beta_1^2\beta_{14}]\underline{h_{2,0}b_{2,0}\eta_5}$

$\underline{3}\beta_1 x_{692}$

754 $[\beta_1^2\beta_{15/5}]\underline{b_{2,0}\eta_6}$

755 $[\alpha_1\beta_2\beta_{14}]\underline{h_{2,0}\eta_{10}}$

$[\alpha_1\beta_1\beta_{15}]\alpha_1 h_{11}\beta_{15/1,2}$

756 $\underline{2}\beta_1^8\gamma_2$

$[\beta_1^5\beta_{12}]\underline{\eta_{11}}$

758 $\underline{\beta}_{16}$

759 $[\alpha_1\beta_1 x_{714}]\alpha_1 b_{2,0}^2\eta_1$

761 $[\alpha_1\beta_1^2\beta_{15/5}]\alpha_1 b_{2,0}\eta_6$

$\underline{b_{2,0}\beta_2\gamma_2} = \langle \alpha_1\beta_1, \beta_6, \gamma_2\rangle = x_{761}$

$\underline{3}\beta_1^2\beta_{14}$

762 $\beta_1 x_{724}$

763 $\underline{2}\beta_2\beta_{14}$

$\beta_1 x'_{724}$

764 $\alpha_1\eta_{11} = \beta_2\beta_{15/5}$

765 $\alpha_1\beta_{16}$

$\underline{3}\,\underline{5}\beta_1^7\gamma_2$

766 $\beta_1^4\beta_{13}$

768 $\alpha_1 x_{761} = \gamma_2 \underline{2}\beta_1\beta_6$

$\beta_1^2 x_{692}$

769 $\alpha_1\beta_1 x'_{724}$

771 $\underline{\beta_2\gamma_3} = \langle \beta_2, \beta_1^5, \beta_1^{13}\rangle = x_{771}$

776 $\beta_1^3\beta_{14}$

777 $\underline{2}\beta_1 x_{724}$

778 $\alpha_1 x_{771}$

779 $\beta_1^9\gamma_2$

$\underline{2}\beta_2\beta_{15/5}$

780 $\underline{5}\beta_1^8\gamma_2$

786 $\beta_1\beta_2\beta_{14}$

789 $\underline{3}\beta_1^4\beta_{13}$

791 $[\underline{3}\beta_1^3 x_{692}]\beta_{5/5}^4$

796 $\beta_1\beta_{16}$

799 $\underline{3}\beta_1^3\beta_{14}$

$\beta_1 x_{761}$

$[\underline{4}\beta_1^2 x_{692}+?]\underline{b_{2,0}^3\beta_2}$

800 $\beta_1^2 x_{724}$

$\beta_1^2 x'_{724}$

801 $[\underline{2}\beta_1\beta_2\beta_{14}]\underline{b_{2,0}^2\beta_7}$

802 $[\beta_1\beta_2\beta_{15/5}]\underline{b_{2,0}\eta_7}$

803 $\underline{3}\,\underline{5}\beta_1^8\gamma_2$

$[\alpha_1\beta_1\beta_{16}]\underline{b_{2,0}\beta_{12}}$

804 $[\beta_1^5\beta_{13}]\underline{\eta_{12}}$

806 $\underline{\beta_{17}}$

$[\beta_1^3 x_{692} + \alpha_1\beta_1 x_{761}]\alpha_1 b_{2,0}^3\beta_2$

$\alpha_1\beta_1 x_{761}$

807 $\alpha_1\beta_1^2 x'_{724}$

809 $[\underline{3}\beta_1\beta_2\beta_{14}]\underline{b_{2,0}^2 u_2}$

794 $\underline{2}\beta_1^9\gamma_2\ \beta_1 x_{771}$

810 $\alpha_1 b_{2,0}\eta_7 =$

$\langle \alpha_1, \alpha_1, \beta_1\beta_{12}, \beta_1^5\rangle = \beta_2 x'_{724}$

811 $\underline{2}\beta_1\beta_{16}$

812 $\alpha_1\eta_{12} = \beta_3\beta_{15/5}$

813 $\alpha_1\beta_{17}$

814 $\beta_1^4\beta_{14}$

815 $\underline{2}\beta_1^2 x_{724}$

816 $\alpha_1\beta_1 x_{771}$

817 $\beta_1^{10}\gamma_2$

$\underline{4}\beta_1\beta_2\beta_{14}$

$[\underline{2}\beta_1\beta_2\beta_{15/5}]\underline{b_{1,1}b_{2,0}u_3}$ □

For the relation in the 643-stem we have

$$\begin{aligned}
\beta_{p/p-1} &= \langle \alpha_1\beta_1^{p-1}, \beta_1, p, \alpha_1\rangle \quad \text{and}\\
\underline{2}\,\underline{p}\gamma_2 &= \langle \alpha_1, \alpha_1\beta_1, p, \gamma_2\rangle \quad \text{so}\\
\beta_1^{p-1}\underline{2}\,\underline{p}\gamma_2 &= \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_2\rangle\\
&= \alpha_1\langle \beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_2\rangle\\
&= \alpha_1\langle \alpha_1\beta_1^{p-1}, \beta_1, p, \gamma_2\rangle\\
&= \alpha_1\langle \alpha_1\beta_1^{p-1}, \beta_1, p, \alpha_1\rangle\gamma_2\\
&= \beta_{p/p-1}\gamma_2.
\end{aligned}$$

This generalizes immediately to

7.6.11. Proposition. *Let x be an element satisfying $px = 0$, $\langle \alpha_1\beta_1, p, x\rangle = 0$, and $\alpha_1 x \neq 0$. Then $\beta_{p/p-1}x = \beta_1^{p-1}\underline{2}\,\underline{p}x$.* □

For the differentials in dimensions 666 and 673 it suffices to show $\beta_1^2\underline{2}\beta_4\beta_{10/5} = 0$. We have $\beta_4\beta_{10/5} = \beta_9\beta_{5/5} = \langle\beta_9, \alpha_1, \beta_1^5\rangle$ so $\underline{2}\beta_4\beta_{10/5} = \langle\beta_9, \alpha_1, \underline{2}\beta_1^5\rangle$. Then

$$\begin{aligned}\beta_1^2\underline{2}\beta_4\beta_{10/5} &= \langle\beta_1^2\beta_9, \alpha_1, \underline{2}\beta_1^5\rangle \\ &= \langle\alpha_1\beta_1\beta_6, \beta_4, \underline{2}\beta_1^5\rangle = 0.\end{aligned}$$

The differential on γ_3 is explained in 7.6.1. Recall that the key point was that $\alpha_1\beta_{5/5}^4$ in Ext is a linear combination of the three elements $\underline{3}\beta_1^3k_{14}$, β_1x_{761} and $\beta_1^3\gamma_3$. In our setting this relation is given by the differential on $b_{2,0}^3\beta_2$, whose target is some linear combination of the four elements (including $\alpha_1\beta_{5/5}^4$) in question. This target is difficult to compute precisely, but it suffices to show that it includes a nontrivial multiple of $\alpha_1\beta_{5/5}^4$. Knowing then that $\underline{3}\beta_1^3\beta_{14}$ and β_1x_{761} are permanent cycles and $\alpha_1\beta_{5/5}^4$ is not, we can conclude that the linear combination also includes $\beta_1^3\gamma_3$ and that the latter is not a permanent cycle in the Adams–Novikov spectral sequence.

To make this calculation we map to the spectral sequence going from

$$\mathrm{Ext}_{P(1)}(\mathbf{Z}/(p), P(0))$$

(this is the R of 7.5.1 and 7.4.13) to $\mathrm{Ext}_{P(1)}(\mathbf{Z}/(p), \mathbf{Z}/(p))$. The elements $\underline{3}\beta_1^3\beta_{14}$, γ_3 and x_{761} all have trivial images, while $b_{2,0}^3\beta_2$ and $\alpha_1\beta_{5/5}^4$ do not, and it suffices to show that $\alpha_1\beta_{5/5}^4 = h_{10}b_{1,1}^4$ vanishes in $\mathrm{Ext}_{P(1)}$. $h_{11}b_{1,1}$ is killed by $b_{2,0}$, so $\langle b_{1,1}^3, h_{11}, h_{11}, h_{11}\rangle$ is killed by $b_{2,0}^3$ so we have

$$\begin{aligned}0 &= \langle b_{1,1}^3, h_{11}, h_{11}, h_{11}\rangle\langle h_{11}, h_{11}, h_{10}\rangle \\ &= b_{1,1}^3\langle h_{11}, h_{11}, h_{11}, \langle h_{11}, h_{11}, h_{10}\rangle\rangle \\ &= b_{1,1}^3\langle h_{11}, h_{11}, h_{11}, h_{11}, h_{11}\rangle h_{10} \\ &= b_{1,1}^4h_{10}.\end{aligned}$$

Given this situation the target of the differential from $\beta_{5/5}$, $\underline{4}\beta_1^{20}$, is the same as $\underline{3}\beta_1^2x_{692}$, and $\alpha_1\beta_{5/5}^4$ is $\underline{4}\beta_1^2x_{692}$ which accounts for the indicated differentials in dimensions 791 and 799.

The differential in the 752-stem can be recovered from the corresponding spectral sequence for Ext_P. The images of η_1 and γ_2 are the Massey products $\langle h_{11}, \underline{h}, \underline{b}\rangle$ and $\langle h_{12}, \underline{h}, \underline{b}\rangle$ where $\underline{h}$ and $\underline{b}$ denote the matrices

$$\begin{pmatrix} h_{11} & h_{12}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{1,1} \\ b_{1,0}\end{pmatrix},$$

respectively. Then we have $\beta_1\eta_1\gamma_2 = \langle h_{12}b_{1,0}\eta_1, \underline{h}, \underline{b}\rangle = \langle h_{11}b_{1,1}\eta_1, \underline{h}, \underline{b}\rangle = 0$ since $h_{11}\eta_1 = 0$.

7.6.5 (*Fifth installment*)

818 $\beta_1^{10}\beta_{10/5}$

824 $\beta_1^2\beta_2\beta_{14}$

825 $\alpha_1b_{1,1}b_{2,0}u_3 = \underline{2}\beta_2x'_{724}$

826 $\underline{b_{1,1}^4}u_4 = \langle\alpha_1, \beta_1^5, \alpha_1\beta_4, \beta_{10/5}\rangle = x_{826}$

827 $\underline{2}\beta_3\beta_{15/5}$

832 $[\underline{2}\beta_1^{10}\gamma_2]\underline{b_{1,1}^2\gamma_2}$

833 α_1x_{826}

834 $\beta_1^2\beta_{16}$

$\underline{b_{1,1}^2\beta_{10/5}} = \langle\beta_1^4, \underline{2}\beta_1^6, \beta_{10/5}\rangle = x_{834}$

837 $\underline{3}\beta_1^4\beta_{14}$

$\beta_1^2x_{761}$

838 $\beta_1^3 x_{724}$

$[\beta_1^3 x'_{724}]\underline{h_{2,0}b_{1,1}^4}$

840 $\underline{3}\beta_1^{10}\gamma_2$

841 $\alpha_1 x_{834}$

$\underline{3}\beta^{10}\beta_{10/5}$

842 $\underline{b_{1,1}^2\beta_{10/4}} = \langle \underline{2}\beta_1^9, \beta_1, \beta_{10/4}\rangle = x_{842}$

844 $\alpha_1\beta_1^2 x_{761}$

$\beta_1\beta_{17}$

845 $[\alpha_1\beta_1^3 x'_{724}]\alpha_1 h_{2,0}b_{1,1}^4$

847 $\beta_1^2 x_{771}$

848 $[\beta_1 x_{810}]\underline{h_{2,0}b_{2,0}^2\beta_7}$

849 $[\underline{2}\beta_1^2\beta_{16} + \alpha_1 x_{842}]b_{2,0}^2\beta_8$

$2\beta_1^2\beta_{16}$

850 $\beta_1\beta_3\beta_{15/3}$

851 $[\alpha_1\beta_1\beta_{17}]\underline{b_{2,0}\beta_{13}}$

852 $[\beta_1^5\beta_{14}]\underline{\eta_{13}}$

853 $\underline{2}\beta_1^3 x_{724}$

854 $\underline{\beta_{18}}$

$\alpha_1\beta_1^2 x_{771}$

855 $\beta_1^{11}\gamma_2$

$\underline{4}\beta_1^2\beta_2\beta_{14}$

856 $[\underline{4}\,\underline{2}x_{810}]\underline{h_{2,0}b_{2,0}^2u_2}$

856 $\beta_1^{11}\beta_{10/5}$

857 $\alpha_1 b_{2,0}^2\beta_8 = x_{857}$

859 $[\underline{2}\beta_1\beta_{17}]\underline{b_{2,0}u_8}$

860 $\alpha_1\eta_{13} = \beta_4\beta_{15/5}$

861 $\alpha_1\beta_{18}$

862 $\beta_1^3\beta_2\beta_{14}$

853 $[\underline{2}\beta_1 x_{810}]\alpha_1 h_{2,0}b_{2,0}^2u_2$

864 $[\beta_1 x_{826}]\underline{h_{2,0}b_{1,1}b_{2,0}u_3}$

865 $\underline{2}\beta_1\beta_2\beta_{15/5}$

867 $\underline{3}\beta_1\beta_{17}$ □

For the differential in the 838-stem we use the method of 7.6.6. We have maps $f\colon \Sigma^{190}T(0)_1 \to S^0$ and $g\colon \Sigma^{609}T(0)_4 \to S^0$ where f is β_1^5 on the bottom cell, and g is $\alpha_1 x_{602}$ on the bottom cell and x_{617} on the second cell. The smash product vanishes on the bottom cell so we have a map $\Sigma^{807}T(0)_4 \to S^0$ which is $\beta_1^5 x_{617} + 2\beta_1^5 x_{602}$ on the bottom cell. The second term vanishes because $\beta_1^5 x_{602} \in \pi_{792} = 0$. We have

$$x_{617} = \left\langle \alpha_1, (\underline{2}\beta_1\beta_{11}\alpha_1), \begin{pmatrix}\beta_1\\ x_{602}\end{pmatrix}\right\rangle.$$

A routine calculation gives $\beta_1 x_{617} = \underline{3}\beta_1^3\beta_{11}$ and $\beta_1^3 x_{617} = \alpha_1 x'_{724}$. Our map gives $0 = \underline{4}\beta_1^5 x_{617} = \beta_1^3 x'_{724}$, hence the desired differential.

We use a similar argument in the 848-stem. We start with the maps

$$\Sigma^{316}T(0)_1 \to S^0 \qquad \text{and} \qquad \Sigma^{531}T(0)_4 \to S^0$$

carrying $\beta_1\beta_6$, and $\alpha_1\beta_2\beta_{10/5}$ on the bottom cells. The resulting relation is $\beta_1^2 x_{810} = 0$. From 7.6.4 we see that $\mathbf{N}$ is vacuous in dimensions 887 and 856, so the indicated differential is the only one which can give this relation.

The argument in dimension 849 is similar to that in dimension 609.

In dimension 864 we use 7.6.6 again starting with the extensions of β_1^5 and $\underline{2}\beta_4\beta_{10/5}$ to $T(0)_1$ and $T(0)_4$.

7.6.5 (*Sixth installment*)

868 $\underline{b_{1,1}u_9} = x_{868}$

871 $[\alpha_1\beta_1 x_{826}]\alpha_1 h_{2,0}b_{1,1}b_{2,0}u_3$

872 $\beta_1^3\beta_{16}$

$[\beta_1 x_{834}]\underline{h_{2,0}b_{1,1}^2u_4}$

875 $\alpha_1 x_{868}$

$\underline{2}\beta_4\beta_{15/5}$

$\beta_1^3 x_{761} = \beta_{10/5}\gamma_2$

876 $\underline{\beta_{10/5}^2}$

$\beta_1^4 x_{724}$

878 $\underline{3}\beta_1^{11}\gamma_2$

879 $[\alpha_1\beta_1 x_{834}]\alpha_1 h_{2,0} b_{1,1}^2 u_4$

$[\underline{3}\beta_1^{11}\beta_{10/5}]\underline{h_{2,0}b_{1,1}^2\gamma_2}$

880 $[\beta_1 x_{842}]\underline{h_{2,0}b_{1,1}^2\gamma_2}$

882 $[\alpha_1\beta_1^3 x_{761}]\underline{b_{1,1}\gamma_3}$

$\beta_1^2\beta_{17}$

883 $\alpha_1\beta_{10/5}^2$

884 $\underline{\beta_{5/4}\beta_{15/5}} = \beta_{10/4}\beta_{10/5}$

885 $\beta_1^3 x_{771}$

887 $\underline{4}\beta_1^{11}\beta_{10/5}$

$[\underline{2}\beta_1^3\beta_{16}]\alpha_1 h_{2,0} b_{1,1}^2\gamma_2$

888 $[\beta_1^2\beta_3\beta_{15/5}]\underline{h_{2,0}h_{11}^2\beta_{10/4}}$

890 $\alpha_1 b_{1,1}\gamma_3 = \langle\alpha_1, \alpha_1, \beta_1^{10}, \beta_1^{13}\rangle\underline{2}\beta_1^3 x_{761}$

891 $\underline{2}\beta_1^4 x_{724}$

$\alpha_1\beta_{5/4}\beta_{15/5}$

892 $\beta_1\beta_{18}$

$\alpha_1\beta_1^3 x_{771}$

$\underline{\beta_{10/4}^2} + \beta_1\beta_{18} = x_{892}$

893 $\beta_1^{12}\gamma_2$

$\underline{4}\beta_1^3\beta_2\beta_{14}$

894 $\beta_1^{12}\beta_{10/5}$

895 $[\beta_1 x_{857}]\underline{h_{2,0}h_{11}b_{2,0}^2 u_2}$

896 $[\underline{4}\,\underline{2}\beta_1\beta_3\beta_{15/5}]\underline{h_{11}b_{2,0}^2 u_3}$

898 $[\beta_1\beta_4\beta_{15/5}]\underline{h_{2,0}b_{1,0}\beta_{13}}$

899 $[\alpha_1 x_{892}]\underline{h_{2,0}\beta_{14}}$

$\underline{\alpha_1\beta_1\beta_{18}}$

900 $\beta_1^4\beta_2\beta_{14}$

902 $\underline{\beta_{19}}$

903 $\underline{4}\beta_1^3\beta_{16}$

$[\underline{2}\beta_1^2\beta_3\beta_{15/5}]\alpha_1 h_{11}b_{2,0}^2 u_3$

905 $\underline{3}\beta_1^2\beta_{17}$

906 $[\alpha_1 h_{2,0}b_{2,0}\beta_{13}]\underline{h_{2,0}b_{2,0}u_8}$

$\beta_1 x_{868}$

907 $\underline{2}x_{892}$

909 $[\alpha_1\beta_{19}]u_{14}$

910 $\beta_1^4\beta_{16}$

913 $\alpha_1\beta_1 x_{868}$

$\beta_1^4 x_{761}$

$[\underline{2}\beta_1\beta_4\beta_{15/5}]\alpha_1 h_{2,0}b_{2,0}u_8$

914 $\beta_1^5 x_{724}$

$[\beta_1\beta_{10/5}^2]\underline{h_{2,0}b_{1,1}u_9}$

916 $\underline{3}\beta_1^{12}\gamma_2$

917 $\underline{2}\beta_{19}$

918 $\underline{\beta_{20/5}}$

920 $\beta_1^3\beta_{17}$

921 $[\alpha_1\beta_1\beta_{10/5}^2]\alpha_1\underline{h_{2,0}b_{1,1}u_9}$

922 $[\beta_1\beta_{5/4}\beta_{15/5}]\underline{h_{2,0}b_{1,1}\beta_{15/5}}$

923 $\beta_1^4 x_{771}$

925 $\alpha_1\beta_{20/5}$

$\underline{4}\beta_1^{12}\beta_{10/5}$

926 $\underline{\beta_{20/4}}$

928 $\underline{2}\beta_1^4 x_{761}$

929 $[\underline{2}\beta_1^5 x_{724}]\underline{h_{2,0}b_{1,1}\gamma_3}$

$[\alpha_1\beta_1\beta_{5/4}\beta_{15/5}]\alpha_1 h_{2,0}b_{1,1}\beta_{15/5}$

930 $\alpha_1\beta_1^4 x_{771}$

$\beta_1^2\beta_{18}$

$[\beta_1 x_{892}]\underline{h_{2,0}b_{1,1}\beta_{15/4}}$

931 $\beta_1^{13}\gamma_2$

$\underline{4}\beta_1^4\beta_2\beta_{14}$

932 $\beta_1^{13}\beta_{10/5}$?

933 $\underline{\gamma_4}$?

$\alpha_1\beta_{20/4}$

934 $\underline{\beta_{20/3}}$

937 $\underline{3}\beta_1^5 x_{724}$

$[\alpha_1\beta_1\beta_{10/4}^2]\alpha_1 h_{2,0}b_{1,1}\beta_{15/4}$

938 $[\beta_1^5\beta_2\beta_{14}]\underline{\eta_1\beta_{14}}$

940 $\alpha_1\gamma_4$

$\beta_1\beta_{19}$

941 $\alpha_1\beta_{20/3}$

$\underline{4}\beta_1^4\beta_{16}$

942 $\underline{\beta_{20/2}}$

943 $[\underline{3}\beta_1^3\beta_{17}]\underline{h_{2,0}h_{11}b_{2,0}^2u_3}$

944 $[\beta_1^2x_{868}]\underline{b_{2,0}^2\eta_5}$

945 $[\underline{2}\beta_1^2\beta_{18}]\underline{h_{2,0}h_{11}b_{2,0}u_8}$

946 $[\beta_2\beta_4\beta_{15/5}]\underline{b_{2,0}\eta_{10}}$

948 $[\beta_1^5\beta_{16}]\underline{\eta_{15}}$

949 $[\alpha_1\beta_{20/2}]\underline{\beta_{20/1,2}}$

950 $\underline{\beta_{20}}$

951 $\beta_1^5x_{761}$

$\underline{4}\beta_1^3\beta_{17}$

$[\alpha_1\beta_1^2x_{868}]\alpha_1b_{2,0}^2\eta_5$

952 $\beta_1^6x_{724}$

$\underline{b_{2,0}\eta_1\gamma_2} = \langle\beta_1, \beta_1\beta_{10/5} + \underline{5}\gamma_2, \gamma_2\rangle$

$= x_{952}$

953 $\underline{3}\beta_1^2\beta_{18}$

954 $\underline{3}\beta_1^{13}\gamma_2$

$\underline{3}\,\underline{4}\beta_1^4\beta_2\beta_{14} +$

$\langle\alpha_1, \alpha_1, \beta_1\beta_2\beta_{14}, \beta_1^4\rangle = x_{954}$

955 $\underline{2}\beta_1\beta_{19}$

956 $\beta_1\beta_{20/5}$

$[\alpha_1\eta_{15} + \beta_1\beta_{20/5}]\underline{h_{2,0}u_{14}}$

957 $\underline{2}\beta_{20/2}$

$\alpha_1\beta_{20}$

958 $\beta_1^4\beta_{17}$

959 α_1x_{952}

961 $[\beta_1^5x_{771}]\underline{h_{11}b_{2,0}\gamma_3}$

963 $\alpha_1\beta_1\beta_{20/5}$

$\underline{4}\beta_1^{13}\beta_{10/5}$

964 $\alpha_1h_{2,0}u_{14} = \underline{3}\,\underline{4}\beta_1^4\beta_{16} = x_{964}$

$[\beta_1\beta_{20/4}]\underline{h_{2,0}\beta_{20/5}}$

966 $\underline{2}\beta_1^5x_{761}$

968 $\beta_1^3\beta_{18}$

$[\alpha_1\beta_1^5x_{771}]\alpha_1h_{11}b_{2,0}\gamma_3$

969 $\beta_1^{14}\gamma_2$

$\underline{2}x_{954}$

970 $\beta_1^{14}\beta_{10/5}$?

971 $\beta_1\gamma_4$?

$[\alpha_1\beta_1\beta_{20/4}]\alpha_1h_{2,0}\beta_{20/5}$

972 $\underline{5\gamma_4}$

$[\beta_1\beta_{20/3}]\underline{h_{2,0}\beta_{20/4}}$

975 $\underline{3}\beta_1^5x_{724}$

978 $\alpha_1\beta_1\gamma_4$

$\beta_1^2\beta_{19}$

979 $\underline{2}x_{964}$

$[\alpha_1\underline{5}\gamma_4]\underline{h_{2,0}\gamma_4}$

$[\alpha_1\beta_1\beta_{20/3}]\alpha_1h_{2,0}\beta_{20/4}$

980 $[\beta_1\beta_{20/2}]\underline{h_{2,0}\beta_{20/3}}$

987 $\underline{2}\cdot\underline{5}\gamma_4$

988 $\beta_2\beta_{19}$

$[\beta_1\beta_{20}]\underline{h_{11}\beta_{20/1,2}}$

989 $\beta_1^6x_{761}$

$\underline{4}\beta_1^4\beta_{17}$

990 $\beta_1^7x_{724}$

$\beta_1x_{952} = \beta_{5/5}^5$

991 $[\underline{3}\beta_1^3\beta_{18}]\underline{h_{2,0}b_{2,0}^2\eta_5}$

992 $\underline{3}\beta_1^{14}\gamma_2$

$[\beta_1x_{954}]\underline{b_{2,0}^2\eta_6}$

993 $[\underline{2}\beta_1^2\beta_9]h_{2,0}b_{2,0}\eta_{10}$

994 $[\beta_1^2\beta_{20/5}]\underline{b_{2,0}\eta_{11}}$

995 $[\alpha_1\beta_2\beta_{19}]\underline{h_{2,0}\eta_{15}}$

$[\alpha_1\beta_1\beta_{20}]\alpha_1h_{11}\beta_{20/1,2}$

996 $[\beta_1^5\beta_{17}]\underline{\eta_{16}}$

997 $[\alpha_1\beta_1x_{952}]\underline{\beta_{25/25}}$

998 $\underline{\beta_{21}}$

999 $\underline{b_{2,0}^2\beta_2\gamma_2} = \langle\beta_1\beta_2, \gamma_1, \gamma_2\rangle = x_{999}$

$[\underline{4}\beta_1^3\beta_{18}]\underline{b_{2,0}^3u_1}$

1000 $[\alpha_1\beta_1^2\beta_{20/5}]\alpha_1b_{2,0}\eta_{11}$ $\square$

The element x_{868} is constracted as follows. There is a commutative diagram

$$\begin{array}{ccc} S^{860} & \xrightarrow{f} & \Sigma^{837}T(0)_2 \\ \downarrow{\scriptstyle \beta_q} & & \downarrow{\scriptstyle g} \\ S^{438} & \xrightarrow{\beta_{10/5}} & S^0 \end{array}$$

where the cofiber of f is $\Sigma^{837}T(0)_3$ and g is an extension of $\underline{3}\beta_1^4\beta_{14}$. Both f and β_9 extend to $\Sigma^{860}T(0)_1$. The difference of the composite extensions of $\beta_{10/5}\beta_9$ and gf gives x_{868} on the top cell. In other words x_{868} iis the Toda bracket for

$$S^{867} \xrightarrow{\alpha_1} S^{860} \to S^{438} \vee \Sigma^{837}T(0)_2 \to S^0.$$

We will see below that $\beta_1^2 g = 0$ and $\beta_1\beta_{10/5}\beta_9 = 0$ so it follows that $\beta_1^2 x_{868}$ is divisible by α_1 and hence trivial.

For the relation in dimension 875 we have $x_{761} = \langle \alpha_1\beta_1, \beta_6, \gamma_2\rangle$ and $\beta_1\beta_{10/5} = \langle \beta_1^4, \alpha_1\beta_1, \beta_6\rangle$ so $\beta_1^4 x_{761} = \gamma_2\beta_1\beta_{10/5}$.

For dimension 879 we have, using 7.6.11,

$$\begin{aligned} \gamma_2 x_{404} &= \gamma_2\langle \beta_{5/4}, \beta_1, \alpha_1\beta_1^4\rangle = \langle \gamma_2\beta_{5/4}, \beta_1, \alpha_1\beta_1^4\rangle \\ &= \langle \underline{2}\cdot\underline{5}\gamma_2\beta_1^4, \beta_1, \alpha_1\beta_1^4\rangle = \underline{3}\cdot\underline{5}\beta_1\gamma_2 = \underline{3}\beta_1^{10}\beta_{10/5} \end{aligned}$$

so

$$\underline{3}\beta_1^{11}\beta_{10/5} = \gamma_2\beta_1 x_{404} = 0.$$

In dimension 888 we have

$$\beta_1^2\beta_3\beta_{15/5} = \beta_1^2\beta_8\beta_{10/5} = \beta_8\beta_1\underline{5}\gamma_2 = \beta_2\beta_7\underline{5}\gamma_2 = 0.$$

For the 896 stem we have

$$\beta_1\underline{4}\,\underline{2}\beta_1\beta_3\beta_{15/5} = \beta_1\beta_2\,\underline{42}\beta_2\beta_{15/5} = 0,$$

which (by inspection 7.6.4) implies $\underline{4}\,\underline{2}\beta_1\beta_3\beta_{15/5} = 0$.

We are not sure about γ_4. A possible approach to it is this. Extrapolating 7.6.4 slightly we see that $\mathrm{Ext}^{7,1016}$ has two generators, $\beta_1^2\gamma_4$ and $\langle \gamma_3, \gamma_1, \beta_3\rangle$. The latter supports a differential hitting $\beta_1^{15}\beta_{10/5} = \langle \beta_1^{18}, \gamma_1, \beta_2\rangle$. The same Ext group contains $\langle \gamma_2, \gamma_2, \beta_3\rangle$, which is a permanent cycle. Hence if it is nonzero it is neither $\beta_1^2\gamma_4$, in which case γ_4 is a permanent cycle, or $\beta_1^2\gamma_4 + \langle \gamma_3, \gamma_1, \beta_3\rangle$, in which case $d_{25}(\gamma_4) = \beta_1^{13}\beta_{10/5}$.

In the 992-stem we have $\beta_1 x_{954} = \beta_4 x_{810}$ so $\beta_1^2 x_{954} = \beta_1\beta_4 x_{810} = 0$. Extrapolating the pattern in 7.6.4 we find that the only element in the appropriate dimension is $b_{1,1}^3\gamma_2$, which kills $\underline{3}\beta_1^{15}\gamma_2$.