CHAPTER 6

Morava Stabilizer Algebras

In this chapter we develop the theory which is the mainspring of the chromatic spectral sequence. Let $K(n)_* = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ have the BP_*-module structure obtained by sending all v_i, with $i \neq n$ to 0. Then define $\Sigma(n)$ to be the Hopf algebra $K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*$. We will describe this explicitly as a $K(n)_*$-algebra below. Its relevance to the Adams–Novikov spectral sequence is the isomorphism (6.1.1)

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n) \cong \mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*),$$

which is input needed for the chromatic spectral sequence machinery described in Section 5.1. In combination with 6.2.4, this is the result promised in 1.4.9. Since $\Sigma(n)$ is much smaller than $BP_*(BP)$, this result is a great computational aid. We will prove it along with some generalizations in Section 1, following Miller and Ravenel [**5**] and Morava [**2**].

In Section 2 we study $\Sigma(n)$, the nth Morava stabilizer algebra. We will show (6.2.5) that it is closely related to the $\mathbf{Z}/(p)$-group algebra of a pro-p-group S_n (6.2.3 and 6.2.4). S_n is the strict automorphism group [i.e., the group of automorphisms $f(x)$ having leading term x] of the height n formal group law F_n (see A2.2.17 for a description of the corresponding endomorphism ring). We use general theorems from the cohomology of profinite groups to show S_n is either p-periodic (if $(p-1) \mid n$) or has cohomological dimension n^2 (6.2.10).

In Section 3 we study this cohomology in more detail. The filtration of 4.3.24 leads to a May spectral sequence studied in 6.3.3 and 6.3.4. Then we compute H^1 (6.3.12) and H^2 (6.3.14) for all n and p. The section concludes with computations of the full cohomology for $n = 1$ (6.3.21), $n = 2$ and $p > 3$ (6.3.22), $n = 2$ and $p = 3$ (6.3.24), $n = 2$ and $p = 2$ (6.3.27), and $n = 3$, $p > 3$ (6.3.32).

The last two sections concern applications of this theory. In Section 4 we consider certain elements β_{p^i/p^i} in $\mathrm{Ext}^2(BP_*)$ for $p > 2$ analogous to the Kervaire invariant elements $\beta_{2^i/2^i}$ for $p = 2$. We show (6.4.1) that these elements are *not* permanent cycles in the Adams–Novikov spectral sequence. A crucial step in the proof uses the fact that S_{p-1} has a subgroup of order p to detect a lot of elements in Ext. Theorem 6.4.1 provides a test that must be passed by any program to prove the Kervaire invariant conjecture: it must not generalize to odd primes!

In Section 5 we construct ring spectra $T(m)$ satisfying $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$ as comodules. The algebraic properties of these spectra will be exploited in the next chapter. We will show (6.5.5, 6.5.6, 6.5.11, and 6.5.12) that its Adams–Novikov E_2-term has nice properties.

1. The Change-of-Rings Isomorphism

Our first objective is to prove

6.1.1. THEOREM (Miller and Ravenel [**5**]). *Let M be a $BP_*(BP)$-comodule annihilated by $I_n = (p, v_1, \dots, v_{n-1})$, and let $\overline{M} = M \otimes_{BP_*} K(n)_*$. Then there is a natural isomorphism*

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}M) = \mathrm{Ext}_{\Sigma(n)}(K(n)_*, \overline{M}).$$

Here $v_n^{-1}M$ denotes $v_n^{-1}BP_* \otimes_{BP_*} M$, which is a comodule (even though $v_n^{-1}BP_*$ is not) by 5.1.6. □

This result can be generalized in two ways. Let

$$E(n)_* = v_n^{-1}BP_*/(v_{n+i} \colon i > 0)$$

and

$$E(n)_*(E(n)) = E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*.$$

It can be shown, using the exact functor theorem of Landweber [**3**], that $E(n)_* \otimes_{BP_*} BP_*(X)$ is a homology theory on X represented by a spectrum $E(n)$ with $\pi_*(E(n)) = E(n)_*$, and with $E(n)_*(E(n))$ being the object defined above. We can generalize 6.1.1 by replacing $\Sigma(n)$ with $E(n)_*(E(n))$ and relaxing the hypothesis on M to the condition that it be I_n-nil, i.e., that each element (but not necessarily the entire comodule) be annihilated by some power of I_n. For example, N^n of Section 5.1 is I_n-nil. Then we have

6.1.2. THEOREM (Miller and Ravenel [**5**]). *Let M be I_n-nil and let $\overline{M} = M \otimes_{BP_*} E(n)_*$. Then there is a natural isomorphism*

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}M) = \mathrm{Ext}_{E(n)_*(E(n))}(E(n)_*, \overline{M}).$$

There is another variation due to Morava [**2**]. Regard BP_* as a $\mathbf{Z}/2(p^n-1)$-graded object and consider the homomorphism $\theta \colon BP_* \to \mathbf{Z}/(p)$ given by $\theta(v_n) = 1$ and $\theta(v_i) = 0$ for $i \neq n$. Let $I \subset BP_*$ be $\ker\theta$ and let V_θ and VT_θ denote the I-adic completions of BP_* and $BP_*(BP)$. Let $E_\theta = V_\theta(v_{n+i} \colon i > 0)$ and $EH_\theta = E_\theta \otimes_{V_\theta} VT_\theta \otimes_{V_\theta} E_\theta$.

6.1.3. THEOREM (Morava [**2**]). *With notation as above there is a natural isomorphism*

$$\mathrm{Ext}_{VT_\theta}(V_\theta, M) \cong \mathrm{Ext}_{EH_\theta}(E_\theta, \overline{M})$$

where M is a VT_θ-comodule and $\overline{M} = M \otimes_{V_\theta} E_\theta$. □

Of these three results only 6.1.1 is relevant to our purposes so we will not prove the others in detail. However, Morava's proof is more illuminating than that of Miller and Ravenel [**5**] so we will sketch it first.

Morava's argument rests on careful analysis of the functors represented by the Hopf algebroids VT_θ and EH_θ. First we need some general nonsense.

Recall that a groupoid is a small category in which every morphism is invertible. Recall that a Hopf algebroid (A, Γ) over K is a cogroupoid object in the category of commutative K-algebras; i.e., it represents a covariant groupoid-valued functor. Let $\alpha, \beta \colon \mathbf{G} \to \mathbf{H}$ be maps (functors) from the groupoid $\mathbf{G}$ to the groupoid $\mathbf{H}$. Since $\mathbf{G}$ is a category it has a set of objects, $Ob(\mathbf{G})$, and a set of morphisms, $\mathrm{Mor}(\mathbf{G})$, and similarly for $\mathbf{H}$.

6.1.4. Definition. *The functors* $\alpha, \beta\colon G \to H$ *are* equivalent *if there is a map* $\theta\colon Ob(\mathbf{G}) \to \mathrm{Mor}(\mathbf{H})$ *such that for any morphism* $g\colon g_1 \to g_2$ *in* $\mathbf{G}$ *the diagram*

$$\begin{tikzcd} \alpha(g_1) \arrow[r, "\alpha(g)"] \arrow[d, "\theta(g_1)"'] & \alpha(g_2) \arrow[d, "\theta(g_2)"'] \\ \beta(g_1) \arrow[r, "\beta(g)"] & \beta(g_2) \end{tikzcd}$$

commutes. Two maps of Hopf algebroids $a, b\colon (A, \Gamma) \to (B, \Sigma)$ *are* naturally equivalent *if the corresponding natural transformations of groupoid-valued functors are naturally equivalent in the above sense. Two Hopf algebroids* (A, Γ) *and* (B, Σ) *are* equivalent *if there are maps* $f\colon (A, \Gamma) \to (B, \Sigma)$ *and* $h\colon (B, \Sigma) \to (A, \Gamma)$ *such that* hf *and* fh *are naturally equivalent to the appropriate identity maps.* □

Now we will show that a Hopf algebroid equivalence induces an isomorphism of certain Ext groups. Given a map $f\colon (A, \Gamma) \to (B, \Sigma)$ and a left Γ-comodule M, define a Σ-comodule $f^*(M)$ to be $B \otimes_A M$ with coactions

$$B \otimes_A M \to B \otimes_A \Gamma \otimes_A M \to B \otimes_B \Sigma \otimes_A M = \Sigma \otimes_B B \otimes_A M.$$

6.1.5. Lemma. *Let* $f\colon (A, \Gamma) \to (B, \Sigma)$ *a Hopf algebroid equivalence. Then there is a natural isomorphism* $\mathrm{Ext}_\Gamma(A, M) \cong \mathrm{Ext}_\Sigma(B, f^*(M))$ *for any* Γ*-comodule* M*.*

Proof. It suffices to show that equivalent maps induce the same homomorphisms of Ext groups. An equivalence between the maps $a, b\colon (A, \Gamma) \to (B, \Sigma)$ is a homomorphism $\phi\colon \Gamma \to B$ with suitable properties, including $\phi\eta_R = a$ and $\phi\eta_L = b$. Since η_R and η_L are related by the conjugation in Γ, it follows that the two A-module structures on B are isomorphic and that $a^*(M)$ is naturally isomorphic to $b^*(M)$. We denote them interchangeably by M'. The maps a and b induce maps of cobar complexes (A1.2.11) $C_\Gamma(M) \to C_\Sigma(M')$. A tedious routine verification shows that ϕ induces the required chain homotopy. □

Now we consider the functors represented by VT_θ and EH_θ. Recall that an *Artin local ring* is a commutative ring with a single maximal ideal satisfying the descending chain condition, i.e., the maximal ideal is nilpotent. If A is such a ring with finite residue field k then it is $W(k)$-module, where $W(k)$ is the Witt ring of A2.2.15. Let $\mathbf{Art}_\theta$ denote the category of $\mathbf{Z}/(2(p^n - 1))$-graded Artin local rings whose residue field is an $\mathbf{F}_p$-algebra. Now let $m_\theta = \ker\theta \subset BP_*$. Then BP_*/m_θ^n with the cyclic grading is is object in $\mathbf{Art}_\theta$, so $V_\theta = \varprojlim BP_*/m_\theta^n$ is an inverse limit of such objects as is VT_θ. For any $A \in \mathbf{Art}_\theta$, we can consider $\mathrm{Hom}^c(VT_\theta, A)$, the set of continuous ring homomorphisms from VT_θ to A. It is a groupoid-valued functor on $\mathbf{Art}_\theta$ pro-represented by VT_θ. (We have to say “pro-represented” rather than “represented” because VT_θ is not in $\mathbf{Art}_\theta$.)

6.1.6. Proposition. VT_θ *pro-represents the functor* $\mathbf{lifts}_\theta$ *from* $\mathbf{Art}_\theta$ *to the category of groupoids, defined as follows. Let* $A \in \mathbf{Art}_\theta$ *have residue field* k*. The objects in* $\mathbf{lifts}_\theta(A)$ *are* p*-typical liftings to* A *of the formal group law over* k *induced by the composite* $BP_* \xrightarrow{\theta} \mathbf{F}_p \to k$*, and morphisms in* $\mathbf{lifts}_\theta(A)$ *are strict isomorphisms between such liftings.* □

6.1.7. DEFINITION. *Let $m_A \subset A$ be the maximal ideal for $A \in \mathbf{Art}_\theta$. Given a homomorphism $f\colon F \to G$ of formal group laws over A, let $\bar{f}\colon \overline{F} \to \overline{G}$ denote their reductions* mod m_A. *f is a $*-$*isomorphism *if $\bar{f}(x) = x$.*

6.1.8. LEMMA. *Let F and G be objects in $\mathbf{lifts}_\theta(A)$. Then the map $\mathrm{Hom}(F,G) \to \mathrm{Hom}(\overline{F},\overline{G})$ is injective.*

PROOF. Suppose $\bar{f} = 0$, i.e., $f(x) = 0 \mod m_A$. We will show that $f(x) \equiv 0 \mod m_A^r$ implies $f(x) \equiv 0 \mod m_A^{r+1}$ for any $r > 0$, so $f(x) = 0$ since m_A is nilpotent. We have

$$G(f(x), f(y)) \equiv f(x) + f(y) \mod m_A^{2r}$$

since

$$G(x,y) \equiv x + y \mod (x,y)^2.$$

Consequently,

$$[p]_G(f(x)) \equiv pf(x) \mod m_A^{2r} \equiv 0 \mod m_A^{r+1}$$

since $p \in m_A$. On the other hand

$$[p]_G(f(x)) = f([p]_F(x))$$

and we know $[p]_F(x) \equiv x^{p^n} \mod m_A$ by A2.2.4. Hence $f([p]_F(x)) \equiv 0 \mod m_A^{r+1}$ gives the desired congruence $f(x) \equiv 0 \mod m_A^{r+1}$. □

Now suppose $f_1, f_2\colon F \to G$ are $*$-isomorphisms (6.1.7) as is $g\colon G \to F$. Then $gf_1 = gf_2$ by 6.1.8 so $f_1 = f_2$; i.e. $*$-isomorphisms are unique. Hence we can make

6.1.9. DEFINITION. $\mathbf{lifts}_\theta^*(A)$ *is the groupoid of $*$-isomorphism classes of objects in* $\mathbf{lifts}_\theta A$.

6.1.10. LEMMA. *The functors* $\mathbf{lifts}_\theta$ *and* $\mathbf{lifts}_\theta^*$ *are naturally equivalent.*

PROOF. There is an obvious natural transformation $\alpha\colon \mathbf{lifts}_\theta \to \mathbf{lifts}_\theta^*$, and we need to define $\beta\colon \mathbf{lifts}_\theta^* \to \mathbf{lifts}_\theta$, of each $*$-isomorphism class. Having done this, $\alpha\beta$ will be the identity on $\mathbf{lifts}_\theta^*$ and we will have to prove $\beta\alpha$ is equivalent (6.1.4) to the identity on $\mathbf{lifts}_\theta$.

The construction of β is essentially due to Lubin and Tate [**3**]. Suppose $G_1 \in \mathbf{lifts}_\theta(A)$ is induced by $\theta_1\colon BP_* \to A$. Using A2.1.26 and A2.2.5 one can show that there is a unique $G_2 \in \mathbf{lifts}_\theta(A)$ $*$-isomorphic to G_1 and induced by θ_2 satisfying $\theta(v_{n+i}) = 0$ for all $i > 0$. We leave the details to the interested reader. As remarked above, the $*$-isomorphism from G_1 to G_2 is unique. The verification that $\beta\alpha$ is equivalent to the identity is straightforward. □

To prove 6.1.3, it follows from 6.1.5 and 6.1.10 that it suffices to show EH_θ pro-represents $\mathbf{lifts}_\theta^*$. In the proof of 6.1.10 it was claimed that any suitable formal group law over A is canonically $*$-isomorphic to one induced by $\theta\colon BP_* \to A$ which is such that it factors through E_θ. In the same way it is clear that the morphism set of $\mathbf{lifts}_\theta^*(A)$ is represented by EH_θ, so 6.1.3 follows.

Now we turn to the proof of 6.1.1. We have a map $BP_*(BP) \to \Sigma(n)$ and we need to show that it satisfies the hypotheses of the general change-of-rings isomorphism theorem A1.3.12, i.e., of A1.1.19. These conditions are

(6.1.11) (i) the map $\Gamma' = BP_*(BP) \otimes_{BP_*} K(n)_* \to \Sigma(n)$ is onto and

(ii) $\Gamma' \,\square_{\Sigma(n)}\, K(n)_*$ is a $K(n)_*$-summand of Γ'.

Part (i) follows immediately from the definition $\Sigma(n) = K(n)_* \otimes_{BP_*} \Gamma'$. Part (ii) is more difficult. We prefer to replace it with its conjugate,

(ii) $K(n)_* \square_{\Sigma(n)} K(n)_*(BP)$ is a $K(n)_*$ summand of $K(n)_*BP$ which is defined to be $K(n)_* \otimes_{BP_*} BP_*(BP)$. Let $B(n)_*$ denote $v_n^{-1}BP_*/I_n$. Then the right BP_*-module structure on $K(n)_*(BP)$ induces a right $B(n)_*$- module structure.

6.1.12. Lemma. *There is a map*

$$K(n)_*BP \to \Sigma(n) \otimes_{K(n)_*} B(n)_*$$

which is an isomorphism of $\Sigma(n)$*-comodules and of* $B(n)_*$*-modules, and which carries* 1 *to* 1.

Proof. Our proof is a counting argument, and in order to meet requirements of connectivity and finiteness, we pass to suitable "valuation rings". Thus let

$$\begin{aligned} k(0)_* &= \mathbf{Z}_{(p)} \subset K(0)_*, \\ k(n)_* &= \mathbf{F}_p[v_n] \subset K(n)_*, \quad n > 0, \\ k(n)_*BP &= k(n)_* \otimes_{BP_*} BP_*(BP) \subset K(n)_*BP, \\ b(n)_* &= k(n)_*[u_1, u_2, \dots] \subset B(n)_*, \end{aligned}$$

where $u_k = v_n^{-1}v_{n+k}$.

It follows from A2.2.5 that in $k(n)_*BP$,

$$\eta_R(v_{n+k}) \equiv v_n t_k^{p^n} - v_n^{p^k} t_k \mod (\eta_R(v_{n+1}), \dots, \eta_R(v_{n+k-1})). \tag{6.1.13}$$

Hence $\eta_R\colon BP_* \to k(n)_*BP$ factors through an algebra map $b(n)_* \to k(n)_*BP$. It is clear from 6.1.13 that as a right $b(n)_*$-module, $k(n)_*BP$ is free on generators $t^\alpha = t_1^{\alpha_1}t_2^{\alpha_2}\dots$ where $0 \le \alpha_i < p^n$ and all but finitely many α_i are 0; in particular, it is of finite type over $b(n)_*$.

Now define

$$\sigma(n) = k(n)_*BP \otimes_{b(n)_*} k(n)_* \subset \Sigma(n);$$

by the above remarks $\sigma(n) = k(n)_*[t_1, t_2, \dots]/(t_k^{p^n} - v_n^{p^k-1}t_k\colon k \ge 1)$ as an algebra. $(k(n)_*, \sigma(n))$ is clearly a sub-Hopf algebroid of $(K(n)_*, \Sigma(n))$, so $\sigma(n)$ is a Hopf algebra over the principal ideal domain $k(n)_*$.

The natural map $BP_*(BP) \to \sigma(n)$ makes $BP_*(BP)$ a left $\sigma(n)$-comodule, and this induces a left $\sigma(n)$-comodule structure on $k(n)_*BP$. We will show that the latter is an *extended* left $\sigma(n)$-comodule.

Define a $b(n)_*$-linear map $f\colon k(n)_*BP \to b(n)_*$ by

$$f(t^\alpha) = \begin{cases} 1 & \text{if } \alpha = (0,0,\dots) \\ 0 & \text{otherwise.} \end{cases}$$

Then f satisfies the equations

$$\begin{aligned} f\bar\eta_R &= id\colon b(n)_* \to b(n)_*, \\ f \otimes_{b(n)_*} k(n)_* &= \varepsilon\colon \sigma(n) \to k(n)_*. \end{aligned}$$

Now let $\tilde{f}$ be the $\sigma(n)$-comodule map lifting f:

(6.1.14)
$$\begin{tikzcd} k(n)_*BP \arrow[r, "\psi"] \arrow[dr, "\tilde{f}"'] & \sigma(n) \otimes_{k(n)^*} k(n)_*BP \arrow[d, "\sigma(n)\otimes f"] \\ & \sigma(n) \otimes_{k(n)^*} b(n)_* \end{tikzcd}$$

Since $\psi\bar{\eta}_R(x) = 1 \otimes \bar{\eta}_R(x)$, ψ is $b(n)_*$-linear, so $\tilde{f}$ is too. We claim $\tilde{f}$ is an isomorphism. Since both sides are free of finite type over $b(n)_*$ it suffices to prove that $\tilde{f} \otimes_{b(n)_*} k(n)$ is an isomorphism. But 6.1.14 is then reduced to

$$\begin{tikzcd} \sigma(n) \arrow[r, "\Delta"] \arrow[dr, "\tilde{f}\otimes_{b(n)_*}k(n)_*"'] & \sigma(n) \otimes_{k(n)_*} \sigma(n) \arrow[d, "1\otimes\varepsilon"] \\ & \sigma(n) \otimes_{k(n)_*} k(n)_* \end{tikzcd}$$

so the claim follows from unitarity of Δ.

Now the map $K(n)_* \otimes_{k(n)_*} \tilde{f}$ satisfies the requirements of the lemma. $\square$

6.1.15. Corollary. *$\bar{\eta}_R\colon B(n)_* \to K(n)_* \square_{\Sigma(n)} K(n)_*BP$ is an isomorphism of $B(n)_*$-modules.*

Proof. The natural isomorphism

$$B(n)_* \to K(n)_* \square_{\Sigma(n)} (\Sigma(n) \otimes_{K(n)_*} B(n)_*)$$

is $B(n)_*$-linear and carries 1 to 1. Hence

$$\begin{tikzcd} & K(n)_* \square_{\Sigma(n)} (\Sigma(n) \otimes_{K(n)_*} B(n)_*) \\ B(n)_* \arrow[ur, "\cong"] \arrow[dr, "\bar{\eta}_R"'] & \\ & K(n)_* \square_{\Sigma(n)} K(n)_*BP \arrow[uu, "\cong"'] \end{tikzcd}$$

commutes, and $\bar{\eta}_R$ is an isomorphism. $\square$

Hence 6.1.11(ii) follows from the fact that $K(n)_*$ is a summand of $\Sigma(n)$, and 6.1.1 is proved. From the proof of 6.1.12 we get an explicit description of $\Sigma(n)$, namely

6.1.16. Corollary. *As an algebra*

$$\Sigma(n) = K(n)_*[t_1, t_2, \ldots]/(v_n t_i^{p^n} - v_n^{p^i} t_i \colon i > 0).$$

Its coproduct is inherited from $BP_*(BP)$, i.e., a suitable reduction of 4.3.13 *holds.*

2. The Structure of $\Sigma(n)$

To study $\Sigma(n)$ it is convenient to pass to the corresponding object graded over $\mathbf{Z}/2(p^n-1)$. Make $\mathbf{F}_p$ a $K(n)_*$-module by sending v_n to 1, and let $S(n) = \Sigma(n)\otimes_{K(n)_*}\mathbf{F}_p$. For a $\Sigma(n)$-comodule M let $\overline{M} = M\otimes_{K(n)_*}\mathbf{F}_p$, which is easily seen to be an $S(n)$-comodule. The categories of $\Sigma(n)$- and $S(n)$-comodules are equivalent and we have

6.2.1. PROPOSITION. *For a $\Sigma(n)$-comodule M,*

$$\operatorname{Ext}_{\Sigma(n)}(K(n)_*, M)\otimes_{K(n)_*}\mathbf{F}_p \cong \operatorname{Ext}_{S(n)}(\mathbf{F}_p, \overline{M}). \qquad \square$$

We will see below (6.2.5) that if we regard $S(n)$ and $\overline{M}$ as graded merely over $\mathbf{Z}/(2)$, there is a way to recover the grading over $\mathbf{Z}/2(p^n-1)$. If M is concentrated in even dimensions (which it is in most applications) then we can regard $\overline{M}$ and $S(n)$ as ungraded objects. Our first major result is that $S(n)\otimes\mathbf{F}_{p^n}$ (ungraded) is the continuous linear dual of the $\mathbf{F}_{p^n}$-group algebra of a certain profinite group S_n to be defined presently.

6.2.2. DEFINITION. *The topological linear dual $S(n)^*$ of $S(n)$ is as follows. [In Ravenel [**5**] $S(n)^*$ and $S(n)$ are denoted by $S(n)$ and $S(n)_*$, respectively.] Let $S(n)_{(i)}$ be the sub-Hopf algebra of $S(n)$ generated by $\{t,\dots,t_i\}$. It is a vector space of rank p^{ni} and $S(n) = \varinjlim S(n)_{(i)}$. Then $S(n)^* = \varprojlim \operatorname{Hom}(S(n)_{(i)}, \mathbf{F}_p)$, equipped with the inverse limit topology. The product and coproduct in $S(n)$ give maps of $S(n)^*$ to and from the completed tensor product*

$$S(n)^*\mathbin{\widehat{\otimes}} S(n)^* = \varprojlim \operatorname{Hom}(S(n)_{(i)}\otimes S(n)_j, \mathbf{F}_p).$$

To define the group S_n recall the $\mathbf{Z}_p$-algebra E_n of A2.2.16, the endomorphism ring of a height n formal group law. It is a free $\mathbf{Z}_p$-algebra of rank n^2 generated by ω and S, where ω is a primitive (p^n-1)th root of units, $S\omega = \omega^p S$, and $S^n = p$. $S_n \subset E_n^\times$, is the group of units congruent to $1 \mod (S)$, the maximal ideal in E_n. S_n is a profinite group, so its group algebra $\mathbf{F}_{p^n}[S_n]$ has a topology and is a profinite Hopf algebra. S_n is also a p-adic Lie group; such groups are studied by Lazard [**4**].

6.2.3. THEOREM. *$S(n)^*\otimes\mathbf{F}_q \cong \mathbf{F}_q[S_n]$ as profinite Hopf algebras, where $q = p^n$, S_n is as above, and we disregard the grading on $S(n)^*$.*

PROOF. First we will show $S(n)^*\otimes\mathbf{F}_q$, is a group algebra. According to Sweedler [**1**], Proposition 3.2.1, a cocommutative Hopf algebra is a group algebra iff it has a basis of group-like elements, i.e., of elements x satisfying $\Delta x = x\otimes x$. This is equivalent to the existence of a dual basis of idempotent elements $\{y\}$ satisfying $y_i^2 = y_i$, and $y_iy_j = 0$ for $i\neq j$. Since $S(n)\otimes\mathbf{F}_q$, is a tensor product of algebras of the form $R = \mathbf{F}_q[t]/(t^q - t)$, it suffices to find such a basis for R. Let $a\in\mathbf{F}_q^\times$ be a generator and let

$$r_i = \begin{cases} -\sum\limits_{0<i<q} (a^i t)^j & \text{for } 0<i<q,\\ 1 - t^{q-1} & \text{for } i = 0.\end{cases}$$

Then $\{r_i\}$ is such a basis, so $S(n)^*\otimes\mathbf{F}_q$, is a group algebra.

Note that tensoring with $\mathbf{F}_q$ cannot be avoided, as the basis of R is not defined over $\mathbf{F}_p$.

For the moment let G_n denote the group satisfying $\mathbf{F}_p[G_n] \cong S(n)^* \otimes \mathbf{F}_q$. To get at it we define a completed left $S(n)$-comodule structure on $\mathbf{F}_q[[x]]$, thereby defining a left G_n-action. Then we will show that it coincides with the action of S_n as formal group law automorphisms given by A2.2.17.

We now define the comodule structure map

$$\psi\colon \mathbf{F}_q[[x]] \to S(n) \mathbin{\widehat{\otimes}} \mathbf{F}_q[[x]]$$

to be an algebra homomorphism given by

$$\psi(x) = \sum_{i\geq 0}^{F} t_i \otimes x^{p^i},$$

where $t_0 = 1$ as usual. To verify that this makes sense we must show that the following diagram commutes.

$$\begin{tikzcd}
\mathbf{F}_q[[x]] \arrow[r, "\psi"] \arrow[d, "\psi"'] & S(n) \mathbin{\widehat{\otimes}} \mathbf{F}_q[[x]] \arrow[d, "\Delta\otimes 1"] \\
S(n) \mathbin{\widehat{\otimes}} \mathbf{F}_q[[x]] \arrow[r, "1\otimes\psi"] & S(n) \mathbin{\widehat{\otimes}} S(n) \mathbin{\widehat{\otimes}} \mathbf{F}_q[[x]]
\end{tikzcd}$$

for which we have

$$\begin{aligned}
(\Delta\otimes 1)\psi(x) &= (\Delta\otimes 1)\sum_{i\geq 0}^{F} t_i \otimes x^{p^i} \\
&= \sum_{i\geq 0}^{F}\left(\sum_{j-k=i}^{F} t_j \otimes t_k^{p^j}\right)\otimes x^{p^i} \\
&= \sum_{j,k\geq 0}^{F} t_j \otimes t_k^{p^j} \otimes x^{p^{j+k}}
\end{aligned}$$

This can be seen by inserting x as a dummy variable in 4.3.12. We also have

$$\begin{aligned}
(1\otimes\psi)\psi(x) &= (1\otimes\psi)\left(\sum_{j\geq 0}^{F} t_i \otimes x^{p^i}\right) \\
&= \sum_{j\geq 0}^{F} t_i \otimes \left(\sum_{j\geq 0}^{F} t_j \otimes x^{p^i}\right)^{p^i} \\
&= \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i} \otimes x^{p^{i+j}}.
\end{aligned}$$

The last equality follows from the fact that $F(x^p, y^p) = F(x,y)^p$. The linearity of ψ follows from A2.2.20(b), so ψ defines an $S(n)\otimes\mathbf{F}_q$-comodule structure on $\mathbf{F}_q[[x]]$.

We can regard the t_i, as continuous $\mathbf{F}_q$-valued functions on G_n and define an action of G_n on the algebra $\mathbf{F}_q[[x]]$ by

$$g(x) = \sum_{i\geq 0}^{F} t_i(g)x^{p^i}$$

for $g \in G_n$. Hence $G(x) = x$ iff $g = 1$, so our representation is faithful.

We can embed G_n in the set of all power series of the form $\sum_{i\geq 0}^{F} a_i x^{p^i}$ which is E_n by A2.2.20 so the result follows. $\square$

6.2.4. COROLLARY. *If M is an ungraded $S(n)$-comodule, then 6.2.3 gives a continuous S_n-action on $M \otimes \mathbf{F}_q$, and*

$$\operatorname{Ext}^*_{S(n)}(\mathbf{F}_p, M) \otimes \mathbf{F}_q = H^*_c(G_n, M \otimes \mathbf{F}_q)$$

where H^*_c denotes continuous group cohomology. $\square$

To recover the grading on $S(n) \otimes M$, we have an action of the cyclic group of order $q-1$ generated by $\bar{\omega}^i \omega^i$ via conjugation in E_n.

6.2.5. PROPOSITION. *The eigenspace of $S(n) \otimes \mathbf{F}_q$ with eigenvalue $\bar{\omega}^i$ is the component $S(n)_{2i} \otimes \mathbf{F}_q$ of degree $2i$.*

PROOF. The eigenspace decomposition is multiplicative in the sense that if x and y are in the eigenspaces with eigenvalues $\bar{\omega}^i$ and $\bar{\omega}^j$, respectively, the xy is in the eigenspace with eigenvalue $\bar{\omega}^{i+j}$. Hence it suffices to show that t_k is in the eigenspace with eigenvalue $\bar{\omega}^{p^k-1}$.

To see this we compute the conjugation of $t_k S^k \in E_n$ by ω and we have $\omega^{-1}(t_k S^k)\omega = \omega^{-1} t_k \omega^{p^k} S^k = \omega^{p^k-1} t_k S^k$. $\square$

Corollary 6.2.4 enables us to apply certain results from group cohomology theory to our situation. First we give a matrix representation of E_n over $W(\mathbf{F}_q)$.

6.2.6. PROPOSITION. *Let $e = \sum_{0 \le i < n} e_i S^i$ with $e_i \in W(\mathbf{F}_q)$ be an element of E_n. Define an $n \times n$ matrix $(e_{i,j})$ over $W(\mathbf{F}_q)$ by*

$$e_{i+1,j+1} = \begin{cases} e_{j-i}^{\sigma^i} & \text{for } i \le j \\ p e_{j+n-i}^{\sigma^i} & \text{for } i > j. \end{cases}$$

Then (a) *this defines a faithful representation of E_n*; (b) *the determinant $|e_{i,j}|$ lies in $\mathbf{Z}_p$.*

PROOF. Part (a) is straightforward. For (b) it suffices to check that ω and S give determinants in $\mathbf{Z}_p$. $\square$

We can now define homomorphisms $c\colon \mathbf{Z}_p \to S_n$ and $d\colon S_n \to \mathbf{Z}_p$ for $p > 2$, and $c\colon \mathbf{Z}_2^\times \to S_n$ and $d\colon \mathbf{Z}_2^\times$ for $p = 2$ by identifying S_n with the appropriate matrix group. ($\mathbf{Z}_p$ is to be regarded here as a subgroup of $\mathbf{Z}_p^\times$.) Let d be the determinant for all primes. For $p > 2$ let $c(x) = \exp(px)I$, where I is the $n \times n$ identity matrix and $x \in \mathbf{Z}_p$; for $p = 2$ let $c(x) = xI$ for $x \in \mathbf{Z}_2^\times$.

6.2.7. THEOREM. *Let $S_n^1 = \ker d$.*
(a) *If $p > 2$ and $p \nmid n$ then $S_n \cong \mathbf{Z}_p \oplus S_n^1$.*
(b) *If $p = 2$ and n is odd then $S_n \cong S_n^1 \oplus \mathbf{Z}_2^\times$.*

PROOF. In both cases one sees that $\operatorname{im} c$ lies in the center of S_n (in fact $\operatorname{im} c$ is the center of S_n) and is therefore a normal subgroup. The composition dc is multiplication by n which is an isomorphism for $p \nmid n$, so we have the desired splitting. $\square$

We now describe an analogous splitting for $S(n)$. Let $A^* = \mathbf{F}_p[\mathbf{Z}_p]$ for $p > 2$ and $A^* = \mathbf{F}_2[\mathbf{Z}_2^\times]$ for $p = 2$. Let A_* be the continuous linear dual of A.

6.2.8. PROPOSITION. *As an algebra* $A = \mathbf{F}_p[u_1, u_2, \ldots]/(u_i - u_i^p)$. *The coproduct* Δ *is given by*

$$\sum_{i\geq 0}^G \Delta(u_i) = \sum_{i,j\geq 0}^G u_i \otimes u_j$$

where $u_0 = 1$ *and* G *is the formal group law with*

$$\log_G(X) = \sum \frac{x^{p^i}}{p^i}.$$

PROOF. Since $A \cong \mathbf{F}_p[S_1]$, this follows immediately from 6.2.3. □

We can define Hopf algebra homomorphisms $c_*\colon S(n) \otimes \mathbf{F}_q \to A \otimes \mathbf{F}_q$ and $d_*\colon A \otimes \mathbf{F}_q \to S(n) \otimes \mathbf{F}_q$ dual to the group homomorphisms c and d defined above.

6.2.9. THEOREM. *There exist maps* $c_*\colon S(n) \to A$ *and* $d_*\colon A \to S(n)$ *corresponding to those defined above, and for* $p \nmid n$, $S(n) \cong A \otimes B$, *where* $B \otimes \mathbf{F}_q$, *is the continuous linear dual of* $\mathbf{F}_q[S_n^1]$, *where* S_n^1, *is defined in* 6.2.7.

PROOF. We can define c_* explicitly by

$$c_* t_i = \begin{cases} u_{i/n} & \text{if } n \mid i \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that this is a homomorphism corresponding to the c_* defined above. In lieu of defining d_* explicitly we observe that the determinant of $\sum_{i\geq 0} t_i S^i$, where $t_i \in W(\mathbf{F}_q)$ and $t_i = t_i^q$, is a power series in p whose coefficients are polynomials in the t_i over $\mathbf{Z}_p$. It follows that d_* can be defined over $\mathbf{F}_p$. The splitting then follows as in 6.2.7. □

Our next result concerns the size of $\operatorname{Ext}_{S(n)}(\mathbf{F}_p, \mathbf{F}_p)$, which we abbreviate by $H^*(S(n))$.

6.2.10. THEOREM.
(a) $H^*(S(n))$ *is finitely generated as an algebra.*
(b) *If* $(p-1) \nmid n$, *then* $H^i(S(n)) = 0$ *for* $i > n^2$ *and* $H^i(S(n)) = H^{n^2-i}(S(n))$ *for* $0 \leq i \leq n^2$, *i.e.,* $H^*(S(n))$ *satisfies Poincaré duality.*
(c) *If* $(p-1) \mid n$, *then* $H^*(S(n))$ *is p-periodic, i.e., there is some* $x \in H^i(S(n))$ *such that* $H^*(S(n))$ *above some finite dimension is a finitely generated free module over* $\mathbf{F}_p[x]$. □

We will prove 6.2.10(a) below as a consequence of the open subgroup theorem (6.3.6), which states that every sufficiently small open subgroup of S_n has the same cohomology as $\mathbf{Z}_p^{n^2}$. Then (c) and the statement in (b) of finite cohomological dimension are equivalent to saying that the Krull dimension of $H^*(S(n))$ is 1 or 0, respectively. Recall that the Krull dimension of a Noetherian ring R is the largest d such that there is an ascending chain $p_0 \subset p_1 \subset \cdots \subset p_d$ of nonunit prime ideals in R. Roughly speaking, d is the number of generators of the largest polynomial algebra contained in R. Thus $d = 0$ iff every element in R is nilpotent, which in view of (a) implies (b). If $d = 1$ and R is a graded $\mathbf{F}_p$-algebra, then every element in R has a power in $\mathbf{F}_p[x]$ for a fixed $x \in R$. R is a module over $\mathbf{F}_p[x]$, which is a principal ideal domain. Since $H^*(S(n))$ is graded and finitely generated, it is a direct sum of cyclic modules over $\mathbf{F}_p[x]$. More specifically it is a direct sum of a torsion module (where each element is annilhilated by some power of x) and a free module. Since it is finitely generated, the torsion must be confined to low dimensions, and $H^*(S(n))$ is therefore a free $\mathbf{F}_p[x]$-module in high dimensions, so (a) implies (c).

The following result helps determine the Krull dimension.

6.2.11. THEOREM (Quillen [**3**]). *For a profinite group G the Krull dimension of $H^*(G;\mathbf{F}_p)$ is the maximal rank of an elementary abelian p-subgroup of G, i.e., subgroup isomorphic to $(\mathbf{Z}/(p))^d$.* □

To determine the maximal elementary abelian subgroup of S_n, we use the fact that $D_n = E_n \otimes \mathbf{Q}$ is a division algebra over $\mathbf{Q}_p$ (A2.2.16), so if $G \subset S_n$ is abelian, then the $\mathbf{Q}_p$-vector space in D_n spanned by the elements of G is a subfield $K \subset D_n$. Hence the elements of G are all roots of unity, G is cyclic, and the Krull dimension is 0 or 1.

6.2.12. THEOREM. *A degree m extension K of $\mathbf{Q}_p$ embeds in D_n iff $m \mid n$.*

PROOF. See Serre [**1**, p. 138] or Cassels and Fröhlich [**1**, p. 202], □

By 6.2.11 $H^*(S(n))$ has Krull dimension 1 iff S_n contains pth roots of unity. Since the field K obtained by adjoining such roots to $\mathbf{Q}_p$ has degree $p-1$, 6.2.12 gives 6.2.10(c) and the finite cohomological dimension statement in (b). For the rest of (b) we rely on theorem V.2.5.8 of Lazard [**4**], which says that if S_n (being an analytic pro-p-group of dimension n^2) has finite cohomological dimension, then that dimension is n^2 and Poincaré duality is satisfied.

The following result identifies some Hopf algebra quotients of $S(n)\otimes\mathbf{F}_{p^n}$. These are related to the graded Hopf algebras $\Sigma_A(n)$ discussed in Ravenel [**10**]. More precisely, $S(d,f)_a$ is a nongraded form of $\Sigma_A(d/f)$, where A is the ring of integers in an extension K (depending on a) of $\mathbf{Q}_p$ of degree fn/d and residue degree f.

6.2.13. THEOREM. *Let $a \in \overline{\mathbf{F}}_p$ be a (p^n-1)th root of unity, let d divide n, and let f divide d. Then there is a Hopf algebra*

$$S(d,f)_a = \mathbf{F}_{p^n}[t_f, t_{2f}, \dots]/(t_{if}^{p^d} - a_i t_{if} : i > 0)$$

where $a_i = a^{p^{id}-1}$, and a surjective homomorphism

$$\theta : S(n) \otimes \mathbf{F}_{p^n} \to S(d,f)_a$$

given by

$$t_i \mapsto \begin{cases} t_i & \text{if } f|i \\ 0 & \text{otherwise.} \end{cases}$$

The coproduct on $S(d,f)_a$ is determined by the one on $S(n)$. This Hopf algebra is cocommutative when $f = d$.

PROOF. We first show that the algebra structure on $S(d,f)_a$ is compatible with that on $S(n)$. The relation $t_{if}^{p^d} = a_i t_{if}$ implies

$$\begin{aligned}
t_{if}^{p^{2d}} &= (a_i t_{if})^{p^d} = a_i^{(p^{2d}-1)/(p^d-1)} t_{if} = a^{(p^{id}-1)(p^{2d}-1)/(p^d-1)} t_{if} \\
t_{if}^{p^{3d}} &= a^{(p^{id}-1)(p^{3d}-1)/(p^d-1)} t_{if} \\
&\ \vdots \\
t_{if}^{p^n} &= a^{(p^{id}-1)(p^n-1)/(p^d-1)} t_{if} = t_{if},
\end{aligned}$$

so θ exists as an algebra map.

For the coproduct in $S(n)$ we have

$$\sum_{i\geq 0}^{F} \Delta(t_i)x^{p^i} = \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i} x^{p^{i+j}}$$

(where x is a dummy variable) which induces

$$\sum_{i\geq 0}^{F} \Delta(t_{if})x^{p^{if}} = \sum_{i,j\geq 0}^{F} t_{if} \otimes t_{jf}^{p^{if}} x^{p^{(i+j)f}}$$

in $S(d,f)_a$. We need to show that this is compatible with the multiplicative relations. We can write $if = kd + \ell f$ with $0 \leq \ell f < d$, so we can rewrite the above as

$$\begin{aligned}
\sum_{i\geq 0}^{F} \Delta(t_{if})x^{p^{if}} &= \sum_{i,j\geq 0}^{F} t_{if} \otimes t_{jf}^{p^{kd+\ell f}} x^{p^{(i+j)f}} \\
&= \sum_{i,j\geq 0}^{F} a_j^{p^{\ell f}(p^{kd}-1)/(p^d-1)} t_{if} \otimes t_{jf}^{p^{\ell f}} x^{p^{(i+j)f}} \\
&= \sum_{i,j\geq 0}^{F} a^{p^{\ell f}(p^{jd}-1)(p^{kd}-1)/(p^d-1)} t_{if} \otimes t_{jf}^{p^{\ell f}} x^{p^{(i+j)f}},
\end{aligned}$$

which gives a well defined coproduct in $S(d,f)_a$.

If $f = d$ then the right hand side simplifies to

$$\sum_{i,j\geq 0}^{F} a^{(p^{jd}-1)(p^{if}-1)/(p^d-1)} t_{if} \otimes t_{jf} x^{p^{(i+j)f}},$$

which is cocommutative as claimed. $\square$

3. The Cohomology of $\Sigma(n)$

In this section we will use a spectral sequence (A1.3.9) based on the filtration of $\Sigma(n)$ induced by the one on $BP_*(BP)/I_n$ given in 4.3.24. We have

6.3.1. THEOREM. *Define integers $d_{n,i}$ by*

$$d_{n,i} = \begin{cases} 0 & \text{if } i \leq 0 \\ \max(i, pd_{n,i-n}) & \text{for } i > 0. \end{cases}$$

Then there is a unique increasing filtration of the Hopf algebra $S(n)$ with $\deg t_i^{p^j} = d_{n,i}$ for $0 \leq j < n$. $\square$

6.3.2. THEOREM. *Let $E^0S(n)$ denote the associated bigraded Hopf algebra. Its algebra structure is*

$$E^0S(n) = T(t_{i,j} \colon i > 0, \quad j \in \mathbf{Z}/(n)),$$

where $T(\cdot)$ denotes the truncated polynomial algebra of height p on the indicated elements and $t_{i,j}$ corresponds to $t_i^{p^j}$. The coproduct is induced by the one given in

4.3.34. *Explicitly, let $m = pn/(p-1)$. Then*

$$\Delta(t_{i,j}) = \begin{cases} \sum\limits_{0\le k\le i} t_{k,j}\otimes t_{i-k,k+j} & \text{if } i<m,\\ \sum\limits_{0\le k\le i} t_{k,j}\otimes t_{i-k,k+j} + \bar{b}_{i-n,j+n-1} & \text{if } i=m,\\ t_{i,j}\otimes 1 + 1\otimes t_{i,j} + \bar{b}_{i-n,j+n-1} & \text{if } i>m, \end{cases}$$

where $t_{0,j} = 1$ and $\bar{b}_{i,j}$ corresponds to the $b_{i,j}$ of 4.3.14. □

As in the case of the Steenrod algebra, the dual object $E_0S(n)^*$ is primitively generated and is the universal enveloping algebra of a restricted Lie algebra $L(n)$. $L(n)$ has basis $\{x_{i,j} : i > 0,\ j \in \mathbf{Z}/(n)\}$, where $x_{i,j}$ is dual to $t_{i,j}$.

6.3.3. THEOREM. *$E_0S(n)$ is the restricted enveloping algebra on primitives $x_{i,j}$ with bracket*

$$[x_{i,j}, x_{k,l}] = \begin{cases} \delta^l_{i+j} x_{i+k,j} - \delta^j_{k+1} x_{i+k,l} & \text{for } i+k\le m,\\ 0 & \text{otherwise,} \end{cases}$$

where m is the largest integer not exceeding $pn/(p-1)$, and $\delta^s_t = 1$ iff $s \equiv t \mod (n)$ and $\delta^s_t = 0$ otherwise. The restriction ξ is given by

$$\xi(x_{i,j}) = \begin{cases} x_{i+n,j+1} & \text{if } i > n/(p-1)\\ & \qquad \text{or } i = n/(p-1) \text{ and } p>2\\ x_{2n,j} + x_{2n,j+1} & \text{if } i = n \text{ and } p = 2\\ 0 & \text{if } i < n/p-1. \quad \square \end{cases}$$

The formula for the restriction was given incorrectly in the first edition, and this error led to an incorrect description in 6.3.24 of the multiplicative structure of $H^*(S(2))$ for $p = 3$. The correct description is due to Henn [**1**] and will be given below. The corrected restriction formula was given to me privately by Ethan Devinatz.

PROOF OF 6.3.3. The formula for the bracket follows easily from 6.3.2. The restriction requires more care. When $i > m$ we have

$$\begin{aligned} \Delta(t_{i,j}) &= t_{i,j}\otimes 1 + 1\otimes t_{i,j} + \bar{b}_{i-n,j-1}\\ &= t_{i,j}\otimes 1 + 1\otimes t_{i,j} - \sum_{0<\ell<p} p^{-1}\binom{p}{\ell} t^{\ell}_{i-n,j-1}\otimes t^{p-\ell}_{i-n,j-1}, \end{aligned}$$

so for $i > n/(p-1)$,

$$\Delta(t_{i+n,j+1}) = t_{i+n,j+1}\otimes 1 + 1\otimes t_{i+n,j+1} - \sum_{0<\ell<p} p^{-1}\binom{p}{\ell} t^{\ell}_{i,j}\otimes t^{p-\ell}_{i,j}.$$

It follows that in the p-fold iterated coproduct we have

$$t_{i+n,j+1} \mapsto t_{i,j}\otimes\cdots\otimes t_{i,j} + \ldots,$$

which leads to the desired value of $\xi(x_{i,j})$ for $i > n/(p-1)$. The argument for $i = n/(p-1)$ and p odd is similar.

For the case $p = 2$ and $i = n$, 6.3.2 gives

$$\begin{aligned}\Delta(t_{2n,j}) &= \sum_{0\le k\le 2n} t_{k,j}\otimes t_{2n-k,k+j} + \overline{b}_{n,j-1}\\ &= t_{n,j-1}\otimes t_{n,j-1} + \sum_{0\le k\le 2n} t_{k,j}\otimes t_{2n-k,k+j}\\ &= t_{n,j-1}\otimes t_{n,j-1} + t_{n,j}\otimes t_{n,j}\\ &\qquad + \sum_{0\le k<n}(t_{k,j}\otimes t_{2n-k,j+k} + t_{2n-k,j}\otimes t_{k,j-k}),\end{aligned}$$

and the formula for $\xi(x_{n,j})$ follows.

For $i < n/(p-1)$ there are no terms in $\Delta(t_{i+n,k})$ for any k that would lead to a nontrivial restriction on $x_{i,j}$. □

Let $L(n)$ be the Lie algebra without restriction with basis $x_{i,j}$ and bracket as above. We now recall the main results of May [**2**].

6.3.4. THEOREM. *There are spectral sequences*
(a) $E_2 = H^*(L(n))\otimes P(b_{i,j}) \Rightarrow H^*(E_0S(n))$,
(b) $E_2 = H^*(E_0S(n)) \Rightarrow H^*(S(n))$,
where $b_{i,j}\in H^{2pd_i}(E_0S(n))$ *with internal degree* $2p^{j+1}(p^i-1)$ *and* $P(\cdot)$ *is the polynomial algebra on the indicated generators.* □

Now let $L(n,k)$ be the quotient of $L(n)$ obtained by setting $x_{i,j} = 0$ for $i > k$. Then our first result is

6.3.5. THEOREM. *The* E_2*-term of the first May spectral sequence* [6.3.4(a)] *may be replaced by* $H^*(L(n,m))\otimes P(b_{i,j}\colon i\le m-n)$, *where* $m = [pn/(p-1)]$ *as before.*

PROOF. By 6.3.3 $L(n)$ is the product of $L(n,m)$ and an abelian Lie algebra, so

$$H^*(L(n)) \cong H^*(L(n,m))\otimes E(h_{i,j}\colon i>m),$$

where $E(\cdot)$ denotes the exterior algebra on the indicated generators and $h_{i,j}\in H^1L(n)$ is the element corresponding to $x_{i,j}$. It also follows from 6.3.4 that the appropriate differential will send $h_{i,j}$ to $-b_{i-n,j-1}$ for $i > m$. It follows that the entire spectral sequence decomposes as a tensor product of two spectral sequences, one with the E_2-term indicated in the statement of the theorem, and the other having $E_2 = E(h_{i,j})\otimes P(b_{i-n,j})$ with $i > m$ and $E_\infty = \mathbf{F}_p$. □

If $n < p-1$ then 6.3.5 gives a spectral sequence whose E_2-term is $H^*(L(n,n))$, showing that $H^*(S(n))$ has cohomological dimension n^2 as claimed in 6.2.10(b).

In Ravenel [**6**] we claimed erroneously that the spectral sequence of 6.3.4(b) collapses for $n < p-1$. The argument given there is incorrect. For example, we have reason to believe that for $p = 11$, $n = 9$ the element

$$(h_{1,0}h_{2,0}\cdots h_{7,0})(h_{2,8}h_{3,7}\cdots h_{7,3})$$

supports a differential that hits a nonzero multiple of

$$h_{1,0}h_{2,0}(h_{1,8}h_{2,7}\cdots h_{6,3})(h_{2,1}h_{3,1}\cdots h_{6,1}).$$

We know of no counterexample for smaller n or p.

Now we will prove 6.2.10(a), i.e., that $H^*(S(n))$ is finitely generated as an algebra. For motivation, the following is a special case of a result in Lazard [**4**].

6.3.6. Open Subgroup Theorem. *Every sufficiently small open subgroup of S_n is cohomologically abelian in the sense that it has the same cohomology as $\mathbf{Z}_p^{n^2}$, i.e., an exterior algebra on n^2 generators.* □

We will give a Hopf algebra theoretic proof of this for a cofinal set of open subgroups, namely the subgroups of elements in E_n congruent to 1 modulo (S^i) for various $i > 0$. The corresponding quotient group (which is finite) is dual the subalgebra of $S(n)$ generated by $\{t_k \colon k < i\}$. Hence the ith subgroup is dual to $S(n)/(t_k \colon k < i)$, which we denote by $S(n,i)$.

The filtration of 6.3.1 induces one on $S(n,i)$ and analogs of the succeeding four theorems hold for it.

6.3.7. Theorem. *If $i \geq n$ and $p > 2$, or $i > n$ and $p = 2$, then*

$$H^*(S(n,i)) = E(h_{k,j} \colon i \leq k < i+n,\ j \in \mathbf{Z}/(n)).$$

Proof. The condition on i is equivalent to $i > n-1$ and $i > m/2$, where as before $m = pn/(p-1)$. In the analog of 6.3.3 we have $i,k > m/2$ so $i+k > m$ so the Lie algebra is abelian. We also see that the restriction ξ is injective, so the spectral sequence of 6.3.5 has the E_2-term claimed to be $H^*(S(n,i))$. This spectral sequence collapses because $h_{k,j}$ corresponds to $t_k^{p^j} \in S(n,i)$, which is primitive for each k and j. □

Proof of 6.2.10(a). Let $A(i)$ be the Hopf algebra corresponding to the quotient of S_n by the ith congruence subgroup, so we have a Hopf algebra extension (A1.1.15)

$$A(i) \to S(n) \to S(n,i).$$

The corresponding Cartan–Eilenberg spectral sequence (A1.3.14) has

$$E_2 = \operatorname{Ext}_{A(i)}(\mathbf{F}_p, H^*(S(n,i)))$$

and converges to $H^*(S(n))$ with $d_r \colon E_r \to E_r^{s+r,t-r+1}$. Each E_r-term is finitely generated since $A(i)$ and $H^*(S(n,i))$ are finite-dimensional for $i > m/2$. Moreover, $E_{n^2} = E_\infty$, so E_∞ and $H^*(S(n))$ are finitely generated. □

Now we continue with the computation of $H^*(S(n))$. Theorem 6.3.5 indicates the necessity of computing $H^*(L(n,k))$ for $k \leq m$, and this may be done with the Koszul complex, i.e.,

6.3.8. Theorem. *$H^*(L(n,k))$ for $k \leq m$ is the cohomology of the exterior complex $E(h_{i,j})$ on one-dimensional generators $h_{i,j}$ with $i \leq k$ and $j \in \mathbf{Z}/(n)$, with coboundary*

$$d(h_{i,j}) = \sum_{0<s<i} h_{s,j} h_{i-s,s+j}.$$

The element $h_{i,j}$ corresponds to the element $x_{i,j}$ and therefore has filtration degree i and internal degree $2p^j(p^i-1)$.

Proof. This follows from standard facts about the cohomology of Lie algebras (Cartan and Eilenberg [**1**, XII, Section 7]). □

Since $L(n,k)$ is nilpotent its cohomology can be computed with a sequence of change-of-rings spectral sequences analogous to A1.3.14.

6.3.9. THEOREM. *There are spectral sequences with*

$$E_2 = E(h_{k,j}) \otimes H^*(L(n,k-1)) \Rightarrow H^*(L(n,k))$$

and $E_3 = E_\infty$.

PROOF. The spectral sequence is that of Hochschild–Serre (see Cartan and Eilenberg [**1**, pp. 349–351] for the extension of Lie algebras

$$A(n,k) \to L(n,k) \to L(n,k-1)$$

where $A(n,k)$ is the abelian Lie algebra on $x_{k,j}$. Hence $H^*(A(n,k)) = E(h_{k,j})$. The E_2-term, $H^*(L,(n,k-1), H^*(A(n,k))$ is isomorphic to the indicated tensor product since the extension is central.

For the second statement, recall that the spectral sequence can be constructed by filtering the complex of 6.3.8 in the obvious way. Inspection of this filtered complex shows that $E_3 = E_\infty$. □

In addition to the spectral sequence of 6.3.4(a), there is an alternative method of computing $H^*E_0S(n)$. Define $\tilde{L}(n,k)$ for $k \leq m$ to be the quotient of $PE_0S(n)$ by the restricted sub-Lie algebra generated by the elements $x_{i,j}$ for $k < i \leq m$, and define $F(n,k)$ to be the kernel of the extension

$$0 \to F(n,k) \to \tilde{L}(n,k) \to \tilde{L}(n,k-1) \to 0.$$

Let $H^*(\tilde{L}(n,k))$ denote the cohomology of the restricted enveloping algebra of $\tilde{L}(n,k)$. Then we have

6.3.10. THEOREM. *There are change-of-rings spectral sequences converging to* $H^*(\tilde{L}(n,k))$ *with*

$$E_2 = H^*(F(n,k)) \otimes H^*(\tilde{L}(n,k-1))$$

where

$$H^*(F(n,k)) = \begin{cases} E(h_{k,j}) & \text{for } k > m-n \\ E(h_{k,j}) \otimes P(b_{k,j}) & \text{for } k \leq m-n \end{cases}$$

and $H^*(\tilde{L}(n,m)) = H^*(E_0S(n))$.

PROOF. Again the spectral sequence is that given in Theorem XVI.6.1 of Cartan and Eilenberg [**1**]. As before, the extension is cocentral, so the E_2-term is the indicated tensor product. The structure of $H^*(F(n,k))$ follows from 6.3.3 and the last statement is a consequence of 6.3.5. □

We begin the computation of $H^1(S(n))$ with:

6.3.11. LEMMA. $H^1(E_0S(n))$ *is generated by*

$$\zeta_n = \sum_j h_{n,j} \quad \text{and} \quad \rho_n = \sum_j h_{2n,j} \quad \text{for } p = 2;$$

and for $n > 1$, $h_{1,j}$ *for each* $j \in \mathbf{Z}/(n)$.

PROOF. By 6.3.4(a) and 6.3.5 $H^1(E_0S(n)) = H^1L(n,m))$. The indicated elements are nontrivial cycles by 6.3.8. It follows from 6.3.3 that $L(n,m)$ can have no other generators since $[x_{1,j}, x_{i-1,j+1}] = x_{i,j} - \delta^j_{i+j} x_{i,j+1}$. □

In order to pass to $H^1(S(n))$ we need to produce primitive elements in $S(n)_*$ corresponding to ζ_n and ρ_n (the primitive $t_1^{p^j}$ corresponds to $h_{1,j}$). We will do this with the help of the determinant of a certain matrix. Recall from (6.2.3) that $S(n)\otimes\mathbf{F}_{p^n}$ was isomorphic to the dual group ring of S_m which has a certain faithful representation over $W(\mathbf{F}_{p^n})$ (6.2.6). The determinant of this representation gave a homomorphism of $S(n)$ into $\mathbf{Z}_p^\times$, the multiplicative group of units in the p-adic integers. We will see that in H^1 this map gives us ζ_n and ρ_n.

More precisely, let $M=(m_{i,j})$ be the n by n matrix over $\mathbf{Z}_p[t_1,t_2,\dots]/(t_1-t_i^{p^n})$ given by

$$m_{i,j}=\begin{cases}\sum\limits_{k\geq 0} p^k t_{kn+j-1}^{p^i} & \text{for } i\leq j\\ \sum\limits_{k\geq 1} p^k t_{kn+j-1}^{p^i} & \text{for } i>j\end{cases}$$

where $t_0=1$.

Now define $T_n\in S(n)_*$ to be the mod (p) reduction $p^{-1}(\det M-1)$ and for $p=2$ define $U_n\in S(n)_*$ to be the mod (2) reduction of $\frac{1}{8}(\det M^2-1)$. Then we have

6.3.12. THEOREM. *The elements $T_n\in S(n)_*$ and, for $p=2$, $U_n\in S(n)_*$ are primitive and represent the elements ζ_n and $\rho_n+\zeta_n\in H^1(S(n))$, respectively. Hence $H^1(S(n))$ is generated by these elements and for $n>1$ by the $h_{1,j}$ for $j\in\mathbf{Z}/(n)$.*

PROOF. The statement that T_n and U_n are primitive follows from 6.2.6. That they represent ζ_n and $\rho_n+\zeta_n$ follows from the fact that

$$T_n\equiv\sum_j t_n^{p^j} \mod (t_1,t_2,\dots,t_{n-1})$$

and

$$U_n\equiv\sum_j t_{2n}^{2^j}+t_n^{2^j} \mod (t_1,t_2,\dots,t_{n-1}). \qquad \square$$

EXAMPLES.

$$T_1=t_1,\quad U_1=t_1+t_2,\quad T_2=t_2+t_2^p-t_1^{1+p},$$
$$U_2=t_4+t_4^2+t_1t_3^2+t_1^2t_3+t_2+t_2^2t_1^3t_2+t_1^3t_2^2,$$

and

$$T_3=t_3+t_3^p+t_3^{p^2}+t_1^{1+p+p^2}-t_1t_2^p-t_1^pt_2^{p^2}-t_1^{p^2}t_2.$$

Moreira [**1, 3**] has found primitive elements in $BP_*(BP)/I_n$ which reduce to our T_n. The following result is a corollary of 6.2.7.

6.3.13. PROPOSITION. *If $p\nmid n$, then $H^*(S(n))$ decomposes as a tensor product of an appropriate subalgebra with $E(\zeta_n)$ for $p>2$ and $P(\zeta_n)\otimes E(\rho_n)$ for $p=2$.* $\square$

We now turn to the computation of $H^2(S(n))$ for $n>2$. We will compute all of $H^*S((n))$ for $n=2$ below.

6.3.14. THEOREM. *Let $n>2$*

(a) *For $p=2$, $H^2(S(n))$ is generated as a vector space by the elements ζ_n^2, $\rho_n\zeta_n$, ζ_n, $\zeta_nh_{1,j}$, $\rho_nh_{1,j}$, and $h_{1,i}h_{1,j}$ for $i\neq j\pm 1$, where $h_{1,i}h_{1,j}=h_{1,j}h_{1,i}$ and $h_{1,i}^2\neq 0$.*

(b) *For $p > 2$, $H^2(S(n))$ is generated by the elements*

$$\zeta_n h_{1,i},\ b_{1,i},\ g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle, \quad k_i = \langle h_{1,i+1}, h_{1,i+1}, h_{1,i}\rangle$$

and $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, where $h_{1,i}h_{1,j} + h_{1,j}h_{1,i} = 0$. □

Both statements require a sequence of lemmas. We treat the case $p = 2$ first.

6.3.15. LEMMA. *Let $p = 2$ and $n > 2$.*

(a) *$H^1(L(n,2))$ is generated by $h_{1,i}$ for $i \in \mathbf{Z}/(n)$.*

(b) *$H^2(L(n,2))$ is generated by the elements $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, g_i, k_i, and $e_{3,i} = \langle h_{1,i}, h_{1,i+1}, h_{1,i+2}\rangle$. The latter elements are represented by $h_{1,i}h_{2,i}$, $h_{1,i+1}h_{2,i}$, and $h_{1,i}h_{2,i+1} + h_{2,i}h_{1,i+2}$, respectively.*

(c) *$e_{3,i}h_{1,i+1} = h_{1,i}e_{3,i+1} + e_{3,i}h_{1,i}h_{1,i+3} = 0$, and these are the only relations among the elements $h_{1,i}e_{3,j}$.*

PROOF. We use the spectral sequence of 6.3.9 with $E_2 = E(h_{1,i}, h_{2,i})$ and $d_2(h_{2,i}) = h_{1,i}h_{1,i+1}$. All three statements can be verified by inspection. □

6.3.16. LEMMA. *Let $p = 2$, $n > 2$, and $2 < k \leq 2n$.*

(a) *$H^1(L(n,k))$ is generated by the elements $h_{1,i}$ along with ζ_n for $k \geq n$ and ρ_n for $k = 2n$.*

(b) *$H^2(L(n,k))$ is generated by products of elements in $H^1(L(n,k))$ subject to $h_{1,i}h_{1,i+1} = 0$, along with*

$$g_i = \langle h_{1,i}, h_{1,i}, h_{1,i+1}\rangle, \quad k_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i+1}\rangle,$$
$$\alpha_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i+2}, h_{1,i+1}\rangle, \quad \textit{and}$$
$$e_{k+1,i} = \langle h_{1,i}, h_{1,i+1}, \ldots, h_{1,i+k}\rangle.$$

The last two families of elements can be represented by $h_{3,i}h_{1,i+1} + h_{2,i}h_{2,i+1}$ and $\Sigma_s h_{s,i}h_{k+1-s,i+1}$ respectively.

(c) *$h_{1,i}e_{k+1,i+1} + e_{k+1,i}h_{1,i+1+k} = 0$ and no other relations hold among products of the $e_{k+1,i}$ with elements of H^1.*

PROOF. Again we use 6.3.9 and argue by induction on k, using 6.3.15 to start the induction. We have $E_2 = E(h_{k,i}) \otimes H^*(L(n,k-1))$ with $d_2(h_{k,i}) = e_{k,i}$. The existence of the α_i follows from the relation $e_{3,i}h_{1,i+1} = 0$ in $H^3(L(n,2))$ and that of $e_{k+1,i}$ from $h_{1,i}e_{k,i+1}h_{1,i+k} = 0$ in $H^3(L(n,k-1))$. The relation (c) for $k < 2$ is formal; it follows from a Massey product identity A1.4.6 or can be verified by direct calculation in the complex of 6.3.8. No combination of these products can be in the image of d_2 for degree reasons. □

6.3.17. *Let $p = 2$ and $n > 2$. Then $H^2(E_0S(n))$ is generated by the elements $\rho_n\zeta_n$, $\rho_n h_{1,i}$, $\zeta_n h_{1,i}$, $h_{1,i}h_{1,j}$ for $i \neq j \pm 1$, α_i, and $h_{i,j}^2 = b_{i,j}$ for $1 \leq i \leq n, j \in \mathbf{Z}/(n)$.*

PROOF. We use the modified first May spectral sequence of 6.3.5. We have $m = 2n$ and $H^2(L(n,m))$ is given by 6.3.16. By easy direct computation one sees that $d_2(g_i) = b_{1,i}h_{1,i+1}$ and $d_2(k_i) = h_{1,i}b_{1,i+1}$. We will show that $d_2(e_{2n+1,i}) = h_{1,i}b_{n,i} + h_{1,i+n}b_{n,i-1}$.

$$\Delta(t_{2n+1}) = \sum t_j \otimes t_{2n+1-j}^{p^j} + b_{n+1,n-1}$$

modulo terms of lower filtration by 4.3.15. Then by 4.3.22

$$d(b_{n+1,n-1}) = t_1 \otimes b_{n,n} + b_{n,n-1} \otimes t_1$$

modulo terms of lower filtration and the nontriviality of $d_2(e_{2n+1,i})$ follows. □

PROOF OF 6.3.14(a). We now consider the second May spectral sequence (6.3.4(b)). By 4.3.22 we have $d_2(b_{i,j}) = h_{1,j+1}b_{i-1,j+1} + h_{1,i+j}b_{i-1,j} \neq 0$ for $i > 1$. The remaining elements of $H^2E_0S(n)$ survive either for degree reasons or by 6.3.12. □

For $p > 2$ we need an analogous sequence of lemmas. We leave the proofs to the reader.

6.3.18. LEMMA. *Let $n > 2$ and $p > 2$.*
(a) *$H^1(L(n,2))$ is generated by $h_{1,i}$.*
(b) *$H^2(L(n,2))$ is generated by the elements $h_{1,i}h_{1,j}$ (with $h_{1,i}h_{1,i+1} = 0$). $g_i = h_{1,i}h_{2,i}$, $k_i = h_{1,i+1}h_{2,i}$ and $e_{3,i} = h_{1,i}h_{2,i+1}h_{2,i}h_{1,i+2}$.*
(c) *The only relations among the elements $h_{1,i}e_{3,j}$ are $h_{1,i}e_{3,i+1} - e_{3,i}h_{1,i+3} = 0$.* □

6.3.19. LEMMA. *Let $n > 2$, $p > 2$, and $2 < k \leq m$. Then*
(a) *$H^1(L(n,k))$ is generated by $h_{1,i}$ and, for $k \geq n$, ζ_n.*
(b) *$H^2(L(n,k))$ is generated by $h_{1,i}h_{1,j}$ (with $h_{1,i}h_{1,i+1} = 0$), g_i, h_i,*

$$e_{k+1,i} = \sum_{0<j<k+1} h_{j,i}h_{k+1-j,i+j},$$

and, for $k \geq n$, $\zeta_n h_{1,i}$.
(c) *The only relations among products of elements in H^1 with the $e_{k+1,i}$ are $h_{1,i}e_{k+1,i+1} - e_{k+1,i}h_{1,k+1} = 0$.* □

6.3.20. LEMMA. *Let $n > 2$ and $p > 2$. Then $H^2(E_0S(n))$ is generated by the elements $b_{i,j}$ for $i \leq m - n$ and by the elements of $H^2(L(n,m))$.*

PROOF OF 6.3.14(b). Again we look at the spectral sequence of 6.3.4(b). By arguments similar to those for $p = 2$ one can show that

$$d_p(b_{i,j}) = h_{1,i+j}b_{i-1,j} - h_{1,j+1}b_{i-1,j+1} \quad \text{for } i > 1$$

and

$$d_s(e_{m+1,i}) = h_{1,m+1+i-n}b_{m-n,i-1} - h_{1,i}b_{m-n,j} \quad \text{where } s = 1 + pn - (p-1)m,$$

and the remaining elements of $H^2(E_0S(n))$ survive as before. □

Now we will compute $H^*(S(n))$ at all primes for $n \leq 2$ and at $p > 3$ for $n = 3$.

6.3.21. THEOREM.
(a) *$H^*(S(1)) = P(h_{1,0}) \otimes E(\rho_1)$ for $p = 2$;*
(b) *$H^*(S(1)) = E(h_{1,0})$ for $p > 2$*
[note that $S(1)$ is commutative and that $\zeta_1 = h_{1,0}$].

PROOF. This follows immediately from 6.3.3, 6.3.5, and routine calculation. □

6.3.22. THEOREM. *For $p > 3$, $H^*(S(2))$ is the tensor product of $E(\zeta_2)$ with the subalgebra with basis $\{1, h_{1,0}, h_{1,1}, g_0, g_1, g_0h_{1,1}\}$ where*

$$g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle,$$
$$h_{1,0}g_1 = g_0h_{1,1}, \quad h_{1,0}g_0 = h_{1,1}g_1 = 0,$$

and

$$h_{1,0}h_{1,1} = h_{1,0}^2 = h_{1,1}^2 = 0.$$

In particular, the Poincaré series is $(1+t)^2(1+t+t^2)$.

PROOF. The computation of $H^*(L(2,2))$ by 6.3.8 or 6.3.9 is elementary, and there are no algebra extension problems for the spectral sequences of 6.3.9 or 6.3.4(b). □

We will now compute $H^*(S(2))$ for $p = 3$. Our description of it in the first edition was incorrect, as was pointed out by Henn [**1**]. The computation given here is influenced by Henn but self-contained. Henn showed that there are two conjugacy classes of subgroups of order 3 in the group S_2. In each case the centralizer is the group of units congruent to one modulo the maximal ideal in the ring of integers of an embedded copy of the field $K = \mathbf{Q}_3[\zeta]$, where ζ is a primitive cube root of unity. Let C_1 and C_2 denote these two centralizers. Henn showed that the resulting map

$$H^*(S_2) \to H^*(C_1) \oplus H^*(C_2)$$

is a monomorphism.

We will describe this map in Hopf algebraic terms. Choose a fourth root of unity $i \in \mathbf{F}_9$, let $a = \pm i$, and consider the two quotients

$$\overline{S(2)}_+ = S(1,1)_i \qquad \text{and} \qquad \overline{S(2)}_- = S(1,1)_{-i},$$

where $S(1,1)_a$ is the quotient of $S(2) \otimes \mathbf{F}_9$ described in 6.2.13. Henn's map is presumably equivalent to

$$H^*(S(2)) \otimes \mathbf{F}_9 \to H^*(\overline{S(2)}_+) \oplus H^*(\overline{S(2)}_-). \tag{6.3.23}$$

In any case we will show that this map is a monomorphism.

We have the following reduced coproducts in $\overline{S(2)}_\pm$.

$$\begin{array}{rcl} \overline{t}_1 & \mapsto & 0 \\ \overline{t}_2 & \mapsto & a\overline{t}_1 \otimes \overline{t}_1 \\ \overline{t}_3 & \mapsto & \overline{t}_1 \otimes \overline{t}_2 + \overline{t}_2 \otimes \overline{t}_1 - a^3(\overline{t}_1^2 \otimes \overline{t}_1 + \overline{t}_1 \otimes \overline{t}_1^2) \end{array}$$

It follows that $\overline{t}_2 + a\overline{t}_1^2$ and $\overline{t}_3 - \overline{t}_1\overline{t}_2$ are primitive. The filtration of 6.3.1 induces one on $\overline{S(2)}_\pm$, and the methods of this section lead to

$$H^*(\overline{S(2)}_\pm) = E(\overline{h}_{1,0}, \overline{h}_{2,0}, \overline{h}_{3,0}) \otimes P(\overline{b}_{1,0})$$

with the evident notation.

6.3.24. THEOREM. *For $p = 3$, $H^*(S(2))$ is a free module over*

$$E(\zeta_2) \otimes P(b_{1,0})$$

on the generators

$$\{1, h_{1,0}, h_{1,1}, b_{1,1}, \xi, a_0, a_1, b_{1,1}\xi\},$$

where the elements $\xi \in H^2$ and $a_0, a_1 \in H^3$ will be defined below. The algebra structure is indicated in the following multiplication table.

1	$h_{1,0}$	$h_{1,1}$	$b_{1,1}$	ξ	a_0	a_1
$h_{1,0}$	0	0	$-b_{1,0}h_{1,1}$	0	$-b_{1,1}\xi$	$-b_{1,0}\xi$
$h_{1,1}$		0	$b_{1,0}h_{1,0}$	0	$-b_{1,0}\xi$	$b_{1,1}\xi$
$b_{1,1}$			$-b_{1,0}^2$	$b_{1,1}\xi$	$-b_{1,0}a_1$	$b_{1,0}a_0$
ξ				0	0	0
a_0					0	0
a_1						0

In particular, the Poincaré series is

$$(1+t)^2(1+t^2)/(1-t).$$

Moreover the map of (6.3.23) is a monomorphism.

PROOF. Our basic tools are the spectral sequences of 6.3.10 and some Massey product identities from A1.4. We have $H^*(\tilde{L}(2,1)) \cong E(h_{1,0}, h_{1,1}) \otimes P(b_{1,0}, b_{1,1})$, and a spectral sequence converging to $H^*(\tilde{L}(2,2))$ with $E_2 = E(\zeta_2, \eta) \otimes H^*(\tilde{L}(2,1))$, where

$$\begin{aligned} \zeta_2 &= h_{2,0} + h_{2,1}, & \eta &= h_{2,1} - h_{2,0}, \\ d_2(\zeta_2) &= 0, & d_2(\eta) &= h_{1,0}h_{1,1}, \end{aligned}$$

and $E_3 = E_\infty$. Hence E_∞ is a free module over $E(\zeta_2) \otimes P(b_{1,0}, b_{1,1})$ on generators

$$\{1,\, h_{1,0},\, h_{1,1},\, g_0,\, g_1,\, h_{1,0}g_1 = h_{1,1}g_0,\},$$

where $g_i = \langle h_{1,i}, h_{1,i+1}, h_{1,i}\rangle$. This determines the additive structure of $H^*(\tilde{L}(2,2))$, but there are some nontrivial extensions in the multiplicative structure. We know by 6.3.13 that we can factor out $E(\zeta_2)$, and we can write $b_{1,i}$ as the Massey product $-\langle h_{1,i}, h_{1,i}, h_{1,i}\rangle$. Then by A1.4.6 we have $h_{1,i}g_i = -b_{1,i}h_{1,i+1}$, $g_i^2 = -b_{1,i}g_{i+1}$, $g_ig_{i+1} = b_{1,i}b_{1,i+1}$. These facts along with the usual $h_{1,i}^2 = h_{1,0}h_{1,1} = 0$ determine $H^*(\tilde{L}(2,2))$ as an algebra.

This algebra structure allows us to embed $H^*(\tilde{L}(2,2))$ in the ring

$$R = E(\zeta_2,\, h_{1,0},\, h_{1,1}) \otimes P(s_0,\, s_1)/(h_{1,0}h_{1,1},\, h_{1,0}s_1 - h_{1,1}s_0)$$

by sending ζ_2 and $h_{1,i}$ to themselves and

$$\begin{aligned} b_{1,i} &\mapsto -s_i^3 \\ g_0 &\mapsto s_0^2 s_1 \\ g_1 &\mapsto s_0 s_1^2. \end{aligned}$$

Here the cohomological degree of s_i is $2/3$, and $H^*(\tilde{L}(2,2))$ maps isomorphically to the subring of R consisting of elements of integral cohomological degree.

Next we have the spectral sequence of 6.3.10 converging to

$$H^*(\tilde{L}(2,3)) \cong H^*(E^0 S(2))$$

with $E_2 = E(h_{3,0}, h_{3,1}) \otimes H^*(\tilde{L}(2,2))$, and $d_2(h_{3,i}) = g_i - b_{1,i+1}$. We will see shortly that $E_3 = E_\infty$ for formal reasons. Tensoring this over $H^*(\tilde{L}(2,2))$ with R gives a spectral sequence with

$$\begin{aligned} & E_2 &=&\ E(h_{3,0}, h_{3,1}) \otimes R \\ \text{and}\quad & d_2(h_{3,0}) &=&\ s_1(s_0^2 + s_1^2) \\ & d_2(h_{3,1}) &=&\ s_0(s_0^2 + s_1^2). \end{aligned}$$

This can be simplified by tensoring with $\mathbf{F}_9$ (which contains $i = \sqrt{-1}$) and defining

$$\begin{array}{rclcrcl} x_0 & = & h_{1,0} + ih_{1,1} & & x_1 & = & h_{1,0} - ih_{1,1} \\ y_0 & = & s_0 + is_1 & & y_1 & = & s_0 - is_1 \\ z_0 & = & ih_{3,0} + h_{3,1} & & z_1 & = & -ih_{3,0} + h_{3,1} \end{array}$$

The Galois group of $\mathbf{F}_9$ over $\mathbf{F}_3$ acts here by conjugating scalars and permuting the two subscripts. Then we have

$$R \otimes \mathbf{F}_9 = E(\zeta_2,\, x_0,\, x_1) \otimes P(y_0,\, y_1)/(x_0x_1,\, x_0y_1 - x_1y_0),$$

where the cohomological degrees of x_i and y_i are 1 and 2/3 respectively. In the spectral sequence we have

$$d_2(z_0) = y_0^2y_1 \qquad \text{and} \qquad d_2(z_1) = y_0y_1^2. \tag{6.3.25}$$

The image of $H^*(\tilde{L}(2,2)) \otimes \mathbf{F}_9$ in $R \otimes \mathbf{F}_9$ is a free module over the ring

$$B = E(\zeta_2) \otimes P(y_0^3, y_1^3)$$

on the following set of six generators.

$$C = \left\{1,\, x_0,\, x_1,\, y_0^2y_1,\, y_0y_1^2,\, x_0y_0y_1^2 = x_1y_0^2y_1\right\}$$

Hence the image of $E(h_{3,0},\, h_{3,1}) \otimes H^*(\tilde{L}(2,2)) \otimes \mathbf{F}_9$ is a free B-module on the set

$$\{1,\, z_0,\, z_1,\, z_0z_1\} \otimes C,$$

but it is convenient to replace this basis by the set of elements listed in the following table.

1	z_0	z_1	z_0z_1
x_0	x_0z_0	$\beta = x_0z_1 - x_1z_0$	$-x_0z_0z_1$
x_1	$\delta = -x_1z_0 - x_0z_1$	x_1z_1	$x_1z_0z_1$
$y_0^2y_1$	$\alpha_1 = y_0^2y_1z_0 - y_0^3z_1$	$\varepsilon = y_0^2y_1z_1 - y_0y_1^2z_0$	$y_0^2y_1z_0z_1$
$y_0y_1^2$	$\gamma = -y_0y_1^2z_0 - y_0^2y_1z_1$	$\alpha_0 = y_0y_1^2z_1 - y_1^3z_0$	$-y_0y_1^2z_0z_1$
$x_0y_0y_1^2$	$-x_0\varepsilon$	$x_1\varepsilon$	$x_0y_0y_1^2z_0z_1$

This basis is Galois invariant up to sign, i.e., the Galois image of each basis element is another basis element. The elements 1, $x_0y_0y_1^2$, δ, and γ are self-conjugate, while β, ε, z_0z_1 and $x_0y_0y_1^2z_0z_1$ are antiself-conjugate. The remaining elements form eight conjugate pairs.

In the spectral sequence the following twelve differentials (listed as six Poincaré dual pairs) are easily derived from (6.3.25) and account for each of these 24 basis elements.

$$\begin{array}{rclcrcl} d_2(z_0) & = & y_0^2y_1 & & d_2(x_1z_0z_1) & = & x_1\varepsilon \\ d_2(z_1) & = & y_0y_1^2 & & d_2(-x_0z_0z_1) & = & -x_0\varepsilon \\ d_2(z_0z_1) & = & \varepsilon & & d_2(\delta) & = & x_0y_0y_1^2 \\ d_2(x_0z_0) & = & y_0^3(x_1) & & d_2(y_0^2y_1z_0z_1) & = & y_0^3(\alpha_0) \\ d_2(x_1z_1) & = & y_1^3(x_0) & & d_2(-y_0y_1^2z_0z_1) & = & y_1^3(\alpha_1) \\ d_2(\gamma) & = & y_0^3y_1^3(1) & & d_2(x_0y_0y_1^2z_0z_1) & = & y_0^3y_1^3(\beta) \end{array}$$

The spectral sequence collapses from E_3 since there are no elements in $E_3^{*,t}$ for $t > 1$. The image of $H^*(\tilde{L}(2,3)) \otimes \mathbf{F}_9$ in the E_∞-term is the B-module generated by

$$\{1,\, x_0,\, x_1,\, \alpha_0,\, \alpha_1,\, \beta\}$$

subject to the module relations

$$\begin{array}{rclcrcl} y_0^3y_1^3(1) & = & 0, & & y_0^3y_1^3(\beta) & = & 0,\\ y_0^3(x_1) & = & 0, & & y_0^3(\alpha_0) & = & 0,\\ y_1^3(x_0) & = & 0, & \text{and} & y_1^3(\alpha_1) & = & 0. \end{array}$$

The only nontrivial products among these six elements are

$$x_0\alpha_1 = -y_0^3\beta \qquad \text{and} \qquad x_1\alpha_0 = y_1^3\beta.$$

Equivalently the image is the free module over $E(\zeta_2)\otimes P(y_0^3+y_1^3)$ on the eight generators

$$\left\{1,\, x_0,\, x_1,\, y_1^3,\, \beta,\, \alpha_0,\, \alpha_1,\, y_1^3\beta\right\} \tag{6.3.26}$$

with suitable algebra relations.

It follows that $H^*(E^0S(2))$ itself is a free module over $E(\zeta_2)\otimes P(b_{1,0})$ on the eight generators

$$\{1,\, h_{1,0},\, h_{1,1},\, b_{1,1},\, \xi,\, a_0,\, a_1,\, b_{1,1}\xi\}\,.$$

where

$$\xi = i\beta, \qquad a_0 = \alpha_0+\alpha_1, \qquad \text{and} \qquad a_1 = i(\alpha_0-\alpha_1).$$

It also follows that $E^0H^*(S(2))$ has the relations stated in the theorem. The absence of nontrivial multiplicative extensions in $H^*(S(2))$ will follow from the the fact that the map of (6.3.23) is monomorphic and there are no extensions in its target.

Now we will determine the images of the elements of (6.3.26) under the map of (6.3.23). Recall that

$$H^*(\overline{S(2)}_\pm) = E(\overline{h}_{1,0},\, \overline{h}_{2,0},\, \overline{h}_{3,0})\otimes P(\overline{b}_{1,0})$$

As before it is convenient to adjoin a cube root $\overline{s}_0$ of $-\overline{b}_{1,0}$ and let

$$\overline{R}_\pm = E(\overline{h}_{1,0},\, \overline{h}_{2,0})\otimes P(\overline{s}_0).$$

The map

$$H^*(S(2))\otimes \mathbf{F}_9 \to E(\overline{h}_{3,0})\otimes \overline{R}_+ \oplus E(\overline{h}_{3,0})\otimes \overline{R}_-$$

behaves as follows.

$$\begin{array}{rcl@{\qquad\qquad}rcl} x_0 & \mapsto & (0,\, -\overline{h}_{1,0}) & x_1 & \mapsto & (-\overline{h}_{1,0},\, 0)\\ y_0 & \mapsto & (0,\, -\overline{s}_0) & y_1 & \mapsto & (-\overline{s}_0,\, 0)\\ z_0 & \mapsto & (-i\overline{h}_{3,0},\, 0) & z_1 & \mapsto & (0,\, i\overline{h}_{3,0})\\ \beta & \mapsto & (-i\overline{h}_{1,0}\overline{h}_{3,0},\, -i\overline{h}_{1,0}\overline{h}_{3,0}) & & & \\ \alpha_0 & \mapsto & (i\overline{s}_0^3\overline{h}_{3,0},\, 0) & \alpha_1 & \mapsto & (0,\, -i\overline{s}_0^3\overline{h}_{3,0}) \end{array}$$

It follows that Henn's map is a monomorphism. □

We now turn to the case $n = p = 2$. We will only compute $E^0H^*(S(2))$, so there will be some ambiguity in the multiplicative structure of $H^*(S(2))$. In order to state our result we need to define some classes. Recall (6.3.12) that $H^1(S(2))$ is the $\mathbf{F}_2$-vector space generated by $h_{1,0}$, $h_{1,1}$, ζ_2 and ρ_2. Let

$$\alpha_0 \in \langle \zeta_2, h_{1,0}, h_{1,1}\rangle, \quad \beta \in \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1}\rangle, \quad g = \langle h, h^2, h, h^2\rangle,$$

where $h = h_{1,0}+h_{1,1}$, $\tilde{x} = \langle x, h, h^2\rangle$ for $x = \zeta_2,\ \alpha_0,\ \zeta_2^2$, and $\alpha_0\zeta_2$ (more precise definitions of α_0 and β will be given in the proof).

6.3.27. THEOREM. *$E^0H^*(S(2))$ for $p=2$ is a free module over $P(g)\otimes E(\rho_2)$ on 20 generators*: 1, $h_{1,0}$, $h_{1,1}$, $h_{1,0}^2$, $h_{1,1}^2$, $h_{1,0}^3$, β, $\beta h_{1,0}$, $\beta h_{1,1}$, $\beta h_{1,0}^2$, $\beta h_{1,1}^2$, $\beta h_{1,0}^3$, ζ_2, α_0, ζ_2^2, $\alpha_0\zeta_2$, $\tilde{\zeta}_2$, $\tilde{\alpha}_0$, $\tilde{\zeta}_2^2$, $\alpha_0\tilde{\zeta}_2$, *where $\alpha_0\in H^2(S(2))$ and has filtration degree* 4, *$\beta\in H^3(S(2))$ and has filtration degree 8, $g\in H^4(S(2))$ and has filtration degree 8, and the cohomological and filtration degrees of $\tilde{x}$ exceed those of x by 2 and 4, respectively. Moreover $h_{1,0}^3=h_{1,1}^3$, $\alpha_0^2=\tilde{\zeta}_2^2$, and all other products are zero. The Poincaré series is* $(1+t)^2(1-t^5)/(1-t)^2(1+t^2)$.

PROOF. We will use the same notation for corresponding classes in the various cohomology groups we will be considering along the way.

Again our basic tool is 6.3.10. It follows from 6.3.5 that $H^*(E_0S(2))$ is the cohomology of the complex

$$P(h_{1,0},h_{1,1},\zeta_2,h_{2,0})\otimes E(h_{3,0},h_{3,1},\rho_2,h_{4,0})$$

with

$$\begin{aligned} &d(h_{1,i})=d(\zeta_2)=d(\rho_2)=0,\\ &d(h_{3,i})=h_{1,i}\zeta_2,\quad d(h_{2,0})=h_{1,0}h_{1,1},\end{aligned}$$

and

$$d(h_{4,0})=h_{1,0}h_{3,1}+h_{1,1}h_{3,0}+\zeta_2^2.$$

This fact will enable us to solve the algebra extension problems in the spectral sequences of 6.3.10.

For $H^*(\tilde{L}(2,2))$ we have a spectral sequence with $E_2=P(h_{1,0},h_{1,1},\zeta_2,h_{2,0})$ with $d_2(\zeta_2)=0$ and $d_2(h_{2,0})=h_{1,0}h_{1,1}$. It follows easily that

$$H^*(\tilde{L}(2,2))=P(h_{1,0},h_{1,1},\zeta_2,b_{2,0})/(h_{1,0}h_{1,1})$$

where $b_{2,0}=h_{2,0}^2=\langle h_{1,0},h_{1,1},h_{1,0},h_{1,1}\rangle$.

For $H^*(\tilde{L}(2,3))$ we have a spectral sequence with

$$E_2=E(h_{3,0},h_{3,1})\otimes H^*(\tilde{L}(2,2))$$

and $d_2(h_{3,i})=h_{1,i}\zeta_2$. Let

$$\alpha_i=h_{1,i+1}h_{3,i}+\zeta_2h_{2,i}\in\langle\zeta_2,h_{1,i},h_{1,i+1}\rangle.$$

Then $H^*(\tilde{L}(2,3))$ as a module over $H^*(\tilde{L}(2,2))$ is generated by 1, α_0, and α_1 with

$$\zeta_2h_{1,i}=\zeta_2(\alpha_0+\alpha_1+\zeta_2^2)=h_{1,i}\alpha_i=\zeta_2h_{1,i+1}\alpha_i=0$$

and

$$\alpha_0^2=\zeta_2^2b_{2,0},\quad \alpha_1^2=\zeta_2^2(\zeta_2^2+b_{2,0}),\quad \alpha_0\alpha_1=\zeta_2^2(\alpha_0+b_{2,0}).$$

The Poincaré series for $H^*(\tilde{L}(2,3))$ is $(1+t+t^2)/(1-t^2)$.

For $H^*(\tilde{L}(2,4))$ we have a spectral sequence with

$$E_2=E(h_{4,0},\rho_2)\otimes H^*(\tilde{L}(2,3)),$$

$d_2(\rho_2)=0$, and $d_2(h_{4,0})=\alpha_0+\alpha_1$. Define $\beta\in H^3(\tilde{L}(2,4))$ by

$$\beta=h_{4,0}(\alpha_0+\alpha_1+\zeta_2^2)+\zeta_2h_{3,0}h_{3,1}\in\langle h_{1,0},\zeta_2,\zeta_2^2,h_{1,1}\rangle.$$

Then $H^*(\tilde{L}(2,4))$ is a free module over $E(\rho_2)\otimes P(b_{2,0})$ on generators 1, $h_{1,i}^t$, ζ_2, ζ_2^2, α_0, $\alpha_0\zeta_2$, β, and $\beta h_{1,i}^t$, where $t>0$. As a module over $H^*(\tilde{L}(2,3))\otimes E(p_2)$ it is generated by 1 and β, with $(\alpha_0+\alpha_1)1=\zeta_2^3(1)=\alpha_0\zeta_2^2(1)=0$. To solve the algebra extension problem we observe that $\beta\zeta_2 = 0$ for degree reasons; $\beta\alpha_i = \beta\langle\zeta_2, h_{1,i}, h_{1,i+1}\rangle = \langle\beta, \zeta_2, h_{1,i}\rangle h_{1,i+1} = 0$ since $\langle\beta, \zeta_2, h_{1,i}\rangle = 0$ for degree reasons; and $E(\rho_2)$ splits off multiplicatively by the remarks at the beginning of the proof.

This completes the computation of $H^*(E_0S(2))$. Its Poincaré series is $(1+t)^2/(1-t)^2$. We now use the second May spectral sequence [6.3.4(b)] to pass to $E^0H^*(S(2))$. $H^*(E_0S(2))$ is generated as an algebra by the elements $h_{1,0}$, $h_{1,1}$, ζ_2, ρ_2, α_0, $b_{2,0}$, and β. The first four of these are permanent cycles by 6.3.12.

By direct computation in the cobar resolution we have

$$d(t_3 + t_1t_2^2) = \zeta_2 \otimes t_1, \tag{6.3.28}$$

so the Massey product for α_0 is defined in $H^*(S(2))$ and the α_0 is a permanent cycle. We also have

$$d(t_2 \otimes t_2 + t_1 \otimes t_1^2t_2 + t_1t_2 \otimes t_1^2) = t_1 \otimes t_1 \otimes t_1 + t_1^2 \otimes t_1^2 \otimes t_1^2,$$

so $d_2(b_{2,0}) = h_{1,0}^2 + h_{1,1}^3$. Inspection of the E_3 term shows that $b_{2,0}^2 = \langle h, h^2, h, h^2\rangle$, (where $h = h_{1,0} + h_{1,1}$) is a permanent cycle for degree reasons.

We now show that $\beta = \langle h_{1,0}, \zeta_2, \zeta_2^2, h_{1,1}\rangle$ is a permanent cycle by showing that its Massey product expression is defined in $E^0H^*(S(2))$. The products $h_{1,0}\zeta_2$ and $\zeta_2^2h_{1,1}$ are zero by 6.3.28 and we have

$$\begin{aligned} d(\tilde{t}_3 \otimes \tilde{t}_3^2 + T_2\tilde{t}_3 \otimes t_1^2 + T_2 \otimes t_4 + T_2 \otimes t_2^3 + T_2 \otimes t_1^3(1 + t_2 + t_2^2)) \\ = T_2 \otimes T_2 \otimes T_2, \end{aligned} \tag{6.3.29}$$

where $\tilde{t}_3 = t_3 + t_1t_2^2$ and $T_2 = t_2 + t_2^2 + t_1^3$, so $\zeta_2^3 = 0$ in $H^*(S(2))$. Inspection of $H^3(E_0S(2))$ shows there are no elements of internal degree 2 or 4 and filtration degree > 7, so the triple products $\langle h_{1,0}, \zeta_2, \zeta_2^2\rangle$ and $\langle\zeta_2, \zeta_2^2, h_{1,1}\rangle$ must vanish and β is a permanent cycle.

Now the E_3 term is a free module over $E(\rho_2) \otimes P(b_{2,0}^2)$ on 20 generators: 1, $h_{1,0}$, $h_{1,1}$, $h_{1,0}^2$, $h_{1,1}^2$, $h_{1,0}^3 = h_{1,1}^3$, β, $\beta h_{1,0}$, $\beta h_{1,1}$, $\beta h_{1,1}^2$, βj_{10}^2, $\beta h_{1,0}^3$, ζ_2, α_0, ζ_2^2, $\alpha_0\zeta_2$, $\zeta_2 b_{1,0}$, $\alpha_0 b_{2,0}$, $\zeta_2^2 b_{2,0}$, $\zeta_2\alpha_0 b_{2,0}$. The last four in the list now have Massey product expressions $\langle\zeta_2, h, h^2\rangle$, $\langle\alpha_0, h, h^2\rangle$, $\langle\zeta_2^2, h, h^2\rangle$, and $\langle\alpha_0, \zeta_2, h, h^2\rangle$, respectively. These elements have to be permanent cycles for degree reasons, so $E_3 = E_\infty$, and we have determined $E^0H^*(S(2))$. □

We now describe an alternative method of computing $H^*(S(2) \otimes \mathbf{F}_4)$, which is quicker than the previous one, but yields less information about the multiplicative structure. By 6.3.4, this group is isomorphic to $H^*(S_2; \mathbf{F}_4)$, the continuous cohomology of certain 2-adic Lie group with trivial coefficients in $\mathbf{F}_4$, S_2 is the group of units in the degree 4 extension E_2 of $\mathbf{Z}_2$ obtained by adjoining ω and S with $\omega^2 + \omega + 1 = 0$, $S^2 = 2$ and $S\omega = \omega^2 S$.

Let Q denote the quaternion group, i.e., the multiplicative group (with 8 elements) of quaternionic integers of modulus 1.

6.3.30. PROPOSITION. *There is a split short exact sequence of groups*

$$1 \to G \xrightarrow{i} S_2 \xrightarrow{j} Q \to 1. \tag{6.3.31}$$

The corresponding extension of dual group algebras over is

$$Q_* \xrightarrow{j_*} S(2) \xrightarrow{i_*} G_*$$

where $Q_* \cong \mathbf{F}_4[x, y]/(x^4 - x, y^2 - y)$ *and* $G_* \cong S(2)/(t_1, t_2 + \omega t_2^2)$ *as algebras where* $j_*(x) = t_1$, $j_*(y) = \bar{\omega}t_2 + \bar{\omega}x^2t_2^2$, *and* $\bar{\omega}$ *is the residue class of* ω.

PROOF. The splitting follows the theory of division algebras over local fields (Cassels and Fröhlich [**1**, pp. 137–138]]) which implies that $E_2 \otimes \mathbf{Q}_2$ is isomorphic to the 2-adic quaternions. We leave the remaining details to the reader. □

6.3.32.
(a) $H^*(Q; \mathbf{F}_2) = P(h_{1,0}, h_{1,1}, g)/(h_{1,0}h_{1,1}, h_{1,0}^3 + h_{1,1}^3)$.
(b) $H^*(G; \mathbf{F}_2) = E(\zeta_2, \rho_2, h_{3,0}, h_{3,1})$.

PROOF. Part (a) is an easy calculation with the change-to-rings spectral sequence (A1.3.14) for $\mathbf{F}_2[x]/(x^4 + x) \to Q_* \to \mathbf{F}_2[y]/(y^2 + y)$. For (b) the filtration of $S(2)$ induces one on G_*. It is easy to see that E^0G_* is cocommutative and the result follows with no difficulty. □

6.3.33. PROPOSITION. *In the Cartan–Eilenberg spectral sequence for* 6.3.31*,* $E_3 = E_\infty$ *and we get the same additive structure for* $H^*(S(2))$ *as in* 6.3.27.

PROOF. We can take $H^*(G) \otimes H^*(Q)$ as our E_1-term. Each term is a free module over $E(\rho_2) \otimes P(g)$. We leave the evaluation of the differentials to the reader. □

Finally, we consider the case $n = 3$ and $p \geq 5$. We will not make any attempt to describe the multiplicative structure. An explict basis of $E^0H^*(S(3))$ will be given in the proof, from which the multiplication can be read off by the interested reader. It seems unlikely that there are any nontrivial multiplicative extensions.

6.3.34. THEOREM. *For* $p \geq 5$*,* $H^*(S(3))$ *has the following Poincaré series*: $(1 + t)^3(1 + t + 6t^2 + 3t^3 + 6t^4 + t^5 + t^6)$.

PROOF. We use the spectral sequences of 6.3.9 to compute $H^*(L(3, 2))$ and $H^*(L(3, 3))$. For the former the E_2-term is $H(h_{1,i}) \otimes E(h_{2,i})$ with $i \in \mathbf{Z}/(3)$, $d_2(h_{1,i}) = 0$ and $d_2(h_{2,i}) = h_{1,i}h_{1,i+1}$. The Poincaré series for $H^*(L(3, 2))$ is $(1 + t)^2(1 + t + 5t^2 + t^3 + t^4)$ and it is generated as a vector space by the following elements and their Poincaré duals: 1, $h_{1,i}$, $g_i = h_{1,i}h_{2,i}$, $k_i = h_{2,i}h_{1,i+1}$, $e_{3,i} = h_{1,i}h_{2i+1} + h_{2,i}h_{1,i+2}$ (where $\sum_i e_{3,i} = 0$), $g_ih_{1,i+1} = h_{1,i}k_i = h_{1,i}h_{2,i}h_{1,i+1}$, and $h_{1,i}e_{3,i} = g_ih_{1,i+2} = h_{1,i}h_{2,i}h_{1,i+2}$.

For $H^*(L(3, 3))$ we have $E_2 = E(h_{3,i} \otimes H^*(L(3, 2)))$ with $d_2(h_{3,i}) = e_{3,i}$, so $d_2(\sum h_{3,i}) = 0$. $H^*(L(3, 3))$ has the indicated Poincaré series and is a free module over $E(\zeta_3)$, where $\zeta_3 = \sum h_{3,i}$, on the following 38 elements and the duals of their products with ζ_3:

$$1,\ h_{1,i},\ g_i,\ k_i,\ b_{1,i+2} = h_{1,i}h_{3,i} + h_{2,i}h_{2,i+2} + h_{3,i}h_{1,i},$$
$$g_ih_{1,i+1} = h_{1,i},\ k_i,\ h_{1,i}h_{2,i}h_{2,i+2},\ h_{1,i}h_{2,i}h_{2,i+1} + h_{1,i}h_{1,i+1}h_{3,i},$$
$$h_{1,i}h_{2,i}h_{3,i},\ h_{1,i}h_{2,i+2}h_{3,i+1},\ \sum_i(h_{1,i}h_{2,i+1} - h_{1,i+1}h_{2,i+2})h_{3,i},\ h_{1,i}k_ih_{3,j}$$

(where $h_{1,i}k_i \sum_j h_{3,j}$ is divisible by ζ_3), and $h_{1,i+2}h_{1,i}h_{2,i}(h_{3,i} + h_{3,i+1}) \pm h_{1,i}h_{2,0}h_{2,1}h_{2,2}$. □

4. The Odd Primary Kervaire Invariant Elements

The object of this section is to apply the machinery above to show that the Adams–Novikov element $\beta_{p^i/p^i} \in \mathrm{Ext}^2$ (see 5.1.19) is not a permanent cycle for $p > 2$ and $i > 0$. This holds for the corresponding Adams element b_i (4.3.2) for $p > 3$ and $i > 0$; by 5.4.6 we know β_{p^i/p^i} maps to b_i. The latter corresponds to the secondary cohomology operation associated with the Adem relation $P^{(p-1)p^i}P^{p^i} = \cdots$. The analogous relation for $p = 2$ is $Sq^{2^i}Sq^{2^i} = \cdots$, which leads to the element h_i^2, which is related to the Kervaire invariant by Browder's theorem, hence the title of the section. To *stress this analogy we will denote* β_{p^i/p^i} *by* θ_i.

We know by direct calculation (e.g., 4.4.20) that θ_0 is a permanent cycle corresponding to the first element in coker J. By Toda's theorem (4.4.22) we know θ_1 is not a permanent cycle; instead we have $d_{2p-1}(\theta_1) = \alpha_1\theta_0^p$ (up to nonzero scalar multiplication) and this is the first nontrivial differential in the Adams–Novikov spectral sequence. Our main result is

6.4.1. ODD PRIMARY KERVAIRE INVARIANT THEOREM. *In the Adams–Novikov spectral sequence for* $p > 2$ $d_{2p-1}(\theta_{i+1}) \equiv \alpha_1\theta_i^p \mod \ker\theta_0^{\alpha_i}$ *(up to nonzero scalar multiplication) where* $a_i = p(p^i-1)/(p-1)$ *and* $\alpha_1\theta_i^p$ *is nonzero modulo this indeterminacy.* □

Our corresponding result about the Adams spectral sequence fails for $p = 3$, where b_2 is a permanent cycle even though b_1 is not.

6.4.2. THEOREM. *In the Adams spectral sequence for* $p \geq 5$ b_i *is not a permanent cycle for* $i \geq 1$. □

From 6.4.1 we can derive the nonexistence of certain finite complexes which would be useful for constructing homotopy elements with Novikov filtration 2.

6.4.3. THEOREM. *There is no connective spectrum* X *such that*

$$BP_*(X) = BP_*/(p, v_1^{p^i}, v_2^{p^i})$$

for $i > 0$ *and* $p > 2$.

PROOF. Using methods developed by Smith [**1**], one can show that such an X must be an 8-cell complex and that there must be cofibrations

(i) $\Sigma^{2p^i(p^2-1)}Y \xrightarrow{f} Y' \to X$,
(ii) $\Sigma^{2p^i(p-1)}V(0) \xrightarrow{g} V(0) \to Y$,
(iii) $\Sigma^{2p^i(p-1)}V(0) \xrightarrow{g'} V(0) \to Y'$,

where $V(0)$ is the mod (p) Moore spectrum, g and g' induce multiplication by $v_1^{p^i}$ in $BP_*(V(0)) = BP_*/(p)$, and f induces multiplication by $v_2^{p^i}$ in

$$BP_*(Y) = BP_*(Y') = BP_*/(p, v_1^{p^i}).$$

$V(0)$ and the maps g, g' certainly exist; e.g., Smith showed that there is a map

$$\alpha\colon \Sigma^{2(p-1)}V(0) \to V(0)$$

which includes multiplication by v_1, hence α^{p^i} induces multiplication by $v_1^{p^i}$, but it may not be the only map that does so.

Hence we have to show that the existence of f leads to a contradiction. Consider the composite

$$S^{2p^i(p^2-1)} \xrightarrow{j} \Sigma^{2p^i(p^2-1)}Y \xrightarrow{f} Y' \xrightarrow{k} S^{2+2p^i(p-1)},$$

where j is the inclusion of the bottom cell and k is the collapse onto the top cell. We will show that the resulting element in $\pi^s_{2p^{i+1}(p-1)-2}$ would be detected in the

Novikov spectral sequence by θ_i, thus contradicting 6.4.1. The cofibrations (ii) and (iii) induce the following short exact sequence of BP^* modules

$$0 \to \Sigma^{2p^i(p-1)}BP_*/(p) \xrightarrow{v_1^{p^i}} BP_*/(p) \to BP_*/(p, v_1^{p^i}) \to 0,$$

and the cofibration

$$S^0 \xrightarrow{p} S^0 \to V(0)$$

induces

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0.$$

Hence we get connecting homomorphisms

$$\delta_1\colon \operatorname{Ext}^0(BP_*/(p, v_1^{p^i})) \to \operatorname{Ext}^1(BP_*/(p))$$

and

$$\delta_0\colon \operatorname{Ext}^1(BP_*/(p)) \to \operatorname{Ext}^2(BP_*).$$

The element $fj \in \pi_{2p^i(p^2-1)}(Y')$ is detected by $v_2^{p^i} \in \operatorname{Ext}^0(BP_*/(p, v_1^{p^i}))$. We know (5.1.19) that

$$\theta_i = \delta_0\delta_1(v_2^{p^i}) \in \operatorname{Ext}^2(BP_*)$$

detects the element $kfj \in \pi^S_{2p^{i+1}(p-1)-2}$. □

The statement in 6.4.1 that $\alpha_1\theta_i^p$ is nonzero modulo the indeterminacy is a corollary of the following result, which relies heavily on the results of the previous three sections.

6.4.4. Detection Theorem. *In the Adams–Novikov E_2-term for $p > 2$ let θ^I be a monomial in the θ_i. Then each θ^I and $\alpha_1\theta^I$ is nontrivial.* □

We are not asserting that these monomials are linearly independent, which indeed they are not. Certain relations among them will be used below to prove 6.4.1. Assuming 6.4.4, we have

Proof of 6.4.1. We begin with a computation in $\operatorname{Ext}(BP_*/(p))$. We use the symbol θ_i to denote the mod p reduction of the θ_i defined above in $\operatorname{Ext}(BP_*)$. We also let h_i denote the element $-[t_1^{p^i}]$. In the cobar construction we have

$$d[t_2] = -[t_1|t_1^p] + v_1 \sum_{0<j<p} \frac{1}{p}\binom{p}{j}[t_1^j|t_1^{p-j}]$$

so

$$v_1\theta_0 = -h_0h_1. \tag{6.4.5}$$

May [**5**] developed a general theory of Steenrod operations which is applicable to this Ext group (see A1.5). His operations are similar to the classical ones in ordinary cohomology, except for the fact that $P^0 \neq 1$. Rather we have $P^0(h_i) = h_{i+1}$ and $P^0(\theta_i) = \theta_{i+1}$. We also have $\beta P^0(h_i) = \theta_i$, $\beta P^0(\theta_i) = 0$, $\beta P^0(v_1) = 0$, $P^1(\theta_i) = \theta_i^p$ and the Cartan formula implies that $P^{p^j}(\theta_i^{p^j}) = \theta_i^{p^{j+1}}$. Applying βP^0 to (6.4.6) gives

$$0 = \theta_0h_2 - h_1\theta_1. \tag{6.4.6}$$

If we apply the operation $P^{p^{i-1}}P^{p^{i-2}}\cdots P^1$ to (6.4.5) we get

$$h_{1+i}\theta_1^{p^i} = h_{2+i}\theta_0^{p^i}. \tag{6.4.7}$$

Now associated with the short exact sequence

$$0 \to BP_* \xrightarrow{p} BP_* \to BP_*/(p) \to 0$$

there is a connecting homomorphism

$$\delta\colon \operatorname{Ext}^{s,*}(BP_*/(p)) \to \operatorname{Ext}^{s+1,*}(BP_*)$$

with $\delta(h_{i+1}) = \theta_i$. Applying δ to 6.4.7 gives

$$\theta_i\theta_1^{p^i} = \theta_{i+1}\theta_0^{p^i} \in \operatorname{Ext}_{(BP_*BP_*)}(BP_*, BP_*). \tag{6.4.8}$$

We can now prove the theorem by induction on i, using 4.4.22 to start the induction. We have for $i > 0$

$$\begin{aligned} d_{2p-1}(\theta_{i+1})\theta_0^{p^i} &= d_{2p-1}(\theta_{i+1}\theta_0^{p^i}) \\ &= d_{2p-1}(\theta_i\theta_1^{p^i}) \\ &= d_{2p-1}(\theta_i)\theta_1^{p^i} \\ &\equiv h_0\theta_{i-1}^p\theta_1^{p^i} \quad \bmod \ker \theta_0^{a_{i-1}} \\ &\equiv h_0(\theta_{i-1}\theta_1^{p^{i-1}})^p \\ &\equiv h_0(\theta_i\theta_0^{p^{i-1}})^p \\ &\equiv h_0\theta_i^p\theta_0^{p^i} \end{aligned}$$

so

$$d_{2p-1}(\theta_{i+1}) \equiv h_0\theta_i^p \quad \bmod \ker \theta_0^{a_i}. \qquad \square$$

We now turn to the proof of 6.4.4. We map $\operatorname{Ext}(BP_*)$ to $\operatorname{Ext}(v_n^{-1}BP_*/I_n)$ with $n = p-1$. By 6.1.1 this group is isomorphic to $\operatorname{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$, which is essentially the cohomology of the profinite group S_n by 6.2.4. By 6.2.12 S_n has a subgroup of order p since the field K obtained by adjoining pth roots of unity to $\mathbf{Q}_p$ has degree $p-1$. We will show that the elements of 6.4.4 have nontrivial images under the resulting map to the cohomology of $\mathbf{Z}/(p)$. In other words, we will consider the composite

$$BP_*(BP) \to \Sigma(n) \to S(n) \otimes \mathbf{F}_{p^n} \to C,$$

where C is the linear dual of the group ring $\mathbf{F}_{p^n}[\mathbf{Z}/(p)]$.

6.4.9. Lemma. *Let C be as above. As a Hopf algebra*

$$C = \mathbf{F}_{p^n}[t]/(t^p - t) \quad \textit{with} \quad \Delta t = t \otimes 1 + 1 \otimes t.$$

Proof. As a Hopf algebra we have $\mathbf{F}_{p^n}[\mathbf{Z}/(p)] = \mathbf{F}_{p^n}[u]/(u^p - 1)$ with $\Delta u = u \otimes u$, where u corresponds to a generator of the group $\mathbf{Z}/(p)$. We define an element $t \in C$ by its Kronecker pairing $\langle u^i, t\rangle = i$. Since the product in C is dual to the coproduct in the group algebra, we have

$$\langle u^i, t^k\rangle = \langle \Delta(u^i), t \otimes t^{k-1}\rangle = \langle u^i, t\rangle\langle u^i, t^{k-1}\rangle$$

so by induction on k

$$\langle u^i, t^k\rangle = i^k. \tag{6.4.10}$$

We also have $\langle u^i, 1\rangle = 1$.

We show that $\{1, t, t^2, \ldots, t^{p-1}\}$ is a basis for C by relating it to the dual basis of the group algebra. Define $x_j \in C$ by

$$x_j = \sum_{0<k<p} (jt)^k$$

for $0 < j < p$ and $x_0 = 1 + \sum_{0<j<p} x_j$. Then

$$\begin{aligned}\langle u^i, x_j\rangle &= \left\langle u^i, \sum_{0<k<p} (jt)^k \right\rangle = \sum_{0<k<p} j^k i^k \\ &= \sum_{0<k<p} (ij)^k = \begin{cases} -1 & \text{if } ij = 1 \mod p \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

and

$$\langle u^i, x_0\rangle = \left\langle u^i, 1 + \sum_{0<j<p} x_j \right\rangle = \begin{cases} 1 & \text{if } i = 0 \\ 0 & i \neq 0 \end{cases}$$

so $\{x_0, -x_1, -x_2, \ldots, -x_{p-1}\}$ is the dual basis up to permutation.

Moreover, 6.4.10 implies that $t^p = t$ so C has the desired algebra structure.

For the coalgebra structure we use the fact that the coproduct in C is dual the product in the group algebra. We have

$$\langle u^i \otimes u^j, t \otimes 1 + 1 \otimes t\rangle = i + j$$

and

$$\langle u^i \otimes u^j, \Delta(t)\rangle = \langle u^{i+j}, t\rangle = i + j$$

so $\Delta t = t \otimes 1 + 1 \otimes t$. □

To proceed with the proof of 6.4.4; we now show that under the epimorphism

$$f \colon \Sigma(n) \otimes_{K(n)_*} \mathbf{F}_{p^n} \to C \quad (\text{where } n = p-1), \quad f(t_1) \neq 0.$$

From the proof of 6.2.3, t_1 can be regarded as a continuous function from S_n to $\mathbf{F}_{p^n}$. It follows then that the nontriviality of $f(t_1)$ is equivalent to the nonvanishing of the function t_1 on the nontrivial element of order p in S_n. Suppose $x \in S_{p-1}$ is such an element. We can write

$$x = 1 + \sum_{i>0} e_i S^i$$

with $e_i \in W(\mathbf{F}_{p^n})$ and $e_i^{p^n} = e_i$. Recalling that $S^{p-1} = p$, we compute

$$1 = x^p \equiv 1 + pe_1 S + (e_1 S)^p \mod (S)^{1+p}$$

and

$$(e_1 S)^p \equiv e_1^{(p^p-1)/(p-1)} S^p \mod (S)^{1+p}$$

so it follows that

$$e_1 + e_1^{(p^p-1)/(p-1)} \equiv 0 \mod (p).$$

[Remember that $t_1(x)$ is the mod (p) reduction of e_1.] Clearly, one solution to this equation is $e_1 \equiv 0 \mod (p)$ and hence $e_1 = 0$. We exclude this possibility by showing that it implies that $x = 1$. Suppose inductively that $e_i = 0$ for $i < k$. Then $x \equiv 1 + e_k S^k \mod (S^{k+1})$ and $x^p \equiv 1 + pe_k S^k \mod (S^{k+p})$ so $e_k \equiv 0 \mod (p)$. Since $e_k^{p^n} - e_k = 0$, this implies $e_k = 0$.

Hence, f is a map of Hopf algebras, $f(t_1)$ primitive, so $f(t_1) = ct$ where $c \in \mathbf{F}_{p^n}$ is nonzero. Now recall that

$$\mathrm{Ext}_C(\mathbf{F}_{p^n}, \mathbf{F}_{p^n}) = H^*(\mathbf{Z}/(p); \mathbf{F}_{p^n}) = E(h) \otimes P(b),$$

where $E()$ and $P()$ denote exterior and polynomial algebras over $\mathbf{F}_{p^n}$, respectively, $h = [t] \in H^1$, and

$$b = \sum_{0<j<p} \frac{1}{p}\binom{p}{j}[t^j|t^{p-j}] \in H^2.$$

Let f^* denote the composition

$$\begin{aligned}\mathrm{Ext}(BP_*) &\to \mathrm{Ext}(v_n^{-1}BP_*/I_n)\\ &\xrightarrow{\cong} \mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)) \to \mathrm{Ext}_C(\mathbf{F}_{p^n}, \mathbf{F}_{p^n}) \xrightarrow{\cong} H^*(\mathbf{Z}/(p); \mathbf{F}_{p^n}).\end{aligned}$$

Then it follows that $f^*(h_0) = -ch$ and $f^*(b_i) = -c^{p^{i+1}}b$ and 6.4.4 is proved.

Note that the scalar c must satisfy $1 + c^{(p^p-p)/(p-1)} = 0$. Since $c^{p^{p-1}-1} = 1$, the equation is equivalent to $1 + c^{(p^{p-1}-1)/(p-1)} = 0$. It follows that $c = w^{(p-1)/2}$ for some generator w of $\mathbf{F}^{\times}_{p^{p-1}}$, so c is not contained in any proper subfield of $\mathbf{F}_{p^{p-1}}$. Hence tensoring with this field is essential to the construction of the detecting map f.

Now we examine the corresponding situation in the Adams spectral sequence. The relations used to prove 6.4.1 (apart from the assertion of nontriviality) are also valid here, but the machinery used to prove 6.4.4 is, of course, not available. Indeed the monomials vanish in some cases. The following result was first proved by May [**1**].

6.4.11. PROPOSITION. *For $p = 3$, $h_0b_1^3 = 0$ in $\mathrm{Ext}_{A_3}(\mathbf{Z}/(3), \mathbf{Z}/(3))$; i.e., b_2 cannot support the expected nontrivial differential.*

PROOF. We use a certain Massey product identity (A1.4.6) and very simple facts about $\mathrm{Ext}_{A_3}(\mathbf{Z}/(3), \mathbf{Z}/(3))$ to show $h_0b_1^2 = 0$. We have

$$h_0b_1^2 = -h_0\langle h_1, h_1, h_1\rangle b_1 = -\langle h_0, h_1, h_1\rangle h_1b_1.$$

By (6.4.7) $h_1b_1 = h_2b_0$, so

$$h_0b_1^2 = -\langle h_0, h_1, h_1\rangle h_2b_0 = -\langle h_1, h_0, h_1\rangle h_2b_0 = -h_1\langle h_0, h_1, h_2\rangle b_0.$$

The element $\langle h_0, h_1, h_2\rangle$ is represented in the cobar construction by $\xi_1^9|\xi_2 + \xi_2^3|\xi_1$, which is the coboundary of ξ_3, so $h_0b_1^2 = 0$. ☐

The case of b_2 at $p = 3$ is rather peculiar. One can show in the Adams–Novikov spectral sequence that $d_5(\beta_7) = \pm\alpha_1\beta_{3/3}^3$. (This follows from the facts that $d_5(\beta_4) = \pm\alpha_1\beta_1^2\beta_{3/3}$, $\beta_4^2 = \pm\beta_1\beta_7$, $\beta_4\beta_{3/3} = \pm\beta_1\beta_{6/3}$, and $\beta_{3/3}^3 = \pm\beta_1^2\beta_{6/3}$. We leave the details to the reader.) Hence $\beta_{9/9} \pm \beta_7$ is a permanent cycle mapping to b_2. The elements β_7 and $\alpha_1\beta_{3/3}^3 = \pm\alpha_1\beta_1^2\beta_{6/3}$ correspond to Adams elements in filtrations 8 and 10 which are linked by a differential. We do not know the fate of the b_i at $p = 3$ for $i > 2$.

To prove 6.4.2 we will need two lemmas.

6.4.12. LEMMA. *For $p \geq 3$*

(i) $\mathrm{Ext}^{2,qp^{i+2}}(BP_*)$ *is generated by the $[(i+3)/2]$ elements $\beta_{a_{i,j}/p^{i+3-2j}}$, where $j = 1, 2, \ldots, [(i+3)/2]$, $a_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1)$, and $[(i+3)/2]$ is the largest integer $\leq (i+3)/2$. Each of these elements has order p.*

(i) *Each of these elements except* $\beta_{p^{i+1}/p^{i+1}}$ *reduces to zero in*

$$\mathrm{Ext}^{2,qp^{i+2}}(BP^*/I_3). \qquad \square$$

6.4.13. LEMMA. *For* $p \geq 5$, *any element of* $\mathrm{Ext}^{2,qp^{i+2}}(BP_*)$ (*for* $i \geq 0$) *which maps to* b_{i+1} *in the Adams* E_2*-term supports a nontrivial differential* d_{2p-1}. $\square$

We have seen above that 6.4.13 is false for $p = 3$.

Theorem 6.4.2 follows immediately from 6.4.13 because a permanent cycle in the Adams spectral sequence of filtration 2 must correspond to one in the Adams–Novikov spectral sequence of filtration ≤ 2. By sparseness (4.4.2) the Novikov filtration must also be 2, but 6.4.13 says that no element in $\mathrm{Ext}^2(BP_*)$ mapping to b_i for $i \geq 1$ can be a permanent cycle.

PROOF OF 6.4.12. Part (i) can be read off from the description of $\mathrm{Ext}^{2,*}(BP_*)$ given in 5.4.5.

To prove (ii) we recall the definition of the elements in question. We have short exact sequences of $BP_*(BP)$-comodules

$$0 \to BP_* \to BP_* \xrightarrow{p} BP_*/(p) \to 0. \tag{6.4.14}$$

$$0 \to BP_*/(p) \xrightarrow{v_1^{p^{i+3-2j}}} BP_*/(p) \to BP_*/\big(p, v_1^{p^{i+3-2j}}\big) \to 0. \tag{6.4.15}$$

Let δ_0 and δ_1, denote the respective connecting homomorphisms. Then we have $v_2^{a_{i,j}} \in \mathrm{Ext}^0_{BP_*BP}\big(BP_*, BP_*/\big(p, v_1^{i+3-2j}\big)\big)$ and $\beta_{\alpha_{i,j}p^{i+3-2j}} = \delta_0\delta_1(v_2^{a_{1,j}})$. The element $\beta_{p^{i+1}/p^{i+1}}$ the above element for $j = 1$) can be shown to be b_{i+1} as follows. The right unit formula 4.3.21 gives

$$\begin{aligned} \eta_R(v_2) &\equiv v_2 + v_1 t_1^p - v_1^p t_1 \mod (p), \\ \delta_1(v_2^{p^{i+1}}) &= t_1^{p^{i+2}} - v_1^{p^{i+2}-p^{i+1}} t_1^{p^{i+1}} \end{aligned} \tag{6.4.16}$$

and $\delta_0(t_1^{p^{i+2}}) = b_{i+1}$. Moreover 6.4.16 implies that in $\mathrm{Ext}(BP_*/(p))$,

$$v_1^{p^j} t_1^{pj+1} \cong v_1^{p^{j+1}} t_1^{p^j}, \quad \text{so} \quad v_1^{p^{i+2}-p^{i+1}} t_1^{p^{i+1}} \cong v_1^{p^{i+2}-1} t_1.$$

This element is the mod (p) reduction of $p^{-i-2}\delta_0(v_1^{p^{i+2}})$ and is therefore in $\ker \delta_0$. Hence $\delta_0\delta_1(v_2^{p^{i+1}}) = \delta_0(t_1^{p^{i+2}}) = b_{i+1}$.

This definition of $\beta_{p^{i+1}/p^{i+1}}$ differs from that of 5.4.5, where for $i > 0$ it is defined to be $\delta_0\delta_1(v_2^{p^2} - v_1^{p^2-1} v_2^{p^2-p+1})^{p^{i-1}}$.

In principle one can compute this element explicitly in the cobar complex (A1.2.11) and reduce mod I_3, but that would be very messy. A much easier method can be devised using Yoneda's interpretation of elements in Ext groups as equivalence classes of exact sequences (see, for example, Chapter IV of Hilton and Stammbach [**1**]) as in 5.1.20(b). Consider the following diagram.

(6.4.17)

$$\begin{tikzcd}
0 \arrow[r] & BP_* \arrow[r, "p"] \arrow[d, "p_1"] & BP_* \arrow[r, "v_1^{i+3-3j}"] \arrow[d, "p_2"] & BP_*/(p) \arrow[r, "\rho_2"] \arrow[d, "p_3"] & BP_*/(p, v_1^{p^{i+3-2j}}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & BP_*/(p, v_1, v_2) \arrow[r] & M_1 \arrow[r] & M_2 \arrow[r] & BP_*/(p, v_1^{p^{i+3-2j}}) \arrow[r] & 0.
\end{tikzcd}$$

The top row is obtained by splicing 6.4.14 and 6.4.15 and it corresponds to an element in $\mathrm{Ext}^2(BP_*/(p, v_1^{p^{i+3-2j}}), BP_*)$. Composing this element with

$$v_2^{a_{i,j}} \in \mathrm{Ext}^0(BP_*/(p, v_1^{i+3-2j}))$$

gives $\beta_{a_{i,j}/p^{i+3-2j}}$.

We let p_1 be the standard surjection. It follows from Yoneda's result that if we choose BP_*BP-comodules M_1 and M_2, and comodule maps p_2 and p_3 such that the diagram commutes and the bottom row is exact, then the latter will determine the element of

$$\mathrm{Ext}^2_{BP_*BP}(BP_*/(p, v_1^{i+3-2j}), BP_*/(p, v_1, v_2))$$

which, when composed with $v_2^{a_{i,j}}$, will give the mod I_3 reduction of $\beta_{a_{i,j}/p^{i+3-2j}}$. We choose $M_1 = BP_*/(p^2, pv_1, v_1^2, pv_2)$ and $M_2 = BP_*/(p, v_1^{2+p^{i+3-2j}})$ and let p_2 and p_3 be the standard surjections. It is easy to check that M_1 and M_2 are comodules over $BP_*(BP)$, i.e., that the corresponding ideals in BP_* are invariant. (The ideal used to define M_1 is simply $I_2^2 + I_1I_3$.) Moreover, the resulting diagram has the desired properties.

The resulting bottom row of 6.4.17 is the splice of the two following short exact sequences.

$$0 \to BP_*/(p, v_1, v_2) \xrightarrow{p} BP_*/(p^2, pv_1, pv_2, v_1^2) \to BP_*/(p, v_1^2) \to 0, \tag{6.4.18}$$

$$0 \to BP_*/(p, v_1^2) \xrightarrow{v_1^{p^{i+3-2j}}} BP_*/(p, v_1^{2+p^{i+3-2j}}) \to BP_*/(p, v_1^{p^{i+3-2j}}) \to 0. \tag{6.4.19}$$

Let δ_0', δ_1' denote the corresponding connecting homomorphisms. The elements we are interested in then are $\delta_0'\delta_1'(v_2^{a_{i,j}})$.

To compute $\delta_1'(v_2^{a_{i,j}})$ we use the formula $d(v_2^n) = (v_2 + v_1v_1t_1^p - v_1^pt_1)^n - v_2^n$, implied by 6.4.16, in the cobar construction for $BP_*/(p, v_1^{2+p^{i+3-2j}})$. Recall that

$$a_{i,j} = (p^{i+2} + p^{i+3-2j})/(p+1) \quad 1 \le j \le [(i+3)/2].$$

Hence $a_{i,j} = p^{i+3-2j} \mod (p^{i+4-2j})$ and $d(v_2^{a_{i,j}}) = v_2^{b_{i,j}} v_1^{p^{i+3-2j}} [t_1^{p^{i+4-2j}}]$, so

$$\delta_1'(v_2^{a_{i,j}}) = v_2^{b_{i,j}} [t_1^{p^{i+3-2j}}],$$

where $b_{i,j} = a_{i,j} - p^{i+3-2j} = (p^{i+2} - p^{i+4-2j})/(p+1)$.

For $j = 1$, $b_{i,1} = 0$ and

$$\delta_0'\delta_1'(v_2^{a_{i,1}}) = -\sum_{0<k<p} \frac{1}{p}\binom{p}{k}[t_1^{kp^i} | t_1^{(p-k)p^i}] = -b_{i+1}.$$

For $j > 1$, $b_{i,j}$ is divisible by p and $d(v_2^{b_{i,j}}) \equiv 0 \mod (p^2, pv_1, v_1^2)$ and

$$v_2^{b_{i,j}} d(t_1^{p^{i+4-2j}}) \equiv 0 \mod (pv_2),$$

so $\delta_1' v_2^{a_{i,j}} \in \mathrm{Ext}^1(BP_*/(p, v_1^2))$ pulls back in 6.4.17 to an element of

$$\mathrm{Ext}^1(BP_*/(p^2, pv_1, pv_2, v^2)) \quad \text{and} \quad \delta_0'\delta_1'(v_2^{a_{i,j}}) = 0,$$

completing the proof. □

Proof of 6.4.13. Any element of $\mathrm{Ext}^{2,qp^{i+2}}(BP_*)$ can be written uniquely as $cb_{i+1} + x$ where x is in the subgroup generated by the elements $\beta_{a_{i,j}/p^{i+3-2j}}$ for

$j > 1$. In 5.4.6, we showed that x maps to zero in the classical Adams E_2-term. Hence it suffices to show that no such x can have the property

$$d_{2p-1}(x) = d_{2p-1}(b_{i+1})$$

By 5.5.2 for $p \geq 5$ there is an 8-cell spectrum $V(2) = M(p, v_1, v_2)$ with $BP_*(V(2)) = BP_*/(p, v_1, v_2)$, and a map $f\colon S^0 \to V(2)$ inducing a surjection in BP homology. f also induces the standard map

$$f_*\colon \operatorname{Ext}(BP_*) \to \operatorname{Ext}(BP_*/I_3).$$

Lemma 6.4.12 asserts that $f_*(\beta_{a_{i,j}/p^{i+3-2j}}) = 0$ for $j > 1$, so $f_*(d_{2p-1}(x)) = 0$ where x is as above. However, 6.4.1 and the proof of 6.4.4 show that

$$g_*(d_{2p-1}(b_{i+1})) \neq 0,$$

where g_* is induced by the obvious map

$$g\colon BP_* \to v_{p-1}^{-1}BP_*/I_{p-1}.$$

Since g factors through BP_*/I_3, this shows that $f_*(d_{2p-1}(b_{i+1})) \neq 0$, completing the proof. □

5. The Spectra $T(m)$

In this section will we construct certain spectra $T(m)$ and study the corresponding chromatic spectral sequence. $T(m)$ satisfies

$$BP_*(T(m)) = BP_*[t_1, t_2, \ldots, t_m] \subset BP_*(BP_*)$$

as a comodule algebra. These are used in Chapter 7 in a computation of the Adams–Novikov E_2-term. We will see there that the Adams–Novikov spectral sequence for $T(m)$ is easy to compute through a range of dimensions that grows rapidly with m, and here we will show that its chromatic spectral sequence is very regular.

To construct the $T(m)$ recall that $BU = \Omega SU$ by Bott periodicity, so we have maps $\Omega SU(k) \to BU$ for each k. Let $X(k)$ be the Thom spectrum of the corresponding vector bundle over $\Omega SU(k)$. An easy calculation shows that $H_*(X(k)) = \mathbf{Z}[b_1, b_2, \ldots, b_{k-1}] \subset H_*(MU)$. Our first result is

6.5.1. Splitting Theorem. *For any prime p, $X(k)_{(p)}$ is equivalent to a wedge of suspensions of $T(m)$ with m chosen so that $p^m \leq k < p^{m+1}$, and $BP_*(T(m)) = BP_*[t_1, \ldots, t_m] \subset BP_*(BP)$. Moreover $T(m)$ is a homotopy associative commutative ring spectrum.* □

From this we get a diagram

$$S^0_{(p)} = T(0) \to T(1) \to T(2) \to \cdots \to BP.$$

In Ravenel [**8**, §3] we show that after p-adic completion there are no essential maps from $T(i)$ to $T(j)$ if $i > j$ or from BP to $T(i)$.

This theorem is an analog of 4.1.12, which says that $MU_{(p)}$ splits into a wedge of suspensions of BP, as is its proof. We start with the following generalization of 4.1.1.

6.5.2. Definition. *Let E be an associative commutative ring spectrum. A complex orientation of degree k for E is a class $x_E \in \widetilde{E}^2(\mathbf{C}P^k)$ whose restriction to $\widetilde{E}^2(\mathbf{C}P^1) \cong \pi_0(E)$ is 1.* □

A complex orientation as in 4.1.1 is of degree k for all $k > 0$. This notion is relevant in view of

6.5.3. LEMMA. *$X(k)$ admits a complex orientation of degree k.*

PROOF. $X(k)$ is a commutative associative ring spectrum (up to homotopy) because $\Omega SU(k)$ is a double loop space. The standard map $\mathbf{C}P^{k-1} \to BU$ lifts to $\Omega SU(k)$. Thomifying gives a stable map $\mathbf{C}P^k \to X(k)$ with the desired properties. $\square$

$X(k)$ plays the role of MU in the theory of spectra with orientation of degree k. The generalizations of lemmas 4.1.4, 4.1.7, 4.1.8, and 4.1.13 are straightforward. We have

6.5.4. PROPOSITION. *Let E be an associative commutative ring spectrum with a complex orientation $x_E \in \widetilde{E}_2(\mathbf{C}P^k)$ of degree k.*

(a) $E^*(\mathbf{C}P^k) = \pi_*(E)[x_E]/(x_E^{k+1})$.

(b) $E^*(\mathbf{C}P^k \times \mathbf{C}P^k) = \pi_*(E)[x_E \otimes 1, 1 \otimes x_E]/(x_E^{k+1} \otimes 1, 1 \otimes x_E^{k+1})$.

(c) *For $0 < i < k$ the map $t\colon \mathbf{C}P^i \times \mathbf{C}P^{k-i} \to \mathbf{C}P^k$ induces a formal group law k-chunk; i.e., the element $t^*(x_E)$ in the truncated power series ring*

$$\pi_*(E)[x_E \otimes 1, 1 \otimes x_E]/(x_E \otimes 1, 1 \otimes x_E)^{k+1}$$

has properties analogous to an formal group law (A2.1.1).

(d) $E_*(X(k)) = \pi_*(E)[b_1^E, \dots, b_{k-1}^E]$ *where $b_i^E \in E_{2i}(X(k))$ is defined as in* 4.1.7.

(e) *With notation as in* 4.1.8, *in $(E \wedge X(k))^2(\mathbf{C}P^k)$ we have*

$$\hat{x}_{X(k)} = \sum_{0 \le i \le k-1} b_i^E \hat{x}_E^{i+1} \quad \textit{where} \quad b_0 = 1.$$

This power series will be denoted by $g_E(\hat{x}_E)$.

(f) *There is a one-to-one correspondence between degree k orientations of E and multiplicative maps $X(k) \to E$ as in* 4.1.13. $\square$

We do not have a generalization of 4.1.15, i.e., a convenient way of detecting maps $X(k) \to X(k)$, but we can get by without it. By 6.5.4(f) a multiplicative map $g\colon X(k)_{(p)} \to X(k)_{(p)}$ is determined by a polynomial $f(x) = \sum_{0 \le i \le k-1} f_i x^{i+1}$ with $f_0 = 1$ and $f_i \in \pi_{2i}(X(k)_{(p)})$. In this range of dimensions $\pi_*(X(k))$ is isomorphic to $\pi_*(MU)$, so we can take $f(x)$ to be the truncated form of the power series of A2.1.23. Then the calculations of 4.1.12 show that g induces an idempotent in ordinary or BP_*-homology. In the absence of 4.1.15 it does not follow that g itself is idempotent. Nevertheless we can define

$$T(m) = \lim_g X(k)_{(p)},$$

i.e., $T(m)$ is the mapping telescope of g. Then we can compose the map $X(k)_{(p)} \to T(m)$ with various self-maps of $X(k)_{(p)}$ to construct the desired splitting, thereby proving 6.5.1.

Now we consider the chromatic spectral sequence for $T(m)$. Using the change-of-rings isomorphism 6.1.1, the input needed for the machinery of Section 5.1 is $\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m)))$ where $K(n)_*(T(m)) = K(n)_*[t_1, \dots, t_m]$. Using notation as in 6.3.7, let $\Sigma(n, m+1) = \Sigma(n)/(t_1, \dots, t_m)$. Then we have

6.5.5. THEOREM. *With notation as above we have*

$$\operatorname{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m))) = K(n)_*[u_{n+1}, \dots, u_{n+m}] \otimes_{K(n)_*} \operatorname{Ext}_{\Sigma(n,m+1)}(K(n)_*, K(n)),$$

where $\dim u_j = \dim v_j$. *Moreover* u_j *maps to* v_j *under the map to* $\operatorname{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(BP)) = B(n)_*$ (6.1.11) *induced by* $T(m) \to BP$. *In other words its image in* $K(n)_*(BP)$ *coincides with that of* $\eta_R(v_j) \in BP_*(BP)$ *under the map* $BP_*(BP) \to K(n)_*(BP)$. □

Applying 6.3.7 gives

6.5.6. COROLLARY. *If* $n < m+2$ *and* $n < 2(p-1)(m+1)/p$ *then*

$$\operatorname{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*(T(m))) = K(n)_*[u_{n+1}, \dots, u_{n+m}] \otimes E(h_{k,j} \colon m+1 \le k \le m+n, \quad j \in \mathbf{Z}/(n)).$$

□

PROOF OF 6.5.5. The images of $\eta_R(v_{n+j})$ (for $1 \le j \le m$) in $K(n)_*(T(m))$ are primitive and give the u_{n+j}. The image of $BP_*(T(m)) \to BP_*(BP) \to \Sigma(n)$ is the subalgebra generated by $\{t_n \colon n \le m\}$. The result follows by a routine argument. □

Now we will use the chromatic spectral sequence to compute $\operatorname{Ext}^s(BP_*(T(m)))$ for $s = 0$ and 1. We assume $m > 0$ since $T(0) = S^0$, which was considered in 5.2.1 and 5.2.6. By 6.5.5 and 6.5.6 we have

$$\begin{aligned} \text{(6.5.7)} \qquad & \operatorname{Ext}_{\Sigma(0)}(K(0)_*, K(0)_*(T(m))) = \mathbf{Q}[u_1, \dots, u_m] \quad \text{and} \\ & \operatorname{Ext}_{\Sigma(1)}(K(1)_*, K(1)_*(T(m))) = K(1)_*[u_2, \dots, u_{m+1}] \otimes E(h_{m+1,0}). \end{aligned}$$

The short exact sequence

$$\text{(6.5.8)} \quad 0 \to M_1^0 \otimes BP_*(T(m)) \xrightarrow{i} M^1 \otimes BP_*(T(m)) \xrightarrow{p} M^1 \otimes BP_*(T(m)) \to 0$$

induces a six-term exact sequence of Ext groups with connecting homomorphism δ. For $j \le m$, $\eta_R(v_j) \in BP_*(T(m)) \subset BP_*(BP)$, so if u is any monomial in these elements then $\delta(u/p^i) = 0$ for all $i > 0$ and $\operatorname{Ext}^0(M^1 \otimes BP_*(T(m)))$ has a corresponding summand isomorphic to $\mathbf{Q}/\mathbf{Z}(p)$. Hence in the chromatic spectral sequence, $E_1^{1,0}$ has a summand isomorphic to $\mathbf{Z}_{(p)}[u_1, \dots, u_m] \otimes \mathbf{Q}/\mathbf{Z}(p)$, which is precisely the image of $d_1 \colon E_1^{0,0} \to E_1^{1,0}$, giving

6.5.9. PROPOSITION.

$$\operatorname{Ext}^0(BP_*(T(m))) = \mathbf{Z}_{(p)}[u_1, \dots, u_m].$$ □

Next we need to consider the divisibility of $u_{m+1}^t/p \in \operatorname{Ext}^0(M^1 \otimes BP_*(T(m)))$. Note that $\eta_R(v_{m+1})$ is not in $BP_*(T(m))$ but $\eta_R(v_{m+1}) - pt_{m+1}$ (where v_{m+1} is Hazewinkel's generator given by A2.2.1) is, so we call this element u_{m+1}. It follows that in the cobar complex $C(BP_*(T(m)))$ (A1.2.11) $d(u_{m+1}) = pt_{m+1}$ and

$$\text{(6.5.10)} \qquad d(u_{m+1}^t) \equiv \mathrm{pt} u_{m+1}^{t-1} t_{m+1} + p^2 \binom{t}{2} u_{m+1}^{t-2} t_{m+1}^2 \mod (p^2 t),$$

where the second term is nonzero only when $p = 2$ and t is even. Thus the situation is similar to that for $m = 0$ where we have $v_1 = u_1$. Recall that in that case the presence of the second term caused Ext^1 to behave differently at $p = 2$. We will show that this does not happen for $m \geq 1$ and we have

6.5.11. THEOREM. *For $m \geq 1$ and all primes p*

$$\mathrm{Ext}^1(BP_*(T(m))) = \mathrm{Ext}^0(BP_*(T(m))) \otimes \{u_{m+1}^t/\mathrm{pt}\colon t > 0\}.$$

PROOF. For $p > 2$ the result follows from 6.5.10 as in 5.2.6. Now recall the situation for $m = 0$, $p = 2$. For $t = 2$, 6.5.10 gives $d(v_1^2) = 4(v_1t_1 + t_1^2)$ and we have $d(4v_1^{-1}v_2) \equiv 4(v_2t_1 + t_1^2) \mod (8)$, so we get a cocycle $(v_1^2 + 4v_1^{-1}v_2)/8$. The analogous cocycle for $m \geq 1$ would be something like

$$(u_{m+1}^2 + 4v_1^{-1}u_{m+2})/q$$

where u_{m+2} is related somehow to v_{m+2}. However, the relevant terms in $\eta_R(v_{m+2})$ mod (2) are $v_1t_{m+1}^2 + v_2^{2^{m+1}}t_{m+1}$, which does not bear the resemblance to 6.5.10 for $m \geq 1$ that it does for $m = 0$. In other words $u_{m+1}^{t-2}t_{m+1}^2$ is not cohomologous mod (2) to $u_{m+1}^{t-1}t_{m+1}$, so the calculation for $p = 2$ can proceed as it does for $p > 2$. $\square$

Our last result is useful for computing the Adams–Novikov E_2-term for $T(m)$ by the method used in Section 4.4.

6.5.12. THEOREM. *For $t < 2(p^{2m+2} - 1)$*

$$\mathrm{Ext}(BP_*(T(m))/I_{m+1}) = \mathbf{Z}/(p)[u_{m+1}, u_{m+2}, \ldots, u_{2m+1}] \otimes E(h_{i,j}) \otimes P(b_{i,j})$$

with $i \geq m+1$, $i + j \leq 2m+2$, $h_{i,j} \in \mathrm{Ext}^{1,2p^j(p^i-1)}$ and $b_{i,j} \in \mathrm{Ext}^{2,2p^{j+1}(p^i-1)}$.

6.5.13. EXAMPLE. For $m = 1$ we have

$$\begin{aligned}\mathrm{Ext}(BP_*(T(1))/I_2) = \mathbf{Z}/(p)[u_2, u_3] &\otimes E(h_{2,0}, h_{2,1}, h_{2,2}, h_{3,0}, h_{3,1}) \\ &\otimes P(b_{2,0}, b_{2,1}, b_{3,0})\end{aligned}$$

in 6.5.1 for $t \leq 2(p^4 - 1)$

PROOF OF 6.5.12. By a routine change-of-rings argument (explained in Section 7.1) the Ext in question is the cohomology of $C_\Gamma(BP_*/I_{m+1})$ (A1.2.11) where $\Gamma = BP_*(BP)/(t_1, \ldots, t_m)$. Then from 4.3.15 and 4.3.20 we can deduce that v_i and t_i are primitive for $m+1 \leq i \leq 2m+1$. $h_{i,j}$ corresponds to $t_i^{p^i}$ and $b_{i,j}$ to $-\sum_{0<k<p} p^{-1}\binom{p}{k} t_i^{kp^j}|t_i^{(p-k)p^j}$. The result follows by routine calculation. $\square$