CHAPTER 3

The Classical Adams Spectral Sequence

In Section 1 we make some simple calculations with the Adams spectral sequence which will be useful later. In particular, we use it to compute $\pi_*(MU)$ (3.1.5), which will be needed in the next chapter. The computations are described in some detail in order to acquaint the reader with the methods involved.

In Sections 2 and 3 we describe the two best methods of computing the Adams spectral sequence for the sphere, i.e., the May spectral sequence and the lambda algebra. In both cases a table is given showing the result in low dimensions (3.2.9 and 3.3.10). Far more extensive charts are given in Tangora [**?, ?**]. The main table in the former is reproduced in Appendix 3.

In Section 4 we survey some general properties of the Adams spectral sequence. We give $E_2^{s,*}$ for $s \leq 3$ (3.4.1 and 3.4.2) and then say what is known about differentials on these elements (3.4.3 and 3.4.4). Then we outline the proof of the Adams vanishing and periodicity theorems (3.4.5 and 3.4.6). For $p = 2$ they say that $E_s^{s,t}$ vanishes roughly for $0 < t - s < 2s$ and has a very regular structure for $t - s < 5s$. The E_∞-term in this region is given in 3.4.16. An elementary proof of the nontriviality of most of these elements is given in 3.4.21.

In Section 5 we survey some other past and current research and suggest further reading.

1. The Steenrod Algebra and Some Easy Calculations

In this section we begin calculating with the classical mod (p) Adams spectral sequence of 2.1.1. We start by describing the dual Steenrod algebra A_*, referring the reader to Milnor [**?**] or Steenrod and Epstein [**?**] for the proof. Throughout this book, $P(x)$ will denote a polynomial algebra (over a field which will be clear from the context) on one or more generators x, and $E(x)$ will denote the exterior algebra on same.

3.1.1. Theorem (Milnor [**?**]). *A_* is a graded commutative, noncocommutative Hopf algebra.*

(a) *For $p = 2$, $A_* = P(\xi_1, \xi_2, \dots)$ as an algebra where $|\xi_n| = 2^n - 1$. The coproduct $\Delta\colon A_* \to A_* \otimes A_*$ is given by $\Delta\xi_n = \sum_{0 \leq i \leq n} \xi_{n-i}^{2^i} \otimes \xi_i$, where $\xi_0 = 1$.*

(b) *For $p > 2$, $A_* = P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots)$ as an algebra, where $|\xi_n| = 2(p^n - 1)$, and $|\tau_n| = 2p^n - 1$. The coproduct $\Delta\colon A_* \to A_* \otimes A_*$ is given by $\Delta\xi_n = \sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} \otimes \xi_i$, where $\xi_0 = 1$ and $\Delta\tau_n = \tau_n \otimes 1 + \sum_{0 \leq i \leq n} \xi_{n-1}^{p^i} \otimes \tau_i$.*

(c) *For each prime p, there is a unit $\eta\colon \mathbf{Z}/(p) \to A_*$, a counit $\varepsilon\colon A_* \to \mathbf{Z}/(p)$ (both of which are isomorphisms in dimension 0), and a conjugation (canonical anti-automorphism) $c\colon A_* \to A_*$ which is an algebra map given recursively by $c(\xi_0) = 1$, $\sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} c(\xi_i) = 0$ for $n > 0$ and $\tau_n + \sum_{0 \leq i \leq n} \xi_{n-i}^{p^i} c(\tau_i) = 0$ for $n \geq 0$. $\bar{A}_*$ will* *denote* $\ker\varepsilon$*; i.e.,* $\bar{A}_*$ *is isomorphic to* A_* *in positive dimensions, and is trivial in dimension* 0. □

A_* is a commutative Hopf algebra and hence a Hopf algebroid. The homological properties of such objects are discussed in Appendix 1.

We will consider the classical Adams spectral sequence formulated in terms of homology (2.2.3) rather than cohomology (2.1.1). The most obvious way of computing the E_2-term is to use the cobar complex. The following description of it is a special case of 2.2.10 and A1.2.11.

3.1.2. PROPOSITION. *The* E_2*-term for the classical Adams spectral sequence for* $\pi_*(X)$ *is the cohomology of the cobar complex* $C^*_{A_*}(H_*(X))$ *defined by*

$$C^s_{A_*}(H_*(X)) = \bar{A}_* \otimes \cdots \otimes \bar{A}_* \otimes H_*(X)$$

(*with* s *tensor factors of* $\bar{A}_*$). *For* $a_i \in A_*$ *and* $x \in H_*(X)$*, the element* $a_1 \otimes \cdots a_s \otimes x$ *will be denoted by* $[a_1|a_2|\cdots|a_s]x$*. The coboundary operator* $d_s\colon C^s_{A_*}(H_*(X)) \to C^{s+1}_{A_*}(H_*(X))$ *is given by*

$$\begin{aligned} d_s[a_1|\cdots|a_s]x = [1|a_1|\cdots|a_s]x + \sum_{i=1}^{s}(-1)^i[a_1|\cdots|a_{i-1}|a_i'|a_i''|a_{i+1}|\cdots|a_s]x \\ + (-1)^{s+1}[a_1|\cdots|a_s|x']x'', \end{aligned}$$

where $\Delta a_i = a_i' \otimes a_i''$ *and* $\psi(x) = x' \otimes x'' \in A_* \otimes H_*(X)$. [*A priori this expression lies in* $A_*^{\otimes s+1} \otimes H_*(X)$*. The diligent reader can verify that it actually lies in* $\bar{A}_*^{\otimes s+1} \otimes H_*(X)$.] □

This E_2-term will be abbreviated by $\mathrm{Ext}(H_*(X))$.

Whenever possible we will omit the subscript A_*.

The following result will be helpful in solving group extension problems in the Adams spectral sequence. For $p > 2$ let $a_0 \in \mathrm{Ext}^{1,1}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ be the class represented by $[\tau_0] \in C(\mathbf{Z}/(p))$. The analogous element for $p = 2$ is represented by $[\xi_1]$ and is denoted by a_0, $h_{1,0}$, or h_0.

3.1.3. LEMMA.

(a) *For* $s \geq 0$, $\mathrm{Ext}^{s,s}(H_*(S^0))$ *is generated by* a_0^s.

(b) *If* $x \in \mathrm{Ext}(H_*(X))$ *is a permanent cycle in the Adams spectral sequence represented by* $\alpha \in \pi_*(X)$*, then* a_0x *is a permanent cycle represented by* $p\alpha$. [*The pairing* $\mathrm{Ext}(H_*(S^0)) \otimes \mathrm{Ext}(H_*(X)) \to \mathrm{Ext}(H_*(X))$ *is given by* 2.3.3.] □

PROOF. Part (a) follows from inspection of $C^*(\mathbf{Z}/(p))$; there are no other elements in the indicated bidegrees. For (b) the naturality of the smash product pairing (2.3.3) reduces the problem to the case $x = 1 \in \mathrm{Ext}(H_*(S^0))$, where it follows from the fact that $\pi_0(S^0) = \mathbf{Z}$. □

The cobar complex is so large that one wants to avoid using it directly at all costs. In this section we will consider four spectra (MO, MU, bo, and bu) in which the change-of-rings isomorphism of A1.1.18 can be used to great advantage. The most important of these for our purposes is MU, so we treat it first. The others are not used in the sequel. Much of this material is covered in chapter 20 of Switzer [**?**].

The computation of $\pi_*(MU)$ is due independently to Milnor [**?**] and Novikov [**?, ?**]. For the definition and basic properties of MU, including the following lemma, we refer the reader to Milnor [**?**] or Stong [**?**] or to Section 4.1.

3.1.4. Lemma.
(a) *$H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_1, b_2, \dots]$, where $b_i \in H_{2i}$.*
(b) *Let $H/(p)$ denote the* mod (p) *Eilenberg–Mac Lane spectrum for a prime p and let $u\colon MU \to H/(p)$ be the Thom class, i.e., the generator of $H^0(MU;\mathbf{Z}/(p))$. Then $H_*(u)$ is an algebra map and its image in $H_*(H/(p)) = A_*$ is $P(\xi_1^2, \xi_2^2, \dots)$ for $p = 2$ and $P(\xi_1, \xi_2, \dots)$ for $p > 2$.* □

The main result concerning MU is the following.

3.1.5. Theorem (Milnor [**?**], Novikov [**?, ?**]).
(a) *$\pi_*(MU) = \mathbf{Z}[x_1, x_2, \dots]$ with $x_i \in \pi_{2i}(MU)$.*
(b) *Let $h\colon \pi_*(MU) \to H_*(MU;\mathbf{Z})$ be the Hurewicz map. Then modulo decomposables in $H_*(MU;\mathbf{Z})$,*

$$h(x_i) = \begin{cases} -pb_i & \text{if } i = p^k - 1 \text{ for some prime } p \\ -b_i & \text{otherwise.} \end{cases} \qquad \square$$

We will prove this in essentially the same way that Milnor and Novikov did. After some preliminaries on the Steenrod algebra we will use the change-of-rings isomorphisms A1.1.18 and A1.3.13 to compute the E_2-term (3.1.10). It will follow easily that the spectral sequence collapses; i.e., it has no nontrivial differentials.

To compute the E_2-term we need to know $H_*(MU;\mathbf{Z}/(p))$ as an A_*-comodule algebra. Since it is concentrated in even dimensions, the following result is useful.

3.1.6. Lemma. *Let M be a left A_*-comodule which is concentrated in even dimensions. Then M is a comodule over $P_* \subset A_*$ defined as follows. For $p > 2$, $P_* = P(\xi_1, \xi_2, \dots)$ and for $p = 2$, $P_* = P(\xi_1^2, \xi_2^2, \dots)$.*

Proof. For $m \in M$, let $\psi(m) = \Sigma m' \otimes m''$. Then each $m' \in A_s$ must be even-dimensional and by coassociativity its coproduct expansion must consist entirely of even-dimensional factors, which means it must lie in P_*. □

3.1.7. Lemma. *As a left A_*-comodule, $H_*(MU) = P_* \otimes C$, where $C = P(u_1, u_2, \dots)$ with $\dim u_i = 2i$ and i is any positive integer not of the form $p^k - 1$.*

Proof. $H_*(MU;\mathbf{Z}/(P))$ is a P_*-comodule algebra by 3.1.4 and 3.1.6. It maps onto P_* by 3.1.4(b), so by A1.1.18 it is $P_* \otimes C$, where $C = \mathbf{Z}/(p) \,\square_{P_*}\, H_*(MU)$. An easy counting argument shows that C must have the indicated form. □

3.1.8. Lemma. *Let M be a comodule algebra over A_* having the form $P_* \otimes N$ for some A_*-comodule algebra N. Then*

$$\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), M) = \operatorname{Ext}_E(\mathbf{Z}/(p), N)$$

where

$$E = A_* \otimes_{P_*} \mathbf{Z}/(p) = \begin{cases} E(\xi_1, \xi_2, \dots) & \text{for } p = 2 \\ E(\tau_0, \tau_1, \dots) & \text{for } p > 2. \end{cases}$$

In particular,

$$\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU)) = \operatorname{Ext}_E(\mathbf{Z}/(p), \mathbf{Z}/(p)) \otimes C.$$

PROOF. The statement about $H_*(MU)$ follows from the general one by 3.1.7. For the latter we claim that $M = A_* \square_E N$. We have $A_* = P_* \otimes E$ as vector spaces and hence as E-comodules by A1.1.20, so

$$A_* \square_E N = P_* \otimes E \square_E N = P_* \otimes N = M,$$

and the result follows from A1.3.13. $\square$

Hence we have reduced the problem of computing the Adams E_2-term for MU to that of computing $\mathrm{Ext}_E(\mathbf{Z}/(p), \mathbf{Z}/(p))$. This is quite easy since E is dual to an exterior algebra of finite type.

3.1.9. LEMMA. *Let Γ be a commutative, graded connected Hopf algebra of finite type over a field K which is an exterior algebra on primitive generators $x_1, x_2, \dots$, each having odd degree if K has characteristic other than 2 (e.g., let $\Gamma = E$). Then*

$$\mathrm{Ext}_\Gamma(K, K) = P(y_1, y_2, \dots),$$

where $y_i \in \mathrm{Ext}^{1,|x_i|}$ is represented by $[x_i]$ in $C_\Gamma(K)$ (the cobar complex of A1.2.11*).*

PROOF. Let $\Gamma_i \subset \Gamma$ be the exterior algebra on x_i. Then an injective Γ_i-resolution of K is given by

$$0 \to K \to \Gamma_i \xrightarrow{d} \Gamma_i \to \Gamma_i \to \cdots$$

where $d(x_i) = 1$ and $d(1) = 0$ applying $\mathrm{Hom}_{\Gamma_i}(K, \ )$ gives a complex with trivial boundary operator and shows $\mathrm{Ext}_{\Gamma_i}(K, K) = P(Y_i)$. Tensoring all the R_i together gives an injective Γ-resolution of K and the result follows from the Kunneth theorem. $\square$

Combining the last three lemmas gives

3.1.10. COROLLARY.

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU)) = C \otimes P(a_0, a_1, \dots),$$

where C is as in 3.1.7 *and $a_i \in \mathrm{Ext}^{1,2p^i-1}$ is represented by $[\tau_i]$ for $p > 2$ and $[\xi_i]$ for $p = 2$ in $C_{A_*}(H_*(MU))$.* $\square$

Thus we have computed the E_2-term of the classical Adams spectral sequence for $\pi_*(MU)$. Since it is generated by even-dimensional classes, i.e., elements in $E_2^{s,t}$ with $t - s$ even, there can be no nontrivial differentials, i.e., $E_2 = E_\infty$.

The group extension problems are solved by 3.1.3; i.e., all multiples of a_0^s are represented in $\pi_*(MU)$ by multiples of p^s. It follows that $\pi_*(MU) \otimes \mathbf{Z}_{(p)}$ is as claimed for each p; i.e., 3.1.5(a) is true locally. Since $\pi_i(MU)$ is finitely generated for each i, we can conclude that it is a free abelian group of the appropriate rank.

To get at the global ring structure note that the mod (p) indecomposable quotient in dimension $2i$, $Q_{2i}\pi_*(MU) \otimes \mathbf{Z}/(p)$ is $\mathbf{Z}/(p)$ for each $i > 0$, so $Q_{2i}\pi_*(MU) = \mathbf{Z}$. Pick a generator x_i in each even dimension and let $R = \mathbf{Z}[x_1, x_2, \dots]$. The map $R \to \pi_*(MU)$ gives an isomorphism after tensoring with $\mathbf{Z}_{(p)}$ for each prime p, so it is isomorphism globally.

To study the Hurewicz map

$$h\colon \pi_*(MU) \to H_*(MU; \mathbf{Z}),$$

recall $H_*(X; \mathbf{Z}) = \pi_*(X \wedge H)$, where H is the integral Eilenberg–Mac Lane spectrum. We will prove 3.1.5(b) by determining the map of Adams spectral sequences induced by $i\colon MU \to MU \wedge H$. We will assume $p > 2$, leaving the obvious changes for $p = 2$ to the reader. The following result on $H_*(H)$ is standard.

3.1.11. LEMMA. *The* mod (p) *homology of the integer Eilenberg–Mac Lane spectrum*

$$H_*(H) = P_* \otimes E(\bar{\tau}_1, \bar{\tau}_2, \dots)$$

as an A_* comodule, where $\bar{\tau}_i$ denotes the conjugate τ_i, i.e., its image under the conjugation c. □

Hence we have

$$H_*(H) = A_* \,\square_{E(\tau_0)}\, \mathbf{Z}/(p)$$

and an argument similar to that of 3.1.8 shows

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X \wedge H)) = \mathrm{Ext}_{E(\tau_0)}(\mathbf{Z}/(p), H_*(X)). \tag{3.1.12}$$

In the case $X = MU$ the comodule structure is trivial, so by 3.1.11,

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU \wedge H)) = H_*(MU) \otimes P(a_0).$$

To determine the map of Ext groups induced by i, we consider three cobar complexes, $C_{A_*}(H_*(MU))$, $C_E(C)$, and $C_{E(\tau_0)}(H_*(MU))$. The cohomologies of the first two are both $\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU))$, by 3.1.2 and 3.1.8, respectively, while that of the third is $\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(MU \wedge H))$ by 3.1.12. There are maps from $C_{A_*}(H_*(MU))$ to each of the other two.

The class $A_n \in \mathrm{Ext}_{A_*}^{1,2p^n-1}(\mathbf{Z}/(p), H_*(MU))$ is represented by $[\tau_n] \in C_E(C)$. The element $-\sum_i [\bar{\tau}_i]\bar{\xi}_{n-i}^{p^i} \in C_{A_*}(H_*(MU))$ [using the decomposition of $H_*(MU)$ given by 3.1.7] is a cycle which maps to $[\tau_n]$ and therefore it also represents a_n. Its image in $C_{E(\tau_0)}(H_*(MU))$ is $[\tau_0]\bar{\xi}_n$, so we have $i_*(a_n) = a_0\xi_n$. Since $\xi_n \in H_*(MU)$ is a generator it is congruent modulo decomposables to a nonzero scalar multiple of b_{p^n-1}, while u_i (3.1.9) can be chosen to be congruent to b_i. It follows that the $x_i \in \pi_{2i}(MU)$ can be chosen to satisfy 3.1.5(b).

We now turn to the other spectra in our list, MO, bu, and bo. The Adams spectral sequence was not used originally to compute the homotopy of these spectra, but we feel these calculations are instructive examples. In each case we will quote without proof a standard theorem on the spectrum's homology as an A_*-comodule and proceed from there.

For similar treatments of MSO, MSU, and MSp see, respectively, Pengelley [**?**], Pengelley [**?**], and Kochman [**?**].

To following result on MO was first proved by Thom [**?**]. Proofs can also be found in Liulevicius [**?**] and Stong [**?**, p. 95].

3.1.13. THEOREM. *For $p = 2$, $H_*(MO) = A_* \otimes N$, where N is a polynomial algebra with one generator in each degree not of the form $2^k - 1$. For $p > 2$, $H_*(MO) = 0$.* □

It follows immediately that

$$\mathrm{Ext}_{A_*}^s(\mathbf{Z}/(2), H_*(MO)) = \begin{cases} N & \text{if } s = 0 \\ 0 & \text{if } s > 0, \end{cases} \tag{3.1.14}$$

the spectral sequence collapses and $\pi_*(MO) = N$.

For bu we have

3.1.15. THEOREM (Adams [**?**]).

$$H_*(bu) = \bigoplus_{0 \le i < p-1} \Sigma^{2i} M$$

where

$$M = P_* \otimes E(\bar{\tau}_2, \bar{\tau}_3, \dots) \qquad \text{for } p > 2$$
$$M = P_* \otimes E(\bar{\xi}_3, \bar{\xi}_4, \dots) \qquad \text{for } p = 2$$

where $\bar{\alpha}$ *for* $\alpha \in A_*$ *is the conjugate* $c(\alpha)$. □

Using 3.1.8 we get

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), M) = \mathrm{Ext}_E(\mathbf{Z}/(p), E(\tau_2, \tau_3, \dots))$$

(again we assume for convenience that $p > 2$) and by an easy calculation A1.3.13 gives

$$\mathrm{Ext}_E(\mathbf{Z}/(p), E(\tau_2, \tau_3, \dots)) = \mathrm{Ext}_{E(\tau_0,\tau_1)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) = P(a_0, a_1)$$

by 3.1.11, so we have

3.1.16. Theorem.

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(bu)) = \bigoplus_{i=0}^{p-2} \Sigma^{2i} P(a_0, a_1)$$

where $a_0 \in \mathrm{Ext}^{1,1}$ *and* $a_1 \in \mathrm{Ext}^{1,2p-1}$. □

As in the MU case the spectral sequence collapses because the E_2-term is concentrated in even dimensions. The extensions can be handled in the same way, so we recover the fact that

$$\pi_i(bu) = \begin{cases} \mathbf{Z} & \text{if } i \geq 0 \text{ and } i \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

The bo spectrum is of interest only at the prime 2 because at odd primes it is a summand of bu (see Adams [?]). For $p = 2$ we have

3.1.17. Theorem (Stong [?]). *For* $p = 2$, $H_*(bo) = P(\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \bar{\xi}_4, \dots)$ *where* $\bar{\xi}_i = c(\xi_i)$. □

Let $A(1)_* = A_*/(\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \bar{\xi}_4, \dots)$. We leave it as an exercise for the reader to show that $A(1)_*$ is dual to the subalgebra $A(1)$ of A generated by Sq^1 and Sq^2, and that

$$H_*(bo) = A_* \,\square_{A(1)_*}\, \mathbf{Z}/(2),$$

so by A1.3.13,

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), H_*(bo)) = \mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2)). \tag{3.1.18}$$

$A(1)$ is not an exterior algebra, so 3.1.9 does not apply. We have to use the Cartan–Eilenberg spectral sequence A1.3.15. The reader can verify that the following is an extension (A1.1.15)

$$\Phi \to A(1)_* \to E(\bar{\xi}_2), \tag{3.1.19}$$

where $\Phi = P(\xi_1)/(\xi_1^4)$. Φ is isomorphic as a coalgebra to an exterior algebra on elements corresponding to ξ_1 and ξ_1^2, so by 3.1.9

$$\mathrm{Ext}_\Phi(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{10}, h_{11})$$

and

$$\mathrm{Ext}_{E(\bar{\xi}_2)}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{20}),$$

where $h_{i,j}$ is represented by $[\bar{\xi}_i^{2^j}]$ in the appropriate cobar complex. Since $P(h_{20})$ has only one basis element in each degree, the coaction of Φ on it is trivial, so by A1.3.15 we have a Cartan–Eilenberg spectral sequence converging to $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ with

$$E_2 = P(h_{10}, h_{11}, h_{20}) \tag{3.1.20}$$

where $h_{1i} \in E_2^{1,0}$ and $h_{20} \in E_2^{0,1}$. We claim

$$d_2(h_{20}) = h_{10}h_{11}. \tag{3.1.21}$$

This follows from the fact that

$$d(\xi_2) = \xi_1 \otimes \xi_1^2$$

in $C_{A(1)_*}(\mathbf{Z}/(2))$. It follows that

$$E_3 = P(u, h_{10}, h_{11})/(h_{10}h_{11}) \tag{3.1.22}$$

where $u \in E_3^{0,2}$ corresponds to h_{20}^2. Next we claim

$$d_3(u) = h_{11}^3. \tag{3.1.23}$$

We have in $C_{A(1)_*}(\mathbf{Z}/(2))$,

$$d(\bar{\xi}_2 \otimes \bar{\xi}_2) = \bar{\xi}_2 \otimes \xi_1 \otimes \xi_1^2 + \xi_1 \otimes \xi_1^2 \otimes \bar{\xi}_2.$$

In this E_2 this gives

$$d_2 h_{20}^2 = h_{10}h_{11}h_{20} + h_{20}h_{10}h_{11} = 0$$

since E_2 is commutative. However, the cobar complex is not commutative and when we add correcting terms to $\bar{\xi}_2 \otimes \bar{\xi}_2$ in the hope of getting a cycle, we get instead

$$d(\bar{\xi}_2 \otimes \bar{\xi}_2 + \xi_1 \otimes \xi_1^2\bar{\xi}_2 + \xi_1\bar{\xi}_2 \otimes \xi_1^2) = \xi_1^2 \otimes \xi_1^2 \otimes \xi_1^2,$$

which implies 3.1.23. It follows that

$$E_4 = P(h_{10}, h_{11}, v, w)/(h_{10}h_{11}, h_{11}^3, v^2 + h_{10}^2 w, vh_{11}), \tag{3.1.24}$$

where $v \in E_4^{1,2}$ and $w \in E_4^{0,4}$ correspond to $h_{10}h_{20}^2$ and h_{20}^4, respectively.

Finally, we claim that $E_4 = E_\infty$; inspection of E_4 shows that there cannot be any higher differentials because there is no $E_r^{s,t}$ for $r \geq 4$ which is nontrivial and for which $E_r^{s+r,t-r+1}$ is also nontrivial. There is also no room for any nontrivial extensions in the multiplicative structure. Thus we have proved

3.1.25. THEOREM. *The E_2-term for the* mod (2) *Adams spectral sequence for $\pi_*(bo)$,*

$$\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), H_*(bo)) = \mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

is

$$P(h_{10}, h_{11}, v, w)/(h_{10}h_{11}, h_{11}^3, v^2 + h_{10}^2 w, vh_{11}),$$

where

$$h_{10} \in \mathrm{Ext}^{1,1}, \quad h_{11} \in \mathrm{Ext}^{1,2}, \quad v \in \mathrm{Ext}^{3,7}, \quad \textit{and} \quad w \in \mathrm{Ext}^{4,12}. \qquad \square$$

This E_2-term is displayed in the accompanying figure. A vertical arrow over an element indicates that $h_{10}^s x$ is also present and nontrivial for all $s > 0$.

Now we claim that this Adams spectral sequence also collapses, i.e., $E_2 = E_\infty$. Inspection shows that the only possible nontrivial differential is $d_r(w^n h_{11}) =$

$w^n h_{10}^{n+r}$. However, bo is a ring spectrum so by 2.3.3 the differentials are derivations and we cannot have $d_r(h_{11}) = h_{10}^{r+1}$ because it contradicts the relation $h_{10}h_{11} = 0$. The extension problem is solved by 3.1.3, giving

3.1.26. THEOREM (Bott [?]).

$$\pi_*(bo) = \mathbf{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

with $\eta \in \pi_1$, $\alpha \in \pi_4$, $\beta \in \pi_8$, *i.e., for* $i \geq 0$

$$\pi_i(bo) = \begin{cases} \mathbf{Z} & \text{if } i \equiv 0 \mod 4 \\ \mathbf{Z}/2 & \text{if } i \equiv 1 \text{ or } 2 \mod 8 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

For future reference we will compute $\mathrm{Ext}_{A(1)}(\mathbf{Z}/(2), M)$ for $M = A(0)_* \equiv E(\xi_1)$ and $M = Y \equiv P(\xi_1)/(\xi_1^4)$. Topologically these are the Adams E_2-terms for the mod (2)-Moore spectrum smashed with bo and bu, respectively. We use the Cartan–Eilenberg spectral sequence as above and our E_2-term is

$$\mathrm{Ext}_\Phi(\mathbf{Z}/(2), \mathrm{Ext}_{E(\bar{\xi}_2)}(\mathbf{Z}/(2), M)).$$

An easy calculation shows that

$$E_2 = P(h_{11}, h_{20}) \quad \text{for } M = A(0)_*$$

and

$$E_2 = P(h_{20}) \quad \text{for } M = Y.$$

In the latter case the Cartan–Eilenberg spectral sequence collapses. In the former case the differentials are not derivations since $A(0)_*$ is not a comodule algebra. From 3.1.23 we get $d_3(h_{20}^2) = h_{11}^3$, so

$$E_\infty = E_4 = P(w) \otimes \{1, h_{11}, h_{11}^2, h_{20}, h_{20}h_{11}, h_{20}h_{11}^2\}.$$

This Ext is not an algebra but it is a module over $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$. We will show that there is a nontrivial extension in this structure, namely $h_{10}h_{20} = h_{11}^2$. We do this by computing in the cobar complex $C_{A(1)_*}(A(0)_*)$. There the class h_{20} is represented by $[\xi_2] + [\xi_1^2]\xi_1$, so $h_{10}h_{20}$ is represented by $[\xi_1|\xi_2] + [\xi_1|\xi_1^2]\xi_1$. The sum of this and $[\xi_1^2|\xi_1^2]$ (which represents $h_{1,1}^2$) is the coboundary of $[\xi_1\xi_2] + [\xi_1^3 + \xi_2]\xi_1$.

From these considerations we get

3.1.27. THEOREM. *As a module over* $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ (3.1.25) *we have*

(a) $\mathrm{Ext}_{A(1)_*}, (\mathbf{Z}/(2), A(0)_*)$ *is generated by* $1 \in \mathrm{Ext}^{0,0}$ *and* $h_{20} \in \mathrm{Ext}^{1,3}$ *with* $h_{10} \cdot 1 = 0$, $h_{10}h_{20} = h_{11}^2 \cdot 1$, $v \cdot 1 = 0$, *and* $vh_{20} = 0$.

(b) $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}(2), Y)$ *is generated by* $\{h_{20}^i \colon 0 \leq i \leq 3\}$ *with* $h_{10}h_{20}^i = h_{11}h_{20}^i = vh_{20}^i = 0$. $\square$

We will also need an odd primary analog of 3.1.27(a). $A(1) = E(\tau_0, \tau_1) \otimes P(\xi_1)/(\xi_1^p)$ is the dual to the subalgebra of A generated by the Bockstein β and the Steenrod reduced power P^1. Instead of generalizing the extension 3.1.19 we use

$$P(0)_* \to A(0)_* \to E(1)_*,$$

where $P(0)_* = P(\xi_1)/(\xi_1^p)$ and $E(1)_* = E(\tau_0, \tau_1)$. The Cartan–Eilenberg spectral sequence E_2-term is therefore

$$\mathrm{Ext}_{p(0)_*}(\mathbf{Z}/(p), \mathrm{Ext}_{E(1)_*}(\mathbf{Z}/(p), A(0)_*)),$$

where $A(0)_* = E(\tau_0)$. An easy calculation gives

3.1.28. THEOREM. *For $p > 2$*

$$\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(p), A(0)_*) = E(h_0) \otimes P(a_1, b_0),$$

where $h_0 \in \mathrm{Ext}^{1,q}$, $a_1 \in \mathrm{Ext}^{1,q+1}$, and $b_0 \in \mathrm{Ext}^{2,pq}$ are represented by $[\xi_1]$, $[\xi_1]\tau_0 + [\tau_1]$, and $\sum_{0<i<p} p^{-1}\binom{p}{i}[\xi_1^i|\xi_1^{p-i}]$, respectively. □

2. The May Spectral Sequence

In this section we discuss a method for computing the classical Adams E_2-term, $\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$, which we will refer to simply as Ext. For the reader hoping to understand the classical Adams spectral sequence we offer two pieces of advice. First, do as many explicit calculations as possible yourself. Seeing someone else do it is no substitute for the insight gained by firsthand experience. The computations sketched below should be reproduced in detail and, if possible, extended by the reader. Second, the E_2-term and the various patterns within it should be examined and analyzed from as many viewpoints as possible. For this reason we will describe several methods for computing Ext. For reasons to be given in Section 4.4, we will limit our attention here to the prime 2.

The most successful method for computing Ext through a range of dimensions is the spectral sequence of May [**?**]. Unfortunately, crucial parts of this material have never been published. The general method for computing Ext over a Hopf algebra is described in May [**?**], and the computation of the differentials in the May spectral sequence for the Steenrod algebra through dimension 70 is described by Tangora [**?**]. A revised account of the May E_2-term is given in May [**?**].

In our language May's approach is to filter A_* by copowers of the unit coideal (A1.3.10) and to study the resulting spectral sequence. Its E_2-term is the Ext over the associated graded Hopf algebra E^0A_*. The structure of this Hopf algebra is as follows.

3.2.1. THEOREM (May [**?**]). (a) *For $p = 2$,*

$$E^0A_* = E(\xi_{i,j} \colon i > 0, j \geq 0)$$

with coproduct

$$\Delta(\xi_{i,j}) = \sum_{0 \leq k \leq i} \xi_{i-k,j+k} \otimes \xi_{k,j},$$

where $\xi_{0,j} = 1$ and $\xi_{i,j} \in E_i^0 A_*$ is the projection of $\xi_i^{2^j}$.

(b) *For $p > 2$,*

$$E^0A_* = E(\tau_i \colon i \geq 0) \otimes T(\xi_{i,j} \colon i > 0, j \geq 0)$$

with coproduct given by

$$\Delta(\xi_{i,j}) = \sum_{0 \leq k \leq i} \xi_{i-k,j+k} \otimes \xi_{k,j}$$

and

$$\Delta(\tau_i) = \tau_i \otimes 1 + \sum_{0 \leq k \leq i} \xi_{i-k,i} \otimes \tau_i,$$

where $T(\ )$ denotes the truncated polynomial algebra of height p on the indicated generators, $\tau_i \in E_{i+1}^0 A_*$ is the projection of $\tau_i \in A_*$, and $\xi_{i,j} \in E_i^0 A_p^*$ is the projection of $\xi_i^{p^j}$. □

May actually filters the Steenrod algebra A rather than its dual, and proves that the associated bigraded Hopf algebra E_0A is primitively generated, which is dual to the statement that each primitive in $E^0A_p^*$ is a generator. A theorem of Milnor and Moore [**?**] says that every graded primitively generated Hopf algebra is isomorphic to the universal enveloping algebra of a restricted Lie algebra. For $p = 2$ let $x_{i,j} \in E_0A$ be the primitive dual to $\xi_{i,j}$. These form the basis of a Lie algebra under commutation, i.e.,

$$[x_{i,j}, x_{k,m}] \equiv x_{i,j}x_{k,m} - x_{k,m}x_{i,j} = \delta_k^i x_{i,m} - \delta_i^m x_{k,j}$$

where δ_j^i is the Kronecker δ. A *restriction* in a graded Lie algebra L is an endomorphism ξ which increases the grading by a factor of p. In the case at hand this restriction is trivial. The universal enveloping algebra $V(L)$ of a restricted Lie algebra L (often referred to as the restricted enveloping algebra) is the associative algebra generated by the elements of L subject to the relations $xy - yx = [x, y]$ and $x^p = \xi(x)$ for $x, y \in L$.

May [**?**] constructs an efficient complex (i.e., one which is much smaller than the cobar complex) for computing Ext over such Hopf algebras. In particular, he proves

3.2.2. Theorem (May [**?**]). *For $p = 2$, $\operatorname{Ext}_{E^0A_*}^{***}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ (the third grading being the May filtration) is the cohomology of the complex*

$$V^{***} = P(h_{i,j} \colon i > 0,\ j \geq 0)$$

with $d(h_{i,j}) = \sum_{0<k<i} h_{k,j}h_{i-k,k+j}$, where $h_{i,j} \in V^{1,2^j(2^i-1),i}$ corresponds to $\xi_{i,j} \in A_2^*$. $\square$

Our $h_{i,j}$ is written R_i^j by May [**?**] and R_{ji} by Tangora [**?**], but as $h_{i,j}$ (in a slightly different context) by Adams [**?**]. Notice that in $C^*(\mathbf{Z}/(2))$ one has $d[\xi_i^{2^j}] = \sum_{0<k<i}[\xi_{k-i}^{2^{i+j}}|\xi_k^{2^j}]$, which corresponds to the formula for $d(h_{i,j})$ above. The theorem asserts that $E^0C^*(\mathbf{Z}/(2))$ is chain homotopy equivalent to the polynomial algebra on the $[\xi_{i,j}]$. We will see below (3.2.7) that $C^*(\mathbf{Z}/(2))$ itself does not enjoy the analogous property and that the May differentials are a measure of its failure to do so.

From 3.2.2 May derives a spectral sequence of the following form.

3.2.3. Theorem (May [**?**]). *There is a spectral sequence converging to*

$$\operatorname{Ext}_{A_*}^{**}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

with $E_1^{***} = V^{***}$ and $d_r \colon E_r^{s,t,u} \to E_r^{s+1,t,u+1-r}$.

Proof of 3.2.2 and 3.2.3. The spectral sequence is a reindexed form of that of A1.3.9, so 3.2.3 follows from 3.2.2. We will show that the same spectral, sequence can be obtained more easily by using a different increasing filtration of A_*. An increasing filtration is defined by setting $|\xi_i^{2^j}| = 2i - 1$. Then it follows easily that this E^0A_* has the same algebra structure as in 3.2.1 but with each $\xi_{i,j}$ primitive. Hence E^0A_* is dual to an exterior algebra and its Ext is V^{***} (suitably reindexed) by 3.1.11. A1.3.9 gives us a spectral sequence associated to this filtration. In particular, it will have $d_1(h_{i,j}) = \sum h_{k,j}h_{i-k,j+k}$ as in 3.2.2. Since all of the $h_{i,j}$ have odd filtration degree, all of the nontrivial differentials must have odd index. It follows that this spectral sequence can be reindexed in such a way that each d_{2r-1} gets converted to a d_r and the resulting spectral sequence is that of 3.2.3. $\square$

For $p > 2$ the spectral sequence obtained by this method is not equivalent to May's but is perhaps more convenient as the latter has an E_1-term which is nonassociative. In the May filtration one has $|\tau_{i-1}| = |\xi_i^{p^j}| = i$. If we instead set $|\tau_{i-1}| = |\xi_i^{p^j}| = 2i - 1$, then the resulting $E^0 A_*$ has the same algebra structure (up to indexing) as that of 3.2.1(b), but all of the generators are primitive. Hence it is dual to a product of exterior algebras and truncated polynomial algebras of height p. To compute its Ext we need, in addition to 3.1.11, the following result.

3.2.4. LEMMA. *Let $\Gamma = T(x)$ with $\dim x = 2n$ and x primitive. Then*

$$\operatorname{Ext}_\Gamma(\mathbf{Z}/(p), \mathbf{Z}/(p)) = E(h) \otimes P(b),$$

where

$$h \in \operatorname{Ext}^1 \quad \textit{is represented in } C_\Gamma(\mathbf{Z}/(p)) \textit{ by } [x]$$

and

$$b \in \operatorname{Ext}^2 \quad \textit{by} \quad \sum_{0<i<p} \frac{1}{p}\binom{p}{i}[x^i|x^{p-i}]. \qquad \square$$

The proof is a routine calculation and is left to the reader.

To describe the resulting spectral sequence we have

3.2.5. THEOREM. *For $p > 2$ the dual Steenrod algebra* (3.1.1) *A_* can be given an increasing filtration with $|\tau_{i-1}| = |\xi_i^{p^j}| = 2i - 1$ for $i - 1$, $j \geq 0$. The associated bigraded Hopf algebra $E^0 A_*$ is primitively generated with the algebra structure of* 3.2.1(b). *In the associated spectral sequence* (A1.3.9)

$$E_1^{***} = E(h_{i,j} \colon i > 0,\ j \geq 0) \otimes P(b_{i,j} \colon i > 0,\ j \geq 0) \otimes P(a_i \colon i \geq 0),$$

where

$$\begin{aligned} h_{i,j} &\in E_1^{1,2(p^i-1)p^j,2i-1}, \\ b_{i,j} &\in E_1^{2,2(p^i-1)p^{1+j},p(2i-1)}, \end{aligned}$$

and

$$a_i \in E_1^{1,2p^i-1,2i+1}$$

($h_{i,j}$ *and* a_i *correspond respectively to* $\xi_i^{p^j}$ *and* τ_i). *One has $d_r \colon E_r^{s,t,u} \to E_r^{s-1,t,u-r}$, and if $x \in E_r^{s,t,u}$ then $d_r(xy) = d_r(x)y + (-1)^s x d_r(y)$. d_1 is given by*

$$\begin{aligned} d_1(h_{i,j}) &= -\sum_{0<k<i} h_{k,j} h_{i-k,k+j}, \\ d_1(a_i) &= -\sum_{0\leq k<i} a_k h_{i-k,k}, \\ d_1(b_{i,j}) &= 0. \end{aligned} \qquad \square$$

In May's spectral sequence for $p > 2$, indexed as in 3.2.3, the E_1-term has the same additive structure (up to indexing) as 3.2.5 and di is the same on the generators, but it is a derivation with respect to a different multiplication, which is unfortunately nonassociative.

We will illustrate this nonassociativity with a simple example for $p = 3$.

3.2.6. EXAMPLE. In the spectral sequence of 3.2.5 the class $h_{10}h_{20}$ corresponds to a nontrivial permanent cycle which we call g_0. Clearly $h_{10}g_0=0$ in E_∞, but for $p=3$ it could be a nonzero multiple of $h_{11}b_{10}$ in Ext. The filtration of $h_{10}g_0$ and $h_{11}b_{10}$ are 5 and 4, respectively. Using Massey products (A1.4), one can show that this extension in the multiplicative structure actually occurs in the following way. Up to nonzero scalar multiplication we have $b_{10}=\langle h_{10},h_{10},h_{10}\rangle$ and $g_0=\langle h_{10},h_{10},h_{11}\rangle$ (there is no indeterminacy), so

$$\begin{aligned} h_{10}g_0 &= h_{10}\langle h_{10},h_{10},h_{11}\rangle \\ &= \langle h_{10},h_{10},h_{10}\rangle h_{11} \\ &= b_{10}h_{11}. \end{aligned}$$

Now in the May filtration, both $h_{10}g_0$ and $b_{10}h_{11}$ have weight 4, so this relation must occur in E_1, i.e., we must have

$$0\neq h_{10}g_0=h_{10}(h_{10}g_0)\neq(h_{10}h_{10})g_0=0,$$

so the multiplication is nonassociative.

To see a case where this nonassociativity affects the behavior of May's d_1, consider the element $h_{10}h_{20}h_{30}$. It is a d_1 cycle in 3.2.5. In E_2 the Massey product $\langle h_{10},h_{11},h_{12}\rangle$ is defined and represented by $\pm(h_{10}h_{21}+h_{20}h_{12})=\pm d_1(h_{30})$. Hence in Ext we have

$$\begin{aligned} 0 &= g_0\langle h_{10},h_{11},h_{12}\rangle \\ &= \langle g_0h_{10},h_{11},h_{12}\rangle \\ &= \pm\langle h_{11}b_{10},h_{11},h_{12}\rangle \\ &= \pm b_{10}\langle h_{11},h_{11},h_{12}\rangle. \end{aligned}$$

The last bracket is represented by $\pm h_{11}h_{21}$, which is a permanent cycle g_1. This implies (A1.4.12) $d_2(h_{10}h_{20}h_{30})=\pm b_{11}g_1$. In May's grading this differential is a d_1.

Now we return to the prime 2.

3.2.7. EXAMPLE. The computation leading to 3.1.25, the Adams E_2-term for bo, can be done with the May spectral sequence. One filters $A(1)_*$ (see 3.1.18) and gets the sub-Hopf algebra of E^0A_* generated by ξ_{10}, ξ_{11}, and ξ_{22}. The complex analogous to 3.2.2 is $P(h_{10},h_{11},h_{20})$ with $d(h_{20})=h_{10}h_{11}$. Hence the May E_2-term is the Cartan–Eilenberg E_3-term (3.1.22) suitably reindexed, and the d_3 of 3.1.23 corresponds to a May d_2.

We will illustrate the May spectral sequence for the mod (2) Steenrod algebra through the range $t-s\leq 13$. This range is small enough to be manageable, large enough to display some nontrivial phenomena, and is convenient because no May differentials originate at $t-s=14$. May [?, ?] was able to describe his E_2-term (including d_2) through a very large range, $t-s\leq 164$ (for $t-s\leq 80$ this description can be found in Tangora [?]). In our small range the E_2-term is as follows.

3.2.8. LEMMA. *In the range $t-s\leq 13$ the E_2-term for the May spectral sequence* (3.2.3) *has generators*

$$h_j=h_{1,j}\in E_2^{1,2^j,1},$$

$$b_{i,j}=h_{i,j}^2\in E_2^{2,2^j(2^i-1),2i},$$

and

$$x_7 = h_{20}h_{21} + h_{11}h_{30} \in E_2^{2,9,4}$$

with relations

$$h_j h_{j+1} = 0,$$
$$h_2 b_{20} = h_0 x_7,$$

and

$$h_2 x_7 = h_0 b_{21}. \qquad \square$$

This list of generators is complete through dimension 37 if one adds x_{16} and x_{34}, obtained from x_7 by adding 1 and 2 to the second component of each index. However, there are many more relations in this larger range.

FIGURE 3.2.9. The May E_2-term for $p = 2$ and $t - s \leq 13$

The E_2-term in this range is illustrated in FIG. 3.2.9. Each dot represents an additive generator. If two dots are joined by a vertical line then the top element is h_0 times the lower element; if they are joined by a line of slope $\frac{1}{3}$ then the right-hand element is h_2 times the left-hand element. Vertical and diagonal arrows mean that the element has linearly independent products with all powers of h_0 and h_1, respectively.

3.2.10. LEMMA. *The differentials in* 3.2.3 *in this range are given by*
(a) $d_r(h_j) = 0$ *for all* r,
(b) $d_2(b_{2,j}) = h_j^2 h_{j+2} + h_{j+1}^3$,
(c) $d_2(x_7) = h_0 h_2^2$,
(d) $d_2(b_{30}) = h_1 b_{21} + h_3 b_{20}$, *and*
(e) $d_4(b_{20}^2) = h_0^4 h_3$.

PROOF. In each case we make the relevant calculation in the cobar complex $C_{A_*}(\mathbf{Z}/(2))$ of 3.1.2. For (a), $[\xi_i^{2^j}]$ is a cycle. For (b) we have

$$d([\xi_2|\xi_2] + [\xi_1^2|\xi_1\xi_2] + [\xi_2\xi_1^2|\xi_1]) = [\xi_1^2|\xi_1^2|\xi_1^2] + [\xi_1^4|\xi_1|\xi_1].$$

For (c) we have

$$d([(\xi_1^3 + \xi_2)|\xi_2^2] + [(\xi_3 + \xi_1^4\xi_2 + \xi_1\xi_2^2 + \xi_1^7)|\xi_1^2] + [\xi_1|\xi_1^2\xi_2^2]) = [\xi_1|\xi_1^4|\xi_1^4].$$

For (d) we use the relation $x_7^2 = h_1^2 b_{30} + b_{20}b_{21}$ (which follows from the definition of the elements in question); the right-hand term must be a cycle in E_2 and we can use this fact along with (b) to calculate $d_2(b_{30})$.

Part (e) follows from the fact that $h_0^4 h_3 = 0$ in Ext, for which three different proofs will be given below. These are by direct calculation in the Λ-algebra (Section 3.3), by application of a Steenrod squaring operation to the relation $h_0 h_1 = 0$, and by the Adams vanishing theorem (3.4.5). $\square$

It follows by inspection that no other differentials can occur in this range. Since no May differentials originate in dimension 14 we get

3.2.11. THEOREM. $\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \mathbf{Z}(2))$ *for* $t - s \leq 13$ *and* $s \leq 7$ *is generated as a vector space by the elements listed in the accompanying table.* (*There are no generators for* $t - s = 12$ *and* 13, *and the only generators in this range with* $s > 7$ *are powers of* h_0.)

In the table c_0 *corresponds to* h_1x_7*, while* Px *corresponds to* $b_{2,0}^2x$*. There are relations* $h_1^3 = h_0^2h_2$*,* $h_2^3 = h_1^2h_3$*, and* $Ph_1^3 = Ph_0^2h^2 = h_0^2Ph_2$*.* $\square$

Inspecting this table one sees that there are no differentials in the Adams spectral sequence in this range, and all of the group extensions are solved by 3.1.3 and we get

3.2.12. COROLLARY. *For $n \leq 13$ the 2-component of $\pi_n(S^0)$ are given by the following table.*

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_n(S^0)$	$\mathbf{Z}_{(2)}$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(8)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(16)$	$(\mathbf{Z}/(2))^2$	$(\mathbf{Z}/(2))^3$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(8)$	0	0

In general the computation of higher May differentials is greatly simplified by the use of algebraic Steenrod operations (see Section A1.5). For details see Nakamura [**?**].

Now we will use the May spectral sequence to compute $\mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$, where $A(n)_* = P(\xi_1, \xi_2, \ldots, \xi_{n+1})/(\xi_i^{2^{n+2-1}})$ is dual to the subalgebra $A(n) \subset A$ generated by $Sq^1, Sq^2, \ldots, Sq^{2^n}$. We filter $A(2)_*$ just as we filter A_*. The resulting May E_1-term is $P(h_{11}, h_{12}, h_{20}, h_{21}, h_{30})$ with $d_1(h_{1,i}) = 0 = d_1(h_{20})$, $d_1(h_{21}) = h_{11}h_{12}$, and $d_1(h_{30}) = h_{20}h_{12}$. This gives

$$E_2 = P(b_{21}, b_{30}) \otimes ((P(h_{11}, h_{20}) \otimes E(x_7)) \oplus \{h_{12}^i \colon i > 0\}), \tag{3.2.13}$$

where $b_{21} = h_{21}^2$, $b_{30} = h_{30}^2$, and $x_7 = h_{11}h_{30} + h_{20}h_{21}$. The d_2's are trivial except for

$$d_2(h_{20}^2) = h_{11}^3, \quad d_2(b_{21}) = h_{12}^3, \quad \text{and} \quad d_2(b_{30}) = h_{11}b_{21}. \tag{3.2.14}$$

Since $A(0)_*$ is not a comodule algebra, this is not a spectral sequence of algebras, but there is a suitable pairing with the May spectral sequence of 3.2.3.

Finding the resulting E_3-term requires a little more ingenuity. In the first place we can factor out $P(b_{30}^2)$, i.e., $E_2 = E_2/(b_{30}^2) \otimes P(b_{30}^2)$ as complexes. We denote $E_2/(b_{30}^2)$ by $\overline{E}_2$ and give it an increasing filtration as a differential algebra by letting $F_0 = P(h_{11}, h_{20}) \otimes E(x_7) \oplus \{h_{12}^i \colon i > 0\}$ and letting $b_{21}, b_{30} \in F_1$. The cohomology of the subcomplex F_0 is essentially determined by 3.1.27(a), which gives $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), A(0)_*)$. Let B denote this object suitably regraded for the present purpose. Then we have

$$H^*(F_0) = B \otimes E(x_7) \oplus \{h_{12}^i \colon i > 0\}. \tag{3.2.15}$$

For $k > 0$ we have $F_k/F_{k-1} = \{b_{21}^k, b_{21}^{k-1}b_{30}\} \otimes F_0$ with $d_2(b_{21}^{k-1}b_{30}) = b_{21}^k h_{11}$. Its cohomology is essentially determined by 3.1.27(b), which describes $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), Y)$. Let C denote this object suitably regraded, i.e., $C = P(h_{20})$. Then we have for $k > 0$

$$H^*(F_k/F_{k-1}) = C\{b_{21}^k\} \otimes E(X_7) \oplus \{b_{21}^k h_{12}^i, b_{30}b_{21}^{k-1}h_{12}^i \colon i > 0\}. \tag{3.2.16}$$

This filtration leads to a spectral sequence converging to $\overline{E}_3$ in which the only nontrivial differential sends

$$b_{21}^k b_{30}^\varepsilon h_{12}^i \quad \text{to} \quad k b_{21}^{k-1} b_{30}^\varepsilon h_{12}^{i+3}$$

for $\varepsilon = 0, 1$, $k > 0$ and $i \geq 1$. This is illustrated in FIG. 3.2.17(a), where a square indicates a copy of B and a large circle indicates a copy of C. Arrows pointing to the left indicate further multiplication by h_{12}, and diagonal lines indicate differentials. Now b_{21} supports a copy of C and a differential. This leads to a copy of C in $\overline{E}_3$ supported by $h_{20}b_{21}$ shown in 3.2.17(b). There is a nontrivial multiplicative extension $h_{20}h)_{12}b_{30} = x_7 b_{21}$ which we indicate by a copy of C in place of $h_{12}b_{30}$ in (b). Fig. 3.2.17(b) also shows the relation $h_{11}b_{21}^2 = h_{12}^3 b_{30}$.

FIGURE 3.2.17. The May spectral sequence for $\mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$. (a) The spectral sequence for $\overline{E}_3$; (b) the $\overline{E}_3$-term; (c) differentials in E_3; (d) E_∞

The differentials in E_3 are generated by $d_3(b_{30}^2) = h_{12}b_{21}^2$ and are shown in 3.2.17(c). The resulting $E_4 = E_\infty$ is shown in 3.2.17(d), where the symbol in place of b_{30}^2 indicates a copy of B with the first element missing.

3. The Lambda Algebra

In this section we describe the lambda algebra of Bousfield et al. [**?**] at the prime 2 and the algorithm suggested by it for computing Ext. For more details, including references, see Tangora [**?, ?**] and Richter [**?**] . For most of this material we are indebted to private conversations with E. B. Curtis. It is closely related to that of Section 1.5.

The lambda algebra Λ is an associative differential bigraded algebra whose cohomology, like that of the cobar complex, is Ext. It is much smaller than the cobar complex; it is probably the smallest such algebra generated by elements of cohomological degree one with cohomology isomorphic to Ext. Its greatest attraction, which will not be exploited here, is that it contains for each $n > 0$ a subcomplex $\Lambda(n)$ whose cohomology is the E_2-term of a spectral sequence converging to the 2-component of the *unstable* homotopy groups of S^n. In other words $\Lambda(n)$ is the E_1-term of an unstable Adams spectral sequence.

More precisely, Λ is a bigraded $\mathbf{Z}/(2)$-algebra with generators $\lambda_n \in \Lambda^{1,n+1}$ $(n \geq 0)$ and relations

$$\lambda_i\lambda_{2i+1+n} = \sum_{j\geq 0}\binom{n-j-1}{j}\lambda_{i+n-j}\lambda_{2i+1+j} \quad \text{for } i,n \geq 0 \tag{3.3.1}$$

with differential

$$d(\lambda_n) = \sum_{j\geq 1}\binom{n-j}{j}\lambda_{n-j}\lambda_{j-1}. \tag{3.3.2}$$

Note that d behaves formally like left multiplication by λ_{-1}.

3.3.3. Definition. *A monomial $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s} \in \Lambda$ is admissible if $2i_r \geq i_{r+1}$ for $1 \leq r < s$. $\Lambda(n) \subset \Lambda$ is the subcomplex spanned by the* admissible *monomials with $i_1 < n$.*

The following is an easy consequence of 3.3.1 and 3.3.2.

3.3.4. Proposition.
(a) *The admissible monomials constitute an additive basis for Λ.*
(b) *There are short exact sequences of complexes*

$$0 \to \Lambda(n) \to \Lambda(n+1) \to \Sigma^n\Lambda(2n+1) \to 0. \qquad \square$$

The significant property of Λ is the following.

3.3.5. Theorem (Bousfield *et al.* [**?**]). (a) $H(\Lambda) = \mathrm{Ext}_{A_*}(\mathbf{Z}/(2), \mathbf{Z}/(2))$, *the classical Adams E_2-term for the sphere.*

(b) $H(\Lambda(n))$ *is the E_2-term of a spectral sequence converging to $\pi_*(S^n)$.*

(c) *The long exact sequence in cohomology* (3.3.6) *given by* 3.3.4(b) *corresponds to the EHP sequence, i.e., to the long exact sequence of homotopy groups of the fiber sequence* (*at the prime* 2)

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1} \quad \text{(see 1.5.1)}. \qquad \square$$

The spectral sequence of (b) is the *unstable Adams spectral sequence*. The long exact sequence in (c) above is

$$\text{(3.3.6)} \qquad \to H^{s,t}(\Lambda(n)) \xrightarrow{E} H^{s,t}(\Lambda(n+1)) \xrightarrow{H} H^{s-1,t-n-1}(\Lambda(2n+1)) \xrightarrow{P} H^{s+1,t}(\Lambda(n)) \to .$$

The letters E, H, and P stand respectively for suspension (*Einhängung* in German), Hopf invariant, and Whitehead product. The map H is obtained by dropping the first factor of each monomial. This sequence leads to an inductive method for calculating $H^{s,t}(\Lambda(n))$ which we will refer to as the *Curtis algorithm*.

Calculations with this algorithm up to $t = 51$ (which means up to $t - s = 33$) are recorded in an unpublished table prepared by G. W. Whitehead. Recently, Tangora [**?**] has programmed a computer to find $H^{s,t}(\Lambda)$ at $p = 2$ for $t \leq 48$ and $p = 3$ for $t \leq 99$. Some related machine calculations are described by Wellington [**?**].

For the Curtis algorithm, note that the long exact sequences of 3.3.6 for all n constitute an exact couple (see Section 2.1) which leads to the following spectral sequence, similar to that of 1.5.7.

3.3.7. PROPOSITION (Algebraic EHP spectral sequence).
(a) *There is a trigraded spectral sequence converging to* $H^{s,t}(\Lambda)$ *with*

$$E_1^{s,t,n} = H^{s-1,t-n}(\Lambda(2n-1)) \quad \text{for } s > 0$$

and

$$E_1^{0,t,n} = \begin{cases} \mathbf{Z}/(2) & \text{for } t = n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and $d_r\colon E_r^{s,t,n} \to E_r^{s+1,t,n-r}$.

(b) *For each* $m > 0$ *there is a similar spectral sequence converging to* $H^{s,t}(\Lambda(m))$ *with*

$$E_1^{s,t,n} = \begin{cases} \text{as above} & \text{for } n \leq m \\ 0 & \text{for } n > m. \end{cases} \qquad \square$$

The EHP sequence in homotopy leads to a similar spectral sequence converging to stable homotopy filtered by sphere of origin which is described in Section 1.5.

At first glance the spectral sequence of 3.3.7 appears to be circular in that the E_1-term consists of the same groups one is trying to compute. However, for $n > 1$ the groups in $E_1^{s,t,n}$ are from the $(t-s-n+1)$-stem, which is known by induction on $t - s$. Hence 3.3.7(b) for odd values of m can be used to compute the E_1-terms. For $n = 1$, we need to know $H^*(\Lambda(1))$ at the outset, but it is easy to compute. $\Lambda(1)$ is generated simply by the powers of λ_0 and it has trivial differential. This corresponds to the homotopy of S^1.

Hence the EHP spectral sequence has the following properties,

3.3.8. LEMMA. *In the spectral sequence of* 3.3.7(a),
(a) $E_1^{s,t,n} = 0$ *for* $t - s < n - 1$ (vanishing line);
(b) $E_1^{s,t,n} = \mathbf{Z}/(2)$ *for* $t - s = n - 1$ *and all* $s \geq 0$ *and if in addition* $n - 1$ *is even and positive,* $d_1\colon E_1^{s,t,n} \to E_1^{s+1,t,n-1}$ *is nontrivial for all* $s \geq 0$ (*diagonal groups*);
(c) $E_1^{s,t} = H^{s-1,t-n}(\Lambda)$ *for* $t - s < 3n$ (*stable zone*); *and*
(d) $E_1^{s,t,1} = 0$ *for* $t > s$.

Proof. The groups in (a) vanish because they come from negative stems in $\Lambda(2n-1)$. The groups in (b) are in the 0-stem of $\Lambda(2n-1)$ and correspond to $\lambda_{n-1}\lambda_0^{s-1} \in \Lambda$. If $n-1$ is even and positive, 3.3.2 gives

$$d(\lambda_{n-1}\lambda_0^{s-1}) \equiv \lambda_{n-2}\lambda_0^s \mod \Lambda(n-2),$$

which means d_1 behaves as claimed. The groups in (c) are independent of n by 3.3.6. The groups in (d) are in $\Lambda(1)$ in positive stems. □

The above result leaves undecided the fate of the generators of $E_1^{0,n-1,n}$ for $n-1$ odd, which correspond to the λ_{n-1}. We use 3.3.2 to compute the differentials on these elements. (See Tangora [?] for some helpful advice on dealing with these binomial coefficients.) We find that if n is a power of 2, λ_{n-1} is a cycle, and if $n = k \cdot 2^j$ for odd $k > 1$ then

$$d(\lambda_{n-1}) \equiv \lambda_{n-1-2^j}\lambda_{2^j-1} \mod \Lambda(n-1-2^j).$$

This equation remains valid after multiplying on the right by any cycle in Λ, so we get

3.3.9. Proposition. *In the spectral sequence of* 3.3.7(a) *every element in* $E_1^{s,t,2^j}$ *is a permanent cycle. For* $n = k2^j$ *for* $k > 1$ *odd, then every element in* $E_r^{s,t,k2^j}$ *is a* d_r*-cycle for* $r < 2^j$ *and*

$$d_{2^j} : E_{2^j}^{0,k\cdot 2^j-1,k2^j} \to E_2^{1,k\cdot 2^j-1,(k-1)2^j}$$

is nontrivial, the target corresponding to λ_{2^j-1} *under the isomorphism of* 3.3.7. *The cycle* λ_{2^j-1} *corresponds to* $h_j \in \mathrm{Ext}^{1,2^j}$. □

Before proceeding any further it is convenient to streamline the notation. Instead of $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}$ we simply write $i_1 i_2 \dots i_s$, e.g., we write 411 instead of $\lambda_4\lambda_1\lambda_1$. If an integer ≥ 10 occurs we underline all of it but the first digit, thereby removing the ambiguity; e.g., $\lambda_{15}\lambda_3\lambda_{15}$ is written as $1\underline{5}31\underline{5}$. Sums of monomials are written as sums of integers, e.g., $d(9) = 71 + 53$ means $d(\lambda_9) = \lambda_7\lambda_1 + \lambda_5\lambda_3$; and we write ϕ for zero, e.g., $d(15) = \phi$ means $d(\lambda_{15}) = 0$.

We now study the EHP spectral sequence [3.3.7(a)] for $t - s \leq 14$. It is known that no differentials or unexpected extensions occur in this range in any of the unstable Adams spectral sequences, so we are effectively computing the 2-component of $\pi_{n+k}(S^n)$ for $k \leq 13$ and all n.

For $t-s = 0$ we have $E_1^{s,s,1} = \mathbf{Z}/(2)$ for all $s \geq 0$ and $E_1^{s,s,n} = 0$ for $n > 1$. For $t - s = 1$ we have $E_2^{1,2,2} = \mathbf{Z}/(2)$, corresponding to λ_1 or h_1, while $E_2^{s,1+s,n} = 0$ for all other s and n. From this and 3.3.8(c) we get $E_1^{2,n+2,n} = \mathbf{Z}/(2)$ generated by $\lambda_{n-1}\lambda_1$ for all $n \geq 2$, while $E_1^{s,t,t-s} = 0$ for all other s, t. The element 11 cannot be hit by a differential because 3 is a cycle, so it survives to a generator of the 2-stem, and it gives generators of $E_1^{3,n+4,n}$ (corresponding to elements with Hopf invariant 11) for $n \geq 2$, while $E_1^{s,t,t-s-1} = 0$ for all other s and t.

This brings us to $t - s = 3$. In addition to the diagonal groups $E_1^{s,s+3,4}$ given by 3.3.8(b) we have $E_1^{2,5,3}$ generated by 21 and $E_1^{3,6,2}$ generated by 111, with no other generators in this stem. These two elements are easily seen to be nontrivial permanent cycles, so $H^{s,s+3}(\Lambda)$ has three generators; 3, 21, and 111. Using 3.3.1 one sees that they are connected by left multiplication by 0 (i.e., by λ_0).

Thus for $t-s \leq 3$ we have produced the same value of Ext as given by the May spectral sequence in 3.2.11. The relation $h_0^2 h_2 = h_1^3$ corresponds to the relation $003 = 111$ in Λ, the latter being easier to derive. It is also true that 300 is cohomologous in Λ to 111, the difference being the coboundary of $40+22$. So far no differentials have occurred other than those of 3.3.8(b).

FIGURE 3.3.10. The EHP spectral sequence (3.3.7) for $t-s \leq 14$

These and subsequent calculations are indicated in FIG. 3.3.10, which we now describe. The gradings $t-s$ and n are displayed; we find this more illuminating than the usual practice of displaying $t-s$ and s. All elements in the spectral sequence in the indicated range are displayed except the infinite towers along the diagonal described in 3.3.8(b). Each element (except the diagonal generators) is referred to by listing the leading term of its Hopf invariant with respect to the left lexicographic ordering; e.g., the cycle $4111+221+1123$ is listed in the fifth row as 111. An important feature of the Curtis algorithm is that it *suffices to record the leading term of each element.* We will illustrate this principle with some examples. For more discussion see Tangora [**?**]. The arrows in the figure indicate differentials in the spectral sequence. Nontrivial cycles in Λ for $0 < t-s < 14$ are listed at the bottom. We do not list them for $t-s = 14$ because the table does not indicate which cycles in the 14th column are hit by differentials coming from the 15th column.

3.3.11. EXAMPLE. Suppose we are given the leading term 4111 of the cycle above. We can find the other terms as follows. Using 3.3.1 and 3.3.2 we find $d(4111) = 21111$. Refering to Fig. 3.3.10 we find 1111 is hit by the differential from 221, so we add 2221 to 4111 and find that $d(4111+2221) = 11121$. The figure shows that 121 is killed by 23, so we add 1123 to our expression and find that $d(4111+2221+1123) = \phi$ i.e., we have found all of the terms in the cycle.

Now suppose the figure has been completed for $t-s < k$. We wish to fill in the column $t-s = k$. The box for $n = 1$ is trivial by 3.3.8(d) and the boxes for $n \geq 3$ can be filled in on the basis of previous calculations. (See 3.3.12.) The elements in the box for $n = 2$ will come from the *cycles* in the box for $n = 3$, $t-s = k-1$, and the elements in the box for $n = 2$, $t-s = k-1$ which are *not hit* by d_1's. Hence before we can fill in the box for $b = 2$, $t-s = k$, we must find the d_1's originating in the box for $n = 3$. The procedure for computing differentials will be described below. Once the column $t-s = k$ has been filled in, one computes the differentials for successively larger values of n.

The above method is adequate for the limited range we will consider, but for more extensive calculations it has a drawback. One could work very hard to show that some element is a cycle only to find at the next stage that it is hit by an easily computed differential. In order to avoid such redundant work one should work by induction on t, then on s and then on n; i.e., one should compute differentials originating in $E_r^{s,t,n}$ only after one has done so for all $E_r^{s',t',n'}$ with $t' < t$, with $t' = t$ and $s' < s$, and with $s' = s$, $t' = t$, and $n' < n$. This triple induction is awkward to display on a sheet of paper but easy to write into a computer program. On the other hand Tangora [**?**, last paragraph starting on page 48] used downward rather than upward induction on s because given knowledge of what happens at all lower values of t, the last group needed for the $(t-s)$-stem is the one with the largest value of s possible under the vanishing line, the unstable analog of 3.4.5. There are advantages to both approaches.

The procedure for finding differentials in the EHP spectral sequence (3.3.7) is the following. We start with some sequence α in the $(n+1)$th row. Suppose inductively that some correcting terms have already been added to $\lambda_n\alpha$, in the manner about to be described, to give an expression x. We use 3.3.1 and 3.3.2 to find the leading term $i_1 i_2 \dots i_{s+1}$ of $d(x)$. If $d(x) = 0$, then our α is a permanent cycle in the spectral sequence. If not, then beginning with $u = 0$ we look in the table for the sequence $i_{s-u+1} i_{s-u+2} \dots i_{s+1}$ in the $(i_{s-u}+1)$th row until we find one that is hit by a differential from some sequence β in the $(m+1)$th row or until $u = s-1$. In the former event we add $\lambda_{i_1} \dots \lambda_{i_{s-u-1}} \lambda_m \beta$ to x and repeat the process. The coboundary of the new expression will have a smaller leading term since we have added a correcting term to cancel out the original leading coboundary term.

If we get up to $u = s-1$ without finding a target of a differential, then it follows that our original α supports a d_{n-i_1} whose target is $i_2 \cdots i_{s+1}$.

It is not necessary to add all of the correcting terms to x to show that our a is a permanent cycle. The figure will provide a finite list of possible targets for the differential in question. As soon as the leading term of $d(x)$ is smaller (in the left lexicographic ordeninng) than any of these candidates then we are done.

In practice it may happen that one of the sequences $i_{s-u+1} \cdots i_{s+1}$ in the $(i_{s-u}+1)$th row supports a nontrivial differential. This would be a contradiction indicating the presence of an error, which should be found and corrected before proceeding further. Inductive calculations of this sort have the advantage that mistakes usually reveal themselves by producing contradictions a few stems later. Thus one can be fairly certain that a calculation through some range that is free of contradictions is also correct through most of that range. In publishing such computations it is prudent to compute a little beyond the stated range to ensure the accuracy of one's results.

We now describe some sample calculations in 3.2.11.

3.3.12. Example. Filling in the table. Consider the boxes with

$$t - s - (n-1) = 8.$$

To fill them in we need to know the 8-stem of $H(\Lambda(2n-1))$. For convenience the values of $2n-1$ are listed at the extreme left. The first element in the 8-stem is 233, which originates on S^3 and hence appears in all boxes for $n \geq 2$. Next we have the elements 53, 521, and 5111 originating on S^6. The latter two are trivial on S^7 and so do not appear in any of our boxes, while 53 appears in all boxes with $n \geq 4$. The element 611 is born on S^7 and dies on S^9 and hence appears only in the box for $n = 4$. Similarly, 71 appears only in the box for $n = 5$.

3.3.13. Example. Computing differentials We will compute the differentials originating in the box for $t - s = 11$, $n = 11$. To begin we have $d(1\underline{0}1) = (90 + 72 + 63 + 54)1 = 721 + 631 + 541$. The table shows that 721 is hit by 83 and we find

$$d(83) = (70 + 61 + 43)3 = 721 + 433.$$

Hence

$$d(1\underline{0}1 + 83) = 631 + 541 + 433.$$

The figure shows that 31 is hit by 5 so we compute

$$d(65) = 631 + (50 + 32)5 = 631 + 541,$$

so

$$d(101 + 83 + 65) = 433,$$

which is the desired result.

Even in this limited range one can see the beginnings of several systematic phenomena worth commenting on.

3.3.14. REMARK. JAMES PERIODICITY. (Compare 1.5.18.) In a neighborhood of the diagonal one sees a certain in the differentials in addition to that of 3.3.9. For example, the leading term of $d(\lambda_n\lambda_1)$ is $\lambda_{n-2}\lambda_1\lambda_1$ if $n \equiv 0$ or $1 \mod (4)$ and $n \geq 4$, giving a periodic family of d_2's in the spectral sequence. The differential computed in 3.3.13 can be shown to recur every 8 stems; add any positive multiple of 8 to the first integer in each sequence appearing in the calculation and the equation remains valid modulo terms which will not affect the outcome.

More generally, one can show that $\Lambda(n)$ is isomorphic to

$$\Sigma^{-2^m}\Lambda(n + 2^m)/\Lambda(2^m)$$

through some range depending on n and m, and a general result on the periodicity of differentials follows. It can be shown that $H^*(\Lambda(n + k)/\Lambda(n))$ is isomorphic in the stable zone [3.3.8(c)] to the Ext for $H^*(RP^{n+k-1}/RP^{n-1})$ and that this periodicity of differentials corresponds to James periodicity. The latter is the fact that the stable homotopy type of RP^{n+k}/RP^n depends (up to suspension) only on the congruence class of n modulo a suitable power of 2. For more on this subject see Mahowald [**?, ?, ?, ?**].

3.3.15. REMARK. THE ADAMS VANISHING LINE. Define a collection of admissible sequences (3.3.3) a_i for $i > 0$ as follows.

$$a_1 = 1, \quad a_2 = 11, \quad a_3 = 111, \quad a_4 = 4111,$$
$$a_5 = 24111, \quad a_6 = 124111, \quad a_7 = 1124111, \quad a_8 = 41124111, \text{ etc.}$$

That is, for $i > 1$

$$a_i = \begin{cases} (1, a_{i-1}) & \text{for } i \equiv 2, 3 \mod (4) \\ (2, a_{i-1}) & \text{for } i \equiv 1 \mod (4) \\ (4, a_{i-1}) & \text{for } i \equiv 0 \mod (4) \end{cases}$$

It can be shown that all of these are nontrivial permanent cycles in the EHP spectral sequence and that they correspond to the elements on the Adams vanishing line (3.4.5). Note that $H(a_{i+1}) = a_i$. All of these elements have order 2 (i.e., are killed by λ_0 multiplication) and half of them, the a_i for $i \equiv 3$ and $0 \mod (4)$, are divisible by 2. The a_{4i+3} are divisible by 4 but not by 8; the sequences obtained are $(2, a_{4i+2})$ and $(4, a_{4i+1})$ except for $i = 1$, when the latter sequence is 3. These little towers correspond to cyclic summands of order 8 in π^S_{8i+3} (see 5.3.7). The a_{4i} are the tops of longer towers whose length depends on i. The sequences in the tower are obtained in a similar manner; i.e., sequences are contracted by adding the first two integers; e.g., in the 7-stem we have 4111, 511, 61, and 7. Whenever i is a power of 2 the tower goes all the way down to filtration 1; i.e., it has $4i$ elements, of which the bottom one is $8i - 1$. The table of Tangora [**?**] shows that the towers in the 23-, 29-, and 55-stems have length 6, while that in the 47-stem has length 12.

Presumably this result generalizes in a straightforward manner. These towers are also discussed in 3.4.21 and following 4.4.47.

3.3.16. Remark. d_1's. It follows from 3.3.9 that all d_1's originate in rows with n odd and that they can be computed by left multiplication by λ_0. In particular, the towers discussed in the above remark will appear repeatedly in the E_1-term and be almost completely cancelled by d_1's, as one can see in Fig. 3.3.10. The elements cancelled by d_1's do not appear in any $H^*(\Lambda(2n-1))$, so if one is not interested in $H^*(\Lambda(2n))$ they can be ignored. This indicates that a lot of repetition could be avoided if one had an algorithm for computing the spectral sequence starting from E_2 instead of E_1.

3.3.17. Remark. S^3. As indicated in 3.3.5, Λ gives unstable as well as stable Ext groups. From a figure such as 3.3.11 one can extract unstable Adams E_2-terms for each sphere. For the reader's amusement we do this for S^3 for $t-s\leq 28$ in Fig. 3.3.18. One can show that if we remove the infinite tower in the 0-stem, what remains is isomorphic above a certain line of slope $\frac{1}{5}$ to the stable Ext for the mod (2) Moore spectrum. This is no accident but part of a general phenomenon described by Mahowald [?].

It is only necessary to label a few of the elements in Fig. 3.3.18 because most of them are part of certain patterns which we now describe. There are clusters of six elements known as *lightning flashes*, the first of which consists of 1, 11, 111, 21, 211, 2111. Vertical and diagonal lines as usual represent right multiplication by λ_0 and λ_1, i.e., by h_0 and h_0 respectively. This point is somewhat delicate. For example the element with in the 9-stem with filtration 4 has leading term (according to 3.3.10) 1233, not 2331. However these elements are cohomologous, their difference being the coboundary of 235.

If the first element of a lightning flash is x, the others are $1x$, $11x$, $2x$, $21x$, and $211x$. In the clusters containing 23577 and 233577, the first elements are missing, but the others behave as if the first ones were 4577 and 43577, respectively. For example, the generator of $E_2^{5,30}$ is 24577. In these two cases the sequences $1x$ and $11x$ are not admissible, but since $14 = 23$ by 3.3.1, we get the indicated values for $1x$.

If $x\in E_2^{s,t}$ is the first element of a lightning flash, there is another one beginning with $Px\in E_2^{s+4,t+12}$. The sequence for Px is obtained from that for x by adding 1 to the last integer and then adjoining 4111 on the right, e.g., $P(233) = 2344111$. This operator P can be iterated any number of times, is related to Bott periodicity, and will be discussed more in the next section.

There are other configurations which we will call *rays* begining with 245333 and 235733. Successive elements in a ray are obtained by left multiplication by λ_2. This operation is related to complex Bott periodicity.

In the range of this figure the only elements in positive stems not part of a ray or lightning flash are 23333 and 2335733. This indicates that the Curtis algorithm would be much faster if it could be modified in some way to incorporate this structure.

Finally, the figure includes Tangora's labels for the stable images of certain elements. This unstable Adams spectral sequence for $\pi_*(S^3)$ is known to have nontrivial d_2's originating on 245333, 222245333, and 2222245333, and d_3's on 2235733 and 22235733. Related to these are some exotic additive and multiplicative

FIGURE 3.3.18. The unstable Adams E_2-term for S^3.

extensions: the homotopy element corresponding to $Ph_1d_0 = 243344111$ is twice any representative of $h_0h_2g = 235733$ and η (the generator of the 1-stem) times a representative of 2245333. Hence the permanent cycles 2245333, 24334111, 235733, 22245333, 224334111, and the missing element 35733 in some sense constitute an exotic lightning flash.

4. Some General Properties of Ext

In this section we abbreviate $\mathrm{Ext}_{A_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$ by Ext. First we describe Ext^s for small values of s. Then we comment on the status of its generators in homotopy. Next we give a vanishing line, i.e., a function $f(s)$ such that $\mathrm{Ext}^{s,t} = 0$ for $0 < t - s < f(s)$. Then we give some results describing $\mathrm{Ext}^{s,t}$ for t near $f(s)$.

3.4.1. Theorem. *For $p = 2$*

(a) $\mathrm{Ext}^0 = \mathbf{Z}/(2)$ *generated by* $1 \in \mathrm{Ext}^{0,0}$.

(b) Ext^1 *is spanned by* $\{h_i : i \geq 0\}$ *with* $h_i \in \mathrm{Ext}^{1,2^i}$ *represented by* $[\xi_1^{2^i}]$.

(c) (Adams [?]) Ext^2 *is spanned by* $\{h_ih_j : 0 \leq i \leq j,\ j \neq i+1\}$.

(d) (Wang [?]) Ext^3 *is spanned by* $h_ih_jh_k$, *subject to the relations*

$$h_ih_j = h_jh_i, \quad h_{ih_{i+1}} = 0 \quad h_1h_{i+2}^2 = 0 \quad h_i^2h_{i+2} = h_{i+1}^3,$$

along with the elements

$$c_i = \langle h_{i+1}, h_i, h_{i+2}^2 \rangle \in \mathrm{Ext}^{3,11\cdot 2^i}. \qquad \square$$

3.4.2. Theorem. *For $p = 2$*

(a) $\mathrm{Ext}^0 = \mathbf{Z}/(p)$ *generated by* $1 \in \mathrm{Ext}^{0,0}$.

(b) Ext^1 *is spanned by* a_0 *and* $\{h_i : i \geq 0\}$ *where* $a_0 \in \mathrm{Ext}^{1,1}$ *is represented by* $[\tau_0]$ *and* $h_i \in \mathrm{Ext}^{1,qp^i}$ *is represented by* $[\xi_i^{p^i}]$.

(c) (Liulevicius [?]) Ext^2 *is spanned by* $\{h_ih_j : 0 \leq i < j-1\}$, a_0^2, $\{a_0h_i : i > 0\}$, $\{g_i : i \geq 0\}$, $\{k_i : i \geq 0\}$, $\{b_i : i \geq 0\}$, *and* Π_0h_0, *where*

$$g_i = \langle h_i, h_i, h_{i+1} \rangle \in \mathrm{Ext}^{2,(2+p)p^iq}, \quad k_i = \langle h_i, h_{i+1}, h_{i+1} \rangle \in \mathrm{Ext}^{2,(2p+1)p^iq},$$

$$b_i = \langle h_i, h_i, \ldots, h_i \rangle \in \mathrm{Ext}^{2,qp^{i+1}} \quad \text{(with } p \text{ factors } h_i),$$

and

$$\Pi_0h_0 = \langle h_0, h_0, a_0 \rangle \in \mathrm{Ext}^{2,1+2q}. \qquad \square$$

Ext^3 for $p > 2$ has recently been computed by Aikawa [?].

The behavior of the elements in Ext^1 in the Adams spectral sequence is described in Theorems 1.2.11–1.2.14.

We know that most of the elements in Ext^2 cannot be permanent cycles, i.e.,

3.4.3. Theorem. (a) (Mahowald and Tangora [?]). *With the exceptions* h_0h_2, h_0h_3, *and* h_2h_4 *the only elements in* Ext^2 *for* $p = 2$ *which can possibly be permanent cycles are* h_j^2 *and* h_1h_j.

(b) (Miller, Ravenel, and Wilson [?]). *For* $p > 2$ *the only elements in* Ext^2 *which can be permanent cycles are* a_0^2, Π_0h_0, k_0, h_0h_i, *and* b_i. $\square$

Part (b) was proved by showing that the elements in question are the only ones with preimages in the Adams–Novikov E_2-term. A similar proof for $p = 2$ is possible using the computation of Shimomura [?]. The list in Mahowald and Tangora [?] includes h_2h_5 and h_3h_6; the latter is known not to come from the Adams–Novikov spectral sequence and the former is known to support a differential.

The cases h_0h_i and b_i, for $p > 3$ and h_1h_i for $p = 2$ are now settled.

3.4.4. Theorem. (a) (Browder [?]). *For $p = 2$ h_j^2 is a permanent cycle iff there is a framed manifold of dimension $2^{j+1} - 2$ with Kervaire invariant one. Such are known to exist for $j \leq 5$. For more discussion see* 1.5.29 *and* 1.5.35.

(b) (Mahowald [?]). *For $p = 2$ h_1h_j is a permanent cycle for all $j \geq 3$.*

(c) (Ravenel [?]). *For $p > 3$ and $i \geq 1$, b_i is not a permanent cycle.* (*At $p = 3$ b_1 is not permanent but b_2 is*; *b_0 is permanent for all odd primes.*)

(d) (R. L. Cohen [?]). *For $p > 2$ h_0b_i is a permanent cycle corresponding to an element of order p for all $i \geq 0$.* □

The proof of (c) will be given in Section 6.4.

Now we describe a vanishing line. The main result is

3.4.5. Vanishing Theorem (Adams [?]). (a) *For $p = 2$ $\mathrm{Ext}^{s,t} = 0$ for $0 < t - s < f(s)$, where $f(s) = 2s - \varepsilon$ and $\varepsilon = 1$ for $s \equiv 0, 1 \mod (4)$, $\varepsilon = 2$ for $s \equiv 2$ and $\varepsilon = 3$ for $s \equiv 3$.*

(b) (May [?]). *For $p > 2$ $\mathrm{Ext}^{s,t} = 0$ for $0 < t - s < sq - \varepsilon$, where $\varepsilon = 1$ if $s \not\equiv 0$* mod (p) *and $\varepsilon = 2$ if $s \equiv 0$.* □

Hence in the usual picture of the Adams spectral sequence, where the x and y coordinates are $t - s$ and s, the E_2-term vanishes above a certain line of slope $1/q$ (e.g., $\frac{1}{2}$ for $p = 2$). Below this line there are certain periodicity operators Π_n which raise the bigrading so as to move elements in a direction parallel to the vanishing line. In a certain region these operators induce isomorphisms.

3.4.6. Periodicity Theorem (Adams [?], May [?]).

(a) *For $p = 2$ and $n \geq 1$ $\mathrm{Ext}^{s,t} \simeq \mathrm{Ext}^{s+2^{n+1},t+3\cdot 2^{n+1}}$ for*

$$0 < t - s < \min(g(s) + 2^{n+2}, h(s)),$$

where $g(s) = 2s - 4 - \tau$ with $\tau = 2$ if $s \equiv 0, 1 \mod (4)$, $\tau = 1$ if $s \equiv 3$, and $\tau = 0$ if $s \equiv 2$, and $h(s)$ is defined by the following table:

s	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	≥ 9
$h(s)$	*1*	*1*	*7*	*10*	*17*	*22*	*25*	*32*	$5s - 7$

(b) *For $p > 2$ and $n \geq 0$ $\mathrm{Ext}^{s,t} \simeq \mathrm{Ext}^{s+p^n,s+(q+1)/p^n}$ for*

$$0 < t - s < \min(g(s) + p^n q, h(s)),$$

where $g(s) = qs - 2p - 1$ and $h(s) = 0$ for $s = 1$ and $h(s) = (p^2 - p - 1)s - \tau$ with $\tau = 2p^2 - 2p + 1$ for even $s > 1$ and $\tau = p^2 + p - 2$ for odd $s > 1$. □

These two theorems are also discussed in Adams [?].

For $p = 2$ these isomorphisms are induced by Massey products (A1.4) sending x to $\langle x, h_0^{2^{n+1}}, h_{n+2}\rangle$. For $n = 1$ this operator is denoted in Tangora [?] and elsewhere in this book by P. The elements x are such that $h_0^{2^{n+1}}x$ is above the vanishing line of 3.4.5, so the Massey product is always defined. The indeterminacy of the product has the form $xy + h_{n+2}z$ with $y \in \mathrm{Ext}^{2^{n+1},3\cdot 2^{n+1}}$ and $z \in \mathrm{Ext}^{s-1+2^{n+1},t+2^{n+1}}$. The group containing y is just below the vanishing line and we will see below that it is always trivial. The group containing z is above the vanishing line so the indeterminacy is zero.

Hence the theorem says that any group close enough to the vanishing line [i.e., satisfying $t - s < 2^{n+2} + g(s)$] and above a certain line with slope $\frac{1}{5}[t - s < h(s)]$

is acted on isomorphically by the periodicity operator. In Adams [?] this line had slope $\frac{1}{3}$. It is known that $\frac{1}{5}$ is the best possible slope, but the intercept could probably be improved by pushing the same methods further. The odd primary case is due entirely to May [?]. We are grateful to him for permission to include this unpublished material here.

Hence for $p = 2$ $\mathrm{Ext}^{s,t}$ has a fairly regular structure in the wedge-shaped region described roughly by $2s < t - s < 5s$. Some of this (partially below the line of slope $\frac{1}{5}$ given above) is described by Mahowald and Tangora [?] and an attempt to describe the entire structure for $p = 2$ is made by Mahowald [?].

However, this structure is of limited interest because we know that almost all of it is wiped out by differentials. All that is left in the E_∞-term are certain few elements near the vanishing line related to the J-homomorphism (1.1.12). We will not formulate a precise statement or proof of this fact, but offer the following explanation. In the language of Section 1.4, the periodicity operators Π_n in the Adams spectral sequence correspond to v_1-periodicity in the Adams–Novikov spectral sequence. More precisely, Π_n corresponds to multiplication by $v_1^{p^n}$. The behavior of the v_1-periodic part of the Adams–Novikov spectral sequence is analyzed completely in Section 5.3. The v_1-periodic part of the Adams–Novikov E_∞-term must correspond to the portion of the Adams spectral sequence E_∞-term lying above (for $p = 2$) a suitable line of slope $\frac{1}{5}$. Once the Adams–Novikov spectral sequence calculation has been made it is not difficult to identify the corresponding elements in the Adams spectral sequence. The elements in the Adams–Novikov spectral sequence all have low filtrations, so it is easy to establish that they cannot be hit by differentials. The elements in the Adams spectral sequence are up near the vanishing line so it is easy to show that they cannot support a nontrivial differential. We list these elements in 3.4.16 and in 3.4.21 give an easy direct proof (i.e., one that does not use BP-theory or K-theory) that most (all for $p > 2$) of them cannot be hit by differentials.

The proof of 3.4.5 involves the comodule M given by the short exact sequence

$$0 \to \mathbf{Z}/(p) \to A_* \,\square_{A(0)_*} \mathbf{Z}/(p) \to M \to 0, \tag{3.4.7}$$

where $A(0)_* = E(\tau_0)$ for $p > 2$ and $E(\xi_1)$ for $p = 2$. M is the homology of the cofiber of the map from S^0 to H, the integral Eilenberg–Mac Lane spectrum. The E_2-term for H was computed in 2.1.18 and it gives us the tower in the 0-stem. Hence the connecting homomorphism of 3.4.7 gives an isomorphism

$$\mathrm{Ext}_{A_*}^{s-1,t}(\mathbf{Z}/(p), M) \simeq \mathrm{Ext}^{s,t} \tag{3.4.8}$$

for $t - s > 0$.

We will consider the subalgebras $A(n) \subset A$ generated by $\{Sq^1, Sq^2, \ldots, Sq^{2^n}\}$ for $p = 2$ and $\{\beta, P^1, P^p, \ldots, P^{p^{n-1}}\}$ for $p > 2$. Their duals $A(n)_*$ are $P(\xi_1, \xi_2, \ldots, \xi_{n+1})/(\xi_i^{2^{n+2-i}})$ for $p = 2$ and

$$E(\tau_0, \ldots, \tau_n) \otimes P(\xi_1, \ldots, \xi_n)/(\xi_i^{p^{n+1-i}})$$

for $p > 2$.

We will be considering A_*-comodules N which are free over $A(0)_*$ and (-1)-connexted. $\Sigma^{-1}M$ is an example. Unless stated otherwise N *will be assumed to have these properties for the rest of the section.*

Closely related to the questions of vanishing and periodicity is that of approximation. For what (s,t) does $\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(p),N)=\mathrm{Ext}_{A(n)_*}^{s,t}(\mathbf{Z}/(p),N)$? This relation is illustrated by

3.4.9. Approximation Lemma. *Suppose that there is a nondecreasing function $f_n(s)$ defined such that for any N as above, $\mathrm{Ext}_{A(n)_*}^{s,t}(\mathbf{Z}/(p),N)=0$ for $t-s<f_n(s)$. Then for $r\geq n$ this group is isomorphic to $\mathrm{Ext}_{A(r)_*}^{s,t}(\mathbf{Z}/(p),N)$ for $t-s<p^nq+f_n(s-1)$, and the map from the former to the latter is onto for $t-s=p^nq+f_n(s)$.* □

Hence if $f_n(s)$ describes a vanishing line for $A(n)$-cohomology then there is a parallel line below it, above which it is isomorphic to A-cohomology. For $n=1$ such a vanishing line follows easily from 3.1.27(a) and 3.1.28, and it has the same slope as that of 3.4.5.

Proof of 3.4.9. The comodule structure map $N\to A(r)_*\otimes N$ gives a monomorphism $N\to A(r)_*\,\square_{A(n)_*}N$ with cokernel C. Then C is $A(0)_*$-free and (p^nq-1)-connected. Then we have

$$\begin{tikzcd}
\mathrm{Ext}^{s-1}_{A(r)_*}(C) \arrow[r] & \mathrm{Ext}^{s}_{A(r)_*}(N) \arrow[r] \arrow[dr] & \mathrm{Ext}^{s}_{A(r)_*}(A(r)_*\,\square_{A(n)_*}N) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^{s}_{A(r)_*(C)} \\
& & \mathrm{Ext}^{s}_{A(n)_*}(N) &
\end{tikzcd}$$

where $\mathrm{Ext}_{A(r)_*}(-)$ is an abbreviation for $\mathrm{Ext}_{A(r)_*}(\mathbf{Z}/(p),-)$. The isomorphism is given by A1.1.18 and the diagonal map is the one we are considering. The high connectivity of C and the exactness of the top row give the desired result. □

Proof of 3.4.5. We use 3.4.9 with $N=M$ as in 3.4.7. An appropriate vanishing line for M will give 3.4.5 by 3.4.8. By 3.4.9 it suffices to get a vanishing line for $\mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(p),M)$. We calculate this by filtering M skeletally as an $A(0)_*$-comodule. Then E^0M is an extended $A(0)_*$-determined by 3.1.27(a) or 3.1.28 and the additive structure of M. Considering the first two (three for $p=2$) subquotients is enough to get the vanishing line. We leave the details to the reader. □

The periodicity operators in 3.4.6 which raise s by p^n correspond in $A(n)$-cohomology to multiplication by an element $\omega_n\in\mathrm{Ext}^{p^n,(q+1)p^n}$. In view of 3.4.9, 3.4.6 can be proved by showing that this multiplication induces an isomorphism in the appropriate range. For $p=2$ our calculation of $\mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2),A(0)_*)$ (3.2.17) is nessesary to establish periodicity above a line of slope $\frac{1}{5}$. To get these ω_n we need

3.4.10. Lemma. *There exist cochains $c_n\in C_{A_*}$ satisfying the following.*

(a) *For $p=2$ $c_n\equiv[\xi_2|\cdots|\xi_2]$ with 2^n factors modulo terms involving ξ_1, and for $p>2$ $c_n\equiv[\tau_1|\cdots|\tau_1]$ with p^n factors.*

(b) *For $p=2$ $d(c_1)=[\xi_1|\xi_1|\xi_1^4]+[\xi_1^2|\xi_1^2|\xi_1^2]$ and for $n>1$ $d(c_n)=[\xi_1|\cdots|\xi_1|\xi_1^{2^{n+1}}]$ factors ξ_1; and for $p>2$ $d(c_n)=-[\tau_0|\cdots|\tau_0|\xi_1^{p^n}]$.*

(c) *c_n is uniquely determined up to a coboundary by* (a) *and* (b).

(d) *For $n\geq 1$ $(p>2)$ or $n\geq 2$ $(p=2)$ c_n projects to a cocycle in $C_{A(n)_*}$ representing a nontrivial element $\omega_n\in\mathrm{Ext}_{A(n)_*}^{p^n,(q+1)p^n}(\mathbf{Z}/(p),\mathbf{Z}/(p))$.*

(e) *For $p=2$, ω_2 maps to ω as in 3.1.27, and in general ω_{n+1} maps to ω_n^p.*

PROOF. We will rely on the algebraic Steenrod operations in Ext described in Section A1.5. We treat only the case $p = 2$. By A1.5.2 there are operations $Sq^i \colon \mathrm{Ext}^{s,t} \to \mathrm{Ext}^{s+i,2t}$ satisfying a Cartan formula with $Sq^0(h_i) = h_{i+1}$ (A1.5.3) and $Sq^1(h_i) = h_1^2$. Applying Sq^1 to the relation $h_0h_1 = 0$ we have

$$\begin{aligned} 0 &= Sq^1(h_0h_1) = Sq^0(h_0)Sq^1(h_1) + Sq^1(h_0)Sq^0(h_1) \\ &= h_1^3 + h_0^2h_2. \end{aligned}$$

Applying Sq^2 to this gives $h_1^4h_2 + h_0^4h_3 = 0$. Since $h_1h_2 = 0$ this implies $h_0^4h_3 = 0$. Applying Sq^4 to this gives $h_0^8h_4 = 0$. Similarly, we get $h_0^{2^i}h_{i+1} = 0$ for all $i \geq 2$. Hence there must be cochains c_n satisfying (b) above.

To show that these cochains can be chosen to satisfy (a) we will use the Kudo transgression theorem A1.5.7. Consider the cocentral extension of Hopf algebras (A1.1.15)

$$P(\xi_1) \to P(\xi_1, \xi_2) \to P(\xi_2).$$

In the Cartan–Eilenberg spectral sequence (A1.3.14 and A1.3.17) for

$$\mathrm{Ext}_{P(\xi_1,\xi_2)}(\mathbf{Z}/(2), \mathbf{Z}/(2))$$

one has $E_2 = P(h_{1j}, h_{2j} \colon y \geq 0)$ with $h_{1j} \in E_2^{1,0}$ and $h_{2j} \in E_2^{0,1}$. By direct calculation one has $d_2(h_{20}) = h_{10}h_{11}$. Applying Sq^2Sq^1 one gets $d_5(h_{20}^4) = h_{10}^4h_{13} + h_{11}^4h_{12}$. The second term was killed by $d_2(h_{11}^3h_{21})$ so we have $d_5(h_{20}^4) = h_{10}h_{13}^4$. Applying appropriate Steenrod operations gives $d_{2^n+1}(h_{20}^{2^n}) = h_{10}^{2^n}h_{1n+1}$. Hence our cochain c_n can be chosen in $C_{P(\xi_1,\xi_2)}$ so that its image in $C_{P(\xi_2)}$ is $[\xi_2|\cdots|\xi_2]$ representing $h_{20}^{2^n}$, so (a) is verified.

For (c), note that (b) determines c_n up to a cocycle, so it suffices to show that each cocycle in that bidegree is a coboundary, i.e., that $\mathrm{Ext}^{2^n, 3\cdot 2^n} = 0$. This group is very close to the vanishing line and can be computed directly by what we already know.

For (d), (a) implies that c_n projects to a cocycle in $C_{A(n)_*}$ which is nontrivial by (b); (e) follows easily from the above considerations. $\square$

For $p = 2$ suppose $x \in \mathrm{Ext}$ satisfies $h_0^{2^n}x = 0$. Let $\hat{x} \in C_{A_*}$ be a cocycle representing x and let y be a cochain with $d(y) = \hat{x}[\xi_1|\cdots|\xi_1]$ with 2^n factors. Then $\hat{x}c_n + y[\xi_1^{2^{n+1}}]$ is a cocycle representing the Massey product $\langle x, h_0^{2^n}, h_{n+1}\rangle$, which we define to be the nth periodicity operator Π_n. This cocycle maps to $\hat{x}c_n$ in $c_{A(n)_*}$, so Π_n corresponds to multiplication by ω_n as claimed. The argument for $p > 2$ is similar.

Now we need to examine ω_1 multiplication in $\mathrm{Ext}(A(1)_*)(\mathbf{Z}/(p), A(0)_*)$ for $p > 2$ using 3.1.28 and ω_2 multiplication in $\mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2), A(0)_*)$ using 3.2.17. The result is

3.4.11. LEMMA.

(a) *For $p = 2$, multiplication by ω_2 in $\mathrm{Ext}_{A(2)_*}^{s,t}(\mathbf{Z}/(2), A(0)_*)$ is an isomorphism for $t - s < v(s)$ and an epimorphism for $t - s < w(s)$, where $v(s)$ and $w(s)$ are given in the following table.*

s	0	1	2	3	4	5	≥ 6
$v(s)$	1	8	6	18	18	21	$5s+3$
$w(s)$	1	8	10	18	23	25	$5s+3$

(b) *For $p > 2$ multiplication by ω_1 in* $\mathrm{Ext}^{s,t}_{A(1)_*}(\mathbf{Z}/(p), A(0)_*)$ *is a monomorphism for all $s \geq 0$ and an epimorphism for $t - s < w(s)$ where*

$$w(s) = \begin{cases} (p^2 - p - 1)s - 1 & \text{for } s \text{ even} \\ (p^2 - p - 1)s + p^2 - 3p & \text{for } s \text{ odd} \end{cases} \qquad \square$$

Next we need an analogous result where $A(0)_*$ is replaced by a (-1)-connected comodule N free over $A(0)_*$. Let $N^0 \subset N$ be the smallest free $A(0)_*$-subcomodule such that N/N^0 is 1-connected. Then

$$0 \to N^0 \to N \to N/N^0 \to 0$$

is an short exact sequence of $A(0)_*$-free comodules inducing an long exact sequence of $A(n)$–Ext groups on which ω_n acts. Hence one can use induction and the 5-lemma to get

3.4.12. Lemma. *Let N be a connective $A(n)_*$-comodule free over $A(0)_*$.*

(a) *For $p = 2$ multiplication by ω_2 in* $\mathrm{Ext}^{s,t}_{A(2)_*}(\mathbf{Z}/(2), M)$ *is an isomorphism for $t - s < \tilde{v}(s)$ and an epimorphism for $t - s < \widetilde{w}(s)$, where these functions are given by the following table*

s	0	1	2	3	4	5	6	≥ 7
$\tilde{v}(x)$	-4	1	6	10	18	21	25	$5s - 2$
$\widetilde{w}(s)$	1	7	10	18	22	25	33	$5s + 3$

(b) *For $p > 2$ a similar result holds for ω_1-multiplication where*

$$\tilde{v}(s) = \begin{cases} (p^2 - p - 1)s - 2p + 1 & \textit{for } s \textit{ even} \\ (p^2 - p - 1)s - p^2 + p & \textit{for } s \textit{ odd} \end{cases}$$

and

$$\widetilde{w}(s) = \begin{cases} (p^2 - p - 1)s - 1 & \textit{for } s \textit{ even} \\ (p^2 - p - 1)s - p^2 + 2p - 1 & \textit{for } s \textit{ odd.} \end{cases} \qquad \square$$

3.4.13. Remark. If N/N^0 is $(q-1)$-connected, as it is when $N = \Sigma^{-q} M$ (3.4.7), then the function $\tilde{v}(s)$ can be improved slightly. This is reflected in 3.4.6 and we leave the details to the reader.

The next step is to prove an analogous result for ω_n-multiplication. We sketch the proof for $p = 2$. Let N be as above and define $\overline{N} = A(n)_* \square_{A(2)_*} N$, and let $C = \overline{N}/N$. Then C is 7-connected if N is (-1)-connected, and $\mathrm{Ext}_{A(n)_*}(\mathbf{Z}/(2), \overline{N}) = \mathrm{Ext}_{A(2)_*}(\mathbf{Z}/(2), N)$. Hence in this group $\omega_n = \omega_2^{2^{n-2}}$ and we know its behavior by 3.4.12. We know the behavior of ω_n on C by induction, since C is highly connected, so we can argue in the usual way by the 5-lemma on the long exact sequence of Ext groups. If N satisfies the condition of 3.4.13, so will $\overline{N}$ and $\overline{C}$, so we can use the improved form of 3.4.12 to start the induction. The result is

3.4.14. Lemma. *Let N be as above and satisfy the condition of* 3.4.13. *Then multiplication by ω_n* (3.4.10) *in* $\mathrm{Ext}^{s,t}_{A(n)_*}(\mathbf{Z}/(p), N)$ *is an isomorphism for $t - s < h(s+1) - 1$ and an epimorphism for $t - s < h(s) - 1$, where $h(s)$ is as in* 3.4.6. $\square$

Now the periodicity operators Π_n, defined above as Massey products, can be described in terms of the cochains c_n of 3.4.10 as follows. Let x represent a class in Ext (also denoted by x) which is annihilated by $h_0^{2^n}$ and let y be a cochain whose coboundary is $x[\xi_1|\xi_1|\cdots|\xi_1]$ with 2^n factors ξ_1. Then $y[\xi_1^{2^{n+1}}]+xc_n$ is a cochain representing $\Pi_n(x)$.

Hence it is evident that the action of Π_n in Ext corresponds to multiplication by ω_n in $A(n)_*$-cohomology. Hence 3.4.14 gives a result about the behavior of Π_n in $\mathrm{Ext}_{A_*}(\mathbf{Z}/(p),M)$ with M as in 3.4.7, so 3.4.6 follows from the isomorphism 3.4.8.

Having proved 3.4.6 we will list the periodic elements in Ext which survive to E_∞ and correspond to nontrivial homotopy elements. First we have

3.4.15. LEMMA. *For $p=2$ and $n\geq 2$, $\Pi_n(h_0^{2^n-1}h_{n+1})=h_0^{2^{n+1}-1}h_{n+2}$. For $p>2$ and $n\geq 1$, $\Pi_n(a_0^{p^n-1}h_n)=a_0^{p^{n+1}-1}h_{n+1}$ up to a nonzero scalar.* [*It is not true that $\Pi_0(h_0)=a_0^{p-1}h_1$.*]

PROOF. We do not know how to make this computation directly. However, 3.4.6 says the indicated operators act isomorphically on the indicated elements, and 3.4.21 below shows that the indicated image elements are nontrivial. Since the groups in question all have rank one the result follows. (3.4.6 does not apply to Π_0 acting on h_0 for $p>2$.) □

3.4.16. THEOREM.

(a) *For $p>2$ the set of elements in the Adams E_∞-term on which all iterates of some periodicity operator Π_n are nontrivial is spanned by $\Pi_n^i(a_0^{p^n-j}h_n)$ with $n\geq 0$, $0<j\leq n+1$ and $i\not\equiv -1 \mod (p)$.* (*For $i\equiv -1$ these elements vanish for $n=0$ and are determined by* 3.4.15 *for $n>0$.*) *The corresponding subgroup of $\pi_*(S^0)$ is the image of the J-homomorphism* (1.1.12). (*Compare* 1.5.19.)

(b) *For $p=2$ the set is generated by all iterates of Π_2 on h_1, h_1^2, $h_1^3=h_0^2h_2$, h_0h_2, h_2, c_0, and h_1c_0 (where $c_0=\langle h_1,h_0,h_2^2\rangle\in\mathrm{Ext}^{3,11}$) and by $\Pi_n^ih_nh_0^{2^{n-1}-j}$ with $n\geq 3$, i odd, and $0<j\leq n+1$.* (*For even i these elements are determined by* 3.4.15.) *The corresponding subgroup of $\pi_*(S^0)$ is $\pi_*(J)$* (1.5.22). *In particular,* $\operatorname{im} J$ *corresponds to the subgroup of E_∞ spanned by all of the above except $\Pi_2^ih_1$ for $i>0$ and $\Pi_2^ih_1^2$ for $i\geq 0$.* □

This can be proved in several ways. The cited results in Section 1.5 are very similar and their proofs are sketched there; use is made of K-theory. The first proof of an essentially equivalent theorem is the one of Adams [**?**], which also uses K-theory. For $p=2$ see also Mahowald [**?**] and Davis and Mahowald [**?**]. The computations of Section 5.3 can be adapted to give a BP-theoretic proof.

The following result is included because it shows that most (all if $p>2$) of the elements listed above are not hit by differentials, and the proof makes no use of any extraordinary homology theory. We will sketch the construction for $p=2$. It is a strengthened version of a result of Maunder [**?**]. Recall (3.1.9) the spectrum bo (representing real connective K-theory) with $H_*(bo)=A_*\,\square_{A(1)_*}\,\mathbf{Z}/(2)=P(\xi_1^4,\bar\xi_2^2,\bar\xi_3,\dots)$. For each $i\geq 0$ there is a map to $\Sigma^{4i}H$ (where H is the integral Eilenberg–Mac Lane spectrum) under which ξ_1^{4i} has a nontrivial image. Together these define a map f from bo to $W=\bigvee_{i\geq 0}\Sigma^{4i}H$. We denote its cofiber by $\overline{W}$. There is a map of cofiber sequences

(3.4.17)
$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & \overline{H} \arrow[d] \\ bo \arrow[r, "f"] & W \arrow[r] & \overline{W} \end{tikzcd}$$

in which each row induces an short exact sequence in homology and therefore an long exact sequence of Ext groups. Recall (3.1.26) that the Ext group for bo has a tower in every fourth dimension, as does the Ext group for W. One can show that the former map injectively to the latter. Then it is easy to work out the Adams E_2-term for $\overline{W}$, namely

(3.4.18)
$$\mathrm{Ext}^{s,t}(H_*(\overline{W})) = \begin{cases} \mathrm{Ext}^{s+1,t}(H_*(bo)) & \text{if } t-s \not\equiv 0 \mod (4) \\ \mathbf{Z}/(2) & \text{if } t-s \equiv 0 \text{ and } \mathrm{Ext}^{s,t}(H_*(bo)) = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{Ext}(M)$ is an abbreviation for $\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), M)$. See FIG. 3.4.20. Combining 3.4.17 and 3.4.8 gives us a map

(3.4.19)
$$\mathrm{Ext}^{s,t}(\mathbf{Z}/(2)) \to \mathrm{Ext}^{s-1,t}(H_*(\overline{W})) \quad \text{for } t-s>0$$

Since this map is topologically induced it commutes with Adams differentials. Hence any element in Ext with a nontrivial image in 3.4.19 cannot be the target of a differential.

FIGURE 3.4.20. $\mathrm{Ext}^{s-1,t}\, H_*(\overline{W})$.

One can show that each h_n for $n>0$ is mapped monomorphically in 3.4.19, so each h_n supports a tower going all the way up to the vanishing line as is required in the proof of 3.4.15. Note that the vanishing here coincides with that for Ext given in 3.4.5.

A similar construction at odd primes detects a tower going up to the vanishing line in every dimension $\equiv -1 \mod (2p-2)$.

To summarize

3.4.21. THEOREM.

(a) *For $p=2$ there is a spectrum $\overline{W}$ with Adams E_2-term described in* 3.4.18 *and* 3.4.20. *The resulting map* 3.4.19 *commutes with Adams differentials and is nontrivial on h_n for all $n>0$ and all Π_2 iterates of h_1, h_1^2, $h_1^3 = h_0^2h_2$, h_2, and $h_0^3h_3$. Hence none of these elements is hit by Adams differentials.*

(b) *A similar construction for $p>2$ gives a map as above which is nontrivial on h_n for all $n \geq 0$ and on all the elements listed in* 3.4.16(a). □

The argument above does not show that the elements in question are permanent cycles. For example, all but a few elements at the top of the towers built on h_n for large n support nontrivial differentials, but map to permanent cycles in the Adams spectral sequence for $\overline{W}$.

We do not know the image of the map in 3.4.19. For $p=2$ it is clearly onto for $t-s = 2^n-1$. For $t-s+1 = (2k+1)2^n$ with $k>0$ the image is at least as big as it is for $k=0$, because the appropriate periodicity operator acts on h_n. However, the actual image appears to be about $\frac{3}{2}$ as large. For example, the towers in Ext in dimensions 23 and 39 have 6 elements instead of the 4 in dimension 7, while the one in dimension 47 has 12. We leave this as a research question for the interested reader.

5. Survey and Further Reading

In this section we survey some other research having to do with the classical Adams spectral sequence, published and unpublished. We will describe in sequence results related to the previous four sections and then indicate some theorems not readily classified by this scheme.

In Section 1 we made some easy Ext calculations and thereby computed the homotopy groups of such spectra as MU and bo. The latter involved the cohomology of $A(1)$, the subalgebra of the mod (2) Steenrod algebra generated by Sq^1 and Sq^2. A pleasant partial classification of $A(1)$-modules is given in section 3 of Adams and Priddy [**?**]. They compute the Ext groups of all of these modules and show that many of them can be realized as bo-module spectra. For example, they use this result to analyze the homotopy type of $bo \wedge bo$.

The cohomology of the subalgebra $A(2)$ was computed by Shimada and Iwai [**?**]. Recently, Davis and Mahowald [**?**] have shown that $A//A(2)$ is *not* the cohomology of any connective spectrum. In Davis and Mahowald [**?**] they compute $A(2)$–Ext groups for the cohomology of stunted real projective spaces.

More general results on subalgebras of A can be found in Adams and Margolis [**?**] and Moore and Peterson [**?**].

The use of the Adams spectral sequence in computing cobordism rings is becoming more popular. The spectra MO, MSO, MSU, and $MSpin$ were originally analyzed by other methods (see Stong [**?**] for references) but in theory could be analyzed with the Adams spectral sequence; see Pengelley [**?, ?**] and Giambalvo and Pengelley [**?**].

The spectrum $MO\langle 8\rangle$ (the Thom spectrum associated with the 7-connected cover of BO) has been investigated by Adams spectral sequence methods in Giambalvo [**?**], Bahri [**?**], Davis [**?, ?**], and Bahri and Mahowald [**?**].

In Johnson and Wilson [**?**] the Adams spectral sequence is used to compute the bordism ring of manifolds with free G-action for an elementary abelian p-group G.

The most prodigious Adams spectral sequence calculation to date is that for the symplectic cobordism ring by Kochman [**?, ?, ?**]. He uses a change-of-rings isomorphism to reduce the computation of the E_2-term to finding Ext over the coalgebra

$$B = P(\xi_1, \xi_2, \dots)/(\xi_i^4) \tag{3.5.1}$$

for which he uses the May spectral sequence. The E_2-term for MSp is a direct sum of many copies of this Ext and these summands are connected to each other by higher Adams differentials. He shows that MSp is indecomposable as a ring spectrum and that the Adams spectral sequence has nontrivial d_r's for arbitrarily large r.

In Section 2 we described the May spectral sequence. The work of Nakamura [**?**] enables one to use algebraic Steenrod operations (A1.5) to compute May differentials.

The May spectral sequence is obtained from an increasing filtration of the dual Steenrod algebra A_*. We will describe some decreasing filtrations of A_* for $p = 2$ and the spectral sequences they lead to. The method of calculation these results suggest is conceptually more complicated than May's but it may have some practical advantages. The E_2-term (3.5.2) can be computed by another spectral sequence (3.5.4) whose E_2-term is the $A(n)$ cohomology (for some fixed n) of a certain trigraded comodule T. The structure of T is given by a third spectral sequence (3.5.10) whose input is essentially the cohomology of the Steenrod algebra through a range of dimensions equal to 2^{-n-1} times the range one wishes to compute.

This method is in practice very similar to Mahowald's unpublished work on "Koszul resolutions".

3.5.2. PROPOSITION. *For each $n \geq 0$, A_* has a decreasing filtration* (A1.3.5) $\{F^s A_*\}$ *where F^s is the smallest possible subgroup satisfying $\bar{\xi}_i^{2^j} \in F^{2^{i+j-n-1}-1}$ for $j \leq n+1$.*

□

In particular, $F^0/F^1 = A(n)_*$, so $A(n)_* \subset E_0 A_*$ where

$$A(n)_* = A_*/(\bar{\xi}_1^{2^{n+1}}, \bar{\xi}_2^{2^n}, \ldots, \bar{\xi}_n^2, \bar{\xi}_{n+1}\bar{\xi}_{n+2}, \ldots).$$

We also have $\bar{\xi}_i^{2^j} \in F^{2^{j-n-1}(2^i-1)}$ for $j \geq n+1$. Hence there is a spectral sequence (A1.3.9) converging to $\operatorname{Ext}_{A_*}(\mathbf{Z}/c(2), M)$ with $E_1^{s,t,u} = \operatorname{Ext}_{E_0 A_*}^{s,t}(\mathbf{Z}/(2), E_0 M)$ and $d_r \colon E_r^{s,t,u} \to E_r^{s+1,t,u+r}$, where the third grading is that given by the filtration, M is any A_*-comodule, and $E_0 M$ is the associated $E_0 A_*$-comodule (A1.3.7).

Now let $G(n)_* = E_0 A_* \,\square_{A(n)_*}\, \mathbf{Z}/(2)$. It inherits a Hopf algebra structure from $E_0 A_*$ and

$$A(n)_* \to E_0 A_* \to G(n)_* \tag{3.5.3}$$

is an extension of Hopf algebras (A1.1.15). Hence we have a Cartan–Eilenberg spectral sequence (A1.3.14), i.e.,

3.5.4. LEMMA. *Associated with the extension* 3.5.3 *there is a spectral sequence with*

$$E_2^{s_1,s_2,t,u} = \operatorname{Ext}_{A(n)_*}^{s_1}(\mathbf{Z}/(2), \operatorname{Ext}_{G(n)_*}^{s_2,t,u}(\mathbf{Z}/(2), M))$$

with $d_r \colon E_r^{s_1,s_2,t,u} \to E_r^{s_1+r,s_2-r+1,t,u}$ converging to $\operatorname{Ext}_{E_0 A_*}^{s_1+s_2,t,u}$ for any $E_0 A_*$ comodule M. [$\operatorname{Ext}_{G(n)_*}(\mathbf{Z}/(2), M)$ *is the T referred to above.*] □

3.5.5. REMARK. According to A1.3.11(a) the cochain complex W used to compute Ext over $G(n)_*$ can be taken to be one of $A(n)_*$-comodules. The E_2-term of the spectral sequence is the $A(n)_*$ Ext of the cohomology of W, and the E_∞-term is the cohomology of the double complex obtained by applying $C^*_{A(n)_*}(\ )$ (A1.2.11) to W. This W is the direct sum [as a complex of $A(n)_*$-comodules] of its components for various u (the filtration grading). The differentials are computed by analyzing this W.

Next observe that $E_0 A_*$ and $G(n)_*$ contain a sub-Hopf algebra $A_*^{(n+1)}$ isomorphic up to regrading to A_*; i.e., $A_*^{(n+1)} \subset E_0 A_*$ is the image of $P(\bar{\xi}_i^{2^{n+1}}) \subset A_*$. The isomorphism follows from the fact that the filtration degree $2^i - 1$ of $\bar{\xi}_i^{2^{n+1}}$ coincides with the topological degree of $\bar{\xi}_i$. Hence we have

$$\operatorname{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = \operatorname{Ext}_{A_*^{(n+1)}}^{s,2^{n+1}t,t}(\mathbf{Z}/(2), \mathbf{Z}/(2)) \tag{3.5.6}$$

and we can take these groups as known inductively.

Let $L(n)_* = G(n)_* \otimes_{A_*^{(n+1)}} \mathbf{Z}/(2)$ and get an extension

$$A_*^{(n+1)} \to G(n)_* \to L(n)_*. \tag{3.5.7}$$

$L(n)_*$ is easily seen to be cocommutative with

$$\operatorname{Ext}_{L(n)_*}^{s,t,u}(\mathbf{Z}/(2), \mathbf{Z}/(2)) = P(h_{i,j} \colon 0 \le j \le n,\ i \ge n+2-j), \tag{3.5.8}$$

where $h_{i,j} \in \operatorname{Ext}^{1,2^j(2^i-1),2^{i+j-n-1}-1}$ corresponds as usual to $\bar{\xi}_i^{2^j}$. This Ext is a comodule algebra over $A_*^{(n+1)}$ (A1.3.14) with coaction given by

$$\psi(h_{i,j}) = \sum_{k>0} \xi_k^{2^{i+j-k}} \otimes h_{i-k,j} \tag{3.5.9}$$

Hence by A1.3.14 we have

3.5.10. Lemma. *The extension* 3.5.7 *leads to a spectral sequence as in* 3.5.4 *with*

$$E_2^{s_1,s_2,t,u} = \operatorname{Ext}_{A_*^{(n+1)}}^{s_1}(\mathbf{Z}/(2), \operatorname{Ext}_{L(n)_*}^{s_2,t,u}(\mathbf{Z}/(2), M))$$

converging to $\operatorname{Ext}_{G(n)_*}^{s_1+s_2,t,u}(\mathbf{Z}/(2), M)$ *for any* $G(n)_*$*-comodule* M*. For* $M = \mathbf{Z}/(2)$*, the* Ext *over* $L(n)_*$ *and its comodule algebra structure are given by* 3.5.8 *and* 3.5.9*. Moreover, this spectral sequence collapses from* E_2*.*

Proof. All is clear but the last statement, which we prove by showing that $G(n)_*$ possesses an extra grading which corresponds to s_2 in the spectral sequence. It will follow that differentials must respect this grading so $d_r = 0$ for $r \ge 2$. Let $\bar{\xi}_{i,j} \in G(n)_*$ be the element corresponding to $\bar{\xi}_i^{2^j}$. The extra grading is defined by

$$|\bar{\xi}_{i,j}| = \begin{cases} 1 & \text{if } j \le n \\ 0 & \text{if } j > n. \end{cases}$$

Since the $\bar{\xi}_{i,j}$ for $j \le n$ are all exterior generators, the multiplication in $G(n)_*$ respects this grading. The coproduct is given by

$$\Delta(\bar{\xi}_{i,j}) = \sum_k \bar{\xi}_{k,j} \otimes \bar{\xi}_{i=k,k+j}.$$

If $j \ge n+1$ then all terms have degree 0, and if $j \le n$, we have $k+j \ge n+2$ so all terms have degree 1, so Δ also respects the extra grading. $\square$

We now describe how to use these results to compute Ext. If one wants to compute through a fixed range of dimensions, the isomorphism 3.5.6 reduces the calculation of the spectral sequence of 3.5.10 to a much smaller range, so we assume inductively that this has been done. The next step is to compute in the spectral sequence of 3.5.4. The input here is the trigraded $A(n)_*$-comodule $\operatorname{Ext}_{G(n)_*}^{s,t,u}(\mathbf{Z}/(2), \mathbf{Z}/(2))$. We began this discussion by assuming we could compute Ext over $A(n)_*$, but in practice we cannot do this directly if $n > 1$. However, for $0 \le m < n$ we can reduce an $A(n)_*$ calculation to an $A(m)_*$ calculation by proceeding as above, starting with the mth filtration of $A(n)_*$ instead of A_*. We leave the precise formulation to the reader. Thus we can compute the $A(n)_*$ Ext of $\operatorname{Ext}_{G(n)_*}^{s,t,u}(\mathbf{Z}/(2), \mathbf{Z}/(2))$ separately for each u; the slogan here is divide and conquer.

This method can be used to compute the cohomology of the Hopf algebra B (3.5.1) relevant to MSp. Filtering with $n = 1$, the SS analogous to 3.5.4 has

$$E_2 = \mathrm{Ext}_{A(1)_*}(\mathbf{Z}/(2), P(h_{21}, h_{30}, h_{31}, h_{40}, \dots))$$

with $\psi(h_{i+1,0}) = \xi_1 \otimes h_{i,1} + 1 \otimes h_{i+1,0}$ and $\psi(h_{i,1}) = 1 \otimes h_{i,1}$ for $i \geq 2$. This Ext is easy to compute. Both this spectral sequence and the analog of the one in 3.5.2 collapse from E_2. Hence we get a description of the cohomology of B which is more concise though less explicit than that of Kochman [**?**].

In Section 3 we described Λ and hinted at an unstable Adams spectral sequence. For more on this theory see Bousfield and Kan [**?**], Bousfield and Curtis [**?**], Bendersky, Curtis, and Miller [**?**], Curtis [**?**], and Singer [**?, ?, ?**]. A particularly interesting point of view is developed by Singer [**?**].

In Mahowald [**?**] the double suspension homomorphism

$$\Lambda(2n-1) \to \Lambda(2n+1)$$

is studied. He shows that the cohomology of its cokernel $W(n)$ is isomorphic to $\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), \Sigma^{2n-1}A(0)_*)$ for $t - s < 5s + k$ for some constant k, i.e., above a line with slope $\frac{1}{5}$. This leads to a similar isomorphism between $H^*(\Lambda(2n+1)/\Lambda(1))$ and $\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), \widetilde{H}_*(RP^{2n}))$. In Mahowald [**?**] he proves a geometric analog, showing that a certain subquotient of $\pi_*(S^{2n+1})$ is isomorphic to that of $\pi_*^S(RP^{2n})$. The odd primary analog of the algebraic result has been demonstrated by Harper and Miller [**?**]. The geometric result is very likely to be true but is still an open question. This point was also discussed in Section 1.5.

Now we will describe some unpublished work of Mahowald concerning generalizations of Λ. In 3.3.3 we defined subcomplexes $\Lambda(n) \subset \Lambda$ by saying that an admissible monomial $\lambda_{i_1} \cdots \lambda_{i_k}$ is in $\Lambda(n)$ if $i_1 < n$. The short exact sequence

$$\Lambda(n-1) \to \Lambda(n) \to \Sigma^n \Lambda(2n-1)$$

led to the algebraic EHP spectral sequence of 3.3.7. Now we define quotient complexes $\Lambda(n)$ by $\Lambda\langle n\rangle = \Lambda/\Lambda(\lambda_0, \dots, \lambda_{n-1})$, so $\Lambda(0) = \Lambda$ and $\varinjlim M\Lambda\langle n\rangle = \mathbf{Z}/(2)$. Then there are short exact sequences

$$0 \to \Sigma^n \Lambda\langle (n+1)/2\rangle \to \Lambda\langle n\rangle \to \Lambda\langle n+1\rangle \to 0 \tag{3.5.11}$$

where the fraction $(n+1)/2$ is taken to be the integer part thereof. This leads to a spectral sequence similar to that of 3.3.7 and an inductive procedure for computing $H_*(\Lambda)$.

Next we define A_*-comodules B_n as follows. Define an increasing filtration on A_* (different from those of 3.5.2) by $\xi_i \in F_{2^i}$ and let $B_n = F_n$. The B_n is realized by the spectra of Brown and Gitler [**?**]. They figure critically in the construction of the η_j's in Mahowald [**?**] and in the Brown–Peterson–Cohen program to prove that every closed smooth n-manifold immerses in $\mathbf{R}^{2n-\alpha(n)}$, where $\alpha(n)$ is the number of ones in the dyadic expansion of n. Brown and Gitler [**?**] show that $\mathrm{Ext}_{A_*}(\mathbf{Z}/(2), B_n) = H^*(\Lambda\langle n\rangle)$ and that the short exact sequence 3.5.11 is realized by a cofibration. It is remarkable that the Brown–Gitler spectra and the unstable spheres both lead in this way to Λ.

Now let $N = (n_1, n_2, \dots)$ be a nonincreasing sequence of nonnegative integers. Let $A(N) = A_*/(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots)$. This is a Hopf algebra. Let $M(N) = A_* \,\square_{A(N)}\, \mathbf{Z}/(2)$, so $M(N) = P(\xi_1^{2^{n_1}}, \xi_2^{2^{n_2}}, \dots)$. The filtration of A_* defined above induces one on $M(N)$ and we have

$$F_iM(N)/F_{i-1}M(N) = \begin{cases} \Sigma^i F_{[i/2]}M(N^1) & \text{if } 2^{n_1} \mid i \\ 0 & \text{otherwise} \end{cases} \tag{3.5.12}$$

where N^k is the sequence $(n_{k+1}, n_{k+2}, \dots)$. For $N = (0, 0, \dots)$ $A(N) = A_*$ and this is equivalent to 3.5.11.

3.5.13. PROPOSITION. *The short exact sequence*

$$0 \to F_{i-1}M(N) \to F_iM(N) \to F_i/F_{i-1} \to 0$$

is split over $A(N)$. □

This result can be used to construct an long exact sequence of A_*-comodules

$$0 \to \mathbf{Z}/(2) \to C_N^0 \to C_N^1 \to C_N^2 \to \cdots \tag{3.5.14}$$

such that C_N^k is a direct sum of suspensions of $M(N^k)$ indexed by sequences $(i_1, i_2, \dots, i_k)$ satisfying $1 + i_j \equiv 0 \mod 2^{n_{1+k-j}}$ and $i_j \le 2i_{j-1}$. Equation 3.5.14 leads to a spectral sequence (A1.3.2) converging to Ext with

$$E_1^{k,s} = \operatorname{Ext}^s_{A_*}(\mathbf{Z}/(2), C_N^k). \tag{3.5.15}$$

The splitting of C_N^k and the change-of-rings isomorphism A1.3.13 show that $E_1^{k,*}$ is a direct sum of suspensions $\operatorname{Ext}_{A(N^k)}(\mathbf{Z}/(2), \mathbf{Z}/(2))$.

The E_1-term of this spectral sequence is a "generalized Λ" in that it consists of copies of $A(N^k)$ Ext groups indexed by certain monomials in Λ. The d_1 is closely related to the differential in Λ.

We will describe the construction of 3.5.14 in more detail and then discuss some examples. Let $\overline{M}(N)$ be the quotient in

$$0 \to \mathbf{Z}/(2) \to M(N) \to \overline{M}(N) \to 0.$$

In 3.5.14 we want $C_N^0 = M(N)$ and $C_N^1 = \bigoplus_{i>0} \Sigma^{i2^{n_1}} M(N^1)$, so we need to embed $\overline{M}(N)$ in this putative C_N^1. The filtration on $M(N)$ induces ones on $M(N)$ and C_N^s; in the latter F_i should be a direct sum of suspensions of $M(N^1)$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F_{i-1}\overline{M}(N) \arrow[r] \arrow[d] & F_i\overline{M}(N) \arrow[r] \arrow[d] \arrow[dl, dashed] & F_i/F_{i-1}\overline{M}(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_{i-1}C_N^1 \arrow[r] \arrow[d, equal] & X_i \arrow[r] \arrow[d] & F_i/F_{i-1}\overline{M}(N) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{i-1}C_N^1 \arrow[r] & F_iC_N^i \arrow[r] & \Sigma^? N(N^1) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The upper short exact sequence splits over $A(N)$ (3.5.13) and hence over $A(N^1)$. Since $F_{i-1}C_N^1$ splits as above, the change-of-rings isomorphism A1.3.13 implies that the map

$$\operatorname{Hom}_{A_*}(F_i\overline{M}(N), F_{i-1}C_N^1) \to \operatorname{Hom}_{A_*}(F_{i-1}\overline{M}(N), F_{i-1}C_N^1)$$

is onto, so the diagonal map exists. It can be used to split the middle short exact sequence, so the lower short exact sequence can be taken to be split and C_N^1 is as claimed.

The rest of 3.5.14 can be similarly constructed.

Now we consider some examples. If $N = (0, 0, \cdots)$ the spectral sequence collapses and we have the Λ-algebra. If $N = (1, 1, \dots)$ we have $\mathrm{Ext}_{A(N)} = P(a_0, a_1, \dots)$ as computed in 3.1.9, and the E_1-term is this ring tensored with the subalgebra of Λ generated by λ_i with i odd, which is isomorphic up to regrading with Λ itself. This is also the E_1-term of a spectral sequence converging to the Adams–Novikov E_2-term to be discussed in Section 4.4. The SS of 3.5.15 in this case can be identified with the one obtained by filtering Λ by the number of λ_i, with i odd occurring in each monomial.

For $N = (2, 2, \cdots)$ we have $A(N) = B$ as in 3.5.1, so the E_1-term is Ext_B tensored with a regraded Λ.

Finally, consider the case $N = (2, 1, 0, 0, \dots)$. We have $E_1^{0,s} = \mathrm{Ext}^s_{A(1)_*}$ and $E_1^{1,s} = \bigoplus_{i>0} \Sigma^{4i} \mathrm{Ext}^s_{A(0)_*}$. One can study the quotient spectral sequence obtained by setting $E_1^{k,s} = 0$ for $k > 1$. The resulting $E_2 = E_\infty$ is the target of a map from Ext, and this map is essentially the one given in 3.4.19. More generally, the first few columns of the spectral sequence of 3.5.15 can be used to detect elements in Ext.

In Section 4 we gave some results concerning vanishing and periodicity. In particular we got a vanishing line of slope $\frac{1}{2}$ (for $p = 2$) for any connective comodule free over $A(0)_*$. This result can be improved if the comodule is free over $A(n)_*$ for some $n > 0$; e.g., one gets a vanishing line of slope $\frac{1}{5}$ for $n = 1$, $p = 2$. See Anderson and Davis [**?**] and Miller and Wilkerson [**?**].

The periodicity in Section 4 is based on multiplication by powers of h_{20} $(p = 2)$ or a_1 $(p > 2)$ and these operators act on classes annihilated by some power of h_{10} or a_0. As remarked above, this corresponds to v_1-periodicity in the Adams–Novikov spectral sequence (see Section 1.4). Therefore one would expect to find other operators based on multiplication by powers of $h_{n+1,0}$ or a_n corresponding to v_n-periodicity for $n > 1$. A v_n-periodicity operator should be a Massey product defined on elements annihilated by some v_{n-1}-periodicity operator. For $n = 2$, $p = 2$ this phenomenon is investigated by Davis and Mahowald [**?**] and Mahowald [**?, ?, ?**].

More generally one can ask if there is an Adams spectral sequence version of the chromatic SS (1.4.8). For this one would need an analog of the chromatic resolution (1.4.6), which means inverting periodicity operators. This problem is addressed by Miller [**?, ?**].

A v_n-periodicity operator in the Adams spectral sequence for $p = 2$ moves an element along a line of slope $1/(2^{n+1} - 2)$. Thus v_n-periodic families of stable homotopy elements would correspond to families of elements in the Adams spectral sequence lying near the line through the origin with this slope. We expect that elements in the E_∞-term cluster around such lines.

Now we will survey some other research with the Adams spectral sequence not directly related to the previous four sections. For $p = 2$ and $t - s \leq 45$, differentials and extensions are analyzed by Mahowald and Tangora [**?**], Barratt, Mahowald, and Tangora [**?**], Tangora [**?**], and Bruner [**?**]. Some systematic phenomena in the E_2-term are described in Davis [**?**], Mahowald and Tangora [**?**], and Margolis, Priddy, and Tangora [**?**]. Some machinery useful for computing Adams spectral sequence differentials involving Massey products is developed by Kochman [**?**] and Section 12 of Kochman [**?**]. See also Milgram [**?**] and Kahn [**?**] and Bruner *et al* [**?**], and Makinen [**?**].

The Adams spectral sequence was used in the proof of the Segal conjecture for $\mathbf{Z}/(2)$ by Lin [**?**] and Lin *et al.* [**?**]. Computationally, the heart of the proof is the startling isomorphism

$$\mathrm{Ext}_{A_*}^{s,t}(\mathbf{Z}/(2), M) = \mathrm{Ext}_{A_*}^{s,t+1}(\mathbf{Z}/(2), \mathbf{Z}/(2)),$$

where M is dual to the A-module $\mathbf{Z}/(2)[x, x^{-1}]$ with $\dim x = 1$ and $Sq^k x^i = \binom{i}{k} x^{i+k}$ (this binomial coefficient makes sense for any integer i). This isomorphism was originally conjectured by Mahowald (see Adams [**?**]). The analogous odd primary result was proved by Gunawardena [**?**]. The calculation is streamlined and generalized to elementary abelian p-groups by Adams, Gunawardena, and Miller [**?**]. This work makes essential use of ideas due to Singer [**?**] and Li and Singer [**?**].

In Ravenel [**?**] we proved the Segal conjecture for cyclic groups by means of a modified form of the Adams spectral sequence in which the filtration is altered. This method was used by Miller and Wilkerson [**?**] to prove the Segal conjecture for periodic groups.

The general Segal conjecture, which is a statement about the stable homotopy type of the classifying space of a finite group, has been proved by Gunnar Carlsson [**?**]. A related result is the Sullivan conjecture, which concerns says among other things that there are no nontrivial maps to a finite complex from such a classifying space. It was proved by Haynes Miller in [**?**] . New insight into both proofs was provided by work of Jean Lannes on unstable modules over the Steenrod algebra, in particular his T-functor, which is an adjoint to a certain tensor product. See Lannes [**?**], Lannes [**?**] and Lannes and Schwartz [**?**]. An account of this theory is given in the book by Lionel Schwartz [**?**].

Recent work of Palmieir (Palmieri [**?**] and Palmieri [**?**]) gives a global description of Ext over the Steenrod algebra modulo nilpotent elements.

Finally, we must mention the Whitehead conjecture. The n-fold symmetric product $Sp^n(X)$ of a space X is the quotient of the n-fold Cartesian product by the action of the symmetric group Σ_n. Dold and Thom [**?**] showed that $Sp^\infty(X) = \varinjlim Sp^n(X)$ is a product of Eilenberg–Mac Lane spaces whosw homotopy is the homotopy of X. Symmetric products can be defined on spectra and we have $Sp^\infty(S^0) = HJ$, the integer Eilenbergh–Mac Lane spectrum. After localizing at the prime p one considers

$$S^0 \to Sp^p(S^0) \to Sp^{p^2}(S^0) \to \cdots$$

and

$$H \leftarrow S^0 \leftarrow \Sigma^{-1} Sp^p(S^0)/S^0 \leftarrow \Sigma^{-2} Sp^{p^2}(S^0)/Sp^p(S^0) \leftarrow \cdots . \tag{3.5.16}$$

Whitehead conjectured that this diagram induces an long exact sequence of homotopy groups. In particular, the map $\Sigma^{-1} Sp^p(S^0)/S^0 \to S^0$ shouls induce a surjection in homotopy in positive dimensions; this is the famous theorem of Kahn and Priddy [**?**]. The analogous statement about Ext groups was proved by Lin [**?**]. Miller [**?**] generalized this to show that 3.5.16 induces an long exact sequence of Ext groups. The long exact sequence of homotopy groups was established by Kuhn [**?**]. The spectra in 3.5.16 were studied by Welcher [**?, ?**]. He showed that $H_*(Sp^{p^{n+1}}(S^0)/Sp^{p^n}(S^0))$ is free over $A(n)_*$, so its Ext groups has a vanishing line given by Anderson and Davis [**?**] and Miller and Wilkerson [**?**] and the long exact sequence of 3.5.16 is finite in each bigrading.