CHAPTER 2

Setting up the Adams Spectral Sequence

In this chapter we introduce the spectral sequence that will be our main object of study. We do not intend to give a definitive account of the underlying theory, but merely to make the rest of the book intelligible. Nearly all of this material is due to Adams. The classical Adams spectral sequence [i.e., the one based on ordinary mod (p) cohomology] was first introduced in Adams [**3**] and a most enjoyable exposition of it can be found in Adams [**7**]. In Section 1 we give a fairly self-contained account of it, referring to Adams [**4**] only for standard facts about Moore spectra and inverse limits. We include a detailed discussion of how one extracts differentials from an exact couple and a proof of convergence.

In Section 2 we describe the Adams spectral sequence based on a generalized homology theory E_* satisfying certain assumptions (2.2.5). We rely heavily on Adams [**4**], referring to it for the more difficult proofs. The E_*-Adams resolutions (2.2.1) and spectral sequences (2.2.4) are defined, the E_2-term is identified, and the convergence question is settled (2.2.3). We do not give the spectral sequence in its full generality; we are only concerned with computing $\pi_*(Y)$, not $[X, Y]$ for spectra X and Y. Most of the relevant algebraic theory, i.e., the study of Hopf algebroids, is developed in Appendix 1.

In Section 3 we study the pairing of Adams spectral sequences induced by a map $\alpha\colon X' \wedge X'' \to X$ and the connecting homomorphism associated with a cofibration realizing a short exact sequence in E-homology. Our smash product result implies that for a ring spectrum the Adams spectral sequence is one of differential algebras. To our knowledge these are the first published proofs of these results in such generality.

Throughout this chapter and the rest of the book we assume a working knowledge of spectra and the stable homotopy category as described, for example, in the first few sections of Adams [**4**].

1. The Classical Adams Spectral Sequence

In this section we will set up the Adams spectral sequence based on ordinary mod (p) cohomology for the homotopy groups of a spectrum X. Unless otherwise stated all homology and cohomology groups will have coefficients in $\mathbf{Z}/(p)$ for a prime number p, and X will be a connective spectrum such that $H^*(X)$ (but not necessarily X itself) has finite type.

Recall that $H^*(X)$ is a module over the mod (p) Steenrod algebra A, to be described explicitly in the next chapter. Our object is to prove

2.1.1. Theorem (Adams [**3**]). *Let X be a spectrum as above. There is a spectral sequence*

$$E_*^{**}(X) \quad \text{with} \quad d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$$

such that

(a) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(X), \mathbf{Z}/(p))$.

(b) *if X is of finite type, E_∞^{**} is the bigraded group associated with a certain filtration of $\pi_*(X) \otimes \mathbf{Z}_p$, where $\mathbf{Z}_p$ denotes the ring of p-adic integers.* □

Let $E = H\mathbf{Z}/(p)$, the mod (p) Eilenberg–Mac Lane spectrum. We recall some of its elementary properties.

2.1.2. Proposition.

(a) $H_*(X) = \pi_*(E \wedge X)$.

(b) $H^*(X) = [X, E]$.

(c) $H^*(E) = A$.

(d) *If K is a locally finite wedge of suspensions of E, i.e., a generalized* mod (p) *Eilenberg–Mac Lane spectrum, then $\pi_*(K)$ is a graded $\mathbf{Z}/(p)$-vector space with one generator for each wedge summand of K. More precisely, $\pi_*(K) = \mathrm{Hom}_A(H^*(K), \mathbf{Z}/(p))$.*

(e) *A map from X to K is equivalent to a locally finite collection of elements in $H^*(X)$ in the appropriate dimensions. Conversely, any locally finite collection of elements in $H^*(X)$ determines a map to such a K.*

(f) *If a locally finite collection of elements in $H^*(X)$ generate it as an A-module, then the corresponding map $f\colon X \to K$ induces a surjection in cohomology.*

(g) *$E \wedge X$ is a wedge of suspensions of E with one wedge summand for each $\mathbf{Z}/(p)$ generator of $H^*(X)$. $H^*(E \wedge X) = A \otimes H^*(X)$ and the map $f\colon X \to E \wedge X$ (obtained by smashing X with the map $S^0 \to E$) induces the A-module structure map $A \otimes H^*(X) \to H^*(X)$ in cohomology. In particular $H^*(F)$ is a surjection.* □

The idea behind the Adams spectral sequence is to use maps such as those of (f) or (g) and our knowledge of $\pi_*(K)$ or $\pi_*(E \wedge X)$ to get information about $\pi_*(X)$. We enlist the aid of homological algebra to make the necessary calculations.

More specifically, we have

2.1.3. Definition. *A* mod (p) *Adams resolution (X_s, g_s) for X is a diagram*

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"] & X_3 \arrow[l, "g_2"'] \\ K_0 & K_1 & K_2 & \end{tikzcd}$$

where each K_s, is a wedge of suspensions of E, $H^*(f_s)$ is onto and X_{s+1} is the fiber of f_s. □

Proposition 2.1.2(f) and (g) enable us to construct such resolutions for any X, e.g., by setting $K_s = E \wedge X_s$. Since $H^*(f_s)$ is onto we have short exact sequences

$$0 \leftarrow H^*(X_s) \leftarrow H^*(K_s) \leftarrow H^*(\Sigma X_{s+1}) \leftarrow 0.$$

We can splice these together to obtain a long exact sequence

$$0 \leftarrow H^*(X) \leftarrow H^*(K_0) \leftarrow H^*(\Sigma K_1) \leftarrow H^*(\Sigma^2 K_2) \leftarrow \cdots . \tag{2.1.4}$$

Since the maps are A-module homomorphisms and each $H^*(K_s)$ is free over A_p, 2.1.4 is a free A-resolution of $H^*(X)$.

Unfortunately, the relation of $\pi_*(K_s)$ to $\pi_*(X)$ is not as simple as that between the corresponding cohomology groups. Life would be very simple if we knew $\pi_*(f_s)$

was onto, but in general it is not. We have instead long exact sequences

$$\begin{tikzcd} \pi_*(X_{s+1}) \arrow[r, "\pi_*(g_s)"] & \pi_*(X_s) \arrow[r, "\pi_*(f_s)"] & \pi_*(K_s) \arrow[ll, bend left, "\partial_{s,*}"] \end{tikzcd} \tag{2.1.5}$$

arising from the fibrations

$$X_{s+1} \xrightarrow{g_s} X_s \xrightarrow{f_s} K_s.$$

If we regard $\pi_*(X_s)$ and $\pi_*(K_s)$ for all s as bigraded abelian groups D_1 and E_1, respectively [i.e., $D_1^{s,t} = \pi_{t-s}(X_s)$ and $E_1^{s,t} = \pi_{t-s}(K_s)$] then 2.1.5 becomes

$$\begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd} \tag{2.1.6}$$

where

$$i_1 = \pi_{t-s}(g_s) \colon D_1^{s+1,t+1} \to D_1^{s,t},$$
$$j_1 = \pi_{t-s}(f_s) \colon D_1^{s,t} \to E_1^{s,t},$$

and

$$k_1 = \partial_{s,t-s} \colon E_1^{s,t} \to D_1^{s+1,t}.$$

The exactness of 2.1.5 translates to $\ker i_1 = \operatorname{im} k_1$, $\ker j_1 = \operatorname{im} i_1$, and $\ker k_1 = \operatorname{im} j_1$. A diagram such as 2.1.6 is known as an *exact couple*. It is standard homological algebra that an exact couple leads one to a spectral sequence; accounts of this theory can be found in Cartan and Eilenberg [**1**, Section XV.7], Mac Lane [**1**, Section XI.5], and Hilton and Stammbach [**1**, Chapter 8] as well as Massey [**2**].

Briefly, $d_1 = j_1 k_1 \colon E_1^{s,t} \to E_1^{s+1,t}$ has $(d_1)^2 = j_1 k_1 j_1 k_1 = 0$ so (E_1, d_1) is a complex and we define $E_2 = H(E_1, d_1)$. We get another exact couple, called the derived couple,

$$\begin{tikzcd} D_2 \arrow[rr, "i_2"] & & D_2 \arrow[dl, "j_2"] \\ & E_2 \arrow[ul, "k_2"] & \end{tikzcd} \tag{2.1.7}$$

where $D_2^{s,t} = i_1 D_1^{s,t}$, i_2 is induced by i_1, $j_2(i_1 d) = j_1 d$ for $d \in D_1$, and $k_2(e) = k_1(e)$ for $e \in \ker d, \subset E_1$. Since 2.1.7 is also an exact couple (this is provable by a diagram chase), we can take its derived couple, and iterating the procedure gives a sequence of exact couples

$$\begin{tikzcd} D_r \arrow[rr, "i_r"] & & D_r \arrow[dl, "j_r"] \\ & E_r \arrow[ul, "k_r"] & \end{tikzcd}$$

where $D_{r+1} = i_r D_r$, $d_r = j_r k_r$, and $E_{r+1} = H(E_r, d_r)$. The sequences of complexes $\{(E_r, d_r)\}$ constitutes a spectral sequence. A close examination of the indices will reveal that $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. It follows that for $s < r$, the image of d_r in $E_r^{s,t}$ is trivial so $E_{r+1}^{s,t}$ is a subgroup of $E_r^{s,t}$, hence we can define

$$E_\infty^{s,t} = \bigcap_{r>s} E_r^{s,t}.$$

This group will be identified (2.1.12) in certain cases with a subquotient of $\pi_{t-s}(X)$, namely, $\operatorname{im}\pi_{t-s}(X_s)/\operatorname{im}\pi_{t-s}(X_{s+1})$. The subgroups $\operatorname{im}\pi_*(X_s) = F^s\pi_*(X)$ form a *decreasing filtration* of $\pi_*(X)$ and E_∞ is the *associated bigraded group.*

2.1.8. DEFINITION. *The* mod (p) *Adams spectral sequence for X is the spectral sequence associated to the exact couple 2.1.6.* $\square$

We will verify that $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ by chasing diagram 2.1.9, where we write $\pi_*(X_*)$ and $\pi_*(K_*)$ instead of D_1 and E_1, with $u = t - s$.
(2.1.9)

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & & \\
{} \arrow[r] & \pi_u(X_{s+2}) \arrow[r, "\pi_u(f_{s+2})"] \arrow[d, "\pi_u(g_{s+1})"'] & \pi_u(K_{s+2}) \arrow[r, "\partial_{s+2,u}"] & \pi_{u-1}(X_{s+3}) \arrow[r, "\pi_{u-1}(f_{s+3})"] \arrow[d, "\pi_{u-1}(g_{s+2})"'] & \pi_{u-1}(K_{s+3}) \arrow[r] & {} \\
{} \arrow[r] & \pi_u(X_{s+1}) \arrow[r, "\pi_u(f_{s+1})"] \arrow[d, "\pi_u(g_s)"'] & \pi_u(K_{s+1}) \arrow[r, "\partial_{s+1,u}"] & \pi_{u-1}(X_{s+2}) \arrow[r, "\pi_{u-1}(f_{s+2})"] \arrow[d, "\pi_{u-1}(g_{s+1})"'] & \pi_{u-1}(K_{s+2}) \arrow[r] & {} \\
{} \arrow[r] & \pi_u(X_s) \arrow[r, "\pi_u(f_s)"] \arrow[d] & \pi_u(K_s) \arrow[r, "\partial_{s,u}"] & \pi_{u-1}(X_{s+1}) \arrow[r, "\pi_{u-1}(f_{s+1})"] \arrow[d] & \pi_{u-1}(K_{s+1}) \arrow[r] & {} \\
 & {} & & {} & &
\end{tikzcd}$$

The long exact sequences 2.1.5 are embedded in this diagram; each consists of a vertical step $\pi_*(g_*)$ followed by horizontal steps $\pi_*(f_*)$ and ∂_{**} and so on. We have $E_1^{s,t} = \pi_u(K_s)$ and $d_1^{s,t} = (\pi_{u-1}(f_{s+1}))(\partial_{s,u})$. We have $E_2^{s,t} = \ker d_1^{s,t}/\operatorname{im} d_1^{s-1,t}$. Suppose an element in $E_2^{s,t}$ is represented by $x \in \pi_u(K_s)$. We will now explain how $d_2[x]$ (where $[x]$ is the class represented by x) is defined. x is a d_1 cycle, i.e., $d_1x = 0$, so exactness in 2.1.4 implies that $\partial_{s,u}x = (\pi_{u+1}(g_{s+1}))(y)$ for some $y \in \pi_{u-1}(X_{s+2})$. Then $(\pi_{u-1}(f_{s+2}))(y)$ is a d_1 cycle which represents $d_2[x] \in E_2^{s+2,t-1}$. If $d_2[x] = 0$ then $[x]$ represents an element in $E_3^{s,t}$ which we also denote by $[x]$. To define $d_3[x]$ it can be shown that y can be chosen so that $y = (\pi_{u-1}(g_{s+2}))(y')$ for some $y' \in \pi_{u-1}(X_{s+3})$ and that $(\pi_{u-1}(f_{s+3}))(y')$ is a d_1 cycle representing a d_2 cycle which represents an element in $E^{s+3,t+2}$ which we define to be $d_3[x]$. These assertions may be verified by drawing another diagram which is related to the derived couple 2.1.7 in the same way that 2.1.9 is related to the original exact couple 2.1.6. The higher differentials are defined in a similar fashion. In practice, even the calculation of d_2 is a delicate business.

Before identifying $E_\infty^{s,t}$ we need to define the homotopy inverse limit of spectra.

2.1.10. DEFINITION. *Given a sequence of spectra and maps*

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \longleftarrow \cdots,$$

$\varprojlim X_i$, *is the fiber of the map*

$$g\colon \prod X_i \to \prod X_i$$

whose i*th component is the difference between the projection* $p_i\colon \prod X_j \to X_i$ *and the composite*

$$\prod X_j \xrightarrow{p_{i+1}} X_{i+1} \xrightarrow{f_{i+1}} X_i. \qquad \square$$

For the existence of products in the stable category see 3.13 of Adams [**4**]. This $\varprojlim$ is not a categorical inverse limit (Mac Lane [**1**, Section III.4] because a compatible collection of maps to the X_i, does not give a unique map to $\varprojlim X_i$. For this reason some authors (e.g., Bousfield and Kan [**1**]) denote it instead by $\underleftarrow{\mathrm{holim}}$. The same can be said of the direct limit, which can be defined as the cofiber of the appropriate self-map of the coproduct of the spectra in question. However this $\varprojlim$ has most of the properties one would like, such as the following.

2.1.11. LEMMA. *Given spectra* $X_{i,j}$ *for* $i, j \geq 0$ *and maps* $f\colon X_{i,j} \to X_{i-1,j}$ *and* $g\colon X_{i,j} \to X_{i,j-1}$ *such that* fg *is homotopic to* gf,

$$\varprojlim_i \varprojlim_j X_{i,j} = \varprojlim_j \varprojlim_i X_{i,j}.$$

PROOF. We have for each i a cofibre sequence

$$\varprojlim_j X_{i,j} \to \prod_j X_{i,j} \to \prod_j X_{i,j}.$$

Next we need to know that products preserve cofiber sequences. For this fact, recall that the product of spectra $\prod Y_i$, is defined via Brown's representability theorem (Adams [**4**], Theorem 3.12) as the spectrum representing the functor $\prod[-, Y_i]$. Hence the statement follows from the fact that a product (although not the inverse limit) of exact sequences is again exact.

Hence we get the following homotopy commutative diagram in which both rows and columns are cofiber sequences.

$$\begin{tikzcd}
\varprojlim_i \varprojlim_j X_{i,j} \arrow[r] \arrow[d] & \varprojlim_i \prod_j X_{i,j} \arrow[r] \arrow[d] & \varprojlim_i \prod X_{i,j} \arrow[d] \\
\prod_i \varprojlim_j X_{i,j} \arrow[r] \arrow[d] & \prod_i \prod_j X_{i,j} \arrow[r] \arrow[d] & \prod_i \prod_j X_{i,j} \arrow[d] \\
\prod_i \varprojlim_j X_{i,j} \arrow[r] & \prod_i \prod_j X_{i,j} \arrow[r] & \prod_i \prod_j X_{i,j}
\end{tikzcd}$$

Everything in sight is determined by the two self-maps of $\prod_i \prod_j X_{i,j}$ and the homotopy that makes them commute. Since the product is categorical we have $\prod_i \prod_j X_{i,j} = \prod_j \prod_i X_{i,j}$. It follows that $\prod_i \varprojlim_j X_{i,j} = \varprojlim_j \prod_i X_{i,j}$ because they are each the fiber of the same map.

Similarly

$$\prod_j \varprojlim_i X_{i,j} = \varprojlim_i \prod_j X_{i,j}$$

so one gets an equivalent diagram with $\varprojlim_j \varprojlim_i X_{i,j}$ in the upper left corner. $\square$

Now we will show that for suitable X, $E_\infty^{s,t}$ is a certain subquotient of $\pi_u(X)$.

2.1.12. Lemma. *Let X be a spectrum with an Adams resolution (X_s, g_s) such that $\varprojlim X_s = \mathrm{pt}$. Then $E_\infty^{s,t}$ is the subquotient $\operatorname{im}\pi_u(X_s)/\operatorname{im}\pi_u(X_{s+1})$ of $\pi_u(X)$ and $\bigcap \operatorname{im}\pi_*(X_s) = 0$.*

Proof. For the triviality of the intersection we have $\varprojlim \pi_*(X_s) = 0$ since $\varprojlim X_s = \mathrm{pt}$. Let $G_s = \pi_*(X_s)$ and

$$G_s^t = \begin{cases} G_s & \text{if } s \geq t \\ \operatorname{im} G_t < G_s & \text{if } t \geq s. \end{cases}$$

We have injections $G_s^t \to G_s^{t-1}$ and surjections $G_s^t \to G_{s-1}^t$, so $\varprojlim_t G_s^t = \bigcap_t G_s^t$ and $\varprojlim_s G_s^t = G_t$. We are trying to show $\varprojlim_t G_0^t = 0$. $\varprojlim_t G_s^t$ maps onto $\varprojlim_t G_{s-1}^t$, so $\varprojlim_s \varprojlim_t G_s^t$ maps onto $\varprojlim_t G_0^t$. But $\varprojlim_s \varprojlim_t G_s^t = \varprojlim_t \varprojlim_s G_s^t = \varprojlim_t G_t = 0$.

For the identification of $E_\infty^{s,t}$, let $0 \neq [x] \in E_\infty^{s,t}$.

First we show $\partial_{s,u}(x) = 0$. Since $d_r[x] = 0$, $\partial_{s,u}(x)$ can be lifted to $\pi_{u-1}(X_{s+r+1})$ for each r. It follows that $\partial_{s,u}(x) \in \operatorname{im}\varprojlim \pi_{u-1}(X_{s+r}) = 0$, so $\partial_{s,u}(x) = 0$.

Hence we have $x = \pi_u(f_s)(y)$ for $y \in \pi_u(X_s)$. It suffices to show that y has a nontrivial image in $\pi_u(X)$. If not, let r be the largest integer such that y has a nontrivial image $z \in \pi_u(X_{s-r+1})$. Then $z = \partial_{s-r,u}(w)$ for $w \in \pi_u(K_{s-r})$ and $d_r[w] = [x]$, contradicting the nontriviality of $[x]$. $\square$

Now we prove 2.1.1(a), the identification of the E_2-term.

By 2.1.2(d), $E_1^{s,t} = \operatorname{Hom}_A(H^{t-s}(K_s), \mathbf{Z}/(p))$. Hence applying $\operatorname{Hom}_A(-, \mathbf{Z}/(p))$ to 2.1.4 gives a complex

$$E_1^{0,t} \xrightarrow{\delta} E_1^{1,t} \xrightarrow{\delta} E_1^{2,t} \to \cdots .$$

The cohomology of this complex is by definition the indicated Ext group. It is straightforward to identify the coboundary δ with the d_1 in the spectral sequence and 2.1.1(a) follows.

2.1.13. Corollary. *If $f\colon X \to Y$ induces an isomorphism in* mod (p) *homology then it induces an isomorphism (from E_2 onward) in the* mod (p) *Adams spectral sequence.* $\square$

2.1.14. Definition. *Let G be an abelian group and X a spectrum. Then $XG = X \wedge SG$, where SG is the Moore spectrum associated with G (Adams* [**4**], p. 200]. *Let $\widehat{X} = X\mathbf{Z}_p$ (the p-aidc completion of X), where $\mathbf{Z}_p$ is the p-adic integers, and $X^m = X\mathbf{Z}/(p^m)$.* $\square$

2.1.15. Lemma. (a) *The map $X \to \widehat{X}$ induces an isomorphism of* mod (p) *Adams spectral sequences.*

(b) $\pi_*(\widehat{X}) = \pi_*(X) \otimes \mathbf{Z}_p$.

(c) $\widehat{X} = \varprojlim X^m$, *if x has finite type.*

Proof. For (a) it suffices by 2.1.11 to show that the map induces an isomorphism in mod (p) homology. For this see Adams [**4**], proposition 6.7, which also shows (b).

Part (c) does not follow immediately from the fact that $S\mathbf{Z}_p = \varprojlim S\mathbf{Z}/(p^m)$ because inverse limits do not in general commute with smash products. Indeed our assertion would be false for $X = S\mathbf{Q}$, but we are assuming that X has finite type.

By 2.1.10 there is a cofibration

$$S\mathbf{Z}_p \to \prod_m S\mathbf{Z}/(p^m) \to \prod_m S\mathbf{Z}/(p^m),$$

so it suffices to show that

$$X \wedge \prod S\mathbf{Z}/(p^m) \simeq \prod X\mathbf{Z}/(p^m).$$

This is a special case (with $X = E$ and $R = \mathbf{Z}$) of Theorem 15.2 of Adams [**4**]. $\square$

2.1.16. Lemma. *If X is a connective spectrum with each $\pi_i(X)$ a finite p-group, then for any* mod (p) *Adams resolution (X_s, g_s) of X, $\varprojlim X_s =$* pt.

Proof. Construct a diagram

$$X = X'_0 \leftarrow X'_1 \leftarrow X'_2 \leftarrow \cdots$$

(not an Adams resolution) by letting X'_{s+1} be the fiber in

$$X'_{s+1} \to X'_s \to K_s,$$

where the right-hand map corresponds [2.1.2(e)] to a basis for the bottom cohomology group of X_s. Then the finiteness of $\pi_i(X)$ implies that for each i, $\pi_i(X'_s) = 0$ for large s. Moreover, $\pi_*(X'_{s+1}) \to \pi_*(X'_s)$ is monomorphic so $\varprojlim X'_s =$ pt.

Now if (X_s, g_s) is an Adams resolution, the triviality of g_s in cohomology enables us to construct compatible maps $X_s \to X'_s$. It follows that the map $\varprojlim \pi_*(X_s) \to \pi_*(X)$ is trivial. Each X_s also satisfies the hypotheses of the lemma, so we conclude that $\varprojlim \pi_*(X_s)$ has trivial image in each $\pi_*(X_s)$ and is therefore trivial. Since $\pi_i(X_s)$ is finite for all i and s, $\varprojlim^1 \pi_*(X_s) = 0$ so $\varprojlim X_s =$ pt. $\square$

We are now ready to prove 2.1.1(b), i.e., to identify the E_∞-term. By 2.1.15(a) it suffices to replace X by $\widehat{X}$. Note that since $S\mathbf{Z}_p \wedge S\mathbf{Z}/(p^m) = S\mathbf{Z}/(p^m)$, $X^m = \widehat{X}^m$. It follows that given a mod (p) Adams resolution (X_s, g_s) for X, smashing with $S\mathbf{Z}_p$ and $S\mathbf{Z}/(p^m)$ gives resolutions $(\widehat{X}_s, \hat{g}_s)$ and (X^m_s, g^m_s) for $\widehat{X}$ and X^m, respectively. Moreover, X^m satisfies 2.1.16 so $\varprojlim_s X^m_s =$ pt. Applying 2.1.15(c) to each X_s, we get $\widehat{X}_s = \varprojlim_m X^m_s$, so

$$\begin{aligned}
\varprojlim_s X_s &= \varprojlim_s \varprojlim_m X^m_s \\
&= \varprojlim_m \varprojlim_s X^m_s \quad \text{by 2.1.11} \\
&= \text{pt}.
\end{aligned}$$

Hence the result follows from 2.1.12. $\square$

2.1.17. Remark. The E_∞ term only gives us a series of subquotients of $\pi_*(X) \otimes \mathbf{Z}_p$, not the group itself. After computing E_∞ one may have to use other methods to solve the extension problem and recover the group.

We close this section with some examples.

2.1.18. EXAMPLE. Let $X = H\mathbf{Z}$, the integral Eilenberg–Mac Lane spectrum. The fundamental cohomology class gives a map $f\colon X \to E$ with $H^*(f)$ surjective. The fiber of f is also X, the inclusion map $g\colon X \to X$ having degree p. Hence we get an Adams resolution (2.1.3) with $X_s = X$ and $K_s = E$ for all s, the map $X = X_s \to X_0 = X$, having degree p^s. We have then

$$E_1^{s,t} = \begin{cases} \mathbf{Z}/(p) & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

There is no room for nontrivial differentials so the spectral sequence collapses, i.e., $E_\infty = E_1$. We have $E_\infty^{s,s} = \mathbf{Z}/(p) = \operatorname{im}\pi_0(X_s)/\operatorname{im}\pi_0(X_{s+1})$. In this case $\widehat{X} = H\mathbf{Z}_p$, the Eilenberg–Mac Lane spectrum for $\mathbf{Z}_p$.

2.1.19. EXAMPLE. Let $X = H\mathbf{Z}/(p^i)$ with $i > 1$. It is known that $H^*(X) = H^*(Y) \oplus \Sigma H^*(Y)$ as A-modules, where $Y = H\mathbf{Z}$. This splitting arises from the two right-hand maps in the cofiber sequence

$$Y \to Y \to X \to \Sigma Y,$$

where the left-hand map has degree p^i. Since the E_2-term of the Adams spectral sequence depends only on $H^*(X)$ as an A-module, the former will enjoy a similar splitting. In the previous example we effectively showed that

$$\operatorname{Ext}_A^{s,t}(H^*(Y), \mathbf{Z}/(p)) = \begin{cases} \mathbf{Z}/(p) & \text{if } t = s \\ 0 & \text{if } t \neq s. \end{cases}$$

It follows that in the spectral sequence for X we have

$$E_2^{s,t} = \begin{cases} \mathbf{Z}/(p) & \text{if } t - s = 0 \text{ or } 1 \\ 0 & \text{otherwise} \end{cases}$$

In order to give the correct answer we must have $E_\infty^{s,t} = 0$ if $t - s = 1$ and $E_\infty^{s,t} = 0$ if $t = s$ for all but i values of s. Multiplicative properties of the spectral sequence to be discussed in Section 3 imply that the only way we can arrive at a suitable E_∞ term is to have $d_i\colon E_i^{s,s+1} \to E_i^{s+i,s+i}$ nontrivial for all $s \geq 0$. A similar conclusion can be drawn by chasing the relevant diagrams.

2.1.20. EXAMPLE. Let X be the fiber in $X \to \widehat{S}^0 \to H\mathbf{Z}_p$ where the right-hand map is the fundamental integral cohomology class on S^0. Smashing the above fibration with X we get

$$X \wedge X \xrightarrow{g_0} X \xrightarrow{f_0} X \wedge H\mathbf{Z}$$

It is known that the integral homology of X has exponent p, so $X \wedge H\mathbf{Z}$ is a wedge of E and $H^*(f_0)$ is surjective. Similar statements hold after smashing with X any number of times, so we get an Adams resolution (2.1.3) with $K_s = X_s \wedge H\mathbf{Z}$ and $X_s = X^{(s+1)}$, the $(s+1)$-fold smash product of X with itself, i.e., one of the form

$$\begin{tikzcd}
X \arrow[d] & X \wedge X \arrow[l] \arrow[d] & X \wedge X \wedge X \arrow[l] & \cdots \arrow[l] \\
X \wedge H\mathbf{Z} & X \wedge X \wedge H\mathbf{Z}. & &
\end{tikzcd}$$

Since X is $(2p-4)$-connected X_s, is $((s+1)(2p-3)-1)$-connected, so $\varprojlim X_s$, is contractible.

2. The Adams Spectral Sequence Based on a Generalized Homology Theory

In this section we will define a spectral sequence similar to that of 2.1.1 (the classical Adams spectral sequence) in which the mod (p) Eilenberg–Mac Lane spectrum is replaced by some more general spectrum E. The main example we have in mind is of course $E = BP$, the Brown–Peterson spectrum, to be defined in 4.1.12. The basic reference for this material is Adams [**4**] (especially Section 15, which includes the requisite preliminaries on the stable homotopy category.

Our spectral sequence should have the two essential properties of the classical one: it converges to $\pi_*(X)$ localized or completed at p and its E_2-term is a functor of $E^*(X)$ (the generalized cohomology of X) as a module over the algebra of cohomology operations $E^*(E)$; i.e., the E_2-term should be computable in some homological way, as in 2.1.1. Experience has shown that with regard to the second property we should dualize and consider instead $E_*(X)$ (the generalized homology of X) as a comodule over $E_*(E)$ (sometimes referred to as the coalgebra of cooperations). In the classical case, i.e., when $E = H\mathbf{Z}/(p)$, $E_*(E)$ is the dual Steenrod algebra A_*.

Theorem 2.1.1(a) can be reformulated as $E_2 = \mathrm{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X))$ using the definition of Ext in the category of comodules given in A1.2.3. In the case $E = BP$ substantial technical problems can be avoided by using homology instead of cohomology. Further discussion of this point can be found in Adams [**6**, pp. 51–55].

Let us assume for the moment that we have known enough about E and $E_*(E)$ to say that $E_*(X)$ is a comodule over $E_*(E)$ and we have a suitable definition of $\mathrm{Ext}_{E_*(E)}(E_*(S^0), E_*(X))$, which we abbreviate as $\mathrm{Ext}(E_*(X))$. Then we might proceed as follows.

2.2.1. Definition. *An E_*-Adams resolution for X is a diagram*

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"'] & \cdots \arrow[l] \\ K_0 & K_1 & K_2 & \end{tikzcd}$$

such that for all $s \geq 0$ the following conditions hold.

(a) *X_{s+1} is the fiber of f_s.*

(b) *$E \wedge X_s$ is a retract of $E \wedge K_s$, i.e., there is a map $h_s\colon E \wedge K_s \to E \wedge X_s$ such that $h_s(E \wedge f_s)$ is an identity map of $E \wedge X_s$. particular $E_*(f_s)$ is a monomorphism.*

(c) *K_s is a retract of $E \wedge K_s$.*

(d)

$$\mathrm{Ext}^{t,u}(E_*(K_s)) = \begin{cases} \pi_u(K_s) & \text{if } t = 0 \\ 0 & \text{if } t > 0. \end{cases} \qquad \square$$

As we will see below, conditions (b) and (c) are necessary to insure that the spectral sequence is natural, while (d) is needed to give the desired E_2-term. As before it is convenient to consider a spectrum with the following properties.

2.2.2. Definition. *An E-completion $\widehat{X}$ of X is a spectrum such that*

(a) *There is a map $X \to \widehat{X}$ inducing an isomorphism in E_*-homology.*

(b) *$\widehat{X}$ has an E_*-Adams resolution $\{\widehat{X}_s\}$ with $\varprojlim \widehat{X}_s = \mathrm{pt}$.* $\square$

This is not necessarily the same as the $\widehat{X}$ of 2.1.14, which will be denoted in this section by X_p (2.2.12). Of course, the existence of such a spectrum (2.2.13) is not obvious and we will not give a proof here. Assuming it, we can state the main result of this section.

2.2.3. THEOREM (Adams [**4**]). *An E_*-Adams resolution for X* (2.2.1) *leads to a natural spectral sequence $E_*^{**}(X)$ with $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that*

(a) $E_2^{s,t} = \mathrm{Ext}(E_*(X))$.

(b) E_∞^{**} *is the bigraded group associated with a certain filtration of $\pi_*(\widehat{X})$, in other words, the spectral sequence converges tothelatter.* (*This filtration will be described in* 2.2.14.)

2.2.4. DEFINITION. *The spectral sequence of 2.2.3 is the* Adams spectral sequence for X based on E-homology. □

2.2.5. ASSUMPTION. We now list the assumptions on E which will enable us to define Ext and $\widehat{X}$.

(a) E is a commutative associative ring spectrum.

(b) E is *connective*, i.e., $\pi_r(E) = 0$ for $r < 0$.

(c) The map $\mu_*\colon \pi_0(E) \otimes \pi_0(E) \to \pi_0(E)$ induced by the multiplication $\mu\colon E \wedge E \to E$ is an isomorphism.

(d) E is *flat*, i.e., $E_*(E)$ is flat as a left module over $\pi_*(E)$.

(e) Let $\theta\colon \mathbf{Z} \to \pi_0(E)$ be the unique ring homomorphism, and let $R \subset \mathbf{Q}$ be the largest subring to which θ extends. Then $H_r(E;R)$ is finitely generated over R for all r.

2.2.6. PROPOSITION. *$H\mathbf{Z}/(p)$ and BP satisfy* 2.2.5(a)–(e) □

The flatness condition 2.2.5(d) is only necessary for identifying E_2^{**} as an Ext. Without it one still has a spectral sequence with the specified convergence properties. Some well-known spectra which satisfy the remaining conditions are $H\mathbf{Z}$, *bo*, *bu*, and MSU. In these cases $E \wedge E$ is not a wedge of suspensions of E as it is when $E = H\mathbf{Z}/(p)$, BP, or MU. $H\mathbf{Z} \wedge H\mathbf{Z}$ is known to be a certain wedge of suspensions of $H\mathbf{Z}/(p)$ and $H\mathbf{Z}$, $bo \wedge bo$ is described by Milgram [**1**], $bu \wedge bu$ by Adams [**4**], Section 17, and $MSU \wedge MSU$ by Pengelley [**1**].

We now turn to the definition of Ext. It follows from our assumptions 2.2.5 that $E_*(E)$ is a ring which is flat as a left $\pi_*(E)$ module. Moreover, $E_*(E)$ is a $\pi_*(E)$ bimodule, the right and left module structures being induced by the maps

$$E = S^0 \wedge E \to E \wedge E \quad \text{and} \quad E = E \wedge S^0 \to E \wedge E,$$

respectively. In the case $E = H\mathbf{Z}/(p)$ these two module structures are identical, but not when $E = BP$. Following Adams [**4**], Section 12, let $\mu\colon E \wedge E$ be the multiplication on E and consider the map

$$(E \wedge E) \wedge (E \wedge X) \xrightarrow{1\wedge\mu\wedge 1} E \wedge E \wedge X.$$

2.2.7. LEMMA. *The above map induces an isomorphism*

$$E_*(E) \otimes_{\pi_*(E)} E_*(X) \to \pi_*(E \wedge E \wedge X).$$

PROOF. The result is trivial for $X = S^n$. It follows for X finite by induction on the number of cells using the 5-lemma, and for arbitrary X by passing to direct limits. □

Now the map

$$E \wedge X = E \wedge S^0 \wedge X \to E \wedge E \wedge X$$

induces

$$\psi\colon E_*(X) \to \pi_*(E \wedge E \wedge X) = E_*(E) \otimes_{\pi_*(E)} E_*(X).$$

In particular, if $X = E$ we get

$$\Delta\colon E_*(E) \to E_*(E) \otimes_{\pi^*(E)} E_*(E).$$

Thus $E_*(E)$ is a coalgebra over $\pi_*(E)$ as well as an algebra, and $E_*(X)$ is a comodule over $E_*(E)$. One would like to say that $E_*(E)$, like the dual Steenrod algebra, is a commutative Hopf algebra, but that would be incorrect since one uses the bimodule structure in the tensor product $E_*(E) \otimes_{\pi_*(E)} E_*(E)$ (i.e., the product is with respect to the right module structure on the first factor and the left module structure on the second). In addition to the coproduct Δ and algebra structure, it has a right and left unit $\eta_R, \eta_L\colon \pi_*(E) \to E_*(E)$ corresponding to the two module structures, a counit $\varepsilon\colon E_*(E) \to \pi_*(E)$ induced by $\mu\colon E \wedge E \to E$, and a conjugation $c\colon E_*(E) \to E_*(E)$ induced by interchange the factors in $E \wedge E$.

2.2.8. PROPOSITION. *With the above structure maps $(\pi_*(E), E_*(E))$ is a Hopf algebroid* (A1.1.1), *and E-homology is a functor to the category of left $E_*(E)$-comodules* (A1.1.2)*, which is abelian* (A1.1.3). $\square$

The problem of computing the relevant Ext groups is discussed in Appendix 1, where an explicit complex (the cobar complex A1.2.11) for doing so is given. This complex can be realized geometrically by the canonical E_*-Adams resolution defined below.

2.2.9. LEMMA. *Let $K_s = E \wedge X_s$, and let X_{s+1} be the fiber of $f_s\colon X_s \to K_s$. Then the resulting diagram* (2.2.1) *is an E_*-Adams resolution for X.*

PROOF. Since E is a ring spectrum it is a retract of $E \wedge E$, so $E \wedge X_s$, is a retract of $E \wedge K_s = E \wedge E \wedge X_s$ and 2.2.1(b) is satisfied. $E \wedge X_s$ is an E-module spectrum so 2.2.1(c) is satisfied. For 2.2.1(d) we have $E_*(K_s) = E_*(E) \otimes_{\pi_*(E)} E_*(X_s)$ by 2.2.7 and $\mathrm{Ext}(E_*(K_s))$ has the desired properties by A1.2.1 and A1.2.4. $\square$

2.2.10. DEFINITION. *The* canonical *E_*-Adams resolution for X is the one given by 2.2.9.*

Note that if E is not a ring spectrum then the above f_s need not induce a monomorphism in E-homology, in which case the above would not be an Adams resolution.

Note also that the canonical resolution for X can be obtained by smashing X with the canonical resolution for S^0.

2.2.11. PROPOSITION. *The E_1-term of the Adams spectral sequence associated with the resolution of* 2.2.9 *is the cobar complex $C^*(E_*(X))$* (A1.2.11). $\square$

Next we describe an E-completion $\widehat{X}$ (2.2.2). First we need some more terminology.

2.2.12. DEFINITION. $X_{(p)} = X\mathbf{Z}_{(p)}$*, where $\mathbf{Z}_{(p)}$ denotes the integers localized at p, and $X_p = X\mathbf{Z}_p$ (see 2.1.14).*

2.2.13. THEOREM. *If X is connective and E satisfies* 2.2.5(a)–(e) *then an E-completion* (2.2.2) *of X is given by*

$$\widehat{X} = \begin{cases} X\mathbf{Q} & \text{if } \pi_0(E) = \mathbf{Q} \\ X_{(p)} & \text{if } \pi_0(E) = \mathbf{Z}_{(p)} \\ X & \text{if } \pi_0(E) = \mathbf{Z} \\ X_p & \text{if } \pi_0(E) = \mathbf{Z}/(p) \text{ and } \pi_n(X) \\ & \quad \text{is finitely generated for all } n. \end{cases} \qquad \square$$

These are not the only possible values of $\pi_0(E)$, but the others will not concern us. A proof is given by Adams [**4**], Theorem 14.6 and Section 15. We will sketch a proof using the additional hypothesis that $\pi_1(E) = 0$, which is true in all of the cases we will consider in this book.

For simplicity assume that $\pi_0(X)$ is the first nonzero homotopy group. Then in the cases where $\pi_0(E)$ is a subring of $\mathbf{Q}$ we have $\pi_i(\widehat{X} \wedge \overline{E}^{(s)}) = 0$ for $i < s$, so by setting $\widehat{X}_s = \widehat{X} \wedge \overline{E}^{(s)}$ we get $\varprojlim \widehat{X}_s = \mathrm{pt}$.

The remaining case, $\pi_0(E) = \mathbf{Z}/(p)$ can be handled by an argument similar to that of the classical case. We show $X\mathbf{Z}/(p^m)$ is its own E-completion by modifying the proof of 2.1.16 appropriately. Then X_p can be shown to be E-complete just as in the proof of 2.1.1(b) (following 2.1.16).

Now we are ready to prove 2.2.3(a). As in Section 1 the diagram 2.2.1 leads to an exact couple which gives the desired spectral sequence. To identify the E_2-term, observe that 2.2.1(a) implies that each fibration in the resolution gives a short (as opposed to long) exact sequence in E-homology. These splice together to give a long exact sequence replacing 2.1.3,

$$0 \to E_*(X) \to E_*(K_0) \to E_*(\Sigma K_1) \to \cdots .$$

Condition 2.2.1(c) implies that the E_2-term of the spectral sequence is the cohomology of the complex

$$\mathrm{Ext}^0(E_*(K_0)) \to \mathrm{Ext}^0(E_*(\Sigma K_1)) \to \cdots .$$

By A1.2.4 this is $\mathrm{Ext}(E_*(X))$.

For 2.2.3(b) we know that the map $X \to \widehat{X}$ induces a spectral sequence isomorphism since it induces an E-homology isomorphism. We also know that $\varprojlim \widehat{X}_s = \mathrm{pt}$, so we can identify E_∞^{**} as in 2.1.12.

We still need to show that the spectral sequence is natural and independent (from E_2 onward) of the choice of resolution. The former implies the latter as the identity map on X induces a map between any two resolutions and standard homological arguments show that such a map induces an isomorphism in E_2 and hence in E_r for $r \geq 2$. The canonical resolution is clearly natural so it suffices to show that any other resolution admits maps to and from the canonical one.

We do this in stages as follows. Let $\{f_s \colon X_s \to K_s\}$ be an arbitrary resolution and let R^0 be the canonical one. Let $R^n = \{f_s^n \colon X_s^n \to K_s^n\}$ be defined by $X_s^n = X_s$, and $K_s^n = K_n$ for $s < n$ and $K_s^n = E \wedge X_s^n$; for $s \geq n$. Then R^∞ is the arbitrary resolution and we construct maps $R^0 \leftrightarrow R^\infty$ by constructing maps $R^n \leftrightarrow R^{n+1}$, for which it suffices to construct maps between K_s and $E \wedge X_s$ compatible with the map from X_s. By 2.2.1(b) and (c), K_s and $E \wedge X_s$ are both retracts of $E \wedge K_s$, so we have a commutative diagram

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] & K_s \arrow[d] \arrow[dr] & \\
E \wedge X_s \arrow[r] \arrow[dr] & E \wedge K_s \arrow[r] \arrow[d] & E \wedge X_s \\
& K_s &
\end{tikzcd}$$

in which the horizontal and vertical composite maps are identities. It follows that the diagonal maps are the ones we want.

The Adams spectral sequence of 2.2.3 is useful for computing $\pi_*(X)$, i.e., $[S^0, X]$. With additional assumptions on E one can generalize to a spectral sequence for computing $[W, X]$. This is done in Adams [**4**] for the case when $E_*(W)$ is projective over $\pi_*(E)$. We omit this material as we have no need for it.

Now we describe the filtration of 2.2.3(b), which will be referred to as the E_*-*Adams filtration* on $\pi_*(\widehat{X})$.

2.2.14. Filtration Theorem. *The filtration on* $\pi_*(\widehat{X})$ *of* 2.2.3(b) *is as follows. A map* $f\colon S^n \to X$ *has filtration* $\geq s$ *if* f *can be factored into* s *maps each of which becomes trivial after smashing the target with* E.

Proof. We have seen above that $F^s\pi_*(\widehat{X}) = \operatorname{im}\pi_*(X_s)$. We will use the canonical resolution (2.2.10). Let $\overline{E}$ be the fiber of the unit map $S^0 \to E$. Then $X_2 = \overline{E}^{(s)} \wedge X$, where $\overline{E}^{(s)}$ is the s-fold smash product of $\overline{E}$. $X_{i+1} \to X_i \to X_i \wedge E$ is a fiber sequence so each such composition is trivial and a map $S^n \to X$ which lifts to X_s clearly satisfies the stated condition. It remains to show the converse, i.e., that if a map $f\colon S^n \to X$ factors as

$$S^n \to Y_s \xrightarrow{g_s} Y_{s-1} \xrightarrow{g_{s-1}} \cdots \to Y_0 = X,$$

where each composite $Y_i \xrightarrow{g_i} Y_{i-1} \to Y_{i-1} \wedge E$ is trivial, then it lifts to X_s. We argue by induction on i. Suppose $Y_{i-1} \to X$ lifts to X_{i-1} (a trivial statement for $i = 1$). Since Y_i maps trivially to $Y_{i-1} \wedge E$, it does so to $X_{i-1} \wedge E$ and therefore lifts to X_i. $\square$

3. The Smash Product Pairing and the Generalized Connecting Homomorphism

In this section we derive two properties of the Adams spectral sequence which will prove usefull in the sequel. The first concerns the structure induced by a map

$$\alpha\colon X' \wedge X'' \to X, \tag{2.3.1}$$

e.g., the multiplication on a ring spectrum. The second concerns a generalized connecting homomorphism arising from a cofiber sequence

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W \tag{2.3.2}$$

when $E_*(h) = 0$. Both of these results are folk theorems long known to experts in the field but to our knowledge never before published in full generality. The first property in the classical case was proved in Adams [**3**], while a weaker form of the second property was proved by Johnson, Miller, Wilson, and Zahler [**1**].

Throughout this section the assumptions 2.2.5 on E will apply. However, the flatness condition [2.2.5(d)] is only necessary for statements explicitly involving Ext, i.e., 2.3.3(e) and 2.3.4(a). For each spectrum X let $E_*^{**}(X)$ be the Adams spectral sequence for X based on E-homology (2.2.3). Our first result is

2.3.3. THEOREM. *Let $2 \le r \le \infty$. Then the map a above induces a natural pairing*

$$E_r^{**}(X') \otimes E_r^{**}(X'') \to E_r^{**}(X)$$

such that

(a) *for $a' \in E_r^{s',t'}(X')$, $a'' \in E_r^{s'',t''}(X'')$,*

$$d_r(a', a'') = d_r(a')a'' + (-1)^{t'-s'} a' d_r(a'');$$

(b) *the pairing on E_{r+1}, is induced by that on E_r;*

(c) *the pairing on E_∞, corresponds to $a_*\colon \pi_*(X') \otimes \pi_*(X'') \to \pi_*(X)$;*

(d) *if $X' = X'' = X$ and $E_*(\alpha)\colon E_*(X) \otimes E_*(X) \to E_*(X)$ is commutative or associative, then so is the pairing [modulo the usual sign conventions, i.e., $a'a'' = (-1)^{(t'-s')(t''-s'')} a''a'$];*

(e) *for $r = 2$ the pairing is the external cup product* (A1.2.13)

$$\mathrm{Ext}(E_*(X')) \otimes \mathrm{Ext}(E_*(X'')) \to \mathrm{Ext}(E_*(X') \otimes_{\pi_*(E)} E_*(X''))$$

composed with the map in Ext *induced by the composition of canonical maps*

$$E_*(X') \otimes_{\pi_*(E)} E_*(X'') \to E_*(X' \wedge X'') \xrightarrow{\alpha_*} E_*(X).$$

In particular, by setting $X' = S^0$ and $X'' = X$ we find that the spectral sequence for X is a module (in the appropriate sense) over that for the sphere S^0. □

The second result is

2.3.4. THEOREM. *Let $E_*(h) = 0$ in 2.3.2. Then for $2 \le r \le \infty$ there are maps $\delta_r\colon E_r^{s,*}(Y) \to E_r^{s+1,*}(W)$ such that*

(a) *δ_2 is the connecting homomorphism associated with the short exact sequence*

$$0 \to E_*(W) \to E_*(X) \to E_*(Y) \to 0,$$

(b) *$\delta_r d_r = d_r \delta_r$ and δ_{r+1} induced by δ_r,*

(c) *δ_∞ is a filtered form of the map $\pi_*(h)$.*

The connecting homomorphism in Ext *can be described as the Yoneda product* (Hilton and Stammbach [**1**, p. 155] *with the element of* $\mathrm{Ext}^1_{E_*(E)}(E_*(Y), E_*(W))$ *corresponding to the short exact sequence*

$$0 \to E_*(W) \to E_*(X) \to E_*(Y) \to 0.$$

Similarly, given a sequence of maps

$$X_0 \xrightarrow{f_0} \Sigma X_1 \xrightarrow{f_1} \Sigma^2 X_2 \to \cdots \to \Sigma^n X_n$$

with $E_*(f_i) = 0$ one gets maps

$$\delta_r\colon E_r^{s,*}(X_0) \to E_r^{s+n*}(X_n)$$

commuting with differentials where δ_2 can be identified as the Yoneda product with the appropriate element in

$$\mathrm{Ext}^n_{E(E_*)}(E_*(X_0), E_*(X_n)).$$ □

If one generalizes the spectral sequence to source spectra other than the sphere one is led to a pairing induced by composition of maps. This has been studied by Moss [**1**], where it is assumed that one has Adams resolutions satisfying much stronger conditions than 2.2.1. In the spectral sequence for the sphere it can be shown that the composition and smash product pairings coincide, but we will not need this fact.

To prove 2.3.3 we will use the canonical resolutions (2.2.9) for X', X'' and X. Recall that these can be obtained by smashing the respective spectra with the canonical resolution for S^0. Let $K_{s,s+r}$ be the cofiber in

$$\overline{E}^{(s+r)} \to \overline{E}^{(s)} \to K_{s,s+r}, \tag{2.3.5}$$

where E is the fiber of $S^0 \to E$.

These spectra have the following properties.

2.3.6. Lemma.

(a) *There are canonical fibrations*

$$K_{s+i,s+i+j} \to K_{s,s+i+j} \to K_{s,s+i}.$$

(b) $E_1^{s,*}(X) = \pi_*(X \wedge K_{s,s+1})$.

Let $Z_r^{s,*}(X)$, $B_r^{s,*}(X) \subset E_1^{s,*}(X)$ *be the images of* $\pi_*(X \wedge K_{s,s+r})$ *and* $\pi_*(X \wedge \Sigma^{-1} K_{s-r+1,s})$, *respectively. Then* $E_r^{s,*}(X) = Z_r^{s,*}(X)/B_r^{s,*}(X)$ *and* d_r *is induced by the map*

$$X \wedge K_{s,s+r} \to X \wedge \Sigma K_{s+r,s+2r}.$$

(c) α *induces map* $X'_s \wedge X''_t \to X_{s+1}$ *(where these are the spectra in the canonical resolutions) compatible with the maps* g'_s, g''_t, *and* g_{s+t} *of* 2.2.1.

(d) *The map*

$$K_{s,s+1} \wedge K_{t,t+1} \to K_{s+t,s+t+1},$$

given by the equivalence

$$K_{n,n+1} = E \wedge \overline{E}^{(n)}$$

and the multiplication on E, *lifts to maps*

$$K_{s,s+r} \wedge K_{t,t+r} \to K_{s+t,s+t+r}$$

for $r > 1$ *such that the following diagram commutes*

$$\begin{tikzcd} K_{s,s+r+1} \wedge K_{t,t+r+1} \arrow[r] \arrow[d] & K_{s+t,s+t+r+1} \arrow[d] \\ K_{s,s+r} \wedge K_{t,t+r} \arrow[r] & K_{s+t,s+t+r} \end{tikzcd}$$

where the vertical maps come from (a).

(e) *The following diagram commutes*

$$\begin{tikzcd} K_{s,s+r} \wedge K_{t,t+r} \arrow[r] \arrow[d] & (\Sigma K_{s+r,s+2r} \wedge K_{t,t+r}) \vee (K_{s,s+r} \wedge \Sigma K_{t+r,t+2r}) \arrow[d] \\ K_{s+t,s+t+r} \arrow[r] & \Sigma K_{s+t+r,s+t+2r} \end{tikzcd}$$

where the vertical maps are those of (d) *and the horizontal maps come from* (a), *the maps to and from the wedge being the sums of the maps to and from the summands.*

PROOF. Part (a) is elementary. For (b) we refer the reader to Cartan and Eilenberg [**1**], Section XV.7, where a spectral sequence is derived from a set of abelian groups $H(p,q)$ satisfying certain axioms. Their $H(p,q)$ in this case is our $\pi_*(K_{p,q})$, and (a) guarantees that these groups have the appropriate properties. For (c) we use the fact that $X'_s = X' \wedge \overline{E}^{(s)}$, $X''_t = X'' \wedge \overline{E}^{(t)}$, and $X_{s+t} = X \wedge \overline{E}^{(s+t)}$.

For (d) we can assume the maps $\overline{E}^{(s+1)} \to \overline{E}^{(s)}$ are all inclusions with $K_{s,s+r} = \overline{E}^{(s)}/\overline{E}^{(s+r)}$. Hence we have

$$K_{s,s+r} \wedge K_{t,t+r} = E^{(s)} \wedge \overline{E}^{(t)}/(\overline{E}^{(s+r)} \wedge \overline{E}^{(t)} \cup \overline{E}^{(s)} \wedge \overline{E}^{(t+r)})$$

and this maps naturally to

$$\overline{E}^{(s+t)}/\overline{E}^{(s+t+r)} = K_{s+t,s+t+r}.$$

For (e) if $\overline{E}^{(s+2r)} \to \overline{E}^{(s+r)} \to \overline{E}^{(s)}$ are inclusions then so is $K_{s+r,s+2r} \to K_{s,s+2r}$ so we have $K_{s,s+r} = K_{s,s+2r}/K_{s+r,s+2r}$ and $K_{t,t+r} = K_{t,t+2r}/K_{t+r,t+2r}$. With this in mind we get a commutative diagram

$$\begin{tikzcd}
K_{s,s+r} \wedge K_{t+r,t+2r} \cup K_{s+r,s+2r} \wedge K_{t,t+r} \arrow[r] \arrow[d] & K_{s+t+r,s+t+2r} \arrow[d] \\
K_{s,s+2r} \wedge K_{t,t+2r} \arrow[r] \arrow[d] & K_{s+t,s+t+2r} \arrow[d] \\
K_{s,s+r} \wedge K_{t,t+r} \arrow[r] \arrow[d] & K_{s+t,s+t+r} \arrow[d] \\
\Sigma(K_{s,s+r} \wedge K_{t+r,t+2r} \cup K_{s+r,s+2r} \wedge K_{t,t+r}) \arrow[r] & \Sigma K_{s+t+r,s+t+2r}
\end{tikzcd}$$

where the horizontal maps come from (d) and the upper vertical maps are inclusions. The lower left-hand map factors through the wedge giving the desired diagram. □

We are now ready to prove 2.3.3. In light of 2.3.6(b), the pairing is induced by the maps of 2.3.6(d). Part 2.3.3(a) then follows from 2.3.6(e) as the differential on $E_r^{**}(X') \otimes E_r^{**}(X'')$ is induced by the top map of 2.3.6(e). Part 2.3.3(b) follows from the commutative diagram in 2.3.6(d). Part 2.3.3(c) follows from the compatibility of the maps in 2.3.6(c) and (d).

Assuming 2.3.3(e), (d) is proved as follows. The pairing on Ext is functorial, so if $E_*(X)$ has a product which is associative or commutative, so will $E_2^{**}(X)$. Now suppose inductively that the product on $E_r^{**}(X)$ has the desired property. Since the product on E_{r+1} is induced by that on E_r the inductive step follows.

It remains then to prove 2.3.3(e). We have $E_*(X' \wedge K_{s,s+1}) = D^s(E_*(X'))$ (A1.2.11) and similarly for X'', so our pairing is induced by a map

$$E_*(X' \wedge K_{s,s+1}) \otimes_{\pi_*(E)} E_*(X'' \wedge K_{t,t+1}) \to E_*(X \wedge K_{s+t,s+t+1}),$$

i.e., by a pairing of resolutions. Hence the pairing on E_2 coincides with the specified algebraic pairing by the uniqueness of the latter (A1.2.14).

We prove 2.3.4 by reducing it to the following special case.

2.3.7. LEMMA. *Theorem* 2.3.4 *holds when* X *is such that* $\mathrm{Ext}^s(E_*(X)) = 0$ *for* $s > 0$ *and* $\pi_*(X) = \mathrm{Ext}^0(E_*(X))$. □

PROOF OF 2.3.4. Let W' be the fiber of the composite

$$W \xrightarrow{f} X \to X \wedge E.$$

Since Σfh is trivial, h lifts to a map $h'\colon Y \to \Sigma W'$. Now consider the cofiber sequence

$$W \to X \wedge E \to \Sigma W' \to \Sigma W.$$

Lemma 2.3.7 applies here and gives maps

$$\delta_r\colon E_r^{s,*}(\Sigma W') \to E_r^{s+1,*}(\Sigma W).$$

Composing this with the maps induced by h' gives the desired result. □

PROOF OF 2.3.7. Disregarding the notation used in the above proof, let $W' = \Sigma^{-1}Y$, $X' = \Sigma^{-1}Y \wedge E$, and $Y' = Y \wedge \overline{E}$. Then we have a commutative diagram in which both rows and columns are cofiber sequences

$$\begin{tikzcd}
X \arrow[d] & W \arrow[l] \arrow[d] & W' \arrow[l] \arrow[d] \\
X \vee (Y \wedge E) \arrow[d] & X \arrow[l] \arrow[d] & X' \arrow[l] \arrow[d] \\
Y \wedge E & Y \arrow[l] & Y' \arrow[l]
\end{tikzcd}$$

Each row is the beginning of an Adams resolution (possibly noncanonical for W and X) which we continue using the canonical resolutions (2.2.9) for W', X', and Y'. Thus we get a commutative diagram

(2.3.8)
$$\begin{tikzcd}
W \arrow[d] & W' \arrow[l] \arrow[d] & W' \wedge \overline{E} \arrow[l] \arrow[d] & W' \wedge \overline{E}^{(2)} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
X \arrow[d] & X' \arrow[l] \arrow[d] & X' \wedge \overline{E} \arrow[l] \arrow[d] & X' \wedge \overline{E}^{(2)} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
Y & Y' \arrow[l] & Y' \wedge \overline{E} \arrow[l] & Y' \wedge \overline{E}^{(2)} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

in which each column is a cofiber sequence. The map $Y \xrightarrow{\simeq} \Sigma W'$ induces maps $\delta_r\colon E_r^{s,*}(Y) \to E_r^{s+1,*}(W)$ which clearly satisfy 2.3.4(a) and (b), so we need only to verify that δ_2 is the connecting homomorphism. The resolutions displayed in 2.3.8 make this verification easy because they yield a short exact sequence of E_1-terms which is additively (though not differentially) split. For $s = 0$ we have

$$E_1^{0,*}(W) = \pi_*(X), \quad E_1^{0,*}(X) = \pi_*(X \vee (Y \wedge E)),$$
$$E_1^{0,*}(Y) = \pi_*(Y \wedge E), \quad E_1^{1,*}(W) = \pi_*(Y \wedge E),$$
$$E_1^{1,*}(X) = \pi_*(Y \wedge E \wedge E) \quad \text{and} \quad E_1^{1,*}(Y) = \pi_*(\Sigma Y \wedge \overline{E} \wedge E),$$

so the relevant diagram for the connecting homomorphism is

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \vee (Y \wedge E) \arrow[d, "d"'] \arrow[r, shift right] & Y \wedge E \arrow[l, shift right, "b"'] \arrow[d] \\
Y \wedge E \arrow[r, shift right] & Y \wedge E \wedge E \arrow[l, shift right, "a"'] \arrow[r] & \Sigma Y \wedge \overline{E} \wedge \overline{E}
\end{tikzcd}$$

where a and b are splitting maps. The connecting homomorphism is induced by adb, which is the identity on $Y \wedge E$, which also induces δ_2.

For $s > 0$ we have

$$E_1^{s,*}(W) = \pi_*(\Sigma^{s-1} Y \wedge E \wedge \overline{E}^{(s-1)}),$$
$$E_1^{s,*}(X) = \pi_*(\Sigma^{s-1} Y \wedge E^{(2)} \wedge \overline{E}^{(s-1)}),$$

and

$$E_1^{s,*}(Y) = \pi_*(\Sigma^{s} Y \wedge E \wedge \overline{E}^{(s)}),$$

so the relevant diagram is

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & E \wedge E \arrow[d] \arrow[r, shift right] & E \wedge \overline{E} \arrow[l, shift right] \arrow[d] \\
\Sigma E \wedge \overline{E} \arrow[r, shift right] & \Sigma E \wedge E \wedge \overline{E} \arrow[l, shift right] \arrow[r] & \Sigma^2 E \wedge \overline{E}^2
\end{tikzcd}$$

and again the connecting homomorphism is induced by the identity on $\Sigma^s Y \wedge E \wedge \overline{E}^s$. $\square$