CHAPTER 1

An Introduction to the Homotopy Groups of Spheres

This chapter is intended to be an expository introduction to the rest of the book. We will informally describe the spectral sequences of Adams and Novikov, which are the subject of the remaining chapters. Our aim here is to give a conceptual picture, suppressing as many technical details as possible.

In Section 1 we list some theorems which are classical in the sense that they do *not* require any of the machinery described in this book. These include the Hurewicz theorem 1.1.2, the Freudenthal suspension theorem 1.1.4, the Serre finiteness theorem 1.1.8, the Nishida nilpotence theorem 1.1.9, and the Cohen–Moore–Neisendorfer exponent theorem 1.1.10. They all pertain directly to the homotopy groups of spheres and are not treated elsewhere here. The homotopy groups of the stable orthogonal group SO are given by the Bott periodicity theorem 1.1.11. In 1.1.12 we define the J-homomorphism from $\pi_i(SO(n))$ to $\pi_{n+i}(S^n)$. Its image is given in 1.1.13, and in 1.1.14 we give its cokernel in low dimensions. Most of the former is proved in Section 5.3.

In Section 2 we describe Serre's method of computing homotopy groups using cohomological techniques. In particular, we show how to find the first element of order p in $\pi_*(S^3)$ 1.2.4. Then we explain how these methods were streamlined by Adams to give his celebrated spectral sequence 1.2.10. The next four theorems describe the Hopf invariant one problem. A table showing the Adams spectral sequence at the prime 2 through dimension 45 is given in 1.2.15. In Chapter 2 we give a more detailed account of how the spectral sequence is set up, including a convergence theorem. In Chapter 3 we make many calculations with it at the prime 2.

In 1.2.16 we summarize Adams's method for purposes of comparing it with that of Novikov. The basic idea is to use complex cobordism (1.2.17) in place of ordinary mod (p) cohomology. Fig. 1.2.19 is a table of the Adams–Novikov spectral sequence for comparison with Fig. 1.2.15.

In the next two sections we describe the algebra surrounding the E_2-term of the Adams–Novikov spectral sequence. To this end formal group laws are defined in 1.3.1 and a complete account of the relevant theory is given in Appendix 2. Their connection with complex cobordism is the subject of Quillen's theorem (1.3.4) and is described more fully in Section 4.1. The Adams–Novikov E_2-term is described in terms of formal group law theory (1.3.5) and as an Ext group over a certain Hopf algebra (1.3.6).

The rest of Section 3 is concerned with the Greek letter construction, a method of producing infinite periodic families of elements in the E_2-term and (in favorable cases) in the stable homotopy groups of spheres. The basic definitions are given in

1.3.17 and 1.3.19 and the main algebraic fact required is the Morava–Landweber theorem (1.3.16). Applications to homotopy are given in 1.3.11, 1.3.15, and 1.3.18. The section ends with a discussion of the proofs and possible extensions of these results. This material is discussed more fully in Chapter 5.

In Section 4 we describe the deeper algebraic properties of the E_2-term. We start by introducing BP and defining a Hopf algebroid. The former is a minimal wedge summand of MU localized at a prime. A Hopf algebroid is a generalized Hopf algebra needed to describe the Adams–Novikov E_2-term more conveniently in terms of BP (1.4.2). The algebraic and homological properties of such objects are the subject of Appendix 1.

Next we give the Lazard classification theorem for formal group laws (1.4.3) over an algebraically closed field of characteristic p, which is proved in Section A2.2. Then we come to Morava's point of view. Theorem 1.3.5 describes the Adams–Novikov E_2-term as the cohomology of a certain group G with coefficients in a certain polynomial ring L. $\mathrm{Spec}(L)$ (in the sense of abstract algebraic geometry) is an infinite dimensional affine space on which G acts. The points in $\mathrm{Spec}(L)$ can be thought of as formal group laws and the G-orbits as isomorphism classes, as described in 1.4.3. This orbit structure is described in 1.4.4. For each orbit there is a stabilizer or isotropy subgroup of G called S_n. Its cohomology is related to that of G (1.4.5), and its structure is known. The theory of Morava stabilizer algebras is the algebraic machinery needed to exploit this fact and is the subject of Chapter 6. Our next topic, the chromatic spectral sequence (1.4.8, the subject of Chapter 5), connects the theory above to the Adams–Novikov E_2-term. The Greek letter construction fits into this apparatus very neatly.

Section 5 is about unstable homotopy groups of spheres and is not needed for the rest of the book. Its introduction is self-explanatory.

1. Classical Theorems Old and New

Homotopy groups. The Hurewicz and Freudenthal theorems. Stable stems. The Hopf map. Serre's finiteness theorem. Nishida's nilpotence theorem. Cohen, Moore and Neisendorfer's exponent theorem. Bott periodicity. The J-homomorphism.

We begin by recalling some definitions. The nth homotopy group of a connected space X, $\pi_n(X)$, is the set of homotopy classes of maps from the n-sphere S^n to X. This set has a natural group structure which is abelian for $n \geq 2$.

We now state three classical theorems about homotopy groups of spheres. Proofs can be found, for example, in Spanier [**1**].

1.1.1. Theorem. *$\pi_1(S^1) = \mathbf{Z}$ and $\pi_m(S^1) = 0$ for $m > 1$.* □

1.1.2. Hurewicz's Theorem. *$\pi_n(S^n) = \mathbf{Z}$ and $\pi_m(S^n) = 0$ for $m < n$. A generator of $\pi_n(S^n)$ is the class of the identity map.* □

For the next theorem we need to define the suspension homomorphism $\sigma\colon \pi_m(S^n) \to \pi_{m+1}(S^{n+1})$.

1.1.3. Definition. *The kth suspension $\Sigma^k X$ of a space X is the quotient of $I^k \times X$ obtained by collapsing $\partial I^k \times X$ onto ∂I^k, ∂I^k being the boundary of I^k, the k-dimensional cube. Note that $\Sigma^i \Sigma^j X = \Sigma^{i+j} X$ and $\Sigma^k f\colon \Sigma^k X \to \Sigma^k Y$ is the quotient of $1 \times f\colon I^k \times X \to I^k \times Y$. In particular, given $f\colon S^m \to S^n$ we have $\Sigma f\colon S^{m+1} \to S^{n+1}$, which induces a homomorphism $\pi_m(S^n) \to \pi_{m+1}(S^{n+1})$.* □

1.1.4. FREUDENTHAL SUSPENSION THEOREM. *The suspension homomorphism* $\sigma\colon \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+q})$ *defined above is an isomorphism for* $k < n-1$ *and a surjection for* $k = n-1$. □

1.1.5. COROLLARY. *The group* $\pi_{n+k}(S^n)$ *depends only on* k *if* $n > k+1$. □

1.1.6. DEFINITION. *The stable* k*-stem or* k*th stable homotopy group of spheres* π_k^S *is* $\pi_{n+k}(S^n)$ *for* $n > k+1$. *The groups* $\pi_{n+k}(S^n)$ *are called stable if* $n > k+1$ *and unstable if* $n \leq k+1$. *When discussing stable groups we will not make any notational distinction between a map and its suspensions.* □

The subsequent chapters of this book will be concerned with machinery for computing the stable homotopy groups of spheres. Most of the time we will not be concerned with unstable groups. The groups π_k^S are known at least for $k \leq 45$. See the tables in Appendix 3, along with Theorem 1.1.13. Here is a table of π_k^S for $k \leq 15$:

k	0	1	2	3	4	5	6	7	8
π_k^S	$\mathbf{Z}$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(24)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(240)$	$(\mathbf{Z}/(2))^2$

k	9	10	11	12	13	14	15
π_k^S	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/6$	$\mathbf{Z}/(504)$	0	$\mathbf{Z}/(3)$	$(\mathbf{Z}/(2))^2$	$\mathbf{Z}/(480) \oplus \mathbf{Z}/(2)$

This should convince the reader that the groups do not fall into any obvious pattern. Later in the book, however, we will present evidence of some deep patterns not apparent in such a small amount of data. The nature of these patterns will be discussed later in this chapter.

When homotopy groups were first defined by Hurewicz in 1935 it was hoped that $\pi_{n+k}(S^n) = 0$ for $k > 0$, since this was already known to be the case for $n = 1$ (1.1.1). The first counterexample is worth examining in some detail.

1.1.7. EXAMPLE. $\pi_3(S^2) = \mathbf{Z}$ generated by the class of the Hopf map $\eta\colon S^3 \to S^2$ defined as follows. Regard S^2 (as Riemann did) as the complex numbers $\mathbf{C}$ with a point at infinity. S^3 is by definition the set of unit vectors in $\mathbf{R}^4 = \mathbf{C}^2$. Hence a point in S^3 is specified by two complex coordinates (z_1, z_2). Define η by

$$\eta(z_1, z_2) = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0 \\ \infty & \text{if } z_2 = 0. \end{cases}$$

It is easy to verify that η is continuous. The inverse image under η of any point in S^2 is a circle, specifically the set of unit vectors in a complex line through the origin in $\mathbf{C}^2$, the set of all such lines being parameterized by S^2. Closer examination will show that any two of these circles in S^3 are linked. One can use quaternions and Cayley numbers in similar ways to obtain maps $\nu\colon S^7 \to S^4$ and $\sigma\colon S^{15} \to S^8$, respectively. Both of these represent generators of infinite cyclic summands. These three maps (η, ν, and σ) were all discovered by Hopf [**1**] and are therefore known as the Hopf maps.

We will now state some other general theorems of more recent vintage.

1.1.8. FINITENESS THEOREM (Serre [**3**]). $\pi_{n+k}(S^n)$ *is finite for* $k > 0$ *except when* $n = 2m$, $k = 2m-1$, *and* $\pi_{4m-1}(S^{2m}) = \mathbf{Z} \oplus F_m$, *where* F_m *is finite.* □

The next theorem concerns the ring structure of $\pi_*^S = \bigoplus_{k\geq 0}\pi_k^S$ which is induced by composition as follows. Let $\alpha \in \pi_i^S$ and $\beta \in \pi_j^S$ be represented by $f\colon S^{n+i} \to S^n$ and $g\colon S^{n+i+j} \to S^{n+i}$, respectively, where n is large. Then $\alpha\beta \in \pi_{i+j}^S$ is defined to be the class represented by $f \cdot g\colon S^{n+i+j} \to S^n$. It can be shown that $\beta\alpha = (-1)^{ij}\alpha\beta$, so π_*^S is an anticommutative graded ring.

1.1.9. NILPOTENCE THEOREM (Nishida [**1**]). *Each element $\alpha \in \pi_k^S$ for $k > 0$ is nilpotent, i.e., $\alpha^t = 0$ for some finite t.* □

For the next result recall that 1.1.8 says $\pi_{2i+1+j}(S^{2i+1})$ is a finite abelian group for all $j > 0$.

1.1.10. EXPONENT THEOREM (Cohen, Moore, and Neisendorfer [**1**]). *For $p \geq 5$ the p-component of $\pi_{2i+1+j}(S^{2i+1})$ has exponent p^i, i.e., each element in it has order $\leq p^i$.* □

This result is also true for $p = 3$ (Neisendorfer [**1**]) as well, but is known to be false for $p = 2$. For example, the 2-component of 3-stem is cyclic of order 4 (see Fig. 3.3.18) on S^3 and of order 8 on S^8 (see Fig. 3.3.10). It is also known (Gray [**1**]) to be the best possible, i.e., $\pi_{2i+1+j}(S^{2i+1})$ is known to contain elements of order p^i for certain j.

We now describe an interesting subgroup of π_*^S, the image of the Hopf–Whitehead J-homomorphism, to be defined below. Let $SO(n)$ be the space of $n\times n$ special orthogonal matrices over $\mathbf{R}$ with the standard topology. $SO(n)$ is a subspace of $SO(n+1)$ and we denote $\bigcup_{n>0} SO(n)$ by SO, known as the stable orthogonal group. It can be shown that $\pi_i(SO) = \pi_i(SO(n))$ if $n > i+1$. The following result of Bott is one of the most remarkable in all of topology.

1.1.11. BOTT PERIODICITY THEOREM (Bott [**1**]; see also Milnor [**1**]).

$$\pi_i(SO) = \begin{cases} \mathbf{Z} & \text{if } i \equiv -1 \mod 4 \\ \mathbf{Z}/(2) & \text{if } i = 0 \text{ or } 1 \mod 8 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

We will now define a homomorphism $J\colon \pi_i(SO(n)) \to \pi_{n+i}(S^n)$. Let $\alpha \in \pi_i(SO(n))$ be the class of $f\colon S^i \to SO(n)$. Let D^n be the n-dimensional disc, i.e., the unit ball in $\mathbf{R}^n$. A matrix in $SO(n)$ defines a linear homeomorphism of D^n to itself. We define $\hat{f}\colon S^i \times D^n \to D^n$ by $\hat{f}(x,y) = f(x)(y)$, where $x \in S^i$, $y \in D^n$, and $f(x) \in SO(n)$. Next observe that S^n is the quotient of D^n obtained by collapsing its boundary S^{n-1} to a single point, so there is a map $p\colon D^n \to S^n$, which sends the boundary to the base point. Also observe that S^{n+i}, being homeomorphic to the boundary of $D^{i+1} \times D^n$, is the union of $S^i \times D^n$ and $D^{i+1} \times S^{n-1}$ along their common boundary $S^i \times S^{n-1}$. We define $\tilde{f}\colon S^{n+i} \to S^n$ to be the extension of $p\hat{f}\colon S^i \times D^n \to S^n$ to S^{n+i} which sends the rest of S^{n+i} to the base point in S^n.

1.1.12. DEFINITION. *The Hopf–Whitehead J-homomorphism $J\colon \pi_i(SO(n)) \to \pi_{n+i}(S^n)$ sends the class of $f\colon S^i \to SO(n)$ to the class of $\tilde{f}\colon S^{n+i} \to S^n$ as described above.* □

We leave it to the skeptical reader to verify that the above construction actually gives us a homomorphism.

Note that both $\pi_i(SO(n))$ and $\pi_{n+i}(S^n)$ are stable, i.e., independent of n, if $n > i+1$. Hence we have $J\colon \pi_k(SO) \to \pi_k^S$. We will now describe its image.

1.1.13. THEOREM (Adams [**1**] and Quillen [**1**]). *$J\colon \pi_k(SO) \to \pi_k^S$ is a monomorphism for $k \equiv 0$ or $1 \mod 8$ and $J(\pi_{4k-1}(SO))$ is a cyclic group whose 2-component is $\mathbf{Z}_{(2)}/(8k)$ and whose p-component for $p \geq 3$ is $\mathbf{Z}_{(p)}/(pk)$ if $(p-1) \mid 2k$ and 0 if $(p-1) \nmid 2k$, where $\mathbf{Z}_{(p)}$ denotes the integers localized at p. In dimensions 1, 3, and 7, $\operatorname{im} J$ is generated by the Hopf maps (1.1.7) η, ν, and σ, respectively. If we denote by x_k the generator in dimension $4k-1$, then ηx_{2k} and $\eta^2 x_{2k}$ are the generators of $\operatorname{im} J$ in dimensions $8k$ and $8k+1$, respectively.* □

The image of J is also known to a direct summand; a proof can be found for example at the end of Chapter 19 of Switzer [**1**]. The order of $J(\pi_{4k-1}(SO))$ was determined by Adams up to a factor of two, and he showed that the remaining ambiguity could be resolved by proving the celebrated Adams conjecture, which Quillen and others did. Denote this number by a_k. Its first few values are tabulated here.

k	1	2	3	4	5	6	7	8	9	10
a_k	24	240	504	480	264	65,520	24	16,320	28,728	13,200

The number a_k has interesting number theoretic properties. It is the denominator of $B_k/4k$, where B_k, is the kth Bernoulli number, and it is the greatest common divisor of numbers $n^{t(n)}(n^{2k}-1)$ for $n \in \mathbf{Z}$ and $t(n)$ sufficiently large. See Adams [**1**] and Milnor and Stasheff [**5**] for details.

Having determined $\operatorname{im} J$, one would like to know something systematic about coker J, i.e., something more than its structure through a finite range of dimensions. For the reader's amusement we record some of that structure now.

1.1.14. THEOREM. *In dimensions ≤ 15, the 2-component of* coker J *has the following generators, each with order 2:*

$$\eta^2 \in \pi_2^S, \quad \nu^2 \in \pi_6^S, \quad \bar{\nu} \in \pi_8^S, \quad \eta\bar{\nu} = \nu^3 \in \pi_9^S, \quad \mu \in \pi_9^S,$$
$$\eta\mu \in \pi_{10}^S, \quad \sigma^2 \in \pi_{14}^S, \quad \kappa \in \pi_{14}^S \quad \text{and} \quad \eta\kappa \in \pi_{15}^S.$$

(There are relations $\eta^3 = 4\nu$ and $\eta^2\mu = 4x_3$). For $p \geq 3$ the p-component of coker J *has the following generators in dimensions $\leq 3pq-6$ (where $q = 2p-2$), each with order p:*

$$\beta_1 \in \pi_{pq-2}^S, \qquad \alpha_1\beta_1 \in \pi_{(p+1)q-3}^S$$

where $\alpha_1 = x_{(p-1)/2} \in \pi_{q-1}^S$ is the first generator of the p-component of $\operatorname{im} J$,

$$\beta_1^2 \in \pi_{2pq-4}^S, \quad \alpha_1\beta_1^2 \in \pi_{(2p+1)q-5}^S, \quad \beta_2 \in \pi_{(2p+1)q-2}^S,$$
$$\alpha_1\beta_2 \in \pi_{(2p+2)q-3}^S, \quad \text{and} \quad \beta_1^3 \in \pi_{3pq-6}^S.$$ □

The proof and the definitions of new elements listed above will be given later in the book, e.g., in Section 4.4.

2. Methods of Computing $\pi_*(S^n)$

Eilenberg–Mac Lane spaces and Serre's method. The Adams spectral sequence. Hopf invariant one theorems. The Adams–Novikov spectral sequence. Tables in low dimensions for $p = 3$.

In this section we will informally discuss three methods of computing homotopy groups of spheres, the spectral sequences of Serre, Adams, and Novikov. A fourth method, the EHP sequence, will be discussed in Section 5. We will not give any proofs and in some cases we will sacrifice precision for conceptual clarity, e.g., in our identification of the E_2-term of the Adams–Novikov spectral sequence.

The Serre spectral sequence (circa 1951) (Serre [**2**]) is included here mainly for historical interest. It was the first systematic method of computing homotopy groups and was a major computational breakthrough. It has been used as late as the 1970s by various authors (Toda [**1**], Oka [**1, 2, 3**]), but computations made with it were greatly clarified by the introduction of the Adams spectral sequence in 1958 in Adams [**3**]. In the Adams spectral sequence the basic mechanism of the Serre spectral sequence information is organized by homological algebra.

For the 2-component of $\pi_*(S^n)$ the Adams spectral sequence is indispensable to this day, but the odd primary calculations were streamlined by the introduction of the Adams–Novikov spectral sequence (Adams–Novikov spectral sequence) in 1967 by Novikov [**1**]. It is the main subject in this book. Its E_2-term contains more information than that of the Adams spectral sequence; i.e., it is a more accurate approximation of stable homotopy and there are fewer differentials in the spectral sequence. Moreover, it has a very rich algebraic structure, as we shall see, largely due to the theorem of Quillen [**2**], which establishes a deep (and still not satisfactorily explained) connection between complex cobordism (the cohomology theory used to define the Adams–Novikov spectral sequence; see below) and the theory of formal group laws. Every major advance in the subject since 1969, especially the work of Jack Morava, has exploited this connection.

We will now describe these three methods in more detail. The starting point for Serre's method is the following classical result.

1.2.1. THEOREM. *Let X be a simply connected space with $H_i(X) = 0$ for $i < n$ for some positive integer $n \geq 2$. Then*

(a) (Hurewicz [**1**]). $\pi_n(X) = H_n(X)$.

(b) (Eilenberg and Mac Lane [**2**]). *There is a space $K(\pi, n)$, characterized up to homotopy equivalence by*

$$\pi_i(K(\pi, n)) = \begin{cases} \pi & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

If X is above and $\pi = \pi_n(X)$ then there is a map $f\colon X \to K(\pi, n)$ such that $H_n(f)$ and $\pi_n(f)$ are isomorphisms. □

1.2.2. COROLLARY. *Let F be the fiber of the map f above. Then*

$$\pi_i(F) = \begin{cases} \pi_i(X) & \text{for } i \geq n+1 \\ 0 & \text{for } i \leq n. \end{cases}$$ □

In other words, F has the same homotopy groups as X in dimensions above n, so computing $\pi_*(F)$ is as good as computing $\pi_*(X)$. Moreover, $H_*(K(\pi, n))$ is known, so $H_*(F)$ can be computed with the Serre spectral sequence applied to the fibration $F \to X \to K(\pi, n)$.

Once this has been done the entire process can be repeated: let $n' > n$ be the dimension of the first nontrivial homology group of F and let $H_{n'}(F) = \pi'$. Then $\pi_{n'}(F) = \pi_{n'}(X) = \pi'$ is the next nontrivial homotopy group of X. Theorem 1.2.1 applied to F gives a map $f'\colon F \to K(\pi', n')$ with fiber F', and 1.2.2 says

$$\pi_i(F') = \begin{cases} \pi_i(X) & \text{for } i > n' \\ 0 & \text{for } i \leq n'. \end{cases}$$

Then one computes $H_*(F')$ using the Serre spectral sequence and repeats the process.

As long as one can compute the homology of the fiber at each stage, one can compute the next homotopy group of X. In Serre [**3**] a theory was developed which allows one to ignore torsion of order prime to a fixed prime p throughout the calculation if one is only interested in the p-component of $\pi_*(X)$. For example, if $X = S^3$, one uses 1.2.1 to get a map to $K(\mathbf{Z}, 3)$. Then $H_*(F)$ is described by:

1.2.3. LEMMA. *If F is the fibre of the map $f\colon S^3 \to K(\mathbf{Z}, 3)$ given by* 1.2.1, *then*

$$H_i(F) = \begin{cases} \mathbf{Z}/(m) & \text{if } i = 2m \text{ and } m > 1 \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

1.2.4. COROLLARY. *The first p-torsion in $\pi_*(S^3)$ is $\mathbf{Z}/(p)$ in $\pi_{2p}(S^3)$ for any prime p.* $\square$

PROOF OF 1.2.3. (It is so easy we cannot resist giving it.) We have a fibration

$$\Omega K(\mathbf{Z}, 3) = K(\mathbf{Z}, 2) \to F \to S^3$$

and $H^*(K(\mathbf{Z}, 2)) = H^*(\mathbf{C}P^\infty) = \mathbf{Z}[x]$, where $x \in H^2(\mathbf{C}P^\infty)$ and $\mathbf{C}P^\infty$ is an infinite-dimensional complex projective space. We will look at the Serre spectral sequence for $H^*(F)$ and use the universal coefficient theorem to translate this to the desired description of $H_*(F)$. Let u be the generator of $H^3(S^3)$. Then in the Serre spectral sequence we must have $d_3(x) = \pm u$; otherwise F would not be 3-connected, contradicting 1.1.2. Since d_3 is a derivation we have $d_3(x^n) = \pm nux^{n-1}$. It is easily seen that there can be no more differentials and we get

$$H^i(F) = \begin{cases} \mathbf{Z}/(m) & \text{if } i = 2m+1,\ m > 1 \\ 0 & \text{otherwise} \end{cases}$$

which leads to the desired result. $\square$

If we start with $X = S^n$ the Serre spectral sequence calculations will be much easier for $\pi_{k+n}(S^n)$ for $k < n-1$. Then all of the computations are in the stable range, i.e., in dimensions less than twice the connectivity of the spaces involved. This means that for a fibration $F \xrightarrow{i} X \xrightarrow{f} K$, the Serre spectral sequence gives a long exact sequence

$$(1.2.5) \qquad \cdots \to H_j(F) \xrightarrow{i_*} H_j(X) \xrightarrow{f_*} H_j(K) \xrightarrow{d} H_{j-1}(F) \to \cdots,$$

where d corresponds to Serre spectral sequence differentials. Even if we know $H_*(X)$, $H_*(K)$, and f_*, we still have to deal with the short exact sequence

$$(1.2.6) \qquad 0 \to \operatorname{coker} f_* \to H_*(F) \to \ker f_* \to 0.$$

It may lead to some ambiguity in $H_*(F)$, which must be resolved by some other means. For example, when computing $\pi_*(S^n)$ for large n one encounters this problem in the 3-component of $\pi_{n+10}(S^n)$ and the 2-component of $\pi_{n+14}(S^n)$. This difficulty is also present in the Adams spectral sequence, where one has the possibility of a nontrivial differential in these dimensions. These differentials were first calculated by Adams [**12**], Liulevicius [**2**], and Shimada and Yamanoshita [**3**] by methods involving secondary cohomology operations and later by Adams and Atiyah [**13**] by methods involving K-theory

The Adams spectral sequence of Adams [**3**] begins with a variation of Serre's method. One works only in the stable range and only on the p-component. Instead of mapping X to $K(\pi, n)$ as in 1.2.1, one maps to $K = \prod_{j>0} K(H^j(X; \mathbf{Z}/(p)), j)$ by a certain map g which induces a surjection in mod (p) cohomology. Let X_1 be the fiber of g. Define spaces X_i and K_i inductively by $K_i = \prod_{j>0} K(H^j(X_i; \mathbf{Z}/(p)), j)$ and X_{i+1} is the fiber of $g\colon X_i \to K_i$ (this map is defined in Section 2.1, where the Adams spectral sequence is discussed in more detail). Since $H^*(g_i)$ is onto, the analog of 1.2.5 is an short exact sequence in the stable range

$$0 \leftarrow H^*(X_i) \leftarrow H^*(K_i) \leftarrow H^*(\Sigma X_{i+1}) \leftarrow 0, \tag{1.2.7}$$

where all cohomology groups are understood to have coefficients $\mathbf{Z}/(p)$. Moreover, $H^*(K_i)$ is a free module over the mod (p) Steenrod algebra A, so if we splice together the short exact sequences of 1.2.7 we get a free A-resolution of $H^*(X)$

$$0 \leftarrow H^*(X) \leftarrow H^*(K) \leftarrow H^*(\Sigma^1 K_1) \leftarrow H^*(\Sigma^2 K_2) \leftarrow \cdots \tag{1.2.8}$$

Each of the fibration $X_{i+1} \to X_i \to K_i$ gives a long exact sequence of homotopy groups. Together these long exact sequences form an exact couple and the associated spectral sequence is the Adams spectral sequence for the p-component of $\pi_*(X)$. If X has finite type, the diagram

$$K \to \Sigma^{-1} K_1 \to \Sigma^{-2} K_2 \to \cdots \tag{1.2.9}$$

(which gives 1.2.8 in cohomology) gives a cochain complex of homotopy groups whose cohomology is $\mathrm{Ext}_A(H^*(X); \mathbf{Z}/(p))$. Hence one gets

1.2.10. Theorem (Adams [**3**]). *There is a spectral sequence converging to the p-component of $\pi_{n+k}(S^n)$ for $k < n - 1$ with*

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbf{Z}/(p), \mathbf{Z}/(p)) =: H^{s,t}(A)$$

and $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. Here the groups $E_\infty^{s,t}$ for $t - s = k$ form the associated graded group to a filtration of the p-component of $\pi_{n+k}(S^n)$. □

Computing this E_2-term is hard work, but it is much easier than making similar computations with Serre spectral sequence. The most widely used method today is the spectral sequence of May [**1, 2**] (see Section 3.2). This is a trigraded spectral sequence converging to $H^{**}(A)$, whose E_2-term is the cohomology of a filtered form of the Steenrod algebra. This method was used by Tangora [**1**] to compute $E_2^{s,t}$ for $p = 2$ and $t - s \leq 70$. Most of his table is reproduced here in Fig. A3.1a–c. Computations for odd primes can be found in Nakamura [**2**].

As noted above, the Adams E_2-term is the cohomology of the Steenrod algebra. Hence $E_2^{1,*} = H^1(A)$ is the indecomposables in A. For $p = 2$ one knows that A is generated by Sq^{2^i} for $i \geq 0$; the corresponding elements in $E_2^{1,*}$ are denoted by $h_i \in E_2^{1,2^i}$. For $p > 2$ the generators are the Bockstein β and $\mathcal{P}^{p^i}$ for $i \geq 0$ and the corresponding elements are $a_0 \in E_2^{1,1}$ and $h_i \in E_2^{1,qp^i}$, where $q = 2p - 2$.

For $p = 2$ these elements figure in the famous Hopf invariant one problem.

1.2.11. Theorem (Adams [**12**]). *The following statements are equivalent.*

(a) *S^{2^i-1} is parallelizable, i.e., it has $2^i - 1$ globally linearly independent tangent vector fields.*

(b) *There is a division algebra (not necessarily associative) over $\mathbf{R}$ of dimension 2^i.*

(c) *There is a map* $S^{2\cdot 2^i-1} \to S^{2^i}$ *of Hopf invariant one* (*see* 1.5.2).

(d) *There is a* 2*-cell complex* $X = S^{2^i} \cup e^{2^{i+1}}$ [*the cofiber of the map in* (*c*)] *in which the generator of* $H^{2^{i+1}}(X)$ *is the square of the generator of* $H^{2^i}(X)$.

(e) *The element* $h_i \in E_2^{1,2^i}$ *is a permanent cycle in the Adams spectral sequence.* □

Condition (b) is clearly true for $i = 0$, 1, 2 and 3, the division algebras being the reals $\mathbf{R}$, the complexes $\mathbf{C}$, the quaternions $\mathbf{H}$ and the Cayley numbers, which are nonassotiative. The problem for $i \geq 4$ is solved by

1.2.12. THEOREM (Adams [**12**]). *The conditions of* 1.2.11 *are false for* $i \geq 4$ *and in the Adams spectral sequence one has* $d_2(h_i) = h_0 h_{i-1}^2 \neq 0$ *for* $i \geq 4$. □

For $i = 4$ the above gives the first nontrivial differential in the Adams spectral sequence. Its target has dimension 14 and is related to the difficulty in Serre's method referred to above.

The analogous results for $p > 2$ are

1.2.13. THEOREM (Liulevicius [**2**] and Shimada and Yamanoshita [**3**]). *The following are equivalent.*

(a) *There is a map* $S^{2p^{i+1}-1} \to \widehat{S}^{2p^i}$ *with Hopf invariant one* (*see* 1.5.3 *for the definition of the Hopf invariant and the space* $\widehat{S}^{2m}$).

(b) *There is a* p*-cell complex* $X = S^{2p^i} \cup e^{4p^i} \cup e^{6p^i} \cup \cdots \cup e^{2p^{i+1}}$ [*the cofiber of the map in* (*a*)] *whose* mod (p) *cohomology is a truncated polynomial algebra on one generator.*

(c) *The element* $h_i \in E_2^{1,qp^i}$ *is a permanent cycle in the Adams spectral sequence.* □

The element h_0 is the first element in the Adams spectral sequence above dimension zero so it is a permanent cycle. The corresponding map in (a) suspends to the element of $\pi_{2p}(S^3)$ given by 1.2.4. For $i \geq 1$ we have

1.2.14. THEOREM (Liulevicius [**2**] and Shimada and Yamanoshita [**3**]). *The conditions of* 1.2.13 *are false for* $i \geq 1$ *and* $d_2(h_i) = a_0 b_{i-1}$, *where* b_{i-1} *is a generator of* E_2^{2,qp^i} (*see Section* 5.2). □

For $i = 1$ the above gives the first nontrivial differential in the Adams spectral sequence for $p > 2$. For $p = 3$ its target is in dimension 10 and was referred to above in our discussion of Serre's method.

FIG. 1.2.15 shows the Adams spectral sequence for $p = 3$ through dimension 45. We present it here mainly for comparison with a similar figure (1.2.19) for the Adams–Novikov spectral sequence. $E_2^{s,t}$ is a $\mathbf{Z}/(p)$ vector space in which each basis element is indicated by a small circle. Fortunately in this range there are just two bigradings [(5,28) and (8,43)] in which there is more than one basis element. The vertical coordinate is s, the cohomological degree, and the horizontal coordinate is $t - s$, the topological dimension. These extra elements appear in the chart to the right of where they should be, and the lines meeting them should be vertical. A d_r is indicated by a line which goes up by r and to the left by 1. The vertical lines represent multiplication by $a_0 \in E_2^{1,1}$ and the vertical arrow in dimension zero indicates that all powers of a_0 are nonzero. This multiplication corresponds to multiplication by p in the corresponding homotopy group. Thus from the figure one can read off $\pi_0 = \mathbf{Z}$, $\pi_{11} = \pi_{45} = \mathbf{Z}/(9)$, $\pi_{23} = \mathbf{Z}/(9) \oplus \mathbf{Z}/(3)$, and $\pi_{35} = \mathbf{Z}/(27)$. Lines that go up 1 and to the right by 3 indicate multiplication by $h_0 \in E_2^{1,4}$, while those that go to the right by 7 indicate the Massey product $\langle h_0, h_0, -\rangle$ (see A1.4.1). The elements a_0 and h_i for $i = 0$, 1, 2 were defined above and the elements $b_0 \in E_2^{2,12}$, $k_0 \in E_2^{2,28}$, and $b_1 \in E_2^{2,36}$ are up to the sign the Massey products $\langle h_0, h_0, h_0\rangle$, $\langle h_0, h_1, h_1\rangle$, and $\langle h_1, h_1, h_1\rangle$, respectively. The unlabeled elements in $E_2^{i,5i-1}$ for $i \geq 2$ (and $h_0 \in E_2^{1,4}$) are related to each other by the Massey product $\langle h_0, a_0, -\rangle$. This accounts for all of the generators except those in $E_2^{3,26}$, $E_2^{7,45}$ and $E_2^{8,50}$, which are too complicated to describe here.

We suggest that the reader take a colored pencil and mark all of the elements which survive to E_∞, i.e., those which are not the source or target of a differential. There are in this range 31 differentials which eliminate about two-thirds of the elements shown.

Now we consider the spectral sequence of Adams and Novikov, which is the main object of interest in this book. Before describing its construction we review the main ideas behind the Adams spectral sequence. They are the following.

1.2.16. Procedure. (i) Use mod (p)-cohomology as a tool to study the p-component of $\pi_*(X)$. (ii) Map X to an appropriate Eilenberg–Mac Lane space K, whose homotopy groups are known. (iii) Use knowledge of $H^*(K)$, i.e., of the Steenrod algebra, to get at the fiber of the map in (ii). (iv) Iterate the above and codify all information in a spectral sequence as in 1.2.10. □

An analogous set of ideas lies behind the Adams–Novikov spectral sequence, with mod p cohomology being replaced by complex cobordism theory. To elaborate, we first remark that "cohomology" in 1.2.16(i) can be replaced by "homology" and 1.2.10 can be reformulated accordingly; the details of this reformulation need not be discussed here. Recall that singular homology is based on the singular chain complex, which is generated by maps of simplices into the space X. Cycles in the chain complex are linear combinations of such maps that fit together in an appropriate way. Hence $H_*(X)$ can be thought of as the group of equivalence classes of maps of certain kinds of simplicial complexes, sometimes called "geometric cycles," into X.

Our point of departure is to replace these geometric cycles by closed complex manifolds. Here we mean "complex" in a very weak sense; the manifold M must be smooth and come equipped with a complex linear structure on its stable normal bundle, i.e., the normal bundle of some embedding of M into a Euclidean space of even codimension. The manifold M need not be analytic or have a complex structure on its tangent bundle, and it may be odd-dimensional.

The appropriate equivalence relation among maps of such manifolds into X is the following.

1.2.17. Definition. *Maps $f_i\colon M \to X$ $(i = 1, 2)$ of n-dimensional complex (in the above sense) manifolds into X are bordant if there is a map $g\colon W \to X$ where W is a complex mainfold with boundary $\partial W = M_1 \cup M_2$ such that $g|M_i = f_i$. (To be correct we should require the restriction to M_2 to respect the complex structure on M_2 opposite to the given one, but we can ignore such details here.)* □

One can then define a graded group $MU_*(X)$, the *complex bordism* of X, analogous to $H_*(X)$. It satisfies all of the Eilenberg–Steenrod axioms except the dimension axiom, i.e., $MU_*(\mathrm{pt})$, is not concentrated in dimension zero. It is by definition

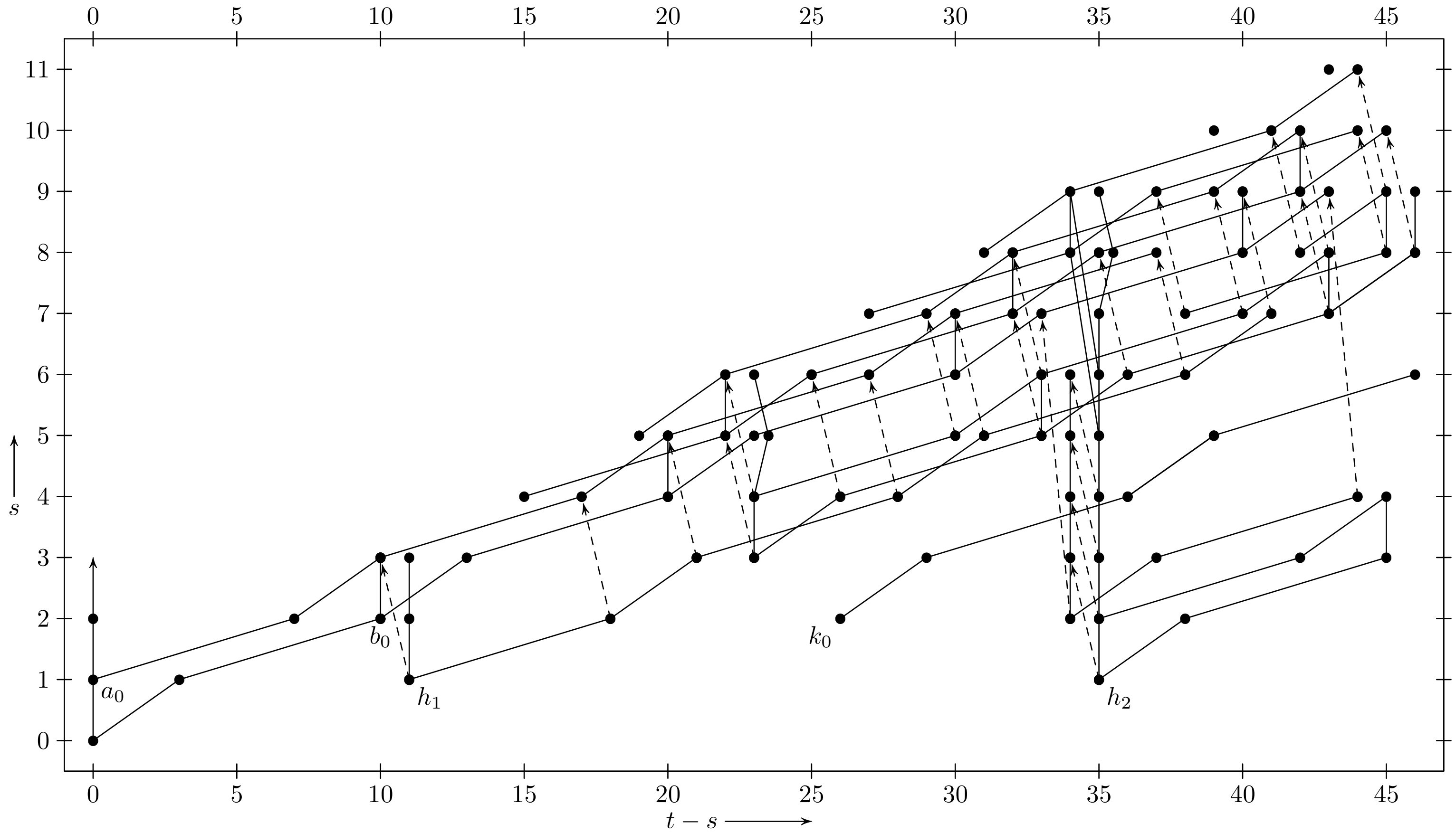

FIGURE 1.2.15. The Adams spectral sequence for $p = 3$, $t - s \leq 45$.

the set of equivalence classes of closed complex manifolds under the relation of 1.2.17 with $X = \mathrm{pt}$, i.e., without any condition on the maps. This set is a ring under disjoint union and Cartesian product and is called the *complex bordism ring.* as are the analogous rings for several other types of manifolds; see Stong [**1**].

1.2.18. THEOREM (Thom [**1**], Milnor [**4**], Novikov [**2**]). *The complex bordism ring,* $MU_*(\mathrm{pt})$, *is* $\mathbf{Z}[x_1, x_2, \dots]$ *where* $\dim x_i = 2i$. □

Now recall 1.2.16. We have described an analog of (i), i.e., a functor $MU_*(-)$ replacing $H_*(-)$. Now we need to modify (ii) accordingly, e.g., to define analogs of the Eilenberg–Mac Lane spaces. These spaces (or rather the corresponding spectrum MU) are described in Section 4.1. Here we merely remark that Thom's contribution to 1.2.18 was to equate $MU_i(\mathrm{pt})$ with the homotopy groups of certain spaces and that these spaces are the ones we need.

To carry out the analog of 1.2.16(iii) we need to know the complex bordism of these spaces, which is also described (stably) in Section 4.1. The resulting spectral sequence is formally introduced in Section 4.4, using constructions given in Section 2.2. We will not state the analog of 1.2.10 here as it would be too much trouble to develop the necessary notation. However we will give a figure analogous to 1.2.15.

The notation of FIG. 1.2.19 is similar to that of FIG. 1.2.15 with some minor differences. The E_2-term here is not a $\mathbf{Z}/(3)$-vector space. Elements of order > 3 occur in $E_2^{0,0}$ (an infinite cyclic group indicated by a square), and in $E_2^{1,12t}$ and $E_2^{3,48}$, in which a generator of order 3^{k+1} is indicated by a small circle with k parentheses to the right. The names α_t, β_t, and $\beta_{s/t}$ will be explained in the next section. The names α_{3t} refer to elements of order 3 in, rather than generators of, $E_2^{1,12t}$. In $E_2^{3,48}$ the product $\alpha_1\beta_3$ is divisible by 3.

One sees from these two figures that the Adams–Novikov spectral sequence has far fewer differentials than the Adams spectral sequence. The first nontrivial Adams–Novikov differential originates in dimension 34 and leads to the relation $\alpha_1\beta_1^3$ in $\pi_*(S^0)$. It was first established by Toda [**2, 3**].

3. The Adams–Novikov E_2-term, Formal Group Laws, and the Greek Letter Construction

Formal group laws and Qillen's theorem. The Adams–Novikov E_2-term as group cohomology. Alphas, beta and gamma. The Morava–Landweber theorem and higher Greek letters. Generalized Greek letter elements.

In this section we will describe the E_2-term of the Adams–Novikov spectral sequence introduced at the end of the previous section. We begin by defining formal group laws (1.3.1) and describing their connection with complex cobordism (1.3.4). Then we characterize the E_2-term in terms of them (1.3.5 and 1.3.6). Next we describe the Greek letter construction, an algebraic method for producing periodic families of elements in the E_2-term. We conclude by commenting on the problem of representing these elements in $\pi_*(S)$.

Suppose T is a one-dimensional commutative analytic Lie group and we have a local coordinate system in which the identity element is the origin. Then the group operation $T \times T \to T$ can be described locally as a real-valued analytic function of two variables. Let $F(x, y) \in \mathbf{R}[[x, y]]$ be the power series expansion of this function about the origin. Since 0 is the identity element we have

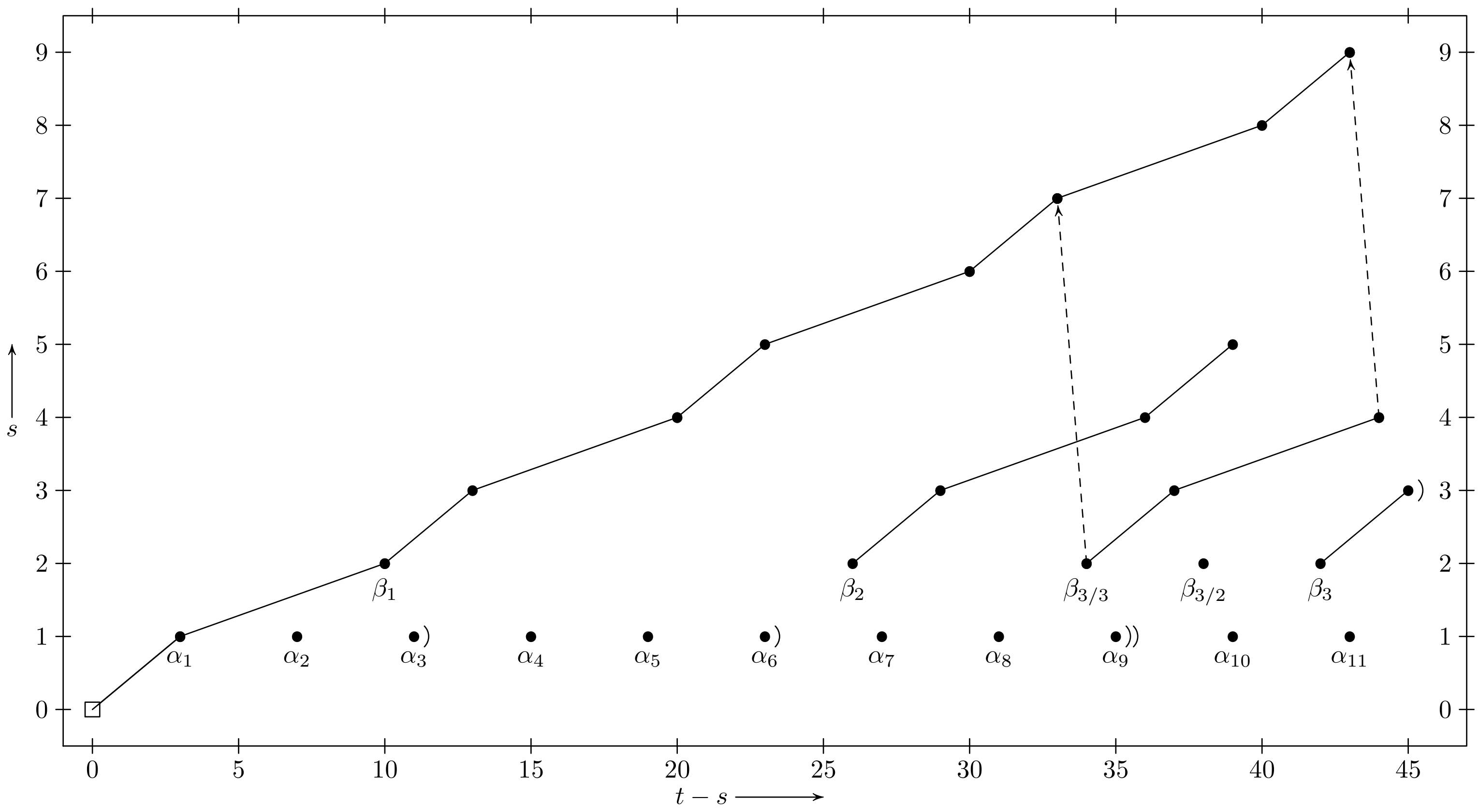

FIGURE 1.2.19. The Adams–Novikov spectral sequence for $p = 3$, $t - s \leq 45$

$F(x,0) = F(0,x) = x$. Commutativity and associativity give $F(x,y) = F(y,x)$ and $F(F(x,y),z) = F(x,F(y,z))$, respectively.

1.3.1. DEFINITION. *A* formal group law *over a commutative ring with unit R is a power series $F(x,y) \in R[[x,y]]$ satisfying the three conditions above.* □

Several remarks are in order. First, the power series in the Lie group will have a positive radius of convergence, but there is no convergence condition in the definition above. Second, there is no need to require the existence of an inverse because it exists automatically. It is a power series $i(x) \in R[[x]]$ satisfying $F(x, i(x)) = 0$; it is an easy exercise to solve this equation for $i(x)$ given F. Third, a rigorous self-contained treatment of the theory of formal group laws is given in Appendix 2.

Note that $F(x,0) = F(0,x) = x$ implies that $F \equiv x + y \mod (x,y)^2$ and that $x+y$ is therefore the simplest example of an formal group law; it is called the *additive* formal group law and is denoted by F_{a}. Another easy example is the *multiplicative* formal group law, $F_{\mathrm{m}} = x+y+rxy$ for $r \in \mathbf{R}$. These two are known to be the only formal group laws which are polynomials. Other examples are given in A2.1.4.

To see what formal group laws have to do with complex cobordism and the Adams–Novikov spectral sequence, consider $MU^*(\mathbf{C}P^\infty)$, the complex cobordism of infinite-dimensional complex projective space. Here $MU^*(-)$ is the cohomology theory dual to the homology theory $MU_*(-)$ (complex bordism) described in Section 2. Like ordinary cohomology it has a cup product and we have

1.3.2. THEOREM. *There is an element $x \in MU^2(\mathbf{C}P^\infty)$ such that*

$$MU^*(\mathbf{C}P^\infty) = MU^*(\mathrm{pt})[[x]]$$

and

$$MU^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = MU^*(\mathrm{pt})[[x \otimes 1, 1 \otimes x]].$$

□

Here $MU^*(\mathrm{pt})$ is the complex cobordism of a point; it differs from $MU_*(\mathrm{pt})$ (described in 1.2.18) only in that its generators are negatively graded. The generator x is closely related to the usual generator of $H^2(\mathbf{C}P^\infty)$, which we also denote by x. The alert reader may have expected $MU^*(\mathbf{C}P^\infty)$ to be a polynomial rather than a power series ring since $H^*(\mathbf{C}P^\infty)$ is traditionally described as $\mathbf{Z}[x]$. However, the latter is really $\mathbf{Z}[[x]]$ since the cohomology of an infinite complex maps onto the inverse limit of the cohomologies of its finite skeleta. [$MU^*(\mathbf{C}P^n)$, like $H^*(\mathbf{C}P^n)$, is a truncated polynomial ring.] Since one usually considers only homogeneous elements in $H^*(\mathbf{C}P^\infty)$, the distinction between $\mathbf{Z}[x]$ and $\mathbf{Z}[[x]]$ is meaningless. However, one can have homogeneous infinite sums in $MU^*(\mathbf{C}P^\infty)$ since the coefficient ring is negatively graded.

Now $\mathbf{C}P^\infty$ is the classifying space for complex line bundles and there is a map $\mu\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ corresponding to the tensor product; in fact, $\mathbf{C}P^\infty$ is known to be a topological abelian group. By 1.3.2 the induced map μ^* in complex cobordism is determined by its behavior on the generator $x \in MU^2(\mathbf{C}P^\infty)$ and one easily proves, using elementary facts about line bundles,

1.3.3. PROPOSITION. *For the tensor product map $\mu\colon \mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$, $\mu^*(x) = F_U(x \otimes 1, 1 \otimes x) \in MU^*(\mathrm{pt})[[x \otimes 1, 1 \otimes x]]$ is an formal group law over $MU^*(\mathrm{pt})$.* □

A similar statement is true of ordinary cohomology and the formal group law one gets is the additive one; this is a restatement of the fact that the first Chern class of a tensor product of complex line bundles is the sum of the first Chern classes of the factors. One can play the same game with complex K-theory and get a multiplicative formal group law.

$\mathbf{C}P^\infty$ is a good test space for both complex cobordism and K-theory. One can analyze the algebra of operations in both theories by studying their behavior in $\mathbf{C}P^\infty$ (see Adams [**5**]) in the same way that Milnor [**2**] analyzed the mod (2) Steenrod algebra by studying its action on $H^*(\mathbf{R}P^\infty; \mathbf{Z}/(2))$. (See also Steenrod and Epstein [**1**].)

The formal group law of 1.3.3 is not as simple as the ones for ordinary cohomology or K-theory; it is complicated enough to have the following universal property.

1.3.4. Theorem (Quillen [**2**]). *For any formal group law F over any commutative ring with unit R there is a unique ring homomorphism $\theta\colon MU^*(\mathrm{pt}) \to R$ such that $F(x,y) = \theta F_U(x,y)$.* □

We remark that the existence of such a universal formal group law is a triviality. Simply write $F(x,y) = \sum a_{i,j} x^i y^i$ and let $L = \mathbf{Z}[a_{i,j}]/I$, where I is the ideal generated by the relations among the $a_{i,j}$ imposed by the definition 1.3.1 of an formal group law. Then there is an obvious formal group law over L having the universal property. Determining the explicit structure of L is much harder and was first done by Lazard [**1**]. Quillen's proof of 1.3.4 consisted of showing that Lazard's universal formal group law is isomorphic to the one given by 1.3.3.

Once Quillen's Theorem 1.3.4 is proved, the manifolds used to define complex bordism theory become irrelevant, however pleasant they may be. All of the applications we will consider follow from purely algebraic properties of formal group laws. This leads one to suspect that the spectrum MU can be constructed somehow using formal group law theory and without using complex manifolds or vector bundles. Perhaps the corresponding infinite loop space is the classifying space for some category defined in terms of formal group laws. Infinite loop space theorists, where are you?

We are now just one step away from a description of the Adams–Novikov spectral sequence E_2-term. Let $G = \{f(x) \in \mathbf{Z}[[x]] \mid f(x) \equiv x \mod (x)^2\}$. Here G is a group under composition and acts on the Lazard/complex cobordism ring $L = MU_*(\mathrm{pt})$ as follows. For $g \in G$ define an formal group law F_f over L by $F_g(x,y) = g^{-1}F_U(g(x), g(y))$. By 1.3.4 F_g is induced by a homomorphism $\theta_g\colon L \to L$. Since g is invertible under composition, θ_g is an automorphism and we have a G-action on L.

Note that $g(x)$ defines an isomorphism between F and F_g. In general, isomorphisms between formal group laws are induced by power series $g(x)$ with leading term a unit multiple (not necessarily one) of x. An isomorphism induced by a g in G is said to be *strict*.

1.3.5. Theorem. *The E_2-term of the Adams–Novikov spectral sequence converging to π_*^S is isomorphic to $H^{**}(G; L)$.* □

There is a difficulty with this statement: since G does not preserve the grading on L, there is no obvious bigrading on $H^*(G; L)$. We need to reformulate in terms of L as a comodule over a certain Hopf algebra B defined as follows.

Let $g \in G$ be written as $g(x) = \sum_{i \geq 0} b_i x^{i+1}$ with $b_0 = 1$. Each b_i for $i > 0$ can be thought of as a $\mathbf{Z}$-valued function on G and they generate a graded algebra of such functions

$$B = \mathbf{Z}[b_1, b_2, \dots] \quad \text{with } \dim b_i = 2i.$$

(Do not confuse this ring with L, to which it happens to be isomorphic.) The group structure on G corresponds to a coproduct $\Delta\colon B \to B \otimes B$ on B given by $\Delta(b) = \sum_{i \geq 0} b^{i+1} \otimes b_i$, where $b = \sum_{i \geq 0} b_i$ and $b_0 = 1$ as before. To see this suppose $g(x) = g^{(1)}(g^{(2)}(x))$ with $g^{(k)}(x) = \sum b_i^{(k)} x^{i+1}$ Then we have

$$\sum b_i x^{i+1} = \sum b_i^{(1)} \left(\sum b_j^{(2)} x^{j+1} \right)^{i+1}$$

from which the formula for Δ follows. This coproduct makes B into a graded connected Hopf algebra over which L is a graded comodule. We can restate 1.3.5 as

1.3.6. THEOREM. *The E_2-term of the Adams–Novikov spectral sequence converging to $\pi_*(S)$ is given by $E_2^{s,t} = \mathrm{Ext}_B^{s,t}(\mathbf{Z}, L)$.* □

The definition of this Ext is given in A1.2.3; all of the relevant homological algebra is discussed in Appendix 1.

Do not be alarmed if the explicit action of G (or coaction of B) on L is not obvious to you. It is hard to get at directly and computing its cohomology is a very devious business.

Next we will describe the *Greek letter construction*, which is a method for producing lots (but by no means all) of elements in the E_2-term, including the α_t's and β_t's seen in 1.2.19. We will use the language suggested by 1.3.5; the interested reader can translate our statements into that of 1.3.6. Our philosophy here is that group cohomology in positive degrees is too hard to comprehend, but $H^0(G; M)$ (the G-module M will vary in the discussion), the submodule of M fixed by G, is relatively straightforward. Hence our starting point is

1.3.7. THEOREM. *$H^0(G; L) = \mathbf{Z}$ concentrated in dimension 0.* □

This corresponds to the 0-stem in stable homotopy. Not a very promising beginning you say? It does give us a toehold on the problem. It tells us that the only principal ideals in L which are G-invariant are those generated by integers and suggests the following. Fix a prime number p and consider the short exact sequence of G-modules

$$0 \to L \xrightarrow{p} L \to L/(p) \to 0. \tag{1.3.8}$$

We have a connecting homomorphism

$$\delta_0\colon H^i(G; L/(p)) \to H^{i+1}(G; L).$$

1.3.9. THEOREM. *$H^0(G; L/(p)) = \mathbf{Z}/(p)[v_1]$, where $v_1 \in L$ has dimension $q = 2(p-1)$.* □

1.3.10. DEFINITION. *For $t > 0$ let $\alpha_t = \delta_0(v_1^t) \in E_2^{1,qt}$.* □

It is clear from the long exact sequence in cohomology associated with 1.3.8 that $\alpha_t \neq 0$ for all $t > 0$, so we have a collection of nontrivial elements in the Adams–Novikov E_2-term. We will comment below on the problems of constructing corresponding elements in $\pi_*(S)$; for now we will simply state the result.

1.3.11. THEOREM. (a) (Toda [**4**, IV]) *For $p > 2$ each α_t is represented by an element of order p in $\pi_{qt-1}(S)$ which is in the image of the J-homomophism* (1.1.12).

(b) *For $p = 2$ α_t is so represented provided $t \not\equiv 3 \mod (4)$. If $t \equiv 2 \mod (4)$ then the element has order 4; otherwise it has order 2. It is in* $\operatorname{im} J$ *if t is even.* $\square$

Theorem 1.3.9 tells us that

$$0 \to \Sigma^q L/(p) \xrightarrow{v_1} L/(p) \to L/(p, v_1) \to 0 \tag{1.3.12}$$

is an short exact sequence of G-modules and there is a connecting homomorphism

$$\delta_1 \colon H^i(G; L/(p, v_1)) \to H^{i+1}(G; L/(p)).$$

The analogs of 1.3.9 and 1.3.10 are

1.3.13. THEOREM. *$H^0(G; L/(p, v_1)) = \mathbf{Z}/(p)[v_2]$ where $v_2 \in L$ has dimension $2(p^2 - 1)$.* $\square$

1.3.14. DEFINITION. *For $t > 0$ let $\beta_t = \delta_0\delta_1(v_2^t) \in E_2^{2,t(p+1)q-q}$.* $\square$

More work is required to show that these elements are nontrivial for $p > 2$, and $\beta_1 = 0$ for $p = 2$. The situation in homotopy is

1.3.15. THEOREM (Smith [**1**]). *For $p \geq 5$ β_t is represented by a nontrivial element of order p in $\pi_{(p+1)tq-q-2}(S^0)$.* $\square$

You are probably wondering if we can continue in this way and construct γ_t, δ_t, etc. The following results allow us to do so.

1.3.16. THEOREM (Morava [**3**], Landweber [**4**]). (a) *There are elements $v_n \in L$ of dimension $2(p^n - 1)$ such that $I_n = (p, v_1, v_2, \ldots, v_{n-1}) \subset L$ is a G-invariant prime ideal for all $n > 0$.*

(b) *$0 \to \Sigma^{2(p^n-1)} L/I_n \xrightarrow{v_n} L/I_n \to L/I_{n+1} \to 0$ is an short exact sequence of modules with connecting homorphism*

$$\delta \colon H^i(G; L/I_{n+1}) \to H^{i+1}(G; L/I_n).$$

(c) *$H^0(G; L/I_n) = \mathbf{Z}/(p)[v_n]$.*

(d) *The only G-invariant prime ideals in L are the I_n for $0 < n \leq \infty$ for all primes p.* $\square$

Part (d) above shows how rigid the G-action on L is; there are frightfully many prime ideals in L, but only the I_n for various primes are G-invariant. Using (b) and (c) we can make

1.3.17. DEFINITION. *For $t, n > 0$ let $\alpha_t^{(n)} = \delta_0\delta_1 \ldots \delta_{n-1}(v_n^t) \in E_2^{n,*}$.* $\square$

Here $\alpha^{(n)}$ stands for the nth letter of the Greek alphabet, the length of which is more than adequate given our current state of knowledge. The only other known result comparable to 1.3.11 or 1.3.15 is

1.3.18. THEOREM. (a) (Miller, Ravenel, and Wilson [**1**]) *The element $\gamma_t \in E_2^{3,tq(p^2+p+1)-q(p+2)}$ is nontrivial for all $t > 0$ and $p > 2$.*

(b) (Toda [**1**]) *For $p \geq 7$ each γ_t is represented by a nontrivial element of order p in $\pi_{tq(p^2+p+1)-q(p+2)-3}(S^0)$.* $\square$

It is known that not all γ_t exist in homotopy for $p = 5$ (see 7.6.1). Part (b) above was proved several years before part (a). In the intervening time there was a controversy over the nontriviality of γ_1 which was unresolved for over a year, ending in 1974 (see Thomas and Zahler [**1**]). This unusual state of affairs attracted the attention of the editors of *Science* [**1**] and the *New York Times* [**1**], who erroneously cited it as evidence of the decline of mathematics.

We conclude our discussion of the Greek letter construction by commenting briefly on *generalized* Greek letter elements. Examples are $\beta_{3/3}$ and $\beta_{3/2}$ (and the elements in $E_2^{1,*}$ of order > 3) in 1.2.19. The elements come via connecting homomorphisms from $H^0(G; L/J)$, where J is a G-invariant *regular* (instead of prime) ideal. Recall that a regular ideal $(x_0, x_1, \ldots, x_{n-1}) \subset L$ is one in which each x_i is not a zero divisor modulo $(x_0, \ldots, x_{i-1})$. Hence G-invariant prime ideals are regular as are ideals of the form $(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$. Many but not all G-invariant regular ideals have this form.

1.3.19. DEFINITION. *$\beta_{s/t}$ (for appropriate s and t) is the image of $v_2^s \in H^0(G; L/(p, v_1^t))$ and $\alpha_{s/t}$ is the image of $v_1^s \in H^0(G; L/(p^t))$.* □

Hence $p\alpha_{s/t} = \alpha_{s/t-1}$, $\alpha_{s/1} = \alpha_s$, and $\beta_{t/1} = \beta_t$ by definition.

Now we will comment on the problem of representing these elements in the E_2-term by elements in stable homotopy, e.g., on the proofs of 1.3.11, 1.3.15, and 1.3.18(b). The first thing we must do is show that the elements produced are actually nontrivial in the E_2-term. This has been done only for α's, β's, and γ's. For $p = 2$, β_1 and γ_1 are zero but for $t > 1$ β_t and γ_t are nontrivial; these results are part of the recent computation of $E_2^{2,*}$ at $p = 2$ by Shimomura [**1**], which also tells us which generalized β's are defined and are nontrivial. The corresponding calculation at odd primes was done in Miller, Ravenel, and Wilson [**1**], as was that of $E_2^{1,*}$ for all primes.

The general strategy for representing Greek letter elements geometrically is to realize the relevant short exact sequences [e.g., 1.3.8, 1.3.12, and 1.3.16(b)] by cofiber sequences of finite spectra. For any connective spectrum X there is an Adams–Novikov spectral sequence converging to $\pi_*(X)$. Its E_2-term [denoted by $E_2(X)$] can be described as in 1.3.5 with $L = MU_*(S^0)$ replaced by $MU_*(X)$, which is a G-module. For 1.3.8 we have a cofiber sequence

$$S^0 \xrightarrow{p} S^0 \to V(0),$$

where $V(0)$ is the mod (p) Moore spectrum. It is known (2.3.4) that the long exact sequence of homotopy groups is compatible with the long exact sequence of E_2-terms. Hence the elements v_1^t of 1.3.9 live in $E_2^{0,qt}(V(0))$ and for 1.3.11(a) [which says α_t is represented by an element of order p in $\pi_{qt-1}(S^0)$ for $p > 2$ and $t > 0$] it would suffice to show that these elements are permanent cycles in the Adams–Novikov spectral sequence for $\pi_*(V(0))$ with $p > 0$. For $t = 1$ (even if $p = 2$) one can show this by brute force; one computes $E_2(V(0))$ through dimension q and sees that there is no possible target for a differential coming from $v_1 \in E_2^{0,q}$. Hence v_1 is realized by a map

$$S^q \to V(0)$$

If we can extend it to $\Sigma^q V(0)$, we can iterate and represent all powers of v_1. We can try to do this either directly, using obstruction theory, or by showing that $V(0)$ is a ring spectrum spectrum. In the latter case our extension α would be the composite

$$S^q \wedge V(0) \to V(0) \wedge V(0) \to V(0),$$

where the first map is the original map smashed with the identity on $V(0)$ and the second is the multiplication on $V(0)$. The second method is generally (in similar situation of this sort) easier because it involves obstruction theory in a lower range of dimensions.

In the problem at hand both methods work for $p > 2$ but both fail for $p = 2$. In that case $V(0)$ is not a ring spectrum and our element in $\pi_2(V(0))$ has order 4, so it does not extend to $\Sigma^2 V(0)$. Further calculations show that v_1^2 and v_1^3 both support nontrivial differentials (see 5.3.13) but v_1^4 is a permanent cycle represented by map $S^8 \to V(0)$, which *does* extend to $\Sigma^8 V(0)$. Hence iterates of this map produce the homotopy elements listed in 1.3.11(b) once certain calculation have been made in dimensions ≤ 8.

For $p > 2$ the map $\alpha\colon \Sigma^q V(0) \to V(0)$ gives us a cofibre sequence

$$\Sigma^q V(0) \xrightarrow{\alpha} V(0) \to V(1),$$

realizing the short exact sequence 1.3.12. Hence to arrive at 1.3.15 (which describes the β's in homotopy) we need to show that $v_2 \in E_2^{0,(p+1)q}(V(1))$ is a permanent cycle represented by a map which extends to $\beta\colon \Sigma^{(p+1)q} V(1) \to V(1)$. We can do this for $p \geq 5$ but not for $p = 3$. Some partial results for β's at $p = 3$ and $p = 2$ are described in Section 5.5.

The cofiber of the map β (corresponding to v_2) for $p \geq 5$ is called $V(2)$ by Toda [**1**]. In order to construct the γ's [1.3.18(b)] one needs a map

$$\gamma\colon \Sigma^{2(p^3-1)} V(2) \to V(2)$$

corresponding to v_3. Toda [**1**] produces such a map for $p \geq 7$ but it is known not to exist for $p = 5$ (see 7.6.1).

Toda [**1**] first considered the problem of constructing the spectra $V(n)$ above, and hence of the representation of Greek letter elements in $\pi_*(S)$, although that terminology (and 1.3.16) was not available at the time. While the results obtained there have not been surpassed, the methods used leave something to be desired. Each positive result is proved by brute force; the relevant obstruction groups are shown to be trivial. This approach can be pushed no further; the obstruction to realizing v_4 lies in a nontrivial group for all primes (5.6.13). Homotopy theorists have yet to learn how to compute obstructions in such situations.

The negative results of Toda [**1**] are proved by ingenious but *ad hoc* methods. The nonexistence of $V(1)$ for $p = 2$ follows easily from the structure of the Steenrod algebra; if it existed its cohomology would contradict the Adem relation $Sq^2Sq^2 = Sq^1Sq^2Sq^1$. For the nonexistence of $V(2)$ at $p = 3$ Toda uses a delicate argument involving the nonassociativity of the mod (3) Moore spectrum, which we will not reproduce here. We will give another proof (5.5.1) which uses the multiplicative structure of the Adams–Novikov E_2-term to show that the nonrealizability of $\beta_4 \in E_2^{2,60}$, and hence of $V(2)$, is a formal consequence of that of $\beta_{3/3} \in E_2^{2,36}$. This was shown by Toda [**2, 3**] using an extended power construction, which will also not be reproduced here. Indeed, all of the differentials in the Adams–Novikov spectral sequence for $p = 3$ in the range we consider are formal consequences of that one in dimension 34. A variant of the second method used for $V(2)$ at $p = 3$ works for $V(3)$ (the cofiber of γ) at $p = 5$.

4. More Formal Group Law Theory, Morava's Point of View, and the Chromatic Spectral Sequence

The Brown–Peterson spectrum. Classification of formal group laws. Morava's group action, its orbits and stabilizers. The chromatic resolution and the chromatic spectral sequence. Bockstein spectral sequences. Use of cyclic subgroups to detect Arf invariant elements. Morava's vanishing theorem. Greek letter elements in the chromatic spectral sequence.

We begin this section by introducing BP-theory, which is essentially a p-local form of MU-theory. With it many of the explicit calculations behind our results become a lot easier. Most of the current literature on the subject is written in terms of BP rather than MU. On the other hand, BP is not essential for the overall picture of the E_2-term we will give later, so it could be regarded as a technicality to be passed over by the casual reader. Next we will describe the classification of formal group laws over an algebraically closed field of characteristic p. This is needed for Morava's point of view, which is a useful way of understanding the action of G on L (1.3.5). The insights that come out of this approach are made computationally precise in the chromatic spectral sequence , which is the pivotal idea in this book. Technically the chromatic spectral sequence is a trigraded spectral sequence converging to the Adams–Novikov E_2-term; heuristically it is like a spectrum in the astronomical sense in that it resolves the E_2-term into various components each having a different type of periodicity. In particular, it incorporates the Greek letter elements of the previous section into a broader scheme which embraces the entire E_2-term.

BP-theory began with Brown and Peterson [**1**] (after whom it is named), who showed that after localization at any prime p, the MU spectrum splits into an infinite wedge suspension of identical smaller spectra subsequently called BP. One has

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \dots], \tag{1.4.1}$$

where $\mathbf{Z}_{(p)}$ denotes the integers localized at p and the v_n's are the same as the generators appearing in the Morava–Landweber theorem 1.3.16. Since $\dim v_n = 2(p^n - 1)$, this coefficient ring, which we will denote by BP_*, is much smaller than $L = \pi_*(MU)$, which has a polynomial generator in every even dimension.

Next Quillen [**2**] observed that there is a good formal group law theoretic reason for this splitting. A theorem of Cartier [**1**] (A2.1.18) says that every formal group law over a $\mathbf{Z}_{(p)}$-algebra is canonically isomorphic to one in a particularly convenient form called a p-typical formal group law (see A2.1.17 and A2.1.22 for the definition, the details of which need not concern us now). This canonical isomorphism is reflected topologically in the above splitting of the localization of MU. This fact is more evidence in support of our belief that MU can somehow be constructed in purely formal group law theoretic terms.

There is a p-typical analog of Quillen's theorem 1.3.4; i.e., $BP^*(\mathbf{C}P^\infty)$ gives us a p-typical formal group law with a similar universal property. Also, there is a BP analog of the Adams–Novikov spectral sequence, which is simply the latter tensored with $\mathbf{Z}_{(p)}$; i.e., its E_2-term is the p-component of $H^*(G; L)$ and it converges to the p-component of $\pi_*(S)$ However, we encounter problems in trying to write an analog of our metaphor 1.3.5 because *there is no p-typical analog of the group G.*

In other words there is no suitable group of power series over $\mathbf{Z}_{(p)}$ which will send any p-typical formal group law into another. Given a p-typical formal group law F over $\mathbf{Z}_{(p)}$ there is a set of power series $g \in \mathbf{Z}_{(p)}[[x]]$ such that $g^{-1}F(g(x), g(y))$ is also p-typical, but this set depends on F. Hence $\mathrm{Hom}(BP_*, K)$ the set of p-typical formal group laws over a $\mathbf{Z}_{(p)}$-algebra K, is acted on not by a group analogous to G, but by a *groupoid*.

Recall that a groupoid is a small category in which every morphism is an equivalence, i.e., it is invertible. A groupoid with a single object is a group. In our case the objects are p-typical formal group laws over K and the morphisms are isomorphisms induced by power series $g(x)$ with leading term x.

Now a Hopf algebra, such as B in 1.3.6, is a *cogroup object* in the category of commutative rings R, which is to say that $\mathrm{Hom}(B, R) = G_R$ is a group-valued functor. In fact G_R is the group (under composition) of power series $f(x)$ over R with leading term x. For a p-typical analog of 1.3.6 we need to replace b by *co-groupoid object* in the category of commutative $\mathbf{Z}_{(p)}$-algebras K. Such an object is called a *Hopf algebroid* (A1.1.1) and consists of a pair (A, Γ) of commutative rings with appropriate structure maps so that $\mathrm{Hom}(A, K)$ and $\mathrm{Hom}(\Gamma, K)$ are the sets of objects and morphisms, respectively, of a groupoid. The groupoid we have in mind, of course, is that of p-typical formal group laws and isomorphisms as above. Hence BP_* is the appropriate choice for A; the choice for Γ turns out to be $BP_*(BP)$, the BP-homology of the spectrum BP. Hence the p-typical analog of 1.3.6 is

1.4.2. Theorem. *The p-component of the E_2-term of the Adams–Novikov spectral sequence converging to $\pi_*(S)$ is*

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*). \qquad \square$$

Again this Ext is defined in A1.2.3 and the relevant homological algebra is discussed in Appendix 1.

We will now describe the classification of formal group laws over an algebraically closed field of characteristic p. First we define power series $[m]_F(x)$ associated with an formal group law F and natural numbers m. We have $[0]_F(x) = 0$, $[1]_F(x) = x$, and $[m]_F(x) = F(x, [m-1]_F(x))$. An easy lemma (A2.1.6) says that if F is defined over a field of characteristic p, then $[p]_F(x)$ is in fact a power series over x^{p^n} with leading term ax^{p^n}, $a \neq 0$, for some $n > 0$, provided F is not isomorphic to the additive formal group law, in which case $[p]_F(x) = 0$. This integer n is called the *height* of F, and the height of the additive formal group law is defined to be ∞. Then we have

1.4.3. Classification Theorem (Lazard [**2**]).

(a) *Two formal group laws defined over the algebraic closure of $\mathbf{F}_p$ are isomorphic iff they have the same height.*

(b) *If F is nonadditive, its height is the smallest n such that $\theta(v_n) \neq 0$, where $\theta\colon L \to K$ is the homomorphism of* 1.3.4 *and $v_n \in L$ is as in* 1.3.16, *where K is finite field.* $\square$

Now we come to Morava's point of view. Let $K = \overline{\mathbf{F}}_p$, the algebraic closure of the field with p elements, and let $G_K \subset K[[x]]$ be the group (under composition) of power series with leading term x. We have seen that G_K acts on $\mathrm{Hom}(L, K)$, the set formal group laws defined over K. Since L is a polynomial ring, we can think of $\mathrm{Hom}(L, K)$ as an infinite-dimensional vector space V over K; a set of polynomial generators of L gives a topological basis of V. For a vector $v \in V$, let F_v be the corresponding formal group law.

Two vectors in V are in the same orbit iff the corresponding formal group laws are strictly isomorphic (strict isomorphism was defined just prior to 1.3.5), and the stabilizer group of $v \in V$ (i.e., the subgroup of G_K leaving V fixed) is the strict automorphism group of F_v. This group S_n (where n is the height) can be described explicitly (A2.2.17); it is a profinite group of units in a certain p-adic division algebra, but the details need not concern us here. Theorem 1.4.3 enables us to describe the orbits explicitly.

1.4.4. THEOREM. *There is one G_K-orbit of V for each height as in* 1.4.3. *The height n orbit V_n is the subset defined by $v_i = 0$ for $i < n$ and $v_n \neq 0$.* □

Now observe that V is the set of closed points in $\mathrm{Spec}(L_n \otimes K)$, and V_n is the set of closed points in $\mathrm{Spec}(L_n \otimes K)$, where $L_n = v_n^{-1}L/I_n$. Here V_n is a homogeneous G_K-space and a standard change-of-rings argument gives

1.4.5. CHANGE-OF-RINGS THEOREM. $H^*(G_K; L_n \otimes K) = H^*(S_n; K)$. □

We will see in Chapter 6 that a form of this isomorphism holds over $\mathbf{F}_p$ as well as over K. In it the right-hand term is the cohomology of a certain Hopf algebra [called the nth Morava stabilizer algebra $\Sigma(n)$] defined over $\mathbf{F}_p$, which, when tensored with $\mathbf{F}_{p^n}$, becomes isomorphic to the dual of $\mathbf{F}_{p^n}[S_n]$, the $\mathbf{F}_{p^n}$-group algebra of S_n.

Now we are ready to describe the central construction of this book, the chromatic spectral sequence, which enables us to use the results above to get more explicit information about the Adams–Novikov E_2-term. We start with a long exact sequence of G-modules, called the chromatic resolution

$$0 \to L \otimes \mathbf{Z}_{(p)} \to M^0 \to M^1 \to \cdots \tag{1.4.6}$$

defined as follows. $M^0 = L \otimes \mathbf{Q}$, and N^1 is the cokernel in the short exact sequence

$$0 \to L \otimes \mathbf{Z}_{(p)} \to M^0 \to N^1 \to 0.$$

M^n and N^n are defined inductively for $n > 0$ by short exact sequences

$$0 \to N^n \to M^n \to N^{n+1} \to 0, \tag{1.4.7}$$

where $M^n = v_n^{-1}N^n$. Hence we have

$$N^1 = L \otimes \mathbf{Q}/\mathbf{Z}_{(p)} = \varinjlim L/(p^i) = L/(p^\infty)$$

and

$$N^{n+1} = \varinjlim L/(p^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n}) = L/(p^\infty, v_1^\infty, \ldots, v_n^\infty).$$

The fact that these are short exact sequences of G-modules is nontrivial. The long exact sequence 1.4.6 is obtained by splicing together the short exact sequences 1.4.7. In Chapter 5, where the chromatic spectral sequence is described in detail, M^n and N^n denote the corresponding objects defined in terms of BP_*. In what follows here $\mathrm{Ext}_B(\mathbf{Z}, M)$ will be abbreviated by $\mathrm{Ext}(M)$ for a B-module (e.g., G-module) M. Standard homological algebra (A1.3.2) gives

1.4.8. PROPOSITION. *There is a spectral sequence converging to* $\mathrm{Ext}(L \otimes \mathbf{Z}_{(p)})$ *with* $E_1^{n,s} = \mathrm{Ext}^s(M^n)$, $d_r\colon E_r^{n,s} \to E_r^{n+r,s-r+1}$, *and* $d_1\colon \mathrm{Ext}(M^n) \to \mathrm{Ext}(M^{n+1})$ *being induced by the maps* $M^n \to M^{n+1}$ *in* 1.4.6. [$E_\infty^{n,s}$ *is a subquotient of* $\mathrm{Ext}^{n+s}(L \otimes \mathbf{Z}_{(p)})$.] □

This is the *chromatic spectral sequence.* We can use 1.4.5 to get at its E_1 term as follows. Define G-modules M_i^n for $0 \le i \le n$ by $M_0^n = M^n$, and M_i^n is the kernel in the short exact sequence

$$0 \to M_i^n \to M_{i-1}^n \xrightarrow{v_{i-1}} M_{i-1}^n \to 0, \tag{1.4.9}$$

where $v_0 = p$. This gives $M_n^n = L_n = v_n^{-1}L/I_n$, so the $\mathbf{F}_p$-analog of 1.4.5 describes $\mathrm{Ext}(M_n^n)$ in terms of the cohomology of the stabilizer group S_n. Equation 1.4.9 gives a long exact sequence of Ext groups of a Bockstein spectral sequence computing $\mathrm{Ext}(M_{i-1}^n)$ in terms of $\mathrm{Ext}(M_i^n)$. Hence in principle we can get from $H^*(S_n)$ to $\mathrm{Ext}(M^n)$, although the Bockstein spectral sequences are difficult to handle in practice.

Certain general facts about $H^*(S_n)$ are worth mentioning here. If $(p-1)$ divides n then this cohomology is periodic (6.2.10); i.e., there is an element $c \in H^*(S_n; \mathbf{F}_p)$ such that $H^*(S_n; \mathbf{F}_p)$ is a finitely generated free module over $\mathbf{F}_p[c]$. In this case S_n has a cyclic subgroup of order p to whose cohomology c restricts nontrivially. This cohomology can be used to detect elements in the Adams–Novikov E_2-term of high cohomological degree, e.g., to prove

1.4.10. THEOREM. *For $p > 2$, all monomials in the β_{p^i/p^i} (1.3.19) are nontrivial.* □

If n is not divisible by $p-1$ then S_n has cohomological dimension n^2; i.e., $H^i(S_n) = 0$ if $i > n^2$, and $H^*(S_n)$ has a certain type of Poincaré duality (6.2.10). It is essentially the cohomology of a certain n-stage nilpotent Lie algebra (6.3.5), at least for $n < p-1$. The cohomological dimension implies

1.4.11. MORAVA VANISHING THEOREM. *If $(p-1) \nmid n$, then in the chromatic spectral sequence* (1.4.8) *$E_1^{n,s} = 0$ for $s > n^2$.* □

It is also known (6.3.6) that every sufficiently small open subgroup of S_n has the same cohomology as a free abelian group of rank n^2. This fact can be used to get information about the Adams–Novikov spectral sequence E_2-term for certain Thom spectra (6.5.6).

Now we will explain how the Greek letter elements of 1.3.17 and 1.3.19 appear in the chromatic spectral sequence. If J is a G-invariant regular ideal with n generators [e.g., the invariant prime ideal $I_n = (p, v_1, \ldots, v_{n-1})$], then L/J is a submodule of N^n and M^n, so $\mathrm{Ext}^0(L/J) \subset \mathrm{Ext}^0(N^n) \subset \mathrm{Ext}^0(M^n) = E_1^{n,0}$. Recall that the Greek letter elements are images of elements in $\mathrm{Ext}^0(J)$ under the appropriate composition of connecting homomorphisms. This composition corresponds to the edge homomorphism $E_2^{n,0} \to E_\infty^{n,0}$ in the chromatic spectral sequence. [Note that every element in the chromatic $E_2^{n,0}$ is a permanent cycle; i.e., it supports no nontrivial differential although it may be the target of one. Elements in $E_1^{n,0}$ coming from $\mathrm{Ext}(L/J)$ lift to $\mathrm{Ext}(N^n)$ are therefore in $\ker d_1$ and live in $E_2^{n,0}$.] The module N^n is the union of the L/J over all possible invariant regular ideals J with n generators, so $\mathrm{Ext}^0(N^n)$ contains all possible nth Greek letter elements.

To be more specific about the particular elements discussed in Section 3 we must introduce *chromatic notation* for elements in N^n and M^n. Such elements will be written as fractions $\frac{x}{y}$ with $x \in L$ and $y = p^{i_0}v^{i_1} \ldots v_{n-1}^{i_{n-1}}$ with all exponent positive, which stands for the image of y in $L/J \subseteq N^n$ where $J = (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$. Hence x/y is annihilated by J and depends only on the mod J reduction of x. The usual rules of addition, subtraction, and cancellation of fractions apply here.

1.4.12. PROPOSITION. *Up to sign the elements* $\alpha_t^{(n)}$ (1.3.17), $\alpha_{s/t}$ *and* $\beta_{s/t}$ (1.3.19) *are represented in the chromatic spectral sequence by* $v_n^t/pv_1\cdots v_{n-1}\in E_2^{n,0}$, $v_1^s/p^t\in E_2^{1,0}$, *and* $v_2^s/pv_1^t\in E_2^{2,0}$, *respectively.* □

The signs here are a little tricky and come from the double complex used to prove 1.4.8 (see 5.1.18). The result suggests elements of a more complicated nature; e.g., $\beta_{s/i_2,i_1}$ stands for $v_2^s/p^{i_1}v_1^{i_2}$, with the convention that if $i_1=1$ it is omitted from the notation. The first such element with $i_1>1$ is $\beta_{p^2/p,2}$. We also remark that some of these elements require correcting terms in their numerators; e.g., $(v_1^4+8v_1v_2)/2^4$ (but not $v_1^4/2^4$) is in $\mathrm{Ext}^0(N^1)$ and represents $\alpha_{4/4}$, which corresponds to the generator $\sigma\in\pi_*(S^0)$.

We will describe $E_1^{n,*}$ for $n\le 1$ at $p>2$. For all primes $E_1^{0,0}=\mathbf{Q}$ (concentrated in dimension 0) and $E_1^{0,s}=0$ for $s>0$. For $p>2$, $E_1^{1,s}=0$ for $s>1$ and $E_1^{1,1}=\mathbf{Q}/\mathbf{Z}_{(p)}$ concentrated in dimension 0, and $E_1^{1,0}$ is trivial in dimensions not divisible by $q=2(p-1)=\dim v_1$ and is generated by all elements of the form v_1^t/pt for $t\in\mathbf{Z}$. Hence if p^i is the largest power of p dividing t, then $E_1^{1,0}\approx\mathbf{Z}/(p^{i+1})$ in dimension qt, and in dimension 0, $E_1^{1,0}=\mathbf{Q}/\mathbf{Z}_{(p)}$.

The differential $d_1\colon E_1^{0,0}\to E_1^{1,0}$ is the usual map $\mathbf{Q}\to\mathbf{Q}/\mathbf{Z}_{(p)}$. Its kernel $\mathbf{Z}_{(p)}$ is $\mathrm{Ext}^0(L\otimes\mathbf{Z}_{(p)})$. On $E_1^{1,1}=\mathbf{Q}/\mathbf{Z}_{(p)}$ the kernel of d_1 is trivial, so $E_2^{1,1}=E_2^{0,2}=0$ and $\mathrm{Ext}^2(L\otimes\mathbf{Z}_{(p)})=E_2^{2,0}$. On $E_1^{1,0}$, the kernel of d_1 consists of all elements in nonnegative dimensions. Since the $\mathbf{Q}/\mathbf{Z}_{(p)}$ in dimension 0 is hit by d_1, $E_2^{1,0}$ consists of the positive dimensional elements in $E_1^{1,0}$ and this group is $\mathrm{Ext}^1(L\otimes\mathbf{Z}_{(p)})$. In $\pi_*(S^0)$ it is represented by the p-component of $\mathrm{im}\,J$.

Now the chromatic E_1-term is periodic in the following sense. By defintion,

$$M^n=\varinjlim v_n^{-1}L/J,$$

where the direct limit is over all invariant regular ideals J with n generators. For each J, $\mathrm{Ext}^0(v_n^{-1}L/J)$ contains some power of v_n, say v_n^k. Then $\mathrm{Ext}(v_n^{-1}L/J)$ is a module over $\mathbf{Z}_{(p)}[v_n^k,v_n^{-k}]$, i.e., multiplication by v_n^k is an isomorphism, so we say that this Ext is v_n-*periodic*. Hence $E_1^{n,*}=\mathrm{Ext}(M^n)$ is a direct limit of such groups. We may say that an element in the Adams–Novikov spectral sequence E_2-term is v_n-*periodic* if it represents an element in $E_\infty^{n,*}$ of the chromatic spectral sequence.

Hence the chromatic spectral sequence E_∞-term is the trigraded group associated with the filtration of $\mathrm{Ext}(L\otimes\mathbf{Z}_{(p)})$ by v_n-periodicity. This filtration is decreasing and has an infinite number of stages in each cohomological degree. One sees this from the diagram

$$\mathrm{Ext}^s(N^0)\leftarrow\mathrm{Ext}^{s-1}(N^1)\leftarrow\ldots\leftarrow\mathrm{Ext}^0(N^s)$$

where $N^0=L\otimes\mathbf{Z}_{(p)}$; the filtration of $\mathrm{Ext}(N^0)$ is by images of the groups $\mathrm{Ext}(N^n)$. This local finiteness allows us to define an increasing filtration on $\mathrm{Ext}(N^0)$ by $F_i\,\mathrm{Ext}^s(N^0)=\mathrm{im}\,\mathrm{Ext}^i(N^{s-i})$ for $0\le i\le s$, and $F_0\,\mathrm{Ext}(N)$ is the subgroup of Greek letter elements in the most general possible sense.

5. Unstable Homotopy Groups and the EHP Spectral Sequence

The EHP sequences. The EHP spectral sequence. The stable zone. The inductive method. The stable EHP spectral sequence. The Adams vector field theorem. James periodicity. The J-spectrum. The spectral sequence for $J_*(RP^\infty)$ and $J_*(B\Sigma_p)$. Relation to the Segal conjecture. The Mahowald root invariant.

In this section we will describe the EHP sequence, which is an inductive method for computing $\pi_{n+k}(S^n)$ beginning with our knowledge of $\pi_*(S^1)$ (1.1.7). We will explain how the Adams vector field theorem, the Kervaire invariant problem, and the Segal conjecture are related to the unstable homotopy groups of spheres. We will not present proofs here or elsewhere in the book, nor will we pursue the topic further except in Section 3.3. We are including this survey here because no comparable exposition exists in the literature and we believe these results should be understood by more than a handful of experts. In particular, this section could serve as an introduction to Mahowald [**4**]. For computations at the prime 3, see Toda [**8**], which extends the known range for unstable 3-primary homotopy groups from 55 to 80.

The EHP sequences are the long exact sequences of homotopy groups associated with certain fibration constructed by James [**1**] and Toda [**6**]. There is a different set of fibrations for each prime p. All spaces and groups are assumed localized at the prime in question. We start with $p = 2$. There we have a fibration

(1.5.1) $$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1},$$

which gives the long exact sequence

(1.5.2)
$$\cdots \to \pi_{n+k}(S^n) \xrightarrow{E} \pi_{n+k+1}(S^{n+1}) \xrightarrow{H} \pi_{n+k+1}(S^{2n+1}) \xrightarrow{P} \pi_{n+k-1}(S^n) \to \cdots .$$

Here E stands for Einhängung (suspension), H for Hopf invariant, and P for Whitehead product. If n is odd the fibration is valid for all primes and it splits at odd primes, so for $p > 2$ we have

$$\pi_{2m+k}(S^{2m}) = \pi_{2m+k-1}(S^{2m-1}) \oplus \pi_{2m+k}(S^{4m-1}).$$

This means that even-dimensional spheres at odd primes are uninteresting. Instead one considers the fibration

(1.5.3) $$\widehat{S}^{2m} \to \Omega S^{2m+1} \to \Omega S^{2pm+1},$$

where the second map is surjective in $H_*(\ ;\mathbf{Z}_{(p)})$, and $\widehat{S}^{2m}$ is the $(2mp-1)$-skeleton of ΩS^{2m+1}, which is a CW-complex with $p-1$ cells of the form $S^{2m} \cup e^{4m} \cup \cdots \cup e^{2(p-1)m}$. The corresponding long exact sequence is

(1.5.4) $$\cdots \to \pi_i(\widehat{S}^{2m}) \xrightarrow{E} \pi_{i+1}(S^{2m+1}) \xrightarrow{H} \pi_{i+1}(S^{2pm+1}) \xrightarrow{P} \pi_{i-1}(\widehat{S}^{2m}) \to \cdots .$$

There is also a fibration

(1.5.5) $$S^{2m-1} \to \Omega\widehat{S}^{2m} \to \Omega S^{2pm-1},$$

which gives

(1.5.6) $$\cdots \to \pi_{i-1}(S^{2m-1}) \xrightarrow{E} \pi_i(\widehat{S}^{2m}) \xrightarrow{H} \pi_i(S^{2pm-1}) \xrightarrow{P} \pi_{i-2}(S^{2m-1}) \to \cdots .$$

1.5.4 and 1.5.6 are the EHP sequences for odd primes. Note that for $p = 2$, $\widehat{S}^{2m} = S^{2m}$ and both sequences coincide with (1.5.2).

For each prime these long exact sequences fit together into an exact couple (2.1.6) and we can study the associated spectral sequence, namely

1.5.7. PROPOSITION.

(a) *For $p = 2$ there is a spectral sequence converging to π_*^S (stable homotopy) with*

$$E_1^{k,n} = \pi_{k+n}(S^{2n-1}) \qquad \text{and} \qquad d_r\colon E_r^{k,n} \to E_r^{k-1,n-r}.$$

$E_\infty^{n,k}$ is the subquotient $\operatorname{im}\pi_{n+k}(S^n)/\operatorname{im}\pi_{n+k-1}(S^{n-1})$ of π_k^S. There is a similar spectral sequence converging to $\pi_*(S^j)$ with $E_1^{k,n}$ as above for $n \le j$ and $E_1^{k,n} = 0$ for $n > j$.

(b) *For $p > 2$ there are similar spectral sequences with*

$$E_1^{k,2m+1} = \pi_{k+2m+1}(S^{2pm+1}) \qquad \text{and} \qquad E_1^{k,2m} = \pi_{k+2m}(S^{2pm-1}).$$

The analogous spectral sequence with $E_1^{k,n} = 0$ for $n > j$ converges to $\pi_*(S^j)$ if j is odd and to $\pi_*(\widehat{S}^j)$ if j is even. □

This is the *EHP spectral sequence*. We will explain below how it can be used to compute $\pi_{n+k}(S^n)$ [or $\pi_{n+k}(\widehat{S}^n)$ if n is even and p is odd] by double induction on n and k. First we make some easy general observations.

1.5.8. Proposition.

(a) *For all primes $E_1^{k,1} = \pi_{1+k}(S^1)$, which is $\mathbf{Z}_{(p)}$ for $k = 0$ and 0 for $k > 0$.*

(b) *For $p = 2$, $E_1^{k,n} = 0$ for $k < n - 1$.*

(c) *For $p = 2$, $E_1^{k,n} = \pi_{k-n+1}^S$ for $k < 3n - 3$.*

(d) *For $p > 2$, $E_1^{k,2m+1} = 0$ for $k < qm$ and $E_1^{k,2m} = 0$ for $k < qm - 1$, where $q = 2(p-1)$.*

(e) *For $p > 2$, $E_1^{k,2m+1} = \pi_{k-qm}^S$ for $k < q(pm + m + 1) - 2$, and $E_1^{k,2m} = \pi_{k+1-qm}^S$ for $k < q(pm + m) - 3$.* □

Part (b) follows from the connectivity of the $(2n-1)$-sphere and similarly for (d); these give us a vanishing line for the spectral sequence. (c) and (e) follow from the fact that $\pi_{2m-1+k}(S^{2m-1}) = \pi_k^S$ for $k < qm - 2$, which is in turn a consequence of 1.5.7. We will refer to the region where $n - 1 \le k$ and $E_1^{k,n}$ is a stable stem as the *stable zone*.

Now we will describe the inductive aspect of the EHP spectral sequence. Assume for the moment that we know how to compute differentials and solve the group extension problems. Also assume inductively that we have computed $E_1^{i,j}$ for all (i,j) with $i < k$ and all (k,j) for $j > n$. For $p = 2$ we have $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$. This group is in the $(k-n+1)$-stem. If $n = 1$, this group is $\pi_{1+k}(S^1)$, which is known, so assume $n > 1$. If $n = 2$ this group is $\pi_{2+k}(S^3)$, which is 0 for $k = 0$, $\mathbf{Z}$ for $k = 1$, and for $k > 1$ is the middle term in the short exact sequence

$$0 \to E_2^{k-1,2} \to \pi_{k+2}(S^3) \to \ker d_1 \subset E_2^{k-1,3} \to 0.$$

Note that $E_2^{k-1,2}$ is the cokernel of the d_1 coming from $E_1^{k,3}$ and is therefore known by induction. Finally, if $n > 2$, $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$ can be read off from the already computed portion of the EHP spectral sequence as follows. As in 1.5.7 one obtains a spectral sequence for $\pi_*(S^{2n-1})$ by truncating the EHP spectral sequence, i.e., by setting all $E_1^{j,m} = 0$ for $m > 2n - 1$. The group $\pi_{n+k}(S^{2n-1})$ lies in a stem which is already known, so we have $E_1^{k,n}$. Similar remarks apply to odd primes.

We will illustrate the method in detail for $p = 2$ by describing what happens for $0 \le k \le 7$ in Fig. 1.5.9. By 1.5.8(c) we have $E_1^{k,k+1} = \pi_0^S = \mathbf{Z}$. Let x_k denote the standard generator of this group. We will see below (1.5.13) that $d_1(x_k) = 2x_{k-1}$

	π_k^S	$\mathbf{Z}$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(2)$	$\mathbf{Z}/(8)$	0	0	$\mathbf{Z}/(2)$	$\mathbf{Z}/(16)$
	$n \backslash k$	0	1	2	3	4	5	6	7
S^1	1	x_0							
S^3	2		x_1	1	11	111 21	211	2111	
S^5	3		2	x_2	1	11	111 21 3	31	311
S^7	4				x_3	1	11	111 21 3	
S^9	5				2	x_4	1	11	111 21 3
S^{11}	6						x_5	1	11
S^{13}	7						2	x_6	1
S^{15}	8								x_7

FIGURE 1.5.9. The EPSS for $p = 2$ and $k \leq 7$.

for even positive k and $d_1(x_k) = 0$ otherwise. Hence $E_2^{1,2} = E_\infty^{1,2} = \pi_1^S = \mathbf{Z}/(2)$, so $E_1^{k,k} = \mathbf{Z}/(2)$ for all $k \geq 2$. We denote the generator of each of these groups by 1 to indicate that, if the generator is a permanent cycle, it corresponds to an element whose Hopf invariant suspends to the element corresponding to x_1. Now the first such generator, that of $E_1^{2,2}$, is not hit by a differential, so we have $E_1^{k,k-1} = \pi_{2k-1}(S^{2k-3}) = \mathbf{Z}/(2)$ for all $k \geq 3$. We denote these generators by 11, to indicate that their Hopf invariants each desuspend to elements with Hopf invariant x_1.

In general we can specify an element $\alpha \in \pi_{n+k}(S^n)$ by a sequence of integers adding up to k as follows. Desuspend α as far as possible, say to S^{m+1} integer is then m (necessarily $\leq k$) and the desuspended d has a Hopf invariant $\beta \in \pi_{m+1+k}(S^{2m+1})$. To get the second integer we desuspend β, and so forth. After a finite number of steps we get an element with Hopf invariant in the zero stem and stop the process. Of course there is some indeterminacy in desuspending but we can ignore it for now. We call this sequence of integers the *serial number* of α. In FIG. 1.5.9 we indicate each element of $E_1^{k,n} = \pi_{n+k}(S^{2n-1})$ by its serial number. In almost all cases if $p\alpha \neq 0$, its serial number differs from that of α itself.

To get back to FIG. 1.5.9, we now have to determine the groups $E_1^{k,k-t} = \pi_{2k-2}(S^{2k-5})$ for $k \geq 4$, which means examining the 3-stem in detail. The groups $E_1^{3,2}$ and $E_1^{3,3}$ are not touched by differentials, so there is an short exact sequence

$$0 \to E_1^{3,2} \to \pi_6(S^3) \to E_1^{3,3} \to 0.$$

The two end terms are $\mathbf{Z}/(2)$ and the group extension can be shown to be nontrivial, so $E_1^{4,2} = \pi_6(S^3) = \mathbf{Z}/(4)$. Using the serial number notation, we denote the generator by 21 and the element of order 2 by 111. Similarly one sees $\pi_5(S^2) = \mathbf{Z}/(2)$, $\pi_7(S^4) = \mathbf{Z} \oplus \mathbf{Z}/(4)$ and there is an short exact sequence

$$0 \to \pi_6(S^3) \to \pi_8(S^5) \to E_2^{3,4} \to 0.$$

Here the subgroup and cokernel are $\mathbf{Z}/(4)$ and $\mathbf{Z}/(2)$, respectively, and the group extension is again nontrivial, so $\pi_8(S^5) = E_1^{k,k-2} = \mathbf{Z}/(8)$ for $k \geq 5$. The generator of this group is the suspension of the Hopf map $\nu\colon S^7 \to S^4$ and is denoted by 3.

To determine $E_1^{k,k-3} = \pi_{2k-3}(S^{2k-7})$ for $k \geq 5$ we need to look at the 4-stem, i.e., at the column $E_r^{4,*}$. The differentials affecting those groups are indicated on the chart. Hence we have $E_2^{4,2} = 0$ so $\pi_7(S^3) = E_1^{5,2} = \mathbf{Z}/(2)$; the d_2 hitting $E_1^{4,3}$ means that the corresponding element *dies* (i.e., becomes null homotopic) when suspended to $\pi_9(S^5)$; since it first appears on S^3 we say it is *born* there. Similarly, the generator of $E_1^{4,4}$ corresponds to an element that is born on S^4 and dies on S^6 and hence shows up in $E_1^{6,3} = \pi_9(S^5)$. We leave it to the reader to determine the remaining groups shown in the chart, assuming the differentials are as shown.

We now turn to the problem of computing differentials and group extensions in the EHP spectral sequence. For the moment we will concentrate on the prime 2. The fibration 1.5.1 can be looped n times to give

$$\Omega^n S^n \to \Omega^{n+1} S^{n+1} \to \Omega^{n+1} S^{2n+1}.$$

In Snaith [**1**] a map is constructed from $\Omega^n S^n$ to $Q\mathbf{R}P^{n-1}$ which is compatible with the suspension map $\Omega^n S^n \to \Omega^{n+1} S^{n+1}$. (Here QX denotes $\varinjlim \Omega^k \Sigma^k X$.) Hence we get a commutative diagram

$$\text{(1.5.10)}\qquad \begin{array}{ccccc} \Omega^n S^n & \longrightarrow & \Omega^{n+1} S^{n+1} & \longrightarrow & \Omega^{n+1} S^{2n+1} \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathbf{R}P^{n-1} & \longrightarrow & Q\mathbf{R}P^n & \longrightarrow & QS^n \end{array}$$

where both rows are fibre sequences and the right-hand vertical map is the standard inclusion. The long exact sequence in homotopy for the bottom row leads to an exact couple and a spectral sequence as in 1.5.7. We call it the *stable EHP spectral sequence.*

There is an odd primary analog of 1.5.10 in which $\mathbf{R}P^n$ is replaced by an appropriate skeleton of $B\Sigma_p$, the classifying space for the symmetric group on p letters. Recall that its mod (p) homology is given by

$$\text{(1.5.11)}\qquad H_i(B\Sigma_p; \mathbf{Z}/(p)) = \begin{cases} \mathbf{Z}/(p) & \text{if } i \equiv 0 \text{ or } -1 \mod (q) \\ 0 & \text{otherwise.} \end{cases}$$

1.5.12. PROPOSITION. (a) *For $p = 2$ there is a spectral sequence converging to $\pi_*^S(\mathbf{R}P^\infty)$ (stable homotopy of $\mathbf{R}P^\infty$) with $E_1^{k,n} = \pi_{k-n+1}^S$ for $n \geq 2$ and $d_r\colon E_r^{k,n} \to E_r^{k-1,n-r}$. Here $E_\infty^{k,n}$ is the subquotient $\operatorname{im}\pi_k^S(\mathbf{R}P^{n-1})/\operatorname{im}\pi_k^S(\mathbf{R}P^{n-2})$ of $\pi_k^S(\mathbf{R}P^\infty)$. There is a similar spectral sequence converging to $\pi_*^S(\mathbf{R}P^{j-1})$ with $E_1^{k,n}$ as above for $n \leq j$ and $E_1^{k,n} = 0$ for $n > j$.*

(b) *For $p > 2$ there is a similar spectral sequence converging to $\pi_*^S(B\Sigma_p)$ with $E_1^{k,2m+1} = \pi_k^S$ and $E_1^{k,2m} = \pi_{k+1-mq}$. There is a similar spectral sequence with $E_1^{k,n} = 0$ for $n > j$ converging to $\pi_*^S(B\Sigma_p^{(q)j-1})$ if j is even and to $\pi_*(B\Sigma_p^{(q)(j-1)})$ if j is odd.*

(c) *There are homomorphisms to these from the corresponding EHP spectral sequences of 1.5.7 induced by suspension on the E_1 level, e.g., at $p = 2$ by the suspension map $\pi_{k+n}(S^{2n-1}) \to \pi_{k-n+1}^S$. Hence the E_1-terms are isomorphic in the stable zone.*

We remark that this stable EHP spectral sequence is nothing but a reindexed form of the Atiyah–Hirzebruch spectral sequence (see Adams [**4**], Section 7) for $\pi_*^S(B\Sigma_p)$. In the latter one has $E_2^{s,t} = H_s(B\Sigma_p; \pi_t^S)$ and this group is easily seen to be $E_2^{s+t,f(s)}$ in the EHP spectral sequence where

$$f(s) = \begin{cases} s/(p-1) + 1 & \text{if } s \equiv 0 \mod (2p-2) \\ (s+1)/(p-1) & \text{if } s \equiv -1 \mod (2p-2). \end{cases}$$

Since everything in 1.5.12 is stable one can use stable homotopy theoretic methods, such as the Adams spectral sequence and K-theory, to compute differentials and group extensions. This is a major theme in Mahowald [**1**]. Differentials originating $E_r^{k,k+1}$ for $p = 2$ correspond to attaching maps in the cellular structure of $\mathbf{R}P^\infty$, and similarly for $p > 2$. For example, we have

1.5.13. PROPOSITION. *In the stable EHP spectral sequence* (1.5.12)*, the differential $d_1\colon E_1^{k,n} \to E_1^{k-1,n-1}$ is multiplication by p if k is even and trivial if k is odd.* □

Another useful feature of this spectral sequence is *James periodicity*: for each r there is a finite i and an isomorphism $E_r^{k,n} \approx E_r^{k+qp^i,n+2p^i}$ which commutes with differentials (note that $q = 2$ when $p = 2$). This fact is a consequence of the vector field theorem and will be explained more fully below (1.5.18).

For $p = 2$, the diagram 1.5.10 can be enlarged as follows. An element in the orthogonal group $O(n)$ gives a homeomorphism $S^{n-1} \to S^{n-1}$. Suspension gives a basepoint-preserving map $S^n \to S^n$ and therefore an element in $\Omega^n S^n$. Hence we have a map $J\colon O(n) \to \Omega^n S^n$ (compare 1.1.12). We also have the reflection map $r\colon \mathbf{R}P^{n-1} \to O(n)$ sending a line through the origin in $\mathbf{R}^n$ to the orthogonal matrix corresponding to reflection through the orthogonal hyperplane. Combining these we get

(1.5.14)
$$\begin{tikzcd}
\mathbf{R}P^{n-1} \arrow[r] \arrow[d] & \mathbf{R}P^n \arrow[r] \arrow[d] & S^n \arrow[d, equal] \\
O(n) \arrow[r] \arrow[d] & O(n+1) \arrow[r] \arrow[d] & S^n \arrow[d] \\
\Omega^n S^n \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d] & \Omega^{n+1} S^{2n+1} \arrow[d] \\
Q\mathbf{R}P^{n-1} \arrow[r] & Q\mathbf{R}P^n \arrow[r] & QS^n.
\end{tikzcd}$$

Here the top row is a cofiber sequence while the others are fiber sequences. The right-hand vertical maps are all suspensions, as is the composite $\mathbf{R}P^n \to Q\mathbf{R}P^n$. The second row leads to a spectral sequence (which we call the *orthogonal spectral sequence*) converging to $\pi_*(O)$ which maps to the EHP spectral sequence. The map on $E_1^{k,n} = \pi_k(S^{n-1})$ is an isomorphism for $k < 2n-3$ by the Freudenthal suspension theorem 1.1.10. The middle right square of this diagram only commmutes after a single looping. This blemish does not affect calculations of homotopy groups.

Hence we have three spectral sequences corresponding to the three lower rows of 1.5.14 and converging to $\pi_*(O)$, the 2-component of π_*^S, and $\pi_*^S(\mathbf{R}P^\infty)$. In all three we have generators $x_k \in E_1^{k,k+1} = \mathbf{Z}$ and we need to determine the first nontrivial differential (if any exists) on it for k odd. We will see that this differential always lands in the zone where all three spectral sequences are isomorphic. In the orthogonal spectral sequence x_k survives to E_r iff the projection $O(k+1)/O(k+1-r) \to S^k$ admits a cross section. It is well known (and easy to prove) that such a cross section exists iff S^k admits $r-1$ linearly independent tangent vector fields. The question of how many such vector fields exist is the *vector field problem*, which was solved by Adams [**16**] (see 1.5.16). We can give equivalent formulations of the problem in terms of the other two spectral sequences.

1.5.15. Theorem (James [**2, 3**]). *The following three statements are equivalent*:
(a) S^{k-1} *admits* $r-1$ *linearly independent tangent vector fields.*
(b) *Let* ι *be the generator of* $\pi_{2k-1}(S^{2k-1}) = \mathbf{Z}$*. Then* $P(\iota) \in \pi_{2k-3}(S^{k-1})$ (*see* 1.5.2) *desuspend to* $\pi_{2k-r-2}(S^{k-r})$.
(c) *The stable map* $\mathbf{R}P^{k-1}/\mathbf{R}P^{k-r} \to S^{k-1}$ *admits a cross section.* □

The largest possible r above depends on the largest powers of 2 dividing $k+1$. Let $k = 2^j(2s+1)$,

$$\phi(j) = \begin{cases} 2j & \text{if } j \equiv 1 \text{ or } 2 \mod (4) \\ 2j+1 & \text{if } j \equiv 0 \mod (4) \\ 2j+2 & \text{if } j \equiv 3 \mod (4) \end{cases}$$

and $\rho(k) = \phi(j)$.

1.5.16. THEOREM (Adams [**16**]).

(a) *With notation as above, S^{k-1} admits $\rho(k)-1$ linearly independent tangent vector fields and no more.*

(b) *Let $\bar{\alpha}_0 = 2 \in \pi_0^S$ and for $j > 0$ let $\bar{\alpha}_j$ denote the generator of* $\operatorname{im} J$ *in $\pi^S_{\rho(j)-1}$ (see* 1.5.15 (c)*). Then in the 2-primary EHP spectral sequence* (1.5.7) *$d_{\phi(j)}(x_{k-1})$ is the (nontrivial) image of $\bar{\alpha}_j$ in $E^{k-2,k-j}_{\phi(j)}$.* □

We remark that the $\rho(k)-1$ vector fields on S^k were constructed long ago by Hurwitz and Radon (see Eckmann [**1**]). Adams [**16**] showed that no more exist by using real K-theory to solve the problem as formulated in 1.5.15(c).

Now we turn to the odd primary analog of this problem, i.e., finding differentials on the generators x_{qk-1} of $E_1^{qk-1,2k} = \mathbf{Z}$. We know of no odd primary analog of the enlarged diagram 1.5.14, so we have no analogs of 1.5.15(a) or 1.5.16(a), but we still call this the *odd primary vector field problem.* The solution is

1.5.17. THEOREM (Kambe, Matsunaga and Toda [**1**]). *Let $\bar{\alpha}_j$ generate* $\operatorname{im} J \subset$ *π^S_{qj-1}* (1.1.12)*, let x_{qk-1} generate $E_1^{qk-1,2k}$ in the EHP spectral sequence* (1.5.7) *for an odd prime p (here $q = 2p-2$), and let $k = p^j s$ with s not divisible by p. Then x_{qk-1} lives to E_{2j+2} and $d_{2j+2}(x_{qk-1})$ is the (nontrivial) image of $\bar{\alpha}_{j+1}$ in $E_{2j}^{qk-2,2k-2j-2}$.* □

Now we will explain the James periodicity referred to above. For $p = 2$ let $\mathbf{R}P^n_m = \mathbf{R}P^n/\mathbf{R}P^{m-1}$ for $m \le n$. There is an i depending only on $n-m$ such that $\mathbf{R}P^{n+2^{i+1}}_{m+2^{i+1}} \simeq \Sigma^{2^{i+1}}\mathbf{R}P^n_m$, a fact first proved by James [**3**]. To prove this, let λ be the canonical real line bundle over $\mathbf{R}P^{n-m}$. Then $\mathbf{R}P^n_m$, is the Thom space for $m\lambda$. The reduced bundle $\lambda - 1$ is an element of finite order $2i+1$ in $KO^*(\mathbf{R}P^{n-m})$, so $(2^{i+1}+m)\lambda = m\lambda + 2^{i+1}$ and the respective Thom spaces $\mathbf{R}P^{n+2^{i+1}}_{m+2^{i+1}}$ and $\Sigma^{2^{i+1}}\mathbf{R}P^n_m$ are equivalent. The relevant computations in $KO^*(\mathbf{R}P^{n-m})$ are also central to the proof of the vector field theorem 1.5.16. Similar statements can be made about the odd primary case. Here one replaces λ by the $\mathbf{C}^{p-1}$ bundle obtained by letting Σ_p act via permutation matrices on $\mathbf{C}^p$ and splitting off the diagonal subspace on which Σ_p acts trivially.

For $p = 2$ one can modify the stable EHP spectral sequence to get a spectral sequence converging to $\pi_*(\mathbf{R}P^n_m)$ by setting $E_1^{k,j} = 0$ for $j < m-1$ and $j > n-1$. Clearly the $d_r \colon E_r^{k,n} \to E_r^{k-1,n-r}$ in the stable EHP spectral sequence is the same as that in the spectral sequence for $\pi_*(\mathbf{R}P^{n-1}_{n-r-1})$ and similar statements can be made for $p > 2$, giving us

1.5.18. JAMES PERIODICITY THEOREM. *In the stable EHP spectral sequence* (1.5.12) *there is an isomorphism $E_r^{k,n} \to E_r^{k+qp^i,n+2p^i}$ commuting with d_r, where $i = [r/2]$.* □

Note that 1.5.17 is simpler than its 2-primary analog 1.5.16(b). The same is true of the next question we shall consider, that of the general behavior of elements in $\operatorname{im} J$ in the EHP spectral sequence. It is ironic that most of the published work in this area, e.g., Mahowald [**2, 4**], is concerned exclusively with the prime 2, where the problem appears to be more difficult.

Theorem 1.5.17 describes the behavior of the elements x_{qk-1} in the odd primary EHP spectral sequence and indicates the need to consider the behavior of $\operatorname{im} J$. The elements $\bar{\alpha}_j$ and their multiples occur in the stable EHP spectral sequence in the groups $E_1^{qk-2,2m}$ and $E_1^{qk-1,2m+1}$ for all $k > m$. To get at this question we use the spectrum J, which is the fibre of a certain map $bu \to \Sigma^2 bu$, where bu is the spectrum representing connective complex K-theory, i.e., the spectrum obtained by delooping the space $\mathbf{Z} \times BU$. There is a stable map $S^0 \to J$ which maps $\operatorname{im} J \subset \pi_*^S$ isomorphically onto $\pi_*(J)$. The stable EHP spectral sequence, which converges to $\pi_*^S(B\Sigma_p)$, maps to a similar spectral sequence converging to $J_*(B\Sigma_p) = \pi_*(J \wedge B\Sigma_p)$. This latter spectral sequence is completely understood and gives information about the former and about the EHP spectral sequence itself.

1.5.19. THEOREM.

(a) *For each odd prime p there is a connective spectrum J and a map $S^0 \to J$ sending the p-component of* $\operatorname{im} J$ (1.1.12) *isomorphically onto $\pi_*(J)$, i.e.,*

$$\pi_i(J) = \begin{cases} \mathbf{Z}_{(p)} & \text{if } i = 0 \\ \mathbf{Z}/(p^{j+1}) & \text{if } i = qk-1,\ k > 0,\ k = sp^j \text{ with } p \nmid s \\ 0 & \text{otherwise.} \end{cases}$$

(b) *There is a spectral sequence converging to $J_*(B\Sigma_p)$ with*

$$E_1^{k,2m+1} = \pi_{k-mq}(J) \qquad \text{and} \qquad E_1^{k,2m} = \pi_{k+1-mq}(J);$$

the map $S^0 \to J$ induces a map to this spectral sequence from the stable EHP spectral sequence of 1.5.12.

(c) *The d_1 in this spectral sequence is determined by* 1.5.13. *The resulting E_2-term has the following nontrivial groups and no other*:

$$\begin{aligned} E_2^{qk-1,2k} &= \mathbf{Z}/(p) \quad \text{generated by } x_{qk-1} \text{ for } k > 0, \\ E_2^{q(k+j)-2,2k} &= \mathbf{Z}/(p) \quad \text{generated by } \bar{\alpha}_j \text{ for } k, j > 0, \end{aligned}$$

and

$$E_2^{q(k+j)-1,2k+1} = \mathbf{Z}/(p) \quad \text{generated by } \alpha_j \text{ for } k, j > 0,$$

where α_j is an element of order p in $\pi_{qj-1}(J)$.

(d) *The higher differentials are determined by* 1.5.17 *and the fact that all group extensions in sight are nontrivial, i.e., with k and j as in* 1.5.17,

$$d_{2j+2}(x_{qk-1}) = \bar{\alpha}_{j+1} \in E_{wj+2}^{qk-2,2(k-j-1)}$$

and d_{2j+3} is nontrivial on $E_{2j+3}^{qk-1,2m+1}$ for $j+2 < m < k$.

(e) *The resulting E_∞-term has the following nontrivial groups and no others*: $E_\infty^{qk-2,2m}$ *for $k > m \geq k-j$ and $E_\infty^{qk-1,2m+1}$ for $1 \leq m \leq j+1$. The group* *extensions are all nontrivial and we have for* $i > 0$

$$J_i(B\Sigma_p) = \pi_i(J) \oplus \begin{cases} \mathbf{Z}/(p^j) & \text{for } i = qsp^j - 2 \text{ with } p \nmid s \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

We will sketch the proof of this theorem. We have the fibration $J \to bu \to \Sigma^2 bu$ for which the long exact sequence of homotopy groups is known; actually bu (when localized at the odd prime p) splits into $p-1$ summands each equivalent to an even suspension of $BP\langle 1\rangle$, where $\pi_*(BP\langle 1\rangle) = \mathbf{Z}_{(p)}[v_1]$ with $\dim v_1 = q$. It is convenient to replace the above fibration by $J \to BP\langle 1\rangle \to \Sigma^q BP\langle 1\rangle$. We also have a transfer map $B\Sigma_p \to S^0_{(p)}$, which is the map which Kahn and Priddy [**2**] show induces a surjection of homotopy groups in positive dimensions (see also Adams [**15**]); the same holds for J-homology groups. Let R be the cofiber of this map. One can show that $S^0_{(p)} \to R$ induces a monomorphism in $BP\langle 1\rangle$-homology (or equivalently in bu-homology) and that $BP\langle 1\rangle \wedge R \simeq \bigvee_{j\ge 0} \Sigma^{qj} H\mathbf{Z}_{(p)}$, i.e., a wedge of suspensions of integral Eilenberg–Mac Lane spectra localized at p. Smashing these two fibrations together gives us a diagram

(1.5.20)
$$\begin{tikzcd}
J \wedge R \arrow[r] & BP\langle 1\rangle \wedge R \arrow[r] & \Sigma^q BP\langle 1\rangle \wedge R \\
J \arrow[u] \arrow[r] & BP\langle 1\rangle \arrow[u] \arrow[r, "f"] & \Sigma^q BP\langle 1\rangle \arrow[u] \\
J \wedge B\Sigma_p \arrow[u] \arrow[r] & BP\langle 1\rangle \wedge B\Sigma_p \arrow[u] \arrow[r] & \Sigma^q BP\langle 1\rangle \wedge B\Sigma_p \arrow[u]
\end{tikzcd}$$

in which each row and column is a cofiber sequence. The known behavior of $\pi_*(f)$ determines that of $\pi_*(f \wedge R)$ and enables one to compute $\pi_*(J \wedge B\Sigma_p) = J_*(B\Sigma_p)$. The answer, described in 1.5.19(c), essentially forces the spectral sequence of 1.5.19 to behave in the way it does. The E_2-term [1.5.19(c)] is a filtered form of $\pi_*(BP\langle 1\rangle \wedge B\Sigma_p) \oplus \pi_*(\Sigma^{q-1} BP\langle 1\rangle \wedge B\Sigma_p)$.

Corresponding statements about the EHP spectral sequence are not yet known but can most likely be proven by using methods of Mahowald [**4**]. We surmise they can be derived from the following.

1.5.21. Conjecture.

(a) *The composite* $\pi_k(\Omega^{2n+1} S^{2n+1}) \to \pi_k(QB\Sigma_p^{qn}) \to J_k(B\Sigma_p^{qn})$ *is onto unless* $k = qsp^j - 2$ (*with* $j > 0$, $sp^j > p$ *and* $p \nmid s$) *and* $n = sp^j - i$ *for* $1 \le i \le j$.

(b) *The groups* $E_\infty^{qk-1,2m+1}$ *of* 1.5.19(e) *pull back to the* E_∞*-term of the EHP spectral sequence and correspond to the element* $\alpha_{k/m}$ (1.3.19) *of order* p^m *in* $\operatorname{im} J \in \pi^S_{qk-1}$. *Hence* $\alpha_{k/m}$ *is born in* S^{2m+1} *and has Hopf invariant* α_{k-m} *except for* α_1, *which is born on* $\widehat{S}^2$ *with Hopf invariant one.* (*This was not suspected when the notation was invented*!) $\square$

We will give an example of an exception to 1.5.21(a) for $p = 3$. One has age $\alpha_8 \in E_3^{39,5}$, which should support a d_3 hitting $\bar{\alpha}_9 \in E_3^{38,2}$, but $E_1^{38,2} = \pi_{40}(S^5)$ and α_9 is only born on S^7, so the proposed d_3 cannot exist (this problem does not occur in the stable EHP spectral sequence). In fact, $\alpha_1\alpha_8 \ne 0 \in \pi_{41}(S^7) = E_1^{38,3}$ and this element is hit by a d_2 supported by the $\alpha_8 \in E_2^{39,5}$.

The other groups in 1.5.19(e), $J_{pqi-2}(B\Sigma_p)$, are harder to analyze. $E_\infty^{pq-2,q}$ pulls back to the EHP spectral sequence and corresponds to $\beta_1 \in \pi_{pq-2}^S$ (1.3.14), the first stable element in coker J (1.1.12), so β_1 is born on $\widehat{S}^q$ and has Hopf invariant α_1. Presumably the corresponding generators of $E_r^{piq-2,2pi-2}$ for $i > 1$ each supports a nontrivial d_q hitting a β_1 in the appropriate group. The behavior of the remaining elements of this sort is probably determined by that of the generators of $E_2^{p^jq-2,wp^j-2j}$ for $j \geq 2$, which we now denote by $\tilde{\theta}_j$. These appear to be closely related to the Arf invariant elements $\theta_j = \beta_{p^{j-1}/p^{j-1}}$ (1.4.10) in E_2^{2,p^jq} of the Adams–Novikov spectral sequence. The latter are known not to survive (6.4.1), so presumably the $\tilde{\theta}_j$ do not survive either. In particular we know $d_{2p^2-6}(\tilde{\theta}_2) = \beta_1^p$ in the appropriate group. There are similar elements at $p = 1$ as we shall see below. In that case the θ_j are presumed but certainly not known (for $j > 5$) to exist in $\pi_{2^{j+1}-2}^S$. Hence any program to prove their existence at $p = 2$ is doomed to fail if it would also lead to a proof for $p > 2$.

We now consider the 2-primary analog of 1.5.19 and 1.5.21. The situation is more complicated for four reasons.

(1) $\operatorname{im} J$ (1.5.15) is more complicated at $p = 2$ than at odd primes.

(2) The homotopy of J (which is the fiber of a certain map $bo \to \Sigma^4 bsp$, where bo and bsp are the spectra representing connective real and symplectic K-theory, respectively) contains more than just $\operatorname{im} J$.

(3) Certain additional exceptions have to be made in the analog 1.5.21.

(4) The groups corresponding to the $J_{piq-2}(B\Sigma_p)$ are more complicated and lead us to the elements $\eta_j \in \pi_{2^j}^S$ of Mahowald [**6**] in addition to the hypothetical $\theta_j \in \pi_{2^{j+1}-2}^S$.

Our first job then is to describe $\pi_*(J)$ and how it differs from $\operatorname{im} J$ as described in 1.1.12. We have $\pi_i(bo) = \pi_{i+7}(O)$ and $\pi_i(bsp) = \pi_{i+3}(O)$ for $i \geq 0$ and $\pi_*(O)$ is described in 1.1.11, i.e.,

$$\pi_i(O) = \begin{cases} \mathbf{Z} & \text{if } i = 3 \mod (4) \\ \mathbf{Z}/(2) & \text{if } i = 0 \text{ or } 1 \mod (8) \\ 0 & \text{otherwise.} \end{cases}$$

The map $bo \to \Sigma^4 bsp$ used to define J is trivial on the torsion in $\pi_*(bo)$, so these groups pull back to $\pi_*(J)$. Hence $\pi_{8i+1}(J)$ and $\pi_{8i+2}(J)$ for $i \geq 1$ contain summands of order 2 not coming from $\operatorname{im} J$.

1.5.22. Proposition. *At $p = 2$*

$$\pi_i(J) = \begin{cases} \mathbf{Z}_{(2)} & \text{if } i = 0 \\ \mathbf{Z}/(2) & \text{if } i = 1 \text{ or } 2 \\ \mathbf{Z}/(8) & \text{if } i \equiv 3 \mod (8) \text{ and } i > 0 \\ \mathbf{Z}/(2) & \text{if } i \equiv 0 \text{ or } 2 \mod 8 \text{ and } i \geq 8 \\ \mathbf{Z}/(2) \oplus \mathbf{Z}/(2) & \text{if } i \equiv 1 \mod (8) \text{ and } i \geq 9 \\ \mathbf{Z}/(2^{j+1}) & \text{if } i = 8m-1,\ m \geq 1 \text{ and } 8m = 2^j(2s+1). \end{cases}$$

Here, $\operatorname{im} J \subset \pi_*(J)$ *consists of cyclic summands in* $\pi_i(J)$ *for* $i > 0$ *and* $i \equiv 7,\ 0,\ 1$ *or* $3 \mod (8)$. □

Now we need to name certain elements in $\pi_*(J)$. As in 1.5.16 let $\bar{\alpha}_j$ denote the generator of $\operatorname{im} J$ in dimension $\phi(j) - 1$, where

$$\phi(j) - 1 = \begin{cases} 2j - 1 & \text{if } j \equiv 1 \text{ or } 2 \mod (4) \\ 2j & \text{if } j \equiv 0 \mod (4) \\ 2j + 1 & \text{if } j \equiv 3 \mod (4). \end{cases}$$

We also define elements α_j in $\pi_*(J)$ of order 2 as follows. $\alpha_1 = \eta \in \pi_1(J)$ and $\alpha_{4k+1} \in \pi_{8k+1}(J)$ is a certain element not in $\operatorname{im} J$ for $k \geq 1$. $\alpha_{4k+2} = \eta\alpha_{4k+1}$, $\alpha_{4k+3} = \eta^2\alpha_{4k+1} = 4\bar{\alpha}_{4k+2}$, and $\alpha_{4k} \in \pi_{8k-1}(J)$ is an element of order 2 in that cyclic group.

1.5.23. THEOREM (Mahowald [**4**]). (a) *There is a spectral sequence converging to $J_*(\mathbf{R}P^\infty)$ with $E_1^{k,n} = \pi_{k-n+1}(J)$; the map $S^0 \to J$ induces a homomorphism to this spectral sequence from the stable EHP spectral sequence of* 1.5.12. (*We will denote the generator of $E_1^{k,k+1}$ by x_k and the generator of $E_1^{k,k+1+m}$ for $m > 0$ by the name of the corresponding element in $\pi_m(J)$.*)

(b) *The d_1 in this spectral sequence is determined by* 1.5.13. *The following is a complete list of nontrivial d_2's and d_3's.*

For $k \geq 1$ and $t \geq 0$, d_2 sends

$$\begin{aligned} x_{4k+1} &\in E_2^{4k+1,4k+2} && \text{to} \quad \alpha_1 \\ \bar{\alpha}_{4t+3+i} &\in E_2^{4k+8+i+8t,4k+2} && \text{to} \quad \bar{\alpha}_{4t+i} \quad \text{for } i = 0,\, 1 \\ \alpha_{4t+1} &\in E_2^{4k+2+8t,4k+2} && \text{to} \quad \alpha_{4t+2} \\ \bar{\alpha}_{4t+4} &\in E_2^{4k+1+8t+7,4k+1} && \text{to} \quad \bar{\alpha}_{4t+5} \end{aligned}$$

and

$$\alpha_{4t+i} \in E_2^{4k+i+8t,4k+1} \quad \text{to} \quad \alpha_{4t+i+1} \quad \text{for } i = 1,\, 2.$$

For $k \geq 1$ and $t \geq 1$, d_3 sends

$$\alpha_{4t} \in E_2^{4k+1+8t,4k+3} \quad \text{to} \quad \alpha_{4t+1}$$

and

$$\bar{\alpha}_{4t+1} \in E_2^{4k+8t+1,4k+1} \quad \text{to} \quad \bar{\alpha}_{4t+2}.$$

See FIG. 1.5.24.

(c) *The resulting E_4-term is a $\mathbf{Z}/(2)$-vector space on the following generators for $k \geq 1$, $t \geq 0$.*

$$\begin{gathered} x_1 \in E_4^{1,2}; \quad \bar{\alpha} \in E_4^{4,2}; \quad \alpha_{4t+i} \in E_4^{8t+i+1,2} \quad \text{for } i = 1,\, 2; \\ \bar{\alpha}_{4t+i} \in E_4^{8t+i+5,2} \quad \text{for } i = 3,\, 4,\, 5; \quad \alpha_{4t+1} \in E_4^{8t+3,3}; \quad \alpha_{4t+4} \in E_4^{8t+9,3}; \\ \bar{\alpha}_{4t+4} \in E_4^{8t+10,3}; \quad x_{4k-1} \in E_4^{4k-1,4k}; \quad \bar{\alpha}_{4t+2} \in E_4^{4k+8t+2,4k}; \\ \bar{\alpha}_{4t+3} \in E_4^{4k+8t+6,4k}; \quad \alpha_{4t+3} \in E_4^{4k+8t+3,4k+1}; \quad \alpha_{4t+4} \in E_4^{4k+8t+7,4k+1}; \\ \alpha_{4t+2} \in E_4^{4k+8t+3,4k+2}; \quad \bar{\alpha}_2 \in E_4^{4k+4,4k+2}; \quad \bar{\alpha}_{4t+5} \in E_4^{4k+8t+10,4k+2}; \\ \alpha_{4t+1} \in E_4^{4k+8t+3,4k+3}; \quad \text{and} \quad \bar{\alpha}_{4t+4} \in E_4^{4k+8t+10,4k+3}. \end{gathered}$$

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	x_1	α_1	α_2	$\bar{\alpha}_2$				$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$ α_5	α_6	$\bar{\alpha}_6$		
3			α_1	α_2	α_3				α_4		$\bar{\alpha}_5$ α_5	α_6	α_7	
4			x_3	α_1	α_2					$\bar{\alpha}_3$	$\bar{\alpha}_4$	α_5 $\bar{\alpha}_5$	α_6	$\bar{\alpha}_6$
5					α_1	α_2	α_3				α_4	$\bar{\alpha}_4$	$\bar{\alpha}_5$ α_5	α_6
6					x_5	α_1	α_2	$\bar{\alpha}_2$				$\bar{\alpha}_3$	$\bar{\alpha}_4$	$\bar{\alpha}_5$ α_5
7							α_1	α_2	α_3				α_4	$\bar{\alpha}_4$

FIGURE 1.5.24. A portion of the E_2-term of the spectral sequence of Theorem 1.5.23 converging to $J_*(\mathbf{R}P^\infty)$ and showing the d_2's and d_3's listed in Theorem 1.5.23, part (c).

(d) *The higher differentials are determined by* 1.5.15 *and the fact that most group extensions in sight are nontrivial. The resulting E_∞-term has the following additive generators and no others for $t \geq 0$.*

$$x_1 \in E_\infty^{1,2}; \quad \alpha_{4t+4} \in E_\infty^{8t+9,3}; \quad \alpha_{4t+i} \in E_\infty^{8t+i+1,2} \quad \text{for } i = 1,\ 2;$$
$$\alpha_{4t+1} \in E_\infty^{8t+3,3}; \quad x_3 \in E_\infty^{3,4}; \quad \alpha_{4t+4} \in E_\infty^{8t+11,5};$$
$$\bar{\alpha}_{4t+i} \in E_\infty^{8t+i+5,2} \quad \text{for } i = 3,\ 4; \quad x_7 \in E_\infty^{7,8};$$
$$\alpha_{4t+4} \in E_\infty^{8t+15,9}; \quad \alpha_{4t+i} \in E_\infty^{8t+7,8-i} \quad \text{for } i = 1, 2, 3;$$
$$\alpha_{2^jt+2^j-j-2} \in E_\infty^{2j+1(1+t)-1,*} \quad \text{for } j \geq 3;$$
$$\bar{\alpha}_2 \in E_\infty^{4t+4,4t+2}; \quad \text{and} \quad \bar{\alpha}_j \in E_\infty^{2^{j+1}(t+1)-2,*} \quad \text{for } j \geq 2.$$

(e) *For $i > 0$*

$$J_i(\mathbf{R}P^\infty) = \pi_i(J) \oplus \begin{cases} \mathbf{Z}/(2) & \text{if } i \equiv 0 \mod (4) \\ \mathbf{Z}/(2^j) & \text{if } i = 2^{j+2}s - 2 \text{ for } s \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Note that the portion of the E_∞-term corresponding to the summand $\pi_*(J)$ in 1.5.23(e) [i.e., all but the last two families of elements listed in 1.5.23(d)] is near the line $n = 0$, while that corresponding to the second summand is near the line $n = k$.

The proof of 1.5.23 is similar to that of 1.5.19 although the details are messier. One has fibrations $J \to bo \to \Sigma^4 bsp$ and $\mathbf{R}P^\infty \to S^0_{(2)} \to R$. We have $R \wedge bo \simeq \bigvee_{j\geq 0} \Sigma^{4j} H\mathbf{Z}_{(2)}$ and we can get a description of $R \wedge bsp$ from the fibration $\Sigma^4 bo \to bsp \to H\mathbf{Z}_{(2)}$. The E_4-term in 1.5.22 is a filtered form of $\pi_*(\Sigma^3 bsp \wedge \mathbf{R}P^\infty) \oplus \pi_*(bo \wedge \mathbf{R}P^\infty)$; elements with Hopf invariants of the form $\bar{\alpha}_j$ are in the first summand while the other generators make up the second summand. By studying the analog of 1.5.20 we can compute $J_*(\mathbf{R}P^\infty)$ and again the answer [1.5.23(e)] forces the spectral sequence to behave the way it does.

Now we come to the analog of 1.5.21.

1.5.25. Theorem (Mahowald [**4**]). (a) *The composite*

$$\pi_k(\Omega^{2n+k} S^{2n+1}) \to \pi_k(Q\mathbf{R}P^{2n}) \to J_k(\mathbf{R}P^{2n})$$

is onto unless $k \equiv 0 \mod (4)$ and $k \leq 2n$, or $k \equiv 6 \mod (8)$. It is also onto if $k = 2^j$ for $j \geq 3$ or if $k \equiv 2^j - 2 \mod (2^{j+1})$ and $k \geq 2n + 8 + 2j$. When $k \leq 2n$ is a multiple of 4 *and not a power of* 2 *at least* 8*, then the cokernel is $\mathbf{Z}/(2)$; when $k \leq 2n$ is* 2 *less than a multiple of* 8 *but not* 2 *less than a power of* 2*, then the cokernel is $J_k(\mathbf{R}P^{2n}) = J_k(\mathbf{R}P^\infty)$.*

(b) *All elements in the E_∞-term corresponding to elements in $\pi_*(J)$ pull back to the EHP spectral sequence except some of the $\bar{\alpha}_{4t+i} \in E_\infty^{8t+i+5,2}$ for $i = 3, 4$ and $t \geq 0$. Hence $H(\alpha_1) = H(\bar{\alpha}_2) = H(\bar{\alpha}_3) = 1$, $H(\alpha_{t+1}) = \alpha_t$, and if $2^i x = \alpha_{t+1}$ for $x \in \operatorname{im} J$ then $H(x) = \alpha_{t-i}$.* $\square$

Theorem 1.5.23 leads one to believe that $H(\bar{\alpha}_{4t+i}) = \bar{\alpha}_{4t+i-1}$ for $i = 4$, 5 and $t \geq 0$, and that these elements are born on S^2, but this cannot be true in all cases. If $\bar{\alpha}_4$ were born on S^2, its Hopf invariant would be in $\pi_{10}(S^3)$, but this group does not contain $\bar{\alpha}_3$, which is born on S^4. In fact we find $H(\bar{\alpha}_4) = \bar{\alpha}_2$, $H(\bar{\alpha}_5) = \bar{\alpha}_2^2$, and $H(\bar{\alpha}_8)$ is an unstable element.

1.5.26. REMARK. Theorem 1.5.25(b) shows that the portion of $\operatorname{im} J$ generated by $\bar{\alpha}_{4t+2}$ and $\bar{\alpha}_{4t+3}$, i.e., the cyclic summands of order ≥ 8 in dimensions $4k-1$, are born on low-dimensional spheres, e.g., $\bar{\alpha}_{4t+2}$ is born on S^5. However, simple calculations with 1.5.14 show that the generator of $\pi_{4k-1}(O)$ pulls back to $\pi_{4k-1}(O(2k+1))$ and no further. Hence $\bar{\alpha}_{4t+2} \in \pi_{qt+8}(S^5)$ is not actually in the image of the unstable J-homomorphism until it is suspended to S^{4t+3}.

Now we consider the second summand of $J_*(\mathbf{R}P^\infty)$ of 1.5.23(e). The elements $\bar{\alpha}_2 \in E_\infty^{4k,4k-2}$ for $k \geq 1$ have no odd primary analog and we treat them first. The main result of Mahowald [**6**] says there are elements $\eta_j \in \pi_{2^j}(S^0)$ for $j \geq 3$ with Hopf invariant $v = \bar{\alpha}_2$. This takes care of the case $k = 2^{j-2}$ above.

1.5.27. THEOREM. *In the EHP spectral sequence the element* $\nu = \bar{\alpha}_2 \in E_1^{4k,4k-2}$ *for* $k \geq 2$ *behaves as follows* (*there is no such element for* $k = 1$).

(a) *If* $k = 2^{j-2}$, $j \geq 3$ *then the element is a permanent cycle corresponding to* η_j; *this is proved by Mahowald* [**6**].

(b) *If* $k = 2s+1$ *then* $d_4(\nu) = \nu^2$. □

1.5.28. CONJECTURE. *If* $k = (2s+1)2^{j-2}$ *with* $s > 0$ *then* $d_{2^j-2}(\nu) = \eta_j$. □

The remaining elements in 1.5.23(e) appear to be related to the famous Kervaire invariant problem (Mahowald [**7**], Browder [**1**]).

1.5.29. CONJECTURE. *In the EHP spectral sequence the elements*

$$\bar{\alpha}_j \in E_2^{2^{j+1}(t+1)-2,*} \qquad \text{for } j \geq 2,\ t \geq 0$$

behave as follows:

(a) *If there is a framed* $(2^{j+1}-2)$*-manifold with Kervaire invariant one then* $\bar{\alpha}_j \in E_2^{2^{j+1}-2,*}$ *is a nontrivial permanent cycle corresponding to an element* $\theta_j \in \pi_{2^{j+1}-2}(S^0)$ (*These elements are known* (*Barratt, Jones, and Mahowald* [**2**]) *to exist for* $j \geq 0$.)

(b) *If* (a) *is true then the element* $\bar{\alpha}_j \in E_2^{2^{j+1}(2s+1)-2,*}$ *satisfies* $d_r(\bar{\alpha}_j) = \theta_j$ *where* $r = 2^{j+1} - 1 - \dim(\bar{\alpha}_j)$. □

The converse of 1.5.29(a) is proved by Mahowald [**4**] 7.11.

Now we will describe the connection of the EHP spectral sequence with the Segal conjecture. For simplicity we will limit our remarks to the 2-primary case, although everything we say has an odd primary analog. As remarked above, the stable EHP spectral sequence (1.5.12) can be modified so as to converge to the stable homotopy of a stunted projective space. Let $\mathbf{R}P_j = \mathbf{R}P^\infty/\mathbf{R}P_{j-1}$ for $j > 0$; i.e., $\mathbf{R}P^j$ is the infinite-dimensional stunted projective space whose first cell is in dimension j. It is easily seen to be the Thom spectrum of the j-fold Whitney sum of the canonical line bundle over $\mathbf{R}P^\infty$. This bundle can be defined stably for $j \leq 0$, so we get Thom spectra $\mathbf{R}P_j$ having one cell in each dimension $\geq j$ for any integer j.

1.5.30. PROPOSITION. *For each* $j \in \mathbf{Z}$ *there is a spectral sequence converging to* $\pi_*(\mathbf{R}P_j)$ *with*

$$E_1^{k,n} = \begin{cases} \pi_{k-n+1}(S^0) & \text{if } n-1 \geq j \\ 0 & \text{if } n-1 < j \end{cases}$$

and $d_r\colon E_r^{k,n} \to E_r^{k-1,n-r}$. *For* $j = 1$ *this is the stable EHP spectral sequence of* 1.5.12. *If* $j < 1$ *this spectral sequence maps to the stable EHP spectral sequence, the map being an isomorphism on* $E_1^{k,n}$ *for* $n \geq 2$. □

The Segal conjecture for $\mathbf{Z}/(2)$, first proved by Lin [**1**], has the following consequence.

1.5.31. THEOREM. *For each* $j < 0$ *there is a map* $S^{-1} \to \mathbf{R}P_j$ *such that the map* $S^{-1} \to \mathbf{R}P_{-\infty} = \varprojlim \mathbf{R}P_j$ *is a homotopy equivalent after* 2*-adic completion of the source* (*the target is already* 2*-adically complete since* $\mathbf{R}P_j$ *is for* j *odd*). *Consequently the inverse limit over* j *of the spectral sequences of* 1.5.30 *converges to the* 2*-component of* $\pi_*(S^{-1})$. *We will call this limit spectral sequence the* superstable EHP spectral sequence. □

Nothing like this is stated in Lin [**1**] even though it is an easy consequence of his results. A proof and some generalizations are given in Ravenel [**4**]. Notice that $H_*(\mathbf{R}P_{-\infty}) \neq \varprojlim H_*(\mathbf{R}P_j)$; this is a spectacular example of the failure of homology to commute with inverse limits. Theorem 1.5.31 was first conjectured by Mahowald and was discussed by Adams [**14**].

Now consider the spectrum $\mathbf{R}P_0$. It is the Thom spectrum of the trivial bundle and is therefore $S^0 \vee \mathbf{R}P_1$. Hence for each $j < 0$ there is a map $\mathbf{R}P_j \to S^0$ which is nontrivial in mod (2) homology. The cofiber of this map for $j = -1$ can be shown to be R, the cofiber of the map $\mathbf{R}P_1 \to S^0$ of Kahn and Priddy [**2**]. The Kahn–Priddy theorem says this map is surjective in homotopy in positive dimensions. Using these facts we get

1.5.32. THEOREM. *In the spectral sequence of* 1.5.30 *for* $j < 0$,

(a) *no element in* $E_r^{0,k}$ *supports a nontrivial differential*;

(b) *no element in* $E_r^{1,k}$ *is the target of a nontrivial differential*;

(c) *every element of* $E_1^{0,k} = \pi_{k+1}(S^0)$ *that is divisible by* 2 *is the target of a nontrivial* d_1 *and every element of* $E_2^{0,k}$ *for* $k > -1$ *is the target of some* d_r *for* $r \geq 2$; *and*

(d) *every element in* $E_1^{1,k} = \pi_k(S^0)$ *not of order* 2 *supports a nontrivial* d_1 *and every element of* $E_2^{1,k}$ *supports a nontrivial* d_r *for some* $r \geq 2$. □

PROOF. Parts (a) and (b) follow from the existence of maps $S^{-1} \to \mathbf{R}P_j \to S^0$, (c) follows from the Kahn–Priddy theorem, and (d) follows from the fact that the map $\varprojlim \mathbf{R}P_j \to S^0$ is trivial. □

Now the spectral sequence converges to $\pi_*(S^{-1})$, yet 1.5.32(c) indicates that the map $S^{-1} \to \mathbf{R}P_{-\infty}$ induces a trivial map of E_∞-terms, except for $E_\infty^{-1,0}$, where it is the projection of $\mathbf{Z}$ onto $\mathbf{Z}/(2)$. [Here we are using a suitably indexed, collapsing AHSS for $\pi_*(S^{-1})$.] This raises the following question: what element in $E_\infty^{k,-n}$ (for some $n > 0$) corresponds to a given element $x \in \pi_k(S^{-1})$? The determination of n is equivalent to finding the smallest n such that the composite $S^k \xrightarrow{x} S^{-1} \to \mathbf{R}P_{-n-1}$ is nontrivial. The Kahn–Priddy theorem tells us this composite is trivial for $n = 0$ if $k \geq 0$ or $k = -1$ and x is divisible by 2; and the Segal conjecture (via 1.5.31) says the map is nontrivial for some $n > 0$. Now consider the cofiber sequence $S^{-n-1} \to \mathbf{R}P_{-n-1} \to \mathbf{R}P_{-n}$. The map from S^k to $\mathbf{R}P_{-n}$ is trivial by assumption so we get a map from S^k to S^{-1-n}, defined modulo some indeterminacy. Hence $x \in \pi_{k+1}(S^0)$ gives us a coset $M(x) \subset \pi_{k+1+n}(S^0)$ which does not contain zero.

We call $M(x)$ the Mahowald invariant of x, and note that n, as well as the coset, depends on x. The invariant can be computed in some cases and appears to be very interesting. For example, we have

1.5.33. THEOREM. *Let $\imath$ be a generator of $\pi_0(S^0)$. Then for each $j>0$, $M(2^j\imath)$ contains α_j, a preimage in $\pi_*(S^0)$ of the $\alpha_j \in \pi_*(J)$ of* 1.5.23. □

A similar result holds for odd primes. In 1.5.31 we replace the $\mathbf{R}P_j$ by Thom spectra of certain bundles over $B\Sigma_p$, and $M(p^j\imath) \ni \alpha_j$ for α_j, as in 1.5.19. We also have

1.5.34. CONJECTURE. *$M(\theta_j)$ contains θ_{j+1} for θ_j as in* 1.5.29. □

1.5.35. CONJECTURE. *Whenever the Greek letter elements* (1.3.17) *$\alpha_j^{(n)}$ and $\alpha_j^{(n+1)}$ exist in homotopy, $\alpha_j^{(n+1)} \in M(\alpha_j^{(n)})$.* □

One can mimic the definition of the Mahowald invariant in terms of the Adams spectral sequence or Adams–Novikov E_2-terms and in the latter case prove an analog of these conjectures. At $p=2$ one can show (in homotopy) that $M(\alpha_1) \ni \bar{\alpha}_2$, $M(\bar{\alpha}_2) \ni \bar{\alpha}_3$, and $M(\bar{\alpha}_3) \ni \bar{\alpha}_3^2 = \theta_3$. This suggests using the iterated Mahowald invariant to *define* (up to indeterminacy) Greek letter elements in homotopy, and that θ_j is a special case (namely $\alpha_1^{(j+1)}$) of this definition.

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