Equivariant stable homotopy theory and the Kervaire invariant problem

WORK IN PROGRESS

This book in being written now and revised almost daily. It is nearly complete but still has a lot of flaws. Please do not post any part of it on the web or cite it publicly.

Read at your own risk!

April 5, 2020

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To Tim, Rose, Vivienne and Elizabeth,
the twinkles in our eyes

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Introduction

The purpose of this book is to introduce equivariant stable homotopy theory in a way that will make the methods of [HHR16] accessible to a well informed graduate student and facilitate further research in this area.

The research leading to [HHR16] was an example of the aphorism "computation precedes theory." In 2005 the second two authors set out to study the homotopy fixed point sets of finite subgroups of the Morava stabilizer groups $\mathbf{S}_n$ under their action on the Morava spectra E_n; Hill joined us a short time later. We knew this would be an interesting project, but we did not anticipate that it would lead to a solution to the Kervaire invariant problem, named after Michel Kervaire (1927–2007). We like to say we went hiking in the Alps and found a short cut up Mount Everest.

After making various assumptions about how things work in equivariant stable homotopy theory, we did the computation that led to our main theorem. Upon further reflection we realized that the existing literature on the subject did not provide an adequate framework for our calculations. This led to the lengthy appendices in [HHR16] providing the necessary theoretical infrastructure. Despite their length, they were written as tersely as possible so as to economize on journal space.

A similar account will be given here at a more leisurely pace, with numerous (more than 150) examples illustrating various concepts. In particular we do our best to motivate the definition of the model structure we need on the category of equivariant orthogonal spectra, the subject of Chapter 9.

Other works called *Equivariant stable homotopy theory* are [GM95], [LMSM86] and [Seg71], and the phrase occurs in numerous other titles.

Nearly every item in the bibliography can be found in the third author's online archive.

1.1 The Kervaire invariant theorem and the ingredients of its proof

Very briefly, the Kervaire invariant problem concerns the fate of the elements h_j^2 in the classical Adams spectral sequence at the prime 2, originally introduced by J. Frank Adams FRS (1930–1989) in [Ada58]. We refer the reader to [Rav86] for a description of it. A theorem of William Browder [Bro69] says that h_j^2 is a permanent cycle iff there exists a framed manifold of dimension $2^{j+1}-2$ with nontrivial Kervaire invariant. The hypothetical element in $\pi^S_{2^{j+1}-2}$ represented by such a framed manifold is denoted by θ_j.

Here π_k^S denotes the stable k-stem, the value of $\pi_{n+k}S^n$ for large n. It is also the kth homotopy group of the sphere spectrum, which was often denoted by S^0 in early works on the subject. **In this book we will denote the sphere spectrum by S^{-0} to avoid confusion with the space S^0**; see Remark 1.4.13 below.

After the publication of Browder's theorem in 1969 there were numerous unsuccessful attempts to prove the existence of θ_j for all $j > 0$. Mark Mahowald (1931–2013) named his sailboat "Thetajay." His colleague and coauthor Michael Barratt (1927–2015) referred to the possibility that they did not all exist as the "Doomsday Hypothesis." More precisely, he gave this name to conjecture, originally due to Joel Cohen [Coh70], that in the Adams spectral sequence only a finite number of elements in each filtration were permanent cycles. The first five θ_js were known to exist, the construction of θ_5 being the subject of [BJM84] and recently simplified in [Xu16].

After 1980, interest in the problem faded as the failed attempts of the 1970s convinced the homotopy theory community that it was beyond their reach. In 2009, just before we announced our theorem, Victor Snaith published [Sna09], a witty account of the state of the art at that moment. Three of his statements are worth repeating here.

About the decline of interest in the problem he said,

> As ideas for progress on a particular mathematics problem atrophy it can disappear. Accordingly I wrote this book to stem the tide of oblivion.

About his own involvement in it he wrote,

> For a brief period overnight we were convinced that we had the method to make all the sought after framed manifolds – a feeling which must have been shared by many topologists working on this problem. All in all, the temporary high of believing that one had the construction was sufficient to maintain in me at least an enthusiastic spectator's interest in the problem.

Best of all,

> In the light of the above conjecture and the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might

> turn out to be a book about things which do not exist. This [is] why the quotations which preface each chapter contain a preponderance of utterances from the pen of Lewis Carroll.

1.1A The main theorem

Indeed the sought after framed manifolds (with a small number of exceptions) **do not exist**. The following was first announced by the second author in April, 2009, in a lecture at a conference in Edinburgh honoring the 80th birthday of Sir Michael Atiyah (1929–2019).

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi^S_{2^{j+1}-2}$ do not exist for $j \geqslant 7$.*

The status of θ_6 in the 126-stem remains open.

In [Rav78] (see also [Rav86, §6.4]) the third author showed long ago that the cohomology of the subgroup of order p in $\mathbf{S}_{p-1}$ could be used to show that odd primary analogs of the Kervaire invariant elements do not exist for $p \geqslant 5$.

Here $\mathbf{S}_n$ denotes the *nth* **Morava stabilizer group**, which plays a critical role in chromatic homotopy theory. We refer the reader to [Rav86, Chapter 6] for its definition and properties. It is a pro-p-group that is the strict automorphism group of a height n formal group law over a sufficiently large finite field of characteristic p. Its cohomology in some sense controls the nth chromatic layer of the Adams-Novikov E_2-term , as explained first in [MRW77] and later in [Rav86, Chapter 5]. It is known to have elements of order p^{i+1} precisely when $(p-1)p^i$ divides n. In particular $\mathbf{S}_{p-1}$ has a cyclic subgroup of order p, and for $p = 2$, $\mathbf{S}_4$ has one of order 8.

Figure 1.1
Fenway's dream

This odd primary Kervaire invariant problem was easier (and hence solved thirty years earlier) than the 2-primary case because Hirosi Toda [Tod67, Tod68] had shown a decade earlier that $\theta_2 \in \pi^S_{2p^2(p-1)-2}$ does not exist. This could be reinterpreted as a proof that the corresponding element in the Adams-Novikov spectral sequence, $\beta_{p/p}$, supports a nontrivial differential hitting $\alpha_1\theta_1^p = \alpha_1\beta_1^p$. The cohomology of $C_p \subseteq \mathbf{S}_{p-1}$ then provided a way to leverage this into a proof that $\theta_j = \beta_{p^{j-1}/p^{j-1}}$ supports a differential hitting $\alpha_1\theta_{j-1}^p$ for all $j \geqslant 2$.

At the prime 2 there was no analog of Toda's theorem; there was no θ_j that was known not to exist. We also know that while the θ_js themselves can be detected in the cohomology of $C_8 \subseteq \mathbf{S}_4$, their products cannot be. This means that the leverage of [Rav78] is not available. The methods of [HHR16], which include the use of equivariant stable homotopy theory, are quite different.

We have a much simpler way of defining the action of the group C_8. In chromatic homotopy theory (for background on this topic see Lurie's 2010 Harvard course [Lur10], the 2013 Talbot syllabus with its numerous references [BL13], [Rav86] and [Rav92]) we learn that $\mathbf{S}_n$, the strict automorphism group of a height n formal group law F_n over the field $\mathbf{F}_{p^n}$, acts on the ring over which its universal deformation (lifting to characteristic zero) is defined. The same goes for $\mathbf{G}_n$, the extension of $\mathbf{S}_n$ by the Galois group of $\mathbf{F}_{p^n}$ over $\mathbf{F}_p$. This ring turns out to be $\pi_0 E_n$, where E_n is the nth Morava E-theory, a variant of the Johnson-Wilson spectrum $E(n)$. These considerations leads to an "action" of $\mathbf{S}_n$ on the spectrum E_n, but it is only defined **up to homotopy**.

This awkward state of affairs was the motivating issue for the Goerss-Hopkins-Miller theorem in the early 1990s; see [Rez98] and [GH04]. Morava's E_n was known to be an E_∞-ring spectrum, meaning that it has a multiplication that is homotopy commutative in the strongest possible sense. They showed that for an E_∞-ring spectrum R there is a **space** of E_∞-ring automorphisms $Aut(R)$. This required a deeper understanding of the stable homotopy category than was prevalent at the time. In the case of $R = E_n$, we knew that the set of path components of this space had to be $\mathbf{G}_n$. **They showed that each path component is contractible.**

This means that $Aut(E_n)$ is homotopy equivalent to $\mathbf{G}_n$, and that for any closed subgroup $G \subseteq \mathbf{G}_n$ one can define the homotopy fixed point spectrum E_n^{hG}. In particular $E_n^{h\mathbf{G}_n} = L_{K(n)}S^0$, the Bousfield localization of the sphere spectrum with respect to the nth Morava K-theory. The calculation of [Rav78] could be reinterpreted as a calculation with $E_{p-1}^{hC_p}$.

The proof of this gratifying result is quite technical. **Fortunately we do not have to deal with it here.** We have a much more direct way of mapping $\pi_* S^0$ to the cohomology of a cyclic 2-group using equivariant stable homotopy theory.

1.1B The equivariant approach

The starting point is the action of C_2 on the complex cobordism spectrum MU via complex conjugation. The resulting C_2-spectrum is denoted by $MU_{\mathbf{R}}$, and known as "real cobordism."

This terminology derives from Atiyah's definition of real K-theory in [Ati66]. (The reader hoping for a definition of "reality" as a technical term will be disppointed to find that the word only appears in the title of the paper.) For him a "real" space is a topological space X equipped with an involution τ. For $x \in X$ he denotes $\tau(x)$ by $\overline{x}$. A "real" vector bundle E over a real space X was not a bundle of real vector spaces, but a complex vector bundle equipped with an involution compatible with that on X such that the induced map from the fiber over x to that over $\overline{x}$ is conjugate linear.

A key example of a real space is the set of complex points of an algebraic variety X defined over the real numbers, which comes equipped with an involution related to complex conjugation. Its fixed point set is the space of real points of X. In particular X could be the Grassmannian variety $G_{n,k}$, whose real and complex points are respectively the spaces of linear k-dimensional subspaces of an n-dimensional vector space over the real and complex numbers. Taking the colimit as n and k go to infinity, we get the classifying space BU equipped an an involution induced by complex conjugation. We denote this object by $BU_{\mathbf{R}}$. We can Thomify this and get a C_2-equivariant spectrum $MU_{\mathbf{R}}$, the **real cobordism spectrum.** Its precise construction is the subject of Chapter 12. It was first studied by Peter Landweber in [Lan68], and subsequently by Michikazu Fujii [Fuj76], Shôrô Araki (1930–2005) [Ara79] and by Po Hu and Igor Kriz [HK01].

The next step is to elevate the C_2-spectrum $MU_{\mathbf{R}}$ to a C_{2^n}-spectrum. More generally when H is a subgroup of G, we define a norm functor N_H^G from the category of H-spectra to that of G-spectra; see Definition 9.7.3. Roughly speaking for, an H-spectrum E, the G-spectrum $N_H^G E$ is $E^{\wedge|G/H|}$ with G permuting the H-invariant factors. A recent theorem of Jeremy Hahn and XiaoLin Danny Shi [HS17] implies that there is a map $N_{C_2}^{C_{2^n}} MU_{\mathbf{R}} \to E_{2^{n-1}}$ which is equivariant with respect to the action of C_{2^n} as a subgroup of $\mathbf{S}_{2^{n-1}}$.

Classically there is a way to derive Atiyah's real K-theory spectrum $K_{\mathbf{R}}$ from $MU_{\mathbf{R}}$, and the former is 8-periodic, meaning that $\pi_i K_{\mathbf{R}}$ and its equivariant variants only depend on the congruence class of i modulo 8. It is a retract of a mapping telescope obtained from $MU_{\mathbf{R}}$ by inverting a certain element in its equivariant homotopy group.

There are similar spectra $K_{\mathbf{H}}$ and $K_{\mathbf{O}}$ that are retracts of telescopes related to $N_{C_2}^{C_4} MU_{\mathbf{R}}$ and $N_{C_2}^{C_8} MU_{\mathbf{R}}$ which are respectively 32 and 256-periodic. The use of the symbols $\mathbf{H}$ and $\mathbf{O}$ here is purely a matter of convenience as these spectra have very little to do with the quaternions or octonions. The spectrum $K_{\mathbf{H}}$ is studied extensively in [HHR17c], where it and $K_{\mathbf{O}}$ are denoted by $K_{[2]}$ and $K_{[3]}$.

There is a similar telescope associated with $N_{C_2}^{C_{2^n}} MU_{\mathbf{R}}$ for each $n \geqslant 1$. It is obtained by inverting an element D specified for the case $n = 3$ in Corollary 13.3.25. Theorem 13.3.23 shows that it has periodicity $2^{n+1+2^{n-1}}$. Passing from the telescope to its retract $K_{[n]}$ simplifies explicit calculations of homotopy groups, but is not needed for our current purposes.

1.1C The spectrum Ξ

The fixed point spectrum Ξ (denoted by Ω in [HHR16]) of the telescope for $N_{C_2}^{C_8} MU_{\mathbf{R}}$, which we denote by $\Xi_{\mathbf{O}}$, is the central object in the solution to the

Kervaire invariant problem. It is a nonconnective ring spectrum with a unit map $S^0 \to \Xi$. It has the following properties:

Key properties of the C_8 fixed point spectrum Ξ.

(i) **Detection Theorem.** *It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial.* **This means that if θ_j exists, we will see its image in $\pi_*(\Xi)$.** *es*

(ii) **Periodicity Theorem.** *It is 256-periodic, meaning that $\pi_k(\Xi)$ depends only on the reduction of k modulo 256. As in the case of Bott periodicity, we have a stable equivalence $\Omega^{256}\Xi \simeq \Xi$.*

(iii) **Gap Theorem.** $\pi_k(\Xi) = 0$ *for* $-4 < k < 0$.

These will be proved in Chapter 13, after developing the necessary machinery in the intervening eleven chapters. We will identify Ξ in Definition 13.3.27. Property (iii) is our zinger. Its proof involves a new tool we call the **slice spectral sequence**.

If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in $\pi_{254}(\Xi)$. On the other hand, (ii) and (iii) imply that $\pi_{254}(\Xi) = 0$, so θ_7 cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geqslant 7$. (Historical note: the third author spent part of his undergraduate career living in a rented room at 254 Elm Street near Oberlin College. It was there that he first became acquainted with homotopy theory, but at that time he did not appreciate the significance his street number. In 2002, he lived in a rented house at 62 Eden Street in Cambridge, UK.)

At the present time, the three theorems listed above are just about **all** we know about Ξ, which is just enough to prove the main theorem. If we could show that $\pi_{126}\Xi = 0$, we would know that θ_6 does not exist. This appears to be a daunting calculation. We computed $\pi_* K_{\mathbf{H}}^{C_4}$ in [HHR17c] as a warmup exercise for it.

The reader may wonder **why we chose the group** C_8. Briefly, the argument for the Detection Theorem §1.1C(i) would break down were we to use C_2 or C_4. We will say more about this in §13.4, specifically in Remark 13.4.17. It would go through for any larger cyclic 2-group, but the period would be greater, which would lead to a weaker theorem. For C_{16} the period is 8192, so the resulting theorem would say that θ_j does not exist for $j \geqslant 12$ rather than for $j \geqslant 7$. The Gap Theorem holds for any cyclic 2-group.

1.2 Background and history

1.2A Pontryagin's early work on homotopy groups of spheres

The Arf-Kervaire invariant problem has its origins in the early work of Lev Pontryagin (1908–1988) on a geometric approach to the homotopy groups of spheres, [Pon38], [Pon50] and [Pon55].

Pontryagin's approach to maps $f : S^{n+k} \to S^n$ is to assume that f is smooth and that the base point y_0 of the target is a regular value. (Any continuous f can be continuously deformed to a map with this property.) This means that $f^{-1}(y_0)$ is a closed smooth k-manifold M in S^{n+k}. Let D^n be the closure of an open ball around y_0. If it is sufficiently small, then $V^{n+k} = f^{-1}(D^n) \subset S^{n+k}$ is an $(n+k)$-manifold homeomorphic to $M \times D^n$ with boundary homeomorphic to $M \times S^{n-1}$. It is also a tubular neighborhood of M^k and comes equipped with a map $p : V^{n+k} \to M^k$ sending each point to the nearest point in M. For each $x \in M$, $p^{-1}(x)$ is homeomorphic to a closed n-ball B^n. The pair $(p, f|V^{n+k})$ defines an explicit homeomorphism

$$V^{n+k} \xrightarrow[\approx]{(p,f|V^{n+k})} M^k \times D^n$$

This structure on M^k is called a **framing**, and M is said to be **framed in** $\mathbf{R}^{n+k}$. A choice of basis of the tangent space at $y_0 \in S^n$ pulls back to a set of linearly independent normal vector fields on $M \subset \mathbf{R}^{n+k}$. These will be indicated in Figures 1.2–1.3 below.

Conversely, suppose we have a closed sub-k-manifold $M \subset \mathbf{R}^{n+k}$ with a closed tubular neighborhood V and a homeomorphism h to $M \times D^n$ as above. This is called a **framed sub-k-manifold** of $\mathbf{R}^{n+k}$. Some remarks are in order here.

- The existence of a framing puts some restrictions on the topology of M. All of its characteristic classes must vanish. In particular it must be orientable.
- A framing can be twisted by a map $g : M \to SO(n)$, where $SO(n)$ denotes the group of orthogonal $n \times n$ matrices with determinant 1. Such matrices act on D^n in an obvious way. The twisted framing is the composite

$$\begin{array}{ccccc} V & \xrightarrow{h} & M^k \times D^n & \longrightarrow & M^k \times D^n \\ & & (m, x) & \longmapsto & (m, g(m)(x)). \end{array}$$

  When $M^k = S^k$, this leads to the Hopf-Whitehead J-homomorphism of Remark 1.2.2 below.
- If we drop the assumption that M is framed, then the tubular neighborhood V is a (possibly nontrivial) disk bundle over M. The map $M \to y_0$ needs to be replaced by a map to the classifying space for such bundles, $BO(n)$. This leads to unoriented bordism theory, which was analyzed by René Thom

(1923–2002) in [Tho54]. Two helpful references for this material are the books by Milnor-Stasheff [MS74] and Robert Stong (1936–2008) [Sto68a].

Pontryagin constructs a map $P(M, h) : S^{n+k} \to S^n$ as follows. We regard S^{n+k} as the one point compactification of $\mathbf{R}^{n+k}$ and S^n as the quotient $D^n/\partial D^n$. This leads to a diagram

$$\begin{tikzcd}
(V, \partial V) \arrow[r, "h"] \arrow[d, hook] & M \times (D^n, \partial D^n) \arrow[r, "p_2"] & (D^n, \partial D^n) \arrow[d] \\
(\mathbf{R}^{n+k}, \mathbf{R}^{n+k} - \mathrm{int}V) \arrow[r] & (S^{n+k}, S^{n+k} - \mathrm{int}V) \arrow[r, dotted, "P(M,h)"] & (S^n, \{\infty\})
\end{tikzcd}$$

The map $P(M, h)$ is the extension of $p_2 h$ obtained by sending the compliment of V in S^{n+k} to the point at infinity in S^n. For $n > k$, the choice of the embedding (but not the choice of framing) of M into the Euclidean space is irrelevant. Any two embeddings (with suitably chosen framings) lead to the same map $P(M, h)$ up to continuous deformation.

To proceed further, we need to be more precise about what we mean by continuous deformation. Two maps $f_1, f_2 : X \to Y$ are **homotopic** if there is a continuous map $h : X \times [0, 1] \to Y$ (called a **homotopy between f_1 and f_2**) such that

$$h(x, 0) = f_1(x) \qquad \text{and} \qquad h(x, 1) = f_2(x).$$

Now suppose $X = S^{n+k}$, $Y = S^n$, and the map h (and hence f_1 and f_2) is smooth with y_0 as a regular value. Then $h^{-1}(y_0)$ is a framed $(k+1)$-manifold N whose boundary is the disjoint union of $M_1 = f^{-1}(y_0)$ and $M_2 = g^{-1}(y_0)$. This N is called a **framed cobordism** between M_1 and M_2, and when it exists the two closed manifolds are said to be **framed cobordant**. An example is shown in Figure 1.2.

Let $\Omega^{\mathrm{fr}}_{k,n}$ denote the cobordism group of framed k-manifolds in $\mathbf{R}^{n+k}$. The above construction leads to Pontryagin's isomorphism

$$\Omega^{\mathrm{fr}}_{k,n} \xrightarrow{\approx} \pi_{n+k}(S^n).$$

First consider the case $k = 0$. Here the 0-dimensional manifold M is a finite set of points in $\mathbf{R}^n$. Each comes with a framing which can be obtained from a standard one by an element in the orthogonal group $O(n)$. We attach a sign to each point corresponding to the sign of the associated determinant. With these signs we can count the points algebraically and get an integer called the **degree of f**. Two framed 0-manifolds are cobordant iff they have the same degree.

Now consider the case $k = 1$. M is a closed 1-manifold, i.e., a disjoint union of circles. Two framings on a single circle differ by a map from S^1 to the group

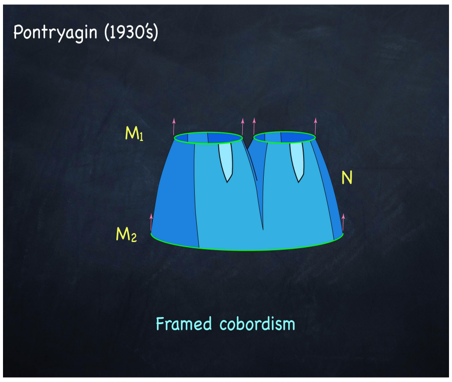

Figure 1.2 A framed cobordism between $M_1 = S^1 \coprod S^1 \subset \mathbf{R}^2$ and $M_2 = S^1 \subset \mathbf{R}^3$ with $N \subset [0,1] \times \mathbf{R}^2$. The normal framings on the circles can be chosen so they extend over N.

$SO(n)$, and it is known that

$$\pi_1(SO(n)) = \begin{cases} 0 & \text{for } n = 1 \\ \mathbf{Z} & \text{for } n = 2 \\ \mathbf{Z}/2 & \text{for } n > 2. \end{cases}$$

It turns out that any disjoint union of framed circles is cobordant to a single framed circle. This can be used to show that

$$\pi_{n+1}(S^n) = \begin{cases} 0 & \text{for } n = 1 \\ \mathbf{Z} & \text{for } n = 2 \\ \mathbf{Z}/2 & \text{for } n > 2. \end{cases}$$

The case $k = 2$ is more subtle. As in the 1-dimensional case we have a complete classification of closed 2-manifolds, and it is only necessary to consider path connected ones. The existence of a framing implies that the surface is orientable, so it is characterized by its genus.

If the genus is zero, namely if $M = S^2$, then there is a framing which extends to a 3-dimensional ball. This makes M cobordant to the empty set, which means that the map is **null homotopic** (or, more briefly, **null**), meaning that it is homotopic to a constant map. Any two framings on S^2 differ by an element in $\pi_2(SO(n))$. This group is known to vanish, so any two framings on S^2 are equivalent, and the map $f : S^{n+2} \to S^n$ is null.

Now suppose the genus is one, as shown in Figure 1.3. Suppose we can find an embedded arc as shown on which the framing extends to a disk. Then there is a cobordism which effectively cuts along the arc and attaches two disks as shown. This process is called **framed surgery**. If we can do this, then we have converted the torus to a 2-sphere and shown that the map $f : S^{n+2} \to S^n$ is null.

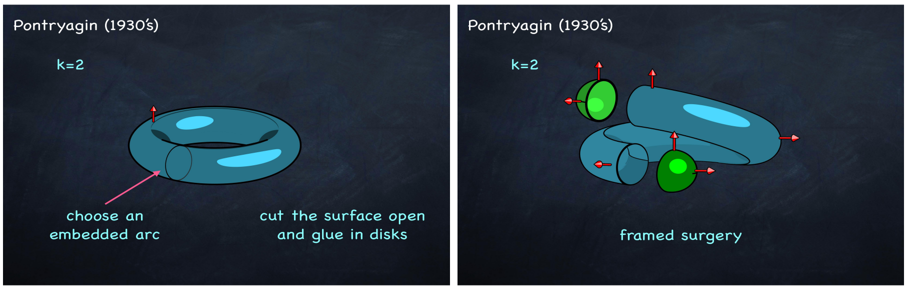

Figure 1.3 The case $k = 2$ and genus 1. If the framing on the embedded arc extends to a disk, then there is a cobordism (called a framed surgery) that converts the torus to a 2-sphere as shown.

When can we find such a closed curve in M? It must represent a generator of $H_1(M)$ and carry a trivial framing. This leads to a map

$$\varphi : H_1(M; \mathbf{Z}/2) \to \mathbf{Z}/2 \tag{1.2.1}$$

defined as follows. Each class in H_1 can be represented by a closed curve which is framed either trivially or nontrivially. It can be shown that homologous curves have the same framing invariant, so φ is well defined. At this point Pontryagin made a famous mistake which went undedected for over a decade: **he assumed that φ was a homomorphism**. We now know this is not the case, and we will say more about it below in §1.2C.

On that basis he argued that φ must have a nontrivial kernel, since the source group is $(\mathbf{Z}/2)^2$. Therefore there is a closed curve along which we can do the surgery shown in Figure 1.3. It follows that M can be surgered into a 2-sphere, leading to the erroneous conclusion that $\pi_{n+2}(S^n) = 0$ for all n. Freudenthal [Fre38] and later George Whitehead [Whi50] both proved that it is $\mathbf{Z}/2$ for $n \geqslant 2$. Pontryagin corrected his mistake in [Pon50], and in [Pon55] he gave a complete account of the relation between framed cobordism and homotopy groups of spheres.

Remark 1.2.2. The Hopf-Whitehead J-homomorphism.

Suppose our framed manifold is S^k with a framing that extends to a D^{k+1}. This will lead to the trivial element in $\pi_{n+k}(S^n)$, but twisting the framing can lead to nontrivial elements. The twist is determined up to homotopy by an element in $\pi_k(SO(n))$. Pontryagin's construction thus leads to the homomorphism

$$\pi_k(SO(n)) \xrightarrow{J} \pi_{n+k}(S^n)$$

introduced by Hopf [Hop35] and Whitehead [Whi42]. Both source and target known to be independent of n for $n > k+1$.

In this case the source group for each k (denoted simply by $\pi_k(SO)$ since *n is irrelevant) was determined by Bott [Bot59] in his remarkable periodicity theorem. He showed*

$$\pi_k(SO) = \begin{cases} \mathbf{Z} & \text{for } k \equiv 3 \text{ or } 7 \bmod 8 \\ \mathbf{Z}/2 & \text{for } k \equiv 0 \text{ or } 1 \bmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Here is a table showing these groups for $k \leqslant 10$.

k	1	2	3	4	5	6	7	8	9	10
$\pi_k(SO)$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0

In each case where the group is nontrivial, the image under J of its generator is known to generate a direct summand; see [Ada66, Theorems 1.1, 1.3, 1.5 and 1.6]. In the jth case we denote this image by β_j and its dimension by $\phi(j)$, which is roughly $2j$. (They will figure in Hypothesis 1.2.4 below.) The first three of these are the Hopf maps $\eta \in \pi_1^S$, $\nu \in \pi_3^S$ and $\sigma \in \pi_7^S$. After that we have $\beta_4 \in \pi_8^S$, $\beta_5 \in \pi_9^S$, $\beta_6 \in \pi_{11}^S$, and so on.

For the case $\pi_{4m-1}(SO) = \mathbf{Z}$, the image under J is known to be a cyclic group whose order a_m is the denominator of $B_m/4m$, where B_m is the mth Bernoulli number. Details can be found in [Ada66, Theorems 1.5 and 1.6] and [MS74, Appendix B]. Here is a table showing these values for $m \leqslant 8$.

m	1	2	3	4	5	6	7	8
a_m	24	240	504	480	264	65,520	24	16,320

1.2B Our main result

Our main theorem can be stated in three different but equivalent ways:

- **Manifold formulation:** It says that a certain geometrically defined invariant $\Phi(M)$ (the Arf-Kervaire invariant, to be defined later) on certain manifolds M is always zero.
- **Stable homotopy theoretic formulation:** It says that certain long sought hypothetical maps between high dimensional spheres do not exist.
- **Unstable homotopy theoretic formulation:** It says something about the EHP sequence (to be defined below), which has to do with unstable homotopy groups of spheres.

The problem solved by our theorem is nearly 50 years old. There were several unsuccessful attempts to solve it in the 1970s. They were all aimed at proving the opposite of what we have proved.

Here again is the stable homotopy theoretic formulation.

Main Theorem. *The Arf-Kervaire elements $\theta_j \in \pi^S_{2^{j+1}-2}$ do not exist for $j \geqslant 7$.*

1.2C The manifold formulation

Let λ be a nonsingular anti-symmetric bilinear form on a free abelian group H of rank $2n$ with mod 2 reduction $\overline{H}$. It is known that $\overline{H}$ has a basis of the form $\{a_i, b_i\colon 1 \leqslant i \leqslant n\}$ with

$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

A **quadratic refinement of** λ is a map $q : \overline{H} \to \mathbf{Z}/2$ satisfying

$$q(x + y) = q(x) + q(y) + \lambda(x, y)$$

Its **Arf invariant** is

$$\mathrm{Arf}(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbf{Z}/2.$$

In 1941 Cahit Arf (1910–1997)[Arf41] proved that this invariant (along with the number n) determines the isomorphism type of q.

An equivalent definition is the "democratic invariant" of Browder. The elements of $\overline{H}$ "vote" for either 0 or 1 by the function q. The winner of the election (which never ends in a tie) is $\mathrm{Arf}(q)$. Here is a table illustrating this for three possible refinements q, q' and q'' when $\overline{H}$ has rank 2.

x	0	a	b	$a+b$	Arf invariant
$q(x)$	0	0	0	1	0
$q'(x)$	0	1	1	1	1
$q''(x)$	0	1	0	0	0

The value each refinement on $a + b$ is determined by those on a and b, and q''

is isomorphic to q. Thus the vote is three to one in each case. When $\overline{H}$ has rank 4, it is 10 to 6.

Let M be a $2m$-connected smooth closed manifold of dimension $4m + 2$ with a framed embedding in $\mathbf{R}^{4m+2+n}$. We saw above that this leads to a map $f : S^{n+4m+2} \to S^n$ and hence an element in $\pi_{n+4m+2}(S^n)$.

Let $H = H_{2m+1}(M; \mathbf{Z})$, the homology group in the middle dimension. Each $x \in H$ is represented by an immersion $i_x : S^{2m+1} \looparrowright M$ with a stably trivialized normal bundle. H has an antisymmetric bilinear form λ defined in terms of intersection numbers.

In 1960 Michel Kervaire (1927–2007) [Ker60] defined a quadratic refinement q on its mod 2 reduction in terms of the trivialization of each sphere's normal bundle. The **Kervaire invariant** $\Phi(M)$ is defined to be the Arf invariant of q. In the case $m = 0$, when the dimension of the manifold is 2, Kervaire's q is Pontryagin's map φ of (3.2.11).

What can we say about $\Phi(M)$?

- Kervaire [Ker60] showed it must vanish when $k = 2$. This enabled him to construct the first example of a topological manifold (of dimension 10) without a smooth structure. This is illustrated in Figure 1.4. N is a smooth 10-manifold with boundary given as the union of two copies of the tangnent disk bundle of S^5. The boundary is homeomorphic to S^9. Thus we can get a closed topological manifold X by gluing on a 10-ball along its common boundary with n, or equivalently collapsing ∂N to a point. X then has nontrivial Kervaire invariant. On the other hand, Kervaire proved that any smooth framed manifold must have trivial Kervaire invariant. Therefore the topological framed manifold X cannot have a smooth structure. Equivalently, the boundary ∂N cannot be diffeomorphic to S^9. It must be an exotic 9-sphere.
- For $k = 0$ there is a framing on the torus $S^1 \times S^1 \subset \mathbf{R}^4$ with nontrivial Kervaire invariant. Pontryagin used it in [Pon50] (after some false starts in the 30s) to show $\pi_{n+2}(S^n) = \mathbf{Z}/2$ for all $n \geqslant 2$.
- There are similar constructions for $k = 1$ and $k = 3$, where the framed manifolds are $S^3 \times S^3$ and $S^7 \times S^7$ respectively. Like S^1, S^3 and S^7 are both parallelizable, meaning that their trivial tangent bundles are trivial. The framings can be twisted in such a way as to yield a nontrivial Kervaire invariant.
- Edgar Brown and Frank Peterson (1930–2000) [BP66] showed that it vanishes for all positive even k. This means that apart from the 2-dimensioanl case, any smooth framed manifold with nontrivail Kervaire invariant must a dimension congruent to 6 modulo 8.
- William Browder [Bro69] showed that it can be nontrivial only if $k = 2^{j-1} - 1$ for some positive integer j. This happens iff the element h_j^2 is a permanent

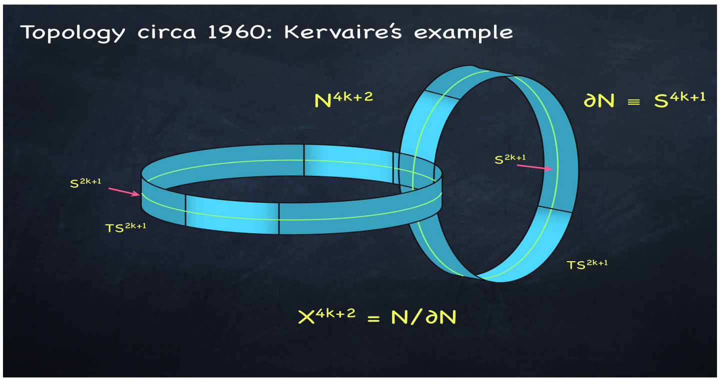

Figure 1.4 Kervaire's example. N is a smooth framed $(4k + 2)$-manifold whose boundary is homeomorphic to S^{4k+1}. The tubular neighbohood of each S^{2k+1} is homeomorphic to its tangent bundle. If ∂N is diffeomorphic to S^{4k+1}, then X is a closed smooth framed $(4k+2)$-manifold with nontrivial Kervaire invariant. We now know this is the case only when $k = 0$, 1, 3, 7, 15 and possibly 31. Otherwise ∂N is an exotic $(4k + 1)$-sphere that is a framed boundary, and collapsing its boundary to a point gives a topological manifold without a smoothness structure. The case $k = 2$ was Kervaire's original example.

cycle in the Adams spectral sequence, which was originally introduced in [Ada58]. (More information about it can be found in [Rav86] and[Rav04].) The corresponding element in $\pi^S_{n+2^{j+1}-2}$ is θ_j, the subject of our theorem. **This is the stable homotopy theoretic formulation of the problem.**

- θ_j is known to exist for $1 \leqslant j \leqslant 3$, i.e., in dimensions 2, 6, and 14. In these cases the relevant framed manifold is $S^{2^j-1} \times S^{2^j-1}$ with a twisted framing as discussed above. The framings on S^{2^j-1} represent the elements h_j in the Adams spectral sequence. The Hopf invariant one theorem of Adams [Ada60] says that for $j > 3$, h_j is not a permanent cycle in the Adams spectral sequence because it supports a nontrivial differential. (His original proof was not written in this language, but had to do with secondary cohomlogy operations.) This means that for $j > 3$, a smooth framed manifold representing θ_j (i.e., having a nontrivial Kervaire invariant) cannot have the form $S^{2^j-1} \times S^{2^j-1}$.
- θ_j is also known to exist for $j = 4$ and $j = 5$, i.e., in dimensions 30 and 62. In both cases the existence was first established by purely homotopy theoretic means, without constructing a suitable framed manifold. For $j = 4$ this was done by Barratt, Mahowald and Tangora in [MT67] and [BMT70]. A framed 30-manifold with nontrivial Kervaire invariant was later constructed by Jones [Jon78]. For $j = 5$ the homotopy theory was done in 1985 by Barratt-Jones-Mahowald in [BJM84]. Their construction was simplified substantially by Zhouli Xu in [Xu16].

- Our theorem says θ_j does **not** exist for $j \geqslant 7$. The case $j = 6$ is still open.

Figure 1.4 illustrates Kervaire's construction of a framed $(4k+2)$-manifold with nontrivial Kervaire invariant. In all cases except $k = 0$, 1 or 3, any framing of this manifold will do because the tangent bundle of S^{2k+1} is nontrivial and leads to a nontrivial invariant. What the picture does not tell us is whether the bounding sphere S^{4k+1} is diffeomorphic to the standard sphere. If it is, then attaching a $(4k+2)$-disk to it will produce a smooth framed manifold with nontrival Kervaire invariant. If it is not, then we have an exotic $(4k+1)$-sphere bounding a framed manifold and hence not detected by framed cobordism.

1.2D The unstable formulation

Assume all spaces in sight are localized and the prime 2. For each $n > 0$ there is a fiber sequence due to Ioan James, [Jam55], [Jam56a], [Jam56b] and [Jam57]

$$S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}. \tag{1.2.3}$$

Here $\Omega X = \Omega^1 X$ where $\Omega^k X$ denotes the space of continuous base point preserving maps to X from the k-sphere S^k, known as the kth loop space of X. This leads to a long exact sequence of homotopy groups

$$\dots \to \pi_{m+n}(S^n) \xrightarrow{E} \pi_{m+n+1}(S^{n+1}) \xrightarrow{H} \pi_{m+n+1}(S^{2n+1}) \xrightarrow{P} \pi_{m+n-1}(S^n) \to \dots$$

Here

- E stands for **E**inhängung, the German word for suspension.
- H stands for **H**opf invariant.
- P stands for Whitehead **p**roduct.

Assembling these for fixed m and various n leads to a diagram

$$\begin{tikzcd}
 & \pi_{m+n+1}(S^{2n-1}) \arrow[d, "P"] & \pi_{m+n+2}(S^{2n+1}) \arrow[d, "P"] & \pi_{m+n+3}(S^{2n+3}) \arrow[d, "P"] & \\
\dots \arrow[r, "E"] & \pi_{m+n-1}(S^{n-1}) \arrow[r, "E"] \arrow[d, "H"] & \pi_{m+n}(S^n) \arrow[r, "E"] \arrow[d, "H"] & \pi_{m+n+1}(S^{n+1}) \arrow[r, "E"] \arrow[d, "H"] & \dots \\
 & \pi_{m+n-1}(S^{2n-3}) & \pi_{m+n}(S^{2n-1}) & \pi_{m+n+1}(S^{2n+1}) &
\end{tikzcd}$$

where

- Sequences of arrows labeled H, P, E, H (or any subset thereof) in that order are exact.
- The groups in the top and bottom rows are inductively known, and we can compute those in the middle row by induction on n.

- The groups in the top and bottom rows vanish for large n, making E an isomorphism.
- An element in the middle row has trivial suspension (is killed by E) iff it is in the image of P.
- It desuspends (is in the image of E) iff its Hopf invariant (image under H) is trivial.

When $m = n - 1$ this diagram is

$$\begin{tikzcd}
 & & \pi_{2n+1}(S^{n+1}) \arrow[d, "H"] & & \\
 & \pi_{2n}(S^{2n-1}) \arrow[d, "P"] & \mathbf{Z} \arrow[d, "P"] & 0 \arrow[d, "P"] & \\
\cdots \arrow[r, "E"] & \pi_{2n-2}(S^{n-1}) \arrow[r, "E"] \arrow[d, "H"] & \pi_{2n-1}(S^n) \arrow[r, "E"] \arrow[d, "H"] & \pi_{2n}(S^{n+1}) \arrow[r, "E"] \arrow[d, "H"] & \cdots \\
 & \pi_{2n-2}(S^{2n-3}) & \mathbf{Z} & 0 &
\end{tikzcd}$$

The image under P of the generator of the upper $\mathbf{Z}$ is called the **Whitehead square**, denoted by $w_n \in \pi_{2n-1}(S^n)$.

- When n is even, $H(w_n) = 2$ and w_n has infinite order.
- w_n is trivial for $n = 1$, 3 and 7. In those cases the generator of the upper $\mathbf{Z}$ is the Hopf invariant (image under H) of one of the three Hopf maps in $\pi_{2n+1}(S^{n+1})$,

$$S^3 \xrightarrow{\eta} S^2, \qquad S^7 \xrightarrow{\nu} S^4 \quad \text{and} \qquad S^{15} \xrightarrow{\sigma} S^8.$$

- For other odd values of n, twice the generator of the upper $\mathbf{Z}$ is $H(w_{n+1})$, so w_n has order 2.
- It turns out that w_n is divisible by 2 iff $n = 2^{j+1} - 1$ and θ_j exists, in which case $w_n = 2\theta_j$.
- Each Whitehead square $w_{2n+1} \in \pi_{4n+1}(S^{2n+1})$ (except the cases $n = 0$, 1 and 3) desuspends to a lower sphere until we get an element with a nontrivial Hopf invariant, which is always some β_j as in Remark 1.2.2. More precisely we have

$$H(w_{(2s+1)2^j-1}) = \beta_j$$

for each $j > 0$ and $s \geqslant 0$. This result is essentially Adams' 1962 solution to the vector field problem [Ada62].

Recall the EHP sequence

$$\cdots \to \pi_{m+n}(S^n) \xrightarrow{E} \pi_{m+n+1}(S^{n+1}) \xrightarrow{H} \pi_{m+n+1}(S^{2n+1}) \xrightarrow{P} \pi_{m+n-1}(S^n) \to \cdots$$

Given some $\beta_j \in \pi_{\phi(j)+2n+1}(S^{2n+1})$ for $\phi(j) < 2n$, one can ask about the

Hopf invariant of its image under P, which vanishes when β_j is in the image of H. In most cases the answer is known and is due to Mahowald, [Mah67] and [Mah82]. They are also discussed in [Rav86, §1.5, especially Theorem 1.5.23].

The remaining cases have to do with θ_j. The answer that he had hoped for is the following, which can be found in [Mah67]. To our knowledge, Mahowald never referred to this as the World Without End Hypothesis. We chose that term to emphasize its contrast with the Doomsday Hypothesis.

World Without End Hypothesis (Mahowald 1967) 1.2.4.

(i) The Arf-Kervaire element $\theta_j \in \pi^S_{2^{j+1}-2}$ *exists for all* $j > 0$.

(ii) It desuspends to $S^{2^{j+1}-1-\phi(j)}$ *and its Hopf invariant is* β_j.

(iii) Let $j, s > 0$ *and suppose that* $m = 2^{j+2}(s+1) - 4 - \phi(j)$ *and* $n = 2^{j+1}(s+1) - 2 - \phi(j)$. *Then* $P(\beta_j)$ *has Hopf invariant* θ_j.

This describes the systematic behavior in the EHP sequence of elements related to the image of J, and the θ_j are an essential part of the picture. Because of our theorem, **we now know that this hypothesis is incorrect**.

Remark 1.2.5. The Doomsday Hypothesis. *In the 1970s, Michael Barratt (1927–2015) wanted very much for Hypothesis 1.2.4 to be true. Thus he gave the name Doomsday Hypothesis to the statement (originally conjectured by Joel Cohen in [Coh70]) that in the Adams spectral sequence only a finite number of elements in each filtration were permanent cycles.*

4/7/19. Check Joel Cohen's book and look up Milgram's list of problems at the 1970 AMS Symposium in Madison.

This was already known to be true for filtration 1. It had been known since roughly 1960 that $E_2^{1,*}$ *(the 1-line of the Adams* E_2*-term) was spanned by the elements*

$$h_j \in E_2^{1,2^j} \qquad \text{for } j \geqslant 0.$$

In [Ada60] Adams had shown that

$$d_2(h_{j+1}) = h_0 h_j^2 \qquad \text{for } j \geqslant 3,$$

meaning that h_0, h_1, h_2 *and* h_3 *are the only surviving elements in filtration 1. They correspond respectively to the degree 2 map and the three Hopf maps*

$$\eta : S^3 \to S^2, \qquad \nu : S^7 \to S^4 \qquad \text{and} \qquad \sigma : S^{15} \to S^8.$$

If it were also true in filtration 2, then only a finite number of the θ_j*s would exist. In [Mah77] Mahowald showed that* $h_1 h_j$ *survives for all* $j \geqslant 3$. *This provides a counter example to the hypothesis stated above but says nothing about the fate of the* θ_j*s.*

In 1995 Minami in [Min95, page 966] proposed a modified form of the statement having to do with the homomorphism

$$\mathrm{Sq}^0 : E_2^{s,t} \to E_2^{s,2t},$$

which is known to send h_j and h_j^2 respectively to h_{j+1} and h_{j+1}^2. His **New Doomsday Conjecture** *is that for each s there is an n such that no element in $E_2^{s,*}$ in the image of $(\mathrm{Sq}^0)^n$ survives. In particular, h_j^2 survives for only finitely many j.*

On the other hand, Mahowald's elements $\eta_j = h_1 h_j$ are not related to each other in this way since $\mathrm{Sq}^0(h_1 h_j) = h_2 h_{j+1}$.

1.2E Questions raised by our theorem

EHP sequence formulation. Hypothesis 1.2.4 was the nicest possible statement of its kind given all that was known prior to our theorem. Now we know it cannot be true since θ_j does not exist for $j \geqslant 7$. **This means the behavior of the indicated elements $P(\beta_j)$ for $j \geqslant 7$ is a mystery.**

Adams spectral sequence formulation. We now know that the h_j^2 for $j \geqslant 7$ are not permanent cycles, so they have to support nontrivial differentials. **We have no idea what their targets are.**

Manifold formulation. Here we are not aware of any new questions raised by our result. It appears to be the final page in the story.

Our method of proof offers a new tool for studying the stable homotopy groups of spheres. We look forward to learning more with it in the future.

1.3 The foundational material in this book

The topics covered in this volume are presented in the most **logical** order possible. This approach differs from the "computation precedes theory" presentation in [HHR16] in which the logical foundations of the calculation were described in two lengthy appendices **after** the description of the calculation itself. A similar approach was used by Aldridge "Pete" Bousfield and Daniel Kan (1928–2013) in the "yellow monster" [BK72], Phillip Hirschhorn's book on model categories [Hir03], and the third author's previous books [Rav86] and [Rav92]. The present approach means that the next five chapters will introduce the requisite tools from category theory including a lengthy description of Quillen model categories and Bousfield localization.

These chapters are designed to present the required tools as clearly as possible. They are not intended to be rigorously self contained. Whenever a lengthy proof is available elsewhere in the literature, we will omit it but tell the reader exactly where she can find it. They are also not intended to be comprehensive introductions to the topics in question. Our choice of definitions and results stated, which may strike some readers as idiosyncratic, is dictated by the needs of the subsequent chapters. We have chosen to ignore some recent developments in these areas, such as the theory of ∞-categories, because we do not need them. On the other hand we have chosen to embrace enriched category theory, the subject of Chapter 3, since it provides the cleanest framework for the definition of equivariant spectra in Chapter 9.

Equivariant homotopy theory, the arena in which our computation is done, first appears in Chapter 8, and our star player, the spectrum $MU_{\mathbf{R}}$, is constructed in Chapter 12. Our main computational tool, the slice spectral sequence, first appears in Chapter 11.

The inexperienced reader may well wonder why we need to devote over two hundred pages to category theory before we even step into the pool of homotopy theory. The answer is that the tools it provides enable us to proceed with far more elegance and rigor than we could without them. This "categorification of algebraic topology" is most apparent in the twenty-first century approach to **spectra**, the fundamental objects of study in stable homotopy theory.

Spectra were first introduced in print [Lim59] in 1959 by Elon Lima (1929–2017), then a student of Edwin Spanier (1921–1996) at the University of Chicago, and later a prominent mathematical educator in Brazil. A spectrum E was defined to be a sequence of pointed spaces E_n for nonnegative integers n, with structure maps

$$\epsilon_n : \Sigma E_n \to E_{n+1}.$$

In the first examples E_n was $(n-1)$-connected, but this was not a formal requirement. The motivation for this definition was the observation that $(n-1)$**-connected spaces behave very nicely in dimensions less than roughly** $2n$. The first theorem in this direction may have been the Freudenthal Suspension Theorem [Fre38] of 1938; see [Rav86, Theorem 1.1.4].

1.3A The hare and the tortoise

Spectra were defined to create a world where n could be arbitrarily large so we could enjoy this nice behavior in **all** dimensions. Perhaps the first extensive account of this new world was a course given by Adams at UC Berkeley in 1961 and published as [Ada64]. In it (pages 22–23), he said the following.

> I want to go ahead and construct a stable category. Now I should warn you that the proper definitions here are still a matter for much pleasurable argumentation

among the experts. The debate is between two attitudes, which I'll personify as the tortoise and the hare. The hare is an idealist: his preferred position is one of elegant and all embracing generality. He wants to build a new heaven and a new earth and no half-measures. If he had to construct the real numbers he'd begin by taking **all** sequences of rationals, and only introduce that tiresome condition about convergence when he was absolutely forced to.

The tortoise, on the other hand, takes a much more restrictive view. He says that his modest aim is to make a cleaner statement of known theorems, and he'd like to put a lot of restrictions on his stable objects so as to be sure that his category has all the good properties he may need. Of course, the tortoise tends to put on more restrictions than are necessary, but the truth is that the restrictions give him confidence.

You can decide which side you're on by contemplating the Spanier-Whitehead dual of an Eilenberg-Mac Lane object. This is a "complex" with "cells" in all stable dimensions from $-\infty$ to $-n$. According to the hare, Eilenberg-Mac Lane objects are good, Spanier-Whitehead duality is good, therefore this is a good object: And if the negative dimensions worry you, he leaves you to decide whether you are a tortoise or a chicken. According to the tortoise, on the other hand, the first theorem in stable homotopy theory is the Hurewicz Isomorphism Theorem, and this object has no dimension at all where that theorem is applicable, and he doesn't mind the hare introducing this object as long as he is allowed to exclude it. Take the nasty thing away!

The resulting homotopy theoretical paradise was described very nicely by Boardman-Vogt in [BV73] about a decade later, but there were some serious technical problems, especially in connection with smash products. For a further account of the adventures of the hare and the tortoise with an assessment of Boardman's work, see [May99b].

It is safe to say now, over half a century later, that **the hare has prevailed**. The technical problems that vexed stable homotopy theorists for a generation have been vanquished. The increasingly sophisticated use of category theory has been instrumental in this triumph. Many of the advances that led to this happy state of affairs occured in the 1990s, the decade following Adams' untimely death in a car crash.

1.3B A letter to Adams

The third author has tried to imagine what it would be like to relate these developments to him.

Dear Frank,

Stable homotopy theory is in much better shape now than when you left us. The definitions are much cleaner and we have a smash product with all of the nice features you could ask for. As you can probably guess, Peter May has been pounding away at this for decades, but you did not live long enough to see just how much success he and his coauthors have had.

Along with Tony Elmendorf, his former student, and Igor Kriz, a Czech immigrant (you may also be interested to know that the Berlin Wall came down, the Soviet

Union collapsed and the Cold War ended, all within three years of your death), he used a lemma due to the second author [EKMM97, Lemma I.5.4] to define a smash product on a certain category of spectra that is **strictly** associative and commutative in 1993. You heard me right, I said strictly, not just up to homotopy (higher or otherwise) or some other convoluted equivalence relation, but pointwise, on the nose! In 1997 (with a fourth coauthor, Mike Mandell, another former student) they published a book about it, [EKMM97].

That construction is complicated and I do not fully understand it. Fortunately Jeff Smith, with help from Mark Hovey and Brooke Shipley, found a simpler way to do it, which they described in their account of symmetric spectra, [HSS00]. May and Mandell used a similar approach in their account of equivariant orthogonal spectra, [MM02]. This one I do understand. It uses a wonderful construction called the **Day convolution**, originally discovered in 1970 [Day70] by the Australian category theorist Brian Day (1945–2012). It is a purely categorical result that happens to be exactly what is needed to define the smash product of spectra. This means the proof that said smash product is strictly commutative and associative is "purely formal." Ironically, Day's first job out of graduate school was a postdoctoral position at the University of Chicago, presumably at the behest of Saunders Mac Lane. As far as I can tell, Brian and Peter did not interact mathematically.

So how do Mandell and May do it? As you know, a spectrum E was originally defined to be a sequence of pointed spaces E_n, one for each integer $n \geqslant 0$, along with pointed structure maps $\epsilon_n : \Sigma E_n \to E_{n+1}$. For them **a spectrum is a functor** from a certain small category $\mathscr{J}$ (the Mandell-May category of Definition 8.9.24) to the category $\mathcal{T}$ of pointed topological spaces. Since $\mathscr{J}$ is small, such a functor could be regarded as a **diagram of pointed spaces**, although it would not be a diagram you could actually draw because it would be infinite. This point of view is developed further in the companion paper to [MM02], [MMSS01] by Mandell, May, Stefan Schwede and Brooke Shipley.

The objects of $\mathscr{J}$ are finite dimensional real orthogonal vector spaces. Since such a vector space is determined up to isomorphism by its dimension, a $\mathcal{T}$-valued functor E on $\mathscr{J}$ gives us a sequence of spaces E_n, as in the orginal definition, but with some additional structure. In order to spell out the additional structure, I need to tell you about the morphisms in $\mathscr{J}$. **This is where things start to get tricky.**

I said the objects of $\mathscr{J}$ are certain vector spaces, but I did not say that $\mathscr{J}$ is **the category** of such vector spaces and inner product preserving maps as usually defined. In order to describe $\mathscr{J}$ we need to generalize what we mean by a category, because $\mathscr{J}$ is not a category in the usual sense. Instead it is an **enriched category**; see Chapter 3. Such things were first studied by Samuel Eilenberg (1913—1998) and Max Kelly (1930–2007) [EK66], and were the subject of Kelly's book [Kel82].

In an ordinary category $\mathcal{C}$ one has a collection (possibly a set) of objects, and for each pair of objects X and Y a set $\mathcal{C}(X, Y)$ (possibly empty) of morphisms $X \to Y$. Of course every object has an identity morphism, and given a third object Z we have a map

$$\mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \tag{1.1}$$

that tells us how to compose morphisms. This map is itself a morphism with suitable properties in $\mathcal{S}et$, the category of sets.

In an enriched category, one has objects as before, but $\mathcal{C}(X, Y)$ is no longer a set or even a class. Instead it is an object in a second category $\mathcal{V}$, which need not be

$\mathcal{S}et$ at all. We say then that $\mathcal{C}$ is **enriched over** $\mathcal{V}$. By this definition, an ordinary category is enriched over $\mathcal{S}et$.

This auxiliary category $\mathcal{V}$ has to have a structure that enables to make sense of the source of the morphism in (1.1). In other words it needs a binary operation, analogous to Cartesian product in $\mathcal{S}et$, that allows us to combine two objects into a third. This binary operation must have a unit analogous to the one element set. A category so endowed is said to be **symmetric monoidal**; see § 2.6 for more information. The relevant example for us is $\mathcal{T}$, the category of pointed topological spaces. Its binary operation is the smash product, for which the unit object is S^0.

Having said what an enriched category is, I can tell you more about the Mandell-May category $\mathscr{J}$, which is enriched over $\mathcal{T}$. This means that for finite dimensional real orthogonal vector spaces V and W, the morphism object $\mathscr{J}(V, W)$ is a pointed topological space, which is defined as follows.

Let $O(V, W)$ denote the (possibly empty) space (also known as a Stiefel manifold, named after Eduard Stiefel (1909–1978)) of orthogonal embeddings of V into W. For each such embedding τ, let $W - \tau(V)$ denote the orthogonal compliment of $\tau(V)$ in W. We can regard it as the fiber of a vector bundle over $O(V, W)$, and **we define** $\mathscr{J}(V, W)$ **to be its Thom space**.

When the dimension of V exceeds that of W, the embedding space $O(V, W)$ is empty, which means the Thom space $\mathscr{J}(V, W)$ is a point. When V and W have the same dimension, the vector bundle has zero dimensional fibers, so $\mathscr{J}(V, W) = O(V)_+$, the orthogonal group with a disjoint base point. When the dimension of W exceeds that of V, we can think of $\mathscr{J}(V, W)$ as a wedge of copies of $S^{W-\tau(V)}$ parametrized by the space of embeddings $O(V, W)$.

Given a third such vector space U, the analog of (1.1) is a suitable map

$$\mathscr{J}(V, W) \wedge \mathscr{J}(U, V) \to \mathscr{J}(U, W). \tag{1.2}$$

It is induced by composition of orthogonal embeddings, i.e., by a map

$$O(V, W) \times O(U, V) \to O(U, W).$$

It is not necessary to think of points in $\mathscr{J}(V, W)$ as maps from V to W. The space $\mathscr{J}(V, W)$ is not a topologized set of ordinary morphisms, but a replacement of the usual morphism set by a morphism object in $\mathcal{T}$. The map of (1.2) tells us how the replacement of composition works.

Mandell-May define an **orthogonal spectrum** E (Definition 9.0.2) to be a functor from $\mathscr{J}$ to $\mathcal{T}$, which happens to be enriched over itself. Since an object of $\mathscr{J}$ is a finite dimensional vector space, which is determined up to isomorphism by its dimension, we denote the image of the functor on $\mathbf{R}^n$ by E_n, as in the orginal definition. Functoriality implies that we have structure maps

$$\epsilon_{n,n+k} : \mathscr{J}(\mathbf{R}^n, \mathbf{R}^{n+k}) \wedge E_n \to E_{n+k}. \tag{1.3}$$

for all $n, k \geqslant 0$.

For $k = 0$ this amounts to a left action on the space E_n of the orthogonal group $O(n)$. That group also acts on $\mathscr{J}(\mathbf{R}^n, \mathbf{R}^{n+k})$ on the right by precomposition. These two actions lead to one on the smash product in (1.3) with $\epsilon_{n,n+k}$ factoring through the orbit space. For $k = 1$ that orbit space is ΣE, so **we have the map**

$$\epsilon_n = \epsilon_{n,n+1} : \Sigma E_n \to E_{n+1}$$

as in the orginal definition. The difference is that now the map does not depend on the choice of orthogonal embedding of $\mathbf{R}^n$ into $\mathbf{R}^{n+1}$ as it did in the classical case. **This coordinate free definition is technically convenient.**

We can define the suspension spectra $\Sigma^\infty X$ for a pointed space X by

$$(\Sigma^\infty X)_n = \Sigma^n X$$

with the evident structure maps. More generally we can define the smash product of a pointed space K with a spectrum E by $(K \wedge E)_n = K \wedge E_n$. We can also define a spectrum E^K (maps from K to E) by

$$(E^K)_n = \mathcal{T}(K, E_n).$$

Since spectra are functors, maps between them are natural transformations. This means a map $f : E \to F$ of spectra is a collection of continuous pointed maps $f_n : E_n \to F_n$ compatible with the structure maps. This is analogous to what you called a **function** in [Ada74b, page 140].

As you pointed out on [Ada74b, page 141], there is no function $f : \Sigma^\infty S^1 \to \Sigma^\infty S^0$ for which $f_2 : S^3 \to S^2$ is the Hopf map η. Since we all love the Hopf map, we would like to have such a function. The fix you suggested is to replace the source spectrum $E = \Sigma^\infty S^1$ by a spectrum E' defined by

$$E'_n = \begin{cases} * & \text{for } n = 0, 1 \\ S^{n+1} & \text{otherwise} \end{cases}$$

Then there is an obvious function $g : E' \to E$ for which g_n is an isomorphism for $n \geqslant 2$, and a function $f' : E' \to \Sigma^\infty S^0$ with $f'_2 = \eta$. You defined a **map** $E \to F$ [Ada74b, page 142] to be an equivalence class of composites of the form $f' = fg$ as above.

You also defined a homotopy between two functions $E \to F$ [Ada74b, page 144] in terms of a map $I \ltimes E \to F$, a homotopy between maps, in similar terms. Finally, you defined a **morphism** in your category [Ada74b, page 143] to be a homotopy class of such maps.

Thus you made a distinction between functions, maps and morphisms. Subsequent experience has led us to approach these issues a little differently. We have learned that the framework provided by Quillen's theory of model categories, the subject of Chapters 4–6 of this book, is very helpful. Among other things, it tells us there are two categories one should consider here. The first is the category of spectra $\mathcal{S}p$ in which the objects are the functors $\mathscr{J} \to \mathcal{T}$ described above, and the morphisms are natural transformations between them, what you called "functions."

Before describing the second category, we need to define stable homotopy groups of spectra and weak equivalences between spectra. This can be done as you did in [Ada74b, §III.3]. Then one gets a **homotopy category** $\operatorname{Ho}\mathcal{S}p$ (see Definition 4.3.16) having the same objects as $\mathcal{S}p$ in which weak equivalences are invertible. Your "morphisms" are morphisms in this category. Your "maps" are equivalence classes of "functions" precomposed with weak equivalences.

Now, at last, I can tell you about smash products. You defined the smash product of two spectra in [Ada74b, §III.4] and spent 30 pages showing that it has the desired properties (commutativity and associativity with the sphere spectrum as unit) **up to homotopy**, that is up to coherent natural weak equivalence. Another way of saying this is that we get a symmetric monoidal structure in the homotopy category $\operatorname{Ho}\mathcal{S}p$. The Mandell-May smash product (Definition 9.1.21), which is based on a very insightful observation by Jeff Smith, leads to such a structure in $\mathcal{S}p$ itself. This smash product has the desired properties up to coherent natural **isomorphism**. Not only is this a huge improvement, it has a much shorter and more elegant proof.

If we have two spectra X and Y, each of which is a functor $\mathscr{J} \to \mathcal{T}$, then together they give us a functor from $\mathscr{J} \times \mathscr{J}$ to $\mathcal{T} \times \mathcal{T}$. Now consider the diagram

$$\begin{tikzcd}
(\mathbf{R}^m, \mathbf{R}^n) \arrow[r, mapsto] & (X_m, Y_n) \arrow[r, mapsto] & X_m \wedge Y_n \\
\mathscr{J} \times \mathscr{J} \arrow[r, "X \times Y"] \arrow[dr, "\oplus"'] & \mathcal{T} \times \mathcal{T} \arrow[r, "\wedge"] & \mathcal{T} \\
& \mathscr{J} \arrow[ur, dashed, "X \wedge Y"] & (X \wedge Y)_{m+n} \\
& \mathbf{R}^{m+n} \arrow[ur, dashed, mapsto] &
\end{tikzcd} \tag{1.4}$$

The smash product we are looking for is a yet to be defined functor

$$X \wedge Y : \mathscr{J} \to \mathcal{T}$$

with suitable properties. We are **not** hoping for the diagram of (1.4) to commute. That would mean

$$X_m \wedge Y_n \cong (X \wedge Y)_{m+n}$$

in all cases, which is not a reasonable thing to expect. On the other hand, we do expect to have maps

$$\eta_{m,n} : X_m \wedge Y_n \to (X \wedge Y)_{m+n}. \tag{1.5}$$

They should be induced by a natural transformation η from the composite functor $\mathscr{J} \times \mathscr{J} \to \mathcal{T}$ on the top of the triangle in (1.4) to the one on the bottom.

It turns out that the right way to define $X \wedge Y$ involves a universal property of this natural transformation. In order to state it, we will replace (1.4) with the following diagram, which will be discussed further in §2.5. Suppose we have categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, with functors F and K as in

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \eta"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, dashed, "L"'] &
\end{tikzcd}$$

We wish to extend the functor F along K to a new functor

$$L : \mathcal{D} \to \mathcal{E}$$

with a natural transformation $\eta : F \Rightarrow LK$. The composite functor LK need not be the same as F. Instead we want L and η to have the following universal property: given another such extension G with a natural transformation $\gamma : F \Rightarrow GK$ as in the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \gamma"] & \mathcal{E} \\
& \mathcal{D}, \arrow[ur, dashed, "G"'] &
\end{tikzcd}$$

there is a unique natural transformation $\alpha : L \Rightarrow G$ with

$$\gamma = (\alpha K)\eta$$

as in the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] \arrow[dr, phantom, shift left=3, "\eta\Downarrow"] & & \mathcal{E} \\
& \mathcal{D} \arrow[ur, dashed, "L"] \arrow[ur, dashed, bend right, "G"'] \arrow[ur, phantom, shift right=2, "\exists!\Searrow\alpha"] &
\end{tikzcd}$$

If such an L exists, it is unique and is called the **left Kan extension of F along K**. It is so named because such functors were first studied by Dan Kan in [Kan58a]; see §2.5. The bottom line is that such an L exists when the categories $\mathcal{C}$ and $\mathcal{D}$ are small and the category $\mathcal{E}$ is closed under colimits. These conditions are met by the categories of (1.4).

There is also an explicit formula for L under these conditions that is described below in § 2.5B. In the case at hand, where $L = X \wedge Y$, it is as follows. Define pointed spaces

$$W_n = \bigvee_{0 \leqslant i \leqslant n} X_i \wedge Y_{n-i}$$

and

$$W'_n = \bigvee_{0 \leqslant i \leqslant n-1} X_i \wedge S^1 \wedge Y_{n-1-i}.$$

Then the maps $\eta_{i,n-i}$ of (1.5) determine a map

$$W_n \to (X \wedge Y)_n.$$

The two maps

$$\begin{tikzcd}
& X_i \wedge S^1 \wedge Y_j \arrow[dl, "\epsilon_{i,i+1} \wedge Y_j"'] \arrow[dr, "X_i \wedge \epsilon_{j,j+1}"] & \\
X_{i+1} \wedge Y_j & & X_i \wedge Y_{j+1}
\end{tikzcd}$$

lead to two maps $\alpha, \beta : W'_n \to W_n$. **Then $(X \wedge Y)_n$ is the coequalizer of these two maps**, meaning the quotient of the space W_n obtained by identifying the two images of W'_n with each other. **This is similar in spirit but not identical to the double telescope you described in [Ada74b, pages 173–176].**

How do we know that this smash product has the properties advertized? This is the subject of the Day Convolution Theorem 3.3.5. Suppose that $\mathcal{D}$ is a small symmetric monoidal category (such as $\mathscr{J}$) enriched over a cocomplete (Definition 2.3.25) closed symmetric monoidal category (Definition 2.6.33) $\mathcal{V}$ such as $\mathcal{T}$. Then we can define a binary operation on the category $[\mathcal{D}, \mathcal{V}]$ (Definition 3.2.18) of functors $\mathcal{D} \to \mathcal{V}$ (the category $\mathcal{S}p$ of orthogonal spectra in our case) using a left Kan extension as in (1.4). The theorem says that this binary operation makes the functor category itself a closed symmetric monoidal category. Its unit is defined in a certain way in terms of the unit objects of $\mathcal{D}$ and $\mathcal{V}$. In the present case this unit is the sphere spectrum as expected.

I hope you agree this is a big improvement over the state of affairs of forty years ago.

In closing I have two additional comments for you.

(i) It is not difficult to adapt this setup to the equivariant case. This is the main point of [MM02]. For a finite group G, let $\mathcal{T}_G$ be the category of pointed G-spaces and continuous (but not necessarily equivariant) pointed maps. Then the mapping space $\mathcal{T}_G(X,Y)$ has a pointed G-action of its own, for which the fixed point set, $\mathcal{T}_G(X,Y)^G$, is the space of all equivariant maps. Hence $\mathcal{T}_G$ is enriched over itself. See Chapter 8 for more discussion.

However, if we want to do homotopy theory, we must limit ourselves to equivariant maps. The reason is that the fiber or cofiber of a map between G-spaces has a well defined G-action only when the map is equivariant. We denote the corresponding category, which is enriched over $\mathcal{T}$, by $\mathcal{T}^G$.

The category $\mathscr{J}_G$ (Definition 8.9.24) has finite dimensional orthogonal representations V of G as objects. The morphism space $\mathscr{J}_G(V,W)$ is the same Thom space as in the nonequivariant case, but now it has a G-action based on the ones on V and W. Hence $\mathscr{J}_G$ is enriched over $\mathcal{T}_G$. **We define a G-spectrum E to be an enriched functor** $\mathscr{J}_G \to \mathcal{T}_G$, and we denote the image of V by E_V and the resulting category by $\mathcal{S}p_G$. The Day Convolution Theorem still applies, so we get a nice smash product as before.

As in the case of spaces, in order to do homotopy theory we must limit ourselves to equivariant maps. We denote the corresponding category by $\mathcal{S}p^G$.

(ii) You might worry that orthogonal spectra are rarer than spectra as originally defined since they appear to have more structure. Fortunately this is not the case. It is shown by Mandell *et al* in [MMSS01] (see Remark 7.2.34) that every ordinary (meaning as defined by Lima) spectrum can be described as an orthogonal one with the help of a left Kan extension. Better yet, all of the computations done with ordinary spectra, in particular everything you did in [Ada64], are still valid in the new category of orthogonal spectra, as well as in various others that have been proposed and studied in recent years. Remarkably, the shifting theoretical foundations of our subject have had no impact on the calculations we actually want to do. **Computation precedes theory. Our intuition about spectra was right all along.**

Thanks for reading, and best of luck in your future travels,

Doug

1.4 Highlights of later chapters

The remaining chapters of this book are written in an order that is logical but not necessarily in the order most convenient for the reader. Our approach differs from that of (in chronological order) [BK72], [Rav86], [Rav92], [Hir03] and [HHR16]. In each of these works the material of greatest interest to the authors was presented first, followed by later chapters or appendices on foundational material needed in the opening chapters. Here we spell out the foundational material **before** treating “the good stuff.”

The proofs of the three statements in §1.1C, and hence of the main theorem, do not appear until the final chapter. Equivariant homotopy theory and orthogonal G-spectra are not discussed until Chapter 8 and Chapter 9 respectively. The star of our show, the real cobordism spectrum $MU_{\mathbf{R}}$, is not introduced until Chapter 12.

Part ONE comprises the next five chapters. Here we collect the relevant definitions from ordinary category theory (Chapter 2), enriched category theory (Chapter 3) and the theory of model categories (Chapter 4, Chapter 5 and Chapter 6). This part of the book should be thought of as a tool box for our study of equivariant stable homotopy theory, which formally begins in Chapter 7. Our choice of topics in these chapters may seem eclectic, but they are dictated by the needs of later chapters. In most cases proofs are provided only when they are not in the literature; when they are, we indicate precisely where they can be found.

Very little in these five chapters is original, and experts are advised to skip them on first reading, only referring back to them when necessary. We include them for the convenience of those who are not experts in these matters, particularly graduate students. Such readers will hopefully find all of the category theoretic definitions and statements needed later in the book **here in one place**.

In Part TWO we use these categorical tools to set up equivariant stable homotopy theory. Some readers may find our approach old fashioned in that no use is made here of ∞-categories. We go to a lot of trouble in Chapter 9, specifically Theorem 9.2.13, to define the model structure on $\mathcal{S}p^G$, the category of orthogonal G-spectra, that suits our purposes. We have heard claims that the theory of ∞-categories could eliminate the need for such effort. However, at the time of this writing we have yet to see anything close to a detailed account of such a shortcut.

We also make no use of operads here.

The fun really begins in Part THREE. In Chapter 11 we introduce the slice filtration in the category $\mathcal{S}p^G$ of orthogonal G-spectra for a finite group G. It is an equivariant generalization of the connectivity filtration in the nonequivariant case. The latter leads to the Postnikov tower (originally due to Mikhail Postnikov (1927–2004)) in which the “layers” are Eilenberg-Mac Lane spectra, that is spectra having a single nontrivial homotopy group. In a similar way the slice filtration leads to a slice tower in which the layers, which we call **slices**, are G-spectra underlain by Eilenberg-Mac Lane spectra. However their equivariant homotopy groups are **not** concentrated in a single dimension. They form the input for the **slice spectral sequence** converging to the equivariant homotopy groups of the spectrum we started with. **It is our main computational tool**.

In Chapter 12 we construct the star of our show, the real cobordism spectrum $MU_{\mathbf{R}}$. It can be thought of as the complex cobordism spectrum MU equipped with an action of the group C_2 induced by complex conjugation. We have a construction called **the norm** that enables us to promote it to a G-spectrum for any group G conaining C_2. We denote it by $N_{C_2}^{G}MU$, and we find that its slice tower has remarkably pleasant properties. Our spectrum Ξ is the fixed point set of a certain telescope associated with $N_{C_2}^{C_8}MU$.

In the climactic Chapter 13 we prove that Ξ satisfies the Gap, Periodicity and Detection Theorems of §1.1C. The Main theorem follows.

1.4A Ordinary category theory

The contents of Chapter 2, which is about ordinary category theory, are listed in its opening paragraphs. They include adjoint functors, limits and colimits, ends and coends, left and right Kan extensions, symmetric monoidal categories and Grothendieck fibrations.

The definitions of categories, functors and natural transformations are given in §2.1. We assume for now that the reader is familiar with them.

Informally there are two types of categories. First there are categories of objects that are of interest for reasons not having to do with category theory. These include the categories of sets, of topological spaces, of groups, and so on. One might say **these categories occur in nature**, as nature is understood by mathematicians. The collections of objects in these categories tend to be proper classes, that is they are too large to be sets. We will refer to them for now as **large categories**; this term does not appear in the literature as far as we know.

Then there are **synthetic categories** (also a term not in the literature) invented by mathematicians primarily for the purpose of studying large categories. They tend to be small (this term is used in the literature), meaning their collections of objects are sets rather than proper classes. Commutative diagrams in a large category $\mathcal{C}$ can be interpreted as $\mathcal{C}$-valued functors on a small category J. The main objects of interest in this book, G-spectra for a finite group G, are best viewed from this perspective.

Isomorphism and equivalence of categories. An isomorphism between categories $\mathcal{C}$ and $\mathcal{D}$ is exactly what one would expect. There are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that the composites FG and GF are each identities.

A more interesting notion is that of categorical equivalence. For each object c in $\mathcal{C}$, we do not require equality with $GF(c)$, but only a natural isomorphism. In other words, we require a natural equivalence (meaning a natural transformation inducing an isomorphism between the images of each obect under the two functors) between GF and the identity functor on $\mathcal{C}$. We also require a natural equivalence between FG and the identity functor on $\mathcal{D}$.

Categories that are wildly different at first glance sometimes turn out to be equivalent. For example the category of topological spaces is known to be equivalent to that of simplicial sets; see Proposition 3.4.10.

The Yoneda lemma and the Yoneda embedding. For an object A in a category $\mathcal{C}$ we define the **Yoneda functor**, denoted by よ^A, to be $\mathcal{C}(A, -)$. The symbol よ is the Japanese hiragana character "yo," the first syllable of Yoneda's name. Its use was suggested to us by Eric Peterson, but we have not seen it elsewhere in the literature. In any case it is a covariant $\mathcal{S}et$-valued functor on $\mathcal{C}$. The Yoneda Lemma 2.2.10 says that the set of natural transformations from よ^A to any other such functor F is naturally isomorphic to the set $F(A)$. There is a similar statement about the **co-Yoneda functor** よ$_B = \mathcal{C}(-, B)$.

The **Yoneda embedding** is a contravariant functor from $\mathcal{C}$ to $[\mathcal{C}, \mathcal{S}et]$ (the category of $\mathcal{S}et$-valued functors on $\mathcal{C}$) that sends A to よ^A. There is a covariant version, a functor from $\mathcal{C}$ to $[\mathcal{C}^{op}, \mathcal{S}et]$ (also known as the ctageory of presheaves on $\mathcal{C}$) sending B to よ$_B$. Both embeddings can be derived from the functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}et$ given by $(A, B) \mapsto \mathcal{C}(A, B)$.

Adjoint functors. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories with functors $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$. For example, $\mathcal{C}$ could be $\mathcal{S}et$, the category of sets, and $\mathcal{D}$ could be $\mathcal{A}b$, the category of abelian groups. F could be the free abelian group functor, which sends a set X to the free abelian group generated by X, and U could be the forgetful functor that sends an abelian group A to its underlying set.

Then we know that for any set X and abelian group A, there is an isomorphism

$$\mathcal{A}b(F(X), A) \cong \mathcal{S}et(X, U(A)). \tag{1.4.1}$$

A homomorphism from a free abelian group generated by a set is determined by its values on the elements of that set, which may be arbitrary elements in the target group. Furthermore, this isomorphism is natural in both X and A. In this case we say that F **is the left adjoint of** U and U **is the right adjoint of** F. The terms "left" and "right" refer to the fact that the domain (which by convention is written on the left) in the left side of (1.4.1) is a value of F while the codomain on the right is a value of U. The notation for this state of affairs is

$$F \dashv U,$$

the symbol $\dashv$ having been used by Dan Kan in his 1958 paper [Kan58a].

An arbitrary functor may or may not have a left or right adjoint, but when either of the latter exists, it is known to be unique up to natural isomorphism. The textbook example above is one of many that we will encounter in this book.

Limits and colimits. Now suppose $\mathcal{C}$ is a category, J is a small category, and

$$\mathcal{D} = \mathcal{C}^J,$$

the category of functors from J to $\mathcal{C}$, that is the category of commutative J-shaped diagrams in $\mathcal{C}$. Then there is a diagonal functor

$$\Delta : \mathcal{C} \to \mathcal{D}$$

that sends an object X to the constant X-valued diagram. Depending on $\mathcal{C}$, **Δ may or may not have a left or a right adjoint**, that is there may or may not be functors that assign to each diagram D in $\mathcal{C}$ (i.e., object in $\mathcal{D}$) objects in $\mathcal{C}$ having a certain universal properties spelled out in §2.3C. When they exist, these two objects are respectively the **colimit** and **limit** of the diagram D.

The simplest nontrivial examples are coequalizers and equalizers, which are colimits and limits of diagrams consisting of the middle two objects in

$$\begin{tikzcd} W \arrow[r, dashed, "f"] & X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & Y \arrow[r, dashed, "g"] & Z \end{tikzcd}$$

The coequalizer is an object Z with a morphism $g : Y \to Z$ such that $g\alpha = g\beta$ and any other morphism $g' : Y \to Z'$ with $g'\alpha = g'\beta$ factors uniquely through g. The equalizer W is dually defined. It turns out (Theorem 2.3.28) that every colimit (respectively limit) is a coequalizer (equalizer).

The category $\mathcal{C}$ is said to be **cocomplete** (respectively **complete**) if colimits (limits) exist for all diagrams (that is, $\mathcal{C}$-valued functors) for all small categories J. It is **bicomplete** if it has both properties. Such categories come equipped with initial and terminal objects; see Example 2.3.35(ii). They are respectively the colimit and limit of the empty diagram, which is the unique $\mathcal{C}$-valued functor on the empty category.

The categories of sets, topological spaces (with or without base points), groups and abelian groups are each known to be bicomplete.

Symmetric monoidal categories. A **monoidal structure** on a category $\mathcal{C}$ is a functor $\otimes : \mathcal{C} \times \mathcal{C}$ with certain properties listed in Definition 2.6.1. It assigns to each pair (x, y) of objects in $\mathcal{C}$ a third object $x \otimes y$, so it is a binary operation on the class of objects. It is required to be associative, unital, and possibly commutatative. Examples include Cartesian product and disjoint union in the categories of sets and of topological spaces, and direct sum and tensor product in the category of abelian groups.

For each object y in a monoidal category $\mathcal{C}$, we get a functor $(\,-\,)\otimes y$ from $\mathcal{C}$ to itself. It may or may not have a right adjoint, which we denote by $\underline{\mathcal{C}}(y, \,-\,)$. If this functor exists, the adjunction isomorphism is

$$\mathcal{C}(x, \underline{\mathcal{C}}(y, z)) \cong \mathcal{C}(x \otimes y, z),$$

which is natural in all three variables. When such a right adjoint exists, we say that the monoidal category $\mathcal{C}$ is **closed** and that $\underline{\mathcal{C}}(\,-\,,\,-\,)$ is its **internal hom functor**. It has similar formal properties to those of the morphism set $\mathcal{C}(x, y)$, but instead of being a set it is an object in $\mathcal{C}$. See Definition 2.6.33 for more details.

Two variable adjunctions. Adjunct functors have the following useful generalization, which is the subject of §2.6C. Suppose we have three categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, along with a functor

$$F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}.$$

Hence F is a functor of two variables, one lying in $\mathcal{C}$ and one lying in $\mathcal{D}$. We could ask for a right adjoint $G : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$, but it is more interesting to think of it as follows.

For each object c in $\mathcal{C}$, the left variable of F, we have a functor $F_c : \mathcal{D} \to \mathcal{E}$ that sends an object d in $\mathcal{D}$ to $F(c, d)$. Suppose that for each c this functor has a right adjoint $G_c : \mathcal{E} \to \mathcal{D}$. It turns out that these functors vary contravariantly with c, so collectively they leads to a functor

$$G_1 : \mathcal{C}^{op} \times \mathcal{E} \to \mathcal{D}.$$

Here the subscript 1 refers to the first variable of F.

Similarly, by fixing the second variable d in $\mathcal{D}$, we get a functor $F_d : \mathcal{C} \to \mathcal{E}$ for which we require a right adjoint $G_d : \mathcal{E} \to \mathcal{C}$. These vary contravariantly with d, so collectively they lead to a functor

$$G_2 : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{C}.$$

The functors F, G_1 and G_2, along with isomorphisms

$$\mathcal{D}(d, G_1(c, e)) \cong \mathcal{E}(F(c, d), e) \cong \mathcal{C}(c, G_2(d, e)), \tag{1.4.2}$$

constitute a **two variable adjunction**, the subject of Definition 2.6.26.

An important example is the case where the three categories are the same and $F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a symmetric monoidal structure, which we denote by $\otimes$. Symmetry implies that the functors G_1 and G_2 are the same up isomorphism, and we denote it by $\underline{\mathcal{C}}$. Then the isomorphisms of (1.4.2) read

$$\mathcal{C}(y, \underline{\mathcal{C}}(x, z)) \cong \mathcal{C}(x \otimes y, z) \cong \mathcal{C}(x, \underline{\mathcal{C}}(y, z)),$$

for objects x, y and z in $\mathcal{C}$. Then the functor $\underline{\mathcal{C}}$ satisfies the definition of the internal hom functor in a closed symmetric monoidal category.

Ends and coends. Let $\mathcal{C}$ be a cocomplete category and J a small category. Instead of a $\mathcal{C}$-valued functor on J, we have a functor

$$H : J^{op} \times J \to \mathcal{C},$$

that is a $\mathcal{C}$-valued functor on two variables in J which is contravariant in the first and covariant in the second. For each morphism $f : j \to j'$ in J we get a diagram

$$\begin{tikzcd} H(j',j) \arrow[r, "f_*"] \arrow[d, "f^*"'] & H(j',j') \\ H(j,j). \end{tikzcd}$$

in $\mathcal{C}$. Taking the coproduct over all morphisms in J leads to

$$\begin{tikzcd} \coprod_{f:j\to j'} H(j',j) \arrow[r, "\varphi_*"] \arrow[d, "\varphi^*"'] & \coprod_{j'} H(j',j') \\ \coprod_{j} H(j,j). \end{tikzcd}$$

It is a coequalizer diagram in $\mathcal{C}$ since the codomains of the morphisms φ_* and φ^* (which are defined in §2.4) are the same. The resulting coequalizer, which exists because $\mathcal{C}$ is cocomplete, is the **coend of** H, denoted by

$$\int_J H(j,j).$$

When $\mathcal{C}$ is complete, there is a dual notion of **the end of** H, denoted by

$$\int^J H(j,j).$$

Our use of subscripts and superscripts here is the reverse of that in [ML71, pages 222–223] and most other works on category theory, but follows that of Jacob Lurie in [Lur09]. See §2.4.

Kan extensions. Suppose we have a cocomplete category $\mathcal{C}$, small categories J and K and functors $F : J \to \mathcal{C}$ and $\lambda : J \to K$, as in the diagram of categories and functors

$$\begin{tikzcd} J \arrow[rr, "F"] \arrow[dr, "\lambda"'] & & \mathcal{C} \\ & K. \arrow[ur, dashed, "G"'] & \end{tikzcd} \tag{1.4.3}$$

We are looking for a functor G such that there is a natural transformation from F to the composite $G\lambda$ with a certain universal property spelled out in §2.5. It is most easily explained in terms of functor categories. Let $\mathcal{C}^J$ and $\mathcal{C}^K$ denoted the categories of $\mathcal{C}$-valued functors on J and K. Then precomposition with λ defines a functor

$$\mathcal{C}^K \xrightarrow{\lambda^*} \mathcal{C}^J \tag{1.4.4}$$

We are seeking its left adjoint, which we denote by

$$\mathcal{C}^J \xrightarrow{\lambda_!} \mathcal{C}^K.$$

If it exists, its value on F (which is by definition an object in $\mathcal{C}^J$) is the desired functor G in (1.4.3), the **left Kan extension of F along λ**. We will see in §2.5B that the cocompleteness of $\mathcal{C}$ leads to a formula for $\lambda_! F$ as a coend, namely for each object k in K

$$(\lambda_! F)(k) = \int_{j \in J} K(\lambda(j), k) \cdot F(j).$$

Note here that $K(\lambda(j), k)$ is a set, namely that of morphisms in K from $\lambda(j)$ to k. The integrand is the coproduct in $\mathcal{C}$ of the object $F(j)$ indexed by this set.

Dually, when $\mathcal{C}$ is complete, the precomposition functor λ^* of (1.4.4) has a right adjoint λ_* and there is a formula for $\lambda_* f$, the **the right Kan extension of F along λ**, as a certain end. In this case the natural transformation in (1.4.3) goes the other way, from $G\lambda$ to F.

Indexed monoidal products. The chapter ends with a more technical discussion of indexed monoidal products. Some constructions there may be new and are needed later in the book. For us the motivating example of an indexed monoidal product is a wedge or smash product of pointed G-spaces (for a finite group G) for which the **indexing set itself has an action of G**. Such a wedge or smash product then has a G-action which differs from that on the wedge or smash product indexed by the same set with trivial G-action. For example, given a subgroup $H \subseteq G$ and a pointed H-space X, we can define a G-space

$$N_H^G X := \bigwedge_{G/H} X,$$

the **norm of X**. The underlying pointed H-space is $X^{\wedge |G/H|}$, the $|G/H|$-fold smash power of X. The larger group G acts on it by permuting the factors. There is an analogous construction on spectra which is pivotal in this book and which is discussed in detail in §9.7.

1.4B Enriched category theory

Chapter 3 concerns enriched category theory. In an ordinary category $\mathcal{C}$, for any two objects X and Y one has a set of morphisms which we denote by $\mathcal{C}(X,Y)$. Given morphisms $f : X \to Y$ and $g : Y \to Z$, one gets a composite morphism $gf : X \to Z$. Thus one has a map of sets

$$\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

with suitable properties. Its domain is the Cartesian product of the indicated two morphism sets. The Cartesian product itself is an example of a symmetric monoidal structure (see §2.6) on $\mathcal{S}et$, the category of sets.

In an enriched category $\mathcal{C}$, the category $\mathcal{S}et$ is replaced by a possibly different symmetric monoidal category $\mathcal{V}$, say the category of widgets. (For those unfamiliar with the term “widget,” it is not a mathematical notion. It can mean whatever you want it to mean.) Then instead of morphism sets in $\mathcal{C}$, we have **morphism widgets**. Then we say that $\mathcal{C}$ is **enriched over** $\mathcal{V}$. Two familiar examples are the category $\mathcal{A}b$ of abelian groups, in which each morphism set has a natural abelian group structure, and the category $\mathcal{T}op$ of topological spaces in which each morphism set has a natural topology. In both cases the categories $\mathcal{C}$ and $\mathcal{V}$ are the same, i.e., $\mathcal{A}b$ and $\mathcal{T}op$ happen to be symmetric monoidal categories that are enriched over themselves. In general $\mathcal{C}$ may be different from $\mathcal{V}$, and it need not be symmetric mondoidal.

Many constructions in ordinary category theory have enriched analogs. These include limits and colimits, ends and coends, and left and right Kan extensions.

The Day convolution. Suppose we have a cocomplete closed symmetric monoidal category $(\mathcal{V}, \wedge, S)$ over which a small symmetric monoidal category $(J, \oplus, \mathbf{0})$ (which need not be closed) is enriched. Now consider the category $[J, \mathcal{V}]$ of enriched $\mathcal{V}$-valued functors on J. Its objects are functors and its morphisms are natural transformations between them. We will denote the value of such a functor X on an object j in J by X_j.

Then the Day Convolution Theorem 3.3.5, proved in 1970 by the Australian category theorist Brian Day, says that this functor category is also closed symmetric monoidal. We will use the same symbol for its binary operation as the one for that on $\mathcal{V}$. Hence for such functors X and Y, we denote their product by $X \wedge Y$. It can be defined as a Kan extension. Consider the diagram

$$\begin{tikzcd} J \times J \arrow[r, "X \times Y"] \arrow[dr, "\oplus"'] & \mathcal{V} \times \mathcal{V} \arrow[r, "\wedge"] & \mathcal{V} \\ & J \arrow[ur, dashed, "X \wedge Y"'] & \end{tikzcd}$$

Thus $X \wedge Y$ is the left Kan extension of the composite functor $\wedge(X \times Y)$ along $\oplus$. There is an explicit formula (3.3.3) for $(X \wedge Y)_j$ as an enriched coend.

This result is pivotal for stable homotopy theory. As we will explain in Chapter 7 and Chapter 9, spectra can be regarded as such functors, where the target is the category $\mathcal{T}$ of pointed topological spaces or some variant thereof. The indexing category J can be one of several defined in §7.2A.

It turns out that the indexing category associated with the original definition of spectra is monoidal but not symmetric. See Remark 7.2.11 for details. This is counterintuitive since the object set is the natural numbers and the monoidal structure is related to addition, which is of course commutative.

This lack of symmetry means that Day's hypotheses are not met, and the original category of spectra does not have a convenient smash product. This was a major headache in the subject for decades.

The first construction of a stable homotopy category with a pointwise symmetric monoidal structure was made by Elmendorf, Kriz and May in 1993 and later published as [EKMM97]. Their smash product was defined by other more complicated means.

The first use of a left Kan extension to define the smash product of spectra is likely due to Jeff Smith in the same decade. His insight led to the publication of [HSS00] with Mark Hovey and Brooke Shipley. The first work in homotopy theory to cite Day's paper [Day70] was [MMSS01].

Simplicial sets and related notions are introduced in §3.4, but relatively little use is made of them in the rest of the book. We prefer topological spaces to simplicial sets because equivariant homotopy theory does not play nicely with the latter. In Corollary 5.6.16 we will see that every topological category is equivalent to a simplicial one. In developing the theory of spectra in §7.2 we need to assume that we are working over a model category (see below) in which every object is fibrant. This is the case for various categories of topological spaces but not for simplicial sets.

1.4C Model categories

The next three chapters concern model categories, Daniel Quillen's (1940–2011) brilliant axiomatization of homotopy theory introduced in [Qui67]. Chapter 4 is an account of the theory roughly as Quillen developed it. Chapter 5 covers some material developed since Quillen's work, and the short Chapter 6 describes the best construction in the subject, that of Bousfield localization.

On this last topic the third author has a confession to make. Earlier in his career he made substantial use of it to develope chromatic homotopy theory; see [Rav84], [Rav92] and the references in the latter. During this time he treated Bousfield's construction as a black box and had no understanding of how it actually works. He has since corrected this deficiency. We will return to this topic below.

Quillen defined a model category to be a bicomplete (meaning it has all limits and colimits) category equipped with three classes of morphisms called **weak equivalences**, **fibrations** and **cofibrations**. Each of them contains all isomorphisms and is closed under retracts. They are required to satisfy certain axioms listed in Definition 4.1.1. Of these the most demanding is the factorization axiom, which says that every morphism can be factored as a cofibration followed by a fibration, either one of which can be required to be a weak equivalence as well. These two factorizations need not be unique and almost never are, but they can be made functorially.

The most widely used axiom concerns liftings. It says that for any commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd} \tag{1.4.5}$$

in which i is a cofibration, p is a fibration and one of them is a weak equivalence, there is a **lifting** h with $hi = \alpha$ and $ph = \beta$. We say that i **has the left lifting property with respect to** p and p **has the right lifting property with respect to** i. Furthermore, one can characterize fibrations and cofibrations, trivial or not, in terms of their lifting properties. We denote this by $i \boxslash p$, where the symbol $\boxslash$ (which we learned from [MP12, Definition 14.1.5]) was chosen for its resemblance to the diagram of (1.4.5).

In the literature there are two adjectives for a fibration or cofibration which is also a weak equivalence: “trivial” and “acyclic.” Quillen used the former in [Qui67], as we do here (but not in [HHR16]). He changed to the latter in [Qui70]. It is also used by Bill Dwyer and Jan Spalinksi in [DS95], which we recommend as a very friendly introduction to the subject. (We are not alone in this endorsement of that paper; it is by far the most widely cited one in [Jam95].)

An object in a model category is **cofibrant** if the unique map to it from the inital object (which exists since the category is cocomplete) is a cofibration. **Fibrant** objects are dually defined. The factorization axiom impies that each object admits a weak equivalence both from a cofibrant one and to a fibrant one. These are called **cofibrant and fibrant approximations**. Since the relevant factorizations can be made functorial, we have **cofibrant and fibrant replacement functors**.

Generally speaking the best behaved maps in a model category are those from a cofibrant object to a fibrant one.

In the category of topological spaces (with or without base points), the cofibrant objects are the CW complexes, and all objects are fibrant. Thus experience with this category gives one the feeling that cofibrant objects are the easiest ones to deal with, but it gives us no insight about fibrant ones.

For fibrancy there is a spectacular example due to Bousfield and Friedlander [BF78]. They defined the first model structure on the category of spectra. **In it the fibrant objects are the Ω-spectra!**

Classical examples. Quillen defined model structures on the categories of topological spaces (with or without base points) and simplicial sets, and on certain categories of chain complexes. These are described in §4.2. The definitions of the morphism classes are familiar in the topological case. Weak equivalences are maps inducing isomorphisms of all homotopy groups. Cofibrations include the inclusion maps

$$i_n : S^{n-1} \to D^n \qquad \text{for all } n \geqslant 0. \tag{1.4.6}$$

All other cofibrations are derived from these by certain operations, namely retractions, coproducts (meaning disjoint union in the unpointed case and wedges in the pointed case), pushouts and transfinite compositions. It follows that if a map has the right lifting property with respect to the maps of (1.4.6), it has it with resepct to **all** cofibrations which makes it a trivial fibration.

Similarly trivial cofibrations include the inclusion maps

$$j_n : I^n \to I^{n+1} \qquad \text{for all } n \geqslant 0, \tag{1.4.7}$$

and a map is a fibration iff it has the right lifting property with respect to these maps. Indeed that is how Serre fibrations were defined long before we had model categories. We will sometimes refer to this as the **Quillen model structure**.

Functors between model categories. It turns out that nearly all such functors worth studying are either left or right adjoints. A left adjoint is a **left Quillen functor** if in addition it preserves cofibrations and trivial cofibrations. **Such functors are not required to preserve weak equivalences**, but they are known to preserve weak equivalences between cofibrant objects. Right Quillen functors are dually defined, and there is a notion of a Quillen adjunction, also known as a Quillen pair. See Definition 4.5.1. There is also a notion of Quillen equivalence in which isomoprhisms are replaced by weak equivalences; see Definition 4.5.14.

Cofibrant generation. When a model category has morphism sets such as those of (1.4.6) and (1.4.7) that generate all cofibrations and trivial cofibrations, we say it is **cofibranty generated**. This property is very convenient, and nearly all of the model catgories we will study in this book have it. In each case we will describe the two sets explicitly.

Suppose we have a bicomplete category $\mathcal{N}$ for which weak equivalences have been defined. The question of when two morphism sets can lead to a model structure as above is the subject of the Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24.

Given such a category $\mathcal{N}$, suppose we have a cofibrantly generated model category $\mathcal{M}$ and a left adjoint functor $F : \mathcal{M} \to \mathcal{N}$ with right adjoint $U : \mathcal{N} \to \mathcal{M}$. Then we can ask if $\mathcal{N}$ has a cofibrantly generated model structure related the one on $\mathcal{M}$. This is the subject of the Crans-Kan Transfer Theorem 5.2.27.

Lack of symmetry. Quillen set up the theory to be entirely self dual. The opposite of a model category (meaning the category having the same objects with all arrows reversed) is also a model category. Many theorems have left and right versions with dual proofs.

However in practice the theory is not entirely symmetric in this sense. For example we saw above that cofibrantly generated model categories are convenient and hence widely studied. In theory one could make similar statements about fibrantly generated model categories, meaning ones in which fibrations and trivial fibrations are generated by morphism sets similar to those of (1.4.6) and (1.4.7). To our knowledge this has never been done due to lack of practical motivation.

The functor category $\mathcal{M}^J$ is the subject of §5.4. Given in model category $\mathcal{M}$ and a small category J, we can define the **projective model structure** on the functor category $\mathcal{M}^J$ as follows. A morphism $f : X \to Y$ in it is a weak equivalence or fibration if the map $f_j : X_j \to Y_j$ is one for each object j in J. Cofibrations in $\mathcal{M}^J$ are defined in terms of lifting properties. While it is true that for a cofibration $i : A \to B$ in $\mathcal{M}^J$, each map $i_j : A_j \to B_j$ is a cofibration in $\mathcal{M}$, this necessary condition is not sufficient. When $\mathcal{M}$ is cofibrantly generated, so is $\mathcal{M}^J$, and we can describe its generating sets in terms of those of $\mathcal{M}$. The description involves Yoneda functors $よ^j$ on J; see Theorem 5.4.10.

An induced model structure. Now suppose K is a full subcatgory of J with inclusion functor α. Then we have the projective model structure on $\mathcal{M}^K$, a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$ and a left Kan extension $\alpha_! : \mathcal{M}^K \to \mathcal{M}^J$. These satisfy the hypotheses of the Crans-Kan Transfer Theorem 5.2.27, giving us a **induced model structure** on $\mathcal{M}^J$. In it a map f is a weak equivalence or a fibration if f_k is one for each object k in K. This condition is weaker than that for the projective model structure. It follows that the induced model structure has more weak equivalences and fibrations and hence fewer cofibrations that the projective one. In the extreme case where K is empty, all maps are weak equivalences and fibrations, and the cofibrations are the isomorphisms. See Theorem 5.4.21.

This type of induction is one of three methods we have of altering the model structure on a functor category. They are summarized in Table 6.1. We need all three to construct the model structure we need on the category of orthogonal G-spectra for a finite group G. See (7.1) and Theorem 9.2.13.

The other two work for more general model categories than functor categories.

Enlarging the class of cofibrations in a model category is the subject of Theorem 5.2.34. We have two cofibrantly generated model categories $\mathcal{M}'$ and $\mathcal{M}$ with an adjunction

$$\begin{tikzcd} \mathcal{M}' \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "U"] \end{tikzcd}$$

that need not be a Quillen pair. Then we consider the composite adjunction

$$\begin{tikzcd} (X, X') \arrow[r, mapsto] & (X, FX') \arrow[r, mapsto] & X \amalg FX' \\ \mathcal{M} \times \mathcal{M}' \arrow[r, shift left, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left, "\mathcal{M}\times U"] \arrow[r, shift left, "\amalg"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "\Delta"] \\ (Y, UY) & (Y, Y) \arrow[l, mapsto] & Y. \arrow[l, mapsto] \end{tikzcd}$$

We use this to transfer the given model structure on $\mathcal{M} \times \mathcal{M}'$ to get a new one on $\mathcal{M}$. It has the same weak equivalences but more cofibrations (including the images of cofibrations in $\mathcal{M}'$ under F) and hence fewer fibrations than the original one.

Bousfield localization is the subject of Chapter 6. In it we want to modify a model category $\mathcal{M}$ by enlarging the class $\mathscr{W}$ of weak equivalences while leaving the class $\mathscr{C}$ of cofibrations unchanged. We will denote the new model category (if it exists) by $\mathcal{M}'$. It has the same underlying category as $\mathcal{M}$. It has has the same class of cofibrations and therefore the same class of trivial fibrations as $\mathcal{M}$, even though the meaning of triviality is different in $\mathcal{M}'$. On the other hand, more of its cofibrations are trivial since there are more weak equivalences. This means that **is has fewer fibrations** than $\mathcal{M}$ and therefore **a more interesting fibrant replacement functor**.

The hardest part of showing that $\mathcal{M}'$ is indeed a model category is verifying the factorization axiom. Recall that it says any morphism can be factored as a cofibration followed by a trivial fibration, **and** as a trivial cofibration followed by a fibration. The first of these is the same as the corresponding factorization in $\mathcal{M}$ since the classes of cofibrations and trivial fibrations are unaltered. The second is far more delicate and its proof involves some set theory, the bane of almost every homotopy theorist, with the notable exception of Pete Bousfield.

There are theorems in §6.3 saying that $\mathcal{M}'$ is a model category if $\mathcal{M}$ satisfies certain technical hypotheses that are met in all cases of interest to us. These theorems do not place any restrictions on how we enlarge $\mathscr{W}$.

We can expand the class $\mathscr{W}$ of weak equivalences by adding a little as a single morphism $f : X \to Y$ to it. Typically we do so by specifying a countable set of such morphisms. If f is a weak equivalence in the new model structure, so are its composites on both sides with old weak equivalences, and retracts thereof. We get many new weak equivalences for the price of a few.

Example 1.4.8. Some instances of Bousfield localization.

(i) **Bousfield's original example in [Bou75].** *Given a generalized homology theory h_*, define a map of spaces to be a weak equivalence if it induces an isomorphism in $h_*(-)$. Then the fibrant objects spaces Y such that for each h_*-equivalence $f : A \to B$, the induced map $f^* : Map(B, Y) \to Map(A, Y)$ is an ordinary weak equivalence. Bousfield calls such spaces* h_***-local**. *Every space is h_*-equivalent to an h_*-local space that is unique up to ordinary weak equivalence.*

The same can be done in the category of spectra as explained in [Bou79]. For examples of this relevant to chromatic homotopy theory, see [Rav84] and [Rav92].

(ii) **Dror Farjoun localization.** *Localizations of the category of topological spaces obtained by adding a single map $f : X \to Y$ to the class of weak equivalences were studied in [Far96a]. Consider the case where the map is $S^{n+1} \to *$ for for some integer $n \geqslant 0$. (The map $* \to S^{n+1}$ would work just as well.) In the resulting model structure, weak equivalences are maps inducing isomorphissms in π_k for $k \leqslant n$. Fibrant objects are spaces Y with $\pi_i Y = 0$ for all $i > n$. The fibrant replacement functor is the nth Postnikov section $P^n(-)$, meaning that $P^n X$ is the space obtained from X by killing all homotopy groups above dimension n.*

(iii) **Stabilization as Bousfield localization**, *the most important example for us, is discussed in Chapter 7.*

(iv) **Localizing subcategories** *τ of a topological model category $\mathcal{M}$ are the subject of Definition 6.3.12. An instructive example is the category of spaces or spectra satisfying a connectivity condition. Another is the smallest subcategory of $\mathcal{M}$ that contains a specified set of objects and is closed under weak equivalence, cofibers, extensions and arbitrary wedges. Given such a subcategory, we can localize by expanding the set of weak equivalences to include all maps $T \to *$ for objects T in τ. When $\mathcal{M} = \mathcal{T}$ and τ is the subcategory generated by S^{n+1}, then the resulting fibrant replacement functor is P^n as in (ii). The slice filtration of $\mathcal{S}p^G$, the subject of Chapter 11, is based on an equivariant generalization of this functor.*

1.4D The theory of spectra

In Chapter 7 we will study spectra from the model category theoretic point of view. For the moment we will use the original definition of a spectrum X as a sequence of pointed spaces (or simplicial sets) X_m for $m \geqslant 0$ with **structure maps** $\epsilon_m^X : \Sigma X_m \to X_{m+1}$. These have adjoints

$$\eta_m^X : X_m \to \Omega X_{m+1}, \tag{1.4.9}$$

the **costructure maps**. The induced map of homotopy groups leads to a diagram

$$\pi_k X_0 \to \pi_{k+1} X_1 \to \pi_{k+2} X_2 \to \cdots ,$$

and we define

$$\pi_k X = \operatorname*{colim}_m \pi_{k+m} X_m, \tag{1.4.10}$$

the kth **stable homotopy group of the spectrum** X. Note that this group is defined for **all integers** k, not just nonnegative ones. For $i < 0$ we define $\pi_i K$ to be 0 for any pointed space K. For each k the homotopy groups on the right above are positively indexed for sufficiently large m.

Let $\mathcal{T}$ denote the category of pointed topological spaces with the Quillen model structure. We will see in §7.2 that spectra as defined above can be regarded as $\mathcal{T}$-valued functors on a certain small pointed topological category (that is a category enriched over $\mathcal{T}$) $\mathscr{J}_{S^1}^{\mathbf{N}}$ whose object set is the natural numbers; see Definition 7.2.4. We will abbreviate it here by $\mathscr{J}^{\mathbf{N}}$. Its morphism spaces are

$$\mathscr{J}^{\mathbf{N}}(\mathbf{m}, \mathbf{n}) \cong \begin{cases} S^{n-m} & \text{for } n \geqslant m \\ * & \text{otherwise.} \end{cases}$$

Functoriality means that the structure maps $\epsilon_m^X : \Sigma X_m \to X_{m+1}$ exist, as do more general maps

$$\epsilon_{m,k}^X : \Sigma^k X_m \to X_{m+k}$$

with appropriate properties.

The category of such enriched functors,

$$\mathcal{S}p^{\mathbf{N}} = [\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$$

has some convenient properties:

- It is bitensored (as in Definition 3.1.31) over $\mathcal{T}$. This means that for a spectrum X and a pointed space K we can define spectra $X \wedge K$ and X^K by

  $$(X \wedge K)_m = X_m \wedge K \qquad \text{and} \qquad (X^K)_m = (X_m)^K = \mathcal{T}(K, X_m).$$

  In other words tensors and cotensors are defined objectwise.

- It is bicomplete as in Definition 2.3.25. For a small category J and a functor $F : J \to \mathcal{S}p^{\mathbf{N}}$ (meaning a J-shaped diagram of spectra in which we denote the image of an object j in J by F_j), we have

$$(\lim_J F)_m = \lim_J (F_j)_m \qquad \text{and} \qquad (\operatorname*{colim}_J F)_m = \operatorname*{colim}_J (F_j)_m.$$

  In other words limits and colimits are also defined objectwise.

$\mathcal{S}p^{\mathbf{N}}$ also has a projective model structure in which a map $f : X \to Y$ is a weak equivalence or a fibration if f_m is one for each $m \geqslant 0$. We know that it is cofibrantly generated since $\mathcal{T}$ is. We can describe its generating sets using the results of §5.4; see Proposition 7.1.33.

Stabilization. Experience has shown that this notion of weak equivalence is too rigid, and it is better to define a stable equivalence to be a map inducing an isomorphism in the stable homotopy groups of (1.4.10). Thus we are enlarging the class of weak equivalences as we do in Bousfirled localization. In §7.3A we will see that this process can be described in terms of adding certain morphisms (which we call **stabilizing maps**) to the class of weak equivalences.

We will describe these maps. For each integer $m \geqslant 0$, let S^{-m} and $S^{-m-1} \wedge S^1$ be the spectra given by

$$(S^{-m})_k = \mathscr{J}^{\mathbf{N}}(\mathbf{m}, \mathbf{k}) \cong \begin{cases} S^{k-m} & \text{for } k \geqslant m \\ * & \text{otherwise} \end{cases} \tag{1.4.11}$$

and

$$(S^{-m-1} \wedge S^1)_k = \mathscr{J}^{\mathbf{N}}(\mathbf{m+1}, \mathbf{k}) \wedge S^1 \cong \begin{cases} S^{k-m} & \text{for } k \geqslant m+1 \\ * & \text{otherwise,} \end{cases}$$

along with obvious structure maps. These two sets of components are the same for each value of k except $k = m$. The m**th stabilizing map** s_m has the form

$$s_m : S^{-m-1} \wedge S^1 \to S^{-m}, \tag{1.4.12}$$

which is the identity in each degree except the mth, where it is the unique base point preserving map $* \to S^0$. One sees easily that it induces an isomorphism of stable homotopy groups. See **??** for more discussion.

Remark 1.4.13. Notation for the sphere spectrum. *In the notation of (1.4.11), S^{-0}* **denotes the sphere spectrum***. To our knowledge, this notation is new. In early literature it was sometimes denoted by S^0, which was potentially confusing since S^0 also denotes 0-sphere. More recently it has been denoted simply by S, possibly written in some fancy font. The symbol S^{-0} has the advantages of being unambiguous and easy to write by hand.*

The spectrum S^{-m} of (1.4.11) is an instance of the **enriched Yoneda functor** $よ^m$ as in the Enriched Yoneda Lemma 3.1.29. For this reason we call it a **Yoneda spectrum**; see Definition 7.1.30. The Enriched Yoneda Lemma 3.1.29 tells us that for an arbitrary spectrum X,

$$\mathcal{S}p^{\mathbf{N}}(S^{-m}, X) \cong X_m.$$

In other words, S^{-m} represents the evaluation functor that sends a spectrum to its mth component. It follows that

$$\mathcal{S}p^{\mathbf{N}}(S^{-m-1} \wedge S^1, X) \cong \mathcal{S}p^{\mathbf{N}}(S^{-m-1}, \Omega X) \cong \Omega X_{m+1}.$$

The stable model structure on $\mathcal{S}p^{\mathbf{N}}$ is the Bousfield localization of the projective one obtained by requiring the maps s_m of (1.4.12) for all $m \geqslant 0$ to be weak equivalences. In it the fibrant objects turn out to be spectra X for which the maps of (1.4.9) are weak equivalences in $\mathcal{T}$ for all n. In other words, they are the Ω-spectra. This remarkable observation is due to Bousfield and Friedlander [BF78].

Cofibrant generation of the stable model structure. The stable model structure on the category of spectra is cofibrantly generated and it would be nice to have an explicit description of its generating sets. Since it has the same cofibrations as the projective structure, we can use the same set of generating cofibrations. On the other hand it has more trivial cofibrations, so we need a **larger** generating set of trivial cofibrations than in the projective model structure. At present there is no general theory about how to describe such a set for the Bousfield localization of cofibrantly generated model category. Fortunately we have such a description for the case at hand in Theorem 7.3.36. The additional trivial cofibrations that it contains are described in terms of generating (nontrivial) cofibrations of $\mathcal{T}$ and the stabilizing maps s_m of (1.4.12).

The positive projective and positive stable model structures. Both the projective and stable model structure on $\mathcal{S}p^{\mathbf{N}}$ can be "positivized" as follows. In the positive projective model structure, we say that a map $f : X \to Y$ is a fibrationor weak equivalence if $f_n : X_n \to Y_n$ is one for all $n > 0$; we ignore the map f_0. Thus we have more such maps than in the projective case, so we have fewer cofibrations and fewer cofibrant objects. If $i : A \to B$ is a cofibration in this new model structure, it must have the left lifting property with respect to all trivial fibrations f. Since f_0 can be arbitrary,, i_0 must be an isomorphism. Surprisingly, it turns out that the sphere spectrum S^{-0} is not positive cofibrant.

The theory behind this modification of the projective model structure is the subject of §5.4C, specifically Theorem 5.4.21, and Theorem 5.6.38 in the enriched case.

Thus we have four model structures on $\mathcal{S}p^{\mathbf{N}}$ the original category of spectra. The projective one can be stabilized, positivized, or both. **Why do we positivize?** Doing it in the original case (where the smash product is problematic) is a warmup for doing it in the symmetric, orthogonal and equivariant cases where, as we will see below, we have a good smash product and can talk about commutative ring objects. We will need to give the category of such commutative ring spectra a model structure of its own. For reasons to be explained in Chapter 10, this can only be done if we replace the projective and stable model structures with their positive analogs. For more discussion, see Remark 7.0.7(ii).

The smash product problem. The original category of spectra $\mathcal{S}p^{\mathbf{N}}$ suffers from a defect that was a major headache for decades: **it lacks a convenient smash product**. With hindsight, we now know that the origin of this problem lies in the indexing category $\mathscr{J}^{\mathbf{N}}$. It is monoidal (under addition), but surprisingly (given that addition is commutative) it is **not symmetric monoidal** as in Definition 2.6.1. See Remark 7.2.11 for an explanation.

Roughly speaking, $\mathscr{J}^{\mathbf{N}}$ is not symmetric monoidal because it does not have enough morphisms. We can solve this problem by replacing it with a category having the same set of objects, but bigger morphism spaces, that **is** symmetric monoidal. We offer two such indexing categories, $\mathscr{J}^{\mathbf{\Sigma}}$ and $\mathscr{J}^{\mathbf{O}}$, in Definition 7.2.4. They lead to the categories of symmetric spectra originally studied by Hovey, Shipley and Smith in [HSS00], and orthogonal spectra studied by Mandell, May, Schwede and Shipley in [MMSS01] and further by Mandell and May in [MM02]. These are defined in Definition 7.2.33. Each of these categories comes with its own Yoneda spectra, given in Definition 7.2.52, which are important theoretical tools.

A fourth type of indexing category, usually having having more objects, is given in Definition 7.2.19. It is also symmetric monoidal, and it will be needed for the orthogonal G-spectra of [MM02] and Chapter 9. We call spectra associated with this type of indexing category **extraorthogonal**. In each case (other than the original one) the Day Convolution Theorem 3.3.5 implies that there is a smash product that makes the category closed symmetric monoidal. We refer to the three flavors (symmetric, orthogonal and extraorthogonal) of spectra with symmetric monoidal indexing categories collectively as **smashable spectra**, referring to the fact that they have convenient smash products. In each of these categories there is a projective and a stable model structure with positive analogs. Cofibrant generating sets for these are identified in Theorem 7.4.52.

1.4E Equivariant homotopy theory

In Chapter 8 we introduce some tools from the homotopy theory of G-spaces for a finite group G that we will need later to study G-spectra. These include the Burnside ring (Definition 8.1.3), Mackey functors (Definition 8.2.3 and Definition 8.2.5) and G-CW complexes (Definition 8.4.13).

When a finite group G acts continuously on a space X, for each subgroup $H \subseteq G$ we have a fixed point space

$$X^H = \{x \in X : \eta(x) = x \text{ for all } \eta \in H\},$$

which is the same as the space of equivariant maps to X from the orbit G/H, $\mathcal{T}op^G(G/H, X)$. This data depends only on the conjugacy class of H. $\mathcal{T}op^G$ denotes the category of G-spaces and equivariant maps, and $\mathcal{T}^G$ denotes its pointed analog. Theorem 8.4.18, due to Glen Bredon (1932–2000), says that an equivariant map $f : X \to Y$ of G-CW complexes is an equivariant homotopy equivalence (meaning a homotopy equivalence in which both maps and both homotopies are equivariant) iff the induced maps $f^H : X^H \to Y^H$ or ordinary homotopy equivalences for all H.

The orbits G/H form a full subcategory of $\mathcal{T}op^G$ which we denote by $\mathcal{O}_G$, the **orbit category** of G. For subgroups $K \subseteq H \subseteq G$, there is a surjective map of G-sets $G/K \to G/H$ that sends the K-coset γK (for $\gamma \in G$) to the H-coset γH. There is also an inclusion map $X^H \to X^K$ since any point fixed by H is also fixed by its subgroup K. Note the change of variance. This means we have a functor

$$\mathcal{O}_G^{op} \to \mathcal{T}op \qquad \text{given by} \qquad G/H \mapsto X^H. \tag{1.4.14}$$

This functor can be composed with any of the usual algebraic functors on $\mathcal{T}op$, such as homotopy and homology. An abelian group valued functor on $\mathcal{O}_G^{op}$ is called a **coefficient system**; see Definition 8.6.24. One example is **equivariant homotopy group**

$$\pi_*^H X := \pi_* X^H \tag{1.4.15}$$

for each subgroup $H \subseteq G$.

In the introduction to Chapter 8 we will explain how the suspension spectrum $\Sigma^\infty G/H_+$ is equivariantly self dual. This means that in addition to the maps $\Sigma^\infty G/K_+ \to \Sigma^\infty G/H_+$ induced by the map of spaces $G/K_+ \to G/H_+$, there is map

$$\Sigma^\infty G/H_+ \to \Sigma^\infty G/K_+$$

going the other way. This means that in the stable analog of (1.4.14) we need to replace $\mathcal{O}_G^{op}$ by a category with more morphisms. An abelian group valued functor on it is called a **Mackey functor**, the subject of §8.2. For a

G-spectrum X (to be defined in Chapter 9) one has the **homotopy Mackey functor** $\underline{\pi}_*X$ given by

$$\underline{\pi}_*X(G/H) := \pi_*^H X \qquad \text{as in (1.4.15).}$$

The homology of a G-space X can be made into a Mackey functor since

$$H_*X \cong H_*\Sigma^\infty X = \pi_* H\mathbf{Z} \wedge X,$$

where $H\mathbf{Z}$ denotes the integer Eilenberg-Mac Lane spectrum.

Remark 1.4.16. Warning. *The homology Mackey functor $\underline{H}_*X$ for a G-space X is* **not** *defined by $\underline{H}_*X(G/H) = H_*X^H$ as one might expect by analogy with the homotopy Mackey functor. Instead we have*

$$\underline{H}_*X(G/H) := H_*(\Sigma^\infty X)^H \cong \pi_*(H\mathbf{Z} \wedge X)^H.$$

The fixed point functor in the stable category behaves badly with respect to both the infinite suspension functor and smash products. The fixed points of a suspension spectrum $(\Sigma^\infty X)^H$ is **not** *the same as the suspension spectrum of the fixed point space, $\Sigma^\infty(X^H)$. Given two spectra A and B, or a spectrum A and a space B, the fixed points of the smash product $(A \wedge B)^H$ is* **not** *the same as the smash product of the fixed point sets $A^H \wedge B^H$.*

Fortunately there is an alternative to the stable fixed point functor which does not suffer from these defects. It is the **geometric fixed point functor** *Φ^G, the subject of §9.11.*

There are four different topological categories associated with G-actions:

- $\mathcal{T}op^G$, the category of G-spaces (which are assumed to compactly generated and weak Hausdorff as in Definition 2.1.46) and continuous equivariant maps. The morphism objects $\mathcal{T}op^G(X,Y)$ are topological spaces.
- $\mathcal{T}op_G$, the category of G-spaces and **all** continuous maps, not just the equivariant ones. The morphisms object $\mathcal{T}op_G(X,Y)$ has a G-action spelled out in Definition 3.1.59. The fixed point set is $\mathcal{T}op^G(X,Y)$.
- $\mathcal{T}^G$, the category of pointed G-spaces and pointed equivariant maps. The basepoint is always fixed by G. The morphism object $\mathcal{T}^G(X,Y)$ is a pointed topological space whose base point in the constant base point valued map $X \to Y$.
- $\mathcal{T}_G$, the category of pointed G-spaces and **all** continuous maps. Morphism objects are pointed G-spaces.

In Definition 8.3.8 we describe four spaces associated with a G-space X, with or without a base point: The **orbit space** X_G, the **fixed point space** X^G and their homotopy analogs X_{hG} (also known as the **Borel construction**) and X^{hG}.

G-CW complexes are the subject of §8.4. They are defined in such a way the group action permutes cells rather than rotating them; see Definition 8.4.13.

Each G-CW complex has a cellular chain complex of modules over the group ring $\mathbf{Z}[G]$. There is an algebraic procedure (Definition 8.5.1) for converting it into a chain complex of Mackey functors. Some illustrative examples are given in §8.5.

Model structures for $\mathcal{T}op^G$ and $\mathcal{T}^G$ are discussed in §8.6. An equivariant map $f : X \to Y$ of G-spaces is a weak equivalence or a fibration if $f^H : X^H \to Y^H$ is one for each subgroup $H \subseteq G$.

The Mandell-May category $\mathscr{J}_G$ is introduced in §8.9C. It is the indexing category for the orthogonal G-spectra of Chapter 9. Its objects are finite dimensional representations of G and its morphism objects $\mathscr{J}_G$ are certain explicitly defined pointed G-spaces. For representations V and W, the morphism space $\mathscr{J}_G(V, W)$ (defined as a certain Thom space in Definition 8.9.22) is a subspace of the pointed G-space $\mathcal{T}_G(S^V, S^W)$ having to do with affine isometric embeddings of V into W, as explained in the proof of Proposition 8.9.27.

1.4F Orthogonal G-spectra

In Chapter 9, after more than 500 pages of preparation, we introduce our main objects of study, orthogonal G-spectra. They are smashable spectra as in Definition 7.2.33. This means they are functors with values in a closed symmetric monoidal topological model category from a small symmetric monoidal category (the indexing category) that is enriched over the same model category. In this case the model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant pointed maps, with the Bredon model structure of Definition 8.6.1. The indexing category is the Mandell-May category $\mathscr{J}_G$ introduced in §8.9C, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional representations (actual rather than virtual) V of G and its morphism objects are certain pointed G-spaces described in Definition 8.9.22.

This means that a lot (but not all) of what we need to know about them is a special case of the general theory of spectra developed in Chapter 7. They have a smash product defined using the Day Convolution Theorem 3.3.5. It makes the category $\mathcal{S}p^G$ of such spectra closed symmetric monoidal. It has a positive stable model structure as in Theorem 7.4.52.

The meaning of "positive" here is a less than obvious generalization of its meaning in the nonequivariant case. In the latter the indexing categories $\mathscr{J}^{\mathbf{N}}$, $\mathscr{J}^{\mathbf{\Sigma}}$ and $\mathscr{J}^{\mathbf{O}}$ each have the natural numbers as objects. In the positive projective model structure on the category of enriched $\mathcal{T}$-valued functors one any of them, and map $f : X \to Y$ is a fibration or a weak equivalence of $f_n : X_n \to Y_n$ is one for each $n > 0$.

In the G-equivariant case for a finite group G, we are looking at $\mathcal{T}^G$-valued functors on the Mandell-May category $\mathscr{J}_G$ of Definition 8.9.24. Its objects are finite dimensional orthogonal representations of G as in §8.9B. Such a representation V is defined to be positive (see Definition 8.9.10(v)) if the invariant subspace V^G is nontrivial.

The features of $\mathcal{S}p^G$ that are **not** derived from the general theory of smashable spectra have to do with the interplay between the subgroups of G and their fixed point sets. We get an $RO(G)$-graded Mackey functor worth of homotopy groups spelled out in Definition 9.1.1. A map of G-spectra $f : X \to Y$ is said to be a **stable equivalence** if it induces an isomorphism of $\mathbf{Z}$-graded Mackey functor homotopy groups; see Proposition 9.1.4.

For each subgroup $H \subseteq G$ we have a restriction functor $i_H^G : \mathcal{S}p^G \to \mathcal{S}p^H$. It has a left adjoint

$$X \mapsto G \underset{H}{\ltimes} X, \tag{1.4.17}$$

and there is a similar adjunction between $\mathcal{T}^H$ and $\mathcal{T}^G$. We call these **change of group adjunctions**. The one between $\mathcal{T}^H$ and $\mathcal{T}^G$ is a Quillen adjunction with respect to the Bredon model structure of Definition 8.6.1, meaning among other things the left adjoint functor sends cofibrations of pointed H-spaces to cofibrations of pointed H-spaces. When a model structure on a category of equivariant objects has a similar property, we say it is **equifibrant**; see Remark 8.6.19.

We need to modify the four previously defined model structures on $\mathcal{S}p^G$, and the corresponding ones on $\mathcal{S}p^H$ for each $H \subseteq G$, to make them equifibrant. This condition is needed for the use of wedges and smash products indexed by G-sets, such as the norm functor of §9.7. The general theory of indexed monoidal products is the subject of §2.9.

As it stands, our four model structures on $\mathcal{S}p^G$ are not equifibrant. In particular the functor of (1.4.17) does **not** send cofibrations in $\mathcal{S}p^H$ to cofibrations in $\mathcal{S}p^G$ for $H \subseteq G$. To fix this we need to **enlarge the class of cofibrations** in $\mathcal{S}p^G$ so that it includes morphisms induced up from cofibrations in $\mathcal{S}p^H$. We can do so without altering the class of weak equivalences. The model category theoretic tool for this enlargement is Theorem 5.2.34.

The resulting eight model structures in $\mathcal{S}p^G$ and their cofibrant generating sets are spelled out in Theorem 9.2.13. They involve three different modifications of the projective model structure, stabilization (a form of Bousfield localization), positivization and equifibrant enht (more generally enht of the class of cofibrations), which can be done in any combination. Their properties are summarized in Table 6.1.

1.4G Multiplicative properties of G-spectra

In Chapter 10 we discuss indexed smash products (including the norm) and related constructions of orthogonal G-spectra. We will need some of them to show that the slice spectral sequence, our main computational tool and the subject of Chapter 11, has the expected multiplicative properties. We will need them again in Chapter 12 where we construct and study the real cobordism spectrum $MU_{\mathbf{R}}$ (a C_2-spectrum) along with its norms to larger cyclic 2-groups.

The category of interest here is $\mathcal{S}p^{\mathcal{B}_T G}$, that of functors to orthogonal spectra from the category $\mathcal{B}_T G$, the groupoid associated with a finite G-set T. The first four sections concern indexed smash products, which we get by varying the G-set T. They are not homotopical in general, but we will show in Theorem 10.4.7 that they are on cofibrant objects.

In the next five sections we study indexed symmetric smash powers, which are needed to construct a model strcuture on the category $\mathbf{Comm}^G$ of G-commutative ring spectra. Here there is a surprising technical difficulty that is illustrated in Example 10.5.2. It means that cofibrant objects in $\mathbf{Comm}^G$ (such as the sphere spectrum) are almost never cofibrant after applying the forgetful functor. This means we cannot use Theorem 10.4.7 to work out the homotopy type of their norms. This problem is addressed by Corollary 10.9.10.

In the last section we define and study twisted monoid rings. These are associative algebras that are weakly equivalent to wedges of spheres, and which are needed for some constructions in Chapter 12.

1.4H The slice filtration and slice spectral sequence

In Chapter 11 we introduce our main computational tool, the slice spectral sequence. It is based on the **slice filtration**, which is an equivariant generalization of the Postnikov filtration. For an ordinary space or spectrum X, one forms the nth Postnikov section $P^n X$ by attaching cells to kill the homotopy groups of X above dimension n. The resulting map $X \to P^n X$ is a cofibration whose homotopy theoretic fiber is the n-connected cover $P_{n+1} X$. We also get a diagram

$$\cdots \to P^{n+1} X \to P^n X \to P^{n-1} X \to \cdots, \tag{1.4.18}$$

the **Postnikov tower of** X. Its limit and colimit are X and $*$ respectively. The fiber $P_n^n X$ of the map $P^n X \to P^{n-1} X$, the n**th Postnikov layer**, is an Eilenberg-Mac Lane space or spectrum capturing the nth homotopy group of X.

This construction can be interpreted model theoretically in two different ways.

(i) We are expanding the class of weak equivalences in $\mathcal{T}$ or $\mathcal{S}p$ by defining a weak equivalence to be a map inducing an isomorphism in homotopy groups **in dimensions** $\leqslant n$, rather than in all dimensions. We can use Bousfield localization, the subject of Chapter 6, to get a new model structure on $\mathcal{T}$ or $\mathcal{S}p$ in which fibrant replacement is the functor P^n.

(ii) We can define localizing subcategories (as in Definition 6.3.12) $\tau_{n+1}\mathcal{T}$ or $\tau_{n+1}\mathcal{S}p$, to be the ones generated by the spheres S^m or $S^{-0} \wedge S^m$ for $m > n$. These are the categories of n-connected spaces or spectra. They lead to localization functors obtained by adding the maps $S^m \to *$ for $m > n$ to the class of weak equivalences. This is the same as the localization described above. In the case of spectra we can define subcategories $\tau_n\mathcal{S}p$ for **all integers** n.

In the G-equivariant case for a finite group G, we replace the ordinary spheres by the objects

$$\widehat{S}(m,H) = G \underset{H}{\ltimes} S^{m\rho_H}, \tag{1.4.19}$$

where ρ_H denotes the regular representation of the subgroup $H \subseteq G$. See the paragraph following Definition 11.1.1 for more details. We call these objects **slice spheres**. $\widehat{S}(m,H)$ is underlain by a wedge of $|G/H|$ copies of $S^{m|H|}$, so we say that its **dimension** is $m|H|$. We can define the **localizing subcategory** $\tau_{n+1}\mathcal{S}p^G$ to be the one generated by all slice spheres of dimension greater than n. This leads to a fibrant replacement functor P_G^n, the n**th slice section**, and a diagram analogous to (and underlain by) (1.4.18),

$$\cdots \to P_G^{n+1}X \to P_G^nX \to P_G^{n-1}X \to \cdots,$$

the **slice tower of** X. We denote its nth layer, the fiber of the map $P_G^nX \to P_G^{n-1}X$, by ${}^GP_n^nX$. It is underlain by the nth Postnikov layer P_n^nX, but its equivariant homotopy groups are **not** concentrated in a single dimension.

In the very favorable cases of interest in this book, these groups are computable. They form the input for the **slice spectral sequence** described in §11.2. We say a G-spectrum is **pure** (Definition 11.3.14) if all of its slices are wedges of spectra of the form $\widehat{S}(m,H) \wedge H\mathbf{Z}$, where $\widehat{S}(m,H)$ is as in (1.4.19) with the subgroup $H \subseteq G$ being nontrivial, and $H\mathbf{Z}$ denotes the integer Eilenberg-Mac Lane spectrum. The equivariant homotopy groups of these slices can be computed by methods described in §9.9. One thing we learn from these computations is that the H-equivariant groups for nontivial H always vanish in dimensions strictly between -4 and 0. This fact is behind the Gap Theorem of §1.1C(iii).

For a G-spectrum X, we denote by π_*^uX the homotopy groups of the ordinary underlying spectrum. When π_d^uX is free abelian, we can find a map $c_d^u : W_d \to X$, where W_d is a wedge of d-spheres, that induces an isomorphism in π_d^u. In Definition 11.3.19 we say a **refinement** of π_d^uX is an equivariant map $c_d : \widehat{W}_d \to X$ in which $\widehat{W}_d$ is a wedge of slice spheres of dimension d, with the property that the map $\pi_d^u\widehat{W}_d \to \pi_d^uX$ is an isomorphism. When π_*^uX is free abelian in all dimensions, we can define a refinement of it to be an equivariant map $c : \widehat{W} \to X$ in which $\widehat{W}$ is a wedge of slice spheres of varying dimensions, such that for each d the restriction of c to the d-dimensional summands of $\widehat{W}$ is a refinement of $\pi_d^u X$.

In §11.4 we specialize to the case where X is a connective commutative ring spectrum R. With the help of various technical results from Chapter 10, we show in Theorem 11.4.13 that each slice section $P_G^n R$ inherits a unique commutative multiplication from R. This means that its slice spectral sequence is one of algebras in which the differentials are derivations. This fact is crucial for the calculations of §13.3, where we prove the Periodicity Theorem of §1.1C(ii).

1.4I The construction of $MU_{\mathbf{R}}$, the star of our show

In Chapter 12 we construct a C_2-equivariant commutative ring spectrum $MU_{\mathbf{R}}$ admitting the canonical homotopy presentation (see §7.4F)

$$MU_{\mathbf{R}} \cong \operatorname{hocolim} S^{-n\rho_2} \wedge MU(n),$$

where ρ_2 denotes the regular representation of C_2, and $MU(n)$ is the Thom complex of the universal bundle over $BU(n)$, the classifying space of the group $U(n)$ of $n \times n$ unitary matrices, with a C_2 action given by complex conjugation. Its image under the forgetful functor to ordinary spectra is MU, the usual complex cobordism spectrum.

Happily the slice tower for $MU_{\mathbf{R}}$ is completely accessible and is strikingly similar to the ordinary Postnikov tower for MU. The latter is concentrated in even degrees. Its $(2n)$th layer is a wedge of copies of $\Sigma^{2n} H\mathbf{Z}$ with a summand for each partition of n. The $(2n)$th slice of $MU_{\mathbf{R}}$ is a wedge of the same number of copies of $S^{n\rho_{C_2}} \wedge H\mathbf{Z}$. Thus we have a refinement (as described above and in Definition 11.3.19) of $\pi_* MU$, and $MU_{\mathbf{R}}$ is pure as in Definition 11.3.14. The refinement is easy to construct, but the statement about the slice tower is more delicate. Its proof is the subject of §12.4.

Now let $G = C_{2^{n+1}}$ and consider the spectrum

$$MU^{((G))} = N_{C_2}^G MU_{\mathbf{R}},$$

the norm of $MU_{\mathbf{R}}$ as in Definition 9.7.3. It is underlain by $MU^{\wedge 2^n}$. The group G acts by cyclically permuting the factors. Its subgroup of order two leaves each factor invariant and acts on it by complex conjugation. We can analyze the slice tower of $MU^{((G))}$ and show that it is also pure with contractible slices of odd degree. **This remarkable property makes the computations of Chapter 13 and the proof of the main theorem possible.**

1.4J The proofs of the Gap, Periodicity and Detection Theorems

Chapter 13 is the payoff, the reason for developing all the machinery of the previous eleven chapters. We will prove (in reverse order) the three theorems listed in §1.1C.

As noted above our spectrum Ξ is the C_8-fixed point spectrum of a telescope $\Xi_{\mathbf{O}}$ formed by inverting a certain element

$$D \in \pi^{C_8}_{19\rho_{C_8}} N^{C_8}_{C_2} MU_{\mathbf{R}}.$$

The reason for choosing an element in this degree is spelled out in §13.3. The methods of §12.4 enable us to describe the slices of $\Xi_{\mathbf{O}}$. The methods of §9.9 show that each slice has $\pi^{C_8}_{-2} = 0$, which implies the Gap Theorem.

The computations of § 13.3 show that there is an invertible element in $\pi^{C_8}_{256}\Xi_{\mathbf{O}}$, which implies the Periodicity Theorem.

This leaves the Detection Theorem, the subject of §13.4. It requires a detailed look at the Adams-Novikov spectral sequence and some computations related to the formal group associated with complex cobordism. We need to consider a certain formal A-module (see Definition 13.4.4) specified in (13.4.5) where A is the extension of the 2-adic integers $\mathbf{Z}_2$ obtained by adjoining an eighth root of unity ζ_8. This section includes an explanation of why we need to norm up $MU_{\mathbf{R}}$ to a C_8-spectrum. The Detection Theorem fails if we do a similar construction for C_2 or C_4.

For each of the three theorems the proof is computational in nature, and **we did the computations before we developed the theoretical framework for them.** This is not the first time, and surely will not be the last, that an advance in homotopy theory has been made in this way. **Computation precedes theory!**

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PART ONE

THE CATEGORICAL TOOL BOX

2

Some categorical tools

> Such a consideration of vector spaces and their linear transformations is but one example of many similar mathematical situations; for instance, we may deal with groups and their homomorphisms, with topological spaces and their continuous mappings, with simplicial complexes and their simplicial transformations, with ordered sets and their order preserving transformations. In order to deal in a general way with such situations, we introduce the concept of a category. Thus a category $\mathfrak{A}$ will consist of abstract elements of two types: the objects A (for example, vector spaces, groups) and the mappings α (for example, linear transformations, homomorphisms). For some pairs of mappings in the category there is defined a product (in the examples, the product is the usual composite of two transformations). Certain of these mappings act as identities with respect to this product, and there is a one-to-one correspondence between the objects of the category and these identities. A category is subject to certain simple axioms, so formulated as to include all examples of the character described above.
>
> *Samuel Eilenberg and Saunders Mac Lane, [EM45, page 235]*

This chapter is a light and leisurely introduction to some category theoretic tools that are useful in stable homotopy theory. The object is to save the reader the trouble of looking up these concepts elsewhere to know what they mean, but not to give a textbook introduction to them. Our treatment will be short on proofs and long on examples. Our favorite references for this material are [ML98] and [Rie14], which the reader should consult for a more rigorous treatment. Most of our examples are lifted shamelessly from them. See also [May96, Chapter V].

In §2.1, after spelling out some notational conventions, we discuss (very briefly) compactly generated weak Hausdorff spaces (Remark 2.1.47) and define the comma category and related constructions in Definition 2.1.51. The fun really begins in §2.2 where we state and prove the Yoneda Lemma 2.2.10 and define the Yoneda embedding よ (Definition 2.2.12). The symbol よ is the Japanese character "yo," the first syllable (in hiragana) of Yoneda's name.

The Yoneda Lemma, due to Nobuo Yoneda (1930–1996), was communicated privately to Saunders Mac Lane (1909–2004) around 1954. They had a lengthy discussion about it at a café in the Gare du Nord in Paris while Yoneda was waiting for a train [Kin96]. Mac Lane followed Yoneda onto his coach (without having a ticket himself) to continue the conversation. It is not known whether he got off the train before it left the station. He subsequently promoted the lemma, noting in [ML98, page 77] that "with time, its importance has grown." It was used extensively by Grothendieck in the 1960s. It was not mentioned in [Yon54], contrary to a claim in Wikipedia.

In §2.2D we introduce adjoint functors, which were first defined by Kan in his landmark 1958 paper [Kan58a], where he also introduced limits and colimits (which he called inverse and direct limits, see §2.3C) and Kan extensions (see §2.5). These are followed by monads in §2.2E. They were first studied by Eilenberg and John Moore (1923–2016) in [EM65], where they were called triples. Mac Lane later wrote in [ML98, page 138], "The frequent but unfortunate use of the term **triple** in this sense has caused a maximum of needless confusion ..."

Limits and colimits are discussed in § 2.3C. Special cases include pullbacks/pushouts, fixed point/orbit spaces of group actions, and equalizers/coequalizers. The closely related notion of pushout and pullback corner map is given in Definition 2.3.9. We will see them again in §6.3B and repeatedly in Chapter 7 and Chapter 10. The cordial relationship between limits/colimits and adjoint functors is the subject of Proposition 2.3.36. Reflexive coequalizers are the subject of §2.3F followed by filtered and sifted colimits in §2.3G.

Ends and coends, very powerful notational devices involving the integral sign from calculus, are the subject of §2.4. Kan extensions are discussed in §2.5. In [ML98, X.7] Mac Lane wrote

> The notion of Kan extensions subsumes all the other fundamental concepts of category theory.

In favorable cases left and right Kan extensions can be described explicitly as coends and ends. This is the subject of §2.5B.

Symmetric monoidal categories are the subject of §2.6. These are categories equipped with associative, commutative and unitary binary operations on their object sets. Given such a category $\mathcal{C}$ with binary operation $\otimes$, we can define a monoid R to be an object R equipped with a morphism $R \otimes R \to R$ with suitable properties (Definition 2.6.58) and a left or right R-module M to be an object equipped with a morphism to it from $R \otimes M$ or $M \otimes R$. When R is commutative, it is possible to define $M \otimes_R N$ (Lemma 2.6.61) for R-modules M and N. The extremely useful **two variable adjunction** is introduced in Definition 2.6.26.

In § 2.7 we discuss 2-categories and in § 2.8 we introduce Grothendieck fibrations and opfibrations.

The next section, §2.9 on indexed monoidal products, is more technical. We need it for the constructions of Chapter 10 on G-spectra. To our knowledge, most of this material is new apart from its briefer treatment in [HHR16].

For the moment, suppose we have a symmetric monoidal category $\mathcal{C}$ with binary operation $\otimes$ in which every object has an action by a fixed finite group G. Since $\otimes$ is associative, we can define the product $X^{\otimes T}$ of a collection of objects X_t in $\mathcal{C}$ indexed by a finite set T. When T itself has a G-action, there is a way to incorporate it into the structure of $X^{\otimes T}$, which we call an **indexed monoidal product**.

It is convenient to take a more abstract perspective and replace the finite G-set T by a small category K and consider the category $\mathcal{C}^K$ of functors $K \to \mathcal{C}$, meaning K-shaped diagrams in $\mathcal{C}$. Then certain functors $p : \widetilde{K} \to K$ between small categories, namely the covering categories of Definition 2.8.1, lead to functors $p_*^{\otimes} : \mathcal{C}^{\widetilde{K}} \to \mathcal{C}^K$ called **indexed monoidal products along** p spelled out in Definition 2.9.6.

When $\mathcal{C}$ has two binary operations $\otimes$ and $\oplus$ related by a distributive law, then a product of sums gets identified with a certain sum of products. When the original product and sums are indexed, so are the new sum and products. This is given by the **indexed distributive law** of Proposition 2.9.20.

When we have a pushout diagram in $\mathcal{C}^A$ for an ordinary set A with pushout object Z, we get a technically useful filtration of $Z^{\otimes A}$, the **target exponent filtration**, defined in Definition 2.9.34 and made more explicit in Lemma 2.9.39. We also discuss commutative algebras (§2.9F) and monomial ideals (§2.9G) in this setting.

2.1 Basic definitions and notational conventions

2.1A Notational conventions

We will usually denote a category $\mathcal{C}$ by a symbol in script (`\mathscr`) or calligraphic (`\mathcal`) font. For us a small category, which will usually be denoted by a Roman letter, is one in which the collection of objects is a set rather than a proper class. The value of a functor F on an object j in a small category J will often be denoted by F_j rather than $F(j)$.

The collections of objects and arrows (i.e., morphisms) in $\mathcal{C}$ will be denoted by $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Arr}\,\mathcal{C}$. As is common practice, **we will sometimes abuse notation by writing $c \in \mathcal{C}$ instead of $c \in \mathbf{Ob}\,\mathcal{C}$ and $c \to c' \in \mathcal{C}$ instead of $c \to c' \in \mathbf{Arr}\,\mathcal{C}$.**

The identity morphism on an object X, when it appears in a commutative diagram, will often be denoted simply by X. This is sometimes called the **Princeton convention**, and is likely due to John Moore.

We will often discuss statements about categories that have dual analogs. We will sometimes make both a statement and its dual at the same time, instead of making two separate statements, with the help of parentheses. For example, instead of writing

fibrations preserve widgets, and cofibrations preserve cowidgets, we will write

> fibrations (cofibrations) preserve widgets (cowidgets).

We will do the same with categorical notions that have pointed analogs. Instead of saying

> widgets have bridges, and pointed widgets have pointed bridges,

we will say

> (pointed) widgets have (pointed) bridges.

2.1B Categories

Definition 2.1.1. *A* **category** $\mathcal{C}$ *consists of*

(i) A collection $Ob\,\mathcal{C}$ *of* **objects**. *This collection could be a proper class rather than a set. When it is a set we say the category is* **small.**

(ii) For each ordered pair (X, Y) *of objects a set* $\mathcal{C}(X, Y)$ *of* **morphisms** $f : X \to Y$*, also known as* **arrows**. *The collection of all arrows in a category* $\mathcal{C}$ *is denoted by* $Arr\,\mathcal{C}$.

We denote the collection of all morphisms in $\mathcal{C}$ *by* $Arr\,\mathcal{C}$. *For a morphism* $f : X \to Y$*, we say that its* **source** *or* **domain** *is* $X = Dom\, f$*, and its* **target** *or* **codomain** *is* $Y = Cod\, f$. *Some authors allow* $\mathcal{C}(X, Y)$ *to be a class rather than a set and call a category* **locally small** *if* $\mathcal{C}(X, Y)$ *is always a set.* **In this book all categories are understood to be locally small.**

Each object X *has an* **identity morphism** $1_X \in \mathcal{C}(X, X)$. *For each ordered triple of objects* (X, Y, Z) *one has a* **composition pairing**

$$c_{X,Y,Z} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \tag{2.1.2}$$

and the image of (g, f) *is denoted by* gf*, the* **composite of** f **and** g. *Composition of morphisms is associative, meaning that* $(hg)f = h(gf)$ *for*

$$(f, g, h) \in \mathcal{C}(W, X) \times \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z)$$

Composition of $f \in \mathcal{C}(X, Y)$ *with identity morphisms behaves as expected, namely*

$$f1_X = f = 1_Y f.$$

Definition 2.1.3. *For a category* $\mathcal{C}$*, the* **opposite category** $\mathcal{C}^{op}$ *has the same object collection as* $\mathcal{C}$ *with morphism sets defined by* $\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X)$. *Thus* $\mathcal{C}^{op}$ *is* $\mathcal{C}$ *with its arrows reversed.*

Definition 2.1.4. *A* **subcategory** $\mathcal{C}'$ **of** $\mathcal{C}$ *is a category whose object collection and morphisms sets are contained in those of* $\mathcal{C}$. *It is* **full** *if for each pair of objects* X *and* Y *in* $\mathcal{C}'$, $\mathcal{C}'(X, Y) = \mathcal{C}(X, Y)$. *It is* **wide** *(or* **lluf***) if it contains all objects of* $\mathcal{C}$.

The structure of a category $\mathcal{C}$ is determined by its collection Arr$\mathcal{C}$ of morphisms and its composition law, since one could recover the identity morphisms and hence the objects of $\mathcal{C}$ from the latter.

Definition 2.1.5. The product and coproduct of two categories. *For categories $\mathcal{C}_1$ and $\mathcal{C}_2$, their product $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ is the category with*

$$Ob\,\mathcal{C} = Ob\,\mathcal{C}_1 \times Ob\,\mathcal{C}_2 \qquad \text{and} \qquad Arr\,\mathcal{C} = Arr\,\mathcal{C}_1 \times Arr\,\mathcal{C}_2$$

and composition of morphisms defined by

$$(g_1, g_2)(f_1, f_2) = (g_1 f_1, g_2 f_2).$$

Their coproduct $\mathcal{C}' = \mathcal{C}_1 \amalg \mathcal{C}_2$ is the category with

$$Ob\,\mathcal{C}' = Ob\,\mathcal{C}_1 \amalg Ob\,\mathcal{C}_2 \qquad \text{and} \qquad Arr\,\mathcal{C}' = Arr\,\mathcal{C}_1 \amalg Arr\,\mathcal{C}_2.$$

There are no morphisms between objects in different summands, so composition in $\mathcal{C}'$ is determined by that in $\mathcal{C}_1$ and $\mathcal{C}_2$.

Definition 2.1.6. Two 2-object categories. *Let* **2** *denote the category* $(1 \to 2)$*, sometimes called the* **walking arrow category** *or* **interval category***, and sometimes denoted by* $\Delta[1]$ *or I. (We will reserve the symbol I for the closed unit interval* $[0, 1]$*.)*

Let Eq denote the **equalizer category** $(1 \rightrightarrows 2)$ *with two objects and two morphisms from the first object to the second one.*

Definition 2.1.7. *For a set A, the corresponding* **discrete category** A^{disc} *is the small category with object set A in which the only morphisms are identity morphisms. For a small category J, we denote the discrete category associated with its object set by J^{disc} or $|J|$. In general* **a category is discrete** *if all of its morphisms are isomorphisms and any two morphisms having the same domain and codomain are equal. In particular all of its automorphisms are identities.*

Example 2.1.8. The product of a discrete category with the walking arrow category. *For a set A, the product (as in Definition 2.1.5) $\mathcal{C} = A^{disc} \times$* **2** *(see Definition 2.1.7 and Definition 2.1.6) has the disjoint union of two copies of A as its object set. For each $a \in A$, denote by a_1 and a_2 the two corresponding objects in $Ob\,\mathcal{C}$. For each such a, there is a morphism $a_1 \to a_2$ in $\mathcal{C}$, and these are the only nonidentity morphisms in $\mathcal{C}$.*

Definition 2.1.9. *A category $\mathcal{C}$ is* **concrete** *if it admits a faithful functor to $\mathcal{S}et$, the category of sets. This means the objects of $\mathcal{C}$ can be regarded as sets possibily with additional structure that morphisms are required to preserve.*

Many familiar categories, such as those of groups, rings and topological spaces, are concrete.

Definition 2.1.10. Special morphisms. *A morphism $f : A \to B$ in a category $\mathcal{C}$ is* **monic** *(or an* **monomorphism***) if two morphisms $a_1, a_2 : X \to A$ are equal iff $fa_1 = fa_2$, or equivalently if it is the equalizer (see Definition 2.3.27 below) of the pair of natural inclusions $B \rightrightarrows B \cup_A B$. It is* **split monic** *if there is a morphism $r : B \to A$, called a* **retraction***, such that $rf = 1_A$.*

It is **epi** *(or an* **epimorphism***) if two morphisms $B_1, B_2 : B \to Y$ are equal iff $b_1 f = b_2 f$. It is* **split epi** *if there a morphism $s : B \to A$, called a* **secction***, such that $fs = 1_B$.*

It is an **isomorphism** *if there is a morphism $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$.*

Proposition 2.1.11. Split monic/epi duality. *With notation as above, if the retraction $r : B \to A$ exists, it is split epi with section f. If the section $s : A \to B$ exists it is split monic with retraction f.*

In a concrete category (Definition 2.1.9) such as $\mathcal{S}et$, a monomorphism (epimorphism) is a map that one to one (onto). The class of monomorphisms (epimorphisms) is closed under composition.

2.1C Functors

Definition 2.1.12. *Given two categories $\mathcal{C}$ and $\mathcal{D}$, a* **functor** *$F : \mathcal{C} \to \mathcal{D}$ consists of*

(i) A function $F : Ob\,\mathcal{C} \to Ob\,\mathcal{D}$. The image of an object X will be denoted by $F(X)$, FX or (when $\mathcal{C}$ is small) by F_X.

(ii) For each pair (X, Y) of objects in $\mathcal{C}$ a function

$$F_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y)).$$

It is common practice to drop the subscripts above, using the same symbol for the functor and the two functions associated with it. The image under F of a morphism $f : X \to Y$ in $\mathcal{C}$ is usually denoted by $F(f) : FX \to FY$.

The morphism function is required to satisfy the rules $F(1_X) = 1_{F(X)}$ (it sends identity morphisms to identity morphisms) and $F(gf) = F(g)F(f)$ (it preserves composition of morphisms).

There is an **identity functor** *$1_{\mathcal{C}}$ for which the two functions are identities. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, one defined a* **composite functor** *$GF : \mathcal{C} \to \mathcal{E}$ by composing the object and morphism functions.*

The functor F is **faithful** *if it sends distinct objects in $\mathcal{C}$ to distinct objects in $\mathcal{D}$ and distinct morphisms in $\mathcal{C}$ to distinct morphisms in $\mathcal{D}$. It is* **full** *for each pair (X, Y) of objects in $\mathcal{C}$, the map $F_{X,Y}$ is onto. It is* **fully faithful** *if in addition its image is a full subcategory of $\mathcal{D}$, making $F_{X,Y}$ an isomorphism.*

Functors defined in this way are said to be **covariant***, meaning they preserve* *the direction of arrows. In a* **contravariant functor** F *the morphism function above is replaced by one of the form*

$$F : \mathcal{C}(X, Y) \to \mathcal{D}(F(Y), F(X));$$

a contravariant functor reverses the direction of arrows. A contravariant functor $\mathcal{C} \to \mathcal{D}$ *is the same thing as a covariant functor* $\mathcal{C}^{op} \to \mathcal{D}$*.*

Finally, we denote the **collection of functors** $\mathcal{C} \to \mathcal{D}$ *by* $\mathcal{F}un(\mathcal{C}, \mathcal{D})$*,* $\mathcal{C}AT(\mathcal{C}, \mathcal{D})$ *or* $\mathcal{C}at(\mathcal{C}, \mathcal{D})$*; see* *Definition 2.1.14* *below. It is likely to be a proper class unless* $\mathcal{C}$ *and* $\mathcal{D}$ *are both small.*

Proposition 2.1.13. Morphisms as functors from the walking arrow category. *For any category* $\mathcal{C}$*, a functor* $F : \mathbf{2} \to \mathcal{C}$ *defines a morphism in* $\mathcal{C}$*, namely* $F(\alpha) : F(1) \to F(2)$*, where* α *denotes the nonidentity morphism in* $\mathbf{2}$*.*

All functors between categories are assumed to be covariant unless stated otherwise. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is the same thing as a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$, where $\mathcal{C}^{op}$, the opposite category of $\mathcal{C}$, has the same objects as $\mathcal{C}$ with all arrows reversed.

Definition 2.1.14. *We will denote the* **category of categories** *by* $\mathcal{C}AT$ *and the category of small categories by* $\mathcal{C}at$*. In both cases the objects are categories and the morphisms are functors.*

Thus $\mathcal{C}AT(\mathcal{C}, \mathcal{D})$ denotes the collection of functors $F : \mathcal{C} \to \mathcal{D}$, which may fail to be a set. In most cases we will consider, the source category $\mathcal{C}$ is small, and there are no set theoretic difficulties. For small categories $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C}at(\mathcal{C}.\mathcal{D})$ is always a set.

Definition 2.1.15. Functor categories. *For categories* $\mathcal{C}$ *and* $\mathcal{D}$*,* $\mathcal{C}^{\mathcal{D}}$ *or* $[\mathcal{D}, \mathcal{C}]$ *denotes the category whose objects are functors* $\mathcal{D} \to \mathcal{C}$ *and whose functors are natural transformations. When* $\mathcal{D}$ *is a small category* J*, then* $\mathcal{C}^J$ *is the* **category of** J**-shaped diagrams in** $\mathcal{C}$*.*

An enriched analog of this will be given in Definition 3.2.18.

The third key notion of category theory, that of a **natural transformation**, is the subject of Definition 2.2.1 below.

2.1D Sets

The category of sets, which we denote by $\mathcal{S}et$, is often the first category one should think of when trying to understand a new categorical concept. On the other hand, some of its most elementary features are not enjoyed by categories in general. Among these are the following.

Example 2.1.16. Some sets.

(i) **The empty set** $\varnothing$ *is characterized by the property that there is a unique morphism* **from** *it to any set, including itself. An object in a general category* $\mathcal{C}$ *with this property is called an* **initial object**. *If such an object exists, it is necessarily unique up to unique isomorphism.*

(ii) **The one point set** $*$ *is characterized by the property that there is a unique morphism* **to** *it to any set, including itself. An object in a general category* $\mathcal{C}$ *with this property is called a* **terminal object**. *If such an object exists, it is also necessarily unique up to unique isomorphism.*

(iii) **Cartesian products.** *Given sets* A *and* B, *we have a third set* $A \times B$. *In a general category* $\mathcal{C}$ *one cannot combine two objects to get a third one. This would require additional structure, namely a functor*

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

where $\mathcal{C} \times \mathcal{C}$ *denotes the category whose objects are ordered pairs of objects in* $\mathcal{C}$, *with morphisms similarly defined. Such a functor with suitable properties is called a* **monoidal structure** *and is the subject of Definition 2.6.1 below.*

Another property of the Cartesian product is that forn any set X, *a map* $X \to A \times B$ *is the same thing as a pair of maps* $X \to A$ *and* $X \to B$. *In a general category* $\mathcal{C}$ *one could ask the following question:*

> *Given two objects* A *and* B, *is there a third object* C *(their product) equipped with morphisms*
>
> $$p_1 : C \to A \qquad \text{and} \qquad p_2 : C \to B$$
>
> *such that a morphism* $f : X \to C$ *from any other object* X *is uniqely determined by the composites* $p_1 f$ *and* $p_2 f$*?*

When the answer is affirmative, we say that $\mathcal{C}$ **has products**.

(iv) **Disjoint unions.** *Given sets* A *and* B, *we have a third set* $A \coprod B$, *their disjoint union. This is another monoidal structure on* $\mathcal{S}et$. *The analogous question for a general category* $\mathcal{C}$ *is*

> *Given two objects* A *and* B, *is there a third object* C *(their coproduct) equipped with morphisms*
>
> $$i_1 : A \to C \qquad \text{and} \qquad i_2 : B \to C$$
>
> *such that a morphism* $f : C \to X$ *to any other object* X *is uniqely determined by the composites* $f i_1$ *and* $f i_2$*?*

This question is the same as the one for products, but with the three arrows reversed, hence the term "coproduct." When the answer is affirmative, we say that $\mathcal{C}$ **has coproducts**.

(v) **The evaluation map.**

Given two sets X *and* Y, *the set of maps* $f : X \to Y$ *is* $\mathcal{S}et(X, Y)$ *by definition. This means there is an* **evaluation map**

$$\mathrm{Ev} : X \times \mathcal{S}et(X, Y) \to Y$$

sending (x, f) to $f(x)$. We will sometimes write the domain as $\mathcal{S}et(X, Y) \times X$.

(vi) **The constant multiplication map.** *Given sets A, X and Y let*

$$\mu_{A,X,Y} : A \times \mathcal{S}et(X, Y) \to \mathcal{S}et(X, A \times Y)$$

be defined by

$$\mu_{A,X,Y}(a, f)(x) = (a, f(x)) \in A \times Y$$

for $a \in A$, $x \in X$ and $f : X \to Y$. More generally, X and Y could be objects in a cocomplete category $\mathcal{C}$. This means that sets and their products with objects in $\mathcal{C}$ are also objects in $\mathcal{C}$. Let

$$\mu_{A,X,Y} : A \times \mathcal{C}(X, Y) \to \mathcal{C}(X, A \times Y)$$

be defined as follows. For $a \in A$ and $f \in \mathcal{C}(X, Y)$, the morphism $\mu_{A,X,Y}(a, f) : X \to A \times Y$ is determined by its compositions with the projections $p_A : A \times Y \to A$ and $p_Y : A \times Y \to Y$. Then $p_A \mu_{A,X,Y}(a, f)$ is the constant a-valued function on X, and $p_Y \mu_{A,X,Y}(a, f) = f$.

(vii) **The Cartesian product map of morphism sets.** *Given sets A, A', B and B', there is map*

$$\Pi_{A,A',B,B'} : \mathcal{S}et(A, B) \times \mathcal{S}et(A', B') \to \mathcal{S}et(A \times A', B \times B')$$

defined as follows. As in (vi), a map to the product $B \times B'$ is determined by its compositions with the projections

$$p_B : B \times B' \to B \qquad \text{and} \qquad p_{B'} : B \times B' \to B',$$

and these compositions may be arbitrary. Given maps $f : A \to B$ and $f' : A' \to B'$, and $(a, a') \in A \times A'$, we have

$$p_B \Pi_{A,A',B,B'}(f, f')(a, a') = f(a)$$

and

$$p_{B'} \Pi_{A,A',B,B'}(f, f')(a, a') = f'(a').$$

More briefly, $\Pi_{A,A',B,B'}$ sends (f, f') to $f \times f'$. This definition will be generalized below in Definition 2.6.50.

The constant multiplication map of (vi) is a special case of the Cartesian product map. When $A = *$, the set with one element, then $\mathcal{S}et(A, B) \cong B$ and $A \times A' \cong A'$, so we have

$$\Pi_{*,A',B,B'} \cong \mu_{A',B,B'} : B \times \mathcal{S}et(A', B') \to \mathcal{S}et(A', B \times B').$$

2.1E Groupoids

Historically the motivating example of a groupoid (at least for topologists), and the rationale for several of the related terms we will define, is the fundamental groupoid $\pi(X)$ of a topological space X described in Definition 2.1.19 below. See [Bro06] and [Bro87] for much more discussion, [Hig71] for an early treatment of this subject, and [Mil17] for a contemporary one.

Definition 2.1.17. *A* **groupoid** $\mathcal{G}$ *is a small category in which every morphism is invertible, that is for each morphism $f : x \to y$ there is a unique morphism $g : y \to x$ such that $gf = 1_x$.*

Equivalently a group is a pair of sets $\mathcal{G}_0$ and $\mathcal{G}_1$ (the sets of objects and morphisms in the category $\mathcal{G}$) with structure maps

(i) $s, t : \mathcal{G}_1 \to \mathcal{G}_0$, sending each morphism $f : x \to y$ to its source $s(f) = x$ and target $t(f) = y$,
(ii) $e : \mathcal{G}_0 \to \mathcal{G}_1$ sending each object x to its identity morphism 1_x,
(iii) $i : \mathcal{G}_1 \to \mathcal{G}_1$ sending each morphism f to its inverse f^{-1} and
(iv) $m : C \to \mathcal{G}_1$, where C is the pullback in

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{G}_1 \arrow[d, "s"] \\ \mathcal{G}_1 \arrow[r, "t"] & \mathcal{G}_0, \end{tikzcd}$$

that is the set of composable morphism pairs

$$C = \{(g, f) \in \mathcal{G}_1 \times \mathcal{G}_1 : t(f) = s(g)\},$$

the set of composable pairs of morphisms in $\mathcal{G}$, and $m(g, f)$ is the composite morphism gf.

These are required to satisfy certain axioms which we leave to the reader. In particular, for each object x the set $\mathcal{G}(x, x)$ is a group under composition. We abbreviate it by $\mathcal{G}(x)$, the **automorphism group of** *x.*

Remark 2.1.18. *A* **Hopf algebroid** *over a commutative ring K is a cogroupoid object in (or a copgroupoid internal to as in Definition 2.3.46 below) the category of (graded or bigraded) commutative K-algebras; see [Rav86, Definition A1.1.1]. There is a notion of split (see Definition 2.1.31 below) Hopf algebroid given in [Rav86, Definition A1.1.21], of which $MU_*(MU)$ is an example. The Hopf algebroid $BP_*(BP)$ is not split. See [Rav86, A2.1] for more discussion.*

Definition 2.1.19. The fundamental groupoid $\pi(X)$ **of a topological space** X *is the category whose objects are the points of X and whose morphisms are homotopy classes of paths from the domain point to the codomain point. This groupoid is functorial on X and a covering $p : \widetilde{X} \to X$ induces a covering of groupoids as in Definition 2.1.23 below. The space X is path* *connected iff $\pi(X)$ is connected as in Definition 2.1.21 below. Each path connected component of X is simply connected iff $\pi(X)$ is 1-connected. For each point $x_0 \in X$, the group $\pi(X)(x_0)$ is the fundamental group $\pi_1(X, x_0)$.*

In order to define notions in groupoids similar to those in topology, we start with the following.

Definition 2.1.20. The star $\mathrm{St}_{\mathcal{G}}\gamma$ *of an object γ in a groupoid $\mathcal{G}$ is the set*

$$\mathrm{St}_{\mathcal{G}}\gamma = \bigcup_{\gamma'} \mathcal{G}(\gamma, \gamma')$$

of all morphisms in $\mathcal{G}$ with domain γ.

Definition 2.1.21. Connected groupoids. *A groupoid $\mathcal{G}$ is* **connected** *if the morphism set $\mathcal{G}(\gamma, \gamma')$ is nonempty for each pair of objects γ, γ' in $\mathcal{G}$. A* **connected component** *of a groupoid is a maximal connected subgroupoid.*

A groupoid $\mathcal{G}$ is **1-connected** *if each morphism set $\mathcal{G}(\gamma, \gamma')$ has at most one element. In particular the automorphism group $\mathcal{G}(\gamma, \gamma)$ for each object γ is trivial.*

The following is an exercise for the reader.

Proposition 2.1.22. Automorphism groups in a connected groupoid. *If x and y are objects in the same connected component of a groupoid $\mathcal{G}$, then for any $\alpha \in \mathcal{G}(x, y)$, we get a group isomorphism $\mathcal{G}(x) \to \mathcal{G}(y)$ given by $\gamma \mapsto \alpha\gamma\alpha^{-1}$, so the groups $\mathcal{G}(x)$ and $\mathcal{G}(y)$ are isomorphic.*

Definition 2.1.23. Coverings of groupoids. *A functor $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ between groupoids is a* **covering** *if for each object $\widetilde{\gamma}$ in $\widetilde{\mathcal{G}}$ the map of stars (as in Definition 2.1.20)*

$$p : \mathrm{St}_{\widetilde{\mathcal{G}}}\widetilde{\gamma} \to \mathrm{St}_{\mathcal{G}}p(\widetilde{\gamma})$$

is a bijection. In other words, for each morphism $f : p(\gamma) \to \gamma'$ in $\mathcal{G}$, there is a unique morphism in $\widetilde{\mathcal{G}}$ from $\widetilde{\gamma}$ that maps to it. We say that p is **connected** *if both $\widetilde{\mathcal{G}}$ and $\mathcal{G}$ are connected as in Definition 2.1.21.*

A covering functor $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ need not be surjective on objects if the target is not connected. It is surjective onto each connected component containing $p(\widetilde{\gamma})$ for some object $\widetilde{\gamma}$ of $\widetilde{\mathcal{G}}$.

The lifts of two morphisms $p(\widetilde{\gamma}) \to \gamma'$ in $\mathcal{G}$ to morphisms from $\widetilde{\gamma}$ in $\widetilde{\mathcal{G}}$ need not have the same target even though their images in $\mathcal{G}$ do. The image of the group $\widetilde{\mathcal{G}}(\widetilde{\gamma})$ is a subgroup of $\mathcal{G}(p(\widetilde{\gamma}))$.

The following is proved by Ronald Brown in [Bro06, §10.2].

Proposition 2.1.24. Properties of groupoid coverings.

(i) *Let $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ be a groupoid covering with $\mathcal{G}$ connected as in Definition 2.1.21. Then for any two objects α and β in $\mathcal{G}$, the cardinalities of $p^{-1}(\alpha)$ and $p^{-1}(\beta)$ are the same.*

(ii) *Let $r : \mathcal{K} \to \mathcal{H}$, $q : \mathcal{H} \to \mathcal{G}$ be morphisms of groupoids. If q and r are covering morphisms, so is qr. If q and qr are covering morphisms, then r is on. If r and qr are covering morphisms, then q is a one when r is surjective on objects*

The following is proved as [Bro06, 10.3.3].

Proposition 2.1.25. Lifting to a groupoid covering. *Suppose we have a diagram of pointed groupoids (meaning groupoids with specified objects preserved by the functors in question)*

$$\begin{tikzcd} & (\widetilde{\mathcal{G}}, \tilde{\gamma}) \arrow[d, "p"] \\ (\mathcal{F}, \phi) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (\mathcal{G}, \gamma) \end{tikzcd}$$

where p is a covering and $\mathcal{F}$ is connected. Then the indicated lifting $\tilde{f}$ exists iff the group $\mathcal{F}(\phi)$ maps monomorphically to $p(\widetilde{\mathcal{G}}(\tilde{\gamma}))$, and if it exists it is unique.

Corollary 2.1.26. Relations between connected groupoid coverings. *Suppose that in the diagram of Proposition 2.1.25, the covering p is connected and that f is also a covering. Then the lifting exists iff the subgroup $f(\mathcal{F}(\phi)) \subseteq \mathcal{G}(\gamma)$ is contained in $p(\widetilde{\mathcal{G}}(\tilde{\gamma}))$. In particular it exists if $\mathcal{F}$ is 1-connected, meaning that the group $\mathcal{F}(\phi)$ is trivial.*

This result suggests the following definition.

Definition 2.1.27. *A connected covering of a connected groupoid is* **universal** *if it is 1-connected. The unversal cover of a general groupoid is the coproduct (as in Definition 2.1.5) of the universal covers of its connected components.*

The existence of a unversal covering groupoid follows from the next result where we see that a pointed connected groupoid $(\mathcal{G}, \gamma)$ has a covering for **any** subgroup $H \subseteq \mathcal{G}(\gamma)$, including the trivial one. It is proved by Brown as [Bro06, 10.4.3].

Proposition 2.1.28. A covering groupoid for each subgroup of $\mathcal{G}(\gamma)$. *Let γ be an object a connected groupoid $\mathcal{G}$ and let $H \subseteq \mathcal{G}(\gamma)$ be a subgroup. Let*

$$X = \{fH \subseteq \mathrm{St}_{\mathcal{G}}\gamma\}$$

be the set of left cosets of H, meaning equivalence classes of morhisms with *domain γ where two such morphisms are equivalent if they differ by precomposition with an element of H. Let $w : X \to \mathcal{G}_0$ be given by $w(fH = f(\gamma))$, and let $\mathcal{G}$ act on X by post composition. Then the evident map $p : X \rtimes \mathcal{G} \to \mathcal{G}$ is a groupoid covering, the action of $\mathcal{G}$ on X is transitive, and $p^{-1}(\gamma) = G(\gamma)/H$.*

Definition 2.1.29. ***G*-sets.**

(i) *For a group G, a G-**set** T is a set equipped with an action of G, that is a map $\mu : G \times T \to T$ such that the diagram*

$$\begin{tikzcd} G \times G \times T \arrow[r, "G\times\mu"] \arrow[d, "m\times T"'] & G \times T \arrow[d, "\mu"] \\ G \times T \arrow[r, "\mu"] & T, \end{tikzcd} \tag{2.1.30}$$

where $m : G \times G \to G$ denotes the multiplication in G. We also require that the composite

$$\begin{array}{ccccc} T & \longrightarrow & G \times T & \xrightarrow{\mu} & T \\ t & \longmapsto & (e, t) & \longmapsto & t, \end{array}$$

i.e., the identity element $e \in G$ acts as the identity map on T. We will usually write $\mu(\gamma, t)$ as $\gamma(t)$ or γt. The commutativity of (2.1.30) means that

$$\gamma_1(\gamma_2(t)) = (\gamma_1\gamma_2)t \qquad \text{for } \gamma_1, \gamma_2 \in G \text{ and } t \in T.$$

*We will sometimes refer to μ as a **left** action of G on T and denote it by μ_L to emphasize that it is acting on the left. Right actions can be similarly defined.*

(ii) *Given a G-set T, one has the **fixed point set***

$$T^G = \{t \in T : \gamma(t) = t \text{ for all } \gamma \in G\}.$$

*For each $t \in T$, the **orbit** of t is the set*

$$Gt = \bigcup_{\gamma \in G} \{\gamma(t)\}.$$

*Being in the same orbit defines an equivalence relation on T, and the set of equivalence classes in the **orbit set** T_G.*

*Associated to each element $t \in T$ is the **isotropy group** (also called the **stabilizer group**)*

$$G_t = \{\gamma \in G : \gamma(t) = t\},$$

the subgroup of elements fixing t. One can show that elements in the same orbit have conjugate isotropy groups. The orbit of t is isomorphic to G/G_t as a G-set.

The action of G on T is **free** *if the isotropy subgroup of every element is trivial. In that case each orbit is isomorphic to G (or G/e) as a G-set.*

A subset $S \subseteq T$ is **invariant** *if it is a union of orbits, or equivalently if it is preserved by the action of G. Its isotropy subgroup is*

$$G_S = \{\gamma \in G \colon \gamma(s) = s \text{ for all } s \in S\}.$$

In particular G_T is the subgroup fixing all of T.

For each subgroup $H \subseteq G$, T^H, T_H and H_t are similarly defined, with T^e and T_e each being T.

(iii) A map $f : T \to T'$ of G-sets is **equivariant** *if it is compatible with the G-actions on T and T', that is if the diagram*

$$\begin{tikzcd} G \times T \arrow[r, "G\times f"] \arrow[d, "\mu"'] & G \times T' \arrow[d, "\mu'"] \\ T \arrow[r, "f"] & T' \end{tikzcd}$$

commutes, where $\mu' : G \times T' \to T'$ defines the action of G on T'.

(iv) We denote the category of G-sets and equivariant maps by $\mathcal{S}et^G$, and we denote the category of G-sets and **all** *maps by $\mathcal{S}et_G$. For an arbitrary map $f : T \to T'$ and an element $\gamma \in G$, the diagram*

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\gamma^{-1}"'] & T' \\ T \arrow[r, "f"] & T' \arrow[u, "\gamma"'] \end{tikzcd}$$

need not commute. This means we get an action of G on the morphism set $\mathcal{S}et_G(T, T')$ defined by $\gamma(f) = \gamma f \gamma^{-1}$.

(v) For any subgroup $H \subseteq G$, a G-set is also an H-set. We denote the **forgetful functors** *$\mathcal{S}et^G \to \mathcal{S}et^H$ and $\mathcal{S}et_G \to \mathcal{S}et_H$ both by i_H^G. When H is the trivial group, we denote them by i_e^G.*

Later in this book we will make similar definitions in other categories, such as those of G-spaces and G-spectra.

Definition 2.1.31. Groupoids associated with a group G. *A group G can be regarded as a groupoid with one object in which the morphism set is isomorphic to G. We will denote this category by $\mathcal{B}G$.*

For a G-set T let $\mathcal{B}_T G$ denote the small category with object set T with a morphism $a \to \gamma(a)$ for each $(a, \gamma) \in T \times G$. Such a category is called a **split groupoid**, *a* **translation groupoid**, *a* **transformation groupoid** *or an* **action groupoid**. *The object sets of its connected components (as in Definition 2.1.21) are the orbits of T.*

For a subgroup $H \subseteq G$, the category $\mathcal{B}_T H$ (strictly speaking $\mathcal{B}_{i_H^G T} H$ where

i_H^G denotes the forgetful functor from G-sets to H-sets) is a wide (as in Definition 2.1.4) subcategory of $\mathcal{B}_T G$. We will denote the inclusion functor by j_H^G.

Example 2.1.32. Not all finite groupoids are split. *Let $\mathcal{G}$ have three objects, a, b and c, with an invertible morphism $a \to b$ and a single morphism $c \to c$. The set $T = \{a, b, c\}$ has an action of the group C_2 in which the nontrivial element permutes a and b while fixing c. The split groupoid associated with it has the same object set as $\mathcal{G}$. Unlike $\mathcal{G}$, it has two morphisms from c to c. Hence $\mathcal{G}$ is not split.*

We learned the following from Todd Trimble.

Proposition 2.1.33. A characterization of finite connected groupoids. *Let $\mathcal{G}$ and $\mathcal{G}'$ be finite connected groupoids having the same number of objects and isomorphic automorphism groups. Then $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic as groupoids.*

In particular, suppose that G and G' are finite groups having the same order and each having a subgroup isomorphic to H. Then the split groupoids $\mathcal{B}_{G/H}G$ and $\mathcal{B}_{G'/H}G'$ are isomorphic **even if the groups G and G' are distinct**. When H is trivial, the two groupoids are 1-connected as in Definition 2.1.21 and have the same number of objects. Thus the category $\mathcal{B}_{G/e}G$ **does not remember the group structure of** G, only its cardinality. We invite the reader to compare the groupoids for C_4/C_2 and $(C_2 \times C_2)/C_2$, or those for C_6/C_2 and Σ_3/C_2.

Proof Let $F : \mathcal{G} \to \mathcal{G}'$ be a bijection of object sets, choose an object x_0 in $\mathcal{G}$ and a group isomorphims $\phi : \mathcal{G}(x_0, x_0) \to \mathcal{G}'(F(x_0), F(x_0))$. We will show that F can be made into a functor that is the desired isomorphims by defining it on morphisms in $\mathcal{G}$.

Let the object set of $\mathcal{G}$ be $\{x_0, x_1, \ldots x_n\}$. Choose morphisms $g_j : x_0 \to x_j$ and $g'_j : F(x_0) \to F(x_j)$ for $1 \leqslant j \leqslant n$. Then define $F(g_j) = g'_j$. We know that each morphism $f : x_i \to x_j$ in $\mathcal{G}$ can be written uniquely as $g_i \alpha g_j^{-1}$ for some $\alpha \in \mathcal{G}(x_0, x_0)$. Hence we can define

$$F(f) = g'_i \phi(\alpha)(g'_j)^{-1}.$$

This makes F the desired isomorphism of groupoids. □

Remark 2.1.34. The independence of the groupoid $\mathcal{B}_{G/H}$G of the group structure of G *implied by Proposition 2.1.33 is surprising. Recall the description of the small category $\mathcal{B}_T G$ of Definition 2.1.31 for a G-set T. Its object set is T and its morphism set is identified with $G \times T$; for each $(\gamma, t) \in G \times T$ there is a unique morphism $t \to \gamma(t)$. This identification, which associates an element of G to each morphism, is* **extracategorical** *in that it more information than is needed to describe $\mathcal{B}_T G$ as a category.*

We have an equivalence relation on the object set of $\mathcal{G}$ in which two objects are equivalent iff there is a morphism between them. The equivalence classes then give us full subcategories which are the connected components of $\mathcal{G}$. It follows from Proposition 2.1.33 that the connected component containg an object γ is isomorphic to the split groupoid $\mathcal{B}_{G_\gamma/H_\gamma}G_\gamma$ as in Definition 2.1.31, where G_γ is any group of the appropriate order having $H_\gamma = \mathcal{G}(\gamma,\gamma)$ as a subgroup. This category is known to be equivalent (but not isomorphic) to $\mathcal{B}H_\gamma$ by Proposition 2.1.38. **The groupoid $\mathcal{G}$ is split iff there is a single group G that fits this description for each connected component.**

We can generalize the notion of a group G acting on a sets as in Definition 2.1.29 to a groupoid action as follows.

Definition 2.1.35. The action of a groupoid on a set. *For a groupoid $\mathcal{G}$, a* **$\mathcal{G}$-set** *X is a set equipped with a map $w : X = X_0 \to \mathcal{G}_0$ with*

$$\begin{tikzcd}
X_0 \arrow[d, "w_0=w"'] & X_1 \arrow[l, "s'"'] \arrow[r, "t'"] \arrow[d, "w_1"] \arrow[dl, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near start] & X_0 \arrow[d, "w_0"] \\
\mathcal{G}_0 & \mathcal{G}_1 \arrow[l, "s"'] \arrow[r, "t"] & \mathcal{G}_0
\end{tikzcd}$$

Here the pullback on the left means that X_1 is the set

$$X_1 = \{(x, \gamma : w(x) \to y) \in X \times \mathcal{G}_1\}$$

of pairs consisting of an element $x \in X$ and a morphism γ with domain $w(x) \in \mathcal{G}_0$, and s' sends such a pair to x. The map t' sends it to an element in $w^{-1}(y)$, which makes the right square a pullback as well. We will refer to the map t' as the **action of $\mathcal{G}$ on** *X and denote $t'(x,\gamma)$ by $\gamma(x)$. This action should be compatible with identity morphisms and composition of morphisms in $\mathcal{G}$. Details can be found in [Bro06, §10.4].*

The **automorphism group $\mathcal{G}_x$ of** *x (Brown calls it the* **group of stability**) *is*

$$\mathcal{G}_x = \{\gamma \in \mathcal{G}(w(x)) \colon \gamma(x) = \gamma\},$$

and we say that such a γ **fixes** *x.*

The action of $\mathcal{G}$ on X is **transitive** *if $\mathcal{G}$ is connected and for all $a, b \in \mathcal{G}_0$, $x \in w^{-1}(a)$ and $y \in w^{-1}(b)$, there is a morphims $\gamma \in \mathcal{G}(a,b)$ such that $\gamma(x) = y$.*

Example 2.1.36. Some $\mathcal{G}$-sets.

(i) For a group G, a G-set X as in Definition 2.1.29 is also a $\mathcal{B}G$-set, where $\mathcal{B}G$ is the one object groupoid of Definition 2.1.31.

(ii) If $p : \widetilde{\mathcal{G}} \to \mathcal{G}$ is a groupoid covering as in Definition 2.1.23, then $\widetilde{\mathcal{G}}_0$ is a $\mathcal{G}$-set, with the diagram of Definition 2.1.35 being

$$\begin{tikzcd}
\widetilde{\mathcal{G}}_0 \arrow[d, "p_0"'] & \widetilde{\mathcal{G}}_1 \arrow[l, "\tilde{s}"'] \arrow[r, "\tilde{t}"] \arrow[d, "p_1"] \arrow[dl, phantom, "\urcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near start] & \widetilde{\mathcal{G}}_0 \arrow[d, "p_0"] \\
\mathcal{G}_0 & \mathcal{G}_1 \arrow[l, "s"'] \arrow[r, "t"] & \mathcal{G}_0
\end{tikzcd}$$

The following is essentially [Bro06, 10.4.2]. Brown calls the groupoid associated with the $\mathcal{G}$-set X the **semidirect product** $X \rtimes \mathcal{G}$.

Proposition 2.1.37. Every $\mathcal{G}$-set is a groupoid. *For a $\mathcal{G}$-set X as in Definition 2.1.35, the sets X_0 and X_1 are the object and morphism sets of a groupoid for which the source and target maps are s' and t', and the map w induced a groupoid covering. In other words, the diagram of Definition 2.1.35* **always** *has the form of Example 2.1.36(ii). This groupoid is connected iff the action of $\mathcal{G}$ on X is transitive. For $x \in X$, the automorphism group $(X \rtimes \mathcal{G})(x)$ of Definition 2.1.17 is the automorphism group $\mathcal{G}_x$ of Definition 2.1.35.*

We record the following for future reference. For an H-set T, we have a G-set $G \times_H T$. Its elements are pairs (γ, t) for $\gamma \in G$ and $t \in T$, subject to the relation $(\gamma\eta, t) \sim (\gamma, \eta t)$ for $\eta \in H$.

Proposition 2.1.38. The equivalence between $\mathcal{B}_{G\times_H T}G$ and $\mathcal{B}_T H$. *Let $H \subseteq G$ be finite groups and let T be a finite H-set. Let $j : \mathcal{B}_T H \to \mathcal{B}_{G\times_H T}G$ be the inclusion functor sending $t \in T$ to the equivalence class of (e, t) in $G \times_H T$. It is an equivalence of categories as in Definition 2.2.4 below. In particular (the case $T = H/H$), $\mathcal{B}H$ is equivalent to $\mathcal{B}_{G/H}G$.*

Proof Choose a representative $(\alpha, t) \in G \times T$ for each element of $G \times_H T$ such that (e, t) represents $j(t)$. Define a functor $k : \mathcal{B}_{G\times_H T}G \to \mathcal{B}_T H$ by $(\alpha, t) \mapsto t$. To describe its effect on morphisms, let $\gamma_1 \in G$ and suppose the chosen representative of $(\gamma_1\alpha_0, t_0)$ is (α_1, t_1). This means that $t_1 = \eta_1 t_0$ for some $\eta_1 \in H$ with $\alpha_1\eta_1 = \gamma_1\alpha_0$. Since $(\gamma_1\alpha_0, t_0) \sim (\alpha_1, \eta_1 t_0)$, we find that $\eta_1 = \alpha_1^{-1}\gamma_1\alpha_0$.

Hence our functor k sends the morphism $\gamma_1 : (\alpha_0, t_0) \to (\alpha_1, t_1)$ in $\mathcal{B}_{G\times_H T}G$ to the morphism $\eta_1 = \alpha_1^{-1}\gamma_1\alpha_0 : t_0 \to t_1$ in $\mathcal{B}_T H$. Similarly it sends the morphism $\gamma_2 : (\alpha_1, t_1) \to (\alpha_2, t_2)$ to $\eta_2 = \alpha_2^{-1}\gamma_2\alpha_1 : t_1 \to t_2$. Thus we have a diagram

$$\begin{tikzcd}
(\alpha_0, t_0) \arrow[r, "\gamma_1"] \arrow[d, mapsto] & (\alpha_1, t_1) \arrow[r, "\gamma_2"] \arrow[d, mapsto] & (\alpha_2, t_2) \arrow[d, mapsto] \\
t_0 \arrow[r, "\alpha_1^{-1}\gamma_1\alpha_0"] & t_1 \arrow[r, "\alpha_2^{-1}\gamma_2\alpha_1"] & t_2
\end{tikzcd}$$

where the top row is in $\mathcal{B}_{G\times_H T}G$ and the bottom row is in $\mathcal{B}_T H$. The composite of the two morphisms in the bottom row is

$$\alpha_2^{-1}\gamma_2\alpha_1 \cdot \alpha_1^{-1}\gamma_1\alpha_0 = \alpha_2^{-1}\gamma_2\gamma_1\alpha_0,$$

which is the image under k of the composite of the morphisms in the top row. This means our functor k is well defined.

Then kj is the identity functor on $\mathcal{B}_T H$, and we need a natural transformation $\theta : jk \Rightarrow 1_{\mathcal{B}_{G\times_H T}G}$. Note that $jk(\alpha, t) = (e, t)$, so we can define θ by

$$\theta_{(\alpha,t)} = \alpha : (e, t) \to (\alpha, t). \tag{2.1.39}$$

We leave the remaining details to the reader. □

Note that the equivalence above is not unique. It depends on the choice of an element in $G \times T$ representing each element of the quotient $G \times_H T$.

Corollary 2.1.40. The equivalence between $\mathcal{C}^{\mathcal{B}_{G\times_H T}G}$ and $\mathcal{C}^{\mathcal{B}_T H}$. *Let $H \subseteq G$ be finite groups, let T be a finite H-set and let $\mathcal{C}$ be any category. Then the functor categories $\mathcal{C}^{\mathcal{B}_{G\times_H T}G}$ and $\mathcal{C}^{\mathcal{B}_T H}$ (for example $\mathcal{C}^{\mathcal{B}_{G/H}G}$ and $\mathcal{C}^{\mathcal{B}H}$) are equivalent.*

The following discussion will be used in the proof of Proposition 2.2.31 and taken up again following Proposition 9.3.16.

Example 2.1.41. The equivalence of the categories $\mathcal{B}H$ and $\mathcal{B}_{G/H}G$ for a subgroup $H \subseteq G$, and those of $\mathcal{C}$-valued functors on them. *Consider the case of Proposition 2.1.38 and Corollary 2.1.40 where $T = H/H$, so $G \times_H T \cong G/H$. Then the category $\mathcal{B}_T H = \mathcal{B}H$ has a single object, so the functor $k : \mathcal{B}_{G/H}G \to \mathcal{B}H$ is uniquely determined on objects. The choice made in the proof of Proposition 2.1.38 is of an element $\alpha \in G$ in each coset γH. The functor k depends on this choice.*

To describe the effect of k on morphisms in $\mathcal{B}_{G/H}G$, each of which is determined by its domain and an element of G, consider the one with domain α_0 and associated with $\gamma_1 \in G$. Suppose that $\gamma_1\alpha_0$ lies in the coset represented by α_1. The image of this morphism under k is a morphism in $\mathcal{B}H$, which is to say an element of H, namely $\alpha_1^{-1}\gamma_1\alpha_0$.

The morphism set of $\mathcal{B}_{G/H}G$ is $G/H \times G$, while that of $\mathcal{B}H$ is H. The calculation above shows that $k(\alpha_0, \gamma_1) = \alpha_1^{-1}\gamma_1\alpha_0$, where α_1 is the chosen representative of the coset containing $\gamma_1\alpha_0$.

Now we look at categories of $\mathcal{C}$-valued functors on $\mathcal{B}_{G/H}G$ and $\mathcal{B}H$. The diagram of categories and functors

$$\begin{tikzcd} \mathcal{B}H \arrow[r, shift right, "j"'] & \mathcal{B}_{G/H}G \arrow[l, shift right, "k"'] \end{tikzcd}$$

leads to

$$\begin{tikzcd} \mathcal{C}^{\mathcal{B}H} \arrow[r, shift left, "k^*"] & \mathcal{C}^{\mathcal{B}_{G/H}G}. \arrow[l, shift left, "j^*"] \end{tikzcd} \tag{2.1.42}$$

The category $\mathcal{C}^{\mathcal{B}H}$ is that of objects X in $\mathcal{C}$ equipped with an action of the group H. An object Y in $\mathcal{C}^{\mathcal{B}_{G/H}G}$ is a collection of objects $Y_{\gamma H}$ in $\mathcal{C}$ indexed by the cosets in G/H, each having an action of H and isomorphisms with other such objects induced by elements of G. When $\mathcal{C}$ is complete (cocomplete) as in Definition 2.3.25 below, the product (coproduct) of these objects has a G-action permuting the factors, each of which is invariant under and acted on by the subgroup H.

The functor j^* sends a collection $\{Y_{\gamma H}\}$ as above to the H-object Y_{eH}. The functor k^* sends an H-object X to a collection of isomorphic copies of X indexed by G/H with suitable isomorphisms between them induced by elements of G.

This means the composite j^*k^* is the identity functor on $\mathcal{C}^{\mathcal{B}H}$, but k^*j^* is not the identity functor on $\mathcal{C}^{\mathcal{B}_{G/H}G}$. It sends the collection $\{Y_{\gamma H}\}$ to the one in which each component is Y_{eH}. Our choice of representatives of the cosets of H leads to a natural transformation between k^*j^* and the identity functor on $\mathcal{C}^{\mathcal{B}_{G/H}G}$.

Example 2.1.43. A case of Corollary 2.1.40 where the subgroup is trivial. *A functor $\mathcal{B}_{G/e}G \to \mathcal{C}$ consists of a collection of objects in $\mathcal{C}$ indexed by the elements of G and an isomorphism from each one to every other. The category of such collections is equivalent to $\mathcal{C}$ itself. In this case the equivalence of Proposition 2.1.38 is unique because each coset to the trivial group has a single element.*

Example 2.1.44. G-sets as groupoids. *If K is the category $\mathcal{B}_T G$ associated with a finite G-set T as in Definition 2.1.31, then its equivalence classes are its orbits and each group G_k is isomorphic to G. The subgroups H_k may vary, even up to conjugacy, from orbit to orbit.*

This implies the following.

Proposition 2.1.45. Functors from a finite groupoid. *Let K be a finite groupoid decomposing as above into a finite union of orbits $\mathcal{B}_{G_k/H_k}G_k$. Then for any category $\mathcal{C}$,*

$$\mathcal{C}^K \cong \prod_k \mathcal{C}^{\mathcal{B}_{G_k/H_k}G_k} \simeq \prod_k \mathcal{C}^{\mathcal{B}H_k} \qquad \text{by Corollary 2.1.40.}$$

For each $k \in K$, the Yoneda embedding (Definition 2.2.12)

$$よ : (\mathcal{B}_{G_k/H_k}G_k)^{op} \to \mathcal{S}et^{\mathcal{B}_{G_k/H_k}G_k}$$

sends each object, i.e., each coset, $\gamma_s H_k$ (for $\gamma_s \in G_k$) to the functor which assigns to each object $\gamma_t H_k$ the set $\gamma_t H_k \gamma_s^{-1}$.

2.1F Topological spaces

There are some technical difficulties associated with $\mathcal{T}op$ (respectively $\mathcal{T}$), the category of (pointed) topological spaces. It turns out that for arbitrary (pointed) spaces X and Y, the sets $\mathcal{T}op(X, Y)$ and $\mathcal{T}(X, Y)$ do not have natural topologies with the desired properties. This problem is discussed in detail in [Rie14, 6.1] and in [Hov99, Definition 2.4.21 and Proposition 2.4.22]. One can get around it by making some mild assumptions on the topological spaces one considers. One replaces $\mathcal{T}op$ and $\mathcal{T}$ by certain full subcategories (compactly generated weak Hausdorff spaces, first introduced in [McC69] and described more recently in [Str09]) known to have the desired properties and to include nearly all of the spaces (such as CW complexes and manifolds) a homotopy theorist would ever want to think about.

Definition 2.1.46. *A topological space X is* **weak Hausdorff** *if the image of any map $K \to X$ from a compact Hausdorff space K is closed in X. X is* **compactly generated** *if every closed subspace of X is a union of compact subspaces.*

In particular, in such a space each point is closed.

Remark 2.1.47. *Working with compactly generated weak Hausdorff spaces has many benefits, but it does create some technical problems. Colimits are computed by forming the colimit in topological spaces, replacing the topology by the compactly generated topology, and then forming the universal quotient which is weak Hausdorff; see [Str09, Corollary 2.23]. This last step can alter the underlying point set since a colimit of weak Hausdorff need not be weak Hausdorff; see Example 2.3.65 below. It does not, however, in the case of pushouts along closed inclusions, meaning contiuous monomorphism whose images are clased subest of the codomain. More precisely, given a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

of topological spaces in which $A \to X$ is a closed inclusion, if A, X, and B are compactly generated and weak Hausdorff then so is Y. This follows from [McC69, Proposition 2.5] and the remark about adjunction spaces immediately preceding its statement, and from [Str09, Proposition 2.35]. Among other things this means that the smash product of two compactly generated weak Hausdorff spaces can be computed as the smash product of the underlying compactly generated spaces.

Definition 2.1.48. The categories $\mathcal{T}op$ and $\mathcal{T}$ *are the category of compactly generated weak Hausdorff spaces and its pointed analog.*

We will use the term "compactly generated" with a different meaning in connection with model categories below in Definition 5.2.6.

In [Hov99] these categories are denoted by **T** and $\mathbf{T}_*$ respectively. He uses the notation **Top** and $\mathbf{Top}_*$ for the categories of topological and pointed topological spaces with no additional conditions.

Definition 2.1.49. Smash products and half smash products. *Let A and B be topological spaces with base points a_0 and b_0. Their* **smash product** *is the topological quotient*

$$A \wedge B := (A \times B)/(A \times \{a_0\} \cup \{a_0\} \times B),$$

where the base point is the image of the contracted subspace $A \times \{a_0\} \cup \{a_0\} \times B$.

For an unpointed space X, let X_+ denote its union with a disjoint base point. Then the **left and right half smash products** *are*

$$\begin{aligned} X \ltimes B &:= X_+ \wedge B = X \times B/X \times \{b_0\}, \\ \text{and} \qquad A \rtimes X &:= A \wedge X_+ = A \times X/\{a_0\} \times X, \end{aligned}$$

where the base points are the images of the contracted subspaces $X \times \{b_0\}$ and $\{a_0\} \times X$. Note that the **two sided half smash product** *is*

$$X \bowtie Y := X_+ \wedge Y_+ \cong X \ltimes Y_+ \cong X_+ \rtimes Y \cong (X \times Y)_+.$$

The following is an easy exercise.

Proposition 2.1.50. Connectivities of the smash and half smash products. *Suppose A and B are respectively $(m-1)$- and $(n-1)$-connected. Then $A \wedge B$, $A \rtimes B$ and $A \ltimes B$ are respectively $(m+n-1)$-, $(m-1)$- and $(n-1)$-connected.*

2.1G Miscellaneous definitions

Definition 2.1.51. The comma category and related constructions. *Suppose we have functors $S : \mathcal{A} \to \mathcal{C}$ (the source) and $T : \mathcal{B} \to \mathcal{C}$ (the target). The associated* **comma category** $(S{\downarrow}T)$ **or** $S{\downarrow}T$ *(also denoted by (S/T) and originally by (S, T), hence the name) has as objects triples of the form (α, f, β) where α and β are objects in $\mathcal{A}$ and $\mathcal{B}$ with $f : S(\alpha) \to T(\beta)$ a morphism in $\mathcal{C}$. A morphism from (α, f, β) to (α', f', β') is a pair of morphisms $g : \alpha \to \alpha'$ in $\mathcal{A}$ and $h : \beta \to \beta'$ in $\mathcal{B}$ such that the following diagram commutes in $\mathcal{C}$.*

$$\begin{tikzcd} S(\alpha) \arrow[r, "S(g)"] \arrow[d, "f"'] & S(\alpha') \arrow[d, "f'"] \\ T(\beta) \arrow[r, "T(h)"'] & T(\beta'). \end{tikzcd}$$

This construction has several interesting special cases.

(i) *Let $\mathcal{B}$ be the category* $\mathbf{1}$ *with a single object and a single identity morphism. Then the functor T identifies an object c in $\mathcal{C}$ and the resulting category is denoted by $(S{\downarrow}c)$, the* **category of objects of $\mathcal{A}$ over** *c, also known as the* **overcategory of** *c. Its objects are pairs (α, ω_α) where α is an object in $\mathcal{A}$ and $\omega_\alpha : S(\alpha) \to c$ is a morphism in $\mathcal{C}$. A morphism from (α, ω_α) to $(\alpha', \omega_{\alpha'})$ is a morphism $g : \alpha \to \alpha'$ in $\mathcal{A}$ such that following diagram commutes in $\mathcal{C}$.*

$$\begin{tikzcd} S(\alpha) \arrow[rr, "S(g)"] \arrow[dr, "\omega_\alpha"'] & & S(\alpha') \arrow[dl, "\omega_{\alpha'}"] \\ & c & \end{tikzcd} \tag{2.1.52}$$

(ii) *Dually, let $\mathcal{A} = \mathbf{1}$, so S identifies an object c in $\mathcal{C}$. The resulting category is denoted by $(c{\downarrow}T)$, the* **category of objects of $\mathcal{B}$ under** *c also known as the* **undercategory of** *c. Its objects are pairs (α, ω_α) where α is an object in $\mathcal{A}$ and $\omega_\alpha : S(\alpha) \to c$ is a morphism in $\mathcal{C}$. A morphism from (α, ω_α) to $(\alpha', \omega_{\alpha'})$ is a morphism $g : \alpha \to \alpha'$ in $\mathcal{A}$ such that following diagram commutes in $\mathcal{C}$. . Its objects are pairs (υ_β, β) where β is an object in $\mathcal{B}$ and $\upsilon_\beta : c \to T(\beta)$ is a morphism in $\mathcal{C}$. A morphism from (υ_β, β) to $(\upsilon_{\beta'}, \beta')$ is a morphism $h : \beta \to \beta'$ in $\mathcal{B}$ such that following diagram commutes in $\mathcal{C}$.*

$$\begin{tikzcd} & c \arrow[dl, "\upsilon_\beta"'] \arrow[dr, "\upsilon_{\beta'}"] & \\ T(\beta) \arrow[rr, "T(h)"] & & T(\beta') \end{tikzcd} \tag{2.1.53}$$

(iii) *Now assume $\mathcal{B} = \mathbf{1}$, $\mathcal{A} = \mathcal{C}$ and S is the identity functor. Then the resulting category $(\mathcal{C}{\downarrow}c)$ is called the* **slice category** *or* **overcategory** *of objects over c. Dually, the* **coslice category** *or* **undercategory** *$(c{\downarrow}\mathcal{C})$ is obtained by making T the identity functor and $\mathcal{A} = \mathbf{1}$.*

(iv) *When $\mathcal{C}$ has a terminal object $*$, then $(*{\downarrow}\mathcal{C})$ is the* **category of pointed objects in $\mathcal{C}$.**

(v) *when $\mathcal{A} = \mathcal{C} = \mathcal{B}$, and both S and T are the identity functor, we get the* **arrow category $\mathbf{Arr}\,\mathcal{C}$, which we will sometimes denote by** *$\mathcal{C}_1$, whose objects are morphisms in $\mathcal{C}$. Given morphisms $f : A \to B$ and $g : X \to Y$ in $\mathcal{C}$, which are also objects in $\mathcal{C}_1$, a morphism $f \to g$ in $\mathcal{C}_1$ is a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ X \arrow[r, "g"] & Y \end{tikzcd} \tag{2.1.54}$$

for arbitrary morphisms a and b. **We will sometimes denote the set of such diagrams by $\diamondsuit(f,g)$.**

(vi) There are forgetful functors from $(S{\downarrow}T)$ to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_1$, known as the **domain, codomain and arrow functors**, *sending (α, f, β) to α, β and f respectively.*

Definition 2.1.55. Connected categories. *A category $\mathcal{C}$ is* **connected** *if for any pair of objects X and Y there is a finite sequence of morphisms connecting them,*

$$X \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdots \longleftarrow Y.$$

Note that the first and/or last morphisms in the diagram of Definition 2.1.55 could be identity maps, so X and/or Y could be a target (rather than a source) in the chain of morphisms.

Definition 2.1.56. Retracts. *An object X in a category $\mathcal{C}$ is a* **retract of an object** *Y if there are morphisms $i : X \to Y$ (the* **section***) and $r : Y \to X$ (the* **retraction***) such that $ri = 1_X$. A morphism $f : X \to X'$ is a* **retract of a morphism** *$g : Y \to Y'$ if there is a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\
X' \arrow[r, "i'"] & Y' \arrow[r, "r'"] & X'
\end{tikzcd}$$

where $ri = 1_X$ and $r'i' = 1_{X'}$.

Remark 2.1.57. The idempotent associated with a retraction. *The composite $e = ir : Y \to Y$ is* **idempotent**, *meaning that $e^2 = e$. See Example 2.3.35(vii) below for a description of a retract as a coequalizer as in Definition 2.3.27 below.*

Definition 2.1.58. *A small category $\mathcal{D}$ is* **direct** *if there there is a function d assigning to each object X a natural number $|X|$ called its* **degree**, *such that there are no morphisms that lower degree, and the only degree preserving morphisms are identities. A category $\mathcal{C}$ is* **inverse** *if $\mathcal{D}^{op}$ is direct.*

A **generalized direct category** *$\mathcal{D}$ is one equipped with such a function in which degree preserving morphisms need not be identities but instead need to be invertible.*

The degree of an object in a direct category is sometimes defined to be a more general type of ordinal, but the above notion is adequate for our purposes. Later in this book we will study the category of functors from a generalized direct category $\mathcal{D}$ to a model category $\mathcal{M}$ that has a monoidal structure as in Definition 5.5.9 and $\mathcal{D}$ is enriched over $\mathcal{M}$. See Definition 5.6.31 below.

2.2 Natural transformations, adjoint functors and monads

2.2A Natural transformations and equivalences

Definition 2.2.1. *Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a* **natural transformation** $\theta : F \Rightarrow G$ *is a function assigning to each object X in $\mathcal{C}$ a morphism $\theta_X : F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $f : X \to Y$ in $\mathcal{C}$ the following diagram commutes in $\mathcal{D}$.*

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd} \tag{2.2.2}$$

We denote the set of such natural transformations by $Nat(F, G)$.

Such a $\theta : F \Rightarrow F$ is the **identity natural transformation** *if each θ_X is the identity morphism on $F(X)$.*

Such a θ is a **natural equivalence** *if each θ_X is an isomorphism. In this case there is a natural transformation $\theta^{-1} : G \Rightarrow F$ such that $\theta^{-1}\theta$ and $\theta\theta^{-1}$ are the identity natural transformations on F and G respectively.*

In [ML98] Mac Lane used the symbol $\dot{\to}$ to denote a natural transformation.

Proposition 2.2.3. Composition of functors with a natural transformation. *With notation as in Definition 2.2.1, suppose we also have functors $K : \mathcal{B} \to \mathcal{C}$ and $L : \mathcal{D} \to \mathcal{E}$. Thus we have the following diagram of categories and functors.*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "K"] & \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \arrow[r, "L"] & \mathcal{E} \end{tikzcd}$$

Then we can define natural transformations

- $L_*\theta : LF \Rightarrow LG$ *by requiring that for each object X in $\mathcal{C}$, the morphism $(L_*\theta)_X : LF(X) \to LG(X)$ in $\mathcal{E}$ is the image of the morphism θ_X in $\mathcal{D}$ under the functor L,*
- $K^*\theta : FK \Rightarrow GK$ *by requiring that for each object W in $\mathcal{K}$, the morphism $(K^*\theta)_W : FK(W) \to GK(W)$ in $\mathcal{D}$ is $\theta_{K(W)}$, and*
- $L_*K^*\theta = K^*L_*\theta : LFK \Rightarrow LGK$ *by requiring that for each object W in $\mathcal{K}$, the morphism $(L_*K^*\theta)_W : LFK(W) \to LGK(W)$ in $\mathcal{E}$ is the image of $\theta_{K(W)}$ under L.*

Thus we have the following commutative diagram of sets of natural transfor-

mations and maps between them.

$$\begin{tikzcd} Nat(F,G) \arrow[r, "L_*"] \arrow[d, "K^*"'] & Nat(LF,LG) \arrow[d, "K^*"] \\ Nat(FK,GK) \arrow[r, "L_*"] & Nat(LFK,LGK) \end{tikzcd}$$

Proof For $L_*\theta$ we need to verify that for each morphism $f : X \to Y$ in $\mathcal{C}$ the following diagram commutes in $\mathcal{E}$.

$$\begin{tikzcd} LF(X) \arrow[r, "(L_*\theta)_X"] \arrow[d, "LF(f)"'] & LG(X) \arrow[d, "LG(f)"] \\ LF(Y) \arrow[r, "(L_*\theta)_Y"] & LG(Y) \end{tikzcd}$$

It does because it is the image of (2.2.2) under L.

The arguments for $K^*\theta$ and $L_*K^*\theta$ are similar. □

Definition 2.2.4. *Two categories $\mathcal{C}$ and $\mathcal{D}$ are* **equivalent** *if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural equivalences $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$. We will sometimes denote this state of affairs by $\mathcal{C} \simeq \mathcal{D}$*

Example 2.2.5. The category of finite dimensional vector spaces. *Let k be a field and let $\mathcal{V}ect_k$ be the category of finite dimensional vector spaces over k and linear maps between them. Let $\mathcal{V}ect'_k$ be the full subcategory consisting of the vector spaces k^n for all nonnegative integers n. Let $F : \mathcal{V}ect'_k \to \mathcal{V}ect_k$ be the inclusion functor, and let $G : \mathcal{V}ect_k \to \mathcal{V}ect'_k$ be the functor that sends each vector space V to $k^{\dim V}$. To define the natural transformation $\epsilon : FG \Rightarrow 1_{\mathcal{V}ect_k}$, choose an isomorphism $\epsilon_V : V \to k^{\dim V}$ for each V, with ϵ_{k^n} the identity map on k^n for each n. Having made this choice, given a linear map $f : V \to W$, we can define $G(f)$ to be the unique linear map making the following diagram commute.*

$$\begin{tikzcd} V \arrow[r, "\epsilon_V"] \arrow[d, "f"'] & k^{\dim V} \arrow[d, "G(f)"] \\ W \arrow[r, "\epsilon_W"] & k^{\dim W}. \end{tikzcd}$$

The other composite functor, GF, is the identity on $\mathcal{V}ect'_k$, so we can define η to be the identity natural transformation. Since η_{k^n} and ϵ_V are isomorphisms in each case, we have an equivalence of categories.

Note that the equivalence above is not canonical. It depends on the choice of an isomorphism between V and $k^{\dim V}$ for each V. For another example with a similar flavor, see Proposition 2.1.38 below.

Definition 2.2.6. Composition and precomposition as natural transformations. *Let*

$$H : \mathcal{S}et^{op} \times \mathcal{S}et \to \mathcal{S}et$$

be a functor. (Such a functor could depend on just one of the two variables. Hence we can treat covariant and contravariant functors of a single variable simultaneously.) For a fixed set Y, consider another such functor

$$\begin{array}{ccc} \mathcal{S}et^{op} \times \mathcal{S}et & \xrightarrow{H_Y} & \mathcal{S}et \\ (X, Z) & \longmapsto & \mathcal{S}et(Y, Z) \times H(X, Y) \end{array}$$

Then we define a natural transformation $\theta^Y : H_Y \implies H$ as follows. For an object (X, Z) in $\mathcal{S}et^{op} \times \mathcal{S}et$, the map

$$H_Y(X, Z) = \mathcal{S}et(Y, Z) \times H(X, Y) = \coprod_{g:Y \to Z} H(X, Y) \xrightarrow{\theta^Y_{(X,Z)}} H(X, Z)$$

on the gth copy of $H(X, Y)$ is $g_* : H(X, Y) \to H(X, Z)$. We call this **composition at** Y*. In particular, let $F = H(X, -)$. Then we have*

$$\theta^X_{(Y,Z)} : \mathcal{S}et(Y, Z) \times F(Y) \to F(Z).$$

Similarly, for a set X consider the functor

$$\begin{array}{ccc} \mathcal{S}et^{op} \times \mathcal{S}et & \xrightarrow{H^X} & \mathcal{S}et \\ (W, Y) & \longmapsto & H(X, Y) \times \mathcal{S}et(W, X) \end{array}$$

and define $\kappa^X : H^X \implies H$, **precomposition at** X*, as follows. For an object (W, Y) in $\mathcal{S}et^{op} \times \mathcal{S}et$, the map*

$$H^X(W, Y) = H(X, Y) \times \mathcal{S}et(W, X) = \coprod_{f:W \to X} H(X, Y) \xrightarrow{\kappa^X_{(W,Y)}} H(W, Y)$$

on the fth copy of $H(X, Y)$ is $f^* : H(X, Y) \to H(W, Y)$. In particular, let G be the contravariant functor $H(-, Y)$. Then we have

$$\kappa^Y_{(W,X)} : G(X) \times \mathcal{S}et(W, X) \to G(W).$$

Threefold composition $W \to X \to Y \to Z$ in $\mathcal{S}et$ leads to a commutative diagram

$$\begin{tikzcd}[column sep=small]
 & \mathcal{S}et(Y,Z)\times H(X,Y)\times \mathcal{S}et(W,X) \arrow[dl, "\theta^Y_{(X,Z)}\times \mathcal{S}et(W,X)"'] \arrow[dr, "\mathcal{S}et(Y,Z)\times\kappa^X_{(W,Y)}"] & \\
H(X,Z)\times \mathcal{S}et(W,X) \arrow[dr, "\kappa^X_{(W,Z)}"'] & & \mathcal{S}et(Y,Z)\times H(W,Y) \arrow[dl, "\theta^Y_{(W,Z)}"] \\
 & H(W,Z). &
\end{tikzcd} \quad (2.2.7)$$

An alternate approach to composition and precomposition will be given below in Definition 2.2.24. A generalization to enriched categories will be given in Definition 3.1.40.

Definition 2.2.8. Augmented and coaugmented functors. *An* **augmented functor** (F,ϵ) *(***coaugmented functor** (F,η)*) on a category* $\mathcal{C}$ *is an endofunctor* $F:\mathcal{C}\to\mathcal{C}$ *with a natural transformation* $\epsilon : F \Rightarrow 1_{\mathcal{C}}$ *(*$\eta : 1_{\mathcal{C}} \Rightarrow F$*). For each object* X*, the map* $\epsilon_X : F(X)\to X$ *in the augmented case (*$\eta_X : X \to F(X)$ *in the coaugmented case) is the* **augmentation** *(***coaugmentation***).*

2.2B Functorial factorizations

Let $[0]$, $[1]$ and $[2]$ denote the small categories $\{0\}$, $\{0\to 1\}$ and $\{0\to 1\to 2\}$. For a category $\mathcal{C}$, we denote the categories of $\mathcal{C}$-valued functors on $[1]$ and $[2]$ by $\mathcal{C}^{[1]}$ and $\mathcal{C}^{[2]}$; the category $\mathcal{C}^{[0]}$ is $\mathcal{C}$ itself. Hence an object X in $\mathcal{C}^{[1]}$ is a morphism $X_0\to X_1$ and an object Y in $\mathcal{C}^{[2]}$ is a composable morphism pair $Y_0\to Y_1\to Y_2$. For $0\leqslant j\leqslant 2$ there is a functor $d_j:[1]\to[2]$ (the face maps of §3.4 below) defined to be the order preserving maps of objects for which j is not in the image. We also have $d_0, d_1 : [0]\to[1]$ defined similarly. These induce functors $\delta_j : \mathcal{C}^{[2]}\to\mathcal{C}^{[1]}$ sending $Y_0\to Y_1\to Y_2$ to the morphisms $Y_1\to Y_2$, $Y_0\to Y_2$ and $Y_0\to Y_1$ respectively. The functors $\delta_0,\delta_1 : \mathcal{C}^{[1]}\to\mathcal{C}$ send a morphism $X_0\to X_1$ to its target and source respectively.

The following is needed for the study of model categories starting in Chapter 4 below.

Definition 2.2.9. *A* **functorial factorization in** $\mathcal{C}$ *is a functor (meaning a natural transformation)*

$$F:\mathcal{C}^{[1]}\to\mathcal{C}^{[2]}$$

such that $\delta_1 F = 1_{\mathcal{C}^{[1]}}$*. We denote its image on* $f : X_0\to X_1$ *by*

$$X_0 \xrightarrow{\delta_2 F(f)} X_{1/2} \xrightarrow{\delta_0 F(f)} X_1.$$

2.2C The Yoneda lemma

We turn now to a fundamental result originally due to Yoneda. We will see other formulations of the Yoneda lemma below in Proposition 2.4.20, the Enriched Yoneda Lemma 3.1.29 and Proposition 3.1.70.

Yoneda Lemma 2.2.10. *For an object A in a category $\mathcal{C}$, consider the covariant $\mathcal{S}et$ valued functor $よ^A = \mathcal{C}(A, -)$ (the* **Yoneda functor***) on $\mathcal{C}$. (The symbol よ is the Japanese hiragana character* yo*, the first syllable of Yoneda's name.) Let F be another such functor. Then there is a bijection*

$$\kappa : Nat(よ^A, F) \to F(A)$$

sending a natural transformation θ to the image of $1_A \in よ^A(A)$ under θ_A.

Proof Let $\theta \in \mathrm{Nat}(よ^A, F)$ be such a natural transformation. Then for any morphism $f : A \to X$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{tikzcd}
1_A \arrow[d, mapsto] & よ^A(A) \arrow[r, "\theta_A"] \arrow[d, "よ^A(f)"'] & F(A) \arrow[d, "F(f)"] & \kappa(\theta) \arrow[d, mapsto] \\
f & よ^A(X) \arrow[r, "\theta_X"] & F(X) & \theta_X(f) = F(f)(\kappa(\theta))
\end{tikzcd}$$

If there are no morphisms from A to X, then the set $よ^A(X)$ is empty and θ_X is uniquely determined. It follows that θ_X is determined by $\kappa(\theta)$, so κ is a bijection as claimed. □

The following can be derived from the Yoneda Lemma 2.2.10 by replacing $\mathcal{C}$ with $\mathcal{C}^{op}$.

co-Yoneda Lemma 2.2.11. *For an object B in a category $\mathcal{C}$, consider the functor $よ_B = \mathcal{C}(-, B) : \mathcal{C}^{op} \to \mathcal{S}et$, the* **co-Yoneda functor***, and let G be another $\mathcal{S}et$-valued functor on $\mathcal{C}^{op}$. Then there is a bijection*

$$\kappa : Nat(よ_B, G) \to G(B)$$

sending a natural transformation θ to the image of $1_B \in よ^B(B)$ under θ_B.

Definition 2.2.12. *The* **Yoneda embedding** *よ is the functor from $\mathcal{C}^{op}$ to the category $[\mathcal{C}, \mathcal{S}et]$ of set valued functors (and natural transformations) on $\mathcal{C}$ given by $A \mapsto よ^A = \mathcal{C}(A, -)$. Dually, one has the* **covariant Yoneda embedding** *of $\mathcal{C}$ into $[\mathcal{C}^{op}, \mathcal{S}et]$ (the category of* **presheaves on** *$\mathcal{C}$) given by $B \mapsto よ_B = \mathcal{C}(-, B)$.*

Both functors can be extended (fattened up) to the product of the domain *category with $\mathcal{S}et$ as in the diagram*

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{S}et \arrow[r, "\text{よ}_B \times \mathcal{S}et"] & \mathcal{S}et & \mathcal{C} \times \mathcal{S}et \arrow[l, "\text{よ}^A \times \mathcal{S}et"'] \\
(B, X) \arrow[r, mapsto] & \text{よ}_B(A) \times X \arrow[d, equal] & \\
& \mathcal{C}(A, B) \times X \arrow[d, equal] & \\
& \text{よ}^A(B) \times X & (A, X). \arrow[l, mapsto]
\end{tikzcd}$$

Here we fix an object A in $\mathcal{C}^{op}$ (B in $\mathcal{C}$) for the functor on the left (right). The one on the left is the **tensored Yoneda functor**.

In [Hir03, 11.5.7] the tensored Yoneda functor is called the **free $\mathcal{C}$-diagram at A, $\mathbf{F}^A_*$**. A **free $\mathcal{C}$-diagram of sets** is a coproduct of diagrams of this form. The tensored Yoneda functor is denoted by $\mathbf{F}^{\mathcal{D}}_{-}$.

In [MMSS01, 1.3] a certain enriched analog of it is called the shift desuspension functor because of the role it plays in the theory of spectra. For us it is the generalized suspension spectrum of Definition 7.1.30 and Definition 7.2.52.

Both embeddings of Definition 2.2.12 can be derived from the functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}et$ given by $(A, B) \mapsto \mathcal{C}(A, B)$. It follows from the Yoneda Lemma 2.2.10 that for each object A in $\mathcal{C}$ and each functor F in $[\mathcal{C}, \mathcal{S}et]$,

$$[\mathcal{C}, \mathcal{S}et](\text{よ}^A, F) = F(A).$$

2.2D Adjoint functors

Definition 2.2.13. *A pair (F, G) of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is an* **adjoint pair** *if there is a natural isomorphism of sets*

$$\varphi_{X,Y} : \mathcal{D}(FX, Y) \xrightarrow{\cong} \mathcal{C}(X, UY)$$

for each object X in $\mathcal{C}$ and Y in $\mathcal{D}$. We say that G is the **right adjoint** *of F, F is the* **left adjoint** *of G, and φ is the* **adjunction isomorphism**. *We abbreviate this situation by*

$$F \dashv G.$$

When in addition $G \dashv F$, we say that F and G are **two sided adjoints**.

We sometimes indicate the data for $F \dashv G$ as a triple (F, G, φ), which we call an **adjunction**. *When the isomorphism φ sends a morphism $f : FX \to Y$ in $\mathcal{D}$ to a morphism $g : X \to GY$ in $\mathcal{C}$, we say that f* **is the left adjoint of** *g and g* **is the right adjoint of** *f. Thus we can speak of* **adjoint morphisms** *as well as adjoint functors.*

Remark 2.2.14. Notation for adjoint functors. *When writing an adjoint pair as a pair of arrows as above, we will almost always write the source of the left adjoint functor F on the left and that of the right adjoint G on* *the right. However the reader is warned that* **not all authors follow this convention**. *Moreover when adjoint functors occur in a complicated diagram, it may be impossible to follow it.*

Fortunately it is possible to rotate Kan's symbol $\dashv$ *(sometimes called the* **turnstile***), and it is common practice to have the* **dash***, that is the line whose endpoint intersects the midpoint of the other line, always pointing toward the left adjoint, even if it is above, below or to the right. For example,* $F \vdash G$ *means that* G *is the left adjoint and* F *is the right one.*

We will often denote the situation of Definition 2.2.13 by

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd} \qquad \textit{or} \qquad F : \begin{tikzcd} \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left] \end{tikzcd} : G.$$

In this case Kan's turnstile coincides with the "perp" symbol commony used to denote perpendicularity, such as the orthogonal complement $V^\perp$ *of a vector space* $V \subseteq W$*. We will also see it in Definition 6.3.12 in connection with localizing subcategories.*

For some extravagant use of the rotating turnstile, see (5.4.30) and (6.2.17) below.

Remark 2.2.15. Existence of adjoints. *A given functor may or may not have a left or right adjoint in general.*

Proposition 2.2.16. Adjunctions for opposite categories. *Suppose we have and adjunction* (F, G, φ) *for categories* $\mathcal{C}$ *and* $\mathcal{D}$ *as in Definition 2.2.13, so* $F \dashv G$*. Let* $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ *and* $G^{op} : \mathcal{D}^{op} \to \mathcal{C}^{op}$ *be the corresponding functors between opposite categories. Then* $G^{op} \dashv F^{op}$ *and conversely.*

Proof If for each object X in $\mathcal{C}$ and Y in $\mathcal{D}$.

$$\mathcal{D}(FX, Y) \cong \mathcal{C}(X, GY),$$

then

$$\mathcal{D}^{op}(Y, F^{op}X) \cong \mathcal{C}^{op}(G^{op}Y, X),$$

so $G^{op} \dashv F^{op}$. $\square$

The following was proved in [ML98, page 85] and [Kan58a, Theorems 3.2 and 3.2*].

Proposition 2.2.17. Uniqueness of adjoint functors. *Any two left or right adjoints of a given functor have a unique natural equivalence (Definition 2.2.1) between them.*

Proposition 2.2.18. Products of adjoints. *Suppose we have adjunctions*

$$\begin{tikzcd} \mathcal{C}_i \arrow[r, shift left, "F_i"] \arrow[r, phantom, "\perp"] & \mathcal{D}_i \arrow[l, shift left, "G_i"] \end{tikzcd} \qquad \textit{for } i = 1, 2.$$

Then

$$\begin{tikzcd} \mathcal{C}_1 \times \mathcal{C}_2 \arrow[r, shift left, "F_1 \times F_2"] \arrow[r, phantom, "\perp"] & \mathcal{D}_1 \times \mathcal{D}_2, \arrow[l, shift left, "G_1 \times G_2"] \end{tikzcd}$$

where the product categories $\mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{D}_1 \times \mathcal{D}_2$ are as in Definition 2.1.5.

Proof This follows easily from the fact that

$$(\mathcal{C}_1 \times \mathcal{C}_2)((X_1, X_2), (Y_1, Y_2)) \cong \mathcal{C}_1(X_1, Y_1) \times \mathcal{C}_2(X_2, Y_2)$$

and similarly for $\mathcal{D}_1 \times \mathcal{D}_2$. □

Proposition 2.2.19. Adjunctions are composable and satisfy the two out of three condition. *Suppose we have functors*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F_1"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G_1"] \arrow[r, shift left, "F_2"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "G_2"] \end{tikzcd}$$

with adjunction isomorphisms φ_1 and φ_2 respectively. Then $F_2F_1 \dashv G_1G_2$, and the corresponding adjunction isomorphism φ_{12} is $\varphi_1\varphi_2$.

For functors F_i and G_i as above, if $F_2F_1 \dashv G_1G_2$, then $F_1 \dashv G_1$ iff $F_2 \dashv G_2$.

Proof Let $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $Z \in \mathcal{E}$. Then

$$\mathcal{E}(F_2F_1X, Z) \cong \mathcal{D}(F_1X, G_2Z) \cong \mathcal{C}(X, G_1G_2Z).$$

If φ_{12} and φ_1 exist, then we can define φ_2 to be $\varphi_1^{-1}\varphi_{12}$. Similarly φ_1 exists if φ_{12} and φ_2 exist. □

Definition 2.2.20. The unit and counit of an adjunction. *Suppose we have a pair of adjoint functors*

$$\begin{tikzcd} F : \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D} : G \arrow[l, shift left] \end{tikzcd}$$

so we have the isomorphism

$$\mathcal{C}(X, GY) \cong \mathcal{D}(FX, Y), \tag{2.2.21}$$

that is natural in both X and Y, which are objects in $\mathcal{C}$ and $\mathcal{D}$ respectively. For $Y = FX$ this reads

$$\mathcal{C}(X, GFX) \cong \mathcal{D}(FX, FX).$$

Hence we get a morphism $\eta_X : X \to GFX$ in $\mathcal{C}$ corresponding to the identity morphism on FX in $\mathcal{D}$. This leads to a natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow GF$ called the **unit of the adjunction**.

Similarly setting $X = GY$ in (2.2.21) leads to a morphism $\epsilon_Y : FGY \to Y$ and a natural transformation $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ called the **counit of the adjunction**.

Note the similarity of the above with Definition 2.2.4, in which the natural transformations η and ϵ are required to be natural equivalences. It is also similar to Definition 2.2.8 in which the symbols are used in connection with a functor equipped with a natural transformation from or to the identity functor.

Note that for each object Y in $\mathcal{D}$, $(\eta G)_Y = \eta_{G(Y)}$ as morphisms in $\mathcal{C}$ from $G(Y)$ to $GFG(Y)$, and for each object X in $\mathcal{C}$, and $(F\eta)_X = F(\eta_X)$ as morphisms in $\mathcal{D}$ from $F(X)$ to $FGF(X)$, as indicated in the following diagram, in which the objects on the left and right are in the categories in the corresponding row of the center column.

$$\begin{tikzcd}
& Y \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
G(Y) \arrow[rr, "(\eta G)_Y = \eta_{G(Y)}"] & & GFG(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D} \arrow[d, "G"] \\
\mathcal{C} \arrow[d, bend right, "1_{\mathcal{C}}"'] \arrow[d, bend left, "GF"] \arrow[d, phantom, "\eta \Rightarrow"] \\
\mathcal{C} \arrow[d, "F"] \\
\mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
F(X) \arrow[rr, "(F\eta)_X = F(\eta_X)"] & & FGF(X)
\end{tikzcd}$$

Dually, $(\epsilon F)_X = \epsilon_{F(X)}$ and $(G\epsilon)_Y = G(\epsilon_Y)$ as in the following.

$$\begin{tikzcd}
& Y \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
G(Y) & & GFG(Y) \arrow[ll, "(G\epsilon)_Y = G(\epsilon_Y)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \arrow[d, bend right, "1_{\mathcal{D}}"'] \arrow[d, bend left, "FG"] \arrow[d, phantom, "\Leftarrow \epsilon"] \\
\mathcal{D} \arrow[d, "G"] \\
\mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
F(X) & & FGF(X) \arrow[ll, "(\epsilon F)_X = \epsilon_{F(X)}"']
\end{tikzcd}$$

The functor $T = GF : \mathcal{C} \to \mathcal{C}$ is an example of a **monad** on $\mathcal{C}$; see Definition 2.2.40 below. Dually, $FG : \mathcal{D} \to \mathcal{D}$ is a **comonad** on $\mathcal{D}$.

Theorem 2.2.22. Characterization of adjoint functors in terms of unit and counit. *An adjunction between a pair of funtors*

$$\begin{tikzcd}
F : \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D} : G \arrow[l, shift left]
\end{tikzcd}$$

is determined by natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow GF$ *and* $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ *for which the composities*

$$G \xRightarrow{\eta G} GFG \xRightarrow{G\epsilon} G \qquad \textit{and} \qquad F \xRightarrow{F\eta} FGF \xRightarrow{\epsilon F} F$$

are each the identity.

The following was proved as [ML98, Theorem IV.3.1] and stated as [Rie17, Lemma 4.5.13].

Theorem 2.2.23. Relation between the right (left) adjoint and the counit (unit). *Let*

$$\begin{tikzcd} F : \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D} : G \arrow[l, shift left] \end{tikzcd}$$

be an adjunction as in Theorem 2.2.22. Then

(i) The right adjoint G is faithful as in Definition 2.1.12 iff every component of the counit $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ is epi as in Definition 2.1.10.
(ii) The right adjoint G is full iff each component of ϵ is split monic.

Hence G is fully faithful iff each component of ϵ is an isomorphism. Dually,

(i′) The left adjoint F is faithful iff every component of the unit $\eta : 1_{\mathcal{C}} \Rightarrow GF$ is monic.
(ii′) The left adjoint F is full iff each component of η is split epi.

Hence F is fully faithful iff each component of η is an isomorphism.

Here is another way to approach the maps of Definition 2.2.6.

Definition 2.2.24. Composition and precomposition as counits of adjunctions. *Fix a set Y and consider the isomorphism in $\mathcal{S}et$,*

$$\mathcal{S}et(X \times Y, Z) \cong \mathcal{S}et(X, \mathcal{S}et(Y, Z)),$$

which says that the functor $(- \times Y)$ is left adjoint to $\mathcal{S}et(Y, -)$. One sees easily that this isomorphism is natural in all three variables. Its counit as in Definition 2.2.20 gives a family of maps

$$\epsilon_Z : \mathcal{S}et(Y, Z) \times Y \to Z.$$

sending (f, y) (for $f : Y \to Z$ and $y \in Y$) to $f(y) \in Z$.
For a fixed set Z we have

$$\mathcal{S}et(X, \mathcal{S}et(Y, Z)) \cong \mathcal{S}et(Y, \mathcal{S}et(X, Z)) = \mathcal{S}et^{op}(\mathcal{S}et(X, Z), Y).$$

Definition 2.2.25. The change of group adjunction for G-sets. *Let $H \subseteq G$ be a subgroup and let $i_H^G : \mathcal{S}et^G \to \mathcal{S}et^H$ be the forgetful or* **restriction functor**. *Let*

$$G \underset{H}{\times} T \qquad \textit{for an } H\textit{-set } T$$

be the orbit set of $G \times T$ under the diagonal action of H with H acting on G by right multiplication. It is a G-set via left multiplication on G.

The left adjoint of i_H^G is given by

$$T \mapsto G \underset{H}{\times} T, \qquad \textit{the } \textbf{induction functor}.$$

We will refer to this as the **change of group adjunction**, *and similar adjunctions will appear several times in this book; see Remark 8.6.19 below. In particular when H is the trivial group, it sends an ordinary set T to the free G-set $G \times T$. For a G-set S and an H-set T, the counit and unit of the adjunction (see Definition 2.2.20) give natural maps*

$$\mu_H^G = \epsilon_S : G \underset{H}{\times} i_H^G S \to S \qquad \text{and} \qquad \psi_H^G = \eta_T : T \to i_H^G \left(G \underset{H}{\times} T \right) \tag{2.2.26}$$

in $\mathcal{S}et^G$ and $\mathcal{S}et^H$ respectively, given by

$$\mu_H^G(\gamma, s) = \gamma(s) \qquad \text{and} \qquad \psi_H^G(t) = (e, t)$$

for $\gamma \in G$, $s \in S$ and $t \in T$. We will call these the **relative action** *and* **relative coaction** *maps respectively.*

When S is induced up from an H-set R, the **extended action** *map*

$$\hat{\mu}_H^G : G \underset{H}{\times} i_H^G \left(G \underset{H}{\times} R \right) = \left(G \underset{H}{\times} G \right) \underset{H}{\times} R \to G \underset{H}{\times} R \tag{2.2.27}$$

is given by

$$(\gamma_1 h_1, \gamma_2 h_2, r) = (\gamma_1, h_1\gamma_2, h_2(r)) \mapsto (\gamma_1 h_1 \gamma_2, h_2(r)) = (\gamma_1 h_1 \gamma_2 h_2, r)$$

for $\gamma_1, \gamma_2 \in G$, $h_1, h_2 \in H$ and $r \in R$.

When T is the restriction of a G-set U, the coaction map

$$\psi_H^G : i_H^G U \to i_H^G \left(G \underset{H}{\times} i_H^G U \right)$$

is the image under i_H^G of

$$\widetilde{\psi}_H^G : U \to G \underset{H}{\times} i_H^G U, \tag{2.2.28}$$

the **lifted coaction** *given as before by $u \mapsto (e, u)$,* **which is a map of G-sets**. *The functor i_H^G sends $\widetilde{\psi}_H^G$ to ψ_H^G.*

When H is the trivial group, we will call them simply the **action** *and* **coaction** *maps. We will sometimes omit the indices when they are clear from the context.*

Definition 2.2.29. The second change of group adjunction. *For S and T as in Definition 2.2.25, the right adjoint of i_H^G is given by $T \mapsto \mathcal{S}et^H(G, T)$, the* **coinduction functor**. *The action of G on the target is by procomposition with multiplication in G. The target is underlain by the Cartesian product $T^{|G/H|}$. In particular when H is the trivial group, the right adjoint sends an* *ordinary set T to the Cartesian power T^G, the set of T-valued functions on G. The unit and counit maps are*

$$\mu^* : S \to \mathcal{S}et^H(G, i_H^G S) \qquad \text{and} \qquad \psi^* : i_H^G \mathcal{S}et^H(G,T) \to T.$$

given by $\mu^*(s)(\gamma) = \mu(\gamma, s)$ for $\gamma \in G$ and $s \in S$, and $\psi^*(f) = f(e)$ for $f : G \to T$.

Example 2.2.30. Some other adjunctions.

(i) *The* **free forgetful adjunction.** *Let $U : \mathcal{A}b \to \mathcal{S}et$ be the forgetful functor from the category of abelian groups to the category of sets. Its left adjoint F is the functor that assigns to a set S the free abelian group on S.*

$$\begin{tikzcd} F : \mathcal{S}et \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{A}b : U \arrow[l, shift left] \end{tikzcd}$$

The unit of the adjunction η induces the canonical map from a set S to the set underlying the free abelian group generated by S. The counit ϵ induces the canonical homomorphism to an abelian group A from the free abelian group generated by its underlying set.

Similar adjunctions can be defined with $\mathcal{S}et$ replaced by a more general category $\mathcal{C}$ and $\mathcal{A}b$ replaced by the category of objects in $\mathcal{C}$ with a specified additional structure. Examples include the Eilenberg-Moore adjunction of Theorem 2.2.47, the adjunctions for associative and commutative algberas of Lemma 2.6.66 and the change of group adjunction of Proposition 8.3.19.

(ii) **The cylinder path space adjunction.** *Let $F : \mathcal{T}op \to \mathcal{T}op$ be the functor that assigns to a space X the Cartesian product $I \times X$, a* **cylinder**, *where I denotes the unit interval $[0,1]$. Its right adjoint G is the* **path space functor** *$X \mapsto X^I$, where $X^I = \mathcal{T}op(I, X)$. The adjunction is the identity*

$$\mathcal{T}op(I \times X, Y) \cong \mathcal{T}op(X, Y^I) = \mathcal{T}op(X, \mathcal{T}op(I, Y)).$$

This example is in [Kan58a]. For a pointed analog, see Example 5.6.12 below. We can generalize it by replacing I by any compactly generated weak Housdorff space A. The adjunction isomorphism is then

$$\mathcal{T}op(A \times X, Y) \cong \mathcal{T}op(X, \mathcal{T}op(A, Y)).$$

The counit $\epsilon : GF \Rightarrow 1_{\mathcal{T}op}$ assigns to a space X the evaluation map

$$\mathrm{Ev} : A \times \mathcal{T}op(A, X) \to X$$

defined by $\mathrm{Ev}(a, p) := p(a)$ for $a \in A$ and $p : A \to X$. The unit $\eta : 1_{\mathcal{T}op} \Rightarrow GF$ assigns to X the map $X \to \mathcal{T}op(A, A \times X)$ sending $x \in X$ to the map $p : A \to A \times X$ defined by $p(a) = (a, x)$. For a pointed analog, see Example 5.6.12 below.

(iii) **The orbit and fixed point adjunctions.** *Let G be a group and let $\mathcal{S}et^G$ denote the category of G-sets and equivariant maps. Let $\Delta : \mathcal{S}et \to \mathcal{S}et^G$ be the functor assigning to each set T the same set with trivial G-action. Then its left and right adjoints are the functors $T \mapsto T_G$ and $T \mapsto T^G$ sending a G-set T to its orbit and fixed point sets respectively. We will sometimes denote the orbit set by T/G.*

(iv) **The Yoneda adjunction.** *Recall that for categories $\mathcal{C}$ and $\mathcal{D}$, $[\mathcal{C}, \mathcal{D}]$ denotes the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations, $\mathcal{C} \times \mathcal{D}$ denotes the category whose objects and morphisms are the evident ordered pairs. For a fixed category $\mathcal{C}$ we define functors F and G from the category of categories to itself by $F(-) = \mathcal{C}^{op} \times (-)$ and $G(-) = [\mathcal{C}^{op}, -]$. Then $F \dashv G$, meaning that for all categories $\mathcal{A}$ and $\mathcal{B}$, there is a natural isomorphism*

$$[\mathcal{C}^{op} \times \mathcal{A}, \mathcal{B}] \cong [\mathcal{A}, [\mathcal{C}^{op}, \mathcal{B}]].$$

In particular for $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} = \mathcal{S}et$, we have

$$[\mathcal{C}^{op} \times \mathcal{C}, \mathcal{S}et] \cong [\mathcal{C}, [\mathcal{C}^{op}, \mathcal{S}et]].$$

An object on the left is the $\mathcal{S}et$-valued functor $\mathcal{C}(-,-)$. The corresponding object on the right is the Yoneda embedding $よ$ of Definition 2.2.12, that is the functor sending an obect C of $\mathcal{C}$ to the contravaraint set valued functor $\mathcal{C}(-, C)$.

If we replace $\mathcal{C}$ by its opposite, and let $\mathcal{B} = \mathcal{S}et$, we have a natural isomorphism

$$[\mathcal{C}^{op} \times \mathcal{C}, \mathcal{S}et] \cong [\mathcal{C}^{op}, [\mathcal{C}, \mathcal{S}et]],$$

sending $\mathcal{C}(-,-)$ to the other form of the Yoneda embedding.

(v) **The object set adjunction.** *Let $(-)^{disc} : \mathcal{S}et \to \mathcal{C}at$ (see Definition 2.1.7 and Definition 2.1.14) be the functor that converts a set into the corresponding discrete category. It is left adjoint to the functor $Ob : \mathcal{C}at \to \mathcal{S}et$ that sends a small category to its object set.*

(vi) **The arrow adjunction.** *The functor $(-)^{disc} \times \mathbf{2} : \mathcal{S}et \to \mathcal{C}at$ (see Example 5.4.16) is the left adjoint of the functor $Arr : \mathcal{C}at \to \mathcal{S}et$ that sends a small category to its morphism set.*

The next example requires a proof.

Proposition 2.2.31. Adjunctions related to groupoids.

(i) *The functors*

$$j : \mathcal{B}_T H \to \mathcal{B}_{G \times_H T} G \qquad \text{and} \qquad k : \mathcal{B}_{G \times_H T} G \to \mathcal{B}_T H$$

of Proposition 2.1.38 are two sided adjoints.

(ii) The functors

$$j^* : \mathcal{C}^{\mathcal{B}_{G\times_H T}G} \to \mathcal{C}^{\mathcal{B}_T H} \qquad \text{and} \qquad k^* : \mathcal{C}^{\mathcal{B}_T H} \to \mathcal{C}^{\mathcal{B}_{G\times_H T}G}$$

of Corollary 2.1.40 are two sided adjoints.

Proof (i) Both categories are self-dual, so $j \dashv k$ iff $k \dashv j$. Consider first the case where T has a single orbit H/K for a subgroup $K \subseteq H$. Then the categories are

$$\mathcal{B}_{H/K}H \qquad \text{and} \qquad \mathcal{B}_{G\times_H H/K}G = \mathcal{B}_{G/K}G.$$

Let $\alpha = \eta K \in \mathcal{B}_{H/K}H$ for $\eta \in H$ and $\beta = \gamma K \in \mathcal{B}_{G/K}$ for $\gamma \in G$. Then

$$\mathcal{B}_{G/K}G(j\alpha, \beta) = \mathcal{B}_{G/K}G(\eta K, \gamma K) = \eta^{-1}\gamma K \gamma^{-1}\eta \cong K$$

as sets. The functor k depends on a choice of representatives in $G \times H/K$ of each element in G/K. Let the representative of γK be $(\gamma_1, \eta_1 K)$, so $k(\gamma K) = \eta_1 K$.

$$\mathcal{B}_{H/K}H(\alpha, k\beta) = \mathcal{B}_{H/K}H(\eta K, k(\gamma K)) = \mathcal{B}_{H/K}H(\eta K, \eta_1 K) = \eta^{-1}\eta_1 K \eta_1^{-1}\eta \cong K$$

and the two sets are naturally isomorphic.

For the general case, both sets are empty unless α and β lie in subcategories corresponding to the same orbit in T, in which case the isomorphism follows from the single orbit case.

(ii) The decomposition of T as a union of single orbits leads to product decompositions of the two functor categories which are respected by the functors j^* and k^*. This means it suffices to treat the case $T = H/K$, for which our functor categories are

$$\mathcal{C}^{\mathcal{B}_{H/K}H} \qquad \text{and} \qquad \mathcal{C}^{\mathcal{B}_{G/K}G}.$$

Objects in these categories were described in Example 2.1.41. An object X in $\mathcal{C}^{\mathcal{B}_{H/K}H}$ is a collection of objects $X_{\eta K}$ in $\mathcal{C}$ indexed by the cosets in H/K, each having an action of K and isomorphisms with other such objects induced by elements of H.

Similarly an object Y in $\mathcal{C}^{\mathcal{B}_{G/K}G}$ is a collection of objects $Y_{\gamma K}$ in $\mathcal{C}$ indexed by the cosets in G/K, each having an action of K and isomorphisms with other such objects induced by elements of G. Its image under j^* is obtained by ignoring all components not having subscripts contained in H.

There are adjoint functors

$$j_0 : \mathcal{B}K \to \mathcal{B}_{H/K}H \qquad \text{and} \qquad k_0 : \mathcal{B}_{H/K}H \to \mathcal{B}K,$$

where k_0 depends on a choice of a representative in H of each coset of K. This leads to a diagram like the one of (2.1.42), namely

$$\begin{tikzcd} \mathcal{C}^{\mathcal{B}K} \arrow[r, shift left, "k_0^*"] & \mathcal{C}^{\mathcal{B}_{H/K}H} \arrow[l, shift left, "j_0^*"] \arrow[r, shift left, "k^*"] & \mathcal{C}^{\mathcal{B}_{G/K}G}. \arrow[l, shift left, "j^*"] \end{tikzcd}$$

We will show that j_0^* is the two sided adjoint of k_0^* and $j_0^* j^*$ is the two sided adjoint of $k^* k_0^*$. This will imoply that j^* is the two sided adjoint of k^* by Proposition 2.2.19.

An object W in $\mathcal{C}^{\mathcal{B}K}$ is a single object in $\mathcal{C}$, which we also denote by W, equipped with an action of the group K. The image of Y under the composite functor $j_0^* j^*$ is the object Y_{eK}. Hence

$$\mathcal{C}^{\mathcal{B}K}(W, j_0^* j^* Y) \cong \mathcal{C}(W, Y_{eK})^G,$$

the set of K-equivariant morphisms from W to Y_{eK}. The choices made in defining k_0 and k lead to a choice of representtaive in G for each coset of K. It follows that

$$\mathcal{C}^{\mathcal{B}_{G/K}G}(k^* k_0^* W, Y)$$

has the same description. Hence $k^* k_0^* \dashv j_0^* j^*$.

Similar computations show that

$$j_0^* j^* \dashv k^* k_0^*, \qquad k_0^* \dashv j_0^* \qquad \text{and} \qquad j_0^* \dashv k_0^*.$$

The result follows. □

Definition 2.2.32. *The* **endomorphism category** End_A *of an object A in $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ with a single object A. The set $End_A(A, A) = \mathcal{C}(A, A)$ is a monoid under composition. Its right action on $\mathcal{C}(A, B)$ by precomposition is denoted by*

$$\mu_R : \mathcal{C}(A, B) \times \mathcal{C}(A, A) \to \mathcal{C}(A, B).$$

The left action of $End_B(B, B) = \mathcal{C}(B, B)$ by postcomposition is denoted by

$$\mu_L : \mathcal{C}(B, B) \times \mathcal{C}(A, B) \to \mathcal{C}(A, B).$$

We denote the inclusion functor $End_A \to \mathcal{C}$ by i_A. It induces a **restriction functor**

$$i_A^* : [\mathcal{C}, \mathcal{E}] \to [End_A, \mathcal{E}].$$

Similarly the **automorphism category** Aut_A *of an object A in $\mathcal{C}$ is the (less than full) subcategory of $\mathcal{C}$ with a single object A in which $Aut_A(A, A) \subseteq \mathcal{C}(A, A)$ is the set of* **invertible** *endomorphisms of A. This set is a group under composition, which we abbreviate by $Aut(A)$.*

Definition 2.2.33. Coevaluation. *For a cocomplete category $\mathcal{E}$ and an object A in $\mathcal{C}$, the* **coevaluation functor**

$$F^A : \mathcal{E} \to [\mathcal{C}, \mathcal{E}],$$

sends an object E to the functor $\mathcal{C}(A, -) \times E$ from $\mathcal{C}$ to $\mathcal{E}$. This functor sends an object C in $\mathcal{C}$ to the product of E with the set $\mathcal{C}(A, C)$, which is defined since $\mathcal{E}$ is cocomplete.

Definition 2.2.34. Corestriction. *For a cocomplete category $\mathcal{E}$, the* **corestriction functor**

$$G^A : [End_A, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}],$$

where End_A is as in Definition 2.2.32, is given by

$$X \mapsto よ^A(-) \underset{\mathcal{C}(A,A)}{\times} X = \mathcal{C}(A,-) \underset{\mathcal{C}(A,A)}{\times} X,$$

where $よ^A$ is as in Definition 2.2.12. The functor $X : End_A \to \mathcal{E}$ is the same thing as an object X_A in $\mathcal{E}$ equipped with a left action of the endomorphism monoid of A, meaning a map

$$\mu_L : \mathcal{C}(A,A) \times X \to X$$

with suitable properties. We also have a right action map

$$\mu_R : よ^A(-) \times \mathcal{C}(A,A) \to よ^A(-)$$

defined in terms of precomposition. The functor $よ^A(-) \underset{\mathcal{C}(A,A)}{\times} X$ is the coequalizer (see Definition 2.3.27 below) of

$$\begin{tikzcd} よ^A(-) \times \mathcal{C}(A,A) \times X \arrow[d, shift right, "\mu_R \times X"'] \arrow[d, shift left, "よ^A(-) \times \mu_L"] \\ よ^A(-) \times X \arrow[d, dashed] \\ よ^A(-) \underset{\mathcal{C}(A,A)}{\times} X. \end{tikzcd}$$

Remark 2.2.35. The terms coevaluation and corestriction. *We use the term* **coevaluation** *because F^A as in Definition 2.2.33 is the left adjoint of the evaluation functor*

$$\mathrm{Ev}_A : [\mathcal{C}, \mathcal{E}] \to \mathcal{E} \tag{2.2.36}$$

(for a cocomplete category $\mathcal{E}$) given by $F \mapsto F(A)$, while the **corestriction** *functor G^A of Definition 2.2.34 is the left adjoint of the restriction functor*

$$i_A^* : [\mathcal{C}, \mathcal{E}] \to [End_A, \mathcal{E}].$$

We will refer to the adjunctions $F^A \dashv \mathrm{Ev}_A$, $G^A \dashv i_A^*$ and others like them as **Yoneda adjunctions***. To our limited knowledge, this terminology is new.*

Definition 2.2.37. The global evaluation functor. *For each object A in* *a category C we have the functor* $\mathrm{Ev}_A : [\mathcal{C}, \mathcal{E}] \to \mathcal{E}$ *of (2.2.36). These can be assembled into a functor*

$$\mathrm{Ev} : [\mathcal{C}, \mathcal{E}] \times \mathcal{C} \to \mathcal{E}$$

given by $(F, A) \mapsto F(A)$.

We will use the following in §9.6.

Lemma 2.2.38. *Suppose that* $U : \mathcal{D} \to \mathcal{C}$ *is a functor with a left adjoint* L *and right adjoint* R, *and that* $\tau : L \Rightarrow R$ *is a natural transformation. We denote the units and counits of the two adjunctions by* $\eta_1 : 1_{\mathcal{D}} \Rightarrow RU$, $\epsilon_1 : UR \Rightarrow 1_{\mathcal{C}}$, $\eta_2 : 1_{\mathcal{C}} \Rightarrow UL$ *and* $\epsilon_2 : LU \Rightarrow 1_{\mathcal{D}}$.

If the composition

$$1_{\mathcal{C}} \xRightarrow{\eta_2} UL \xRightarrow{U\tau} UR \xRightarrow{\epsilon_1} 1_{\mathcal{C}} \tag{2.2.39}$$

is the identity, then $\tau : L \Rightarrow R$ *is a retract of* $\tau UR : LUR \Rightarrow RUR$.

Proof We apply $L \Rightarrow R$ on the left to the composition (2.2.39) to get

$$\begin{tikzcd}
L \arrow[r, Rightarrow, "L\eta_2"] \arrow[d, Rightarrow, "\tau"'] & LUL \arrow[r, Rightarrow, "LU\tau"] \arrow[d, Rightarrow, "\tau UL"'] & LUR \arrow[r, Rightarrow, "L\epsilon_1"] \arrow[d, Rightarrow, "\tau UR"'] & L \arrow[d, Rightarrow, "\tau"'] \\
R \arrow[r, Rightarrow, "R\eta_2"'] & RUL \arrow[r, Rightarrow, "RU\tau"'] & RUR \arrow[r, Rightarrow, "R\epsilon_1"'] & R,
\end{tikzcd}$$

which displays the desired retraction since the composite of both rows is the identity. □

2.2E Monads

For more discussion on the following, see [ML98, Chapter VI].

Definition 2.2.40. *A* **monad** *(also known as a* **triple***) on a category* $\mathcal{C}$ *is a functor* $T : \mathcal{C} \to \mathcal{C}$ *equipped with natural transformations* $\eta : 1_{\mathcal{C}} \Rightarrow T$ *(making it a coaugmented functor as in Definition 2.2.8) and* $\mu : T^2 \Rightarrow T$ *such that*

- $\mu \cdot T\mu = \mu \cdot \mu T$ *as natural transformations* $T^3 \Rightarrow T$ *and*
- $\mu \cdot T\eta = \mu \cdot \eta T = 1_T$ *as natural transformations* $T \Rightarrow T$.

Equivalently the following diagrams commute.

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[d, Rightarrow, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T.
\end{tikzcd}$$

When μ *is a natural equivalence, we say that the monad* T *is* **idempotent**.

A **comonad** *(or* **cotriple***) on $\mathcal{C}$ is a monad on $\mathcal{C}^{op}$, namely a functor $U : \mathcal{C} \to \mathcal{C}$ with natural transformations $\epsilon : U \Rightarrow 1_{\mathcal{C}}$ (making it an augmented functor) and $\nu : U \Rightarrow U^2$ with diagrams dual to the ones above. When ν is a natural equivalence, we say that the comonad U is* **idempotent**.

Example 2.2.41. Adjunctions as monads. *As noted above (see Definition 2.2.20), given adjoint functors*

$$F : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : G,$$

we get a monad (T, η, μ) on $\mathcal{C}$ defined by $T = GF$ with η being the unit of the adjunction and $\mu = \epsilon F$, where ϵ is the counit of the adjunction.

Definition 2.2.42. *Given a monad (T, η, μ) on a category $\mathcal{C}$, a* **T-algebra** *(X, h) consists of an object X in $\mathcal{C}$ and a* **structure map** *$h : T(X) \to X$ such that the following diagrams commute.*

$$\begin{tikzcd} T(T(X)) \arrow[r, "T(h)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "h"] \\ T(X) \arrow[r, "h"] & X \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & T(X) \arrow[d, "h"] \\ & X \end{tikzcd}$$

These correspond to the usual associativity and unit laws in the examples below. We denote the category of T-algebras by T-alg or $\mathcal{C}^T$.

Remark 2.2.43. The notation $\mathcal{C}^T$. *Given a category $\mathcal{C}$ and a small category J, we will often denote the category of functors $J \to \mathcal{C}$ (the category of J-shaped diagrams in $\mathcal{C}$) by $\mathcal{C}^J$. It should be clear from the context whether the exponent is a small category J or an endofunctor T.*

Example 2.2.44. Groups. *Let $\mathcal{C} = \mathcal{S}et$ and let T be the functor that assigns to a set X the set underlying free group generated by X. Define η by letting η_X be the usual embedding of X into the free group generated by it, and define μ by letting μ_X be the map underlying the evident group homomorphism $T(T(X)) \to T(X)$. Then (T, η, μ) is a monad on $\mathcal{S}et$ and a T-algebra on a set X is a group structure on it.*

Example 2.2.45. Group actions on sets. *Let $\mathcal{C} = \mathcal{S}et$ and let G be a group with identity element e. Define the monad (T, η, μ) by $T(X) = G \times X$ with η and μ given by*

$$x \mapsto (e, x) \qquad \text{and} \qquad (g_1, (g_2, x)) \mapsto (g_1 g_2, x)$$

for $x \in X$ and $g_1, g_2 \in G$. Then a T-algebra on X is a G-action.

Example 2.2.46. R-modules. *Let $\mathcal{C} = \mathcal{A}b$ and let R be a ring with unit. Define a monad (T, η, μ) by $T(A) = R \otimes A$, $\eta(a) = (1, a)$ and $\mu(r_1(r_2, a)) = (r_1 r_2, a)$ for A an abelian group, $a \in A$ and $r_1, r_2 \in R$. Then a T-algebra on A is an R-module structure.*

The following is due to [EM65]. It is illustrated by each of the three examples above, and is itself an example of a free forgetful adjunction as in Example 2.2.30(i).

Theorem 2.2.47. The Eilenberg-Moore adjunction. *Let (T, η, μ) be a monad on a category $\mathcal{C}$ as in Definition 2.2.40, and let $\mathcal{C}^T$ denote the category of T-algebras. Then the forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{C}^T$ that assigns to each object X the free T-algebra generated by it. The monad associated with this adjunction (see Example 2.2.41) is T itself.*

If

$$F' : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : G$$

is another adjunction whose monad is (T, η, μ), then there is a unique functor $K : \mathcal{D} \to \mathcal{C}^T$ with $F = KF'$ and $UK = G$.

Definition 2.2.48. *A subcategory $\mathcal{D}$ of $\mathcal{C}$ is* **replete** *if any object in $\mathcal{C}$ isomorphic to an object in $\mathcal{D}$ is also in $\mathcal{D}$. The* **repletion** *of an arbitrary subcategory $\mathcal{D}$ of $\mathcal{C}$ is the smallest replete subcategory containing it. Its objects are all objects in $\mathcal{C}$ that are isomorphic to objects in $\mathcal{D}$, and its morphisms are all composites of morphisms in $\mathcal{D}$ with isomorphisms in $\mathcal{C}$.*

The following terminology is taken from [ML98, IV.3, page 91].

Definition 2.2.49. Reflective and coreflective subcategories. *A subcategory $\mathcal{A} \subseteq \mathcal{B}$ is* **reflective** *(***coreflective***) if the inclusion functor $K : \mathcal{A} \to \mathcal{B}$ has a left adjoint $F : \mathcal{B} \to \mathcal{A}$ (right adjoint $F : \mathcal{B} \to \mathcal{A}$), which is called a* **reflector** *(***coreflector***). The corresponding adjunction is called a* **reflection** *(***coreflection***) of $\mathcal{B}$ in its subcategory $\mathcal{A}$.*

The term "reflective" here is not to be confused with "reflexive," to be introduced in §2.3F below.

For Mac Lane [ML71] the inclusion functor was faithful but not necessarily full, but more recent authors, such as Kelly in [Kel82, page 25] and Riehl in [Rie17, Definition 4.5.12], assume that the subcategory is full, making each component of the unit (counit) an isomorphism. In [Rie17, Example 4.5.14] she gives an interesting list of examples of reflexive full subcategories, including that of abelian groups in the category of all groups. In that case the left adjoint of the inclusion functor is the abelianization functor.

Definition 2.2.50. Bireflective subcategories. *A reflective (coreflective) full subcategory $\mathcal{A} \subseteq \mathcal{B}$ as in Definition 2.2.49 is* **bireflective** *if the left (right) adjoint F of the inclusion functor is also a right (left) adjoint. In this case there are two inclusion functors, K_R and K_L, the right and left adjoints of F.*

The following is an exercise for the reader.

Proposition 2.2.51. Products of bireflective subcategories. *Suppose $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A}' \subseteq \mathcal{B}'$ are bireflective subcategories as in Definition 2.2.50. Then $\mathcal{A} \times \mathcal{A}'$ is a bireflective subcategory of $\mathcal{B} \times \mathcal{B}'$.*

The following is proved by Mac Lane and Ieke Moerdijk as [MLM94, Lemma 7.4.1]. We learned it from Emily Riehl.

Proposition 2.2.52. Fully faithful functors. *In the situation of Definition 2.2.50, with $K_L \dashv F \dashv K_R$, the inclusion functor K_L is fully faithful iff K_R is. The unit $\eta^L : 1_{\mathcal{D}} \Rightarrow \alpha^* \alpha_!$ of the adjunction $K_L \dashv F$ is an isomorphism iff the counit $\epsilon^R : \alpha^* \alpha_* \Rightarrow 1_{\mathcal{D}}$ of $F \dashv K_R$ is.*

Remark 2.2.53. Related terms. *The pair of adjunctions $K_L \dashv F \dashv K_R$ is called an* **adjoint cylinder** *by William Lawvere in [Law94]. It is also known as a* **fully faithful adjoint triple**.

Category theorists (for example [EBV02]) have also considered the situation in which the inclusion functor $K : \mathcal{D} \to \mathcal{C}$ of a full subcategory has both left and right adjoints, so we have $L \dashv K \dashv R$ for functors $R, L : \mathcal{C} \to \mathcal{D}$. Thus $\mathcal{D}$ is simultaneously reflective and coreflective as a subcategory $\mathcal{C}$.

Example 2.2.54. Left and right Kan extensions. *Let $\mathcal{C}$ be a bicomplete category and let $\alpha : K \to J$ be a fully faithful functor of small categories usch as the inclusion of a full subcategory. We will see in §2.5 below that the precomposition functor $\alpha^* : \mathcal{C}^J \to \mathcal{C}^K$ has both left and right adjoints $\alpha_!$ and α_* called* **Kan extensions**. *Thus we have a diagram*

$$\begin{tikzcd} \mathcal{C}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{C}^J \arrow[l, shift left, "\alpha^*"] \arrow[r, shift left, "\alpha^*"] \arrow[r, phantom, "\perp"] & \mathcal{C}^K \arrow[l, shift left, "\alpha_*"] \end{tikzcd}$$

making $\mathcal{C}^K$ a bireflective subcategory of $\mathcal{C}^J$. Are both composite endofunctors of $\mathcal{C}^K$ the identity functor? See Proposition 2.5.15.

The following is a consequence of Theorem 2.2.23.

Proposition 2.2.55. The (co)monad of a (co)reflective subcategory. *If $\mathcal{A} \subseteq \mathcal{B}$ is a reflective subcategory as in Definition 2.2.49, then every component of the counit*

$$\epsilon : FK \Rightarrow 1_{\mathcal{A}}$$

is epi as in Definition 2.1.10. If in addition $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, then every component of the counit is an isomorphism, and $T = KF$ is an idempotent monad as in Definition 2.2.40. When the subcategory $\mathcal{A}$ is replete as in Definition 2.2.48, we can choose the left adjoint F so that $FK = 1_{\mathcal{A}}$, and hence $T^2 = T$.

In the coreflective case, every component of the unit $\eta : 1_{\mathcal{A}} \Rightarrow GK$ is monic. When $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, each component of η is an isomorphism and $U = KG$ is an idempotent comonad.

2.3 Limits and colimits as adjoint functors

2.3A Pushouts and pullbacks

The **pushout** (if it exists) of a diagram

$$\begin{tikzcd} & B \\ A \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & C \end{tikzcd} \tag{2.3.1}$$

in a category $\mathcal{C}$ is an object D receiving morphisms from B and D making the diagram

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ A \arrow[ur, "b"] \arrow[dr, "c"'] & & D \\ & C \arrow[ur, "g"'] & \end{tikzcd}$$

commute and having the following universal property. Given another such object D', there is a unique morphism $h : D \to D'$ making the following diagram commute.

$$\begin{tikzcd} & B \arrow[dr, "f"'] \arrow[drr, "f'"] & & \\ A \arrow[ur, "b"] \arrow[dr, "c"'] & & D \arrow[r, dashed, "h"] & D' \\ & C \arrow[ur, "g"] \arrow[urr, "g'"'] & & \end{tikzcd} \tag{2.3.2}$$

For example if $\mathcal{C} = \mathcal{S}et$ and the morphisms b and c are one-to-one, the pushout is the union $B \cup_A C$. A pushout is also called a **cobase change**. When a property of the map b implies the same for g, we say that such maps are **closed under cobase change**. The data consisting of the object D' in (2.3.2) and the morphisms to it is called a **cone under** the diagram (2.3.1), with D being called the **universal cone under** (2.3.1). The notion of a cone will be formalized in Definition 2.3.21 below.

By reversing all the arrows above, we get the dual notion of a **pullback** of the diagram

$$
\begin{tikzcd}
B \arrow[dr, "b"] & \\
& A \\
C, \arrow[ur, "c"'] &
\end{tikzcd}
\tag{2.3.3}
$$

which is an object D having the universal property indicated by the diagram

$$
\begin{tikzcd}
& & B \arrow[dr, "b"] & \\
D' \arrow[urr, "f'"] \arrow[drr, "g'"'] \arrow[r, dashed, "h"] & D \arrow[ur, "f"] \arrow[dr, "g"] & & A \\
& & C. \arrow[ur, "c"'] &
\end{tikzcd}
\tag{2.3.4}
$$

For example the pullback in $\mathcal{S}et$ when A has one element is the Cartesian product $B \times C$. A pullback is also called a **base change**. When a property of the map b implies the same for g, we say that such maps are **closed under base change**. The data consisting of the object D' in (2.3.4) and the morphisms from it is called a **cone over** the diagram (2.3.3), with D being called the **universal cone over** (2.3.3).

Most of our pushout and pullback diagrams will have arrows that are horizontal and vertical, rather than the diagonal arrows shown above. We will sometimes write a pullback or pushout diagram as

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & D.
\end{tikzcd}
$$

When necessary we will rotate the symbol as required. Some authors reverse the meanings of these two symbols, placing our pushout symbol in the upper left corner of a pullback diagram and *vice versa*.

Proposition 2.3.5. The morphism set in the arrow category as a pullback. *For morphisms $f : A \to B$ and $g : X \to Y$ in a category $\mathcal{C}$, and notation as in Definition 2.1.51(v), the following is a pullback diagram.*

$$
\begin{tikzcd}
\Diamond(f,g) \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C}(A,X) \arrow[d, "g_*"] \\
\mathcal{C}(B,Y) \arrow[r, "f^*"] & \mathcal{C}(A,Y),
\end{tikzcd}
$$

where a and b correspond to the maps of the same names in (2.1.54) associated with each element of the set $\Diamond(f,g)$.

We will prove a homotopy analog of the following in Proposition 5.8.48 below.

Proposition 2.3.6. Composition of pullbacks and of pushouts. *Suppose we have commutative diagrams*

$$\begin{tikzcd} A_0 \arrow[r, "a_0"] \arrow[d, "f_0"'] \arrow[dr, phantom, "(\ulcorner)", pos=0.2] & A_1 \arrow[d, "f_1"] \\ B_0 \arrow[r, "b_0"'] \arrow[ur, phantom, "\lrcorner", pos=0.8] & B_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "(\ulcorner)", pos=0.2] & A_2 \arrow[d, "f_2"] \\ B_1 \arrow[r, "b_1"'] \arrow[ur, phantom, "\lrcorner", pos=0.8] & B_2 \end{tikzcd} \tag{2.3.7}$$

in a cocomplete (complete) category $\mathcal{C}$. If both of them are pushouts (pullbacks), so is the composite diagram

$$\begin{tikzcd} A_0 \arrow[rr, "a_1 a_0"] \arrow[d, "f_0"'] \arrow[drr, phantom, "(\ulcorner)", pos=0.1] & & A_2 \arrow[d, "f_2"] \\ B_0 \arrow[rr, "b_1 b_0"'] \arrow[urr, phantom, "\lrcorner", pos=0.9] & & B_2. \end{tikzcd} \tag{2.3.8}$$

Conversely, if (2.3.8) and the first (second) square of (2.3.7) are pushouts (pullbacks), then B_2 (A_0) is the pushout (pullback) of the second (first) square of (2.3.7).

Proof We will prove the statements about pullbacks, leaving the dual statements about pushouts to the reader. For the first one consider a commutative diagram

$$\begin{tikzcd} X \arrow[drrrr, bend left] \arrow[dr, dotted] \arrow[drrr, dotted] \arrow[ddr] & & & & \\ & A_0 \arrow[rr, "a_0"] \arrow[d, "f_0"] \arrow[drr, phantom, "\ulcorner", pos=0.1] & & A_1 \arrow[r, "a_1"] \arrow[d, "f_1"] \arrow[dr, phantom, "\ulcorner", pos=0.1] & A_2 \arrow[d, "f_1"] \\ & B_0 \arrow[rr, "b_0"] & & B_1 \arrow[r, "b_1"] & B_2. \end{tikzcd}$$

Since X maps compatibly (over B_2) to B_1 and A_2, those two maps factor uniquely through the second pullback A_1. Now we have maps from X to B_0 and A_1 that are compatible over B_1, so they factor uniquely through the first pullback A_0. This means that A_0 is the pullback of (2.3.8) as claimed.

The second statement is proved by a diagram chase that can be found in [Hir03, Proposition 7.2.14]. $\square$

Definition 2.3.9. Corner maps. *If the pushout D of the diagram (2.3.1)* *exists and we have a commutative diagram of the form*

$$\begin{tikzcd} & B \arrow[dr, "f'"] & \\ A \arrow[ur, "b"] \arrow[dr, "c"'] & & D', \\ & C \arrow[ur, "g'"'] & \end{tikzcd}$$

the resulting map $h : D \to D'$ is called the **pushout corner map** *or simply* **corner map** *of the diagram above. The* **pullback corner map** *from A to the pullback of f' and g' (if it exists) is similarly defined.*

We will make similar definitions below in Definition 2.3.57, Definition 2.6.12 and Definition 2.9.29.

2.3B Liftings

Definition 2.3.10. Right and left lifting properties. *Suppose we have a commutative diagram in a category $\mathcal{C}$ of the form*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"] \arrow[ur, dashed, "h"] & Y. \end{tikzcd} \tag{2.3.11}$$

This is a **lifting diagram**. *A morphism h satisfying $hi = a$ and $ph = b$ (which may or may not exist in general) is a* **lifting** *for the diagram. If it exists for any a and b making the diagram commute, we say that i* **has the left lifting property with respect to** *p, p* **has the right lifting property with respect to** *i, and (i, p) is a* **lifting pair**. *We will sometimes denote this state of affairs by $i \boxslash p$.*

Given a class of morphisms $\mathcal{X}$ in a category $\mathcal{C}$, let

$$\mathcal{X}\text{-}inj = \mathcal{X}^{\boxslash} = \{p \in Arr\,\mathcal{C} : x \boxslash p \ \forall x \in \mathcal{X}\},$$

the class of morphisms having the right lifting property with respect to each morphism in $\mathcal{X}$, also called the **$\mathcal{X}$-injectives**, *and*

$$\mathcal{X}\text{-}proj = {}^{\boxslash}\mathcal{X} = \{i \in Arr\,\mathcal{C} : i \boxslash x \ \forall x \in \mathcal{X}\},$$

the class of morphisms having the left lifting property with respect to each morphism in $\mathcal{X}$, also called the **$\mathcal{X}$-projectives**. *Let*

$$cofib(\mathcal{X}) = {}^{\boxslash}(\mathcal{X}^{\boxslash}) \qquad and \qquad fib(\mathcal{X}) = ({}^{\boxslash}\mathcal{X})^{\boxslash}, \tag{2.3.12}$$

the **$\mathcal{X}$-cofibrations** *and the* **$\mathcal{X}$-fibrations**. *For two morphism classes $\mathcal{X}$ and $\mathcal{Y}$, we will write $\mathcal{X} \boxslash \mathcal{Y}$ when $\mathcal{X} = {}^{\boxslash}\mathcal{Y}$ and $\mathcal{X}^{\boxslash} = \mathcal{Y}$.*

We learned this use of the symbol $\boxslash$ from [MP12, Definition 14.1.5]. May and Ponto presumably chose it for its resemblance to the diagram of (2.3.11). We will see many such diagrams in this book.

In the context of model categories (see Chapter 4 below, specifically Definition 4.1.10 and Example 4.1.11), the class $\mathcal{X}^{\boxslash}$ is called the class of $\mathcal{X}$-injectives and denoted by $\mathcal{X}$-inj by [DHK97, 7.2], [SS00, §2], [HSS00, Definition 3.2.7], [Hov99, Definition 2.1.7], and [Hir03, Definition 10.5.2]. The classes of (2.3.12) have to do with cofibrations and fibrations.

The two lifting properties mentioned above are equivalent. The following is proved by Riehl as [Rie14, Lemma 11.1.5].

Proposition 2.3.13. Liftings and adjunctions. *Suppose we have an adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

and morphism classes $\mathcal{L}$ in $\mathcal{C}$ and $\mathcal{R}$ in $\mathcal{D}$. Then

$$F\mathcal{L} \boxslash \mathcal{R} \qquad \text{if and only if} \qquad \mathcal{L} \boxslash G\mathcal{R}.$$

Definition 2.3.14. The lifting test map. *Let $i : A \to B$ and $p : X \to Y$ be morphisms in a category $\mathcal{C}$. This leads to a diagram of sets*

$$\begin{tikzcd} \mathcal{C}(B,X) \arrow[r, "p_*"] \arrow[d, "i^*"'] & \mathcal{C}(B,Y) \arrow[d, "i^*"] \\ \mathcal{C}(A,X) \arrow[r, "p_*"] & \mathcal{C}(A,Y) \end{tikzcd} \tag{2.3.15}$$

We denote by $\mathcal{C}_{\diamond}(i,p)$ the resulting from $\mathcal{C}(B,X)$ to the pullback set of Proposition 2.3.5,

$$\diamond(i,p) = \mathcal{C}(B,Y) \underset{\mathcal{C}(A,Y)}{\times} \mathcal{C}(A,X).$$

Remark 2.3.16. Notation for the lifting test map. *In the context of simplicial model categories, Quillen denoted this map by (i^*, p_*) in [Qui67, Definition II.2.2]. In the context of model categories, this map is denoted by $\mathcal{C}(i^*, p_*)$ in [MMSS01, (5.11)] and by $\mathcal{C}_{\square}(i,p)$ in [HSS00, Definition 3.3.6]. We are using the symbol above because in Definition 2.6.12 below we will use $\square$ and $\diamond$ to denote pushout and pullback corner maps respectively.*

Proposition 2.3.17. Special cases of the lifting test map. *With notation as in Definition 2.3.14,*

(i) If $\mathcal{C}$ has an initial object $\varnothing$ and $A = \varnothing$, then

$$\mathcal{C}_{\diamond}(i,p) = p_* : \mathcal{C}(B,X) \to \mathcal{C}(B,Y).$$

(ii) If $\mathcal{C}$ has a terminal object $*$ and $Y = *$, then

$$\mathcal{C}_{\diamondsuit}(i,p) = i^* : \mathcal{C}(B,X) \to \mathcal{C}(A,X).$$

Proof In the first case, the two bottom sets in (2.3.15) are singletons, so the pullback is $\mathcal{C}(B,Y)$. In the second case, the two right sets in (2.3.15) are singletons, so the pullback is $\mathcal{C}(A,X)$. □

We call $\mathcal{C}_{\diamondsuit}(i,p)$ the lifting test map because of the following.

Proposition 2.3.18. The surjectivity of $\mathcal{C}_{\diamondsuit}(i,p)$ and the existence of liftings. *In the commutative diagram (2.3.11), $i \boxslash p$, that is there exists a map h (a* **lifting***) with $hi = a$ and $ph = b$ for any a and b iff the lifting test map $\mathcal{C}_{\diamondsuit}(i,p)$ of Definition 2.3.14 is onto, or equivalently iff it has a section, that is a map $s : \diamondsuit(i,p) \to \mathcal{C}(B,X)$ with $\mathcal{C}_{\diamondsuit}(i,p)s = 1_{\diamondsuit(i,p)}$.*

The following definition is essentially due to Bousfield, [Bou77, Definition 2.1]. See also [JT07, Definition 7.1] and [MP12, Definition 14.1.11].

Definition 2.3.19. *A* **weak factorization system** *in a category $\mathcal{C}$ is a pair of morphism classes $(\mathcal{L}, \mathcal{R})$ such that*

(i) Any morphism in $\mathcal{C}$ can be factored as a morphism in $\mathcal{L}$ followed by one in $\mathcal{R}$.
(ii) $\mathcal{L} \boxslash \mathcal{R}$ as in Definition 2.3.10, that is all maps in $\mathcal{L}$ have the right lifting property with respect to all maps in $\mathcal{R}$ and vice versa.

We say that $\mathcal{L}$ is the **left class** *and $\mathcal{R}$ is the* **right class**.

The term "weak" is used above because the factorization is not required to be unique or functorial.

Proposition 2.3.20. Properties of left and right classes. *The left and right classes in any weak factorization system are closed under composition and include all isomorphisms.*

2.3C Limits and colimits

Pushouts and pullbacks are examples of colimits and limits respectively. Both can be reinterpreted and generalized as follows.

The diagram (2.3.1) is the same thing as a functor $K \to \mathcal{C}$, where the indexing category K ($\bullet \leftarrow \bullet \to \bullet$) has three objects and a single nonidentity morphism from the first object to each of the other two. The category $\mathcal{C}^K$ of such functors is the category of diagrams in $\mathcal{C}$ that look like (2.3.1).

There is a diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^K$ that assigns to each object X in $\mathcal{C}$ the constant X-valued diagram. If pushouts exist in $\mathcal{C}$ (they do in $\mathcal{S}et$ and in $\mathcal{T}op$, the category of topological spaces), they are defined by a functor

$\mathrm{colim}_K : \mathcal{C}^K \to \mathcal{C}$ **which is the left adjoint of** Δ. The functor Δ also has a less interesting right adjoint that assigns the object A to the diagram (2.3.1).

Similarly the diagram (2.3.3) is the same thing as a functor $K^{op} \to \mathcal{C}$. Again we have the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^{K^{op}}$. If pullbacks exist in $\mathcal{C}$ (as they do in $\mathcal{S}et$ and in $\mathcal{T}op$), they are defined by a functor $\lim_{K^{op}} : \mathcal{C}^{K^{op}} \to \mathcal{C}$ **which is the right adjoint of** Δ. In this case there is a less interesting left adjoint whose value on (2.3.3) is A.

That was the reinterpretation; now for the generalization. We can replace K or K^{op} by an arbitrary small category J. Then $\mathcal{C}^J$ is the category of J-shaped diagrams in $\mathcal{C}$. We still have the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^J$ that sends each object X to the constant X-valued diagram. We can ask for its right and left adjoints $\lim_J$ and colim_J.

The following can be found in [Rie17, Definition 3.1.2] and is originally due to [ML98, page 67].

Definition 2.3.21. Cones. *Let $X : J \to \mathcal{C}$ be a functor from a small category J, and denote its value on an object j or morphism $f : j \to j'$ in J by X_j or $X_f : X_j \to X_{j'}$. A* **cone over (under)** *X* **with summit or apex (nadir)** *C is a natural transformation λ to (from) X from (to) the constant C-valued functor on J. More explicitly in the "over" case, it is a collection of morphisms $\lambda_j : C \to X_j$ with $X_f \lambda_j = \lambda_{j'}$ for all morphisms f in J. The morphisms λ_j are the* **legs** *of the cone.*

Definition 2.3.22. *Let $X : J \to \mathcal{C}$ be as in Definition 2.3.21. Its* **colimit** *$\underset{J}{\mathrm{colim}}\, X$, if it exists, is a cone W under X that admits a unique natural transformation to any other cone under X. In other words it is an object W in $\mathcal{C}$ with a morphism $w_j : X_j \to W$ for each j such that*

(i) for each morphism $f : j \to j'$, $w_j = w_{j'} X_f$ and
(ii) given any other object Y in $\mathcal{C}$ with morphisms $y_j : X_j \to Y$ satisfying $y_j = y_{j'} X_f$ in all cases, there is a unique morphism $p : W \to Y$ with $y_j = p w_j$ for all j.

These are shown in the following diagram for each morphism $f : j \to j'$ in J.

$$
\begin{tikzcd}
X_j \arrow[dd, "X_f"'] \arrow[dr, "w_j"] \arrow[drr, bend left, "y_j"] & & \\
& \underset{J}{\mathrm{colim}}\, X \arrow[r, dashed, "\exists! p"] & Y. \\
X_{j'} \arrow[ur, "w_{j'}"] \arrow[urr, bend right, "y_{j'}"'] & &
\end{tikzcd}
\tag{2.3.23}
$$

Its **limit** *$\underset{J}{\lim}\, X$, if it exists, is a cone L over X that admits a unique natural* *transformation from any other cone over X. In other words it is an object L in $\mathcal{C}$ with morphisms $\ell_j : L \to X_j$ for all j such that*

(i) for each morphism $f : j \to j'$, $\ell_{j'} = X_f \ell_j$ and
(ii) given any other object K in $\mathcal{C}$ with morphisms $k_j : K \to X_j$ satisfying $k_{j'} = X_f k_j$ in all cases, there is a unique morphism $q : K \to L$ with $k_j = \ell_j q$ for all j.

These are shown in the following diagram for each morphism $f : j \to j'$ in J.

$$\begin{tikzcd} & & X_j \arrow[dd, "X_f"] \\ K \arrow[r, dashed, "\exists! q"] \arrow[urr, bend left, "k_j"] \arrow[drr, bend right, "k_{j'}"'] & \lim_J X \arrow[ur, "\ell_j"] \arrow[dr, "\ell_{j'}"'] & \\ & & X_{j'}. \end{tikzcd}$$

We will sometimes drop the subscript J when it is clear from the context. The following is an immediate consequence of the definitions.

Proposition 2.3.24. Limits and colimits as adjoint functors. *For a small category J and an arbitrary category $\mathcal{C}$, each object of the functor category $\mathcal{C}^J$. i.e., each J-shaped diagram in $\mathcal{C}$, has a colimit (limit) iff the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C}^J$ has a left (right) adjoint, which we denote by colim_J ($\lim_J$).*

Definition 2.3.25. *A category $\mathcal{C}$ is* **complete** *(***cocomplete***) if all diagrams in $\mathcal{C}$, i.e., all functors to $\mathcal{C}$ from small categories, have limits (colimits). $\mathcal{C}$ is* **bicomplete** *if both conditions hold.*

The following is well known and could be an exercise for the reader.

Theorem 2.3.26. Bicompleteness of familiar categories. *The categories $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{T}$, $\mathcal{A}b$, $\mathcal{C}at$ (the category of small categories) and $\mathcal{G}rp$ (the category of groups) are bicomplete.*

Definition 2.3.27. Equalizers and coequalizers. *Let Eq be the equalizer category of Definition 2.1.6. Hence on object in $\mathcal{C}^{Eq}$ is a pair of morphisms having the same source and target. Its limit (colimit) is called its* **equalizer** *(***coequalizer***).*

See Definition 2.3.60 for a related concept.
The following was proved by Mac Lane in [ML98, Theorem V.2.2].

Theorem 2.3.28. Every limit (colimit) is an equalizer (coequalizer). *Let J be a small category, let $\mathcal{C}$ be a complete one, and let $X : J \to \mathcal{C}$ be a* *functor, that is a J-shaped diagram in $\mathcal{C}$. Then there are morphisms f and g in $\mathcal{C}$ such that the limit of X is the equalizer (as in Definition 2.3.27) of*

$$\prod_{j\in Ob\, J} X_j \underset{g}{\overset{f}{\rightrightarrows}} \prod_{(u:j\to k)\in Arr\, J} X_k. \tag{2.3.29}$$

Since f and g are morphisms to a product indexed over the set of morphisms u in J, they are determined by their composites with the projections p_u, namely $p_u f = p_k$ and $p_u g = X_u p_j$, where X_u denotes the image of the morphism u under the functor X.

Colimits can be described dually as coequalizers. For a functor $X : J \to \mathcal{C}$ from a small category J to a cocomplete category $\mathcal{C}$, we have maps

$$\coprod_{(u:j\to k)\in Arr\, J} X_j \underset{g'}{\overset{f'}{\rightrightarrows}} \coprod_{j\in Ob\, J} X_j \tag{2.3.30}$$

Since f' and g' are morphisms from a coproduct indexed over the set of morphisms u in J, they are determined by their composites with the inclusions i_u, namely $f' i_u = i_j$ and $g' i_u = X_u i_j$.

The following can be used to simplify certain coequalizers and equalizers.

Proposition 2.3.31. A cancellation rule for equalizers and coequalizers. *Suppose we have a commutative diagram in a cocomplete category*

$$\begin{tikzcd}
A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, "i_A"'] & C \arrow[r, dashed, "g"] \arrow[d, "i_C"] & Z \arrow[d] \\
A \coprod B \arrow[r, shift left, "i_C f_1 \coprod k_1"] \arrow[r, shift right, "i_C f_2 \coprod k_2"'] & C \coprod D \arrow[r, dashed, "h"] \arrow[d, "g \coprod D"] & E \arrow[d, dashed, "\cong"] \\
B \arrow[u, "i_B"] \arrow[r, shift left, "f_1'"] \arrow[r, shift right, "f_2'"'] & Z \coprod D \arrow[r, dashed, "h'"] & E'
\end{tikzcd} \tag{2.3.32}$$

in which i_A, i_B and i_C are the evident inclusions, k_1 and k_2 are morphisms from B to $C \coprod D$, and each object in the third column is the coequalizer of the two maps in the same row on the left. The maps f_1' and f_2' are the indicated composites.

Then there is an isomorphism $E \to E'$ that makes the lower right square commute. In other words, we can use the bottom row to find the coequalizer of the middle row.

Dually, suppose we have a commutative diagram in a complete category

$$
\begin{tikzcd}
A & C \arrow[l, shift left, "f_1"'] \arrow[l, shift right, "f_2"] & Z \arrow[l, dashed, "g"'] \\
A \times B \arrow[u, "p_A"] \arrow[d, "p_B"'] & C \times D \arrow[u, "p_C"'] \arrow[l, shift left, "f_1 p_C \times k_1"'] \arrow[l, shift right, "f_2 p_C \times k_2"] & E \arrow[l, dashed, "h"'] \arrow[u] \\
B & Z \times D \arrow[u, "g \times D"'] \arrow[l, shift left, "f_1'"'] \arrow[l, shift right, "f_2'"] & E' \arrow[l, dashed, "h'"'] \arrow[u, dashed, "\cong"']
\end{tikzcd}
\tag{2.3.33}
$$

where the maps p_A, p_B and p_C are coordinate projections, k_1 and k_2 and maps from D to $A \times B$, and each object in the third coloum is the equalizer of the two maps in the same row on the left. The maps f_1' and f_2' are the indicated composites.

Then there is an isomorphism $E' \to E$ that makes the lower right square commute. In other words, we can use the bottom row to find the equalizer of the middle row.

Proof We prove the statement about coequalizers only. Consider the larger diagram in which the third object in each row is a coequalizer.

$$
\begin{tikzcd}
A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, equal] & C \arrow[r, dashed, "g"] \arrow[d, "i_C"] & Z \arrow[d, "i_Z"] \\
A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[d, "i_A"'] & C \coprod D \arrow[r, dashed, "g \coprod D"] \arrow[d, equal] & Z \coprod D \arrow[d] \\
A \coprod B \arrow[r, shift left, "i_C f_1 \coprod k_1"] \arrow[r, shift right, "i_C f_2 \coprod k_2"'] \arrow[d, equal] & C \coprod D \arrow[r, dashed, "h"] \arrow[d, "g \coprod D"] & E \arrow[d, dashed, "\cong"] \\
A \coprod B \arrow[r, shift left, "f_1''"] \arrow[r, shift right, "f_2''"'] & Z \coprod D \arrow[r, dashed, "h''"'] \arrow[ur, dashed, "h'''"'] \arrow[d, equal] & E'' \\
B \arrow[u, "i_B"] \arrow[r, shift left, "f_1'"] \arrow[r, shift right, "f_2'"'] & Z \coprod D \arrow[r, dashed, "h'"'] & E' \arrow[u, "z'"']
\end{tikzcd}
$$

Then we have

$$f_1'' i_A = g f_1 i_A = g f_2 i_A = f_2'' i_A.$$

This means the summand A has no effect on the value of E'', so z' is an isomorphism. The composite $(z')^{-1} z : E \to E'$ is the map we are claiming is an isomorphism.

Since $hf_1 = hf_2$, $hf_1 i_A = hf_2 i_A$, so h factors through $Z \coprod D$ as indicated. This means that h''' factors through E'' by the universal property of the coequalizer. It follows that z is an isomorphism. $\square$

The following is proved by Ando, the second author, and Strickland in [AHS90, Proposition 11.11].

Proposition 2.3.34. The pullback as an equalizer. *Given a pullback diagram as in (2.3.3) in a complete category,*

$$\begin{tikzcd} B \arrow[dr, "b"] & \\ & A \\ C, \arrow[ur, "c"'] & \end{tikzcd}$$

we have two maps

$$B \times C \overset{bp_1}{\underset{cp_2}{\rightrightarrows}} A,$$

where $p_1 : B \times C \to B$ and $p_2 : B \times C \to C$ are projections onto the two factors. Their equalizer is the pullback, which we denote by

$$B \times_A C.$$

Example 2.3.35. More limits and colimits.

(i) If C is an object in a cocomplete category $\mathcal{C}$ and A is a set, we can define an object $A \times C$ in $\mathcal{C}$ to be the colimit of the constant C-valued functor on the discrete category of A as in Definition 2.1.7. Equivalently it is the coproduct of copies of C indexed by A. Similarly for $\mathcal{C}$ complete we can define C^A, the product of copies of C indexed by the set A, to be the limit of the same functor. See Definition 3.1.31 and Example 3.1.49 below.

(ii) Let J be the empty category. Then $\mathcal{C}^J$ has one object, the empty diagram. Its limit and colimit, if they exist, are the **terminal** *and* **initial objects** *respectively of $\mathcal{C}$. In the cases of $\mathcal{S}et$ and $\mathcal{T}op$ these are the empty set and a point. For this reason we denote them by $\varnothing$ and $*$ in general.*

(iii) Let G be a group and let $J = \mathcal{B}G$ be the associated one object category having an invertible morphism for each element of G as in Definition 2.1.31. Let $\mathcal{C}$ be $\mathcal{S}et$ or $\mathcal{T}op$. Then an element in $\mathcal{C}^J$ is a G-action on a set or space X. Its limit and colimit are the fixed point and orbit sets or spaces X^G and X_G (or X/G). Compare with Example 2.2.30(iii). **It follows that passage to fixed points (orbit spaces) commutes with other limits (colimits), and more generally with other right (left) adjoints (Proposition 2.3.36).**

(iv) In particular for $J = \mathcal{B}G$ for a group G and $\mathcal{C} = \mathcal{S}et$, the diagram (2.3.29) for a G-set X reads

$$X \overset{\Delta}{\underset{\psi}{\rightrightarrows}} X^{|G|}$$

where Δ is the diagonal embedding, and for $\gamma \in G$ the γth coordinate of $\psi(x)$ is $\gamma(x)$. The equalizer is the subset of X on which the two maps agree, namely the fixed point set X^G. The dual diagram is

$$G \times X \underset{\mu}{\overset{\nabla}{\rightrightarrows}} X$$

where $\nabla(\gamma, x) = x$ and $\mu(\gamma, x) = \gamma(x)$. The coequalizer is the quotient of X obtained by identifying x wth $\gamma(x)$ in all cases, namely the orbit set X_G.

(v) If J has an initial (terminal) object, then the limit (colimit) of a functor $J \to \mathcal{C}$ is its value on that object. This generalizes the uninteresting cases above.

(vi) Suppose J has an initial (terminal) object j_0 and that the only nonidentity morphisms in J are from (to) j_0. Suppose further that the functor $J \to \mathcal{C}$ sends j_0 to the initial (terminal) object of $\mathcal{C}$. Then its colimit (limit) is the coproduct (product) in $\mathcal{C}$ of the images of the other objects in J. When $\mathcal{C}$ is $\mathcal{S}et$ or $\mathcal{T}op$, these are the disjoint union and Cartesian product of the objects in question.

(vii) Let X be a retract of Y as in Definition 2.1.56. Then X is both the equalizer and the coequalizer of

$$Y \underset{1_Y}{\overset{e=ir}{\rightrightarrows}} Y.$$

The following was proved by Kan in [Kan58a] as Theorems 12.1, 12.4 and 12.4*.

Proposition 2.3.36. Left (right) adjoints preserve colimits (limits). *Let J be a small category and suppose we have a pair of adjoint functors*

$$F : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : G.$$

Then we have an adjunction of functor categories

$$F_* : \mathcal{C}^J \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D}^J : G_*.$$

If $\mathcal{C}$ and $\mathcal{D}$ are cocomplete, F preserves colimits, meaning that for a functor X in $\mathcal{C}^J$, the map $\operatorname{colim}_J F_* X \to F \operatorname{colim}_J X$ in $\mathcal{D}$ is a natural isomorphism.

If $\mathcal{C}$ and $\mathcal{D}$ are complete, G preserves limits.

Proof The adjunction between F_* and G_* can be verified objectwise.

The next two statements are dual to each other, so we only treat the colimit case. Consider the diagram of adjunctions

$$\begin{tikzcd}
\mathcal{C}^J \arrow[rr, shift left=2, "F_*"] \arrow[rr, phantom, "\perp"] \arrow[d, shift left=2, "\mathrm{colim}"] \arrow[d, phantom, "\vdash"] & & \mathcal{D}^J \arrow[ll, shift left=2, "G_*"] \arrow[d, shift left=2, "\mathrm{colim}"] \arrow[d, phantom, "\vdash"] \\
\mathcal{C} \arrow[u, shift left=2, "\Delta"] \arrow[rr, shift left=2, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{D} \arrow[u, shift left=2, "\Delta"] \arrow[ll, shift left=2, "G"]
\end{tikzcd}$$

For objects X in $\mathcal{C}^J$ and Y in $\mathcal{D}$ we have

$$\mathcal{C}^J(X, G_*\Delta Y) \cong \mathcal{D}^J(F_*X, \Delta Y) \cong \mathcal{D}(\operatorname{colim} F_*X, Y)$$

so $\operatorname{colim} F_* \dashv G_*\Delta$, and similarly $F\operatorname{colim} \dashv \Delta G$. It is obvious that $G_*\Delta = \Delta G$, so the map $\operatorname{colim} F_* \to F\operatorname{colim}$ is as claimed by Proposition 2.2.17. $\square$

Example 2.3.37. Left (right) adjoints need not preserves limits (colimits). *Consider the free forgetful adjunction*

$$F : \mathcal{S}et \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{A}b : U$$

of Example 2.2.30(i). The image under the free abelian group functor F of the Cartesian product of two sets S_1 and S_2, which is a kind of limit, is not the product (direct product) of the free abelian groups $F(S_1)$ and $F(S_2)$. Similarly, the image under the forgetful functor U of the coproduct (direct sum) of two abelian groups A_1 and B_2, which is a kind of colimit, is not the coproduct (disjoint union) of the underlying sets $U(A_1)$ and $U(A_2)$.

Proposition 2.3.38. Pullbacks in the category of small categories. *Let A, B, C and D be small categories and let*

$$\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, "P"'] & B \arrow[d, "Q"] \\
C \arrow[r, "G"] & D
\end{tikzcd} \tag{2.3.39}$$

be a commutative diagram of categories and functors. It is a pullback diagram iff the two diagrams in $\mathcal{S}et$

$$\begin{tikzcd}
Ob\,A \arrow[r, "F"] \arrow[d, "P"'] & Ob\,B \arrow[d, "Q"] \\
Ob\,C \arrow[r, "G"] & Ob\,D
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
Arr\,A \arrow[r, "F"] \arrow[d, "P"'] & Arr\,B \arrow[d, "Q"] \\
Arr\,C \arrow[r, "G"] & Arr\,D
\end{tikzcd} \tag{2.3.40}$$

are also pullbacks.

Proof We know that the functors Ob and Arr are right adjoints by Example 2.2.30(v) and Example 2.2.30(vi), so they preserve limits and hence pullbacks by Proposition 2.3.36. This means that if (2.3.39) is a pullback diagram, so are the diagrams of (2.3.40).

For the converse we use the description of the pullback of Proposition 2.3.34, which in this case reads

$$A \cong C \times_D B.$$

The product category $C \times B$ is as in Definition 2.1.5, so we have

$$\mathrm{Ob}\,(C \times B) = \mathrm{Ob}\,C \times \mathrm{Ob}\,B \quad \text{and} \quad \mathrm{Arr}\,(C \times B) = \mathrm{Arr}\,C \times \mathrm{Arr}\,B,$$

which implies that

$$\mathrm{Ob}\,(C \times_D B) = \mathrm{Ob}\,C \times_{\mathrm{Ob}\,D} \mathrm{Ob}\,B \quad \text{and} \quad \mathrm{Arr}\,(C \times_D B) = \mathrm{Arr}\,C \times_{\mathrm{Arr}\,D} \mathrm{Arr}\,B.$$

This means that if the diagrams of (2.3.40) are pullbacks, then A has the structure required of a pullback. □

Proposition 2.3.41. Colimits (limits) commute with each other. *Let J and J' be small categories, let the category $\mathcal{C}$ by cocomplete (complete), and let $F : J \times J' \to \mathcal{C}$ by a functor. Then we have functors*

$$\begin{array}{ccc} J' \xrightarrow{F_J} \mathcal{C}^J & \qquad & J \xrightarrow{F_{J'}} \mathcal{C}^{J'} \\ j' \longmapsto F(-, j') & \qquad & j \longmapsto F(j, -). \end{array}$$

In the cocomplete case there are isomorphisms

$$\operatorname*{colim}_{J \times J'} F \cong \operatorname*{colim}_{J} (\operatorname*{colim}_{J'} F_J) \cong \operatorname*{colim}_{J'} (\operatorname*{colim}_{J} F_{J'}).$$

Equivalently the following diagram of categories and functors commutes.

$$\begin{tikzcd}
\mathcal{C}^{J \times J'} \arrow[rrr, "\operatorname*{colim}_J"] \arrow[ddd, "\operatorname*{colim}_{J'}"'] & & & \mathcal{C}^{J'} \arrow[ddd, "\operatorname*{colim}_{J'}"] \\
& F \arrow[r, mapsto] \arrow[d, mapsto] & \operatorname*{colim}_J F_{J'} \arrow[d, mapsto] & \\
& \operatorname*{colim}_{J'} F_J \arrow[r, mapsto] & \operatorname*{colim}_{J \times J'} F & \\
\mathcal{C}^J \arrow[rrr, "\operatorname*{colim}_J"'] & & & \mathcal{C}
\end{tikzcd}$$

There are similar statements about limits in the complete case.

Proof By Proposition 2.3.24, each functor in the diagram is the left adjoint of a suitable diagonal functor, and left adjoints preserve colimits by Proposition 2.3.36. The proof of the dual statement is similar. □

Example 2.3.42. The failure of limits to commute with colimits. *It is not true in general that limits commute with colimits. For a bicomplete category $\mathcal{C}$ with functors as in Proposition 2.3.41, there is a map*

$$\underset{J}{\operatorname{colim}}\, \underset{J'}{\lim}\, F_J \to \underset{J'}{\lim}\, \underset{J}{\operatorname{colim}}\, F_{J'}. \tag{2.3.43}$$

Let $\mathcal{C} = \mathcal{A}b$, the category of ableian groups. Let J be the sequential colimit category N of Definition 2.3.63 below, and let $J' = J^{op}$. Let $F : J \times J' \to \mathcal{A}b$ be the functor that sends each object to $\mathbf{Z}_{(p)}$ and each generating morphism to multiplication by a fixed prime p. The resulting diagram is

$$\begin{tikzcd} \vdots \arrow[d, "p"'] & \vdots \arrow[d, "p"'] & \\ \mathbf{Z}_{(p)} \arrow[r, "p"] \arrow[d, "p"'] & \mathbf{Z}_{(p)} \arrow[r, "p"] \arrow[d, "p"'] & \cdots \\ \mathbf{Z}_{(p)} \arrow[r, "p"] & \mathbf{Z}_{(p)} \arrow[r, "p"] & \cdots \end{tikzcd}$$

The limit of each column is trivial while the colimit of each row is $\mathbf{Q}$. This means that the domain of the map of (2.3.43) is trivialbut the codomain is not.

2.3D Categories internal to another category

Recall that a category J is small as in Definition 2.1.1 if its collection of objects $J_0 = \operatorname{Ob} J$ is a set. It follows that its morphism collection $J_1 = \operatorname{Arr} J$ is also a set. The structure of J is determined by maps

- $s, t : J_1 \to J_0$ sending a morphism to its source and target,
- $e : J_0 \to J_1$ sending an object to the corresponding identity morphism, and
- $c : J_1 \underset{J_0}{\times} J_1 \to \operatorname{Arr} J$ sending a suitable pair of morphisms to their composite. Here $J_1 \underset{J_0}{\times} J_1$ is the pullback in the diagram

$$\begin{tikzcd} J_1 \underset{J_0}{\times} J_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner" very near start] & J_1 \arrow[d, "t"] \\ J_1 \arrow[r, "s"] & J_0, \end{tikzcd} \tag{2.3.44}$$

namely the set of morphism pairs

$$\{(g, f) \in J_1 \times J_1 : \operatorname{Dom}(g) = \operatorname{Cod}(f)\}$$

for which the composite gf is defined. We are using the calculus convention in which gf denotes the composite

$$\operatorname{Dom} f \xrightarrow{\quad f \quad} \operatorname{Cod} f = \operatorname{Dom} g \xrightarrow{\quad g \quad} \operatorname{Cod} g.$$

The small category J is a groupoid if there is also a map $i : J_1 \to J_1$ sending a morphism to its inverse. The maps s, t, e, and c (and i in the case of a groupoid) need to satisfy certain conditions whose formulation we leave to the reader.

Now J_0 and J_1 are objects in $\mathcal{S}et$, and the maps s, t, e and c are morphisms in $\mathcal{S}et$. **We could make a similar definition in which $\mathcal{S}et$ is replaced by an arbitrary category $\mathcal{C}$, provided it has enough pullbacks to make sense of (2.3.44).**

Pullbacks were discussed in §2.3A. Recall that the set $J_1 \underset{J_0}{\times} J_1$ of (2.3.44) has a the following universal property. For any set X equipped with maps to $s', t' : X \to J_1$ with $ss' = tt'$, there is a unique map $h : X \to J_1 \underset{J_0}{\times} J_1$, defined by $h(x) = (s'(x), t'(x))$, such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[dr, "h"] \arrow[drr, bend left, "t'"] \arrow[ddr, bend right, "s'"'] & & \\
& J_1 \underset{J_0}{\times} J_1 \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\ulcorner", very near start] & J_1 \arrow[d, "t"] \\
& J_1 \arrow[r, "s"] & J_0 .
\end{tikzcd} \tag{2.3.45}$$

In a general category $\mathcal{C}$ with objects J_0 and J_1 and morphisms $s, t : J_1 \to J_0$, there may or may not be an object having the properties of $J_1 \underset{J_0}{\times} J_1$ above.

We found the following in [Lin13, Appendix A].

Definition 2.3.46. Categories internal to $\mathcal{C}$. *Let $\mathcal{C}$ be a category with objects J_0 and J_1 and morphisms $s, t : J_1 \to J_0$, such that there is an object $J_1 \underset{J_0}{\times} J_1$ with the universal property of (2.3.45).*

A **category** J **internal to a category** $\mathcal{C}$ *consists of the objects J_0 and J_1 in $\mathcal{C}$, its object and morphism objects, with morphisms s, t, e and c as above such that the following diagrams commute in $\mathcal{C}$.*

- *Source and target of identity maps:*

$$\begin{tikzcd}
J_0 \arrow[r, "e"] \arrow[dr, "J_0"'] & J_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
& J_0
\end{tikzcd}$$

- *Source and target of composites:*

$$\begin{tikzcd} J_1 \arrow[d, "s"'] & J_1 \underset{J_0}{\times} J_1 \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "c"] & J_1 \arrow[d, "t"] \\ J_0 & J_1 \arrow[l, "s"'] \arrow[r, "t"] & J_0 \end{tikzcd}$$

Here $J_1 \underset{J_0}{\times} J_1$ *denotes the pullback as in (2.3.44).*

- *Associativity of composition:*

$$\begin{tikzcd} J_1 \underset{J_0}{\times} J_1 \underset{J_0}{\times} J_1 \arrow[r, "J_1 \times c"] \arrow[d, "c \times J_1"'] & J_1 \underset{J_0}{\times} J_1 \arrow[d, "c"] \\ J_1 \underset{J_0}{\times} J_1 \arrow[r, "c"] & J_1 \end{tikzcd}$$

- *Left and right composition with identity maps:*

$$\begin{tikzcd} J_0 \underset{J_0}{\times} J_1 \arrow[r, "e \times J_1"] \arrow[dr, "p_2"'] & J_1 \underset{J_0}{\times} J_1 \arrow[d, "c"] & J_1 \underset{J_0}{\times} J_0 \arrow[l, "J_1 \times e"'] \arrow[dl, "p_1"] \\ & J_1 & \end{tikzcd}$$

A **groupoid internal to a category** $\mathcal{C}$ *is a category* J *internal to* $\mathcal{C}$ *that is equipped with an inverse morphism* $i : J_1 \to J_1$ *with* $ii = J_1$ *such that the following diagrams commute in* $\mathcal{C}$.

- *Reversal of source and target:*

$$\begin{tikzcd} & J_1 \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[dd, "i"] & \\ J_0 & & J_0 \\ & J_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

- *In the following we need to consider pullbacks similar to that of (2.3.44) but with other combinations of maps* $J_1 \to J_0$*, namely*

$$\begin{tikzcd} J_1 \underset{s,s}{\times} J_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & J_1 \arrow[d, "s"] \\ J_1 \arrow[r, "s"] & J_0 \end{tikzcd} \quad , \quad \begin{tikzcd} J_1 \underset{t,t}{\times} J_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & J_1 \arrow[d, "t"] \\ J_1 \arrow[r, "t"] & J_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} J_1 \underset{t,s}{\times} J_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & J_1 \arrow[d, "s"] \\ J_1 \arrow[r, "t"] & J_0 . \end{tikzcd}$$

The first two receive a diagonal map Δ *from* J_1 *while the third supports an* *opposite composition map $\overline{c}$ to J_1, and we have*

$$\begin{tikzcd}
& J_1 \underset{t,s}{\times} J_1 \arrow[dd, "\overline{c}"'] & \\
J_1 \underset{s,s}{\times} J_1 \arrow[ur, "i\times J_1"] & & J_1 \underset{t,t}{\times} J_1 \arrow[ul, "J_1\times i"'] \\
& J_1 & \\
J_1 \arrow[uu, "\Delta"] \arrow[dr, "t"] & & J_1 \arrow[uu, "\Delta"'] \arrow[dl, "s"'] \\
& J_0. \arrow[uu, "e"] &
\end{tikzcd}$$

When $\mathcal{C}$ has a terminal object $*$ as in Example 2.1.16(ii), a **group internal to** $\mathcal{C}$ *(also known as a* **group object in** $\mathcal{C}$*) is a groupoid J as above in which $J_0 = *$.*

A **cocategory internal to** $\mathcal{C}$ *is a category internal to $\mathcal{C}^{op}$.* **Cogroupoids** *and* **cogroups** *internal to $\mathcal{C}$ are similarly defined.*

Remark 2.3.47. The existence of $J_1 \underset{J_0}{\times} J_1$ *could be guaranteed by requiring $\mathcal{C}$ to be complete as in Definition 2.3.25, or just to have finite limits, but we can get by with less. On the other hand, the most common case we will consider is $\mathcal{C} = \mathcal{T}op$, which is complete.*

Example 2.3.48. *A group internal to $\mathcal{T}op$ is a topological group. A group internal to the category of smooth manifilds is a Lie group.*

Example 2.3.49. *A cogroupoid internal to the category of commutative algebras over a commutative ring K is a Hopf algebroid over K.*

Note that J_1 and J_0 are both objects over $J_0 \times J_0$ via the maps

$$(s,t) : J_1 \to J_0 \times J_0 \qquad \text{and} \qquad \Delta : J_0 \to J_0 \times J_0.$$

When $\mathcal{C}$ has a terminal object $*$ as in Example 2.1.16(ii), we can think of an object on J as a morphism $x : * \to J_0$. Given two "objects" $x, y : * \to J_0$, we can define a "morphism object" $J(x,y)$ to be the pullback in the diagram

$$\begin{tikzcd}
J(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & J_1 \arrow[d, "(s,t)"] \\
* \arrow[r, "(x,y)"'] & J_0 \times J_0
\end{tikzcd} \tag{2.3.50}$$

This coincides with the usual morphism set $J(x,y)$ when J is a small category and $\mathcal{C} = \mathcal{S}et$. In Chapter 3 below we will discuss enriched categories, in which morphisms sets are replaced by morphism objects in a ground category with suitable structure. See Remark 3.1.8.

Definition 2.3.51. Left and right J-modules. *Let J be a category internal to $\mathcal{C}$ as in Definition 2.3.46. For objects A and B in $\mathcal{C}$ over J_0, let $A \underset{J}{\times} B$ be the pullback*

$$\begin{tikzcd} A \underset{J}{\times} B \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ A \arrow[r] & J_0. \end{tikzcd}$$

A **left (right)** J**-module** *is an object X in $\mathcal{C}$ with a morphism $t : X \to J_0$ ($s : X \to J_0$) and an action map*

$$\lambda : J_1 \underset{J}{\times} X \to X \qquad (\rho : X \underset{J}{\times} J_1 \to X)$$

that is associative and unital. In the left case this means the diagrams

$$\begin{tikzcd} J_1 \underset{J}{\times} J_1 \underset{J}{\times} X \arrow[r, "J_1 \times \lambda"] \arrow[d, "c \times X"'] & J_1 \underset{J}{\times} X \arrow[d, "\lambda"] \\ J_1 \underset{J}{\times} X \arrow[r, "\lambda"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} J_0 \underset{J}{\times} X \arrow[r, "e \times X"] \arrow[dr, equal] & J_1 \underset{J}{\times} X \arrow[d, "\lambda"] \\ & X \end{tikzcd}$$

both commute.

Example 2.3.52. *X could be J_1 equipped with the target morphism t (source morphism s) and λ (ρ) could be the composition morphism c. Hence J_1 is both a left and a right J-module.*

The same is true of the object

$$J_1 \underset{J_0}{\times} J_1 \underset{J_0}{\times} \cdots \underset{J_0}{\times} J_1$$

with n factors for some positive integer n. When $\mathcal{C} = \mathcal{S}et$, this is the set of diagrams in J of the form

$$j_0 \to j_1 \to \cdots \to j_n.$$

When $\mathcal{C} = \mathcal{T}op$, it is the **space** *of such diagrams suitably topologized.*

2.3E n-Cartesian diagrams

We will use the following result for $|S| = 3$ in the proof of Proposition 3.1.53 below. We leave its proof as an exercise for the reader. A proof of the statement for the case where the target category is $\mathcal{T}op$ can be found in [MV15, Lemmas 5.2.8 and 5.6.7]. Further discussion of such diagrams can be found in [ACB14, §2].

Proposition 2.3.53. Limits and colimits of n-Cartesian diagrams. *Let S be a finite set with n elements for $n \geqslant 2$. Let $\mathscr{P}(S)$, $\mathscr{P}_0(S)$ and $\mathscr{P}_1(S)$ be the categories of subsets, nonempty subsets and proper subsets of S respectively, each with inclusion maps as morphisms. For each $s \in S$ let S_s denote the complement of $\{s\}$ and define fully faithful functors $A_s, B_s : \mathscr{P}(S_s) \to \mathscr{P}(S)$ where the image of A_s (B_s) is the subcategory of all subsets of S containing (not containing) s.*

*Let F be a functor from $\mathscr{P}(S)$ to a complete category $\mathcal{C}$, i.e., a diagram in $\mathcal{C}$ shaped like an n-cube. Then the n-**fold pullback** $\lim\limits_{\mathscr{P}_0(S)} F$ can be described as a simple (2-fold) pullback in n different ways. For each s it is the limit of the diagram*

$$F(\{s\}) = \lim_{\mathscr{P}(S_s)} FA_s \longrightarrow \lim_{\mathscr{P}_0(S_s)} FA_s \longleftarrow \lim_{\mathscr{P}_0(S_s)} FB_s$$

where the arrow on the left is induced by the inclusion functor of $\mathscr{P}_0(S_s)$ into $\mathscr{P}(S_s)$, and the one on the right is induced by the functor

$$A_s\mathscr{P}_0(S_s) \to B_s\mathscr{P}_0(S_s)$$

given by sending a set S properly containing $\{i\}$ to the nonempty set obtained by removing s from S.

*Dually, let G be a functor from $\mathscr{P}(S)$ to a cocomplete category $\mathcal{D}$. Then for each s, the n-**fold pushout** $\operatorname*{colim}\limits_{\mathscr{P}_1(S)} G$ is the simple (2-fold) pushout of the diagram*

$$G(S_s) = \operatorname*{colim}_{\mathscr{P}(S_s)} GB_s \longleftarrow \operatorname*{colim}_{\mathscr{P}_1(S_s)} GB_s \longrightarrow \operatorname*{colim}_{\mathscr{P}_1(S_s)} GA_s. \tag{2.3.54}$$

where the arrow on the left is induced by the inclusion of $\mathscr{P}_1(S_s)$ into $\mathscr{P}(S_s)$, and the one on the right is induced by the functor $B_s\mathscr{P}_1(S_s) \to A_s\mathscr{P}_1(S_s)$ given by sending a set S not containing s and at least one other element to the proper subset obtained by adding s to S.

Remark 2.3.55. The case $n = 2$ of Proposition 2.3.53. *For $n = 2$ the two specified simple pullbacks (pushouts) are the same. For the pullback case, the functor FA_i (for either i) sends the single object of $\mathscr{P}_0(\mathbf{1})$ to $F(\mathbf{2})$. Since $\mathscr{P}(\mathbf{1})$ has an initial object, the value of $\lim\limits_{\mathscr{P}(\mathbf{1})} FA_i$ is $F(\{i\})$. The right hand limit is the value of F on the unique singleton not containing i.*

For $0 \leqslant i \leqslant j \leqslant n$, let $\mathscr{P}_i^j(S)$ denote the subcategory of $\mathscr{P}(S)$ consisting of subsets T with $i \leqslant |T| \leqslant j$. In particular $\mathscr{P}(S) = \mathscr{P}_0^n(S)$, $\mathscr{P}_0(S) = \mathscr{P}_1^n(S)$ and $\mathscr{P}_1(S) = \mathscr{P}_0^{n-1}(S)$. We leave the proof of the following, and the formulation of the dual statement, as an exercise for the reader.

Proposition 2.3.56. The n-fold pushout as a coequalizer. *With notation as above, let G be a functor from $\mathscr{P}(S)$ to a cocomplete category $\mathcal{D}$. Then*

$$\operatorname*{colim}_{\mathscr{P}_1(S)} G = \operatorname*{colim}_{\mathscr{P}_{n-2}^{n-1}(S)} G.$$

Then colimit on the right is the coequalizer in

$$\coprod_{|T|=n-2} G_T \rightrightarrows \coprod_{|T'|=n-1} G_{T'} \to \operatorname*{colim}_{\mathscr{P}_1(S)} G,$$

where for each subset T with $n-2$ elements, the two maps from G_T are induced by the two inclusions of T into a subset T' with $n-1$ elements.

Here is a illustration of Proposition 2.3.56 for $S = \mathbf{3} = \{1, 2, 3\}$. Consider the following diagram in $\mathcal{D}$.

$$\begin{tikzcd}
 & G_{\{1\}} \arrow[r] \arrow[dr] & G_{\{1,2\}} \arrow[dr] & \\
G_{\varnothing} \arrow[ur] \arrow[r] \arrow[dr] & G_{\{2\}} \arrow[ur] \arrow[dr] & G_{\{1,3\}} \arrow[r] & \operatorname*{colim}_{\mathscr{P}_1(\mathbf{3})} G \\
 & G_{\{3\}} \arrow[r] \arrow[ur] & G_{\{2,3\}} \arrow[ur] &
\end{tikzcd}$$

Each map to the colimit (the 3-fold pushout) factors through an object in the third column. If we have a set of maps from the objects in the third column such that the two composite maps from each object in the second column agree, then the six composite maps from $G_{\varnothing}$ will also agree. This means we could omit the first column without changing the value of the colimit.

The following is a generalization of Definition 2.3.9.

Definition 2.3.57. Boundaries and corner maps. *Let $G : \mathscr{P}(S) \to \mathcal{D}$ for a finite set S and a cocomplete category $\mathcal{D}$ with $X = G_S$. Then the* **boundary of X with respect to G** *is*

$$\partial_G X := \operatorname*{colim}_{\mathscr{P}_1(S)} G,$$

and the **corner map of G** *is the map $\partial_G X \to X$ induced by the inclusion functor $\mathscr{P}_1(S) \to \mathscr{P}(S)$.*

We will make closely related definitions below in Definition 2.6.12 and Definition 2.9.29. This terminology is motivated by the following.

Example 2.3.58. Manifolds with corners. *Let $\mathcal{D} = \mathcal{T}op$, let S be a finite set and define a functor $G : \mathscr{P}(S) \to \mathcal{T}op$ as follows. For each $s \in S$, let M_s*

be a manifold with boundary. For each $T \subseteq S$, let

$$G_T = \prod_{t \in T} M_t \times \prod_{t \notin T} \partial M_t.$$

Then $G_S = \prod_{s \in S} M_s =: X$, the boundary of the manifold X is $\partial_G X$ as in Definition 2.3.57, and the corner map of G is the inclusion map $\partial X \to X$.

Remark 2.3.59. The category $\mathscr{P}(S)^{op}$. *Note that the category $\mathscr{P}(S)$ is self dual with $\mathscr{P}_0(S)^{op}$ isomorphic to $\mathscr{P}_1(S)$ and* vice versa. *Thus a functor $F : \mathscr{P}_0(S)^{op} \to \mathcal{D}$ for cocomplete $\mathcal{D}$ has a colimit which is an n-fold pushout, where n is the cardinality of S. Hence it can be described as an ordinary pushout in n different ways as in Proposition 2.3.53 and as a coequalizer as in Proposition 2.3.56.*

2.3F Reflexive coequalizers

Definition 2.3.60. *A* **reflexive coequalizer** *is the colimit of a functor to a cocomplete category from the category $\tilde{J}$ having two objects A and B and morphisms*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[l, "s"'] \end{tikzcd},$$

where $fs = gs = 1_B$; there is no condition on sf and sg. We will refer to s as the **section** *since it splits the morphisms f and g. Dually, a* **coreflexive equalizer** *is the limit of a functor to a complete category from $\tilde{J}^{op}$.*

The term "reflexive" here is not to be confused with "reflective," as in Definition 2.2.49.

The category $\tilde{J}$ has two subcategories of interest.

(i) Let $\mathrm{End}_A \subseteq \tilde{J}$ be the full subcategory having a single object A and hence two morphisms sf and sg which need not be the identity morphism on A, and let $F : \mathrm{End}_A \to \tilde{J}$ be the inclusion functor. Then for a functor $X : \tilde{J} \to \mathcal{C}$ to a cocomplete category $\mathcal{C}$, $\operatorname{colim} XF$ is the coequalizer of the maps X_{sf} and X_{sg} from X_A to itself.

(ii) Let $J \subseteq \tilde{J}$ be the subcategory obtained by omitting the section s (as in Example 2.3.35(iii)), and let $G : J \to \tilde{J}$ be the inclusion functor. Then for a functor $X : \tilde{J} \to \mathcal{C}$ to a cocomplete category $\mathcal{C}$, $\operatorname{colim} XG$ is the coequalizer of the maps X_f and X_g.

We will show that the colimits of X, XF and XG are all the same. The fact that $\operatorname{colim} XF$ is the coequalizer of two self-maps of A is the origin of the term "reflexive" coequalizer.

Proposition 2.3.61. Reflexive coequalizers are ordinary coequalizers. *Let $F : End_A \to \tilde{J}$ and $G : J \to \tilde{J}$ be as above and let $\mathcal{C}$ be a cocomplete category. Then for any functor $X : \tilde{J} \to \mathcal{C}$, the objects $\operatorname{colim} X$, $\operatorname{colim} XF$ and $\operatorname{colim} XG$ in $\mathcal{C}$ are all the same.*

Proof Applying the functor X to the diagram of Definition 2.3.60 gives

$$\begin{tikzcd} X_A \arrow[r, bend left, "X_f"] \arrow[r, bend right, "X_g"'] & X_B, \arrow[l, "X_s"'] \end{tikzcd}$$

where

$$X_f X_s = X_g X_s = 1_{X_B}.$$

The ordinary coequalizers $\operatorname{colim} XF$ and $\operatorname{colim} XG$ support unique maps λ' and λ to the reflexive coequalizer $\operatorname{colim} X$ with appropriate properties by the universal property of colimits. We have maps

$$\begin{tikzcd} \operatorname{colim} XF \arrow[ddrr, "\lambda'"'] & X_A \arrow[l, "\alpha'"'] \arrow[rr, bend left, "X_f"] \arrow[rr, bend right, "X_g"'] \arrow[dr, "\alpha"] \arrow[ddr, "\tilde{\alpha}"'] & & X_B \arrow[ll, "X_s"'] \arrow[lll, bend right=25, "\beta'"'] \arrow[dl, "\beta"'] \arrow[ddl, "\tilde{\beta}"] \\ & & \operatorname{colim} XG \arrow[d, "\lambda"'] & \\ & & \operatorname{colim} X & \end{tikzcd}$$

The maps to $\operatorname{colim} X$ are required to satisfy

$$\tilde{\alpha} = \tilde{\beta} X_f = \tilde{\beta} X_g \qquad \text{and} \qquad \tilde{\alpha} X_s = \tilde{\beta}.$$

The map α' to $\operatorname{colim} XF$ is required to satisfy

$$\alpha' X_{sf} = \alpha' X_{sg}$$

and we denote the composite $\alpha' X_s$ by β'. Hence we have

$$\beta' X_f = \beta' X_g = \alpha',$$

which are the same conditions required of $\tilde{\alpha}$ and $\tilde{\beta}$, so $\operatorname{colim} X = \operatorname{colim} XF$.

The maps to $\operatorname{colim} XG$ are required to satisfy

$$\alpha = \beta X_f = \beta X_g,$$

which implies

$$\alpha X_s = \beta X_f X_s = \beta X_g X_s = \beta 1_{X_B} = \beta.$$

These are the same properties satisfied by $\tilde{\alpha}$ and $\tilde{\beta}$, so $\operatorname{colim} X = \operatorname{colim} XG$. $\square$

Remark 2.3.62. Functors between indexing categories may alter colimits. *The previous result is equivalent to the commutativity of the following diagram of categories and functors.*

$$\begin{tikzcd} \mathcal{C}^J \arrow[dr, "\underset{J}{\operatorname{colim}}"'] & \mathcal{C}^{\tilde{J}} \arrow[l, "F^*"'] \arrow[r, "G^*"] \arrow[d, "\underset{\tilde{J}}{\operatorname{colim}}"] & \mathcal{C}^{End_A} \arrow[dl, "\underset{End_A}{\operatorname{colim}}"] \\ & \mathcal{C} & \end{tikzcd}$$

Lest the reader get the wrong idea, such diagrams do **not** *commute in general. For example let D be the discrete category (Definition 2.1.7) having the same set of objects as an arbitrary small category J and let $K : D \to J$ be the inclusion functor. Then for a cocomplete category $\mathcal{C}$, a functor $X : J \to \mathcal{C}$, K induces a map*

$$\underset{D}{\operatorname{colim}} XK \to \underset{J}{\operatorname{colim}} X.$$

Here the source is the coproduct of the objects X_j for $j \in J$, so the map need not be an isomorphism. We will discuss this more below in §2.3H.

2.3G Filtered and sifted limits and colimits

Definition 2.3.63. *A small category J is* **filtered** *if*

(i) for each pair of objects j_1 and j_2 in J, there is a third object j_3 with morphisms $j_1 \to j_3$ and $j_2 \to j_3$ and

(ii) for each pair of morphisms $f, g : j_1 \to j_2$ in J there is an object j_3 and morphism $h : j_2 \to j_3$ such that $hf = hg$.

A **filtered colimit** *is a colimit indexed by a filtered category. A* **sequential colimit** *is the colimit of a diagram of the form*

$$X_0 \to X_1 \to X_2 \to X_3 \to \cdots . \qquad (2.3.64)$$

We will denote the corresponding indexing category by N, the **sequential colimit category**. *Its objects are natural numbers $n \geqslant 0$ and it has a unique morphism $m \to n$ whenever $m \leqslant n$.*

An object A in a cocomplete category $\mathcal{C}$ is **finitely presented** *or* **finite** *if the Yoneda functor (Definition 2.2.33) $\text{よ}^A = \mathcal{C}(A, -)$ preserves sequential colimits.*

A small category J is **cofiltered** *if J^{op} is filtered. A* **cofiltered limit** *is a limit indexed by a cofiltered category. A* **sequential limit** *is the limit of a diagram of the form*

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots .$$

Its indexing category is N^{op}, the **sequential limit category**.

Example 2.3.65. A curious sequential colimit of topological spaces. *Let X_0 be the disjoint union of two copies of the real line, $\{a,b\} \times \mathbf{R}$. For $n > 0$ let X_n be the quotient of X_0, obtained by identifying (a,x) with (b,x) for $|x| \geqslant 1/n$. Hence each X_n for $n > 0$ has the homotopy type of S^1 and each map $x_n : X_n \to X_{n+1}$ (which preserves the real coordinate and is not a closed inclusion) is a homotopy equivalence. However the colimit (in the category of* **arbitrary** *topological spaces) is the quotient of $\{a,b\} \times \mathbf{R}$ obtained by identifying (a,x) with (b,x) for $x \neq 0$. It is* **not Hausdorff** *because the distinct points $(a,0)$ and $(b,0)$ do not have disjoint neighborhoods. It is not weak Hausdorff (see Definition 2.1.46) because the closure of any neighborhood of $(a,0)$ contains $(b,0)$, but there is a map from I whose image contains $(a,0)$ but not $(b,0)$.*

In the category of compactly generated weak Hausdorff spaces, this colimit is simply $\mathbf{R}$. It has the "wrong" homotopy type in that it is homotopically distinct from each X_n.

If we choose a base point in X_0 and replace each space in sight by its loop space, then we get a colimit which does not preserve π_0.

Now suppose we replace the map $x_n : X_n \to X_{n+1}$ for $n > 0$ above by x'_n defined by

$$x'_n(\epsilon, x) = \left(\epsilon, \frac{nx}{n+1}\right).$$

It is a homeomorphism that is homotopic to x_n. It follows that the corresponding colimit is homeomorphic to X_1 and thus homotopy equivalent to S^1. Thus we see that the homotopy type of a colimit is **not** *determined by the homotopy classes of the maps in the diagram.*

Example 2.3.66. A curious sequential limit of topological spaces. *For each integer $n \geqslant 0$, let Y_n be the quotient of $\{a,b\} \times \mathbf{R}$ obtained by identifying (a,x) with (b,x) for $|x| \geqslant n$, and let $p_n : \{a,b\} \times \mathbf{R} \to Y_n$ be the projection map. Let $y_n : Y_n \to Y_{n-1}$ be the evident surjection preserving the real coordinate, so $y_n p_n = p_{n-1}$. As in Example 2.3.65, each Y_n for $n > 0$ has the homotopy type of S^1 and each map y_{n+1} is a homotopy equivalence.*

However the limit is $\{a,b\} \times \mathbf{R}$, the disjoint union of two copies of $\mathbf{R}$. It has the "wrong" homotopy type in that it is homotopically distinct from each Y_n. It is not path connected even though each Y_n is.

If we replace y_n by the homotopic map y'_n defined by

$$y'_n(\epsilon, y) = \left(\epsilon, \frac{(n-1)y}{n}\right).$$

Like x'_n in Example 2.3.65, it is a homeomorphism. The corresponding limit is homeomorphic to Y_1 and thus homotopically equivalent to S^1. Thus we see that the homotopy type of a limit is **not** *determined by the homotopy classes of the maps in the diagram.*

Example 2.3.67. Sequential limits as equalizers. *Recall that every limit is an equalizer by Theorem 2.3.28. In the case of a sequential limit, (2.3.29) reads*

$$\prod_{n\geqslant 0} X_n \overset{f}{\underset{g}{\rightrightarrows}} \prod_{n\geqslant m\geqslant 0} X_m,$$

where $p_{m,n}f = p_m$ and $p_{m,n}g = s_{n,m}p_n$, where $s_{n,m}$ is is the morphism $X_n \to X_m$. In this case the product on the right can be replaced by the smaller one in which we only have factors for which $n = m+1$ so we have

$$\prod_{n\geqslant 0} X_n \overset{f}{\underset{g}{\rightrightarrows}} \prod_{n\geqslant 0} X_n, \tag{2.3.68}$$

where $p_n f = p_n$, so f is the identity map, and $p_n g = s_{n+1,n}p_{n+1}$, so g is a shift map. If each X_n is a set, then this equalizer is

$$\left\{(x_0, x_1, \dots) \in \prod_{n\geqslant 0} X_n \colon x_m = s_{m+1,m}(x_{m+1})\right\},$$

the set of sequences of x_is compatible under the maps in the diagram X.

There is a similar description of a sequential colimit as a coequalizer which we leave to the reader.

The above and the following will be repeated below in Definition 4.8.8.

Definition 2.3.69. Relative finiteness. *An object A in a cocomplete category $\mathcal{C}$ is* **finitely presented (or finite) relative to a subcategory** $\mathcal{D}$ *if the Yoneda functor (Yoneda Lemma 2.2.10) $よ^A = \mathcal{C}(A, -)$ preserves sequential colimits when the diagram of (2.3.64) is in $\mathcal{D}$.*

Proposition 2.3.70. Morphisms from finitely presented objects to sequential colimits. *If an object A in a cocomplete category $\mathcal{C}$ is finitely presented realative to a subcategory $\mathcal{D}$ (which could be all of $\mathcal{C}$), then any morphism $A \to \operatorname{colim}_N X$ factors through some X_n.*

Proof Since A is finitely presented, the map

$$\operatorname{colim}_N \mathcal{C}(A, X_n) \to \mathcal{C}(A, \operatorname{colim}_N X_n)$$

is an isomorphism. Each element in the set on the left is the image of a morphism $A \to X_n$ for some n, so the same is true for each morphism $A \to \operatorname*{colim}_{N} X_n$. $\square$

The following example is due to [Hov99, page 49], and is discussed further in [Hov01a].

Example 2.3.71. A two point space which is not finitely presented. *Let $A = \{0, 1\}$ with the trivial topology, meaning that the only nonempty open subset is A. It is* **not** *a weak Hausdorff space since its points are not closed.*

Let $X_n = [n, \infty) \times A$ be topologized as follows. The collection of nonempty open subsets is

$$\{([n, \infty) \times \{0\}) \cup ([x, \infty) \times \{1\}) \colon x \geqslant n\}\,.$$

This means a continuous map $A \to X_n$ must send both points of A to the subset $[n, \infty) \times \{0\}$, or to the same point in $[n, \infty) \times \{1\}$. The mapping space $Map(A, X_n)$ is the disjoint union of $[n, \infty)^2$ and $[n, \infty)$ with the trivial topology on each component.

We define a continuous map $X_n \to X_{n+1}$ by

$$(x, \epsilon) \mapsto \begin{cases} (n+1, \epsilon) & \text{for } n \leqslant x \leqslant n+1 \\ (x, \epsilon) & \text{otherwise.} \end{cases}$$

Then $\operatorname*{colim}_{N} X_n \cong A$, but the identity map to it from A does not factor through any X_n.

Example 2.3.72. Finitely generated abelian groups. *In $\mathcal{A}b$, the category of abelian groups, the finitely presented objects are finitely generated groups. Consider the infinitely generated group* $\mathbf{Q}$*. It is the colimit of the sequential diagram*

$$\mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{3} \cdots$$

The image of the nth group is the additive subgroup generated by $1/n!$. Any homomorphism to $\mathbf{Q}$ *from a finitely generated abelian group A factors through one of these subgroups. However the identity morphism, whose domain is not finitely presented, does not factor through any of them.*

There is a similar notion for an enriched category that is the subject of Definition 3.2.6 and Proposition 3.2.7 below.

It follows that for J filtered, any functor $D \to J$ from a finite category D extends to D^+, the category obtained from D by adjoining a terminal object.

Definition 2.3.73. *A small category J is* **sifted** *if colimits of sets indexed by*

J commute with finite products, i.e., for every finite discrete category (Definition 2.1.7) S and every functor $F : J \times S \to \mathcal{S}et$, the canonical morphism

$$\underset{j\in J}{colim} \prod_{s\in S} F(j,s) \to \prod_{s\in S} \underset{j\in J}{colim} F(j,s)$$

is an isomorphism. A **sifted colimit** *is a colimit indexed by a sifted category.*

Every filtered category is sifted since filtered colimits commute with finite products in $\mathcal{S}et$, the category of sets. The latter was proved by Mac Lane in [ML98, Theorem IX.2.1].

We learned the following from https://ncatlab.org/nlab/show/filtered+limit, which was contributed by John Baez.

Remark 2.3.74. Warning. *It is* **not** *true that sifted colimits and finite limits commute in any category in which they are defined, such as a bicomplete category. Here is a counterexample.*

Let $\widehat{\mathbf{N}}_0$ be the one point compactification of the discretely topologized natural numbers $\mathbf{N}_0$. In it the neighborhoods of the point ∞ are complements of finite subsets of $\mathbf{N}_0$. The closed subspaces of $\widehat{\mathbf{N}}_0$ are the finite subsets of $\mathbf{N}_0$ and all subsets containing ∞.

Let $\mathcal{C}$ be the poset of closed subspaces of $\widehat{\mathbf{N}}_0$, meaning the category whose object are such closed subspaces and whose morphisms are inclusion maps. It has terminal and initial objects, namely $\widehat{\mathbf{N}}_0$ and $\varnothing$. This means the categorical Cartesian product and coproduct in $\mathcal{C}$ are respectively the intersection of the closed subspaces and the closure of their union. Thus $\mathcal{C}$ has arbitrary products and coproducts.

Now consider the product (meaning intersection) of $\{\infty\}$ with the coproduct of all the finite subsets B_α of $\mathbf{N}_0$. The latter is a colimit indexed by a sifted category. This coproduct, being the closure of $\mathbf{N}_0$, is the entire space, so its intersection with $\{\infty\}$ is $\{\infty\}$ again. On the other hand, the intersection of $\{\infty\}$ with each finite B_α is empty, so the closure of the union of these intersections is also empty. Hence

$$\{\infty\} = \{\infty\} \cap colim\, B_\alpha \neq colim\,(\{\infty\} \cap B_\alpha) = \varnothing.$$

The following characterization is due to [GU71, 15.2.c] and can also be found (in English) in [ARV11, Theorem 2.15].

Theorem 2.3.75. The diagonal map of a sifted category. *A nonempty small category J is sifted iff the diagonal functor $\Delta : J \to J \times J$ is final, meaning that it induces an isomorphism of colimits for any functor from $J \times J$ to a cocomplete category.*

Final functors are more explicitly defined below in Definition 2.3.80 and are the subject of Theorem 2.3.82.

Example 2.3.76. A sifted category that is not filtered. *Let $\tilde{J}$ be the category of Definition 2.3.60. It is not filtered because there is no morphism coequalizing f and g. See [ARV10, Example 2.2] for a proof that it is sifted.*

We record the following observation for future use.

Proposition 2.3.77. Some sifted colimits. *Reflexive coequalizers and filtered colimits are both sifted colimits.*

In some sense, these two types of colimits generate sifted ones. See [ARV10] and [ARV11] for more discussion.

2.3H Changing the indexing category

The discussion here applies to both limits and colimits. We will treat colimits only, leaving the dual statements about limits to the reader.

Suppose we have small categories J and K, a cocomplete category $\mathcal{C}$, and functors

$$J \xrightarrow{\alpha} K \xrightarrow{X} \mathcal{C}.$$

Then we have a diagram

$$\begin{tikzcd} \mathcal{C}^J \arrow[dr, "\operatorname{colim}_J"'] & & \mathcal{C}^K \arrow[ll, "\alpha^*"'] \arrow[dl, "\operatorname{colim}_K"] \\ & \mathcal{C} & \end{tikzcd} \tag{2.3.78}$$

and thus a morphism

$$\phi_\alpha : \operatorname*{colim}_J X\alpha \to \operatorname*{colim}_K X \tag{2.3.79}$$

in $\mathcal{C}$. As noted in Remark 2.3.62, the diagram (2.3.78) does not commute in general.

However there are some cases in which ϕ_α is an isomorphism. For example, K could be the category whose objects are pairs of natural numbers (m, n) with a single morphism $(m, n) \to (m', n')$ whenever $m \leqslant m'$ and $n \leqslant n'$, and J could be the subcategory of pairs (m, m). In that case the two colimits are the same. We know that for each pair of elements of the larger category K there is one in the subcategory J and theta they both map uniquely to. Therefore J has enough information to determine the colimit. This situation is discussed in [ML98, §IX.3] and [Dug17, §I.6.1].

Definition 2.3.80. Final functors. *For small categories J and K, a functor $\alpha : J \to K$ is* **final** *(or* **terminal***, or* **left cofinal***) if for each object $k \in K$ the undercategory $(k{\downarrow}\alpha)$ as in Definition 2.1.51 is non-empty and connected as in Definition 2.1.55. When α is the inclusion of a subcategory, we say that*

J **is final in** *K. A* **cofinal** *or* **initial functor** *$J \to K$ is one that induces a final functor $J^{op} \to K^{op}$.*

For more details, see [KS06, §2.5], where the term "co-cofinal" is used for final.

The nonemptiness of $(k{\downarrow}\alpha)$ means that for each object k in K there is an object j in J such that there is a morphism $k \to \alpha(j)$. Its connectivity means that for any two such js there is a finite commutative diagram in K of the form

$$\begin{tikzcd} & & k \arrow[dll] \arrow[dl] \arrow[d] \arrow[drr] & & \\ \alpha(j_0) \arrow[r] & \cdot & \cdot \arrow[l] \arrow[r] & \cdots & \alpha(j_n). \arrow[l] \end{tikzcd} \tag{2.3.81}$$

where the morphisms in the bottom row are in the image of α, and the left and right morphisms from k are given.

The following was proved by Mac Lane as [ML98, Theorem IX.3.1].

Theorem 2.3.82. Colimit maps induced by final functors. *For a final functor $\alpha : J \to K$ as in Definition 2.3.80, if $X : K \to \mathcal{C}$ is a functor for which $\operatorname*{colim}_J X\alpha$ exists, then $\operatorname*{colim}_K X$ also exists and the induced map ϕ_α of (2.3.79) is an isomorphism.*

Corollary 2.3.83. Colimits indexed by categories with terminal objects. *Suppose the small category K has a terminal object k as in Example 2.1.16(ii) and $X : K \to \mathcal{C}$ is a functor. Then $\operatorname*{colim}_K X$ exists and is equal to the value of X on k.*

Proof Let J be the trivial category and let $\alpha : J \to K$ send its one object to k. This functor is easily seen to be final as in Definition 2.3.80, so the result is a special case of Theorem 2.3.82. □

2.4 Ends and coends

Yoneda originally introduced ends and coends in the context of functors enriched (see §3.1 below) over $\mathcal{A}b$ in [Yon60, §4, page 545]. He called them the "integration" and "cointegration" and denoted them by

$$\int_J H \qquad \text{and} \qquad \int_J^* H$$

or a functor $H : J^{op} \times J \to \mathcal{C}$ from a small category J to a complete or cocomplete category $\mathcal{C}$. In this book we will denote the end and coend by

$$\int^J H \qquad \text{and} \qquad \int_J H$$

respectively. **We will use a superscript for an end and a subscript for a coend.** This differs from the notation of [ML71, pages 222–223] and most other works in category theory, where the opposite convention is used. However it agrees with the notation used for coends by Jacob Lurie in [Lur09, Chapter 2 and Appendix A], and in some papers on factorization homology such as [AF19].

Thus H is a functor of two variables in J, contravariant in the first and covariant in the second. For example we could have $\mathcal{C} = \mathcal{S}et$ and $H(j, j') := J(j, j')$, the set of morphisms $j \to j'$.

Given such a functor H, for each morphism $f : j \to j'$ in J we have a diagram in $\mathcal{C}$,

$$\begin{tikzcd} & H(j,j) \arrow[d, "f_*"] \\ H(j',j') \arrow[r, "f^*"] & H(j,j'). \end{tikzcd}$$

which has a limit (the pullback) when $\mathcal{C}$ is complete. We use the Yoneda's symbol

$$\int^J H(j, j),$$

now called an **end**, to denote the limit obtained by considering such diagrams for **all** morphisms f in J, assuming that the target category is complete. More explicitly, for each morphism $f \in \operatorname{Arr} J$ we get a morphism

$$H(\operatorname{Dom} f, \operatorname{Dom} f) \xrightarrow{f_*} H(\operatorname{Dom} f, \operatorname{Cod} f)$$

in $\mathcal{C}$. Hence we get a morphism to the product of such sets over all f having domain j,

$$H(j, j) \xrightarrow{\phi_*} \prod_{\substack{f \in \operatorname{Arr} J \\ \operatorname{Dom} f = j}} H(j, \operatorname{Cod} f).$$

given by $p_f \phi_* = f_*$, where p_f denotes the projection of the product onto the factor corresponding to f. Now we take the product of these morphisms over all objects j in J and get

$$\prod_{j \in \operatorname{Ob} J} H(j, j) \xrightarrow{\phi_*} \prod_{f \in \operatorname{Arr} J} H(\operatorname{Dom} f, \operatorname{Cod} f). \tag{2.4.1}$$

In a similar fashion the morphism

$$H(\operatorname{Cod} f, \operatorname{Cod} f) \xrightarrow{f^*} H(\operatorname{Dom} f, \operatorname{Cod} f)$$

leads to

$$\prod_{j \in \mathrm{Ob}\, J} H(j,j) \xrightarrow{\ \phi^*\ } \prod_{f \in \mathrm{Arr}\, J} H(\mathrm{Dom}\, f, \mathrm{Cod}\, f). \tag{2.4.2}$$

In other words, we have the following diagram in which the products on the right are over all objects or all morphisms in J.

$$\begin{tikzcd} & H(j,j) \arrow[d, "f_*"'] & & & \prod_j H(j,j) \arrow[d, "\phi_*"] \\ H(j',j') \arrow[r, "f^*"] & H(j,j') & \arrow[r, squiggly] & \prod_{j'} H(j',j') \arrow[r, "\phi^*"] & \prod_{f:j\to j'} H(j,j'). \end{tikzcd} \tag{2.4.3}$$

Dually, when $\mathcal{C}$ is cocomplete, we have a similar diagram with coproducts over all objects or all morphisms in J.

$$\begin{tikzcd} H(j',j) \arrow[r, "f_*"] \arrow[d, "f^*"'] & H(j',j') & \arrow[r, squiggly] & \coprod_{f:j\to j'} H(j',j) \arrow[r, "\varphi_*"] \arrow[d, "\varphi^*"] & \coprod_{j'} H(j',j') \\ H(j,j) & & & \coprod_j H(j,j) & \end{tikzcd} \tag{2.4.4}$$

Definition 2.4.5. *For a functor $H : J^{op} \times J \to \mathcal{C}$ for a small category J to a complete category $\mathcal{C}$, the* **end**

$$\int^J H(j,j)$$

is the equalizer of

$$\begin{tikzcd} \int^J H(j,j) \arrow[r, dashed] & \prod_{j \in Ob\, J} H(j,j) \arrow[r, shift left, "\phi^*"] \arrow[r, shift right, "\phi_*"'] & \prod_{f \in Arr\, J} H(Dom\, f, Cod\, f). \end{tikzcd}$$

for ϕ_* and ϕ^* as in (2.4.1) and (2.4.2).

For a similar functor to a cocomplete category $\mathcal{C}$, the **coend**

$$\int_J H(j,j)$$

is the coequalizer of

$$\begin{tikzcd} \coprod_{f \in Arr\, J} H(Cod\, f, Dom\, f) \arrow[r, shift left, "\varphi^*"] \arrow[r, shift right, "\varphi_*"'] & \coprod_{j \in Ob\, J} H(j,j) \arrow[r, dashed] & \int_J H(j,j), \end{tikzcd} \tag{2.4.6}$$

with φ_* and φ^* as in (2.4.4).

In both cases the "variable of integration" j appears twice in the "integrand" and could be replaced by any other symbol for an object in J.

Alternatively, for each morphism $f : j \to j'$ in J, we have a diagram in $\mathcal{C}$,

$$\begin{tikzcd} H(j',j) \arrow[r, "f^*"] \arrow[d, "f_*"'] & H(j,j) \arrow[d, "f_*"] \\ H(j',j') \arrow[r, "f^*"] & H(j,j'). \end{tikzcd}$$

Suppose for the moment that $\mathcal{C}$ is bicomplete. For a fixed pair of objects (j,j') in J we could combine the above for all morphisms $j \to j'$ and get

$$\begin{tikzcd} \coprod_{J(j,j')} H(j',j) \arrow[r, "\varphi^*"] \arrow[d, "\varphi_*"'] & H(j,j) \arrow[d, "\phi_*"] \\ H(j',j') \arrow[r, "\phi^*"] & \prod_{J(j,j')} H(j,j'). \end{tikzcd} \tag{2.4.7}$$

For cocomplete $\mathcal{C}$ this leads to a coequalizer diagram

$$\coprod_{f\in \mathrm{Arr}\,J} H(j',j) \underset{\varphi_*}{\overset{\varphi^*}{\rightrightarrows}} \coprod_{k\in \mathrm{Ob}\,J} H(k,k) \dashrightarrow \int_J H(k,k),$$

and for complete $\mathcal{C}$ we have an equalizer diagram

$$\int^J H(k,k) \dashrightarrow \prod_{k\in \mathrm{Ob}\,J} H(k,k) \underset{\phi_*}{\overset{\phi^*}{\rightrightarrows}} \prod_{f\in \mathrm{Arr}\,J} H(j,j').$$

Proposition 2.4.8. Ends and coends on the walking arrow category. *Let J be walking arrow category $(0 \to 1)$ as in Definition 2.1.6, let $\mathcal{C}$ be a cocomplete category and let : $H : J^{op} \times J \to \mathcal{C}$ be a functor. Then*

$$\int_J H(j,j) \cong H(0,0) \underset{H(1,0)}{\amalg} H(1,1),$$

the pushout of the diagram

$$\begin{tikzcd} & H(1,0) \arrow[dl, "\alpha^*"'] \arrow[dr, "\alpha_*"] & \\ H(0,0) & & H(1,1), \end{tikzcd} \tag{2.4.9}$$

where $\alpha : 0 \to 1$ denotes the unique nonidentity morphism in J.

Dually, for complete $\mathcal{C}$,

$$\int^J J(c,c) \cong H(0,0) \underset{H(0,1)}{\times} H(1,1),$$

the pullback of the diagram

$$\begin{tikzcd} H(0,0) \arrow[dr, "\alpha_*"'] & & H(1,1) \arrow[dl, "\alpha^*"] \\ & H(0,1). & \end{tikzcd}$$

Proof The diagram of (2.4.6) is

$$\begin{tikzcd} H(0,0) \amalg H(1,0) \amalg H(1,1) \arrow[d, shift right, "\varphi_*"'] \arrow[d, shift left, "\varphi^*"] \\ H(0,0) \amalg H(1,1) \arrow[d, dashed] \\ \displaystyle\int_J H(j,j). \end{tikzcd}$$

The restrictions of both φ^* and φ_* to $H(0,0)$ send it identically to $H(0,0)$, and similarly for their restrictions to $H_{1,1}$. This means that they contribute nothing to the coend, which is therefore the pushout of (2.4.9).

The dual case is similar. □

For a related result, see Proposition 2.4.18 below.

The following are immediate consequences of the definitions.

Proposition 2.4.10. Functoriality of ends and coends. *Given two functors $H, H' : J^{op} \times J \to \mathcal{C}$, a natural transformation $\theta : H \Rightarrow H'$ induces morphisms*

$$\int_J \theta : \int_J H \to \int_J H' \qquad \text{and} \qquad \int^J \theta : \int^J H \to \int^J H'$$

with composition of natural transformations inducing composition of such morphisms.

Proposition 2.4.11. Limits (colimits) as ends (coends). *When the functor H is constant on the first variable, then its end (coend) is the usual limit (colimit) of H as a functor of the second variable for complete (cocomplete) $\mathcal{C}$.*

Remark 2.4.12. Ends (coends) as limits (colimits). *Every end (coend) is a limit (colimit) since it an equalizer (coequalizer) by definition. The statement at hand concerns the case when an end (coend) over a small category J is also an ordinary limit (colimit) over J.*

Proof This follows from the definitions and the calculation of Example 2.3.35(iii). □

Given a functor $H : J^{op} \times J \to \mathcal{C}$ and objects X and Y in $\mathcal{C}$, there are $\mathcal{S}et$-valued functors on $J^{op} \times J$,

$$J^{op} \times J \xrightarrow{t} J \times J^{op} \xrightarrow{H^{op}} \mathcal{C}^{op} \xrightarrow{\mathcal{C}(-,Y)} \mathcal{S}et \tag{2.4.13}$$

and

$$J \times J^{op} \xrightarrow{H} \mathcal{C} \xrightarrow{\mathcal{C}(X,-)} \mathcal{S}et. \tag{2.4.14}$$

The following is immediate from the definitions.

Proposition 2.4.15. End/coend duality. *Given a functor H from $J^{op} \times J$ (for a small category J) to a cocomplete category $\mathcal{C}$, and an object Y in $\mathcal{C}$, there is a natural isomorphism*

$$\mathcal{C}\left(\int_J H, Y\right) \cong \int^J \mathcal{C}(H, Y),$$

where the expression on the left is the set of morphisms from the indicated coend to Y, and the expression on the right is the end of the $\mathcal{S}et$-valued functor of (2.4.13).

For an object X in $\mathcal{C}$, there is a natural isomorphism

$$\mathcal{C}\left(X, \int^J H\right) \cong \int^J \mathcal{C}(X, H),$$

where the expression on the left is the set of morphisms from X to the indicated end, and that on the right is the end for the functor of (2.4.14).

An enriched version of the above is Proposition 3.2.16 below.

There is a converse to Proposition 2.4.11. It is taken from [ML98, IX.5] where it is stated for ends and limits. We will construct a new small category $J_\S$ (Mac Lane's notation for the opposite category is $J^\S$) such that the coend of Definition 2.4.5 is the colimit of a certain $\mathcal{C}$-valued functor on $J_\S$.

Definition 2.4.16. The cosubdivision category of a small category. *For a small category J, let $J_\S$ be the category whose objects are symbols $j_\S$ and $f_\S$ for objects j and arrows f in J. Note that $j_\S$ and $(1_j)_\S$ are different objects. The only nonidentity morphisms are arrows*

$$j_\S \leftarrow f_\S \rightarrow j'_\S$$

for each arrow $f : j \to j'$ in J.

Given a functor $H : J^{op} \times J \to \mathcal{C}$, let $H_\S : J_\S \to \mathcal{C}$ be the functor indicated by the following diagram.

$$\begin{tikzcd}
j_\S \arrow[d, mapsto] & & f_\S \arrow[ll] \arrow[rr] \arrow[d, mapsto] \arrow[dl, phantom, "\mapsto"] \arrow[dr, phantom, "\mapsto"] & & j'_\S \arrow[d, mapsto] \\
H(j,j) & & H(j',j) \arrow[ll, "f^*"] \arrow[rr, "f_*"'] & & H(j',j')
\end{tikzcd}$$

Dually, let the **subdivision category of** J *be* $J^\S = (J_\S)^{op}$*. We denote the corresponding objects in it by* $j^\S$ *and* $f^\S$*, and the only nonidentity morphisms are arrows*

$$j^\S \to f^\S \leftarrow (j')^\S$$

for each arrow $f : j \to j'$ *in* J*. The functor* $H^\S : J^\S \to \mathcal{C}$ *is indicated by*

$$\begin{tikzcd} j^\S \arrow[rr] \arrow[d, mapsto] & {} \arrow[d, mapsto] & f^\S \arrow[d, mapsto] & {} \arrow[d, mapsto] & (j')^\S \arrow[ll] \arrow[d, mapsto] \\ H(j,j) \arrow[rr, "f_*"'] & {} & H(j,j') & {} & H(j',j'). \arrow[ll, "f^*"] \end{tikzcd}$$

The following is stated for coends only. Its proof and that of its dual can be found in [ML98, IX.8]

Proposition 2.4.17. Fubini theorem for coends. *Let*

$$H : J_1^{op} \times J_2^{op} \times J_1 \times J_2 \to \mathcal{C}$$

for small categories J_1 *and* J_2 *and a cocomplete category* $\mathcal{C}$*. Then for any pair* $(a, b) \in J_1^{op} \times J_1$*, we have the functor*

$$H(a, -, b, -) : J_2^{op} \times J_2 \to \mathcal{C},$$

and its coend

$$\int_{J_2} H(a, c, b, c)$$

is a functor on $J_1^{op} \times J_1$*, so the double coend*

$$\int_{J_1} \int_{J_2} H(a, c, a, c)$$

is defined. Similarly we can define the double coend

$$\int_{J_2} \int_{J_1} H(a, c, a, c).$$

We can also define the coend on the product category

$$\int_{J_1 \times J_2} H(a, c, a, c).$$

These three objects in $\mathcal{C}$ *are naturally isomorphic.*

Note that if the functor H above is constant on the contravariant variables, then Proposition 2.4.17 reduces to the statement that colimits over different diagrams commute with each other. The corresponding result about ends reduces to the commuting of limits.

The following is the double coend version of Proposition 2.4.8.

Proposition 2.4.18. Double coends on the walking arrow category. *Let J_1 and J_2 each be the walking arrow category $J = (0 \to 1)$ of Definition 2.1.6, and let*

$$H : J_1^{op} \times J_2^{op} \times J_1 \times J_2 \to \mathcal{C}$$

be a functor to a cocomplete category $\mathcal{C}$.

For each $(a, b) \in J^{op} \times J$, let

$$P(a,b) = \int_{c \in J} H(a,c,b,c),$$

which was identified as a certain pushout in Proposition 2.4.8. Then

$$\int_{J \times J} H(a,c,a,c) \cong \int_J P(a,a) \cong P(0,0) \underset{P(1,0)}{\amalg} P(1,1).$$

The following are special cases.

(i) When the value of H is nontrivial (meaning not equal to $\varnothing$) only when both contravariant variables are 0, then the double coend is $H(0,0,0,0)$.
(ii) When the value of H is trivial when both contravariant variables are 1, then the double coend is the pushout of the diagram

$$\begin{tikzcd}
 & H(0,1,0,0) \arrow[dl, "H(0,1,0,\alpha)"'] \arrow[dr, "H(0,\alpha,0,0)"] & & H(1,0,0,0) \arrow[dl, "H(\alpha,0,0,0)"'] \arrow[dr, "H(1,0,\alpha,0)"] & \\
H(0,1,0,1) & & H(0,0,0,0) & & H(1,0,1,0).
\end{tikzcd}$$

(iii) When the functor H is independent of the contravariant variables, then the double coend is $H(-,-,1,1,)$.

Proof Using Proposition 2.4.17, we have

$$\begin{aligned}
\int_{(a,c) \in J \times J} H(a,c,a,c) &\cong \int_{a \in J} \int_{c \in J} H(a,c,a,c) \\
&\cong \int_{a \in J} P(a,a) \\
&\cong P(0,0) \underset{P(1,0)}{\amalg} P(1,1).
\end{aligned}$$

For (i),

$$\begin{aligned}
\int_J H(0,0,b,0) &= H(0,0,0,0) \underset{H(0,0,1,0)}{\amalg} \varnothing \\
&= H(0,0,0,0)
\end{aligned}$$

so

$$\int_{J \times J} H(a,c,a,c) = H(0,0,0,0).$$

For (ii), since

$$P(a,b) = H(a,0,b,0) \underset{H(a,1,b,0)}{\amalg} H(a,1,b,1),$$

we have

$$\begin{aligned}
P(1,1) &= H(1,0,1,0) \underset{H(1,1,1,0)}{\amalg} H(1,1,1,1) \\
&= H(1,0,1,0) \underset{\varnothing}{\amalg} \varnothing = H(1,0,1,0), \\
P(1,0) &= H(1,0,0,0) \underset{H(1,1,0,0)}{\amalg} H(1,1,0,1) = H(1,0,0,0) \\
\text{and} \qquad P(0,0) &= H(0,0,0,0) \underset{H(0,1,0,0)}{\amalg} H(0,1,0,1) \\
&= H(0,1,0,1) \underset{H(0,1,0,0)}{\amalg} H(0,0,0,0).
\end{aligned}$$

It follows that the double coend is

$$\begin{aligned}
&P(0,0) \underset{P(1,0)}{\amalg} P(1,1) \\
&\qquad = \left(H(0,1,0,1) \underset{H(0,1,0,0)}{\amalg} H(0,0,0,0)\right) \underset{H(1,0,0,0)}{\amalg} H(1,0,1,0) \\
&\qquad = H(0,1,0,1) \underset{H(0,1,0,0)}{\amalg} H(0,0,0,0) \underset{H(1,0,0,0)}{\amalg} H(1,0,1,0),
\end{aligned}$$

which is the indicated pushout.

For (iii), when the functor H of Proposition 2.4.17 is independent of the contravariant variables, the coend is an ordinary colimit by Proposition 2.4.11. Since $J \times J$ has terminal object $(1,1)$, the coend in this case is $H(-,-,1,1,)$. □

Proposition 2.4.19. The set of natural transformations as an end. *Suppose we have two functors $F, G : J \to \mathcal{E}$ where J is small and $\mathcal{E}$ is complete. Let $H : J^{op} \times J \to \mathcal{S}et$ be*

$$H(C, C') = \mathcal{E}(F(C), G(C')).$$

Then the end

$$\int^J H(C,C) = \int^J \mathcal{E}(F(C), G(C))$$

is the set of natural transformations from F to G,

$$Nat(F,G) = [J, \mathcal{E}](F,G).$$

Proof By Definition 2.4.5 the end is the equalizer of two morphisms from the product

$$\prod_{X \in J} \mathcal{E}(F(X), G(X)).$$

A natural transformation $\theta : F \to G$ assigns to each object X of J a morphism $\theta_X \in \mathcal{E}(F(X), G(X))$, so θ defines an element in the same product. The requirement that the diagrams (2.2.2) all commute is equivalent to requiring this element to be in the equalizer. $\square$

When $\mathcal{E} = \mathcal{S}et$ and $F = よ^A$, Proposition 2.4.19 reads

$$\int^{B\in J} \mathcal{S}et(よ^A(B), G(B)) = \int^{B\in J} \mathcal{S}et(J(A,B), G(B)) = \mathrm{Nat}(よ^A, G).$$

The right hand side is $G(A)$ by the Yoneda Lemma 2.2.10, so we have the following.

Proposition 2.4.20. The Yoneda reduction. *Let J be a small category and $F : J \to \mathcal{S}et$. Then for each object A of J,*

$$\int^{B\in J} \mathcal{S}et(J(A,B), F(B)) \cong F(A).$$

Now

$$\mathcal{S}et(J(A,B), F(B)) = F(B)^{J(A,B)},$$

the Cartesian power of the set $F(B)$ indexed by the set $J(A,B)$. The right hand side is defined more generally for a functor F with valued in a complete category $\mathcal{E}$, and Proposition 2.4.20 has the following generalization.

Proposition 2.4.21. The generalized Yoneda reduction. *Let $F : J \to \mathcal{E}$ be a functor from a small category J to a complete category $\mathcal{E}$. Then for each object A of J,*

$$\int^{B\in J} F(B)^{J(A,B)} \cong F(A).$$

Proof For each $f \in J(A,B)$ we get a map $F(f) : F(A) \to F(B)$. Collecting these for all f gives an evaluation map

$$i_B : F(A) \to F(B)^{J(A,B)}. \tag{2.4.22}$$

Collecting these for all objects B in the small category J defines a map

$$i : F(A) \longrightarrow \prod_{B\in J} F(B)^{J(A,B)}.$$

The end in question also supports a morphism to this product. It is by Definition 2.4.5 the equalizer of

$$\prod_{B\in \mathrm{Ob}\, J} F(B)^{J(A,B)} \overset{\phi^*}{\underset{\phi_*}{\rightrightarrows}} \prod_{h:B\to B'} F(B')^{J(A,B)}.$$

The equalizer is $F(A)$ because for each morphism $h : B \to B'$ in J, the following diagram commutes:

$$\begin{tikzcd}
 & F(B')^{J(A,B')} \arrow[dr, "F(B')^{h*}"] & \\
F(A) \arrow[ur, "i_{B'}"] \arrow[dr, "i_B"'] & & F(B')^{J(A,B)}. \\
 & F(B)^{J(A,B)} \arrow[ur, "F(h)^{J(A,B)}"'] &
\end{tikzcd} \tag{2.4.23}$$

□

There is a dual formula for coends, which is sometimes called the **co-Yoneda lemma**. We will formulate and prove it simultaneously by dualizing the proof of Proposition 2.4.21.

For a $\mathcal{S}et$-valued functor F, map i_B of (2.4.22) is adjoint to

$$j_A : J(A, B) \times F(A) \to F(B).$$

The Cartesian product on the left, the disjoint union of copies of $F(A)$ indexed by the set $J(A, B)$, is defined whenever F takes values in a **cocomplete** category $\mathcal{E}$. We can take the coproduct of such things over all objects A of J and get a map

$$j : \coprod_{A \in J} J(A, B) \times F(A) \to F(B).$$

Then for each morphism $g : A' \to A$ in J, following diagram, which is dual to (2.4.23), commutes:

$$\begin{tikzcd}
 & J(A', B) \times F(A') \arrow[dl, "j_{A'}"'] & \\
F(B) & & J(A, B) \times F(A') \arrow[ul, "g^* \times F(A')"'] \arrow[dl, "J(A,B) \times F(g)"] \\
 & J(A, B) \times F(A) \arrow[ul, "j_A"] &
\end{tikzcd}$$

This means that $F(B)$ can be described as a coend, and we have proved the following.

Proposition 2.4.24. The generalized Yoneda coreduction. *Let $F : J \to \mathcal{E}$ be a functor from small category J to a cocomplete category $\mathcal{E}$. Then for each object B of J,*

$$\int^{A \in J} J(A, B) \times F(A) \cong F(B).$$

We will describe another approach to this for $\mathcal{S}et$-valued functors below in Example 2.5.14.

Remark 2.4.25. The case of a bicomplete category $\mathcal{E}$. *By interchanging A and B, we can rewrite Proposition 2.4.21 as*

$$F(B) \cong \int^{A \in J} F(A)^{J(B,A)},$$

while Proposition 2.4.24 gives

$$F(B) \cong \int_{A \in J} J(A,B) \times F(A).$$

Note that the first formula for $F(B)$ involves $J(B,A)$ while the second involves $J(A,B)$. It has to be this way because both expressions must be covariant in B, which $J(A,B)$ is. The expression in the end is contravariant in $J(B,A)$, which itself is contravariant in B.

2.5 Kan extensions

> The notion of Kan extensions subsumes all the other fundamental concepts of category theory.
>
> *Saunders Mac Lane, [ML98, X.7]*

2.5A Definitions and examples

Suppose we have functors F and K as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \eta"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dashed, "L"'] & \end{tikzcd} \tag{2.5.1}$$

and we wish to extend the functor F along K to a new functor $L : \mathcal{D} \to \mathcal{E}$ with a natural transformation $\eta : F \Rightarrow LK$. We do **not** require L to be an actual extension of F, meaing we do not require that $LK = F$, We only require that the two be related by the natural transformation $\eta : F \Rightarrow LK$. We want it to have the following universal property: given another such extension G and natural transformation $\gamma : F \Rightarrow GK$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \gamma"] & \mathcal{E} \\ & \mathcal{D}, \arrow[ur, "G"'] & \end{tikzcd}$$

there is a unique natural transformation $\lambda : L \Rightarrow G$ with $\gamma = (\lambda K)\eta$. If such an L exists, it is unique and is called the **left Kan extension of F along K**. We will denote it from now on by $(Lan_K F, \eta)$ or simply $Lan_K F$.

One can also define the **right Kan extension of F along** K, $(Ran_K F, \epsilon)$ in a similar way with the direction of the natural transformations (but not the functors) reversed. We will see in §2.5B that for small $\mathcal{C}$ and cocomplete (complete) $\mathcal{E}$, $Lan_K F$ $(Ran_K F)$ exists and can be expressed as a certain coend (end).

Equivalently, K induces a precomposition functor (natural transformation)

$$\mathcal{E}^{\mathcal{D}} \xrightarrow{K^*} \mathcal{E}^{\mathcal{C}} \tag{2.5.2}$$

for which we are seeking left and right adjoints Lan_K and Ran_K that could be applied to any F.

Given a right Kan extension as above, one can ask if it is preserved by a functor out of $\mathcal{E}$, that is (in the right case), given a functor $G : \mathcal{E} \to \mathcal{F}$, if the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{E} \arrow[r, "G"] & \mathcal{F} \\
& \mathcal{D} \arrow[ur, dashed, "Ran_K F"'] \arrow[urr, dashed, "Ran_K GF"'] \arrow[u, phantom, "\Uparrow \epsilon"] & &
\end{tikzcd}$$

where the natural transformation associated with $Ran_K GF$ is $G_*\epsilon$ as in Proposition 2.2.3. The following definition is taken from [Rie14, 1.3.4].

Definition 2.5.3. Pointwise Kan extensions. *For locally small $\mathcal{E}$ the right Kan extension $Ran_K F$ is a* **pointwise right Kan extension** *if it is preserved by all representable functors $\text{よ}^e : \mathcal{E} \to \mathcal{S}et$, namely the functors given by $e' \mapsto \mathcal{E}(e, e')$ for some object e of $\mathcal{E}$. Dually, the left Kan extension $Lan_K F$ is a* **pointwise left Kan extension** *if it is preserved by all corepresentable functors $\text{よ}_{e'} : \mathcal{E} \to \mathcal{S}et^{op}$, namely the functors given by $e \mapsto \mathcal{E}(e, e')$ for some object e' of $\mathcal{E}$.*

The following is an immediate consequence of the definitions.

Proposition 2.5.4. Kan extensions as adjoints to precomposition. *The left (right) Kan extension Lan_K (Ran_K) induces a functor*

$$K_! : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}} \qquad (K_* : \mathcal{E}^{\mathcal{C}} \to \mathcal{E}^{\mathcal{D}})$$

which is the left (right) adjoint of the precomposition functor K^* of (2.5.2).

Some authors (for example [Lur17, Construction 6.1.6.4]) denote the right adjoint above by K_*. We prefer to use that notation for a covariant functor induced by K, such as $K_* : \mathcal{C}^J \to \mathcal{D}^J$.

All of the Kan extensions we will consider in this book are pointwise Kan extensions. The name comes from the fact that they can be computed on any element d of $\mathcal{D}$ as the limit or colimit of the functor

$$(d{\downarrow}\mathcal{C}) \xrightarrow{\ U\ } \mathcal{C} \xrightarrow{\ F\ } \mathcal{E}, \tag{2.5.5}$$

where $(d{\downarrow}\mathcal{C})$ is as in Definition 2.1.51 and U is the codomain functor. This means the objects of $(d{\downarrow}\mathcal{C})$ are pairs (f, c) with c an object in $\mathcal{C}$ and $f : K(c) \to d$ a morphism in $\mathcal{D}$. The functor FU sends such an object to $F(c)$. The following converse was proved by Mac Lane as [ML98, X.5.3].

Theorem 2.5.6. Pointwise Kan extensions as limits or colimits. *The right (left) Kan extension of F along K is a pointwise Kan extension iff its value on each object d of $\mathcal{D}$ is the limit (colimit) of the functor FU of (2.5.5), in which case, in particular, this limit (colimit) exists.*

In the following examples we will often refer to the diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \eta"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dashed, "Lan_K F"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow \epsilon"] & \mathcal{E} \\ & \mathcal{D}. \arrow[ur, dashed, "Ran_K F"'] & \end{tikzcd} \tag{2.5.7}$$

Example 2.5.8. Some Kan extensions.

(i) *Suppose that $\mathcal{E} = \mathcal{C}$ and F is the identity functor $1_{\mathcal{C}}$. Then its left (right) Kan extension along K is the left adjoint L (right adjoint R) of K with unit η (counit ϵ).*

(ii) *Suppose $\mathcal{C}$ is small and $\mathcal{D}$ is the terminal category, meaning it has just one object and one morphism. Hence the right and left Kan extension of F along the constant functor K are choices of an object in $\mathcal{E}$ with suitable properties. They are the limit and colimit of the diagram in $\mathcal{E}$ defined by F.*

(iii) *For each object d let $\mathcal{C}_d = K^{-1}d$, the subcategory of $\mathcal{C}$ whose objects map to d under K, and let $F_d = F|\mathcal{C}_d$, the restriction of F to that subcategory. Then the value of the left (right) Kan extension on d is the colimit (limit) of F_d.*

(iv) *Let $\mathcal{E}$ be the category of vector spaces over a field k. It is bicomplete. Let $\mathcal{D}$ and $\mathcal{C}$ be the one object categories corresponding to a group G and a subgroup H, and let $K : \mathcal{C} \to \mathcal{D}$ be the inclusion functor. Then a functor $V : \mathcal{C} \to \mathcal{E}$ is a representation of H over k. Equivalently it is a $k[H]$-module, where $k[H]$ denotes the group ring of H over k. Its left Kan extension $Lan_K V$ is*

the induced representation

$$\mathrm{Ind}_H^G V := k[G] \otimes_{k[H]} V.$$

Its right Kan extension is known as the coinduced representation of G, defined to be the $k[G]$-module

$$\mathrm{Coind}_H^G V := \mathrm{Hom}_{k[H]}(k[G], V).$$

(v) *Let $\phi : \tilde{G} \to G$ be a surjective group homomorphism with kernel N, and let $\mathcal{C}$ be a cocomplete category. The ϕ induces a precomposition functor $\phi^* : \mathcal{C}^G \to \mathcal{C}^{\tilde{G}}$. Its left adjoint $i_! : \mathcal{C}^{\tilde{G}} \to \mathcal{C}^G$ sends a $\tilde{G}$-object X to its orbit object X/N with the residual G-action. To see this let Y denote a G-object. Then the adjunction means that there is an isomorphism*

$$\mathcal{C}^{\tilde{G}}(X, \phi^* Y) \cong \mathcal{C}^G(i_! X, Y) \tag{2.5.9}$$

The subgroup N acts trivially on $\phi^* Y$, so any $\tilde{G}$-equivariant map to it from X factors uniquely through the orbit object X/N, which means that the latter is $i_! X$.

In particular if G is trivial and $\tilde{G} = N$, then the functor ϕ^* is the functor Δ of Example 2.2.30(iii), which endows an object in $\mathcal{C}$ with the trivial N-action, and (2.5.9) is the orbit adjunction.

2.5B A formula for Kan extensions

Before giving the formula for a Kan extension below, we offer the following.

Example 2.5.10. A cautionary toy example. *Suppose $\mathcal{E}$ is the category with two objects a and b and two nonidentity morphisms $a \rightrightarrows b$. It is neither complete nor cocomplete because there is no equalizer or coequalizer for the pair of morphisms, and it has no initial or terminal object.*

Let $\mathcal{C}$ be the empty category and let $\mathcal{D}$ be the trivial category, meaning it has one object with only an identity morphism. Thus functors from $\mathcal{C}$ to $\mathcal{E}$ and natural transformations between them are vacuous, while a functor $\mathcal{D} \to \mathcal{E}$ amounts to a choice of object in $\mathcal{E}$. This means that a left (right) Kan extension is an initial (terminal) object in $\mathcal{E}$. Since $\mathcal{E}$ has neither, **the two Kan extensions do not exist**.

We now give a formula for the left (right) Kan extension for small $\mathcal{C}$ and cocomplete (complete) $\mathcal{E}$ as a coend (end). In the former case for each object d in $\mathcal{D}$ the formula is (see [ML98, X.4.1-2])

$$Lan_K F(d) = \int_{\mathcal{C}} \mathcal{D}(K(c), d) \cdot F(c). \tag{2.5.11}$$

Some explanation is in order. Part of the "integrand" is $\mathcal{D}(K(c), d)$, the set of morphisms in $\mathcal{D}$ from $K(c)$ to d. As a set valued functor it is contravariant in c. This set gets "multiplied" by the object $F(c)$ in the cocomplete category $\mathcal{E}$. This means we take the coproduct of $F(c)$ with itself indexed by the set $\mathcal{D}(K(c), d)$. This is a tensor product in the sense of Example 3.1.49 below. When the set is empty, this coproduct is the terminal object of $\mathcal{E}$. The integrand is thus the $\mathcal{E}$-valued functor

$$\mathcal{D}(K(-), d) \cdot F(-)$$

evaluated on the object (c, c) of $\mathcal{C}^{op} \times \mathcal{C}$. Hence we are describing $Lan_K F(d)$ as a certain colimit in $\mathcal{E}$, namely that of the functor (2.5.5).

Similarly for small $\mathcal{C}$ and complete $\mathcal{E}$ the formula for the right Kan extension is

$$Ran_K F(d) = \int^{\mathcal{C}} F(c)^{\mathcal{D}(d, K(c))}. \tag{2.5.12}$$

Here "multiplication" is the product in $\mathcal{E}$. We take the product of the object $F(c)$ with itself indexed by the set $\mathcal{D}(d, K(c))$, which we write as $F(c)^{\mathcal{D}(d,K(c))}$ as in Definition 3.1.31 below. This end is the limit in $\mathcal{E}$ of the functor FU of (2.5.5).

We will give enriched analogs of (2.5.11) and (2.5.12) below in Proposition 3.2.35.

Example 2.5.13. Adjoints, limits and colimits. *Applying this formula to the first two cases of Example 2.5.8, we find that*

$$\begin{aligned} L(d) &= \int_{\mathcal{C}} \mathcal{D}(K(c), d) \cdot c \\ R(d) &= \int^{\mathcal{C}} \mathcal{D}(K(c), d) \cdot c \\ colim\, F &= \int_{\mathcal{C}} F(c) \\ lim\, F &= \int^{\mathcal{C}} F(c). \end{aligned}$$

Example 2.5.14. The Yoneda reduction and coreduction again. *In the diagrams of (2.5.7), let $\mathcal{D} = \mathcal{C}$ with $K = 1_{\mathcal{C}}$, and let $\mathcal{E} = \mathcal{S}et$. Then $Ran_{1_{\mathcal{C}}} F = F$, so (2.5.12) reads*

$$F(d) = \int^{\mathcal{C}} \mathcal{S}et(\mathcal{C}(d, c), F(c)) = \int^{\mathcal{C}} \mathcal{S}et(よ^d(c), F(c)) = Nat(よ^d, F)$$

Thus we recover the Yoneda Reduction of Proposition 2.4.20.

We also have $Lan_{1_{\mathcal{C}}} F = F$, so (2.5.11) reads

$$F(d) = \int_{\mathcal{C}} \mathcal{C}(c, d) \times F(c).$$

This is the Yoneda coreduction of Proposition 2.4.24.

The following can be found in [MMSS01, Proposition 3.2, proved in §23] and in [Kel82, Proposition 4.23].

Proposition 2.5.15. Kan extensions along fully faithful functors. *Let $\alpha : K \to J$ be a fully faithful functor between small categories, and let $\mathcal{C}$ be a cocomplete (complete) category. Let $\alpha^* : \mathcal{C}^J \to \mathcal{C}^K$ be the precomposition functor as in (2.5.2), and let $\alpha_! : \mathcal{C}^K \to \mathcal{C}^J$ ($\alpha_* : \mathcal{C}^K \to \mathcal{C}^J$) be the functor induced by left (right) Kan extension as in Proposition 2.5.4. Then the unit $\eta : 1_{\mathcal{C}^K} \Rightarrow \alpha^*\alpha_!$ (counit $\epsilon : \alpha^*\alpha_* \Rightarrow 1_{\mathcal{C}^K}$) of the adjunction of Proposition 2.5.4 is a natural equivalence.*

More precisely, for any object k in K and any functor $F : K \to \mathcal{C}$, we have

$$(\alpha^*\alpha_!F)_k \cong F_k \qquad ((\alpha^*\alpha_*F)_k \cong F_k)$$

with the unit (counit) of the adjunction inducing the identity map. Thus $\mathcal{C}^K$ is a full coreflective (reflective) subcategory of $\mathcal{C}^J$ as in Definition 2.2.49.

If in addition $\mathcal{C}$ is bicomplete, then right (left) adjoint α^* is also a left (right) adjoint, namely that of the right (left) Kan extension, and $\mathcal{C}^K$ is a bireflective subcategory of $\mathcal{C}^J$ as in Definition 2.2.50.

We will use this in the proof of Theorem 5.4.21 below.

Proof We will prove this in the cocomplete case, the proof of the dual statement being similar. Given an object in $\mathcal{C}^K$, meaning a functor $F : K \to \mathcal{C}$, and an object j in J, we have

$$(\alpha_!F)_j = \int_{k\in K} J(\alpha(k), j) \cdot F_k \qquad \text{by (2.5.11).}$$

It follows that for an object k' in K,

$$\begin{aligned}(\alpha^*\alpha_!F)_{k'} &= \int_{k\in K} J(\alpha(k), \alpha(k')) \cdot F_k \\ &\cong \int_{k\in K} K(k, k') \cdot F_k && \text{since } \alpha \text{ is fully faithful} \\ &= F_{k'} && \text{by Proposition 2.4.24,}\end{aligned}$$

so the functors $\alpha^*\alpha_!F$ and F are naturally isomorphic. □

2.6 Monoidal and symmetric monoidal categories

2.6A Basic definitions

A monoidal category is a monoid object in the category of categories. More explicitly, we have the following, which is taken from [ML98, VII.1].

The symbol $\square$ below is meant to denote a generic binary operation. Hence it could be replaced by symbols such as $\otimes$, $\oplus$, $\wedge$ and $\vee$, which could refer to specific binary operations in certain categories. We will also use the symbol $\Box$ for a specific binary operation in Definition 2.6.12 below.

Definition 2.6.1. *A category $\mathcal{C}$ is* **monoidal** *if it has a binary operation $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called a* **monoidal structure***) and a unit object* $\mathbf{1}$ *with natural isomorphisms*

$$(X \Box Y) \Box Z \xrightarrow[\cong]{a_{X,Y,Z}} X \Box (Y \Box Z) \ ,$$

$$X \Box \mathbf{1} \xrightarrow[\cong]{\rho_X} X \ , \quad \text{and} \quad \mathbf{1} \Box X \xrightarrow[\cong]{\lambda_X} X$$

for all objects X, Y and Z, called the **associator**, **right unitor** *and* **left unitor**. *They are required to satisfy the following coherence conditions:*

(i) The isomorphisms $\lambda_{\mathbf{1}}$ and $\rho_{\mathbf{1}}$ from $\mathbf{1} \Box \mathbf{1}$ to $\mathbf{1}$ are the same.
(ii) For all objects X and Y the following diagram commutes.

$$\begin{tikzcd}
X \Box (\mathbf{1} \Box Y) \arrow[rr, "a_{X,\mathbf{1},Y}"] \arrow[dr, "X \Box \lambda_Y"'] & & (X \Box \mathbf{1}) \Box Y \arrow[dl, "\rho_X \Box Y"] \\
& X \Box Y &
\end{tikzcd}$$

(iii) For all objects W, X, Y and Z the following diagram (the **Stasheff pentagon***) of isomorphisms commutes. Here we omit the symbol $\Box$ in object names to save space.*

$$\begin{tikzcd}
& (WX)(YZ) \arrow[dr, "a_{WX,Y,Z}"] & \\
W(X(YZ)) \arrow[ur, "a_{W,X,YZ}"] \arrow[d, "W \Box a_{X,Y,Z}"'] & & ((WX)Y)Z \\
W((XY)Z) \arrow[r, "a_{W,XY,Z}"] & (W(XY))Z \arrow[ur, "a_{W,X,Y} \Box Z"'] &
\end{tikzcd}$$

While the isomorphisms and coherence diagrams are part of the structure, they are typically omitted from the notation, the monoidal category in question being denoted by $(\mathcal{C}, \Box, \mathbf{1})$. A monoidal category that is also complete (cocomplete) is said to be **Cartesian** *(***coCartesian***) if $\Box$ is the categorical product (coproduct) and $\mathbf{1}$ is the terminal (initial) object.*

The category $\mathcal{C}$ is **symmetric monoidal** *if it also has a natural isomorphism*

$$X \Box Y \xrightarrow[\cong]{\tau_{X,Y}} Y \Box X, \tag{2.6.2}$$

the **twist isomorphism**, *with* $\tau_{Y,X} = (\tau_{X,Y})^{-1}$ *such that the following diagrams commute for all* X, Y *and* Z*: the* **triangle identity**

$$\begin{tikzcd} \mathbf{1} \,\Box\, X \arrow[rr, "\tau_{\mathbf{1},X}"] \arrow[dr, "\lambda_X"'] & & X \,\Box\, \mathbf{1} \arrow[dl, "\rho_X"] \\ & X & \end{tikzcd}$$

and the **first hexagon identity**

$$\begin{tikzcd} (X \,\Box\, Y) \,\Box\, Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "\tau_{X,Y} \,\Box\, Z"'] & X \,\Box\, (Y \,\Box\, Z) \arrow[r, "\tau_{X,Y \,\Box\, Z}"] & (Y \,\Box\, Z) \,\Box\, X \arrow[d, "a_{Y,Z,X}"] \\ (Y \,\Box\, X) \,\Box\, Z \arrow[r, "a_{Y,X,Z}"] & Y \,\Box\, (X \,\Box\, Z) \arrow[r, "Y \,\Box\, \tau_{X,Z}"] & Y \,\Box\, (Z \,\Box\, X). \end{tikzcd} \tag{2.6.3}$$

We will often omit the subscripts of τ.

Equivalently there is a natural transformation τ *between the two functors*

$$\begin{array}{c} (X,Y) \longmapsto X \,\Box\, Y \\ \mathcal{C} \times \mathcal{C} \rightrightarrows \mathcal{C} \\ (X,Y) \longmapsto Y \,\Box\, X \end{array}$$

with suitable properties.

There is a weaker notion of a **braided** monoidal category in which there is a twist isomorphism, but $\tau_{Y,X}\tau_{X,Y}$ is not required to be the identity on $X \,\Box\, Y$. Its definition requires a **second hexagon identity**, namely (2.6.3) with X and Y reversed.

Definition 2.6.4. *A monoidal category* $\mathcal{C}$ *as in Definition 2.6.1, symmetric or not, is* **strict** *(or strictly monoidal) if the isomorphisms* $a_{X,Y,Z}$, ρ_X *and* λ_X *are identity morphisms in all cases.*

In a strict symmetric monoidal category, we do **not** requite the twist isomorphism $\tau_{X,Y}$ of (2.6.2) to be an identity morphism.

Remark 2.6.5. Natural equivalences. *The natural isomorphisms* $a_{X,Y,Z}$, ρ_X, λ_X *and* $\tau_{X,Y}$ *in the above definitions are components of natural equivalences between the relevant functors. We leave this formulation as an exercise for the reader.*

Definition 2.6.6. Addition functors and morphisms. *Let* $(\mathcal{C}, \oplus, \mathbf{0})$ *be a symmetric monoidal category, so* $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *is a functor with two covariant variables. By setting one of them equal to an object* A *in* $\mathcal{C}$ *we get two* **addition functors**

$$\alpha_A = A \oplus (-) : \mathcal{C} \to \mathcal{C} \text{ given by } X \mapsto A \oplus X$$

$$\text{and} \qquad \omega_A = (-) \oplus A : \mathcal{C} \to \mathcal{C} \text{ given by } X \mapsto X \oplus A.$$

These functors are naturally isomorphic to each other but not identical. When the objects $A \oplus X$ and $X \oplus A$ are not merely isomorphic, but equal, the two functors induce the same maps on objects but not necessarily on morphisms.

For each pair of objects X and Y in $\mathcal{C}$, these functors induce an **addition morphisms** *of morphism sets*

$$\begin{aligned} &\alpha_{A,X,Y} : \mathcal{C}(X,Y) \to \mathcal{C}(A \oplus X, A \oplus Y) \\ \text{and} \qquad &\omega_{A,X,Y} : \mathcal{C}(X,Y) \to \mathcal{C}(X \oplus A, Y \oplus A) \end{aligned}$$

given by $f \mapsto \alpha_A(f)$ and $f \mapsto \omega_A(f)$ respectively.

The following is an immediate consequence of this definition.

Proposition 2.6.7. Relations between α and ω. *Let $\mathcal{C}$ as in Definition 2.6.6, and let U, V and W be any three objects in $\mathcal{C}$. Then the following diagrams of sets commute.*

$$\begin{tikzcd} \mathcal{C}(\mathbf{0}, W) \arrow[r, "\omega_{U,\mathbf{0},W}"] \arrow[d, "\alpha_{V,\mathbf{0},W}"'] & \mathcal{C}(\mathbf{0} \oplus U, W \oplus U) \arrow[d, "\alpha_{V,\mathbf{0}\oplus U, W\oplus U}"] \\ \mathcal{C}(V \oplus \mathbf{0}, V \oplus W) \arrow[r, "\omega_{U,V\oplus\mathbf{0},V\oplus W}"] & \mathcal{C}(V \oplus \mathbf{0} \oplus U, V \oplus W \oplus U) \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small] \mathcal{C}(\mathbf{0}, W) \times \mathcal{C}(U, V \oplus U) \arrow[rr, "\tau"] \arrow[d, "\omega_{V\oplus U,\mathbf{0},W} \times \alpha_{\mathbf{0},U,V\oplus U}"'] & & \mathcal{C}(U, V \oplus U) \times \mathcal{C}(\mathbf{0}, W) \arrow[d, "\alpha_{W,U,V\oplus U} \times \omega_{U,\mathbf{0},W}"] \\ \begin{array}{c} \mathcal{C}(\mathbf{0} \oplus V \oplus U, W \oplus V \oplus U) \\ \times \\ \mathcal{C}(\mathbf{0} \oplus U, \mathbf{0} \oplus V \oplus U) \end{array} \arrow[dr, "j'"'] & & \begin{array}{c} \mathcal{C}(W \oplus U, W \oplus V \oplus U) \\ \times \\ \mathcal{C}(\mathbf{0} \oplus U, W \oplus U) \end{array} \arrow[dl, "j''"] \\ & \mathcal{C}(\mathbf{0} \oplus U, W \oplus V \oplus U), & \end{tikzcd}$$

where τ permutes the two factors, and j' and j'' are composition morphisms.

Proof Let $f : \mathbf{0} \to W$ be a morphism in $\mathcal{C}$. Chasing it around the first diagram, we have

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & f \oplus U \arrow[d, mapsto] \\ V \oplus f \arrow[r, mapsto] & V \oplus f \oplus U \end{tikzcd}$$

Now let $g : U \to V \oplus U$ be a second morphism in $\mathcal{C}$. The chasing (f, g) around the second diagram gives

$$\begin{tikzcd}
(f,g) \arrow[rr, mapsto] \arrow[d, mapsto] & & (g,f) \arrow[d, mapsto] \\
(f \oplus V \oplus U, \mathbf{0} \oplus g) \arrow[dr, mapsto] & & (W \oplus g, f \oplus U) \arrow[dl, mapsto] \\
& f \oplus g. &
\end{tikzcd}$$

□

Proposition 2.6.7 also holds in the enriched case after suitable reinterpretation. See Remark 3.1.52 below.

Remark 2.6.8. Isomorphic objects need not be equal. *Note that we have taken care* **not** *to equate $A \oplus B$ with $B \oplus A$ and A with either $A \oplus 0$ or $0 \oplus A$. In a general symmetric monoidal category they are naturally isomorphic but not equal. When working in a strict symmetric monoidal category as in Definition 2.6.4, we may identity $A \oplus \mathbf{0}$ and $\mathbf{0} \oplus A$ with A. See Remark 7.2.13 below.*

Definition 2.6.9. Ideals in a symmetric monoidal category. *Let $(\mathcal{C}, \oplus, \mathbf{0})$ be a symmetric monoidal category. An* **ideal** *in $\mathcal{C}$ is a full subcategory $\mathcal{D}$ such that for any object c in $\mathcal{C}$ and d in $\mathcal{D}$, $c \oplus d$ is also in $\mathcal{D}$. The* **principal ideal** *$\mathcal{C}d$ generated by an object d in $\mathcal{C}$ consists of all objects of the isomorphic to $c \oplus d$ for some object c.*

Without symmetry, one could speak of left, right and two sided ideals, but we will not need such notions here.

Example 2.6.10. The category $[\mathcal{C}, \mathcal{C}]$ of endofunctors of a category $\mathcal{C}$ *is the category whose objects are endofunctors $E : \mathcal{C} \to \mathcal{C}$ and whose morphisms are natural transformations between such functors. It is monoidal (but not symmetric monoidal) under composition with the identity functor as unit.*

Proposition 2.6.11. A coend reduction for small monoidal categories. *Let $(\mathcal{D}, \oplus, \mathbf{0})$ be a small monoidal category. Then for any two objects X and Y of $\mathcal{D}$,*

$$\int_{\mathcal{D}} \mathcal{D}(W \oplus X, Y) \times \mathcal{D}(\mathbf{0}, W) \cong \mathcal{D}(X, Y).$$

Proof The argument is similar to to that of Proposition 2.4.24. Given

$$(f, g) \in \mathcal{D}(W \oplus X, Y) \times \mathcal{D}(\mathbf{0}, W),$$

we have

$$X = \mathbf{0} \oplus X \xrightarrow{g \oplus X} W \oplus X \xrightarrow{f} Y.$$

Hence for each object W in $\mathcal{D}$ we have a map

$$\mathcal{D}(W \oplus X, Y) \times \mathcal{D}(\mathbf{0}, W) \to \mathcal{D}(X, Y)$$

and therefore a map

$$\coprod_{W \in \mathcal{D}} \mathcal{D}(W \oplus X, Y) \times \mathcal{D}(\mathbf{0}, W) \to \mathcal{D}(X, Y).$$

To show that the target is the desired coequalizer it suffices to observe that for every morphism $\beta : W \to W'$, $f' : W' \oplus X \to Y$ and $g : \mathbf{0} \to W$ in $\mathcal{D}$, the following diagram commutes.

$$\begin{tikzcd}
 & & (f'(\beta \oplus X), g) \arrow[ddrr, mapsto, bend left] & & \\
 & & \begin{array}{c}\mathcal{D}(W \oplus X, Y)\\ \times\mathcal{D}(\mathbf{0}, W)\end{array} \arrow[dr] & & \\
(f', g) \arrow[uurr, mapsto, bend left] \arrow[ddrr, mapsto, bend right] & \begin{array}{c}\mathcal{D}(W' \oplus X, Y)\\ \times\mathcal{D}(\mathbf{0}, W)\end{array} \arrow[ur, "(\beta\oplus X)^{*}\times\mathcal{D}(\mathbf{0},W)"] \arrow[dr, "\mathcal{D}(W'\oplus X,Y)\times\beta_{*}"'] & & \mathcal{D}(X, Y) & \begin{array}{c}f'(\beta \oplus X)(g \oplus X)\\ = f'(\beta g \oplus X)\end{array} \\
 & & \begin{array}{c}\mathcal{D}(W' \oplus X, Y)\\ \times\mathcal{D}(\mathbf{0}, W')\end{array} \arrow[ur] & & \\
 & & (f', \beta g) \arrow[uurr, mapsto, bend right] & &
\end{tikzcd}$$

□

The following should be compared with Definition 2.3.9.

Definition 2.6.12. Pushout and pullback corner maps. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ categories with a functor $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ where $\mathcal{E}$ is cocomplete. In particular we could have $\mathcal{C} = \mathcal{D} = \mathcal{E}$, a monoidal category with pushouts.*

(i) Let $f : A \to B$ and $g : X \to Y$ be morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively. Consider the diagram in $\mathcal{E}$

$$\begin{tikzcd}
& A\otimes Y \arrow[dr] \arrow[drr, "f\otimes Y"] & & \\
A\otimes X \arrow[ur, "A\otimes g"] \arrow[dr, "f\otimes X"'] & \arrow[r, phantom, "\rangle"] & P \arrow[r, "f\,\square\, g"] & B\otimes Y, \\
& B\otimes X \arrow[ur] \arrow[urr, "B\otimes g"'] & &
\end{tikzcd}$$

where P is the pushout of $A\otimes g$ and $f\otimes X$, and $f\square g$ is the canonical map from it to $B\otimes Y$. Then $f\,\square\,g$ is the **pushout corner map formed by tensoring f and g**, *also known as the* **pushout product of f with g**.

(ii) *Given sets of maps $\mathcal{I}$ in $\mathcal{C}$ and $\mathcal{I}'$ in $\mathcal{D}$, define a set of maps $\mathcal{I}\,\square\,\mathcal{I}'$ in $\mathcal{E}$ by*

$$\mathcal{I}\,\square\,\mathcal{I}' = \left\{f\,\square\,g\colon f\in\mathcal{I}, g\in\mathcal{I}'\right\}. \tag{2.6.13}$$

Compare this with the corner map of Definition 2.3.57.

(iii) *More generally suppose we have an n-fold product functor*

$$\bigotimes : \mathcal{C}_1\times\mathcal{C}_2\cdots\times\mathcal{C}_n\to\mathcal{E}$$

and morphisms $f_i : A_i\to B_i$ in $\mathcal{C}_i$ for $1\leqslant i\leqslant n$. Tensoring these maps together leads to a functor $F:\mathscr{P}(\mathbf{n})\to\mathcal{E}$ for $\mathscr{P}(\mathbf{n})$ the poset category of Proposition 2.3.53. When $\mathcal{E}$ is cocomplete, we get a canonical map from the n-fold pushout $\operatorname*{colim}_{\mathscr{P}_1(\mathbf{n})} G$ (which is described as a simple pushout in n different ways for $n>2$ in Proposition 2.3.53) to the tensor product of all the B_i. This is the **n-fold pushout corner map** *denoted by $f_1\,\square\cdots\square\,f_n$. Again compare this with Definition 2.3.57.*

When the maps f_i are all the same map $f : A\to B$, we denote this map by $f^{\square n}$, the **nth pushout power of f**. *The symmetric group Σ_n acts on both its domain and codomain, and the map is equivariant. We denote the induced map on orbit objects by $f^{\square n}/\Sigma_n$.*

(iv) *Dually, suppose that $\mathcal{E}$ is complete and consider the diagram*

$$\begin{tikzcd}
 & & A \otimes Y \arrow[dr, "f\otimes Y"] & \\
A \otimes X \arrow[urr, "A\otimes g"] \arrow[r, "f \diamond g"] \arrow[drr, "f\otimes X"'] & R \arrow[ur] \arrow[dr] \arrow[r, phantom, "\langle"] & & B \otimes Y, \\
 & & B \otimes X \arrow[ur, "B\otimes g"'] &
\end{tikzcd}$$

where R is the pullback of $B \otimes g$ and $f \otimes Y$, and $f \diamond g$ is the canonical map to it to $A \otimes X$. Then $f \diamond g$, the **pullback corner map formed by tensoring f and g**, *also known as the* **pullback product of f with g**.

(v) *When $\mathcal{E}$ is complete, there is a canonical map to the n-fold pullback*

$$\lim_{\mathscr{P}_0(\mathbf{n})} G$$

(which is described as a simple pullback in n different ways for $n > 2$ in Proposition 2.3.53) from the tensor product of all the A_i. This is the **n-fold pullback corner map** *denoted by $f_1 \diamond \cdots \diamond f_n$.*

Example 2.6.14. The pushout (pullback) product with the map from (to) the initial (terminal) object, and with an identity map. *Suppose that the category $\mathcal{C}$ in Definition 2.6.12 has an initial object $\varnothing$ and that f is the map $\varnothing \to B$. Then then the map $f \square g$ is the map $B \otimes g$. Dually, if $\mathcal{C}$ has a terminal object $*$ and f is the map $A \to *$, then $f \diamond g = A \otimes g$.*

If $f = 1_B$ is an identity morphism, and $g : X \to Y$, then $f \square g = 1_{B\otimes Y}$ and $f \diamond g = 1_{B\otimes X}$.

Definition 2.6.15. Pushout corner maps and horns. *In $\mathcal{T}op$, let $\mathcal{I}'$ in (2.6.13) consist of a single map $f : A \to B$, and let*

$$\mathcal{I} = \left\{ i_n : S^{n-1} = \partial D^n \to D^n : n \geqslant 0 \right\}$$

When f is a closed inclusion, $i_n \square f$ is the map

$$D^n \times A \cup_{S^{n-1}\times A} S^{n-1} \times B \to D^n \times B.$$

In $\mathcal{T}$ (the pointed analog), let $\mathcal{I}'$ in (2.6.13) consist of a single pointed map $f : (A, a_0) \to (B, b_0)$, and let

$$\mathcal{I}_+ = \left\{ i_{n+} : S_+^{n-1} \to D_+^n : n \geqslant 0 \right\}$$

where X_+ denotes the space X with a disjoint base point. When f is a closed pointed inclusion, $i_{n+} \square f$ is the map

$$D^n \ltimes A \cup_{S^{n-1}\ltimes A} S^{n-1} \ltimes B \to D^n \ltimes B,$$

where $X \ltimes Y$ is the left half smash product of Definition 2.1.49. Note here that for a pointed space (Y, y_0),

$$X \ltimes Y = (X \times Y)/(X \times \{y_0\}), \tag{2.6.16}$$

where the base point on the right is the image of $X \times \{y_0\}$.

In both cases the set $\mathcal{I} \,\square\, \{f\}$ is called **the set of horns on** f*, denoted by $\Lambda\{f\}$, in [Hir03, Definition 1.3.2].*

See Definition 6.3.8 below for a generalization of the above.

Example 2.6.17. The pushout corner formed by the Cartesian product of two boundary inclusions. *For $\mathcal{C} = \mathcal{D} = \mathcal{E} = \mathcal{T}op$ with $\otimes$ the Cartesian product, let $i_n : S^{n-1} \to D^n$ be the inclusion of the boundary. Then $i_m \,\square\, i_n = i_{m+n}$.*

More generally if M and N are manifolds with boundaries, and with inclusion maps $i_M : \partial M \to M$ and $i_N : \partial N \to N$, then $i_M \,\square\, i_N = i_{M\times N} : \partial(M \times N) \to M \times N$.

Compare this with Example 2.3.58.

Remark 2.6.18. Associativity of the pushout product. *The map $f \square g$ is called* **the pushout product** *of f and g in [Hov99, 4.2.1]. When $\mathcal{C} = \mathcal{D} = \mathcal{E}$ is a cocomplete monoidal category, $\square$ is a binary operation on the category $Arr\mathcal{C}$ of morphisms (and commutative squares) in $\mathcal{C}$. It gives $Arr\mathcal{C}$ itself a monoidal structure in which the unit object is the morphism from the initial object to the unit object of $\mathcal{C}$. We will say more about it below in § 2.6F. Similarly when $\mathcal{C} = \mathcal{D} = \mathcal{E}$ is a complete monoidal category, $\diamond$ is a binary operation on the category $Arr\mathcal{C}$ of morphisms in $\mathcal{C}$ with similar properties.*

We will use this definition below in Proposition 3.1.53, Definition 5.5.3, Definition 5.5.9 and Definition 6.3.8. We invite the reader to generalize it further, replacing the n variable functor $\otimes$ by one which is covariant in some of the variables and contravariant in others. The map of Definition 2.3.14 is an example where $\mathcal{C} = \mathcal{D}$, $\mathcal{E} = \mathcal{S}et$, and the mixed functor $\mathcal{C} \times \mathcal{C} \to \mathcal{E}$ is the morphism set $\mathcal{C}(-,-)$, which is contravariant in the first variable and covariant in the second.

2.6B Functors between monoidal categories

Definition 2.6.19. *Let $(\mathcal{C}, \oplus, \mathbf{0})$ and $(\mathcal{D}, \otimes, \mathbf{1})$ be (symmetric) monoidal categories. A* **lax (symmetric) monoidal functor**

$$F : (\mathcal{C}, \oplus, \mathbf{0}) \to (\mathcal{D}, \otimes, \mathbf{1})$$

is a functor F equipped with a natural transformation

$$\mu : F(-) \otimes F(-) \Rightarrow F(- \oplus -)$$

of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$ *and a morphism* $\iota : F(\mathbf{0}) \to \mathbf{1}$ *in* $\mathcal{D}$ *satisfying the following conditions:*

- *For all objects* X, Y, Z *in* $\mathcal{C}$*, the following diagram commutes in* $\mathcal{D}$*.*

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[rr, "a_{F(X),F(Y),F(Z)}"] \arrow[d, "\mu_{X,Y} \otimes F(Z)"'] & & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "F(X) \otimes \mu_{Y,Z}"] \\
F(X \oplus Y) \otimes F(Z) \arrow[d, "\mu_{X \oplus Y, Z}"'] & & F(X) \otimes F(Y \oplus Z) \arrow[d, "\mu_{X, Y \oplus Z}"] \\
F((X \oplus Y) \oplus Z) \arrow[rr, "F(a_{X,Y,Z})"'] & & F(X \oplus (Y \oplus Z))
\end{tikzcd}$$

- *For each object* X *of* $\mathcal{C}$*, the following diagrams commute in* $\mathcal{D}$*.*

$$\begin{tikzcd}
F(X) \otimes \mathbf{1} \arrow[r, "\rho_{\mathcal{D}}"] \arrow[d, "F(X) \otimes \iota"'] & F(X) \\
F(X) \otimes F(\mathbf{0}) \arrow[r, "\mu_{X,\mathbf{0}}"'] & F(X \otimes \mathbf{0}) \arrow[u, "F(\rho_{\mathcal{C}})"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{1} \otimes F(X) \arrow[r, "\lambda_{\mathcal{D}}"] \arrow[d, "\iota \otimes F(X)"'] & F(X) \\
F(\mathbf{0}) \otimes F(X) \arrow[r, "\mu_{\mathbf{0},X}"'] & F(\mathbf{0} \otimes X). \arrow[u, "F(\lambda_{\mathcal{C}})"']
\end{tikzcd}$$

- *In the symmetric case the following diagram commutes for all objects* X *and* Y *in* $\mathcal{C}$*.*

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[rr, "\tau_{F(X),F(Y)}"] \arrow[d, "\mu_{X,Y}"'] & & F(Y) \otimes F(X) \arrow[d, "\mu_{Y,X}"] \\
F(X \oplus Y) \arrow[rr, "F(\tau_{X,Y})"] & & F(Y \oplus X).
\end{tikzcd}$$

F *is* **oplax** *if the arrows* μ *and* ι *are reversed and the coherence diagrams modified accordingly.*

Definition 2.6.20. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *as in Definition 2.6.19 is* **strong (symmetric) monoidal** *if* ι *is an isomorphism and* $\mu_{X,Y}$ *is a natural isomorphism.* F *is* **strictly (symmetric) monoidal***, or simply* **(symmetric) monoidal***, if* ι *and* $\mu_{X,Y}$ *are identity morphisms. Then we say that* $\mathcal{D}$ **is a** $\mathcal{C}$**-algebra***.*

Recall that a ring homomorphism $R \to S$ makes S into an R-algebra.

Note that a strong monoidal functor is both lax and oplax. Some authors use the term "lax comonoidal" instead of "oplax monoidal."

Proposition 2.6.21. Monoidal adjoints. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be (symmetric) monoidal categories and let*

$$F : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : G$$

be a pair of adjoint functors. Then F *is oplax (symmetric) monoidal iff* G *is* *lax (symmetric) monoidal. In particular G is lax monoidal when F is strong monoidal, and F is oplax monoidal when G is strong monoidal.*

Proof Suppose G is lax monoidal, so for each pair of objects D_1 and D_2 in $\mathcal{D}$ we have a morphism

$$\mu : G(D_1) \oplus G(D_2) \to G(D_1 \otimes D_2)$$

in $\mathcal{C}$, which is adjoint to a morphism

$$\mu' : F(G(D_1) \oplus G(D_2)) \to D_1 \otimes D_2$$

in $\mathcal{D}$. Now suppose each D_i is $F(C_i)$ for an object C_i in $\mathcal{C}$, so we have

$$\mu' : F(GF(C_1) \oplus GF(C_2)) \to F(C_1) \otimes F(C_2).$$

We can precompose this with $F(\eta(-) \oplus \eta(-))$, where η is the unit of the adjunction as in Definition 2.2.20.

Thus we get a morphism

$$\mu'' : F(C_1 \oplus C_2) \to F(C_1) \otimes F(C_2).$$

This is an instance of the natural transformation required for F to be an oplax monoidal functor. We leave the remaining details to the reader.

A dual argument shows that G is lax monoidal when F is oplax monoidal. □

For more discussion of the following, see [JK02].

Definition 2.6.22. A module category over a monoidal category $\mathcal{V}$ *or* **$\mathcal{V}$-module** *is a category $\mathcal{C}$ with a strong monoidal functor (Definition 2.6.19) $A : \mathcal{V} \to [\mathcal{C}, \mathcal{C}]$, where $[\mathcal{C}, \mathcal{C}]$ is the endofunctor category of Example 2.6.10. Given objects V and C in $\mathcal{V}$ and $\mathcal{C}$, we get an object $VC := A(V)(C)$, and hence an action of $\mathcal{V}$ on $\mathcal{C}$.*

Example 2.6.23. Cocomplete categories as $\mathcal{S}et$-modules. *A cocomplete category $\mathcal{C}$ as in Definition 2.3.25 is a module over $(\mathcal{S}et, \times, *)$, i.e., for an object C in $\mathcal{C}$ and a set V, we can make sense of $A \times C$ and there is an endofunctor A of $\mathcal{C}$ given by $A(C) = V \times C$.*

Remark 2.6.24. The word "module." *This is our first use to the word "module" to denote something other than an object M in a monoidal category with a map $R \otimes M \to M$ or $M \otimes R \to M$ where R is a "ring" or "monoid," an object in the same category equipped with a map $R \otimes R \to R$; see Definition 2.6.58 and Example 3.1.67 below.*

Here a module is itself a category with suitable properties in relation to another category with a monoidal structure of its own. The word will be used in a similar way below in Definition 2.6.42, where we define a closed $\mathcal{V}$-module, *which has more structure than a $\mathcal{V}$-module as above. We will define a corresponding notion (Quillen modules over a Quillen ring) for model categories in Definition 5.5.17 below.*

The following is straightforward.

Proposition 2.6.25. Functors into a (symmetric) monoidal category. *Let J be a small category and $(\mathcal{C}, \otimes, \mathbf{1})$ a (symmetric) monoidal categry.*

(i) The functor category $\mathcal{C}^J$ is also a (symmetric) monoidal category in which the product operation is defined objectwise and the unit is the constant $\mathbf{1}$-valued functor on J.
(ii) A functor $F : J \to K$ to a second small category K induces a functor $F^* : \mathcal{C}^K \to \mathcal{C}^J$ which is (symmetric) monoidal (Definition 2.6.19).
(iii) $\mathcal{C}^J$ is tensored over $\mathcal{C}$ (see Definition 3.1.31 below), meaning that for a functor $H : J \to \mathcal{C}$ and an object X in $\mathcal{C}$, we can define a functor $H \otimes X$ by $(H \otimes X)(j) = H(j) \otimes X$.

When J is also symmetric monoidal and enriched over the symmetric monoidal category $\mathcal{C}$, the functor category has an additional symmetric monoidal structure called the Day convolution, which is the subject of §3.3.

2.6C Two variable adjunctions

The following is taken from [Hov99, Definition 4.1.12] and was originally given in [Kan58a, §4.3]. See also [Shu06, Definition 14.2].

Definition 2.6.26. *For categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, a* **two variable adjunction** *is a quintuple $(\otimes, \mathrm{Hom}_\ell, \mathrm{Hom}_r, \varphi_\ell, \varphi_r)$, where*

$$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}, \quad \mathrm{Hom}_\ell : \mathcal{C}^{op} \times \mathcal{E} \to \mathcal{D} \quad \text{and} \quad \mathrm{Hom}_r : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{C}$$

are functors, and φ_ℓ and φ_r are natural isomorphisms

$$\mathcal{D}(D, \mathrm{Hom}_\ell(C, E)) \xleftarrow[\cong]{\varphi_\ell} \mathcal{E}(C \otimes D, E) \xrightarrow[\cong]{\varphi_r} \mathcal{C}(C, \mathrm{Hom}_r(D, E))$$

for objects C, D and E in the categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ respectively. We will sometimes drop the isomorphisms from the notation and write it as

$$(\otimes, \mathrm{Hom}_\ell, \mathrm{Hom}_r) : \mathcal{C} \times \mathcal{D} \to \mathcal{E}.$$

A closed symmetric monoidal category as in Definition 2.6.33 below is a category $\mathcal{C}$ equipped with a two variable adjunction in which $\mathcal{C} = \mathcal{D} = \mathcal{E}$.

Thus we have two ordinary adjunctions,

$$(C \otimes -) \dashv \mathrm{Hom}_\ell(C, -) \tag{2.6.27}$$

as functors between $\mathcal{D}$ and $\mathcal{E}$ for each C in $\mathcal{C}$, and

$$(-\otimes D) \dashv \mathrm{Hom}_r(D, -) \tag{2.6.28}$$

as functors between $\mathcal{C}$ and $\mathcal{E}$ for each D in $\mathcal{D}$.

For each object E in $\mathcal{E}$ we have functors

$$\mathrm{Hom}_\ell(-, E) : \mathcal{C}^{op} \to \mathcal{D} \qquad \text{and} \qquad \mathrm{Hom}_r(-, E) : \mathcal{D}^{op} \to \mathcal{C} \tag{2.6.29}$$

and therefore

$$\mathrm{Hom}_\ell^{op}(E, -) : \mathcal{C} \to \mathcal{D}^{op} \qquad \text{and} \qquad \mathrm{Hom}_r^{op}(E, -) : \mathcal{D} \to \mathcal{C}^{op}. \tag{2.6.30}$$

These lead to two more equivalent ordinary adjunctions, in addition to those of (2.6.27) and (2.6.28). We have not seen them in the literature, but we will use them below in Proposition 5.5.8, Proposition 5.5.21 and Lemma 5.8.51.

Proposition 2.6.31. Two more equivalent adjunctions. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be as in Definition 2.6.26. Then for each object E of $\mathcal{E}$, the functors of (2.6.29) and (2.6.30) form equivalent adjunctions*

$$\mathrm{Hom}_\ell^{op}(E, -) \dashv \mathrm{Hom}_r(-, E) \qquad \textit{and} \qquad \mathrm{Hom}_r^{op}(E, -) \dashv \mathrm{Hom}_\ell(-, E).$$

Proof For the first of these we have

$$\mathcal{D}^{op}(\mathrm{Hom}_\ell^{op}(E, C), D) \xrightarrow{\cong} \mathcal{D}(D, \mathrm{Hom}_\ell(C, E)) \xrightarrow{\varphi_r \varphi_\ell^{-1}} \mathcal{C}(C, \mathrm{Hom}_r(D, E)),$$

so this composite is the desired adjunction isomorphism. The second adjunction is similar. The two are equivalent by Proposition 2.2.16. □

The following result is immediate.

Proposition 2.6.32. Naturality of two variable adjunctions. *Let*

$$(\otimes, \mathrm{Hom}_\ell, \mathrm{Hom}_r, \varphi_\ell, \varphi_r)$$

be as in Definition 2.6.26, and suppose there is an adjunction

$$F : \mathcal{D}' \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : G.$$

Then $(\otimes', \mathrm{Hom}'_\ell, \mathrm{Hom}'_r, \varphi'_\ell, \varphi'_r)$ is a two variable adjunction in which $\mathcal{D}$ is replaced by $\mathcal{D}'$ and for objects C, D' and E in the three categories,

$$\begin{aligned} C \otimes' D' &:= C \otimes F(D') && \textit{in } \mathcal{E} \\ \mathrm{Hom}'_\ell(C, E) &:= G(\mathrm{Hom}_\ell(C, E)) && \textit{in } \mathcal{D}' \\ \mathrm{Hom}'_r(D', E) &:= \mathrm{Hom}_r(F(D'), E) && \textit{in } \mathcal{C} \end{aligned}$$

and the isomorphisms φ'_ℓ *and* φ'_r *are given by the diagram*

$$\begin{tikzcd}
\mathcal{D}(F(D'),\mathrm{Hom}_\ell(C,E)) \arrow[d, "\cong"'] & \mathcal{E}(C\otimes F(D'),E) \arrow[l, "\varphi_\ell"', "\cong"] \arrow[r, "\varphi_r", "\cong"'] \arrow[dd, equal] & \mathcal{C}(C,\mathrm{Hom}_r(F(D'),E)) \arrow[dd, equal] \\
\mathcal{D}'(D',G(\mathrm{Hom}_\ell(C,E))) \arrow[d, equal] & & \\
\mathcal{D}'(D',\mathrm{Hom}'_\ell(C,E)) & \mathcal{E}(C\otimes' D',E) \arrow[l, "\varphi'_\ell"', "\cong"] \arrow[r, "\varphi'_r", "\cong"'] & \mathcal{C}(C,\mathrm{Hom}'_r(D',E)).
\end{tikzcd}$$

Conversely given a two variable adjunction $(\otimes',\mathrm{Hom}'_\ell,\mathrm{Hom}'_r,\varphi'_\ell,\varphi'_r)$ *for* $\mathcal{C}$, $\mathcal{D}'$ *and* $\mathcal{E}$ *and an adjunction* $F\dashv G$ *as above, there is a two variable adjunction* $(\otimes,\mathrm{Hom}_\ell,\mathrm{Hom}_r,\varphi_\ell,\varphi_r)$ *for* $\mathcal{C}$, $\mathcal{D}$ *and* $\mathcal{E}$ *in which, for objects* C, D *and* E *in the three categories,*

$$\begin{aligned}
C\otimes D &:= C\otimes' G(D) && \text{in } \mathcal{E}\\
\mathrm{Hom}_\ell(C,E) &:= F(\mathrm{Hom}'_\ell(C,E)) && \text{in } \mathcal{D}\\
\mathrm{Hom}_r(D,E) &:= \mathrm{Hom}'_r(G(D),E) && \text{in } \mathcal{C}
\end{aligned}$$

and the isomorphisms φ_ℓ *and* φ_r *are determined by* φ'_ℓ *and* φ'_r *in a similar way.*

There are similar statements for an adjunction

$$F:\mathcal{C}' \underset{\perp}{\rightleftarrows} \mathcal{C}:G.$$

2.6D Closed monoidal categories

The simplest example of a closed symmetric monoidal category (to be defined momentarily) is the category $\mathcal{S}et$ of sets, for which the binary operation is Cartesian product. Here we have the identity

$$\mathcal{S}et(X\times Y,Z)\cong \mathcal{S}et(X,\mathcal{S}et(Y,Z)),$$

meaning that a map from the Cartesian product $X\times Y$ is the same thing a family of maps from Y parametrized by X. Equivalently the functor $(-\times Y)$ has a right adjoint, $\mathcal{S}et(Y,-)$. In a general symmetric monoidal category $(\mathcal{C},\otimes,\mathbf{1})$, the functor $-\otimes Y$ may or may not have a right adjoint.

Definition 2.6.33. *A monoidal category (symmetric or not)* $(\mathcal{C},\otimes,\mathbf{1})$ *is* **closed**, *if for each object* Y *in* $\mathcal{C}$, *the functor* $(-)\otimes Y:\mathcal{C}\to\mathcal{C}$ *has a right adjoint denoted by* $(-)^Y$ *or* $\underline{\mathcal{C}}(Y,-)$, *the* **internal Hom functor** *with*

$$\mathcal{C}(\mathbf{1},\underline{\mathcal{C}}(Y,Z))\cong\mathcal{C}(Y,Z).$$

and more generally

$$\mathcal{C}(X,\underline{\mathcal{C}}(Y,Z))\cong\mathcal{C}(X\otimes Y,Z) \tag{2.6.34}$$

with the isomorphism being natural in all three variables.

Equivalently $(\mathcal{C}, \otimes, \mathbf{1})$ *is closed if there is a two variable adjunction as in Definition 2.6.26) with* $\mathcal{C} = \mathcal{D} = \mathcal{E}$, $\mathrm{Hom}_r = \mathrm{Hom}_\ell = \underline{\mathcal{C}}$, *and* $\varphi_r = \varphi_\ell$ *is the isomorphism above with appropriate conditions on the binary operation* $\otimes$.

Remark 2.6.35. Notation for the internal Hom functor. *We will often denote the internal Hom functor by* $\mathcal{C}(-,-)$ *rather than* $\underline{\mathcal{C}}(X, Y)$. *We will use the latter notation only when we need to make a distinction between the morphism set and the morphism object. In most cases the latter will be some sort of topological space, and we will have no need to consider its underlying set.*

Some authors, such as Morten Brun, Bjørn Dundas and Martin Stolz [BDS16], denote the internal Hom functor by $\mathbf{Hom}_{\mathcal{C}}(-,-)$. *[Kel82, §1.5] denotes it by* $[-,-]$ *(where the variables are objects in* $\mathcal{C}$*), the same symbol he used for enriched functor categories, where the variables are* $\mathcal{V}$*-categories.*

The following is proved in [BDS16, Lemma 5.1.8] and by Riehl in [Rie14, Proposition 3.7.10].

Proposition 2.6.36. The adjunction relating tensor product and internal hom. *In a closed monoidal category* $(\mathcal{C}, \otimes, \mathbf{1})$ *there is a natural isomorphism*

$$\kappa_{X,Y,Z} : \underline{\mathcal{C}}(X \otimes Y, Z) \cong \underline{\mathcal{C}}(X, \underline{\mathcal{C}}(Y, Z)),$$

the **closed monoidal adjunction isomorphism**, *and for each object* Y *in* $\mathcal{C}$,

$$(-\otimes Y) \dashv \underline{\mathcal{C}}(Y, -) \tag{2.6.37}$$

as endofunctors on $\mathcal{C}$.

If in addition $\mathcal{C}$ *is symmetric, then there are also natural isomorphisms*

$$\underline{\mathcal{C}}(Y, \underline{\mathcal{C}}(X, Z)) \cong \underline{\mathcal{C}}(Y \otimes X, Z) \cong \underline{\mathcal{C}}(X \otimes Y, Z) \cong \underline{\mathcal{C}}(X, \underline{\mathcal{C}}(Y, Z)), \tag{2.6.38}$$

so

$$(Y \otimes -) \dashv \underline{\mathcal{C}}(Y, -). \tag{2.6.39}$$

We will see variants of this below in Definition 3.1.31 and Proposition 3.2.23.

Corollary 2.6.40. The unit and the internal Hom functor.

(i) For any objects Y *and* Z *in a closed symmetric monoidal category* $(\mathcal{C}, \otimes, \mathbf{1})$, *there are natural isomorphisms*

$$\underline{\mathcal{C}}(Y, Z) \cong \underline{\mathcal{C}}(\mathbf{1}, \underline{\mathcal{C}}(Y, Z)) \cong \underline{\mathcal{C}}(Y, \underline{\mathcal{C}}(\mathbf{1}, Z)).$$

(ii) Suppose we have a morphism $f : Y \to \mathbf{1}$. Then the following diagram commutes up to natural isomorphism.

$$\begin{tikzcd}
\underline{\mathcal{C}}(\mathbf{1}, \underline{\mathcal{C}}(Y,Z)) \arrow[rr, "\cong"] \arrow[dr, "f^*"'] & & \underline{\mathcal{C}}(Y, \underline{\mathcal{C}}(\mathbf{1},Z)) \arrow[dl, "\underline{\mathcal{C}}(Y,f^*)"] \\
& \underline{\mathcal{C}}(Y, \underline{\mathcal{C}}(Y,Z)). &
\end{tikzcd}$$

Proof (i) For the first of these isomorphisms we have

$$\underline{\mathcal{C}}(Y,Z) \cong \underline{\mathcal{C}}(\mathbf{1} \otimes Y, Z) \cong \underline{\mathcal{C}}(\mathbf{1}, \underline{\mathcal{C}}(Y,Z)),$$

the second isomorphism being $\kappa_{\mathbf{1},Y,Z}$. The second stated isomorphism is (2.6.38) for $X = \mathbf{1}$.

(ii) This follows from the naturality of the isomorphism of (2.6.38). □

Example 2.6.41. Some symmetric monoidal categories that are not closed. *The category of vector spaces over a field k and linear embeddings is symmetric monoidal under direct sum, but it has no internal Hom functor. The same goes for the symmetric monoidal categories $\mathscr{J}_K^{\mathbf{\Sigma}}$ of Definition 7.2.4, and $\mathscr{J}_G$ and $\mathscr{I}_G$ of Definition 8.9.24.*

The following is taken from [Shu06, Definition 14.3].

Definition 2.6.42. Closed modules over a closed symmetric monoidal category. *Let $\mathcal{V} = (\mathcal{V}_0, \square, \mathbf{1})$ be a closed symmetric monoidal category as in Definition 2.6.33. A* **closed $\mathcal{V}$-module** *$\mathcal{C}$ consists of a category $\mathcal{C}_0$ with*

(i) a two variable adjunction (Definition 2.6.26)

$$(\otimes, \{-,-\}, \mathrm{Hom}(-,-)) : \mathcal{V}_0 \times \mathcal{C}_0 \to \mathcal{C}_0,$$

where $\{X, N\}$ denotes the cotensor product N^X (see Definition 3.1.31 below) and $\mathrm{Hom}(M,N) \in \mathcal{V}_0$ for $M, N \in \mathcal{C}_0$ (the functor Hom being contravariant in the first variable),

(ii) a natural isomorphism $a : K \otimes (L \otimes M) \to (K \square L) \otimes M$ and

(iii) a natural isomorphism $\ell : \mathbf{1} \otimes M \to M$

such that the diagrams corresponding to the three conditions listed in Definition 2.6.1 each commute. In particular $\mathcal{C}$ is a module category over $\mathcal{V}$ as in Definition 2.6.22 with additional structure.

A **closed topological category** *is a closed $\mathcal{V}$-module for $\mathcal{V} = (\mathcal{T}op, \times, *)$, the category of topological spaces under Cartesian product. A* **pointed closed topological category** *is a closed $\mathcal{V}$-module for $\mathcal{V} = (\mathcal{T}, \wedge, S^0)$, the category of pointed topological spaces under smash product.*

The model category analog of the above is Definition 5.5.17 below.

Remark 2.6.43. The word "closed". *The use of that word in Definition 2.6.33 and Definition 2.6.42 is different from its use in topology. When we say "closed topological category," we are using the word in the monoidal sense rather then the topological one.*

The following should be compared with Example 2.6.23.

Example 2.6.44. Bicomplete categories as closed $\mathcal{S}et$-modules. *Let $\mathcal{V} = (\mathcal{S}et, \times, *)$ in the above definition. Then any bicomplete (Definition 2.3.25) category $\mathcal{C}_0$ has the indicated structure, where $K \otimes X$ and X^K are the coproduct and product indexed by the set K and $\mathrm{Hom}(M, N) = \mathcal{C}_0(M, N)$ is the usual set of morphisms.*

For general $\mathcal{V}$, the coherence diagrams require the Hom functor to have a "composition" morphism

$$v_{M,M',M''} : \mathrm{Hom}(M', M'') \,\square\, \mathrm{Hom}(M, M') \to \mathrm{Hom}(M, M''),$$

with the expected properties. Thus $\mathrm{Hom}(M, N)$ can be thought of as an object in $\mathcal{V}_0$ substituting for the morphism set $\mathcal{C}_0(M, N)$. This idea leads to the theory of **enriched categories**, the subject of Chapter 3. A $\mathcal{V}$-category is one equipped with such a Hom functor, but **not** necessarily with the other parts of the two variable adjunction of Definition 2.6.26. It is the subject of Definition 3.1.1. In this sense an ordinary category is a $\mathcal{S}et$-category. The other two functors of Definition 2.6.42 are the subject of Definition 3.1.31. The two variable adjunction above reappears in Proposition 3.1.47.

For more discussion about the following, see [GM11, §3].

Proposition 2.6.45. Changing symmetric monoidal categories. *Let $\mathcal{C}$ be a closed $\mathcal{V}$-module as in Definition 2.6.42. Suppose there is another closed symmetric monoidal category $\mathcal{V}'$ with an adjunction*

$$F : (\mathcal{V}'_0, \square', \mathbf{1}') = \mathcal{V}' \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{V} = (\mathcal{V}_0, \square, \mathbf{1}) : G$$

in which F is strong symmetric monoidal as in Definition 2.6.19. Then $\mathcal{C}$ is also a closed $\mathcal{V}'$-module.

Note that the roles of $\mathcal{V}$ and $\mathcal{V}'$ are **not** interchangeable here, unlike the statement of Proposition 2.6.32. The closed symmetric monoidal category $\mathcal{V}$ that we start with has to be on the right and the functor F to it has to be strong symmetric monoidal, while the functor G from it need only be lax symmetric monoidal.

Proof We use Proposition 2.6.32 to replace the two variable adjunction of Definition 2.6.42 (i) with one of the form

$$(\otimes', \mathrm{Hom}'(-\,,\,-)), \{-,-\} : \mathcal{C}_0 \times \mathcal{V}'_0 \to \mathcal{C}_0.$$

Our assumption about F means there are isomorphisms $\iota' : F(\mathbf{1}') \to \mathbf{1}$ and $\mu' : F(K') \,\square\, F(L') \to F(K' \,\square'\, L')$ in $\mathcal{V}_0$ for objects K' and L' in $\mathcal{V}_0'$.

We need to verify the existence of natural isomorphisms

$$r' : M \otimes' \mathbf{1}' \to M$$

and

$$a' : (M \otimes' K') \otimes' L' \to M \otimes' (K' \,\square'\, L')$$

in $\mathcal{C}_0$ for objects K' and L' in $\mathcal{V}_0'$ and M in $\mathcal{C}_0$.

For r' we have

$$M \otimes' \mathbf{1}' = M \otimes F(\mathbf{1}')$$

by definition, so we can define $r' = r(M \otimes \iota')$.

For a' we have

$$(M \otimes' K') \otimes' L' = (M \otimes F(K')) \otimes F(L')$$

by definition, so we can define a' to be $(M \otimes \mu')a$ since in that case

$$\begin{aligned} a'((M \otimes' K') \otimes' L') &= (M \otimes \mu')a((M \otimes F(K')) \otimes F(L')) \\ &= (M \otimes \mu')(M \otimes (F(K') \,\square\, F(L')) \\ &= M \otimes F(K' \,\square'\, L') \\ &= M \otimes' (K' \,\square'\, L'). \end{aligned} \qquad \square$$

The following was noted by Shulman as [Shu06, Proposition 14.4]. See [Hov99, Definition 4.1.14] for an alternate definition of a closed $\mathcal{V}$-functor.

Proposition 2.6.46. Closed $\mathcal{V}$-functors. *Let $\mathcal{C}$ and $\mathcal{D}$ be closed $\mathcal{V}$-modules as in Definition 2.6.42, and let $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ be an ordinary functor. Then the following are equivalent.*

(i) A $\mathcal{V}$-functor (meaning a functor between $\mathcal{V}$-modules preserving all structure in sight) $F : \mathcal{C} \to \mathcal{D}$ underlain by F_0.

(ii) A natural transformation $m : (- \otimes F_0(-)) \Rightarrow F_0(- \otimes -)$ of functors $\mathcal{V}_0 \times \mathcal{C}_0 \to \mathcal{D}_0$ that is associative and unital.

(iii) A natural transformation $n : F_0(\{-,-\}) \Rightarrow \{-, F_0(-)\}$ of functors $\mathcal{V}_0^{op} \times \mathcal{C}_0 \to \mathcal{D}_0$ that is associative and unital.

The collection of $\mathcal{C}$-modules, $\mathcal{C}$-module functors and $\mathcal{C}$-module natural transformations forms a 2-category; see §2.7.

2.6E Duality in a closed symmetric monoidal category

The material in taken from [LMSM86, Chapter III.1]. Later in the book (§8.0B and §8.0C) we will apply the ideas here to categories of spectra in which the duality is that of Spanier-Whitehead, hence the notation for the unit S and monoidal operation $\wedge$.

Before proceeding, we should warn the reader that the theory discussed here is valid in any closed symmetric monoidal category, **but it is not always interesting**. For example, let $\mathcal{C}$ be the category of topological spaces $\mathcal{T}op$ under Cartesian product. Then the unit object is a point and the dual (as in Definition 2.6.47(iii)) of any space X is a point. It follows that the only space X for which $DDX \cong X$, that is the only finite object as in Definition 2.6.54 below, is a single point.

Definition 2.6.47. Evaluation, coevaluation, duality and related maps. *Let $(\mathcal{C}, \wedge, S)$ be a closed symmetric monoidal category.*

(i) The **evaluation map** *(compare with Example 2.1.16(v)) map is the counit (see Definition 2.2.20) of the adjunction of (2.6.37), namely*

$$\epsilon_{X,Y} : \underline{\mathcal{C}}(X, Y) \wedge X \to Y.$$

(ii) The **coevaluation map** *is the unit,*

$$\eta_{X,Y} : X \to \underline{\mathcal{C}}(Y, X \wedge Y).$$

(iii) The **duality functor** *$D : \mathcal{C}^{op} \to \mathcal{C}$ is given by*

$$DX := \underline{\mathcal{C}}(X, S),$$

and DX is the **dual of** *X. We also will denote the opposite functor $D^{op} : \mathcal{C} \to \mathcal{C}^{op}$ abusively by D.*

(iv) The natural transformation $\rho : 1_{\mathcal{C}} \Rightarrow DD$ has X-component the **double dual map**

$$\rho_X : X \to \underline{\mathcal{C}}(DX, S) = DDX,$$

which is the adjoint of

$$\epsilon_{X,S}\tau_{X,DX} : X \wedge DX \to S.$$

Thus we have

$$\tilde{\epsilon}_X := \epsilon_{X,S} : DX \wedge X \to S. \tag{2.6.48}$$

Note also that

$$\begin{aligned}\underline{\mathcal{C}}(DY, DX) &= \underline{\mathcal{C}}(DY, \underline{\mathcal{C}}(X, S)) \cong \underline{\mathcal{C}}(DY \wedge X, S)\\ &\cong \underline{\mathcal{C}}(X \wedge DY, S) \cong \underline{\mathcal{C}}(Y, \underline{\mathcal{C}}(DX, S))\\ &= \underline{\mathcal{C}}(Y, DDX),\end{aligned}$$

so we have a map

$$(\rho_X)_* : \underline{\mathcal{C}}(Y, X) \to \underline{\mathcal{C}}(Y, DDX)) \cong \underline{\mathcal{C}}(DY, DX)$$

which is an isomorphism iff ρ_X is one.

Proposition 2.6.49. The maps $\epsilon_{S,X}$ and $\eta_{S,X}$ are isomorphisms inverse to each other, *namely*

$$\epsilon_{S,X} : \underline{\mathcal{C}}(S,X) \wedge S \cong \underline{\mathcal{C}}(S,X) \to X$$

and

$$\eta_{X,S} : X \to \underline{\mathcal{C}}(S, X \wedge S) \cong \underline{\mathcal{C}}(S,X).$$

Definition 2.6.50. The Cartesian product map for a closed monoidal category. *For objects X, X', Y and Y' in a closed monoidal category $\mathcal{C} = (\mathcal{C}_0, \wedge, S)$, there is a natural (in all four variables) morphism*

$$\Pi_{X,X',Y,Y'} : \underline{\mathcal{C}}(X,Y) \wedge \underline{\mathcal{C}}(X',Y') \to \underline{\mathcal{C}}(X \wedge X', Y \wedge Y')$$

generalizing that of Example 2.1.16(vii). It is a component of a natural transforamtion between two $\mathcal{C}$-valued functors on $\mathcal{C}^{op} \times \mathcal{C}^{op} \times \mathcal{C} \times \mathcal{C}$. We leave the details to the reader.

In particular there is a map from the smash product of two dual to the dual of the smash product,

$$\Pi_{X,X',S,S} : DX \wedge DX' \to D(X \wedge X').$$

The **self-duality map** $\delta_{X,Y}$ *is the composite*

$$\begin{tikzcd} DY \wedge X \arrow[r, "\delta_{X,Y}"] \arrow[d, "\tau_{DY,X}"'] & D(DX \wedge Y) \\ X \wedge DY \arrow[r, "\rho_X \wedge DY"] & DDX \wedge DY. \arrow[u, "\Pi_{DX,Y,S,S}"'] \end{tikzcd} \tag{2.6.51}$$

The **internal Hom product map** $\nu_{X,Y,Z}$ *is the composite*

$$\begin{tikzcd} \underline{\mathcal{C}}(X,Y) \wedge Z \arrow[rr, "\nu_{X,Y,Z}"] \arrow[dr, "\cong", "\underline{\mathcal{C}}(X,Y) \wedge \eta_{S,Z}"'] & & \underline{\mathcal{C}}(X, Y \wedge Z). \\ & \underline{\mathcal{C}}(X,Y) \wedge \underline{\mathcal{C}}(S,Z) \arrow[ur, "\Pi_{X,S,Y,Z}"'] & \end{tikzcd}$$

Proposition 2.6.52. The adjoint of the Cartesian product map $\Pi_{X,X',Y,Y'}$ *is the composite*

$$\begin{tikzcd} \underline{\mathcal{C}}(X,Y) \wedge \underline{\mathcal{C}}(X',Y') \wedge X \wedge X' \arrow[d, "\underline{\mathcal{C}}(X,Y) \wedge \tau_{\underline{\mathcal{C}}(X',Y'),X} \wedge X'"] \\ \underline{\mathcal{C}}(X,Y) \wedge X \wedge \underline{\mathcal{C}}(X',Y') \wedge X' \arrow[d, "\epsilon_{X,Y} \wedge \epsilon_{X',Y'}"] \\ Y \wedge Y'. \end{tikzcd}$$

Example 2.6.53. Some closed monoidal categories. *The symmetric monoidal categories* $(\mathcal{S}et, \times, *)$ *and* $(\mathcal{V}ect_k, \otimes, k)$ *are closed. In* $\mathcal{S}et$ *one can define* Z^Y *to be* $\mathcal{S}et(Y, Z)$ *since the latter is a set by definition. In* $\mathcal{V}ect_k$ *the set* $\mathcal{V}ect_k(Y, Z)$ *has a natural structure as a vector space over* k*, which we can take as the definition of* $\underline{\mathcal{V}ect_k}(Y, Z)$*.*

The duality functor D *is the usual linear dual, and we know that the map* $\rho_V : V \to DDV$ *as in Definition 2.6.47(iv) from a vector space to its double dual is an isomophism iff* V *is finite dimensional.*

We also know that for finite dimensional V *there is a map* $k \to DV \otimes V$ *whose composite with* $\epsilon_{V,k}$ *is multiplication by the dimension of* V*.*

The special properties of finite dimensional vector spaces suggests there may be similar objects in a closed symmetric monoidal category. The following is similar to [LMSM86, Definition III.1.1]. In the category of spectra, the objects that are finite in this sense are spectra that are finite in the usual sense, such as in Remark 7.1.31 below.

Definition 2.6.54. Finite objects in a closed symmetric monoidal category. *An object* X *of* $\mathcal{C}$ *is* **finite** *or* **strongly dualizable** *if there is a map* $\tilde{\eta}_X : S \to X \wedge DX$ *such that the diagram*

$$\begin{tikzcd} S \arrow[r, "\tilde{\eta}_X"] \arrow[d, "\eta_{S,X}"'] & X \wedge DX \arrow[d, "\tau_{X,DX}"] \\ \underline{\mathcal{C}}(X, X) & DX \wedge X \arrow[l, "\nu_{X,S,X}"'] \end{tikzcd}$$

commutes.

2.6F The category of arrows in a closed symmetric monoidal category

Given a bicomplete closed symmetric monoidal category $(\mathcal{C}, \wedge, S)$, its morphism category $\mathcal{C}_1$ is the same as the category of functors to $\mathcal{C}$ from the bicomplete category $(0 \to 1)$ with two objects and a single nonidentity morphism. Hence $\mathcal{C}_1$ is also bicomplete, with limits and colimits defined objectwise. We have two functors $\mathcal{C}_1 \to \mathcal{C}$ that send a morphism to its source and target,

$$\begin{tikzcd} \mathcal{C} & \mathcal{C}_1 \arrow[l, "\mathrm{Ev}_0"'] \arrow[r, "\mathrm{Ev}_1"] & \mathcal{C} \\ X_0 & (f : X_0 \to X_1) \arrow[l, mapsto] \arrow[r, mapsto] & X_1. \end{tikzcd}$$

They have left and right adjoints L_i and R_i given by

$$L_0(X) = R_1(X) = 1_X, \quad L_1(X) = (\varnothing \to X) \quad \text{and} \qquad R_0(X) = (X \to *).$$

There are at least two ways to define a closed symmetric monoidal structure on its morphism category $\mathcal{C}_1$.

Definition 2.6.55. Two closed symmetric monoidal structures on the arrow category $\mathcal{C}_1$. *For a closed symmetric monoidal category $(\mathcal{C}, \wedge, S)$, the* **tensor product monoidal structure** *on $\mathcal{C}_1$ is given by*

$$f \otimes g = f \wedge g : X_0 \wedge Y_0 \to X_1 \wedge Y_1$$

for $f : X_0 \to X_1$ and $g : Y_0 \to Y_1$. The unit is the map 1_S and the closed structure is such that $\underline{(\mathcal{C}_1)_{\otimes}}(f, g)$ is the upper horizotal arrow in the pullback diagram

$$\begin{tikzcd}
\underline{\mathcal{C}}(X_0, Y_0) \wedge_{\underline{\mathcal{C}}(X_0,Y_1)} \underline{\mathcal{C}}(X_1, Y_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \underline{\mathcal{C}}(X_1, Y_1) \arrow[d, "f^*"] \\
\underline{\mathcal{C}}(X_0, Y_0) \arrow[r, "g_*"] & \underline{\mathcal{C}}(X_0, Y_1).
\end{tikzcd} \tag{2.6.56}$$

The **pushout product monoidal structure** *on $\mathcal{C}_1$ is the operation $\Box$ of [Definition 2.6.12](). The unit is $\varnothing \to S$ and the closed structure is given by defining $\underline{(\mathcal{C}_1)_{\Box}}(f, g)$ to be the morphism*

$$\underline{\mathcal{C}}(X_1, Y_0) \to \underline{\mathcal{C}}(X_0, Y_0) \wedge_{\underline{\mathcal{C}}(X_0,Y_1)} \underline{\mathcal{C}}(X_1, Y_1).$$

associated with (2.6.56).

See Theorem 3.3.6 below for an alternate description of these two structures.

Remark 2.6.57. Internal and categorical homs.

(i) *If we replace the internal hom objects in (2.6.56) by categorical hom sets, then the pullback set $\mathcal{C}(X_0, Y_0) \underset{\mathcal{C}(X_0,Y_1)}{\times} \mathcal{C}(X_1, Y_1)$ is the set of commutative diagrams of the form*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f"'] & Y_0 \arrow[d, "g"] \\
X_1 \arrow[r] & Y_1,
\end{tikzcd}$$

which is $(\mathcal{C}_1)_{\otimes}(f, g)$ by definition.

(ii) *The object $\underline{(\mathcal{C}_1)_{\Box}}(f, g)$ is similar to the pullback corner map of Definition 2.6.12. When $\mathcal{C}$ is concrete (Definition 2.1.9), its underlying map of sets is the lifting test map $\mathcal{C}_{\diamond}(f, g)$ of Definition 2.3.14.*

One has to verify that these two structures have the required properties. The hardest part is showing that $(f \,\Box\, g) \,\Box\, h = f \,\Box\, (g \,\Box\, h)$ for $h : Z_0 \to Z_1$.

Using the notation of Proposition 2.3.53, both can be shown to be the triple corner map

$$\operatorname*{colim}_{P_3''} F \to \operatorname*{colim}_{P_3} F = X_1 \wedge Y_1 \wedge Z_1$$

where F is the cubical diagram in $\mathcal{C}$ obtained by smashing the maps f, g and h.

2.6G Monoids, modules and algebras in a (symmetric) monoidal category

The following material is discussed further in [ML98, Chapter VII].

Definition 2.6.58. *A* **monoid** *in a monoidal category* $(\mathcal{C}, \otimes, \mathbf{1})$ *is an object* R *equipped with an associative multiplication* $m : R \otimes R \to R$ *and unit* $\eta : \mathbf{1} \to R$ *with appropriate properties spelled out in [ML98, VII.3]. A left or right* R**-module** *is an object* M *equipped with a morphism* $R \otimes M \to M$ *or* $M \otimes R \to M$ *with suitable properties. A* **commutative monoid** *in a symmetric monoidal category* $(\mathcal{C}, \otimes, \mathbf{1})$ *is a monoid in which the multiplication is commutative. We denote the categories of such objects by* $\mathbf{Assoc}\,\mathcal{C}$ *and* $\mathbf{Comm}\,\mathcal{C}$*.*

The following follows immediately from the definitions.

Proposition 2.6.59. Completeness in $\mathbf{Assoc}\,\mathcal{C}$ and $\mathbf{Comm}\,\mathcal{C}$. *If the monoidal category* $(\mathcal{C}, \otimes, \mathbf{1})$ *is complete (cocomplete) and the binary operation* $\otimes$ *preserves limits (colimits) in both variables, then the categories* $\mathbf{Assoc}\,\mathcal{C}$ *and* $\mathbf{Comm}\,\mathcal{C}$ *are complete (cocomplete).*

Example 2.6.60. Some categorical monoids. *A monoid in* $(\mathcal{S}et, \times, *)$ *is an ordinary monoid. A monoid in* $(\mathcal{A}b, \otimes, \mathbf{Z})$ *is an ordinary ring. A monoid in the category of* K*-modules for a ring* K *is a* K*-algebra.*

The following is proved by Brun-Dundas-Stolz in [BDS16, Lemma 5.1.15].

Lemma 2.6.61. The category of modules over a commutative monoid R. *Let* R *be a commutative monoid in a closed symmetric monoidal category* $(\mathcal{C}, \otimes, \mathbf{1})$ *which is bicomplete. Then the category* $(\mathcal{C}_R, \otimes_R, R)$ *of* R*-modules (as in Definition 2.6.58) is also a closed symmetric monoidal category in which the unit is* R*, the product* $M \otimes_R N$ *is the coequalizer of*

$$M \otimes R \otimes N \rightrightarrows M \otimes N \to M \otimes_R N$$

where the two maps are the actions of R *on* M *and* N*, and the internal Hom functor is the equalizer of*

$$\underline{\mathcal{C}_R}(M, N) \to \underline{\mathcal{C}}(M, N) \rightrightarrows \underline{\mathcal{C}}(M \otimes R, N)$$

in which one of the two maps is induced by the action of R *on* M*. For the second, the structure map* $N \otimes R \to N$ *determines a map* $N \to \underline{\mathcal{C}}(R, N)$

under the isomorphism $\mathcal{C}(N\otimes R,N)\cong\mathcal{C}(N,\underline{\mathcal{C}}(R,N))$. *The latter gives a map* $\underline{\mathcal{C}}(M,N)\to\underline{\mathcal{C}}(M,\underline{\mathcal{C}}(R,N))$, *whose target is isomorphic by Proposition 2.6.36 to* $\underline{\mathcal{C}}(M\otimes R,N)$.

If the binary operation $\otimes$ *preserves colimits in both variables, then* $\otimes_R$ *does the same and* $\mathcal{C}_R$ *is cocomplete.*

Remark 2.6.62. Nonsymmetric products of modules. *The product* $M\otimes_R N$ *can be similarly defined in a cocomplete monoidal category* $(\mathcal{C},\otimes,\mathbf{1})$ *(without symmetry) for a right* R*-module* M *and left* R*-module* N.

Definition 2.6.63. Associative and commutative algebras. *For* R *and* $\mathcal{C}$ *as in Lemma 2.6.61, a* **(commutative)** R**-algebra** *is a (commutative) monoid (Definition 2.6.58) in* $\mathcal{C}_R$. *We denote the categories of such objects by* $\mathbf{Assoc}\,\mathcal{C}_R$ *and* $\mathbf{Comm}\,\mathcal{C}_R$. *One has forgetful functors*

$$U_R:\mathbf{Assoc}\,\mathcal{C}_R\to\mathcal{C}_R\qquad\text{and}\qquad U_R:\mathbf{Comm}\,\mathcal{C}_R\to\mathcal{C}_R,$$

the **free associative** R**-algebra functor**

$$X\mapsto T_R(X):=\coprod_{n\geqslant 0}X^{\otimes_R n}. \tag{2.6.64}$$

and the **free commutative** R**-algebra functor**

$$X\mapsto \mathrm{Sym}_R(X):=\coprod_{n\geqslant 0}\left(X^{\otimes_R n}\right)_{\Sigma_n}. \tag{2.6.65}$$

In both cases the unit is the inclusion of the 0th factor of the coproduct, and multiplication is by concatenation of the coproduct factors. We denote the n*th component of the functor of (2.6.65) by* Sym_R^n, *the* n**th symmetric product over** R.

When R *is the unit object, we drop it as a subscript in all cases.*

The following is straightforward and is stated as [BDS16, Lemma 5.1.18].

Lemma 2.6.66. Adjoint functors related to R**-modules and algebras.** *Let* R *be a monoid in the monoidal category* $(\mathcal{C},\otimes,\mathbf{1})$. *Then the left adjoint of the forgetful functor from the category left (right)* R*-modules to* $\mathcal{C}$ *is* $R\otimes(-)$ $((-)\otimes R)$ *where the action of* R *on the target is the multiplication in* R.

For a closed symmetric monoidal category $(\mathcal{C},\otimes,\mathbf{1})$ *that is also cocomplete, the functors* T *of (2.6.64) and* Sym *of (2.6.65) are left adjoints of the forgetful functors, so we have adjunctions*

$$T:\mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Assoc}\,C:U \tag{2.6.67}$$

and

$$\mathrm{Sym}:\mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Comm}\,C:U \tag{2.6.68}$$

and similarly with $\mathcal{C}$ *replaced by* $\mathcal{C}_R$.

The functors T_R *and* Sym_R *commute with sifted colimits (Definition 2.3.73).*

2.7 2-categories and beyond

The language of 2-categories gives a convenient reformulation of some of the concepts of this and subsequent chapters. A friendly introduction to this subject can be found in [Lei04].

A 2-category is a category with certain additional structure. Just as the simplest example of a category is that of sets and maps between them, the archtypical example of a 2-category is that of categories, functors and natural transformations. While it is common in category theory to assume that the collection of morphisms between two given objects is a set (even though the collection of objects is a proper class instead of a set), no such assumption is made about 2-categories. In addition to collections of objects and morphisms (categories and functors in the archtypical example), one also has a collection of 2-morphisms between morphisms having the same source and target.

These are best understood with the help of the following diagrams (which were copied from [Haz00, Bicategories and 2-categories by Ross Street]), in which upper case Roman letters denote objects, lower case ones denote morphisms (or 1-morphisms) and lower case Greek letters denote 2-morphisms. First we have

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & B \end{tikzcd}$$

where $\theta : f \implies g$ is a 2-morphism relating the 1-morphisms $f, g : A \to B$. Such 2-morphisms can be composed in two ways. The first is **vertical**, as in

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\theta"] \arrow[r, phantom, shift right=3, "\Downarrow\phi"] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\phi\theta"] & B. \end{tikzcd}$$

A 2-morphism that is invertible in this sense is called a **2-isomorphism**.

The second form a composition of 2-morphisms is **horizontal**, as in

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & B \arrow[r, bend left, "r"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\Downarrow\psi"] & C \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, bend left, "rf"] \arrow[r, bend right, "sg"'] \arrow[r, phantom, "\Downarrow\psi\circ\theta"] & C. \end{tikzcd} \tag{2.7.1}$$

Both of these compositions of 2-morphisms are required to be associative and unital, as is composition of 1-morphisms. One also has an **interchange law**,

$$(\chi\psi) \circ (\phi\theta) = (\chi \circ \phi)(\psi \circ \theta)$$

in the diagram

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, "g"'] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\theta"] \arrow[r, phantom, shift right=3, "\Downarrow\phi"] & B \arrow[r, bend left=50, "r"] \arrow[r, "s"'] \arrow[r, bend right=50, "t"'] \arrow[r, phantom, shift left=3, "\Downarrow\psi"] \arrow[r, phantom, shift right=3, "\Downarrow\chi"] & C. \end{tikzcd}$$

In a 2-category, objects, morphisms and 2-morphisms are often referred to as 0-cells, 1-cells and 2-cells respectively. The formal definition of a 2-category can be found in [Hov99, Definition 1.4.1], in [Bor94a, Definition 7.1.1] and in [Lei04, §1.4]. Leinster uses the term **strict 2-category** for the above, and the term **weak 2-category** for the bicategory of Remark 2.7.5 below.

Given objects A and B in a 2-category $\mathcal{C}$, the morphism class $\mathcal{C}(A, B)$ is itself a category in which the objects are 1-morphisms $A \to B$ in $\mathcal{C}$ and the morphisms $f \to g$ are 2-morphisms $f \implies g$ in $\mathcal{C}$. Thus we can speak of $\mathcal{C}(A, B)$ as the **hom category**.

In [EK66, page 425] 2-categories and 2-morphisms were called **hypercategories** and **hypermorpsisms**. The “closed categories” of the title were closed symmetric monoidal categories.

In the language of Definition 3.1.1 below, **a 2-category is a category enriched over** $\mathcal{CAT}$, the category of categories. In other words, the set (or possibly a proper class) of morphisms $\mathcal{C}(X, Y)$ is itself a category whose objects are the morphisms $X \to Y$ in $\mathcal{C}$, and whose morphisms are natural transformations between such morphisms in $\mathcal{C}$. See the discussion beginning on [Lur09, page 4], where we find the words

> At this point, we should object that the definition of a strict 2-category violates one of the basic philosophical principles of category theory: one should never demand that two functors F and F' be equal to one another. Instead one should postulate the existence of a natural isomorphism between F and F'.

This description begs for generalization. We could define a 3-category to be a category enriched over 2-categories, and recursively an n-category could be defined to be a category enriched over $(n-1)$-categories. At each stage one can ask for various identities to hold either strictly or up to isomorphism. Requiring strict equality seems to limit the usefulness of the definition, so various ways to weaken it have been studied. There are at least nine such definitions in the literature. A charming overview of them can be found in [CL04]. The one that may have stuck, in that it is the basis of Jacob Lurie's foundational work [Lur09], is that of Andre Joyal [Joy02]. On [Lur09, page 5] Lurie writes

> Fortunately, it turns out that major simplifications can be introduced if we are willing to restrict our attention to ∞-categories in which most of the higher morphisms are invertible.

Example 2.7.2. Some 2-categories. *For each 2-category $\mathcal{C}$ there is an underlying ordinary category $\mathcal{C}_0$ having the same objects and morphisms as $\mathcal{C}$, in which we ignore the 2-morphisms.*

(i) *A 2-category in which all 2-functors are identities is the same thing as an ordinary category.* **Ordinary categories are to 2-categories as discrete categories (Definition 2.1.7) are to categories.**

(ii) *$\mathcal{C}AT$ as in Definition 2.1.14, the category of categories, is itself a 2-category in which the objects are categories, the morphisms are functors, and the 2-morphisms are natural transformations. $\mathcal{C}at$, the category of small categories, is also a 2-category. In the language of Definition 3.1.1 below, the 2-category $\mathcal{C}AT$ ($\mathcal{C}at$) is enriched over $\mathcal{C}AT_0$ ($\mathcal{C}at_0$).*

(iii) *$\mathcal{VC}AT$ ($\mathcal{VC}at$), the category of (small) $\mathcal{V}$-categories or categories enriched over a symmetric monoidal category $\mathcal{V}$; see Chapter 3. Here the objects are enriched (small) categories, the morphisms are enriched functors, and the 2-morphisms are enriched natural transformations. It is also enriched over $\mathcal{C}AT_0$ ($\mathcal{C}at_0$), and, in some circumstances, over $\mathcal{VC}AT_0$ ($\mathcal{VC}at_0$). See Proposition 3.2.20 below.*

(iv) *The* **2-category of adjunctions** *$\mathcal{C}AT_{ad}$ ($\mathbf{Cat}_{ad}$ in [Hov99]) has categories as objects, adjunctions (F, G, φ) (see § 2.2D) as morphisms and natural transformations $\theta : F \Rightarrow F'$ as 2-morphisms $(F, G, \varphi) \Rightarrow (F', G', \varphi')$.*

(v) *The* **2-category of model categories** *$\mathcal{M}od$ ($\mathbf{Mod}$ in [Hov99]) has model categories (see Chapter 4) as objects, Quillen adjunctions (Definition 4.5.1) as morphisms and natural transformations as 2-morphisms. There are set theoretic difficulties associated with $\mathcal{M}od$ as an ordinary category which are discussed by Hovey on [Hov99, page 15].*

(vi) *The* **2-category of monoidal categories** *$\mathcal{M}on\mathcal{C}AT$ has monoidal categories (Definition 2.6.1) as objects, lax monoidal functors (Definition 2.6.19) as 1-morphisms, and natural transformations between them as 2-morphisms.*

Definition 2.7.3. Equivalence of objects in a 2-category. *An equivalence between two objects A and B in a 2-category $\mathcal{C}$ is a pair of morphisms $f : A \to B$ and $g : B \to A$ along with invertible 2-morphisms $\eta : 1_A \Rightarrow gf$ and $\epsilon : fg \Rightarrow 1_B$.*

Example 2.7.4. Various notions of equivalence.

(i) **Equivalence of objects in an ordinary category.** *If $\mathcal{C}$ is an ordinary category, meaning that all 2-morphisms are identities, then Definition 2.7.3 coincides with that of isomorphism of objects.*

(ii) **Equivalence of objects in $\mathcal{C}AT$.** *When the 2-category $\mathcal{C}$ is $\mathcal{C}AT$, the 2-category of categories, Definition 2.7.3 coincides with Definition 2.2.4.*

Remark 2.7.5. Bicategories. *There is a weaker notion of a* **bicategory** *or* **weak 2-category** *in which horizontal composition as in (2.7.1) is only associative and unital up to natural isomorphism. A formal definition is given in [Bor94a, Definition 7.7.1] and in [Lei04, Definition 1.5.1].*

Example 2.7.6. A monoidal category is a bicategory with one object. *Given a monoidal category* $(\mathcal{C}, \square, \mathbf{1})$ *as in Definition 2.6.1, there is a bicategory with a single object whose morphisms (2-morphisms) are the objects (morphisms) of* $\mathcal{C}$*. Horizontal and vertical composition of 2-morphisms correspond to the binary operation* $\square$ *on and ordinary composition of morphisms in* $\mathcal{C}$ *respectively. Thus horizontal composition is associative and unital only up to natural isomorphism, just as in Definition 2.6.1.*

Definition 2.7.7. *A* **2-functor** $F : \mathcal{C} \to \mathcal{D}$ *is a map sending objects, morphisms and 2-morphisms in the 2-category* $\mathcal{C}$ *to those in the 2-category* $\mathcal{D}$ *and preserving domains, codomains, identities and all compositions.*

Example 2.7.8. Some 2-functors.

(i) There is a **forgetful 2-functor** $\mathcal{M}od \to \mathcal{C}AT_{ad}$.
(ii) There are **duality 2-functors**

$$D : \mathcal{C}AT_{ad} \to \mathcal{C}AT_{ad} \qquad \text{and} \qquad D : \mathcal{M}od \to \mathcal{M}od$$

sending a (model) category $\mathcal{C}$ *to its opposite* $\mathcal{C}^{op}$ *and an adjunction* (F, G, φ) *to* (G, F, φ^{-1})*. In the model category case we give* $D\mathcal{C} = \mathcal{C}^{op}$ *the opposite model structure, reversing the roles of fibrations and cofibrations. The image of a natural transformation under* D *is spelled out on [Hov99, page 24].*

Proposition 2.7.9. Equivalences are preserved by 2-functors. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a 2-functor as in Definition 2.7.7. If two objects* A *and* B *in* $\mathcal{C}$ *are equivalent as in Definition 2.7.3, then* $F(A)$ *and* $F(B)$ *are equivalent in* $\mathcal{D}$.

Definition 2.7.10. *A* **weak 2-functor** *or* **pseudofunctor** $\Phi : \mathcal{C} \to \mathcal{D}$ *between bicategories (in particular between 2-categories and more particularly from an ordinary category* $\mathcal{C}$ *to a 2-category* $\mathcal{D}$*) is a map sending objects, morphisms and 2-morphisms in* $\mathcal{C}$ *to those in* $\mathcal{D}$ *and preserving domains and codomains, but not necessarily identities and compositions. More precisely, we have*

(i) for each object A *in* $\mathcal{C}$*, an object* Φ_A *in* $\mathcal{D}$*;*
(ii) for each hom category $\mathcal{C}(A, B)$ *(which is discrete as in Definition 2.1.7 when* $\mathcal{C}$ *is an ordinary category), a functor*

$$\Phi_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(\Phi_A, \Phi_B)$$

which we will abbreviate abusively by Φ *in order to save space in the diagrams below;*
(iii) for each object A *of* $\mathcal{C}$*, an invertible 2-morphism (or 2-cell)* $\Phi_{1_A} : 1_{\Phi_A} \Rightarrow \Phi(1_A)$*;*

(iv) for each composable pair of morphisms $f\colon A \to B$ and $g\colon B \to C$ in $\mathcal{C}$, a natural equivalence

$$\Phi_{A,B,C}(f,g)\colon \Phi_{B,C}(g)\Phi_{A,B}(f) \Rightarrow \Phi_{A,C}(gf),$$

the compositor of f and g, *which we will abbreviate by $\Phi(f,g)$;*

(v) for each object $f : A \to B$ in each hom category $\mathcal{C}(A,B)$, isomorphisms ρ_f, and λ_f in $\mathcal{C}$, and $\rho_{\Phi_{A,B}(f)}$, and $\lambda_{\Phi_{A,B}(f)}$ in $\mathcal{D}$ (comparable to the right and left unitors of Definition 2.6.1) for which the following diagrams commute:

$$\begin{tikzcd}
1_{\Phi_B}\Phi(f) \arrow[r, Rightarrow, "\rho_{\Phi(f)}"] \arrow[d, Rightarrow, "\Phi_{1_B}1_{\Phi(f)}"'] & \Phi(f) \\
\Phi(1_B)\Phi(f) \arrow[r, Rightarrow, "\Phi(f{,}1_B)"] & \Phi(1_B f) \arrow[u, Rightarrow, "\Phi(\rho_f)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Phi(f)1_{\Phi_A} \arrow[r, Rightarrow, "\lambda_{\Phi(f)}"] \arrow[d, Rightarrow, "1_{\Phi(f)}\Phi_{1_A}"'] & \Phi(f) \\
\Phi(f)\Phi(1_A) \arrow[r, Rightarrow, "\Phi(1_A{,}f)"] & \Phi(f\,1_A); \arrow[u, Rightarrow, "\Phi(\lambda_f)"']
\end{tikzcd}$$

(vi) for each composable triple of morphisms $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$, isomorphisms $a_{f,g,h}$ in $\mathcal{C}$ and $a_{\Phi(f),\Phi(g),\Phi(h)}$ in $\mathcal{D}$ (comparable to the associators of Definition 2.6.1) for which the following diagram commutes:

$$\begin{tikzcd}
\Phi(h)(\Phi(g)\Phi(f)) \arrow[r, Rightarrow, "a_{\Phi(f),\Phi(g),\Phi(h)}"] \arrow[d, Rightarrow, "1_{\Phi(h)}\Phi(f{,}g)"'] & (\Phi(h)\Phi(g))\Phi(f) \arrow[d, Rightarrow, "\Phi(g{,}h)1_{\Phi(f)}"] \\
\Phi(h)\Phi(gf) \arrow[d, Rightarrow, "\Phi(gf{,}h)"'] & \Phi(hg)\Phi(f) \arrow[d, Rightarrow, "\Phi(f{,}hg)"] \\
\Phi(h(gf)) \arrow[r, Rightarrow, "\Phi(a_{f,g,h})"] & \Phi((hg)f).
\end{tikzcd}$$

Some examples of pseudofunctors will be given in Chapter 4. See Example 4.5.11.

Remark 2.7.11. Strict, weak and lax functors. *In Definition 2.7.10 the 2-morphism Φ_{1_A} of (iii) is required to be invertible and the natural transformation $\Phi_{A,B,C}(f,g)$ of (iv) is required to be an equivalence. A* **lax functor** *(compare with Definition 2.6.19) is one in which these requirements are dropped. An* **oplax functor** *is one in which the direction of each is reversed with no invertibility requirement. A* **strict functor** *is one in which the two are required to be identities.*

2.8 Grothendieck fibrations and opfibrations

Grothendieck fibrations, also known as covering categories, will be a key tool in §2.9. Some of the following material can be found in [Bor94b, Chapter 8]. It is also discussed in [Rie14, Construction 7.1.9] and [Kel82, §4.7]. An enriched version of it can be found in [Tam09].

Definition 2.8.1. Grothendieck fibrations and opfibrations. *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor. An arrow $\phi : E' \to E$ in $\mathcal{E}$ is* **Cartesian** *if for any arrow $\psi : E'' \to E$ in $\mathcal{E}$ and $g : P(E'') \to P(E')$ in $\mathcal{B}$ such that $P(\phi) \circ g = P(\psi)$, there exists a unique arrow $\chi : E'' \to E'$ such that $\psi = \phi \circ \chi$ and $P(\chi) = g$. In other words, for any ψ, any morphism g in $\mathcal{B}$ of the lower part of the following diagram can be lifted up to a unique χ in $\mathcal{E}$:*

$$
\begin{tikzcd}
E'' \arrow[rr, dashed, "\exists!\chi"] \arrow[dr, "\psi"'] \arrow[dd, mapsto] & & E' \arrow[dl, "\phi"] \arrow[dd, mapsto] \\
& E \arrow[dd, mapsto] & \\
P(E'') \arrow[rr, "g"] \arrow[dr, "P(\psi)"'] & & P(E') \arrow[dl, "P(\phi)=f"] \\
& P(E) &
\end{tikzcd}
\tag{2.8.2}
$$

We say that $P : \mathcal{E} \to \mathcal{B}$ is **fibered** *or is a* **Grothendieck fibration** *or that $\mathcal{E}$* **is a covering category of** *$\mathcal{B}$, if for any object $E \in \mathcal{E}$ and arrow $f : P(E') \to P(E)$ in $\mathcal{B}$, there is a unique Cartesian arrow $\phi : E' \to E$ with $P(\phi) = f$. Such an arrow is called a* **Cartesian lifting** *of f to $\mathcal{E}$, and a choice of Cartesian lifting for every E and f is called a* **cleavage of** *P. A* **splitting of** *P is a cleavage in which the set of arrows in $\mathcal{E}$ is closed under composition and contains all identity maps, and P* **is split** *if it has a splitting. A* **section of** *P is a functor $I : \mathcal{B} \to \mathcal{E}$ with $PI = 1_{\mathcal{B}}$.*

Dually, the functor P above is **opfibered** *or is a* **Grothendieck opfibration** *(or* **cofibration***) if the opposite functor $P^{op} : \mathcal{E}^{op} \to \mathcal{B}^{op}$ is a Grothendieck fibration. The diagram corresponding to (2.8.2) is*

$$
\begin{tikzcd}
E'' \arrow[dd, mapsto] & & E' \arrow[ll, dashed, "\exists!\chi"'] \arrow[dd, mapsto] \\
& E \arrow[ul, "\psi"] \arrow[ur, "\phi"'] \arrow[dd, mapsto] & \\
P(E'') & & P(E') \arrow[ll, "g"'] \\
& P(E) \arrow[ul, "P(\psi)"] \arrow[ur, "P(\phi)"'] &
\end{tikzcd}
\tag{2.8.3}
$$

An arrow $\phi : E \to E'$ in $\mathcal{E}$ is **opCartesian** *(or* **coCartesian***) if for any arrow $\psi : E \to E''$ in $\mathcal{E}$ and $g : P(E') \to P(E'')$ in $\mathcal{B}$ such that $g \circ P(\phi) = P(\psi)$, there exists a unique arrow $\chi : E' \to E''$ such that $\psi = \chi \circ \phi$ and*

$P(\chi) = g$. We require that for each f as above, there is a unique opCartesian ϕ making the diagram commute.

Thus, assuming the axiom of choice, a functor is a fibration iff it admits a cleavage. We learned the terms cleavage and splitting, along with the following, from [Vis05].

Example 2.8.4. Not all Grothendieck fibrations are split or have sections. *Let $\mathcal{B}G$ be the one object category associated with a group G as in Example 2.9.1 below. Then a surjective group homomorphism $G \to H$ induces a functor $\mathcal{B}G \to \mathcal{B}H$ which is a Grothendieck fibration. A cleavage of it is a choice of preimage in G of each element in H. A cleavage is a splitting iff the set of elements so chosen is a subgroup of G. A splitting or section can exist only when the homomorphism is split in the sense of group theory.*

Example 2.8.5. *A* **groupoid covering** *$p : \widetilde{\mathcal{G}} \to \mathcal{G}$ as in Definition 2.1.23 is both a Grothendieck fibration and a Grothendieck opfibration.*

Remark 2.8.6. Analogy with covering spaces. *Let $p : \widetilde{X} \to X$ be a covering of topological spaces. For both X and we have the fundamental groupoid of Definition 2.1.19. This groupoid is connected (1-connected) as in Definition 2.1.21 iff the space is path connected (simply connected).*

Then a path in X and a preimage of one of its endpoints in $\widetilde{X}$ determines a unique path in $\widetilde{X}$. In the case of Grothendieck fibration (opfibration), the end point is the starting point (end point). If we think of objects and arrows in $\mathcal{B}$ as analogs of points and paths in the space X, then the conditions of Definition 2.8.1 are analogous to those for path liftings associated with a covering, hence the term covering category.

Remark 2.8.7. Opfibrations/cofibrations. *Grothendieck opfibrations are sometimes called Grothendieck cofibrations. We find this term misleading since it suggests a functor that is injective rather than surjective on objects. In a model category (see Chapter 4) cofibrations are defined in terms of an extension property. Here an opfibration, like a Grothendieck fibration and a fibration in a model category, is defined in terms of a lifting property.*

Remark 2.8.8. The Grothendieck construction. *Given a Grothendieck fibration $P : \mathcal{E} \to \mathcal{B}$, we obtain a pseudofunctor (see Definition 2.7.10)*

$$\Phi : \mathcal{B}^{op} \to \mathcal{C}AT$$

by sending each $B \in \mathcal{B}$ to the category $\mathcal{E}_B = P^{-1}(B)$ of objects of $\mathcal{E}$ mapping onto B and morphisms mapping to 1_B. To obtain the action of Φ on morphisms in $\mathcal{B}^{op}$, given a morphism $f : A \to B$ in $\mathcal{B}$ and an object $E \in \mathcal{E}_B$, we choose a Cartesian arrow $\phi : E' \to E$ over f and call its source $f^*(E)$. The universal factorization property of Cartesian arrows then makes f^* into a *functor* $\mathcal{E}_B \to \mathcal{E}_A$*, and it is easy to verify that it is a pseudofunctor. We say that* **the Grothendieck fibration is classified by the pseudofunctor** Φ*.*

Conversely, given such a pseudofunctor Φ *on* $\mathcal{B}^{op}$*, there is a Grothendieck fibration over* $\mathcal{B}$ *called* **the Grothendieck construction** *and denoted by* $\int\Phi$*. (This is not to be confused with an end or coend as in §2.4, for which the integral sign is also used.) This yields a strict 2-equivalence of 2-categories between*

- *Grothendieck fibrations over* $\mathcal{B}$*, morphisms of Grothendieck fibrations over* $1_{\mathcal{B}}$*, and 2-cells over* $1_{1_{\mathcal{B}}}$*, and*
- *pseudofunctors* $\mathcal{B}^{op} \to \mathcal{CAT}$ *as in Definition 2.7.10, pseudonatural transformations, and modifications. The latter are 1-cells and 2-cells in the evident pseudofunctor category. Definitions of their lax analogs (see Remark 2.7.11) can be found in [Lei04, Definitions 1.5.10 and 1.5.12].*

Example 2.8.9. The Grothendieck construction on a corepresentable functor. *Suppose* Φ *is the co-Yoneda functor (see the Yoneda Lemma 2.2.10)*

$$\text{よ}_X = \mathcal{B}(-,X) : \mathcal{B}^{op} \to \mathcal{S}et$$

for an object X *of* $\mathcal{B}$*. Here we are regarding* $\mathcal{S}et$ *as the full subcategory of* $\mathcal{CAT}$ *consisting of small discrete categories as in Definition 2.1.7. Hence we get a Grothendieck fibration* $P = \int\Phi$ *with domain category* $\mathcal{E}$

In this case for an object B *in* $\mathcal{B}$*, the small category* $\mathcal{E}_B$ *is discrete, meaning that its only morphisms are identity morphisms. Hence an object* E *of* $\mathcal{E}_B$ *is a morphism* $B \to X$ *in* $\mathcal{B}$*, and a morphism* $f : A \to B$ *in* $\mathcal{B}$ *induces a functor* $f^* : \mathcal{E}_B \to \mathcal{E}_A$ *by precomposition with* f*. It follows that the domain category* $\mathcal{E}$ *of* $\int \text{よ}_B$ *is the slice category* $(\mathcal{B}{\downarrow}X)$ *(Definition 2.1.51) of objects in* $\mathcal{B}$ *equipped with a morphism to* X*.*

Another example (the main one for us) of the Grothendieck construction can be found in Example 2.9.1 below.

2.9 Indexed monoidal products

The material in this lengthy section is more specialized than what we have seen so far, and the reader may want to skip it at first. However it is essential for some constructions we will need later, such as the norm in §9.7B. To our knowledge, most of this material is new apart from its briefer treatment in [HHR16].

Fix a symmetric monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ throughout. We will be considering $\mathcal{V}$-valued functors on a small category J and will denote the category of such functors and natural transformations between them by $\mathcal{V}^J$. It has a symmetric mondoidal structure that is defined objectwise.

For a finite set A we have the category $\mathcal{V}^A$ of functors from the discrete category A (Definition 2.1.7) to $\mathcal{V}$, meaning collections of objects in $\mathcal{V}$ indexed by A. We will denote the value of such a functor X on $\alpha \in A$ by X_α. There is an **iterated monoidal product functor**

$$\otimes^A : \mathcal{V}^A \to \mathcal{V} \quad \text{given by} \quad X \mapsto \otimes_{\alpha \in A} X_a.$$

The symmetry of $\mathcal{V}$ implies that this functor is natural with respect to isomorphisms of the set A.

2.9A G-sets and indexed products

The following example of a Grothendieck fibration (Definition 2.8.1) or covering category is the motivating one for us.

Example 2.9.1. G-sets and covering categories. *Let G be a finite group and T a G-set. Let $\mathcal{B}G$ and $\mathcal{B}_T G$ be the small categories of Definition 2.1.31. The functor category $\mathcal{V}^{\mathcal{B}G}$ is the category of objects in $\mathcal{V}$ with G-action. The map $T \to G/G = *$ induces a pullback functor*

$$U : \mathcal{V}^{\mathcal{B}G} \to \mathcal{V}^{\mathcal{B}_T G}. \tag{2.9.2}$$

Here the image under U of an object X in $\mathcal{V}$ with G-action is the constant X-valued functor on $\mathcal{B}_T G$ in which a morphism associated with $\gamma \in G$ is sent to the corresponding automorphism of X.

Given a map of G-sets $r : S \to T$, the corresponding functor

$$P : \mathcal{B}_S G \to \mathcal{B}_T G$$

is a covering category. Each $t \in T$ is in an orbit of the form G/G_t, where G_t is the stabilizer group of t. The preimage of an orbit in T is a union of orbits in S of the form G/H_α for $H_\alpha \subseteq G_t$ with $r^{-1}(t)$ being the corresponding union of suborbits G_t/H_α.

The classifying functor Φ of P (see Remark 2.8.8) sends t to this union of suborbits. (Note here that the category $\mathcal{B}_T G$ is self dual, so a functor on $\mathcal{B}_T G$ is the same thing as a functor on $\mathcal{B}_T G^{op}$.) A morphism $f \in \mathcal{B}_T G(t, t')$ is an element $\gamma \in G$ with $\gamma(t) = t'$, and such a γ defines a map $G_t/H_\alpha \to G_{t'}/H_{\alpha'}$ for each α. In the Grothendieck construction for this functor, the morphism g is the identity on $P(f)(s) = \gamma(s) \in G_{t'}/H_{\alpha'}$ for $s \in G_t/H_\alpha$.

Remark 2.9.3. The variance of the classifying functor. *Covering categories and the Grothendieck construction are mentioned in [HHR16, §A.3], where a covering category over $\mathcal{C}$ is said to classified by a functor on $\mathcal{C}$ rather than on $\mathcal{C}^{op}$. This error is harmless because the only categories we consider there are the self dual ones of Example 2.9.1*

Definition 2.9.4. Working fiberwise. *Let $F : \mathcal{S}et_{iso} \to \mathcal{C}AT$ be a functor and $P : \widetilde{K} \to K$ a covering category classified by $\Phi : K^{op} \to \mathcal{S}et_{iso}$. Then let $F_{\widetilde{K}}$ be the Grothendieck construction $\int F\Phi$ (see Remark 2.8.8) and let $\mathcal{V}(F, P)$ be the category of sections of $F_P : F_{\widetilde{K}} \to K$. We will say that $\mathcal{V}(F, P)$ is constructed from F by* **working fiberwise***. A natural transformation $F \Rightarrow F'$ induces a functor $\mathcal{V}(F, P) \to \mathcal{V}(F', P)$ which we will also describe as being constructed by* **working fiberwise***.*

In practice working fiberwise means we can study the category $C(F, P)$ by studying the fibers $F_P^{-1}(k)$ for objects k in K.

Example 2.9.5. *$\mathcal{V}^{\widetilde{K}}$ is the category constructed by working fiberwise from the functor F given by $S \mapsto \mathcal{V}^S$ for a fixed category $\mathcal{V}$. The one constructed from the constant functor $S \mapsto \mathcal{V}$ is $\mathcal{V}^K$. The functor $p^* : \mathcal{V}^K \to \mathcal{V}^{\widetilde{K}}$ is induced by the diagonal natural transformation from the constant functor to F.*

Definition 2.9.6. *For a finite covering $p : \widetilde{K} \to K$, the* **indexed monoidal product along** *p is the functor $p_*^{\otimes} : \mathcal{V}^{\widetilde{K}} \to \mathcal{V}^K$ given by*

$$(p_*^{\otimes} X)_k = \bigotimes_{\tilde{k} \in p^{-1}(k)} X_{\tilde{k}}.$$

We will sometimes denote $p_*^{\otimes} X$ by $X^{\otimes \widetilde{K}/K}$ or (when K has only one object) $X^{\otimes \widetilde{K}}$.

Proposition 2.9.7. Properties of the indexed monoidal product. *The functor $p_*^{\otimes}$ is symmetric monoidal. If the structure map*

$$\otimes : \mathcal{V}^2 \to \mathcal{V}$$

commutes with colimits in each variable, then $p_*^{\otimes}$ commutes with sifted colimits (Definition 2.3.73).

Example 2.9.8. Subgroups and induction. *Let $A = G/H$ for a subgroup $H \subseteq G$ and let $p : A \to *$ denote the unique map. Inclusion of the coset of the identity element leads to an equivalence $j : \mathcal{B}H \to \mathcal{B}_{G/H} G$ (Proposition 2.1.38) and therefore to an equivalence of functor categories*

$$\mathcal{V}^{\mathcal{B}_{G/H} G} \to \mathcal{V}^{\mathcal{B}H},$$

with an inverse given by the left Kan extension when $\mathcal{V}$ is cocomplete. It follows that p induces a functor

$$p_*^{\otimes} : \mathcal{V}^{\mathcal{B}H} \to \mathcal{V}^{\mathcal{B}G}. \tag{2.9.9}$$

See Definition 8.3.23 below for an application of this to the cases where $\mathcal{V}$ is $\mathcal{T}op$ (topological spaces) or $\mathcal{T}$ (pointed spaces).

Now let $\mathcal{V}$ be the symmetric monoidal category $(\mathcal{A}b, \oplus, 0)$, the category of *abelian groups under direct sum. Then* $\mathcal{A}b^{\mathcal{B}G}$ *is the category of* $\mathbf{Z}[G]$*-modules and the functor* $p_*^{\oplus}$ *is induction given by* $M \mapsto \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} M$.

If $\mathcal{V}$ *is* $(\mathcal{A}b, \otimes, \mathbf{Z})$ *(abelian groups under tensor product), then* $\mathcal{A}b^{\mathcal{B}G}$ *is the category of* $\mathbf{Z}[G]$*-modules under tensor product over* $\mathbf{Z}$*, and the functor* $p_*^{\otimes}$ *is called* **norm induction***. We will define a similar functor of spectra below in Definition 9.7.3.*

The following result is also proved by working fiberwise.

Proposition 2.9.10. Indexed products and lax monoidal functors. *Suppose that* $(\mathcal{C}, \otimes, \mathbf{1}_{\mathcal{C}})$ *and* $(\mathcal{D}, \wedge, \mathbf{1}_{\mathcal{D}})$ *are symmetric monoidal categories, and that*

$$F : \mathcal{C} \to \mathcal{D}, \qquad T : FX \wedge FY \to F(X \otimes Y), \text{ and} \qquad \phi : \mathbf{1}_{\mathcal{D}} \to F\mathbf{1}_{\mathcal{D}}$$

form a lax monoidal functor as in Definition 2.6.19. If $p : J \to K$ *is a finite covering category (Definition 2.8.1) then* T *gives a natural transformation*

$$p_*^T : p_*^{\wedge} \circ F^J \to F^K \circ p_*^{\otimes}$$

between the two ways of going around

$$\begin{tikzcd} \mathcal{C}^J \arrow[r, "F^J"] \arrow[d, "p_*^{\otimes}"'] & \mathcal{D}^J \arrow[d, "p_*^{\wedge}"] \\ \mathcal{C}^K \arrow[r, "F^J"'] & \mathcal{D}^K . \end{tikzcd}$$

If T *is a natural isomorphism, then* p_*^T *is a natural equivalence.* □

The categories J and K used in this book arise from a left action of a group G on a finite set A as in Example 2.9.1. Given such an A, let $\mathcal{B}_A G$ be the category whose set of objects is A and in which a map $a \to a'$ is an element $\gamma \in G$ with the property that $\gamma a = a'$. When $A = *$ we will abbreviate $\mathcal{B}_A G$ to just $\mathcal{B}G$. For any finite map $A \to B$ of G-sets, the corresponding functor

$$\mathcal{B}_A G \to \mathcal{B}_B G$$

is a covering category.

In the following series of examples we suppose $H \subset G$ is a subgroup, take $A = G/H$ to be the set of right H-cosets, and write $p : A \to *$ for the unique equivariant map. In this case the inclusion of the identity coset gives an equivalence

$$\mathcal{B}H \to \mathcal{B}_A G$$

and hence an equivalence of functor categories

$$\mathcal{C}^{\mathcal{B}_A G} \to \mathcal{C}^{\mathcal{B}H}.$$

An inverse is provided by the left Kan extension when $\mathcal{C}$ is cocomplete.

Example 2.9.11. *Suppose $\mathcal{C}$ is the category of abelian groups, with $\oplus$ as the symmetric monoidal structure. Then $\mathcal{C}^{\mathcal{B}_{G/H}G}$ is equivalent to the category of left H-modules, and the functor $p_*^{\oplus}$ is left additive induction. If the symmetric monoidal structure is taken to be the tensor product, then $p_*^{\otimes}$ is "norm induction" as in Example 2.9.8.*

Example 2.9.12. *Let $\mathcal{S}p$ be the category of orthogonal spectra as in Definition 9.0.2 below. From the above and Theorem 9.3.10 below, the category $\mathcal{S}p^{\mathcal{B}_{G/H}G}$ is equivalent to the category of orthogonal H-spectra, and $\mathcal{S}p^{\mathcal{B}G}$ is equivalent to the category of orthogonal G-spectra. In this case $p_*^{\wedge}$ defines a multiplicative transfer from orthogonal H-spectra to orthogonal G-spectra. This is the* **norm**. *It is discussed more fully in §9.7B and Chapter 10.*

Remark 2.9.13. Weak monoidal products. *When $\mathcal{V}$ has all colimits and the tensor unit $\mathbf{1}$ is the initial object one may form infinite "weak" monoidal products, and the condition that $p : J \to K$ be finite may be dropped. If J is an infinite set and $\{X_j\}$ a collection of objects indexed by $j \in J$, set*

$$\otimes^J X_j = \operatorname*{colim}_{J' \subset J \text{ finite}} \otimes^{J'} X_j$$

in which the transition maps associated to $J' \subset J''$ are given by tensoring with the unit

$$\otimes^{J'} X_j \approx \left(\otimes^{J'} X_j\right) \otimes \left(\otimes^{J''-J'} \mathbf{1}\right) \to \otimes^{J''} X_j.$$

The functor $p_*^{\otimes}$ is constructed by working fiberwise.

2.9B Distributive laws

Now suppose we have two symmetric monoidal structures on $\mathcal{V}$, which we will denote by $\otimes$ and $\oplus$ (with units 1 and 0 respectively), that satisfy a distributive law, meaning a natural isomorphism

$$(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C) \tag{2.9.14}$$

(with coherence conditions spelled out in [Lap72]) expressing the product of sums as a sum of products. We will assume that $\oplus$ is the categorical coproduct $\amalg$ and that $A \otimes (-)$ commutes with colimits.

Suppose we have small categories J, K and L with finite coverings (in the sense of Definition 2.8.1) $p : J \to K$ and $q : K \to L$. Then we have indexed products as in Definition 2.9.6

$$\mathcal{V}^J \xrightarrow{p_*^{\oplus}} \mathcal{V}^K \xrightarrow{q_*^{\otimes}} \mathcal{V}^L, \tag{2.9.15}$$

and we want an identity of the form

$$q_*^{\otimes} \cdot p_*^{\oplus} = r_*^{\oplus} \cdot \varpi_*^{\otimes}.$$

for functors $r_*^{\oplus}$ and $\varpi_*^{\otimes}$ to be defined presently.

We first consider the case where L is the trivial category, meaning it has a single object and a single morphism, and $p : J \to K$ is a map of finite sets. Then the composite of (2.9.15) is a product of $|K|$ sums with varying numbers of terms. The distributive law equates it with a sum of $|K|$-fold products. Each of these products has a factor chosen from the set $p^{-1}(k) \subset J$ for each $k \in K$, and we take the sum of all such products. This sum is indexed by the set $\Gamma = \Gamma(J/K)$ of all sections $s : K \to J$. Then we have evaluation and projection maps

$$\begin{tikzcd} J & K \times \Gamma \arrow[l, "\mathrm{Ev}"'] \arrow[r, "\varpi"] & \Gamma \\ s(k) & (k,s) \arrow[l, mapsto] \arrow[r, mapsto] & s. \end{tikzcd} \tag{2.9.16}$$

For a functor X in $\mathcal{V}^J$, the usual distributive law is

$$\bigotimes_{k\in K}\left(\bigoplus_{p(j)=k} X_j\right) \cong \bigoplus_{s\in\Gamma}\left(\bigotimes_{k\in K} X_{s(k)}\right).$$

The $|K|$-fold product of sums is on the left and the sum (indexed by Γ) of $|K|$-fold products is on the right.

Proposition 2.9.17. The original distributive law. *With notation as above, the following diagram commutes up to canonical natural isomorphism given by the two symmetric monoidal structures on $\mathcal{V}$.*

$$\begin{tikzcd} \mathcal{V}^J \arrow[rr, "\mathrm{Ev}^*"] \arrow[d, "p_*^{\oplus}"'] & & \mathcal{V}^{K\times\Gamma} \arrow[d, "\varpi_*^{\otimes}"] \\ \mathcal{V}^K \arrow[r, "q_*^{\otimes}"'] & \mathcal{V} & \mathcal{V}^{\Gamma} \arrow[l, "r_*^{\oplus}"] \end{tikzcd}$$

We now generalize this to the case where $p : J \to K$ and $q : K \to L$ are covering categories and L may be nontrivial. This time let Γ be the category of pairs (ℓ, s) with ℓ an object of L and s a section of $(q \cdot p)^{-1}(\ell) \to q^{-1}(\ell)$. A morphism $(\ell, s) \to (\ell', s')$ is a map $f : \ell \to \ell'$ in L making the following diagram commute.

$$\begin{tikzcd} (q\cdot p)^{-1}(\ell) \arrow[r, "(q\cdot p)^{-1}(f)"] & (q\cdot p)^{-1}(\ell') \\ q^{-1}(\ell) \arrow[u, "s"] \arrow[r, "q^{-1}(f)"'] & q^{-1}(\ell') \arrow[u, "s'"'] \end{tikzcd} \tag{2.9.18}$$

We replace the product $K \times \Gamma$ of (2.9.16) with the fiber product

$$K \underset{L}{\times} \Gamma = \{(k,(\ell,s)) \in K \times \Gamma : q(k) = \ell\}.$$

Equivalently it is the pullback of the inner rectangle in the diagram

$$
\begin{tikzcd}
(k,(\ell,s)) \arrow[rr, mapsto] \arrow[ddd, mapsto] & & s(k) \arrow[rr, mapsto] & & k \arrow[ddd, mapsto] \\
& K \underset{L}{\times} \Gamma \arrow[r, "\mathrm{Ev}"] \arrow[d, "\varpi"'] \arrow[dr, phantom, very near start, "\ulcorner"] & J \arrow[r, "p"] & K \arrow[d, "q"] & \\
& \Gamma \arrow[rr, "r"'] & & L & \\
(\ell,s) \arrow[rrrr, mapsto] & & & & \ell.
\end{tikzcd}
\tag{2.9.19}
$$

Then the naturality of the original distributive law (Proposition 2.9.17) in J and K implies the following.

Proposition 2.9.20. The indexed distributive law. *With notation as above, the following diagram commutes up to canonical natural isomorphism given by the symmetric monoidal structures.*

$$
\begin{tikzcd}
\mathcal{V}^J \arrow[rr, "\mathrm{Ev}^*"] \arrow[d, "p_*^\oplus"'] & & \mathcal{V}^{K\times_L\Gamma} \arrow[d, "\varpi_*^\otimes"] \\
\mathcal{V}^K \arrow[r, "q_*^\otimes"'] & \mathcal{V}^L & \mathcal{V}^\Gamma \arrow[l, "r_*^\oplus"]
\end{tikzcd}
$$

We can study the diagram above by working fiberwise as in Definition 2.9.4. This means choosing an object $\ell \in L$ and replacing each category in (2.9.19) by the subcategory whose object set is the preimage of ℓ. For each $\ell \in L$ the the diagram corresponding to the one above is

$$
\begin{tikzcd}
\mathcal{V}^{(qp)^{-1}(\ell)} \arrow[rr, "\mathrm{Ev}^*"] \arrow[d, "p_*^\oplus"'] & & \mathcal{V}^{(r\varpi)^{-1}(\ell)} \arrow[d, "\varpi_*^\otimes"] \\
\mathcal{V}^{q^{-1}(\ell)} \arrow[r, "q_*^\otimes"'] & \mathcal{V} & \mathcal{V}^{r^{-1}(\ell)}. \arrow[l, "r_*^\oplus"]
\end{tikzcd}
\tag{2.9.21}
$$

Example 2.9.22. The indexed distributive law for functors to $\mathcal{T}$ from G-orbit categories. *Let G be a finite group with subgroups $H_1 \subseteq H_2 \subseteq H_3 \subseteq G$, and let the sequence of finite covering categories be*

$$
\begin{tikzcd}
J \arrow[r, "p"] \arrow[d, equal] & K \arrow[r, "q"] \arrow[d, equal] & L \arrow[d, equal] \\
\mathcal{B}_{G/H_1}G & \mathcal{B}_{G/H_2}G & \mathcal{B}_{G/H_3}G
\end{tikzcd}
$$

where the category $\mathcal{B}_TG$ for a G-set T is as in Example 2.9.1, and the functors p and q are induced by the surjections of G-sets $G/H_1 \to G/H_2 \to G/H_3$. Let the target category $(\mathcal{V}, \oplus, \otimes)$ be $(\mathcal{T}, \vee, \wedge)$. For counting purposes, write

$$|H_1| = a_1, \qquad |H_2| = a_1a_2, \qquad |H_3| = a_1a_2a_3 \qquad \textit{and} \qquad |G| = a_1a_2a_3b$$

for positive integers a_i and b. For concreteness choose elements

$$\beta_1, \beta_2, \dots \beta_b \in G, \qquad \gamma_1, \gamma_2, \dots \gamma_{a_3} \in H_3 \qquad \text{and} \qquad \delta_1, \delta_2, \dots \delta_{a_2} \in H_2$$

representing each left coset in G/H_3, H_3/H_2 and H_2/H_1 respectively, with β_1, γ_1 and δ_1 being the identity elements of their respective groups.

An object X in $\mathcal{T}^J$ is a set of H_1-spaces

$$\left\{X_{\beta_i\gamma_j\delta_k H_1} : 1 \leqslant i \leqslant b,\ 1 \leqslant j \leqslant a_3,\ 1 \leqslant k \leqslant a_2\right\} \tag{2.9.23}$$

indexed by left cosets of H_1 along with homeomorphisms between them induced by elements of G. For $\gamma \in G$ we have

$$X(\gamma) : X_{\beta_i\gamma_j\delta_k H_1} \to X_{\beta_i\gamma_j\delta_k H_1\gamma} = X_{\beta_{i'}\gamma_{j'}\delta_{k'} H_1},$$

where (i', j', k') is determined by (i, j, k) and γ with

$$i = i' \text{ for } \gamma \in H_3,\ j = j' \text{ for } \gamma \in H_2 \text{ and } k = k' \text{ for } \gamma \in H_1. \tag{2.9.24}$$

It follows that X is determined by a single H_1-space, say X_{H_1}. If G is abelian, then of course $(i', j', k') = (i, j, k)$ for all $\gamma \in G$.

The functor $p_*^\vee : \mathcal{T}^J \to \mathcal{T}^K$ is given by

$$(p_*^\vee X)_{\beta_i\gamma_j H_2} = \bigvee_{1\leqslant k\leqslant a_2} X_{\beta_i\gamma_j\delta_k H_1} = H_{2+} \wedge_{H_1} X_{\beta_i\gamma_j H_1},$$

and this disjoint union of $|H_2/H_1|$ copies of the H_1-space $X_{\beta_i\gamma_j H_1}$ is an H_2-space.

Similarly an object Y in $\mathcal{T}^K$ is a collection of homeomorphic H_2-spaces indexed by G/H_2 and

$$(q_*^\wedge Y)_{\beta_i H_3} = \bigwedge_{1\leqslant j\leqslant a_3} Y_{\beta_i\gamma_j H_2} = Map_*(H_{3+}, Y_{\beta_i H_2})^{H_2},$$

the space of H_2-equivariant pointed maps $H_{3+} \to Y_{\beta_i H2}$, which is a pointed H_3-space and smash product of $|H_3/H_2|$ copies of the space $Y_{\beta_i H_2}$.

It follows that

$$\begin{aligned}(q_*^\wedge p_*^\vee X)_{\beta_i H_3} &= Map_*(H_{3+}, (p_*^\vee X)_{\beta_i H_2})^{H_2} \\ &= Map_*(H_{3+}, H_{2+} \wedge_{H_1} X_{\beta_i H_1})^{H_2}\end{aligned} \tag{2.9.25}$$

which is an $|H_3/H_2|$-fold smash power of the wedge of $|H_2/H_1|$ copies of the space $X_{\beta_i H_1}$. **This gives us one of the two paths around the diagram of Proposition 2.9.20.**

For the other path we need to identify the categories Γ and $K \underset{\Gamma}{\times} L$. An object in Γ is a pair $(\beta_i H_3, s)$ where the coset $\beta_i H_3$ is an element of the G-set G/H_3, and s is a section of $(q \cdot p)^{-1}(\beta_i H_3) \to q^{-1}(\beta_i H_3)$.

Now $q^{-1}(\beta_i H_3)$ consists of $|H_3/H_2| = a_3$ cosets of H_2 while $(q \cdot p)^{-1}(\beta_i H_3)$ consists of $|H_3/H_1| = a_2 a_3$ cosets of H_1. A section s assigns to each of the a_3

H_2-cosets in $q^{-1}(\beta_i H_3)$ one of the a_2 H_1-cosets it contains. Hence the number of sections for each H_3-coset is $a_2^{a_3}$, and

$$|\,Ob\,\Gamma| = |G/H_3||H_2/H_1|^{|H_3/H_2|} = ba_2^{a_3}.$$

We claim the category Γ has the form $\mathcal{B}_T G$ for some finite G-set T. To see this, note that the morphism f in (2.9.18) is determined by an element $\gamma \in G$, so the horizontal arrows are invertible and s' is uniquely determined by s and f.

To describe T, note that the set of cosets in G/H_2 is

$$\{\beta_i\gamma_j H_2 \colon 1 \leqslant i \leqslant b, 1 \leqslant j \leqslant a_3\}$$

and a section $s : \beta_i H_3/H_2 \to \beta_i H_3/H_1$ can be interpreted as a map $\mathbf{a_3} \to \mathbf{a_2}$ which we also denote by s. Then

$$\begin{aligned} T &= \coprod_{1\leqslant i\leqslant b} \left\{(\beta_i\gamma_1\delta_{s(1)}H_1,\, \dots,\, \beta_i\gamma_{a_3}\delta_{s(a_3)}H_1) \colon s \in (H_2/H_1)^{H_3/H_2}\right\} \\ &= \coprod_{1\leqslant i\leqslant b} \beta_i \left\{(\gamma_1\delta_{s(1)}H_1,\, \dots,\, \gamma_{a_3}\delta_{s(a_3)}H_1) \colon s \in (H_2/H_1)^{H_3/H_2}\right\} \\ &\cong G/H_3 \times (H_2/H_1)^{H_3/H_2}. \end{aligned}$$

It follows that an object in $\mathcal{T}^\Gamma$ is a collection of $|G/H_3||H_2/H_1|^{|H_3/H_2|}$ H_3-spaces $X_{\beta_i H_3, s}$ indexed by cosets in G/H_3 and maps s of the form (sections)

$$\begin{tikzcd} \{\gamma_j H_2 \colon 1 \leqslant j \leqslant a_3\} \arrow[d, equal] & \{\gamma_j\delta_k H_1 \colon 1 \leqslant j \leqslant a_3,\, 1 \leqslant k \leqslant a_2\} \arrow[d, equal] \\ q^{-1}(H_3) \arrow[r, shift left, "s"] \arrow[d, leftrightarrow] & (qp)^{-1}(H_3) \arrow[l, shift left, "p"] \arrow[d, leftrightarrow] \\ q^{-1}(H_3\beta_i) \arrow[r, shift left, "s'"] \arrow[d, equal] & (qp)^{-1}(H_3\beta_i) \arrow[l, shift left, "p"] \arrow[d, equal] \\ \{\gamma_j H_2\beta_i \colon 1 \leqslant j \leqslant a_3\} \arrow[d, equal] & \{\gamma_j\delta_k H_1\beta_i \colon 1 \leqslant j \leqslant a_3,\, 1 \leqslant k \leqslant a_2\} \arrow[d, equal] \\ \{\beta_{i'}\gamma_j H_2 \colon 1 \leqslant j \leqslant a_3\} & \{\beta_{i'}\gamma_j\delta_k H_1 \colon 1 \leqslant j \leqslant a_3,\, 1 \leqslant k \leqslant a_2\} \end{tikzcd}$$

with $ps = 1_{q^{-1}(H_3)}$ and $ps' = 1_{q^{-1}(\beta_i H_3)}$. There are homeomorphisms

$$\beta_i : X_{H_3,s} \to X_{H_3\beta_i,s} = X_{\beta_{i'}H_3,s'}$$

for each i. The section s' and value of i' are uniquely determined by s and β_i.

The functor $r_*^\vee : \mathcal{T}^\Gamma \to \mathcal{T}^L$ is given by

$$\begin{array}{c} \left\{X_{\beta_i H_3,s} \colon 1 \leqslant i \leqslant b,\ s \in (H_2/H_1)^{H_3/H_2}\right\} \,:\, 1 \leqslant i \leqslant b. \\ \big\downarrow \\ \left\{\bigvee_s X_{\beta_i H_3,s} \colon 1 \leqslant i \leqslant b\right\}. \end{array} \tag{2.9.26}$$

The functor $\Gamma \to L$ *is the projection* $G/H_3 \times (H_2/H_1)^{H_3/H_2} \to G/H_3$ *on object sets so* $K \underset{L}{\times} \Gamma = \mathcal{B}_{T'}G$ *where*

$$T' = G/H_2 \underset{G/H_3}{\times} G/H_3 \times (H_2/H_1)^{H_3/H_2} = G/H_2 \times (H_2/H_1)^{H_3/H_2}.$$

It follows that an object in $\mathcal{T}^{K \underset{L}{\times} \Gamma}$ *is a collection of* H_2*-spaces* $X_{\beta_i\gamma_j H_2, s}$ *indexed by cosets in* G/H_2 *and maps* s *as before, with homeomorphisms*

$$\gamma : X_{\beta_i\gamma_j H_2, s} \to X_{\beta_i\gamma_j H_2\gamma, s} = X_{\beta_{i'}\gamma_{j'} H_2, s'}$$

for $\gamma \in G$ *with* i'*,* j' *and* s' *uniquely determined by* γ*,* i*,* j *and* s*.*

The functor $\varpi_*^{\wedge} : \mathcal{T}^{K \underset{L}{\times} \Gamma} \to \mathcal{T}^{\Gamma}$ *is given by*

$$\begin{array}{c} \left\{X_{\beta_i\gamma_j H_2, s} : 1 \leqslant i \leqslant b,\ 1 \leqslant j \leqslant a_3,\ s \in (H_2/H_1)^{H_3/H_2}\right\} \\ \downarrow \\ \left\{\bigwedge_{1 \leqslant j \leqslant a_3} X_{\beta_i\gamma_j H_2, s} : 1 \leqslant i \leqslant b, s \in (H_2/H_1)^{H_3/H_2}\right\} \\ \| \\ \left\{Map_*(H_{3+}, X_{\beta_i H_2, s})^{H_2} : 1 \leqslant i \leqslant b, s \in (H_2/H_1)^{H_3/H_2}\right\}. \end{array} \tag{2.9.27}$$

The functor $\mathrm{Ev}^* : \mathcal{T}^J \to \mathcal{T}^{K \underset{L}{\times} \Gamma}$ *is induced by the evaluation functor* $\mathrm{Ev} : K \underset{L}{\times} \Gamma \to J$ *given by*

$$(\beta_i\gamma_j H_2, s) \mapsto \beta_i\gamma_j\delta_{s(j)}H_1$$

and sends X *to the object* Y *defined by*

$$Y_{\beta_i\gamma_j H_2, s} = X_{\beta_i\gamma_j\delta_{s(j)}H_1}. \tag{2.9.28}$$

For $\gamma \in H_2$ *we have, using (2.9.24) and the fact that the section* s *is unchanged by right multiplication by any element of* H_2*,*

$$\gamma : Y_{\beta_i\gamma_j H_2, s} = X_{\beta_i\gamma_j\delta_{s(j)}H_1} \to X_{\beta_i\gamma_j\delta_{s(j)}H_1\gamma} = X_{\beta_i\gamma_j\delta_{s(j)}H_1} = Y_{\beta_i\gamma_j H_2, s}.$$

This means that each of the spaces $Y_{\beta_i\gamma_j H_2, s}$ *has an action of the subgroup* H_2*, so* Y *is indeed an object of* $\mathcal{T}^{K \underset{L}{\times} \Gamma}$*.*

Combining (2.9.28), (2.9.27) and (2.9.26) enables us to follow the upper path around the diagram of Proposition 2.9.20 in this case. For an object X *in* $\mathcal{T}^J$ *as described in (2.9.23) we have*

$$\begin{aligned} r_*^{\vee}\varpi_*^{\wedge}\mathrm{Ev}_*(X) & \\ &= r_*^{\vee}\varpi_*^{\wedge}\mathrm{Ev}_*\left(\left\{X_{\beta_i\gamma_j\delta_k H_1} : 1 \leqslant i \leqslant b,\ 1 \leqslant j \leqslant a_3,\ 1 \leqslant k \leqslant a_2\right\}\right) \\ &= r_*^{\vee}\varpi_*^{\wedge}\left(\left\{X_{\beta_i\gamma_j\delta_{s(j)}H_1} : 1 \leqslant i \leqslant b,\ 1 \leqslant j \leqslant a_3,\ s \in (H_2/H_1)^{H_3/H_2}\right\}\right) \\ &= r_*^{\vee}\left(\left\{Map(H_3, X_{\beta_i\delta_{s(j)}H_1})^{H_2} : 1 \leqslant i \leqslant b,\ s \in (H_2/H_1)^{H_3/H_2}\right\}\right) \end{aligned}$$

$$= \left\{ \bigvee_{s \in (H_2/H_1)^{H_3/H_2}} Map(H_3, X_{\beta_i \delta_{s(j)} H_1})^{H_2} : 1 \leqslant i \leqslant b \right\}$$

Equating this with (2.9.25) we get

$$\begin{aligned} &Map_*(H_{3+}, H_{2+} \wedge_{H_1} X_{\beta_i H_1})^{H_2} \\ &\qquad \cong \bigvee_{s \in (H_2/H_1)^{H_3/H_2}} Map(H_3, X_{\beta_i \delta_{s(j)} H_1})^{H_2} \end{aligned}$$

for $1 \leqslant i \leqslant b$. The left hand side is an a_3-fold product of a_2-fold disjoint unions, while the right hand side is an $a_2^{a_3}$-fold disjoint union of a_3-fold products.

2.9C Indexed monoidal products and pushouts

Thoughout this subsection, $(\mathcal{V}, \otimes, 1)$ **will be a cocomplete symmetric monoidal category**. We will study finite products of pushout diagrams and define a useful technical tool called the **target exponent filtration**. The formal description will be in Definition 2.9.34 below. We will use it later in the proofs of Theorem 3.5.21, Theorem 10.2.4, Proposition 10.3.8, Lemma 10.4.1, Lemma 10.5.18, and Theorem 10.9.9.

First we need to define a morphism associated with such products.

Definition 2.9.29. Indexed corner maps. *Let $(\mathcal{V}, \otimes, 1)$ be a cocomplete symmetric monoidal category, and let A be a finite set. Let $f : X \to Y$ be a morphism in $\mathcal{V}^A$, that is a collection of morphisms $f_\alpha : X_\alpha \to Y_\alpha$ indexed by $\alpha \in A$. Let $\mathscr{P}(A)$ be the category of subsets of A and inclusion maps as in Proposition 2.3.53, and let $\mathscr{P}_1(A)$ denote the full subcategory of proper subsets. Define a functor*

$$F : \mathscr{P}(A) \to \mathcal{V} \quad by \quad T \mapsto X^{\otimes T'} \otimes Y^{\otimes T}$$

where $T \subseteq A$ and T' is its complement in A. Let

$$\partial_X Y^{\otimes A} = \operatorname*{colim}_{\mathscr{P}_1(A)} F \tag{2.9.30}$$

be the boundary of $Y^{\otimes A}$ with respect to F. (Compare this with Definition 2.3.57, where it was denoted by $\partial_F Y^{\otimes A}$.) Then the **indexed corner map** *is*

$$f_A := \square_{\alpha \in A} f_\alpha : \partial_X Y^{\otimes A} \to Y^{\otimes A},$$

the pushout product (as in Definition 2.6.12) of the maps f_α.

Note that the subscript of the indexed corner map denotes a set, while those of its factors are elements in that set. The notation for the domain is meant to suggest its formal similarity with the boundary of an $|A|$-fold product of bounded manifolds; see Example 2.9.50 below. Let $n = |A|$. The functor

F above defines a diagram in $\mathcal{V}$ shaped like an n-cube, and its restriction to $\mathscr{P}_1(A)$ (which lacks a terminal object) gives a diagram shaped like a punctured n-cube. Its colimit, the source of the indexed corner map, is the n-fold pushout described in Proposition 2.3.53.

For $X \in \mathcal{V}^A$ write $X^{\otimes A}$ for the iterated monoidal product. Suppose we are given a pushout diagram

$$\begin{tikzcd} W \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "b"'] & Z \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \tag{2.9.31}$$

in $\mathcal{V}^A$, i.e., a collection of pushout diagrams in $\mathcal{V}$ indexed by A. These diagrams in $\mathcal{V}$ could all be the same, but we need not assume that now. (We will in §2.9D below.) Our filtration will be a sequence of objects interpolating between $X^{\otimes A}$ and $Z^{\otimes A}$.

First note that Z is the coequalizer of

$$W \rightrightarrows X \amalg Y \to Z$$

where the two maps send W to the two summands via f and a. This can be completed to a reflexive coequalizer diagram (see §2.3F)

$$X \amalg W \amalg Y \rightrightarrows X \amalg Y \to Z$$

in which the two maps restrict to the identity on $X \amalg Y$, and the section is the evident inclusion map $X \amalg Y \to X \amalg W \amalg Y$. Proposition 2.9.7 then implies that the sequence

$$(X \amalg W \amalg Y)^{\otimes A} \rightrightarrows (X \amalg Y)^{\otimes A} \to Z^{\otimes A}$$

is also a reflexive coequalizer. Using the distributivity law of Proposition 2.9.20, this can be rewritten as

$$\begin{array}{c} \displaystyle\coprod_{A = A_0 \amalg A_1 \amalg A_2} X^{\otimes A_0} \otimes W^{\otimes A_1} \otimes Y^{\otimes A_2} \\ \downarrow\downarrow \\ \displaystyle\coprod_{A = \hat{A}_0 \amalg \hat{A}_1} X^{\otimes \hat{A}_0} \otimes Y^{\otimes \hat{A}_1} \\ \downarrow \\ Z^{\otimes A}. \end{array} \tag{2.9.32}$$

The first and second coproducts above are over all partitions of A into three and two subsets respectively. The two maps send $X^{\otimes A_0} \otimes W^{\otimes A_1} \otimes Y^{\otimes A_2}$ to

$$(X^{\otimes A_0} \otimes f(W^{\otimes A_1})) \otimes Y^{\otimes A_2} \cong X^{\otimes A_0 \amalg A_1} \otimes Y^{\otimes A_2}$$

and

$$X^{\otimes A_0} \otimes (a(W^{\otimes A_1}) \otimes Y^{\otimes A_2}) \cong X^{\otimes A_0} \otimes Y^{\otimes A_1 \amalg A_2}.$$

The two maps on a summand with $A_1 = \varnothing$ send it identically to the same summand in the target. This means we can drop these summands without changing the value of the coequalizer, so we can replace (2.9.32) with

$$\begin{array}{c} \coprod\limits_{\substack{A=A_0 \amalg A_1 \amalg A_2 \\ |A_1|>0}} X^{\otimes A_0} \otimes W^{\otimes A_1} \otimes Y^{\otimes A_2} \\ \downdownarrows \\ \coprod\limits_{A=\hat{A}_0 \amalg \hat{A}_1} X^{\otimes \hat{A}_0} \otimes Y^{\otimes \hat{A}_1} \\ \downarrow \\ Z^{\otimes A}. \end{array} \tag{2.9.33}$$

Thus the vertical arrows do not preserve the coproduct decompositions. We will see that the sequence can be filtered by the cardinality of the exponent of Y so that the induced maps of the resulting layers do preserve them.

Definition 2.9.34. The target exponent filtration of the product of pushouts. *Referring to the diagrams (2.9.31) and (2.9.33), let* $\mathrm{fil}_0 Z = X^{\otimes A}$ *and define* $\mathrm{fil}_n Z^{\otimes A}$ *for* $n > 0$ *to be the coequalizer in*

$$\begin{array}{c} \coprod\limits_{\substack{A=A_0 \amalg A_1 \amalg A_2 \\ |A_1 \amalg A_2| \leqslant n \\ |A_1|>0}} X^{\otimes A_0} \otimes W^{\otimes A_1} \otimes Y^{\otimes A_2} \\ \downdownarrows \\ \coprod\limits_{\substack{A=\hat{A}_0 \amalg \hat{A}_1 \\ |\hat{A}_1| \leqslant n}} X^{\otimes \hat{A}_0} \otimes Y^{\otimes \hat{A}_1} \\ \downarrow \\ \mathrm{fil}_n Z^{\otimes A} \end{array} \tag{2.9.35}$$

For $0 \leqslant n \leqslant |A|$, these objects interpolate between $\mathrm{fil}_0 Z^{\otimes A} = X^{\otimes A}$, and $\mathrm{fil}_{|A|} Z^{\otimes A} = Z^{\otimes A}$. For $n > |A|$, the two coproducts in (2.9.35) coincide with those in (2.9.33), so $\mathrm{fil}_n Z^{\otimes A} = Z^{\otimes A}$.

Example 2.9.36. A target exponent filtration of the unit m-cube. *Let*

$$(\mathcal{V}, \otimes, 1) = (\mathcal{T}op, \times, *),$$

$A = \{1, 2, \ldots m\}$, *and for* $1 \leqslant i \leqslant m$, *let the ith pushout diagram of (2.9.31)* *be*

$$\begin{tikzcd} {[x_i,\,1/2]} \arrow[r] \arrow[d] & {[x_i,\,1]} \arrow[d] \\ {[0,\,1/2]} \arrow[r] & {[0,\,1]} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \tag{2.9.37}$$

where $0 \leqslant x_i \leqslant 1/2$*, and the maps are the obvious inclusions. The top row of the diagram varies with* i*, but the bottom rows are all the same. Then for any such* x_i *we have*

$$\mathrm{fil}_n I^m = \bigcup_{\substack{A_1 \subseteq \mathbf{m} \\ |A_1| = n}} [0,\,1]^{\times A_1} \times [0,\,1/2]^{\times A_1'} \qquad \text{for } 0 \leqslant n \leqslant m,$$

where A_1' *denotes the complement of* A_1 *in* $\mathbf{m}$*. The extreme cases are*

$$\mathrm{fil}_0 I^m = [0,\,1/2]^m \qquad \text{and} \qquad \mathrm{fil}_m I^m = [0,\,1]^m.$$

For $m = 2$ *we have*

$$\mathrm{fil}_1 I^2 = ([0,\,1/2] \times [0,\,1]) \cup ([0,\,1] \times [0,\,1/2])\,.$$

The fact that the filtration depends only on the bottom rows of the diagrams of (2.9.37) is an illustration of Proposition 2.9.41(ii) below.

We can make the filtration of Definition 2.9.34 more explicit. In the coequalizer diagram (2.9.35) for $\mathrm{fil}_1 Z$, the source is

$$X^{\otimes A} \amalg \coprod_{|A_0| = |A| - 1} X^{\otimes A_0} \otimes W^{\otimes A_0'},$$

where A_0', the complement of A_0 in A, is a singleton. The target is

$$X^{\otimes A} \amalg \coprod_{|A_0| = |A| - 1} X^{\otimes A_0} \otimes Y^{\otimes A_0'},$$

We can ignore the summand $X^{\otimes A}$ in the source since both maps send it identically to an isomorphic summand of the target. This enables us to rewrite (2.9.35) as a pushout diagram

$$\begin{tikzcd} \coprod\limits_{|A_0'|=1} X^{\otimes A_0} \otimes W^{\otimes A_0'} \arrow[r, "\tilde{a}_1"] \arrow[d, "\tilde{f}_0"'] & \coprod\limits_{|A_0'|=1} X^{\otimes A_0} \otimes Y^{\otimes A_0'} \arrow[d, "\tilde{g}_1"] \\ X^{\otimes A} = \mathrm{fil}_0 Z^{\otimes A} \arrow[r] & \mathrm{fil}_1 Z^{\otimes A}, \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

where

$$\tilde{a}_1 = \coprod_{|A_0'|=1} X^{\otimes A_0} \otimes a^{\otimes A_0'} \qquad \text{and} \qquad \tilde{f}_0 = \coprod_{|A_0'|=1} X^{\otimes A_0} \otimes f^{\otimes A_0'}.$$

For $n > 1$, the summands of the source in (2.9.35) with $|A_1 \amalg A_2| < n$ and those of the target with $|\widehat{A}_1| < n$ all map to $\mathrm{fil}_{n-1}Z$, so we can rewrite the diagram as the pushout

$$\begin{tikzcd}
\coprod\limits_{\substack{A=A_0\amalg A_1\amalg A_2\\ |A_0|=|A|-n,\,A_1\neq\varnothing}} X^{\otimes A_0}\otimes W^{\otimes A_1}\otimes Y^{\otimes A_2} \arrow[r, "\tilde{a}_n"] \arrow[d, "\tilde{f}_{n-1}"'] & \coprod\limits_{|A_0|=|A|-n} X^{\otimes A_0}\otimes Y^{\otimes A_0'} \arrow[d, "\tilde{g}_n"] \\
\mathrm{fil}_{n-1}Z^{\otimes A} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathrm{fil}_n Z^{\otimes A},
\end{tikzcd} \tag{2.9.38}$$

where

$$\tilde{a}_n = \coprod X^{\otimes A_0}\otimes a^{\otimes A_1}\otimes Y^{\otimes A_2}, \qquad \tilde{f}_{n-1} = \coprod X^{\otimes A_0}\otimes f^{\otimes A_1}\otimes g^{\otimes A_2},$$

(with each coproduct being over the same partitions of A into three subsets as the one in the upper left of (2.9.38)) and

$$\tilde{g}_n = \coprod_{|A_0|=|A|-n} X^{\otimes A_0}\otimes g^{\otimes A_0'}.$$

The next step is to replace the upper left object of (2.9.38), which we abbreviate here by XWY, with a suitable colimit through which both outgoing maps factor canonically. For each subset $A_0 \subseteq A$ with cardinality $|A| - n$, XWY has $2^n - 1$ summands, one for each proper subset $A_2 \subset A_0'$. They each map to the colimit over such A_2, namely

$$X^{\otimes A_0}\otimes \partial_W Y^{\otimes A_0'},$$

where the second factor is as in (2.9.30).

Thus we have proved

Lemma 2.9.39. The target exponent filtration as a series of pushouts. *With notation as above, for $0 < n \leqslant |A|$ there is a pushout square*

$$\begin{tikzcd}
\coprod\limits_{\substack{A=A_0\amalg A_0'\\ |A_0'|=n}} X^{\otimes A_0}\otimes \partial_W Y^{\otimes A_0'} \arrow[r, "\coprod X^{\otimes A_0}\otimes a_{A_0'}"] \arrow[d, "\tilde{f}_{n-1}"'] & \coprod\limits_{\substack{A=A_0\amalg A_0'\\ |A_0'|=n}} X^{\otimes A_0}\otimes Y^{\otimes A_0'} \arrow[d] \\
\mathrm{fil}_{n-1}Z^{\otimes A} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathrm{fil}_n Z^{\otimes A},
\end{tikzcd}$$

where $\mathrm{fil}_n Z^{\otimes A}$ is as in (2.9.35) and the restriction of $\tilde{f}_{n-1}$ to $X^{\otimes A_0}\otimes W^{\otimes A_1}\otimes Y^{\otimes A_2}$ is $X^{\otimes A_0}\otimes f^{\otimes A_1}\otimes g^{\otimes A_2}$.

In particular, for $n = |A|$ we have

$$\begin{tikzcd} \partial_W Y^{\otimes A} \arrow[r] \arrow[d] & Y^{\otimes A} \arrow[d] \\ \mathrm{fil}_{|A|-1} Z^{\otimes A} \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & Z^{\otimes A}. \end{tikzcd} \tag{2.9.40}$$

Proposition 2.9.41. The independence of the target exponent filtration on W and Y. *For a cocomplete symmetric monoidal category $(\mathcal{V}, \otimes, 1)$ and a finite set A, let*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & Z \end{tikzcd}$$

be a pushout square in $\mathcal{V}^A$ as in (2.9.31). Then

(i) for $\partial_W Y^{\otimes A}$ and $\partial_X Z^{\otimes A}$ as in (2.9.30), the diagram

$$\begin{tikzcd} \partial_W Y^{\otimes A} \arrow[r] \arrow[d] & Y^{\otimes A} \arrow[d] \\ \partial_X Z^{\otimes A} \arrow[r] & Z^{\otimes A} \end{tikzcd}$$

is a pushout square, and

(ii) the filtration of $Z^{\otimes A}$ arising from (2.9.31) coincides with the one arising from

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Z \arrow[d, equal] \\ X \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & Z, \end{tikzcd} \tag{2.9.42}$$

that is, it only depends on the bottom row of (2.9.31).

Proof The proof is by induction on $m = |A|$, the case $m = 1$ being obvious. Let $\mathrm{fil}_n Z^{\otimes A}$ be the filtration computed from the pushout square (2.9.31), and $\mathrm{fil}'_n Z^{\otimes A}$ the one computed from (2.9.42). The evident map of pushout squares gives a natural map $\mathrm{fil}_n Z^{\otimes A} \to \mathrm{fil}'_n Z^{\otimes A}$. Consider the diagram

$$\begin{tikzcd}
\coprod\limits_{\substack{A=A_0\amalg A_1\\ |A_1|=n}} X^{\otimes A_0} \otimes \partial_W Y^{\otimes A_1} \arrow[r] \arrow[d] & \coprod\limits_{\substack{A=A_0\amalg A_1\\ |A_1|=n}} X^{\otimes A_0} \otimes Y^{\otimes A_1} \arrow[d] \\
\coprod\limits_{\substack{A=A_0\amalg A_1\\ |A_1|=n}} X^{\otimes A_0} \otimes \partial_X Z^{\otimes A_1} \arrow[r] \arrow[d] & \coprod\limits_{\substack{A=A_0\amalg A_1\\ |A_1|=n}} X^{\otimes A_0} \otimes Z^{\otimes A_1} \arrow[d] \\
\mathrm{fil}_{n-1} Z^{\otimes A} \arrow[r] \arrow[d] & \mathrm{fil}_n Z^{\otimes A} \arrow[d] \\
\mathrm{fil}'_{n-1} Z^{\otimes A} \arrow[r] & \mathrm{fil}'_n Z^{\otimes A}\,.
\end{tikzcd}$$

If $n < m$, then the induction hypothesis and (i) imply that the upper square is a pushout. The composite of the top two squares is the diagram of Lemma 2.9.39 and therefore a pushout. This makes the middle square a pushout by Proposition 2.3.6. Similarly the composite of the bottom two squares is the diagram of Lemma 2.9.39 associated with (2.9.42) and therefore a pushout. Since the middle square is a pushout, using Proposition 2.3.6 again shows that the bottom square is a pushout.

This shows that the map $\mathrm{fil}_n Z^{\otimes A} \to \mathrm{fil}'_n Z^{\otimes A}$ is an isomorphism for $n < m$. The case $n = m$ then gives an identification

$$\mathrm{fil}_{m-1} Z^{\otimes A} = \partial_X Z^{\otimes A},$$

which, when combined with the pushout square of (2.9.40), gives (i). The second statement (ii) is automatic since

$$\mathrm{fil}_m Z^{\otimes A} = \mathrm{fil}'_m Z^{\otimes A} = Z^{\otimes A}. \qquad \square$$

2.9D The symmetric product and pushout ring filtrations

We will need a symmetric variant of the target exponent filtration for the proof of Lemma 10.7.3 below. We will use it again in §10.9A. The proof of Theorem 11.4.13 will make use of the closely related pushout ring filtration of Definition 2.9.47.

Again thoughout this subsection, $(\mathcal{V}, \otimes, 1)$ will be a cocomplete symmetric monoidal category. Recall that (2.9.31) is a collection of pushout diagrams in $\mathcal{V}$ indexed by a finite set A. We suppose now that that **the diagrams in $\mathcal{V}$**

are all the same. We will write each of them as

$$\begin{tikzcd} W \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "b"'] \arrow[ur, phantom, "\lrcorner"] & Z; \end{tikzcd} \tag{2.9.43}$$

for the rest of this subsection W**,** X**,** Y **and** Z **will denote objects in** $\mathcal{V}$ rather than in $\mathcal{V}^A$.

When we have m copies of (2.9.43), the symmetric group Σ_m acts on the two upper objects in (2.9.35) thorough its action on the n-element indexing set A. The two maps are equivariant, so we get an action on the coequalizer $\mathrm{fil}_n Z^{\otimes A}$. Note that orbit objects and coequalizers are both colimits, so the two constructions commute with each other by Proposition 2.3.41. Thus we get another coequalizer diagram by passing to Σ_n-orbits, namely

$$\begin{gathered} \left(\coprod_{\substack{A=A_0\amalg A_1\amalg A_2 \\ |A_1\amalg A_2|\leqslant n \\ |A_1|>0}} X^{\otimes A_0}\otimes W^{\otimes A_1}\otimes Y^{\otimes A_2} \right)_{\Sigma_m} \\ \downdownarrows \\ \left(\coprod_{\substack{A=\hat{A}_0\amalg \hat{A}_1 \\ |\hat{A}_1|\leqslant n}} X^{\otimes \hat{A}_0}\otimes Y^{\otimes \hat{A}_1} \right)_{\Sigma_m} \\ \downarrow \\ (\mathrm{fil}_n Z^{\otimes n})_{\Sigma_m} \end{gathered} \tag{2.9.44}$$

The objects $(\mathrm{fil}_n Z^{\otimes A})_{\Sigma_m}$ for $0\leqslant n\leqslant m$ interpolate between

$$\left(\mathrm{fil}_0 Z^{\otimes m}\right)_{\Sigma_m} = \mathrm{Sym}^m Y \qquad \text{and} \qquad \left(\mathrm{fil}_m Z^{\otimes m}\right)_{\Sigma_m} = \mathrm{Sym}^m Z,$$

where Sym^n is the nth symmetric product functor of Definition 2.6.63. We can do this for any n, and we have

$$\left(\mathrm{fil}_n Z^{\otimes A}\right)_{\Sigma_m} = \mathrm{Sym}^m Z \qquad \text{for } n>m.$$

Definition 2.9.45. The symmetric product filtration. *For a morphism* $Y\to Z$ *in a cocomplete symmetric monoidal category*

$$\mathrm{fil}_n^{\Sigma} Z = \coprod_m (\mathrm{fil}_n Z^{\otimes m})_{\Sigma_m},$$

where the coproduct summands are the coequalizers of (2.9.44).

These objects interpolate between

$$\mathrm{fil}_0^{\Sigma} Z = \mathrm{Sym}\, Y \qquad \textit{and} \qquad \mathrm{fil}_{\infty}^{\Sigma} Z := \mathop{colim}_n \mathrm{fil}_n^{\Sigma} Z = \mathrm{Sym}\, Z.$$

The group Σ_m acts on the diagram of Lemma 2.9.39 for $|A| = m$. The resulting orbit diagram is

$$\begin{tikzcd} \mathrm{Sym}^{m-n} X \otimes \partial_W \mathrm{Sym}^n Y \arrow[r] \arrow[d] & \mathrm{Sym}^{m-n} X \otimes \mathrm{Sym}^n Y \arrow[d] \\ (\mathrm{fil}_{n-1} Z^{\otimes m})_{\Sigma_m} \arrow[r] \arrow[ur, phantom, "\lrcorner" near end] & (\mathrm{fil}_n Z^{\otimes m})_{\Sigma_m} \end{tikzcd}$$

where

$$\partial_W \mathrm{Sym}^n Y := \left(\partial_W Y^{\otimes n}\right)_{\Sigma_n}$$

for $\partial_W Y^{\otimes n}$ as in (2.9.30). The argument of Proposition 2.9.41 shows that this filtration depends only on X and Z, so we may as well assume that $W = X$ and $Y = Z$. Taking the coproduct of such diagrams for all $m \geqslant 0$ (with the understanding that objects in the upper row are trivial for $m < n$), we get

$$\begin{tikzcd} \mathrm{Sym}\, X \otimes \partial_X \mathrm{Sym}^n Z \arrow[r] \arrow[d] & \mathrm{Sym}\, X \otimes \mathrm{Sym}^n Z \arrow[d] \\ \mathrm{fil}^{\Sigma}_{n-1} Z \arrow[r] \arrow[ur, phantom, "\lrcorner" near end] & \mathrm{fil}^{\Sigma}_n Z. \end{tikzcd} \tag{2.9.46}$$

Hence each $\mathrm{fil}^{\Sigma}_n Z$ is a $\mathrm{Sym}\, X$-submodule of $\mathrm{Sym}\, Z$.

Definition 2.9.47. The pushout ring filtration. *Suppose we have a pushout diagram of commutative rings in $\mathcal{V}$,*

$$\begin{tikzcd} \mathrm{Sym}\, X \arrow[r] \arrow[d] & \mathrm{Sym}\, Z \arrow[d] \\ R \arrow[r] \arrow[ur, phantom, "\lrcorner" near end] & R' \end{tikzcd}$$

Then we define a filtration of R' by R-modules by

$$\mathrm{fil}^R_n R' = R \otimes_{\mathrm{Sym}\, X} \mathrm{fil}^{\Sigma}_n Z$$

These objects interpolate between R and the pushout ring R'. Applying the functor $R \otimes_{\mathrm{Sym}\, X} (-)$ to (2.9.46) gives the following pushout square of R-modules.

$$\begin{tikzcd} R \wedge \partial_X \mathrm{Sym}^n Z \arrow[r] \arrow[d] & R \wedge \mathrm{Sym}^n Z \arrow[d] \\ \mathrm{fil}^R_{n-1} R' \arrow[r] \arrow[ur, phantom, "\lrcorner" near end] & \mathrm{fil}^R_n R' \end{tikzcd} \tag{2.9.48}$$

The map $R \to R'$ is the transfinite composition of the bottom maps above.

2.9E The distributive law in the arrow category

As in §2.9B, let $\mathcal{V}$ be a category with two symmetric monoidal structures $\oplus$ and $\otimes$ with units 0 and 1 related by a distributive law as in (2.9.14). We can define a similar pair of symmetric monoidal structures $\oplus$ and $\square$ on the arrow category $\operatorname{Arr}\mathcal{V}$ as follows. Given objects $f_i : A_i \to B_i$ in $\operatorname{Arr}\mathcal{V}$ for $i = 1, 2$, let $f_1 \oplus f_2$ be the evident map $A_1 \oplus A_2 \to B_1 \oplus B_2$, and let $f_1 \square f_2$ be the pushout corner map with respect to $\otimes$ as in Definition 2.6.12. The units for the two structures are $0 \to 0$ for $\oplus$ and $0 \to 1$ for $\square$.

Then for a third object $g : X \to Y$ in $\operatorname{Arr}\mathcal{V}$ we have

$$(f_1 \oplus f_2) \square g \cong (f_1 \square g) \oplus (f_2 \square g). \tag{2.9.49}$$

This is the simplest instance of the distribute law in $\operatorname{Arr}\mathcal{V}$. It equates a pushout product of sums with a sum of pushout products.

Example 2.9.50. Manifolds with boundary. *Recall Example 2.3.58 and Example 2.6.17 in $\mathcal{T}op$ equipped with disjoint union and Cartesian product. Let M_1, M_2 and N be manifolds with boundary and let*

$$f_i : \partial M_i \to M_i \qquad \textit{and} \qquad g : \partial N \to N$$

be the evident inclusions. We saw in Example 2.6.17 that $(f_1 \amalg f_2) \square g$ is the inclusion of the boundary of $(M_1 \amalg M_2) \times N$.

We also have

$$\begin{aligned}\partial((M_1 \amalg M_2) \times N) &= \partial((M_1 \times N) \amalg (M_2 \times N)) \\ &= \partial(M_1 \times N) \amalg \partial(M_2 \times N) \\ &= ((\partial M_1) \times N \cup_{\partial M_1 \times \partial N} M_1 \times \partial N) \\ &\qquad \amalg ((\partial M_2) \times N \cup_{\partial M_2 \times \partial N} M_2 \times \partial N),\end{aligned}$$

so the inclusion of the boundary is also $(f_1 \square g) \amalg (f_2 \square g)$. This illustrates (2.9.49) in this case.

Given the maps of (2.9.19), the diagram of (2.9.21) (meaning the ℓth component of the diagram of Proposition 2.9.20) with $\mathcal{V}$ replaced by $\operatorname{Arr}\mathcal{V}$ is

$$\begin{tikzcd} (\operatorname{Arr}\mathcal{V})^{(qp)^{-1}(\ell)} \arrow[rr, "\mathrm{Ev}^*"] \arrow[d, "p_*^{\oplus}"'] & & (\operatorname{Arr}\mathcal{V})^{(r\varpi)^{-1}(\ell)} \arrow[d, "\varpi_*^{\square}"] \\ (\operatorname{Arr}\mathcal{V})^{q^{-1}(\ell)} \arrow[dr, "q_*^{\square}"'] & & (\operatorname{Arr}\mathcal{V})^{r^{-1}(\ell)} \arrow[dl, "r_*^{\oplus}"] \\ & \operatorname{Arr}\mathcal{V}. & \end{tikzcd} \tag{2.9.51}$$

Suppose that $f : X \to Z$ is a map in $\mathcal{V}^J$, i.e., an object in $(\operatorname{Arr}\mathcal{V})^J$, and let $\ell \in L$. Then the collection of maps $f_j : X_j \to Z_j$ indexed by the subset $(qp)^{-1}(\ell) \subseteq J$ is an object in the upper left category of (2.9.51). We can analyze its counterclockwise and clockwise images in $\operatorname{Arr}\mathcal{V}$. They are respectively a pushout product of sums of certain f_js, and a sum of certain pushout products of maps in $\operatorname{Arr}\mathcal{V}$. They are naturally isomorphic by the indexed distributive law of Proposition 2.9.20. In particular there is a natural isomorphism between their domains.

Proposition 2.9.52. Indexed corner maps and the distributive law. *With notation as in Proposition 2.9.20), let $f : X \to Z$ be a map in $\mathcal{V}^J$. Then for each $\ell \in L$ there is a natural isomorphism between the following two objects.*

(i) The tensor product indexed by $q^{-1}(\ell)$ of the sources of the indexed corner maps with factors

$$g_k := \bigoplus_{j\in(p)^{-1}k} (X_j \xrightarrow{f_j} Z_j), \tag{2.9.53}$$

namely

$$\partial_{p_*^{\oplus}X}(p_*^{\oplus}Z)^{\otimes q^{-1}(\ell)}.$$

(ii) The sum over all sections s with $(\ell, s) \in r^{-1}(\ell)$ of the objects

$$\partial_X Z^{\otimes T_s},$$

where $T_s = s(q^{-1}(\ell)) \subseteq J$. For each such section s, this is the domain of the pushout product map

$$\mathop{\Box}_{k\in q^{-1}(\ell)} f_{s(k)}.$$

Proof The ℓth component of the counterclockwise image of f in $(\operatorname{Arr}\mathcal{V})^L$ is the pushout product indexed by the set $q^{-1}(\ell) \subset K$ of the maps g_k of (2.9.53) for each $k \in q^{-1}(\ell)$. In short, it is a pushout product of direct sums of certain f_js, namely

$$g_{q^{-1}(\ell)} = \mathop{\Box}_{k\in q^{-1}(\ell)} g_k : \partial_{p_*^{\oplus}X}(p_*^{\oplus}Z)^{\otimes q^{-1}(\ell)} \to (p_*^{\oplus}Z)^{\otimes q^{-1}(\ell)},$$

where the kth (for $k \in q^{-1}(\ell)$) factor involves the sum over the set $p^{-1}(k) \subseteq J$.

The distributive law equates this pushout product of sums with a sum of pushout products. The latter sum is indexed by the set $r^{-1}(\ell)$ of sections s of the map p in the diagram

$$\begin{tikzcd} J \supset (q\cdot p)^{-1}(\ell) \arrow[r, "p"'] & q^{-1}(\ell) \subseteq K. \arrow[l, bend right, "s"'] \end{tikzcd}$$

For each such section we have the pushout product

$$\left(f_{T_s} : \partial_X Z^{\otimes T_s} \to Z^{\otimes T_s}\right) = \mathop{\Box}_{k \in q^{-1}(\ell)} \left(f_{s(k)} : X_{s(k)} \to Z_{s(k)}\right),$$

where $T_s = s(q^{-1}(\ell)) \subseteq J$. The ℓth component of the clockwise image of f in (2.9.51) is the sum of all such pushout products, namely

$$\bigoplus_{s \text{ with } (\ell,s) \in r^{-1}(\ell)} \left(f_{T_s} : \partial_X Z^{\otimes T_s} \to Z^{\otimes T_s}\right).$$

The distributivity isomorphism in the arrow category is given by

$$\begin{tikzcd} \partial_{p_*^{\oplus} X}(p_*^{\oplus} Z)^{\otimes q^{-1}(\ell)} \arrow[r, "g_{q^{-1}(\ell)}"] \arrow[d, "\cong"'] & (p_*^{\oplus} Z)^{\otimes q^{-1}(\ell)} \arrow[d, "\cong"] \\ \bigoplus \partial_X Z^{\otimes T_s} \arrow[r, "\bigoplus f_{T_s}"] & \bigoplus Z^{\otimes T_s} \end{tikzcd} \tag{2.9.54}$$

where the sums in the bottom row are over all sections s with $(\ell, s) \in r^{-1}(\ell)$. The two rows are the images of f in the ℓth component of $(\operatorname{Arr} \mathcal{V})^L$ given by the two ways of going around (2.9.51). The isomorphism on the right is the indexed distributive law in $\mathcal{V}$ (Proposition 2.9.20) applied to the object Z in $\mathcal{V}^{(qp)^{-1}(\ell)}$. The left vertical arrow is the desired isomorphism. □

2.9F Commutative algebras and indexed monoidal products

By Proposition 2.6.59, if $\mathcal{V}$ is a cocomplete closed symmetric monoidal category, then $\mathbf{Comm}\,\mathcal{V}$ is cocomplete. For a covering category $p : J \to K$, the restriction functor

$$p^* : \mathbf{Comm}\,\mathcal{V}^K \to \mathbf{Comm}\,\mathcal{V}^J$$

has a left adjoint $p_!$ given by left Kan extension.

Proposition 2.9.55. Monoidal products of commutative algebras as left Kan extensions. *If $p : J \to K$ is a covering category, the following diagram commutes up to natural isomorphism*

$$\begin{tikzcd} \mathbf{Comm}\,\mathcal{V}^J \arrow[r] \arrow[d, "p_!"] & \mathcal{V}^J \arrow[d, "p_*^{\otimes}"] \\ \mathbf{Comm}\,\mathcal{V}^K \arrow[r] & \mathcal{V}^K. \end{tikzcd}$$

Proof For a commutative algebra $A \in \mathbf{Comm}\,\mathcal{V}^J$, and $k \in K$ the value of $p_! A$ at k is calculated as the colimit over the category $(J \downarrow k)$ (as in Definition 2.1.51(i)) of the restriction of p. Since $p : J \to K$ is a covering category, the category $(J \downarrow k)$ is equivalent to the discrete category $p^{-1}k$, and so

$$(p_! A)_k = \otimes_{p^{-1}k} A,$$

and the result follows. □

2.9G Monomial ideals

Let J be a set and consider the polynomial algebra

$$A = \mathbf{Z}[x_i], \quad i \in J.$$

As an abelian group, it has a basis consisting of the monomials x^f, with

$$f : J \to \{0, 1, 2, \dots\}$$

a function taking the value zero on all but finitely many elements, and

$$x^f = \prod_{i \in J} x_i^{f(i)}.$$

The collection of such f is a monoid under addition, and we denote it $\mathbf{N}_0^J$. If $D \subset \mathbf{N}_0^J$ is a monoid ideal then the subgroup $M_D \subset A$ with basis $\{x^f \mid f \in D\}$ is an ideal. These are the **monomial ideals** and they can be formed in any monoidal product of free associative algebras in any closed symmetric monoidal category.

Let $(\mathcal{V}, \otimes, \mathbf{1})$ be a closed symmetric monoidal category. Fix a set J which we temporarily assume to be finite. Given $X \in \mathcal{V}^J$ let

$$TX = \coprod_{n \geqslant 0} X^{\otimes n}$$

be the free associative algebra generated by X. Write $A = p_*^{\otimes} TX \in \mathcal{V}$, where $p : J \to *$ is the unique map. Then A is an associative algebra in $\mathcal{V}$. The motivating example above occurs when $\mathcal{V}$ is the category of abelian groups and X is the constant diagram $X_j = \mathbf{Z}$.

Using indexed distributive law (Proposition 2.9.20), the object A can be expressed as an indexed coproduct

$$A = \coprod_{f : J \to \mathbf{N}_0} X^{\otimes f}$$

where $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ and

$$X^{\otimes f} = \bigotimes_{j \in J} X_j^{\otimes f(j)}.$$

The set

$$\mathbf{N}_0^J = \{f : J \to \mathbf{N}_0\}$$

is a commutative monoid under addition of functions. The multiplication map in A is the sum of the isomorphisms

$$X^{\otimes f} \otimes X^{\otimes g} \approx X^{\otimes (f+g)} \tag{2.9.56}$$

given by the symmetry of the monoidal product $\otimes$, and the isomorphism

$$X^{\otimes f(i)} \otimes X^{\otimes g(i)} \approx X^{\otimes(f(i)+g(i))}.$$

For a monoid ideal $D \subset \mathbf{N}_0^J$, set

$$M_D = \coprod_{f \in D} X^{\otimes f}.$$

The formula (2.9.56) for the multiplication in A gives M_D the structure of an ideal in A. If $D \subset D'$ then the evident inclusion $M_D \subset M_{D'}$ is an inclusion of ideals.

When $\mathcal{V}$ is pointed (in the sense that the initial object is isomorphic to the terminal object), the map

$$A \to A/M_D$$

is a map of associative algebras, where A/M_D is defined by the pushout diagram

$$\begin{tikzcd} M_D \arrow[r] \arrow[d] & A \arrow[d] \\ * \arrow[r] & A/M_D \arrow[ul, phantom, "\lrcorner"] \end{tikzcd},$$

with $*$ denoting the terminal (and initial) object.

Definition 2.9.57. *The ideal $M_D \subset A$ is the* **monomial ideal** *associated to the monoid ideal D.*

Example 2.9.58. *Suppose that $\dim : \mathbf{N}_0^J \to \mathbf{N}_0$ is any homomorphism. Given $d \in \mathbf{N}_0$ the set*

$$\{f \mid \dim f \geqslant d\}$$

is a monoid ideal. We denote the corresponding monomial ideal M_d. The M_d form a decreasing filtration

$$\cdots \subset M_{d+1} \subset M_d \subset \cdots \subset M_1 \subset M_0 = A.$$

When $\mathcal{V}$ is pointed, the quotient

$$M_d/M_{d+1}$$

is isomorphic as an A bimodule to

$$A/M_1 \otimes \coprod_{\dim f = d} X^{\otimes f},$$

in which A act through its action on the left factor.

Remark 2.9.59. *The quotient module is defined by the pushout square*

$$\begin{tikzcd} M_{d+1} \arrow[r] \arrow[d] & M_d \arrow[d] \\ * \arrow[r] & M_d/M_{d+1} \, . \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

The pushout can be calculated in the category of left A-modules, A-bimodules, or just in $\mathcal{V}$.

Remark 2.9.60. *All of this discussion can be made to be covariant with respect to inclusion in J. Suppose that $J_0 \subset J_1$ is an inclusion of finite sets and $X_1 : J_1 \to \mathcal{V}$ is an J_1-diagram. Define $X_0 : J_1 \to \mathcal{V}$ by*

$$X_0(j) = \begin{cases} X_1(j) & j \in J_0 \\ * & \textit{otherwise.} \end{cases}$$

There is a natural map $X_0 \to X_1$. Let A_0 and A_1 be the associative algebras constructed from the X_i as described above. The algebra A_0 coincides with the one constructed directly from the restriction of X_0 to J_0. A monoid ideal $D_1 \subset \mathbf{N}_0^{J_1}$ defines ideals $M_{D_0} \subset A_0$ and $M_{D_1} \subset A_1$. The monoid ideal D_0 is the same as the one constructed from the intersection of D_0 with $\mathbf{N}_0^{J_0}$, where $\mathbf{N}_0^{J_0}$ is regarded as a subset of $\mathbf{N}_0^{J_1}$ by extension by 0. There is a commutative diagram

$$\begin{tikzcd} M_{D_0} \arrow[r] \arrow[d] & M_{D_1} \arrow[d] \\ A_0 \arrow[r] & A_1 \, . \end{tikzcd}$$

Using this, the construction of monomial ideals can be extended to the case of infinite sets J, by passing to the colimit over the finite subsets. As in the motivating example, when the set J is infinite, the indexing monoid $\mathbf{N}_0^J$ is the set of finitely supported functions.

By working fiberwise, this entire discussion applies to the situation of a (possibly infinite) covering category $p : J \to K$. Associated to $X : J \to \mathcal{V}$ is

$$A = p_*^{\otimes} TX \in \mathbf{Assoc}\,\mathcal{C}^K = (\mathbf{Assoc}\,\mathcal{C})^K.$$

In case J/K is infinite, the algebra A is formed fiberwise by passing to the colimit from the finite monoidal products using the unit map, as described in Remark 2.9.13. As an object of $\mathcal{C}^K$, the algebra A decomposes into

$$A = \coprod_{f \in \Gamma} X^{\otimes f}$$

where Γ is the set of sections of

$$\mathbf{N}_0^{J/K} \to K$$

with $\mathbf{N}_0^{J/K}$ formed from the Grothendieck construction applied to

$$j \mapsto \mathbf{N}_0^{J_j} \qquad (J_j = p^{-1}(j)).$$

The category $\mathbf{N}_0^{J/K}$ is a commutative monoid over K, and associated to any monoid ideal $D \subset \mathbf{N}_0^{J/K}$ over K, is a monomial ideal $M_D \subset A$.

The situation of interest in this book (see §10.10) is when $J \to K$ is of the form

$$\mathcal{B}_K G \to \mathcal{B} G$$

associated to a G-set K, and the unique map $K \to *$. In this case $\mathbf{N}_0^{J/K}$ is the G-set $\mathbf{N}_0^K$ of finitely supported functions $K \to \mathbf{N}_0$. The relative monoid ideals are just the G-stable monoid ideals. A simple algebraic example arises in the case of a polynomial algebra $\mathbf{Z}[x_i]$ in which a group G is acting on the set indexing the variables.

3

Enriched category theory

> Most of the results apply equally to categories and to $\mathcal{V}$-categories, without a word's being changed in the statement or the proof; so that scarcely a word would be saved if we restricted ourselves to ordinary categories alone. Certainly this requires proofs adapted to the case of a general $\mathcal{V}$; but these almost always turn out to be the best proofs in the classical case $\mathcal{V} = \mathcal{S}et$ as well.
>
> *Max Kelly, [Kel82, page 2]*

In § 3.1–§ 3.2 we discuss enriched categories. This is the most convenient framework for the definition of G-spectra, the central objects of study in this book, to be given in Chapter 9. In a category $\mathcal{C}$ enriched over $\mathcal{V}$, or $\mathcal{V}$-category for short, morphism sets are replaced by morphism objects in a symmetric monoidal category $\mathcal{V}_0$ as explained in Definition 3.1.1. An ordinary category is enriched over $(\mathcal{S}et, \times, *)$. A closed $\mathcal{V}$-module as in Definition 2.6.42 is a $\mathcal{V}$-category with additional structure.

We have an enriched Yoneda lemma, Enriched Yoneda Lemma 3.1.29, enriched functor categories (Definition 3.2.18), enriched Yoneda embedding (Definition 3.1.68), enriched limits, colimits, ends and coends (§3.2).

In §3.3 we discuss the most useful (for us) construction of enriched category theory, the Day convolution. It is the formal tool that leads to the definition of smash products in the stable homotopy category in §7.2C and Chapter 9. **Its use simplifies stable homotopy theory considerably.** It is the main motivation for introducing many of the tools we have mentioned thus far.

We discuss simplicial sets and simplicial spaces in § 3.4. The former are combinatorial structures (Definition 3.4.1) having topological spaces (their geometric realizations, Definition 3.4.3) associated with them. These spaces are always CW complexes and hence convenient to work with. Much of homotopy theory can be done in the world of simplicial sets. For example the yellow monster (Kan's nickname for [BK72]) is written entirely in this language; when they say "space" they really mean "simplicial set." We have decided **not** to do the same in this book because simplicial sets are not convenient for describing certain spaces we use repeatedly, namely ones associated with orthogonal representations of finite groups such as Stiefel manifolds and Thom spaces.

3.1 Basic definitions

There are familiar examples of closed symmetric monoidal categories $\mathcal{C}$ (such as $\mathcal{A}b$, $\mathcal{V}ect_k$ and the redefined $\mathcal{T}op$) in which the morphism set $\mathcal{C}(X,Y)$ has a natural structure as an object in $\mathcal{C}$. More generally, the morphism set could have a natural structure as an object in a category other than $\mathcal{C}$.

We will generalize the definition of a category as follows. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as in Definition 2.6.1. The following is originally due to Eilenberg-Kelly [EK66] and is repeated in [Kel82]. In the former, closed symmetric monoidal categorys were called "closed categories," and categories enriched over such a $\mathcal{V}$ were called "categories over $\mathcal{V}$" or "$\mathcal{V}$-categories."

3.1A Enriched categories, functors and natural transformations

Definition 3.1.1. $\mathcal{V}$-categories. *A $\mathcal{V}$-category $\mathcal{C}$ (or a* **category enriched over a symmetric monoidal category** *$\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$) consists of a collection $Ob\,\mathcal{C}$ called the* **objects of** *$\mathcal{C}$, for each pair $X, Y \in Ob\,\mathcal{C}$ a* **morphism object** *$\mathcal{C}(X,Y) \in Ob\,\mathcal{V}_0$ (instead of a set of morphisms $X \to Y$), for each X an* **identity morphism** *$1_X : \mathbf{1} \to \mathcal{C}(X,X)$ in $\mathcal{V}_0$ (instead of an identity morphism $X \to X$) and for each triple (X, Y, Z) of objects of $\mathcal{C}$ a* **composition morphism**

$$c_{X,Y,Z} : \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z), \tag{3.1.2}$$

which is a morphism in $\mathcal{V}_0$. These data are required to satisfy the evident unit and associativity properties as in Definition 2.6.1. **We will usually denote it as above with a lower case Roman letter corresponding to the name of the category having subscripts indicating to the three objects in question.**

There is an ordinary category $\mathcal{C}_0$ underlying the enriched category $\mathcal{C}$ with objects as in $\mathcal{C}$ and morphism sets defined by

$$\mathcal{C}_0(X,Y) := \mathcal{V}_0(\mathbf{1}, \mathcal{C}(X,Y)). \tag{3.1.3}$$

In an ordinary category $\mathcal{C}_0$ the morphism set $\mathcal{C}_0(X,Y)$ could be empty. An ordinary category is enriched over $\mathcal{S}et$, in which the empty set is the initial object. Hence the analogous situation in a $\mathcal{V}$-category $\mathcal{C}$ would be that $\mathcal{C}(X,Y)$ is the initial object in $\mathcal{V}$, if there is one.

In the examples of interest in this book, $\mathcal{V}_0$ is a model category and hence bicomplete as in Definition 2.3.25. This means it has both an initial object and a terminal object, as well as products and coproducts indexed by arbitrary sets. In the language of Definition 3.1.31 below, it is bitensored over $\mathcal{S}et$.

Remark 3.1.4. Ordinary properties of enriched categories. *We will sometimes say that a $\mathcal{V}$-category $\mathcal{C}$ has a certain property if its underlying ordinary category $\mathcal{C}_0$ has the same property.*

For more discussion of the following, see [Rie14, §13.1].

Definition 3.1.5. The enriched arrow category. *Let $\mathcal{C}$ be a category that is bitensored and enriched over a bicomplete symmetric monoidal $\mathcal{V}$-category with unit $*$. The enriched arrow category $\mathcal{C}_1$ has as objects morphisms $f : * \to \mathcal{C}(A, B)$ in $\mathcal{V}$. Given a second object $g : * \to \mathcal{C}(X, Y)$, the morphism object $\diamondsuit(f, g) = \mathcal{C}_1(f, g)$ is the pullback of the following diagram in $\mathcal{V}$.*

$$\begin{tikzcd} \diamondsuit(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C}(A,X) \arrow[d, "g_*"] \\ \mathcal{C}(B,Y) \arrow[r, "f^*"] & \mathcal{C}(A,Y), \end{tikzcd}$$

the analog of the diagram of sets in Proposition 2.3.5.

Then Proposition 2.3.18 suggests the following.

Definition 3.1.6. Enriched lifting. *Let f and g be as in Definition 3.1.5. The* **enriched lifting test map** *$\mathcal{C}_\diamondsuit(f, g)$ is the morphism to the pullback in the diagram*

$$\begin{tikzcd} \mathcal{C}(B,X) \arrow[drr, bend left, "f^*"] \arrow[dr, dashed, "\mathcal{C}_\diamondsuit(f,g)"] \arrow[ddr, "g_*"'] & & \\ & \diamondsuit(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C}(A,X) \arrow[d, "g_*"] \\ & \mathcal{C}(B,Y) \arrow[r, "f^*"] & \mathcal{C}(A,Y). \end{tikzcd}$$

and we say $f \boxslash g$ (the enriched analog of Definition 2.3.10) if $\mathcal{C}_\diamondsuit(f, g)$ has a section.

Remark 3.1.7. Dugger's approach to enrichment. *In [Dug06, 2.1] Dugger gives a variant of the above in which he starts with the ordinary category $\mathcal{C}_0$ and equips it with*

(i) a functor $\tau : \mathcal{C}_0^{op} \times \mathcal{C}_0 \to \mathcal{V}_0$ whose value on (X, Y) is our $\mathcal{C}(X, Y)$,
(ii) for each object X in $\mathcal{C}_0$ an identity morphism $\mathbf{1} \to \mathcal{C}(X, X)$ in $\mathcal{V}_0$,

(iii) a natural transformation inducing the composition morphism

$$\mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y)\to\mathcal{C}(X,Z)$$

and

(iv) for each morphism $f: X\to Y$ in $\mathcal{C}_0$ a commuting diagram in $\mathcal{V}_0$,

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[d] & \mathcal{C}(X,X) \arrow[d, "f_*"] \\
\mathcal{C}(Y,Y) \arrow[r, "f^*"] & \mathcal{C}(X,Y)
\end{tikzcd}$$

with suitable properties.

He then considers what happens when the functor τ is varied.

Remark 3.1.8. $\mathcal{V}$-categories and categories internal to $\mathcal{V}$. *In § 2.3D we discussed categories J internal to a ground category $\mathcal{C}$ in which certain pullbacks can be defined. These are generalizations of small categories in which the object set $J_0 := Ob\,J$ is instead an object in $\mathcal{C}$, as is the morphism set $J_1 := Arr\,J$. The latter comes equipped with a morphism to $J_0\times J_0$ related to the domain and codomain. When the $\mathcal{C}$ has a terminal object $*$, we can speak of points (morphisms from $*$) in J_0, J_1 and $J_0\times J_0$, and hence of the preimage in J_1 of a point (x,y) (an ordered pair of objects in J) in J_1. This preimage is a certain pullback (see (2.3.50)) and an object $J(x,y)$ in $\mathcal{C}$, the morphism object generalizing the set of morphisms $x\to y$. Hence* **if $\mathcal{C}$ were symmetric monoidal, we could say that J is enriched over it***.*

The ground category $\mathcal{C}$ is not required to be symmetric monoidal in Definition 2.3.46 because we only need the existence of certain objects such as $J_1\underset{J_0}{\times}J_1$, which is weaker than a monoidal structure. On the other hand, the requirement that J_0 be an object in the ground category $\mathcal{C}$ is a smallness condition not required of an enriched category.

The enriched analog of Definition 2.1.56 is the following.

Definition 3.1.9. Enriched retracts. *Let $\mathcal{C}$ be a $\mathcal{V}$-category as in Definition 3.1.1. An object X is a* **retract** *of an object Y if there is a lifting of the identity morphism for X in the following diagram in $\mathcal{V}_0$*

$$\begin{tikzcd}
 & \mathcal{C}(Y,X)\otimes\mathcal{C}(X,Y) \arrow[d, "c_{X,Y,X}"] \\
\mathbf{1} \arrow[ur, dashed] \arrow[r, "1_X"'] & \mathcal{C}(X,X).
\end{tikzcd}$$

A closed $\mathcal{V}$-module $\mathcal{C}$ as in Definition 2.6.42 (in which the symmetric monoidal category $\mathcal{V}$ is required to be closed) is a $\mathcal{V}$-category as in Definition 3.1.1

with additional structure. It has tensor and cotensor products over $\mathcal{V}$; these will be defined below in Definition 3.1.31.

Note here that $\mathcal{C}(X,Y)$ is no longer a set endowed with additional structure; it is simply an object in $\mathcal{V}_0$. Hence $\mathcal{C}$ does not have morphisms in the usual sense, but only morphism objects in $\mathcal{V}_0$. In an ordinary category a morphism set could be empty, i.e., it could be the initial object of $\mathcal{S}et$. The analog here is that $\mathcal{C}(X,Y)$ could be the unit object $\mathbf{1}$, the initial object of $\mathcal{V}_0$.

Associativity of composition implies the following, which should be compared with Definition 2.2.34.

Proposition 3.1.10. The reduced composition morphism in a cocomplete category. *For a cocomplete $\mathcal{V}$-category $\mathcal{C}$, let*

$$\mathcal{C}(Y,Z) \otimes_{\mathcal{C}(Y,Y)} \mathcal{C}(X,Y)$$

denote the coequalizer of

$$\begin{tikzcd}
\mathcal{C}(Y,Z) \otimes \mathcal{C}(Y,Y) \otimes \mathcal{C}(X,Y) \arrow[d, shift right, "\mathcal{C}(Y,Z)\otimes c_{X,Y,Y}"'] \arrow[d, shift left, "c_{Y,Y,Z}\otimes\mathcal{C}(X,Y)"] \\
\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \arrow[d, dashed] \\
\mathcal{C}(Y,Z) \otimes_{\mathcal{C}(Y,Y)} \mathcal{C}(X,Y).
\end{tikzcd}$$

Then the composition morphism

$$c_{X,Y,Z} : \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

of (3.1.2) factors uniquely through the **reduced composition morphism**

$$\widetilde{c}_{X,Y,Z} : \mathcal{C}(Y,Z) \otimes_{\mathcal{C}(Y,Y)} \mathcal{C}(X,Y) \to \mathcal{C}(X,Z).$$

Proposition 3.1.11. Reduced composition with endomorphisms. *The reduced composition morphisms*

$$\widetilde{c}_{X,Y,Y} : \mathcal{C}(Y,Y) \otimes_{\mathcal{C}(Y,Y)} \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$$

and

$$\widetilde{c}_{Y,Y,Z} : \mathcal{C}(Y,Z) \otimes_{\mathcal{C}(Y,Y)} \mathcal{C}(Y,Y) \to \mathcal{C}(Y,Z).$$

are isomorphisms.

If either $\mathcal{C}(Y,Z)$ or $\mathcal{C}(X,Y)$ is isomorphic to $\mathcal{C}(Y,Y)$, then $\widetilde{c}_{X,Y,Z}$ is an isomorphism.

Proof For the first isomorphism, note that when $Z = Y$ the diagram of Proposition 3.1.10 fits into

$$\begin{tikzcd}
 & \mathcal{C}(Y,Y)\otimes\mathcal{C}(Y,Y)\otimes\mathcal{C}(X,Y) \arrow[d, shift right, "\mathcal{C}(Y,Y)\otimes c_{X,Y,Y}"'] \arrow[d, shift left, "c_{Y,Y,Y}\otimes\mathcal{C}(X,Y)"] & \\
\mathbf{1}\otimes\mathcal{C}(X,Y) \arrow[r, "1_Y\otimes\mathcal{C}(X,Y)"] & \mathcal{C}(Y,Y)\otimes\mathcal{C}(X,Y) \arrow[r, "c_{X,Y,Y}"] \arrow[d, dashed] & \mathcal{C}(X,Y) \\
 & \mathcal{C}(Y,Y)\otimes_{\mathcal{C}(Y,Y)}\mathcal{C}(X,Y), \arrow[ur, "\tilde{c}_{X,Y,Y}"'] &
\end{tikzcd}$$

in which the composite map $\mathbf{1}\otimes\mathcal{C}(X,Y)\to\mathcal{C}(X,Y)$ is isomorphic to the identity via the left unitor (see Definition 2.6.1) in $\mathcal{V}$. This means that the coequalizer has to be $\mathcal{C}(X,Y)$. In view of this, an isomorphism between $\mathcal{C}(Y,Z)$ and $\mathcal{C}(Y,Y)$ induces one between the colimits

$$\mathcal{C}(Y,Z)\otimes_{\mathcal{C}(Y,Y)}\mathcal{C}(X,Y)$$

and

$$\mathcal{C}(Y,Y)\otimes_{\mathcal{C}(Y,Y)}\mathcal{C}(X,Y)\cong\mathcal{C}(X,Y).$$

The arguments for the other two isomorphisms are similar. □

Definition 3.1.12. Enriched composition and precomposition with an ordinary morphism. *Given a morphism $f\in\mathcal{C}_0(X,Y)$ and an object W in $\mathcal{C}$, the morphism f_*, which we will also denote by $\mathcal{C}(W,f)$, in $\mathcal{V}_0$,* **enriched composition with** *f, is the composite*

$$\mathcal{C}(W,X)\xrightarrow[\cong]{\lambda^{-1}_{\mathcal{C}(W,X)}}\mathbf{1}\otimes\mathcal{C}(W,X)\xrightarrow{f\otimes\mathcal{C}(W,X)}\mathcal{C}(X,Y)\otimes\mathcal{C}(W,X)\longrightarrow\mathcal{C}(W,Y),$$

where $\lambda^{-1}_{\mathcal{C}(W,X)}$ is the inverse of the left unitor of Definition 2.6.1.

Similarly, given an object Z in $\mathcal{C}$, the morphism f^*, which we will also denote by $\mathcal{C}(f,Z)$, in $\mathcal{V}_0$, **enriched precomposition with** *f, is the composite*

$$\mathcal{C}(Y,Z)\xrightarrow[\cong]{\rho^{-1}_{\mathcal{C}(Y,Z)}}\mathcal{C}(Y,Z)\otimes\mathbf{1}\xrightarrow{\mathcal{C}(Y,Z)\otimes f}\mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y)\longrightarrow\mathcal{C}(X,Z),$$

where $\rho^{-1}_{\mathcal{C}(Y,Z)}$ is the inverse of the right unitor of Definition 2.6.1.

Definition 3.1.13. Enriched functors and natural transformations. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories is in Definition 3.1.1. A* **$\mathcal{V}$-functor** *or* **enriched functor** *$F:\mathcal{C}\to\mathcal{D}$ consists of a map F from the objects of $\mathcal{C}$ to those of $\mathcal{D}$ and for each pair of objects X, Y in $\mathcal{C}$ a morphism*

$$F_{X,Y}:\mathcal{C}(X,Y)\to\mathcal{D}(F(X),F(Y)) \tag{3.1.14}$$

in $\mathcal{V}_0$ such that the following diagrams in $\mathcal{V}_0$ commute for all objects X, Y, Z in $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y) \arrow[rr, "c_{X,Y,Z}"] \arrow[d, "F_{Y,Z}\otimes F_{X,Y}"'] & & \mathcal{C}(X,Z) \arrow[d, "F_{X,Z}"] \\ \mathcal{D}(F(Y),F(Z))\otimes\mathcal{D}(F(X),F(Y)) \arrow[rr, "d_{F(X),F(Y),F(Z)}"] & & \mathcal{D}(F(X),F(Z)) \end{tikzcd} \tag{3.1.15}$$

and

$$\begin{tikzcd} \mathbf{1} \arrow[r, "1_X"] \arrow[dr, "1_{F(X)}"'] & \mathcal{C}(X,X) \arrow[d, "F_{X,X}"] \\ & \mathcal{D}(F(X),F(X)). \end{tikzcd} \tag{3.1.16}$$

Given two such functors F and G, a **$\mathcal{V}$-natural transformation** *or* **enriched natural transformation** *$\theta : F \implies G$ consists of a morphism*

$$\theta_X : \mathbf{1} \to \mathcal{D}(F(X), G(X))$$

for each object X of $\mathcal{C}$ such that for all objects X, Y of $\mathcal{C}$ the following diagram in $\mathcal{V}_0$ commutes:

$$\begin{tikzcd} \mathcal{C}(X,Y) \arrow[r, "F_{X,Y}"] \arrow[d, "G_{X,Y}"'] & \mathcal{D}(F(X),F(Y)) \arrow[d, "(\theta_Y)_*"] \\ \mathcal{D}(G(X),G(Y)) \arrow[r, "(\theta_X)^*"] & \mathcal{D}(F(X),G(Y)) \end{tikzcd} \tag{3.1.17}$$

where the morphisms $(\theta_Y)_* = \mathcal{D}(F(X), \theta_Y)$ and $(\theta_X)^* = \mathcal{D}(\theta_X, G(Y))$ are composition and precomposition as in Definition 3.1.12. We say θ is a **$\mathcal{V}$-natural equivalence** *if the image of each θ_X is an isomorphism in $\mathcal{D}_0$.*

Definition 3.1.18. *Two $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ are* **$\mathcal{V}$-equivalent** *if there are $\mathcal{V}$-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and $\mathcal{V}$-natural equivalences $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$.*

The diagram (3.1.17) above is the enriched analog of (2.2.2). It is the same as Kelly's diagram [Kel82, (1.39)].

3.1B Enriched adjunctions

Here is the enriched analog of Definition 2.2.13. See [Kel82, §1.11] for more discussion.

Definition 3.1.19. Enriched adjunctions. *A pair (F, G) of $\mathcal{V}$-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ between $\mathcal{V}$-categories is* **adjoint pair** *or* **$\mathcal{V}$-adjunction** *if there is a natural isomorphism of objects* **in** *$\mathcal{V}$*

$$\varphi : \mathcal{D}(FX, Y) \xrightarrow{\cong} \mathcal{C}(X, UY)$$

for each object X in $\mathcal{C}$ and Y in $\mathcal{D}$. We say that G is the **right adjoint** *of F, F is the* **left adjoint** *of G, and φ is the* **adjunction isomorphism**.

The other notions of Definition 2.2.13 are defined similarly.

Proposition 3.1.20. The 2-category of $\mathcal{V}$-categories. *Recall the 2-categories $\mathcal{VCAT}$ and $\mathcal{VC}at$ (Example 2.7.2(iii)). In them objects (i.e., $\mathcal{V}$-categories) are equivalent as in Definition 2.7.3 if the underlying categories are equivalent as in Definition 2.2.4 with the relevant functors and natural equivalences being $\mathcal{V}$-functors and $\mathcal{V}$-natural equivalences.*

The following is proved by Riehl as [Rie14, Lemma 3.4.3] and by Geoffrey Cruttwell as [Cru08, Proposition 4.2.1].

Proposition 3.1.21. Changing the base of enrichment. *Suppose we have a second closed symmetric monoidal category $\mathcal{W} = (\mathcal{W}_0, \times, *)$ and a lax monoidal functor $L : \mathcal{V} \to \mathcal{W}$ as in Definition 2.6.19. Then for a $\mathcal{V}$-category $\mathcal{C}$ there is a $\mathcal{W}$-category $L_*\mathcal{C}$ having the same objects as $\mathcal{C}$ in which the morphism objects are the images of those in $\mathcal{C}$ (which lie in $\mathcal{V}$) under the functor L.*

Cruttwell proves more than this in [Cru08, Theorem 4.2.4]. Recall that the class of $\mathcal{V}$-categories ($\mathcal{W}$-categories) forms a 2-category $\mathcal{VCAT}$ ($\mathcal{WCAT}$) as in Example 2.7.2(iii).

Proposition 3.1.22. More about change of enrichment base. *A lax monoidal functor $L : \mathcal{V} \to \mathcal{W}$ as in Proposition 3.1.21 induces a functor of 2-categories,*

$$L_* : \mathcal{VCAT} \to \mathcal{WCAT}.$$

This was first proved by Eilenberg and Kelly in [EK66, §6].

Proposition 3.1.21 describes the effect of this functor on objects or 0-cells in $\mathcal{VCAT}$. For 1-cells, Proposition 3.1.22 means that given a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$, we get a $\mathcal{W}$-functor

$$L_*F : L_*\mathcal{C} \to L_*\mathcal{D}$$

with the expected properties. For 2-cells, given a $\mathcal{V}$-natural transformation $\theta : F \Rightarrow G$ we get a $\mathcal{W}$-natural taansformation

$$L_*\theta : L_*F \Rightarrow L_*G$$

with the expected properties.

In particular the functor

$$V = \mathcal{V}_0(\mathbf{1}, -) : \mathcal{V} \to (\mathcal{S}et, \times, *) \tag{3.1.23}$$

is lax monoidal. It converts the $\mathcal{V}$-category $\mathcal{C}$ to the ordinary (meaning enriched over $\mathcal{S}et$) category $\mathcal{C}_0$.

Definition 3.1.24. The free $\mathcal{V}$-category generated by an ordinary category. *Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a closed symmetric monoidal category in which $\mathcal{V}_0$ is cocomplete, and let I be the monoidal functor*

$$I = \coprod_{(\,-\,)} \mathbf{1} : \mathcal{S}et \to \mathcal{V}_0. \tag{3.1.25}$$

that sends a set X to the coproduct of the unit object indexed by X, which we denote by $X \cdot \mathbf{1}$. In particular it sends the empty set to the initial object of $\mathcal{V}_0$, and it sends $X \times Y$ to $(X \cdot \mathbf{1}) \otimes (Y \cdot \mathbf{1})$. Using Proposition 3.1.21, for any ordinary category $\mathcal{C}$, we get a $\mathcal{V}$-category $\mathcal{C}_{\mathcal{V}} = I_*\mathcal{C}$, the **free $\mathcal{V}$-category generated by $\mathcal{C}$**.

The following was proved by Kelly in [Kel82, §2.5].

Proposition 3.1.26. An adjunction of 2-categories. *Let $\mathcal{V} = (\mathcal{V}_0, \times, \mathbf{1})$ be a closed symmetric monoidal category with $\mathcal{V}_0$ cocomplete. Then the functors V of (3.1.23) and I of (3.1.25) induce functors $\mathcal{VCAT} \to \mathcal{CAT}$ ($\mathcal{V}\mathcal{C}at \to \mathcal{C}at$) and $\mathcal{CAT} \to \mathcal{VCAT}$ ($\mathcal{C}at \to \mathcal{V}\mathcal{C}at$), which we denote by $(\,-\,)_0$ and $(\,-\,)_{\mathcal{V}}$. Moreover $(\,-\,)_{\mathcal{V}}$ is the left adjoint of $(\,-\,)_0$.*

Remark 3.1.27. Categories enriched over concrete $\mathcal{V}$. *Following [Kel82, page 8], we denote the functor $\mathcal{V}_0(\mathbf{1}, -) : \mathcal{V}_0 \to \mathcal{S}et$ by V. It may or may not be faithful in general. It* **is** *faithful in the cases of greatest interest in this book, namely when $\mathcal{V}_0$ is a category of topological spaces and continuous maps, possibly with additional structure such as a base point, a group action or both. These categories are* **concrete** *as in Definition 2.1.9.*

If $\mathcal{C}$ is a $\mathcal{V}$-category as in Definition 3.1.1 for concrete $\mathcal{V}_0$, its morphism objects $\mathcal{C}(X, Y)$ **can** *be regarded as sets with additional structure. Composition of morphisms is as in the ordinary category $\mathcal{C}_0$, and it respects the additional structure.* **No information is lost by passing from $\mathcal{C}$ to $\mathcal{C}_0$.**

The following is taken from [Kel82, page 12] where the evident composition rule is spelled out.

Definition 3.1.28. The product of two $\mathcal{V}$-categories. *Let $\mathcal{C}$ and $\mathcal{C}'$ be $\mathcal{V}$-categories (Definition 3.1.1) for $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ as above. Then their product $\mathcal{C} \otimes \mathcal{C}'$ is the $\mathcal{V}$-category whose object class is*

$$Ob\,(\mathcal{C} \otimes \mathcal{C}') = Ob\,\mathcal{C} \times Ob\,\mathcal{C}'$$

and whose morphism objects are

$$(\mathcal{C} \otimes \mathcal{C}')((X, X'), (Y, Y')) = \mathcal{C}(X, Y) \otimes \mathcal{C}'(X', Y')$$

for X and Y in $\mathcal{C}$, and X' and Y' in $\mathcal{C}'$.

If the binary operation in $\mathcal{V}$ were denoted by a symbol other than $\otimes$, we would also use that symbol to denote this product.

The following was proved by Kelly in [Kel82, §2.4] and by Borceaux [Bor94b, Theorem 6.3.5].

Enriched Yoneda Lemma 3.1.29. *For an object K in a $\mathcal{V}$-category $\mathcal{C}$, consider the covariant $\mathcal{V}$-valued functor $よ^K = \mathcal{C}(K, -)$ (the* **enriched Yoneda functor***) on $\mathcal{C}$. Let F be another such functor, so both F and $よ^K$ are objects in the enriched functor category $[\mathcal{C}, \mathcal{V}]$. Then the $\mathcal{V}$ object of natural transformations from $よ^K$ to F, that is $[\mathcal{C}, \mathcal{V}](よ^K, F)$, is $F(K)$.*

Remark 3.1.30. Typo warning. *There appears to be a typo in Kelly's statement of the enriched Yoneda isomorphism, [Kel82, (2.31)]. The right hand side should read $[\mathcal{A}, \mathcal{V}](\mathcal{A}(K, -), F)$, where Kelly's category $\mathcal{A}$ is our $\mathcal{C}$, so his Yoneda functor $\mathcal{A}(K, -)$ is our $よ^K$. The left hand side of [Kel82, (2.31)] is $F(K)$.*

3.1C Tensors and cotensors

Definition 3.1.31. Tensor products and cotensor products. *Let $\mathcal{C}$ be a category enriched over a closed symmetric monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$. Then $\mathcal{C}$* **is tensored (or copowered) over** *$\mathcal{V}$ if for each object (K, X) in $\mathcal{V} \times \mathcal{C}$, there are objects $K \otimes X$, the* **left tensor product***, and $X \otimes K$, the* **right tensor product***, in $\mathcal{C}$, with natural isomorphisms in $\mathcal{V}$,*

$$\mathcal{C}(K \otimes X, Y) \cong \mathcal{C}(X \otimes K, Y) \cong \mathcal{V}(K, \mathcal{C}(X, Y)) \tag{3.1.32}$$

for each object Y of $\mathcal{C}$.

In other words tensoring with X on either side as a functor $\mathcal{V} \to \mathcal{C}$ is left adjoint of the functor $\mathcal{C}(X, -)$ from $\mathcal{C}$ to $\mathcal{V}$. For each object Y in $\mathcal{C}$, the counit (as in Definition 2.2.20) of this adjunction is a map

$$\epsilon_Y : \mathcal{C}(X, Y) \otimes X \to Y, \tag{3.1.33}$$

the **evaluation map***. In the case of an ordinary category it sends the pair $(f : X \to Y, X)$ to Y. For each object K in $\mathcal{V}$, the unit is a map*

$$\eta_K : K \to \mathcal{C}(X, K \otimes X),$$

the **coevaluation map***.*

Dually, $\mathcal{C}$ **is cotensored (or powered) over** *$\mathcal{V}$ if for each object (K, Y) in $\mathcal{V} \times \mathcal{C}$ there is an object Y^K in $\mathcal{C}$ with a composite natural isomorphism in $\mathcal{V}$,*

$$\mathcal{C}^{op}(Y^K, X) \cong \mathcal{C}(X, Y^K) \cong \mathcal{C}(X \otimes K, Y) \cong\cong \mathcal{V}(K, \mathcal{C}(X, Y)) \tag{3.1.34}$$

for each object X of $\mathcal{C}$. In this case there is no chirality as in the tensor product case.

In other words the functor $Y^{(-)} : \mathcal{V}^{op} \to \mathcal{C}$, which can also be thought of as *a functor $\mathcal{V}$ to $\mathcal{C}^{op}$, is the left adjoint of the functor $\mathcal{C}(-,Y) : \mathcal{C}^{op} \to \mathcal{V}$. For each object X in $\mathcal{C}$, the counit of this adjunction, meaning the map on the left adjoint to the identity on the right in the case $K = \mathcal{C}(X,Y)$, is a morphism in $\mathcal{C}^{op}$, and we denote its opposite in $\mathcal{C}$ by*

$$\epsilon_X : X \to Y^{\mathcal{C}(X,Y)}. \tag{3.1.35}$$

For each object K in $\mathcal{V}$, the unit, meaning the right adjoint of the identity on the left in the case $X = Y^K$, is

$$\eta_K : K \to \mathcal{C}(Y^K, Y).$$

When $\mathcal{C}$ is both tensored and cotensored over $\mathcal{V}$, we say it is **bitensored over** $\mathcal{V}$.

A closed $\mathcal{V}$-module $\mathcal{C}$ as in Definition 2.6.42 is bitensored over $\mathcal{V}$.

Remark 3.1.36. Tensor (cotensor) products and colimits (limits). *The tensor (cotensor) product is a colimit (limit). Hence it is often convenient to assume that the categories $\mathcal{V}$ and $\mathcal{C}$ are bicomplete (Definition 2.3.25), as are model categories (to be introduced below in Chapter 4) by definition.*

Proposition 3.1.37. Cotensor commutativity. *Let $\mathcal{C}$ and $\mathcal{V}$ be as in Definition 3.1.31 with $\mathcal{V}$ symmetric monoidal. Then for objects K and L in $\mathcal{V}$ and Y in $\mathcal{C}$, there are natural isomorphisms*

$$(Y^K)^L \cong (Y^{L\otimes K}) \cong (Y^{K\otimes L}) \cong (Y^L)^K.$$

Proof For any object X in $\mathcal{C}$ we have

$$\begin{aligned}
\mathcal{C}(X,(Y^K)^L) &\cong \mathcal{V}(L, \mathcal{C}(X, Y^K)) && \text{by (3.1.34)}\\
&\cong \mathcal{V}(L, \mathcal{V}(K, \mathcal{C}(X,Y)) && \text{by (3.1.34) again}\\
&\cong \mathcal{V}(L\otimes K, \mathcal{C}(X,Y)) && \text{by (2.6.34)}\\
&\cong \mathcal{C}(X, Y^{L\otimes K}) && \text{by (3.1.34) a third time.}
\end{aligned}$$

Now we use the symmetry of $\mathcal{V}$ to interchange K and L to conclude that there is a natural isomorphism

$$\begin{aligned}
&\mathcal{C}(X,(Y^K)^L) \cong \mathcal{C}(X,(Y^L)^K)\\
\text{so}\qquad &\mathcal{C}_0(X,(Y^K)^L) \cong \mathcal{C}_0(X,(Y^L)^K).
\end{aligned}$$

Now let $X = (Y^K)^L$. Then the natural isomorphism above sends the identity morphism in the morphism set on the left to the desired isomorphism in the morphism set on the right. The other isomorphisms follow for similar reasons. □

When $F : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is a $\mathcal{V}$-category that need not be tensored over $\mathcal{V}$, for objects A and B in $\mathcal{D}$ we have the composite

$$\begin{tikzcd} \mathcal{D}(A,B) \otimes F(A) \arrow[r, "F_{A,B} \otimes F(A)"] \arrow[dr, "\epsilon^F_{A,B}"'] & \mathcal{C}(F(A), F(B)) \otimes F(A) \arrow[d, "\epsilon_{F(B)}"] \\ & F(B), \end{tikzcd} \tag{3.1.38}$$

the **composition map** or **structure map**.

For a functor F as above, we have a composite

$$\begin{tikzcd} F(A) \arrow[d, "\eta_{F(A)}"'] \arrow[dr, "\eta^F_{A,B}"] & \\ F(B)^{\mathcal{C}(F(A),F(B))} \arrow[r, "F(B)^{F_{A,B}}"'] & F(B)^{\mathcal{D}(A,B)}, \end{tikzcd} \tag{3.1.39}$$

the **cocomposition map** or **costructure map**.

When $\mathcal{C}$ is bitensored over $\mathcal{V}$, the map $\epsilon^F_{A,B}$ is adjoint to $\eta^F_{A,B}$ under the adjunction

$$\mathcal{C}(\mathcal{D}(A,B) \otimes F(A), F(B)) \cong \mathcal{C}(F(A), F(B)^{\mathcal{D}(A,B)}).$$

Next we give the enriched analog of Definition 2.2.6.

Definition 3.1.40. Enriched composition and precomposition as enriched natural transformations. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories (where $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$) with $\mathcal{C}$ bitensored over $\mathcal{V}$ as in Definition 3.1.31. Let*

$$H : \mathcal{D}^{op} \otimes \mathcal{D} \to \mathcal{C},$$

where the tensor product of categories is as in Definition 3.1.28, be a $\mathcal{V}$-functor as in Definition 3.1.13. For a fixed object B in $\mathcal{D}$, consider another such functor

$$\begin{tikzcd} \mathcal{D}^{op} \otimes \mathcal{D} \arrow[r, "H_B"] & \mathcal{C} \\ (A, C) \arrow[r, mapsto] & \mathcal{D}(B,C) \otimes H(A,B), \end{tikzcd}$$

in which the object in $\mathcal{C}$ is a left tensor product (as in Definition 3.1.31) with the object $\mathcal{D}(B,C)$ in $\mathcal{V}$. Then we define a natural transformation $\theta^B : H_B \implies H$ (as in Definition 3.1.13) by

$$\theta^B_{A,C} = \epsilon^{H(A,-)}_{B,C} : \mathcal{D}(B,C) \otimes H(A,B) \to H(A,C) \tag{3.1.41}$$

for $\epsilon^{H(A,\,-\,)}_{A,C}$ as in (3.1.38), in which the superscript is a functor. We call this **composition at** *B. It is adjoint to a map*

$$\hat{\theta}^B_{A,C} : H(A,B) \to H(A,C)^{\mathcal{D}(B,C)} \tag{3.1.42}$$

Similarly, for an object A in $\mathcal{D}$ consider the functor

$$\begin{tikzcd} \mathcal{D}^{op}\otimes\mathcal{D} \arrow[rr, "H^A"] & & \mathcal{C} \\ (D,B) \arrow[r, mapsto] & H(A,B)\otimes\mathcal{D}(D,A) \end{tikzcd}$$

in which the object in $\mathcal{C}$ is a right tensor product with the object $\mathcal{D}(A,B)$ in $\mathcal{V}$. Then define $\kappa^A : H^A \implies H$, **precomposition at** *A, as follows. For an object (D,B) in $\mathcal{D}^{op}\times\mathcal{D}$, we have*

$$\widehat{\kappa}^A_{D,B} = \eta^{H(-,B)}_{A,D} : H(A,B) \to H(D,B)^{\mathcal{D}^{op}(A,D)} = H(D,B)^{\mathcal{D}(D,A)} \quad (3.1.43)$$

where $H(-,B)$ is a functor $\mathcal{D}^{op}\to\mathcal{C}$ and $\eta^{H(-,B)}_{A,D}$ is as in (3.1.39) with $\mathcal{D}$ replaced by its opposite. We define

$$\kappa^A_{D,B} : \mathcal{D}(D,A)\otimes H(A,B) \cong H(A,B)\otimes\mathcal{D}(D,A) \to H(D,B) \quad (3.1.44)$$

to be the adjoint of the map of (3.1.43).

The enriched analog of (2.2.7) is the following

$$\begin{tikzcd} & \mathcal{D}(C,D)\otimes H(B,C)\otimes\mathcal{D}(A,B) \arrow[dl, "\theta^C_{B,D}\otimes\mathcal{D}(A,B)"'] \arrow[dr, "\mathcal{D}(C,D)\otimes\kappa^B_{A,C}"] & \\ H(B,D)\otimes\mathcal{D}(A,B) \arrow[dr, "\kappa^B_{A,D}"'] & & \mathcal{D}(C,D)\otimes H(A,C) \arrow[dl, "\theta^C_{A,D}"] \\ & H(A,D). & \end{tikzcd} \quad (3.1.45)$$

It is adjoint to the following

$$\begin{tikzcd} & H(B,C) \arrow[dl, "\widehat{\theta}^C_{B,D}"'] \arrow[dr, "\widehat{\kappa}^B_{A,C}"] & \\ H(B,D)^{\mathcal{D}(C,D)} \arrow[d, "\left(\widehat{\kappa}^B_{A,D}\right)^{\mathcal{D}(C,D)}"'] & & H(A,C)^{\mathcal{D}(A,B)} \arrow[d, "\left(\widehat{\theta}^C_{A,D}\right)^{\mathcal{D}(A,B)}"] \\ H(A,D)^{\mathcal{D}(A,B)\otimes\mathcal{D}(C,D)} \arrow[rr, "\cong"] & & H(A,D)^{\mathcal{D}(C,D)\otimes\mathcal{D}(A,B)}, \end{tikzcd} \quad (3.1.46)$$

where the bottom isomorphism that of Proposition 3.1.37.

A $\mathcal{V}$-category $\mathcal{C}$ that is both tensored and cotensored over $\mathcal{V}$ as in Definition 3.1.31 is the same thing as a closed $\mathcal{V}$-module as in Definition 2.6.42. The following, in which the two variable adjunction is the same as that of Definition 2.6.42, is an immediate consequence of these definitions.

Proposition 3.1.47. Tensor and cotensor products as components of a two variable adjunction. *The two structures in Definition 3.1.31 together are equivalent to a two variable adjunction (Definition 2.6.26) with the categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ replaced by $\mathcal{V}$, $\mathcal{C}$ and $\mathcal{C}$ respectively, the three functors given by*

$$\begin{array}{ll}
\mathcal{V} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C} & (K, X) \longmapsto K \otimes X \\
\mathcal{V}^{op} \times \mathcal{C} \xrightarrow{\mathrm{Hom}_\ell} \mathcal{C} & (K, Y) \longmapsto Y^K \\
\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathrm{Hom}_r} \mathcal{V} & (X, Y) \longmapsto \mathcal{C}(X, Y),
\end{array}$$

and the two natural isomorphisms being

$$\mathcal{C}(X, Y^K) \xleftarrow[\cong]{\phi_\ell} \mathcal{C}(K \otimes X, Y) \xrightarrow[\cong]{\phi_r} \mathcal{V}(K, \mathcal{C}(X, Y)). \qquad (3.1.48)$$

Example 3.1.49. Tensoring and cotensoring over $\mathcal{S}et$. *Any cocomplete (complete) category $\mathcal{C}$ is tensored (cotensored) over $\mathcal{S}et$, the tensor (cotensor) product of an object in $\mathcal{C}$ with a set K being the coproduct (product) indexed by K.* **Note the reversal here of the placement of the prefix "co."**

Proposition 3.1.50. Tensoring a G-category with $\mathcal{B}G$ washes out the G-action. *Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a closed symmetric monoidal category (Definition 2.6.1 and Definition 2.6.33) which is bitensored over $\mathcal{S}et$ (Definition 3.1.31). Then there is a similar structure on $\mathcal{V}^{\mathcal{B}G}$, the category of objects in $\mathcal{V}$ with G-action for a finite group G. It is bitensored over $\mathcal{S}et^{\mathcal{B}G}$, the category of G-sets. Then $\mathcal{B}G$, the one object category for G as in Example 2.9.1, is enriched over $\mathcal{S}et^{\mathcal{B}G}$ and therefore over $\mathcal{V}^{\mathcal{B}G}$.*

Let $\mathcal{C}$ be a $\mathcal{V}^{\mathcal{B}G}$-category. Let $\overline{\mathcal{C}}$ be the same category with trivial G-action on its morphism objects. Then the categories $\mathcal{B}G \otimes \mathcal{C}$ (where the product of the two categories is as in Definition 3.1.28) and $\mathcal{B}G \otimes \overline{\mathcal{C}}$ are isomorphic.

See Definition 9.4.10 below for a definition that is similar in spirit.

Proof The categories $\mathcal{B}G \otimes \mathcal{C}$ and $\mathcal{B}G \otimes \overline{\mathcal{C}}$ have the same objects, namely those of $\mathcal{C}$, since $\mathcal{B}G$ has one object. We will define an isomorphism functor $F : \mathcal{B}G \otimes \mathcal{C} \to \mathcal{B}G \otimes \overline{\mathcal{C}}$ which is the identity on objects. Given objects X and Y in $\mathcal{C}$, we will use the same symbols for the corresponding objects in $\mathcal{B}G \otimes \mathcal{C}$ and $\mathcal{B}G \otimes \overline{\mathcal{C}}$. We need to define the morphism

$$(\mathcal{B}G \otimes \mathcal{C})(X, Y) = G \otimes \mathcal{C}(X, Y) \to G \otimes \overline{\mathcal{C}}(X, Y) = (\mathcal{B}G \otimes \overline{\mathcal{C}})(X, Y)$$

in $\mathcal{V}^{\mathcal{B}G}$ induced by F. Since $\mathcal{V}$ is concrete, we can treat $\mathcal{C}(X, Y)$ as a G-set and define the map for $\gamma \in G$ and $z \in \mathcal{C}(X, Y)$ by

$$\gamma \otimes z \mapsto \gamma \otimes \gamma^{-1}(z).$$

The group G acts diagonally on the left, and on the first factor on the right. Thus for $\alpha \in G$ we have

$$F(\alpha(\gamma \otimes z)) = F(\alpha\gamma \otimes \alpha z) = \alpha\gamma \otimes (\alpha\gamma)^{-1}\alpha z = \alpha\gamma \otimes \gamma^{-1}z = \alpha F(\gamma \otimes z),$$

so F is the desired isomorphism. □

3.1D Enriched monoidal categories

Definition 3.1.51. An enriched monoidal category $\mathcal{C} = (\mathcal{C}_0, \oplus, \mathbf{0})$ *is a category* $\mathcal{C}$ *enriched over a symmetric monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ *(Definition 2.6.1) with a* $\mathcal{V}$*-functor (see Definition 3.1.13)*

$$\oplus : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$$

(where $\mathcal{C} \otimes \mathcal{C}$ *is as in Definition 3.1.28) and a unit object* $\mathbf{0}$ *with natural* $\mathcal{V}$*-isomorphisms*

$$a_{X,Y,Z} : (X \oplus Y) \oplus Z \cong X \oplus (Y \oplus Z), \quad \rho_X : X \oplus \mathbf{1} \cong X \quad \text{and} \quad \lambda_X : \mathbf{1} \oplus X \cong X$$

for all objects X*,* Y *and* Z*, called the* **associator**, **right unitor** *and* **left unitor***. The monoidal category* $\mathcal{C}$ *is* **symmetric** *if in addition there is a natural twist isomorphism*

$$\tau_{X,Y} : X \oplus Y \cong Y \oplus X.$$

as in Equation 2.6.2. These $\mathcal{V}$*-natural isomorphisms are components of* $\mathcal{V}$*-natural equivalences as in Definition 3.1.13; see Remark 2.6.5. In the underlying category* $\mathcal{C}_0$ *they are required to satisfy the coherence conditions of Definition 2.6.1.*

Remark 3.1.52. Enriched addition functors and morphisms. *Let* $\mathcal{C} = (\mathcal{C}_0, \oplus, \mathbf{0})$ *be a symmetric monoidal category enriched over* $\mathcal{V}$ *as in Definition 3.1.51. Then for each object* A *in* $\mathcal{C}_0$ *we can define addition functors* α_A *and* ω_A *as in Definition 2.6.6, along with morphisms* $\alpha_{A,X,Y}$ *and* $\omega_{A,X,Y}$ **in** $\mathcal{V}$ *(rather than in* $\mathcal{S}et$*) for each pair of objects* X *and* Y *in* $\mathcal{C}_0$*.*

It follows that we have commutativity of the diagrams of Proposition 2.6.7, which are now **diagrams in** $\mathcal{V}$ *rather than in* $\mathcal{S}et$*.*

3.1E Liftings in enriched categories

Now we will discuss liftings in enriched categories. If $\mathcal{C}$ is a $\mathcal{V}$-category, then we do not have morphisms between objects X and Y in $\mathcal{C}$, but only morphism objects $\mathcal{C}(X, Y)$ in $\mathcal{V}$. We need to replace these by the corresponding morphism sets in the underlying ordinary category $\mathcal{C}_0$ as in (3.1.3). Only then can we speak of morphisms i and p and define a lifting test map $\mathcal{C}_0(i^*, p_*)$ as in

Definition 2.3.14. Proposition 2.3.18 then applies to the analog of (2.3.11) in the ordinary category $\mathcal{C}_0$.

The following result is similar to [MMSS01, Lemma 5.16] and [HSS00, Corollary 3.3.9]. It will be used in the proofs of Theorem 7.3.36 and Theorem 7.4.52 below.

Proposition 3.1.53. The right lifting property with respect to a pushout corner map. *With notation as in Definition 2.6.12, suppose that $\mathcal{C}$ is a closed symmetric monoidal category and that $\mathcal{D} = \mathcal{E}$, which is enriched and bitensored (see Definition 3.1.31) over $\mathcal{C}$. Let $i : A \to B$ be a morphism in $\mathcal{C}_0$ and let $f : X \to Y$ and $g : W \to Z$ be morphisms in $\mathcal{E}_0$.*

Then the following are equivalent:

(i) The morphism g has the right lifting property with respect to $i \,\Box\, f$.

(ii) The pullback corner map for

$$\begin{tikzcd} W^B \arrow[r, "g_*"] \arrow[d, "i^*"'] & Z^B \arrow[d, "i^*"'] \\ W^A \arrow[r, "g_*"] & Z^A, \end{tikzcd}$$

has the right lifting property with respect to f.

(iii) The lifting test map $(\mathcal{E}_0)_\diamond(f, g)$ of Definition 2.3.14 in $\mathcal{C}_0$, meaning the pullback corner map for

$$\begin{tikzcd} \mathcal{E}_0(Y, W) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathcal{E}_0(Y, Z) \arrow[d, "f^*"'] \\ \mathcal{E}_0(X, W) \arrow[r, "g_*"] & \mathcal{E}_0(X, Z), \end{tikzcd}$$

has the right lifting property with respect to i.

The three equivalent statements above each say that there is a lifting pair (see Definition 2.3.10) consisting of one of the maps i, f and g and a map constructed from the other two as either a pushout corner map (in the case of g) or a pullback corner map. The map g is on the right of its lifting pair, while i and f are on the left. Each statement is equivalent to the assertion that the map on the left side has the left lifting property with respect to the one on the right.

Proof Consider the cubical diagram of sets

$$\begin{tikzcd}
 & & \mathcal{E}_0(B\otimes Y, W) \arrow[dll, "g_*"'] \arrow[d, "f^*"] \arrow[drr, "i^*"] & & \\
\mathcal{E}_0(B\otimes Y, Z) \arrow[d, "f^*"'] \arrow[drr, "i^*"] & & \mathcal{E}_0(B\otimes X, W) \arrow[dll, "g_*"'] \arrow[drr, "i^*"] & & \mathcal{E}_0(A\otimes Y, W) \arrow[dll, "g_*"'] \arrow[d, "f^*"] \\
\mathcal{E}_0(B\otimes X, Z) \arrow[drr, "i^*"] & & \mathcal{E}_0(A\otimes Y, Z) \arrow[d, "f^*"] & & \mathcal{E}_0(A\otimes X, W) \arrow[dll, "g_*"'] \\
 & & \mathcal{E}_0(A\otimes X, Z) & &
\end{tikzcd} \tag{3.1.54}$$

Each set in it can be written in three different ways. For example

$$\mathcal{E}_0(B\otimes Y, W) \cong \mathcal{E}_0(Y, W^B) \cong \mathcal{C}_0(B, \mathcal{E}(Y, W)), \tag{3.1.55}$$

where the objects W^B of $\mathcal{E}$ and $\mathcal{E}(Y, W)$ of $\mathcal{C}$ are also objects of $\mathcal{E}_0$ and $\mathcal{C}_0$ respectively.

There is a map from this set, the one in the top row of (3.1.54), to the limit of the diagram obtained from (3.1.54) by removing the top row. This limit is by definition a triple pullback. By Proposition 2.3.53 it can be described as a simple pullback in three different ways, namely those of the three diagrams below, in which $R(\alpha, \beta)$ denotes the pullback of two maps α and β having the same target.

$$\begin{array}{c}
\begin{tikzcd}
R(\mathcal{E}_0(i\otimes X, W), \mathcal{E}_0(A\otimes f, W)) \arrow[dr] & & \mathcal{E}_0(B\otimes Y, Z) \arrow[dl] \\
 & R(\mathcal{E}_0(i\otimes X, Z), \mathcal{E}_0(A\otimes f, Z)) &
\end{tikzcd} \\[2em]
\begin{tikzcd}
R(\mathcal{E}_0(i\otimes Y, Z), \mathcal{E}_0(A\otimes Y, g)) \arrow[dr] & & \mathcal{E}_0(B\otimes X, W) \arrow[dl] \\
 & R(\mathcal{E}_0(i\otimes X, Z), \mathcal{E}_0(A\otimes X, g)) &
\end{tikzcd} \\[2em]
\begin{tikzcd}
R(\mathcal{E}_0(B\otimes f, Z), \mathcal{E}_0(B\otimes X, g)) \arrow[dr] & & \mathcal{E}_0(A\otimes Y, W) \arrow[dl] \\
 & R(\mathcal{E}_0(A\otimes f, Z), \mathcal{E}_0(A\otimes X, g)) &
\end{tikzcd}
\end{array} \tag{3.1.56}$$

These three descriptions of the source in (3.1.55) and the target in (3.1.56) of the triple pullback corner map translate into those of the lifting test maps for the three stated right lifting properties. We leave the remaining details to the reader. □

Example 3.1.57. The topological case with $Z = *$. *Let*

$$\mathcal{C} = (\mathcal{T}op, \times, *),$$

let i be the standard inclusion $S^{n-1} \to D^n$ for some integer $n \geqslant 0$, and suppose that $Z = *$. Then in Proposition 3.1.53 (iii), the pullback corner map is f^*, so g has the right lifting property with respect to the corner map $i \square f$ iff f^* has it with respect to i. In the case $\mathcal{D} = \mathcal{T}op$, this is proved by Hirschhorn as [Hir03, Proposition 1.3.3]. Similarly in (ii) the pullback corner map is i^*, so g has the right lifting property with respect to the $i \square f$ iff i^* has it with respect to f. In the case where $\mathcal{D}$ is a simplicial model category, Proposition 3.1.53 is comparable to [Hir03, Lemma 9.4.7].

Corollary 3.1.58. Formulation in terms of the arrow categories. *With notation as in Proposition 3.1.53, let $\mathcal{C}_1$ and $\mathcal{E}_1$ denote the arrow categories for $\mathcal{C}_0$ and $\mathcal{E}_0$. Consider two variable adjunction as in Proposition 3.1.47 with $\mathcal{V}$ and $\mathcal{C}$ replaced by $\mathcal{C}_1$ and $\mathcal{E}_1$. The isomorphisms of (3.1.48) are*

$$\mathcal{C}_1(i, \mathcal{E}_0(f^*, g_*)) \xleftarrow[\cong]{\phi_r} \mathcal{E}_1(i \square f, g) \xrightarrow[\cong]{\phi_\ell} \mathcal{E}_1(f, \mathcal{E}_1'(i^*, g_*)).$$

If any of these three pairs of morphisms is a lifting pair (Definition 2.3.10), then the other two are as well.

3.1F Continuous group actions

Definition 3.1.59. Enriched categories related to continuous group actions. *Consider the category $\mathcal{C}$ whose objects are compactly generated (pointed) weak Hausdorff spaces equipped with an action of a fixed group G (that fixes the base point). We could define a morphism to be any continuous (pointed) map, not necessarily equivariant. We will sometimes use the term* **nonequivariant** *as shorthand for* **not necessarily equivariant**. *In that case the morphism set is itself a (pointed) G-space. The action of $\gamma \in G$ on a map $f : X \to Y$ is $\gamma(f) = \gamma f \gamma^{-1}$ as indicated in the following diagram.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma^{-1}"'] & Y \\ X \arrow[r, "f"] & Y \arrow[u, "\gamma"'] \end{tikzcd} \tag{3.1.60}$$

We will denote this category by $\mathcal{T}op_G$ ($\mathcal{T}_G$). (In [HHR16] the latter is denoted by $\underline{\mathcal{T}}_G$; we are dropping the underline.) Alternatively we could consider only equivariant maps, which would make the morphism set a topological space without G-action. We will denote this category by $\mathcal{T}op^G$ ($\mathcal{T}^G$).

The notions for G-sets of Definition 2.2.25 carry over to G-spaces, and we will use the same notation for them in this context. For a subgroup $H \subseteq G$ and a pointed H-space X, the induction functor, the left adjoint of the forgetful functor $i_H^G : \mathcal{T}^G \to \mathcal{T}^H$, is given by

$$X \mapsto G \underset{H}{\ltimes} X := (G \ltimes X)_H, \tag{3.1.61}$$

the orbit space for the diagonal action of H on the left half smash product of Definition 2.1.49, on which G acts by left multiplication as in the unpointed case.

Proposition 3.1.62. The fixed point set of $\mathcal{T}op_G(X,Y)$ is $\mathcal{T}op^G(X,Y)$, *and that of $\mathcal{T}_G(X,Y)$ is $\mathcal{T}^G(X,Y)$. For $H \subseteq G$,*

$$\left(i_H^G \mathcal{T}_G(X,Y)\right)^H \cong \mathcal{T}_H(i_H^G X, i_H^G Y)^H \cong \mathcal{T}^H(i_H^G X, i_H^G Y),$$

and similarly in the unpointed case. The action of G on $\mathcal{T}_G(X,Y)$ induces an action of N_H/H (where N_H is the normalizer of H in G) on this fixed point set.

Proof The diagram of (3.1.60) commutes for each $\gamma \in G$, meaning that the map f is fixed by G, iff f is equivariant. We can make a similar arguemnt for each $\eta \in H$ after applying the forgetful functor i_H^G. □

Proposition 3.1.63. Equivariance of composition in $\mathcal{T}op_G$ and $\mathcal{T}_G$. *For G-spaces X, Y and Z, the composition map*

$$\mathcal{T}op_G(Y,Z) \times \mathcal{T}op_G(X,Y) \to \mathcal{T}op_G(X,Z)$$

is equivariant, as is the map

$$\mathcal{T}_G(Y,Z) \wedge \mathcal{T}_G(X,Y) \to \mathcal{T}_G(X,Z)$$

in the pointed case. Hence $\mathcal{T}op_G$ ($\mathcal{T}_G$) is enriched over $\mathcal{T}op^G$ ($\mathcal{T}^G$).

Proof In both the unpointed and pointed cases, the group action on the morphism space is given by

$$\gamma(gf) = \gamma g f \gamma^{-1} = (\gamma g \gamma^{-1})(\gamma f \gamma^{-1}) = \gamma(g)\gamma(f),$$

which gives the desired equivariance. □

Proposition 3.1.64. Closed symmetric monoidal structures. *For any group G, both $\mathcal{T}_G$ and $\mathcal{T}^G$ ($\mathcal{T}op_G$ and $\mathcal{T}op^G$) are closed symmetric monoidal categories for which the internal Hom functor is $\mathcal{T}_G(-,-)$ ($\mathcal{T}op_G(-,-)$).*

Proof We will prove it in the pointed case. We know that $(\mathcal{T}, \wedge, S^0)$ is a closed symmetric monoidal category in which the categorical and internal Hom functors are the same. Thus we have

$$\mathcal{T}(X \wedge Y, Z) \cong \mathcal{T}(X, \mathcal{T}(Y,Z)).$$

If G is acts on the three spaces, this gives

$$\mathcal{T}_G(X \wedge Y, Z) \cong \mathcal{T}_G(X, \mathcal{T}_G(Y, Z)). \tag{3.1.65}$$

This means that $\mathcal{T}_G$ is also a closed symmetric monoidal in which the categorical and internal Hom functors are the same.

We do not expect them to be the same in $\mathcal{T}^G$ since its categorical Hom objects do not have a G-action. Instead we have

$$\begin{aligned}
\mathcal{T}^G(X, \mathcal{T}_G(Y, Z)) &\cong \mathcal{T}_G(X, \mathcal{T}_G(Y, Z))^G && \text{by Proposition 3.1.62} \\
&\cong \mathcal{T}_G(X \wedge Y, Z)^G && \text{by (3.1.65)} \\
&\cong \mathcal{T}^G(X \wedge Y, Z) && \text{by Proposition 3.1.62 again,}
\end{aligned}$$

so $\mathcal{T}_G(-,-)$ is the internal Hom functor for $\mathcal{T}^G$ as well as for $\mathcal{T}_G$. □

Definition 3.1.66. Topological categories and topological G-categories. *A* **(pointed) topological category** *is a category enriched over $\mathcal{T}op$ ($\mathcal{T}$). It is a* **(pointed) topological G-category** *if it is enriched over $\mathcal{T}op^G$ ($\mathcal{T}^G$).*

Thus $\mathcal{T}op$ and $\mathcal{T}op^G$ are both topological categories, $\mathcal{T}op_G$ is a topological G-category and a topological G-category is also a topological category. An ordinary category can be made into a topological category by endowing each of its morphism sets with the discrete topology. Since $\mathcal{T}^G$ is a subcategory of $\mathcal{T}_G$ (having the same objects but fewer morphisms), a category enriched over the former is also enriched over the latter.

Example 3.1.67. Rings and modules as one object $\mathcal{A}b$-categories and $\mathcal{A}b$-functors. *Recall that a group G can be thought of as an ordinary category with one object in which all morphisms are invertible and the set of endomorphisms under composition is isomorphic to G. Similarly a ring R can be thought of one object category $\mathcal{C}_R$ enriched over $(\mathcal{A}b, \otimes, \mathbf{Z})$. Here the endomorphism object is the abelian group underlying R and composition is the morphism $R \otimes R \to R$ given by multiplication.*

A covariant (contravariant) functor $\mathcal{C}_R \to \mathcal{A}b$ defines a left (right) R-module whose underlying abelian group is the image of the functor. A natural transformation between two such functors is equivalent to a homomorphism between the two modules. The enriched functor category $[\mathcal{C}_R, \mathcal{A}b]$ ($[\mathcal{C}_R^{op}, \mathcal{A}b]$) is isomorphic to the category of left (right) R-modules.

3.1G Yoneda this and that

The following are the enriched analogs of Yoneda Lemma 2.2.10 and Definition 2.2.32.

Definition 3.1.68. The enriched Yoneda functor $よ^D$ *of an object D in a $\mathcal{V}$-category $\mathcal{D}$ is the $\mathcal{V}$-functor $\mathcal{D}(D, -)$ in $[\mathcal{D}, \mathcal{V}]$. The* **enriched Yoneda**

embedding $よ : \mathcal{D}^{op} \to [\mathcal{D}, \mathcal{V}]$ *is given by* $D \mapsto よ^D$*; compare with Definition 2.2.12. For a* $\mathcal{V}$*-category* $\mathcal{E}$ *tensored over* $\mathcal{V}$ *(Definition 3.1.31), the* **enriched tensored Yoneda functor** $F^D : \mathcal{E} \to [\mathcal{D}, \mathcal{E}]$ *is given by*

$$X \mapsto \mathcal{D}(D, -) \otimes X$$

for each object X *in* $\mathcal{E}$*.*

Definition 3.1.69. *The* **endomorphism** $\mathcal{V}$**-category** End_D *of an object* D *is the full* $\mathcal{V}$*-subcategory of* $\mathcal{D}$ *with one object* D*. Its right action on* $\mathcal{D}(D, D')$ *by precomposition is denoted by*

$$\mu_R : \mathcal{D}(D, D') \otimes \mathcal{D}(D, D) \to \mathcal{D}(D, D').$$

Similarly the left action of $End_{D'}$ *acts on it by postcomposition is denoted by*

$$\mu_L : \mathcal{D}(D', D') \otimes \mathcal{D}(D, D') \to \mathcal{D}(D, D').$$

For a $\mathcal{V}$*-category* $\mathcal{E}$ *that is tensored over* $\mathcal{V}$*, the* **corestriction functor** $G^D : [End_D, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ *is given by*

$$(G^D X)_{D'} := \mathcal{D}(D, D') \otimes_{\mathcal{D}(D,D)} X_D$$

where the enriched functor $X : End_D \to \mathcal{E}$ *is the same thing as an object* X_D *in* $\mathcal{E}$ *equipped with a left action of the endomorphism monoid of* D*, meaning a map* $\mu_L : \mathcal{D}(D, D) \otimes X_D \to X_D$ *with suitable properties.*

We can restate Enriched Yoneda Lemma 3.1.29 as follows.

Proposition 3.1.70. Enriched Yoneda lemma revisited. *For each functor* F *in* $[\mathcal{D}, \mathcal{V}]$*, the* $\mathcal{V}$ *object of natural transformations from* $よ^D$ *to* F*, i.e.,* $[\mathcal{D}, \mathcal{V}](よ^D, F)$*, is* F_D*.*

As in Proposition 2.4.19, we will identify the object (rather than set) of natural transformations between two such functors as an enriched end, after saying what an enriched end is, below in Definition 3.2.18.

Here is the enriched analog of the Yoneda adjunction of Remark 2.2.35.

Proposition 3.1.71. The enriched Yoneda adjunction. *For an object* D *in* $\mathcal{D}$*, let the coevaluation functor* $F^D : \mathcal{V} \to [\mathcal{D}, \mathcal{V}]$ *be given by* $F^D X = よ^D \otimes X$*. Then we have an adjunction*

$$\begin{tikzcd} X \arrow[r, mapsto] & よ^D \wedge X \\ F^D : \mathcal{V} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & {[\mathcal{D}, \mathcal{V}] : \mathrm{Ev}_D} \arrow[l, shift left] \\ X_D & X. \arrow[l, mapsto] \end{tikzcd}$$

3.2 Limits, colimits, ends and coends in enriched categories

Our source for this material is [Rie14, Chapter 7].

3.2A Weighted limits and colimits

The generalizations of limits and colimits to the enriched setting are called **weighted** limits and colimits. (In [Kel82, Chapter 3] Kelly called them **indexed** limits and colimits.) In order to motivate the definition, we start with a reinterpetation of ordinary limits and colimits. Let $F : J \to \mathcal{C}$ be a functor from a small category J; we will denote its value on an object j by F_j. Then we can define a J-set $\mathcal{C}(c, F)$ by $j \mapsto \mathcal{C}(c, F(j))$ and dually a J^{op}-set $\mathcal{C}(F, c)$ by $j \mapsto \mathcal{C}(F(j), c)$. We also have the constant $*$-valued (where $*$ denotes the set with one element) J-set and J^{op}-set, both of which we also denote by $*$. Then the limit and colimit of the functor F, assuming they exist, are characterized by

$$\mathcal{C}(c, \lim F) \cong \mathcal{S}et^J(*, \mathcal{C}(c, F)) \qquad \text{and} \qquad \mathcal{C}(\operatorname{colim} F, c) \cong \mathcal{S}et^{J^{op}}(*, \mathcal{C}(F, c)).$$

In other words a morphism $c \to \lim F$ in $\mathcal{C}$ is equivalent to a natural transformation $* \Rightarrow \mathcal{C}(c, F)$ of functors $J \to \mathcal{S}et$, i.e., of J-sets.

Example 3.2.1. Pullbacks. *Let $J = (a' \to b \leftarrow a'')$, so a functor $F : J \to \mathcal{C}$ is a pullback diagram $F_{a'} \to F_b \leftarrow F_{a''}$. The J-set $\mathcal{C}(c, F)$ is the diagram of sets*

$$\mathcal{C}(c, F_{a'}) \longrightarrow \mathcal{C}(c, F_b) \longleftarrow \mathcal{C}(c, F_{a''})$$

and $\mathcal{S}et^J(*, \mathcal{C}(c, F))$ is the set of diagrams of the form

$$\begin{tikzcd} * \arrow[r, "a"] \arrow[d, "f"'] & \mathcal{C}(c, F_{a''}) \arrow[d] \\ \mathcal{C}(c, F_{a'}) \arrow[r] & \mathcal{C}(c, F_b) \end{tikzcd}$$

This set of diagrams is the pullback set

$$\mathcal{C}(c, F_{a'}) \underset{\mathcal{C}(c, F_b)}{\times} \mathcal{C}(c, F_{a''}) = \mathcal{C}(c, \underset{F}{lim}).$$

We can generalize this by replacing $*$ by another J-set (or J^{op}-set) W called the **weight** and define the **weighted limit** $\lim^W F$ and by

$$\mathcal{C}(c, \lim{}^W F) \cong \mathcal{S}et^J(W, \mathcal{C}(c, F))$$

and the **weighted colimit** $\operatorname{colim}^W F$ by

$$\mathcal{C}(\operatorname{colim}^W F, c) \cong \mathcal{S}et^{J^{op}}(W, \mathcal{C}(F, c)).$$

This concept is not all that useful in ordinary category theory because every weighted limit or colimit can be rewritten as an ordinary one. For example it can be shown that

$$\lim{}^W F = \int^{j\in J} F_j^{W_j}.$$

When $\mathcal{C} = \mathcal{S}et$, this is the set of natural transformations $W \Rightarrow F$, $\mathrm{Nat}(W, F)$.

We remind the reader that **we are superscripts for ends and subscripts for coends.**

Example 3.2.2. Some ordinary weighted limits.

(i) For the Yoneda functor $よ^j$ *of Definition 3.1.68,* $lim^{よ^j} F = F_j$.

(ii) For $\mathcal{C}$ *complete and* J *small, for any functors* $F : J \to \mathcal{C}$ *and* $K : J \to \mathcal{D}$*, the right Kan extension of* F *along* K *defined by*

$$Ran_K F(d) = \int^{j\in J} F_j^{\mathcal{D}(d, K_j)} \cong lim^{\mathcal{D}(d, K-)} F,$$

the limit of F *weighted by* $\mathcal{D}(d, K-)$.

(iii) For $\mathcal{C}$ *cocomplete and* J *small, for any functors* $F : J \to \mathcal{C}$ *and* $K : J \to \mathcal{D}$*, the left Kan extension of* F *along* K *defined by*

$$Lan_K F(d) = \int_{j\in J} \mathcal{D}(K(j), d) \otimes F(j) \cong colim^{\mathcal{D}(d, K-)} F,$$

the colimit of F *weighted by* $\mathcal{D}(K-, d)$*. For more details, see [Bor94b, Theorem 6.7.7].*

Recall from Definition 2.3.63 that a sequential colimit in an ordinary category $\mathcal{C}_0$ is one for a diagram of the form

$$X_0 \to X_1 \to X_2 \to \cdots, \tag{3.2.3}$$

which is equivalent to a $\mathcal{C}_0$-valued functor X on the category N. When $\mathcal{V}_0$ is cocomplete, there is a $\mathcal{V}$-category $N_{\mathcal{V}}$ as in Definition 3.1.24. The morphism objects in it are

$$N_{\mathcal{V}}(m, n) = \begin{cases} \varnothing & \text{for } m > n \\ \mathbf{1} & \text{for } m \leqslant n, \end{cases}$$

where $\varnothing$ and $\mathbf{1}$ are the initial and unit objects of $\mathcal{V}_0$. An object X in the $\mathcal{V}$-enriched functor category $[N_{\mathcal{V}}, \mathcal{C}]$ is a diagram of the form (3.2.3). For an object A in $\mathcal{C}$ we can apply the enriched Yoneda functor of Enriched Yoneda Lemma 3.1.29, $よ^A = \mathcal{C}(A, -)$ and a diagram

$$\mathcal{C}(A, X_0) \to \mathcal{C}(A, X_1) \to \mathcal{C}(A, X_2) \to \cdots, \tag{3.2.4}$$

in $\mathcal{V}_0$. When $\mathcal{C}_0$ and $\mathcal{V}_0$ are both complete, both diagrams have colimits, and there is a morphism

$$\operatorname*{colim}_{N_\mathcal{V}} \mathcal{C}(A, X_n) \to \mathcal{C}(A, \operatorname*{colim}_{N_\mathcal{V}} X_n). \tag{3.2.5}$$

in $\mathcal{V}_0$.

Definition 3.2.6. Finitely presented objects in a $\mathcal{V}$-category. *Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a closed symmetric monoidal category in which $\mathcal{V}_0$ is cocomplete, and let $\mathcal{C}$ be a cocomplete $\mathcal{V}$-category. An object A in $\mathcal{C}$ is* **finitely presented** *if the enriched Yoneda functor of Enriched Yoneda Lemma 3.1.29, $よ^A = \mathcal{C}(A, -)$, preserves sequential colimits, meaning that the morphism of (3.2.5) is an isomorphism.*

The following is a consequence of this definition.

Proposition 3.2.7. Enriched and ordinary finiteness. *Let $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ be a closed symmetric monoidal category in which $\mathcal{V}_0$ is cocomplete. An object A is a cocomplete $\mathcal{V}$-category $\mathcal{C}$ is finitely presented as in Definition 3.2.6 iff it is finitely presented as in Definition 2.3.63 in the ordinary category $\mathcal{C}_0$.*

In particular each morphism $A \to \operatorname*{colim}_N X$ in $\mathcal{C}_0$ factors through some X_n.

3.2B Enriched ends and coends

Recall from §2.4 that (co)ends are defined as certain (co)limits of diagrams involving (co)products in a (co)complete category $\mathcal{C}$ indexed by either the set of objects or the set of morphisms in a small category J. In the latter case the objects $H(x, y)$ being indexed depend only on the source and target of the morphisms, but not on the morphisms themselves.

We wish to generalize this to the enriched setting where the source category $\mathcal{D}$ (instead of J) and the target category $\mathcal{C}$ are both enriched over a symmetric monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$. This means we no longer have a set of morphisms in J to index over. Instead we have for each pair of objects (x, y) in $\mathcal{D}$ a morphism object $\mathcal{D}(x, y)$ in $\mathcal{V}$.

Hence in Definition 2.4.5 we replace the coproduct

$$\coprod_{f \in \operatorname{Arr} J} H(\operatorname{Cod} f, \operatorname{Dom} f) = \coprod_{x,y \in \operatorname{Ob} J} \coprod_{f \in J(x,y)} H(x, y)$$

by

$$\coprod_{x,y \in \operatorname{Ob}(\mathcal{D})} \mathcal{D}(x, y) \otimes H(x, y),$$

where the tensor product is that of the object $\mathcal{D}(x,y)$ in $\mathcal{V}$ with the object $H(x,y)$ in $\mathcal{C}$, **which we will assume to be tensored over** $\mathcal{V}$ as in Definition 3.1.31.

For an end we need to deal with products instead of coproducts. A set indexed product of copies of the same object is the same thing as a map from the indexing set to the object. Hence in Definition 2.4.5 we replace the product

$$\prod_{f\in \operatorname{Arr} J} H(\operatorname{Dom} f, \operatorname{Cod} f) = \prod_{x,y\in \operatorname{Ob} J}\ \prod_{f\in J(x,y)} H(x,y)$$

by

$$\prod_{x,y\in \operatorname{Ob}\mathcal{D}} H(x,y)^{\mathcal{D}(x,y)},$$

which is an object in $\mathcal{C}$ assuming the latter is cotensored over $\mathcal{V}$ as in (3.1.34).

Note that if $\mathcal{V}$ is a **closed** symmetric monoidal category as in Definition 2.6.33 and $\mathcal{C}$ is a closed $\mathcal{V}$-module as in Definition 2.6.42, then $\mathcal{C}$ is bitensored over $\mathcal{V}$.

When $\mathcal{C}$ is tensored over $\mathcal{V}$, there are morphisms

$$\begin{aligned} \theta^y_{x,z} &: \mathcal{D}(y,z)\otimes H(x,y)\to H(x,z) \\ \text{and}\qquad \kappa^x_{w,y} &: \mathcal{D}(w,x)\otimes H(x,y)\to H(w,y) \end{aligned} \tag{3.2.8}$$

as in (3.1.41) and (3.1.44) for objects w, x, y and z in $\mathcal{D}$. In particular for $z=x$ and for $w=y$ respectively, we have

$$\mathcal{D}(y,x)\otimes H(x,y)\to H(x,x) \quad\text{and}\quad \mathcal{D}(y,x)\otimes H(x,y)\to H(y,y). \tag{3.2.9}$$

When $\mathcal{C}$ is cotensored over $\mathcal{V}$ we have maps

$$\begin{aligned} \widehat{\theta}^z_{x,y} &: H(x,y)\to H(x,z)^{\mathcal{D}(y,z)} \\ \text{and}\qquad \widehat{\kappa}^w_{x,y} &: H(x,y)\to H(w,y)^{\mathcal{D}(w,x)}, \end{aligned} \tag{3.2.10}$$

as in (3.1.42) and (3.1.43). In particular for $y=x$, we have

$$H(x,x)\to H(x,z)^{\mathcal{D}(x,z)} \qquad\text{and}\qquad H(x,x)\to H(w,x)^{\mathcal{D}(w,x)}. \tag{3.2.11}$$

Definition 3.2.12. *Let $\mathcal{D}$ and $\mathcal{C}$ be categories enriched over a symmetric monoidal category $\mathcal{V}=(\mathcal{V}_0,\otimes,\mathbf{1})$. Assume that $\mathcal{D}$ is small and we have an enriched functor $H:\mathcal{D}^{op}\times\mathcal{D}\to\mathcal{C}$. For $\mathcal{C}$ complete and cotensored over $\mathcal{V}$, the* **enriched end**

$$\int^{\mathcal{D}} H(x,x)$$

is the equalizer of

$$\int^{\mathcal{D}} H(x,x) \dashrightarrow \prod_{x\in Ob\,\mathcal{D}} H(x,x) \underset{\phi_*}{\overset{\phi^*}{\rightrightarrows}} \prod_{x,y\in Ob\,\mathcal{D}} H(x,y)^{\mathcal{D}(x,y)}, \tag{3.2.13}$$

where the maps ϕ^* and ϕ_* are products of those of (3.2.11).

Similarly for $\mathcal{C}$ cocomplete and tensored over $\mathcal{V}$, the **enriched coend**

$$\int_{\mathcal{D}} H(x,x)$$

is the coequalizer of

$$\coprod_{x,y\in Ob\,\mathcal{D}} \mathcal{D}(x,y)\otimes H(y,x) \overset{\varphi^*}{\underset{\varphi_*}{\rightrightarrows}} \coprod_{x\in Ob\,\mathcal{D}} H(x,x) \dashrightarrow \int_{\mathcal{D}} H(x,x), \qquad (3.2.14)$$

where φ^* and φ_* are coproducts of the maps of (3.2.9).

Both the enriched end and the enriched coend are objects in $\mathcal{C}$. Again we remind the reader that **we are superscripts for ends and subscripts for coends.**

The enriched analog of (2.4.7) is for $\mathcal{C}$ bitensored over $\mathcal{V}$ is

$$\begin{tikzcd}
\mathcal{D}(y,x)\otimes H(y,x) \arrow[r, "\theta^y_{x,x}"] \arrow[d, "\kappa^x_{y,y}"'] & H(x,x) \arrow[d, "\hat{\kappa}^y_{x,x}"] \\
H(y,y) \arrow[r, "\hat{\theta}^x_{y,y}"] & H(x,y)^{\mathcal{D}(x,y)}
\end{tikzcd}$$

There is one such diagram for each pair (x,y) of objects in $\mathcal{D}$. For complete $\mathcal{C}$ this leads to the equalizer diagram of (3.2.13), and for cocomplete $\mathcal{C}$ this leads to the coequalizer diagram of (3.2.14).

Recall from Proposition 2.4.11 that an ordinary limit (colimit) is a special case of an end (coend) in which the functor H is constant on the first variable. By specializing Definition 3.2.12 to this case, we get the following enriched analog of Theorem 2.3.28.

Proposition 3.2.15. Every enriched limit (colimit) is an equalizer (a coequalizer).

The following isomorphisms are consequences of the definitions. They are stated by Kelly as [Kel82, (3.60) and (3.67)]. It is the enriched analog of Proposition 2.4.15.

Proposition 3.2.16. Morphism objects involving ends or coends. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{V}$ and H be as in Definition 3.2.12 with $\mathcal{V}$ complete.*

(i) When $\mathcal{C}$ is complete and cotensored over $\mathcal{V}$, there are natural isomorphisms

$$\mathcal{C}\left(c, \int^{\mathcal{D}} H(d,d)\right) \cong \int^{\mathcal{D}} \mathcal{C}(c, H(d,d))$$

for each object c in $\mathcal{C}$.

(ii) When $\mathcal{C}$ is cocomplete and tensored over $\mathcal{V}$, there are natural isomorphisms

$$\mathcal{C}\left(\int_{\mathcal{D}} H(d,d), c\right) \cong \int^{\mathcal{D}} \mathcal{C}(H(d,d), c)$$

for each object c in $\mathcal{C}$.

In both cases the end or coend on the left is an object of $\mathcal{C}$, so the expression on the left is in $\mathcal{V}$. On the right we are taking the end of a $\mathcal{V}$-valued functor on $\mathcal{D}^{op} \times \mathcal{D}$, whose value on (d_1, d_2) is either $\mathcal{C}(c, H(d_1, d_2))$ or $\mathcal{C}(H(d_2, d_1), c)$ for a fixed object c of $\mathcal{C}$. This is also an object of $\mathcal{V}$.

Corollary 3.2.17. The morphism object involving both a coend and an end. *Let $\mathcal{C}$ and $\mathcal{V}$ be as in Definition 3.2.12 with $\mathcal{V}$ complete, and $\mathcal{C}$ bicomplete and bitensored over $\mathcal{V}$. Suppose we have small $\mathcal{V}$-categories $\mathcal{D}_1$ and $\mathcal{D}_2$ with functors $H_i : \mathcal{D}_i^{op} \times \mathcal{D}_i \to \mathcal{C}$. Then there is a natural isomorphism*

$$\mathcal{C}\left(\int_{\mathcal{D}_1} H(d_1, d_1), \int^{\mathcal{D}_2} H(d_2, d_2)\right) \cong \int^{\mathcal{D}_1 \times \mathcal{D}_2} \mathcal{C}\left(H_1(d_1, d_1), H_2(d_2, d_2)\right).$$

3.2C Enriched functor categories

Several such categories figure prominently in this book. Various categories of spectra are best thought of in this way. See Definition 7.2.33 and Definition 9.0.2 below.

The following notation is taken from [Kel82, §2.2].

Definition 3.2.18. Enriched functor categories. *For $\mathcal{V}$-categories $\mathcal{D}$ and $\mathcal{C}$ as above with $\mathcal{D}$ small and $\mathcal{V}_0$ complete, $[\mathcal{D}, \mathcal{C}]$ denotes the category whose objects are $\mathcal{V}$-functors $\mathcal{D} \to \mathcal{C}$ as in Definition 3.1.13. We will denote the value of such a functor F on an object D in $\mathcal{D}$ by F_D. For two such functors F and G, we define the morphism object to be the enriched end (Definition 3.2.12)*

$$[\mathcal{D}, \mathcal{C}](F, G) = \int^{D \in ob\mathcal{D}} \mathcal{C}(F(D), G(D)).$$

The enriched end above is the generalization of Proposition 2.4.19 to the enriched case. The completeness assumption on $\mathcal{V}_0$ and the smallness assumption on $\mathcal{D}$ are needed to define it. We are particularly interested in the category $[\mathcal{D}, \mathcal{V}]$ because, as we will see below in §7.2 and Chapter 9, the category of G-spectra has this form.

Remark 3.2.19. The smallness of the source category. *Kelly discussed enriched functor categories extensively in [Kel82, Chapter 2]. He did not want to assume that $\mathcal{D}$ was small, and considered various ways to weaken that assumption including enlarging the set theoretic universe. We will not discuss these matters here because the functor categories of interest to us all have small domain categories.*

Proposition 3.2.20. A 2-category enriched over $\mathcal{V}\mathcal{C}at_0$. *When $\mathcal{V}_0$, the ordinary category underlying the symmetric monoidal category $\mathcal{V}$, is complete, then the 2-category $\mathcal{V}\mathcal{C}at$ as in Example 2.7.2(iii) is enriched over $\mathcal{V}\mathcal{C}at_0$.*

For more discussion of the following, see [Rie14, §13.1].

Definition 3.2.21. The enriched arrow category. *Let $\mathcal{C}$ be enriched over $\mathcal{V}$, and assume that the underlying category $\mathcal{V}_0$ is bicomplete as in Definition 2.3.25. Then its arrow category $\mathcal{C}_1$ is $\mathcal{C}^{\mathbf{2}}$, the category of $\mathcal{C}$-valued functors on the walking arrow category $\mathbf{f}$ of Definition 2.1.6. Thus its objects are arrows $\alpha : a_1 \to a_2$ in $\mathcal{C}_0$. If $\beta : b_1 \to b_2$ is another such arrow, then the morphism object $\mathcal{C}_1(\alpha, \beta)$ is the pullback in the following diagram in $\mathcal{V}$.*

$$\begin{tikzcd}
\mathcal{C}_1(\alpha,\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C}(a_1,b_1) \arrow[d, "\beta_*"] \\
\mathcal{C}(a_2,b_2) \arrow[r, "\alpha^*"] & \mathcal{C}(a_1,b_2)
\end{tikzcd}$$

The reader can check that the description of the morphism object as a pullback in Definition 3.2.21 is consistent with its description as an enriched end in Definition 3.2.18.

Proposition 3.2.22. Bitensored arrow category. *The enriched arrow category $\mathcal{C}_1$ of Definition 3.2.21 is bitensored (as in Definition 3.1.31) over $\mathcal{V}$ if $\mathcal{C}$ is.*

The following is proved in [Kel82, §2.3].

Proposition 3.2.23. The Kelly isomorphism. *For $\mathcal{V}$-categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ with $\mathcal{C}$ and $\mathcal{D}$ small and $\mathcal{V}_0$ complete, there is an isomorphism of $\mathcal{V}$-categories*

$$[\mathcal{C} \otimes \mathcal{D}, \mathcal{E}] \cong [\mathcal{C}, [\mathcal{D}, \mathcal{E}]],$$

where $\mathcal{C} \otimes \mathcal{D}$ is as in Definition 3.1.28. Equivalently there is an enriched adjunction $- \otimes \mathcal{D} \dashv [\mathcal{D}, -]$.

Here is a sketch of Kelly's proof. Given a $\mathcal{V}$-functor $F : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$, we define a functor $\mathcal{C} \to [\mathcal{D}, \mathcal{E}]$ by $A \mapsto F(A, -)$ for each object A in $\mathcal{C}$. To get from a $\mathcal{V}$-functor $G : \mathcal{C} \to [\mathcal{D}, \mathcal{E}]$, that is an object in $[\mathcal{C}, [\mathcal{D}, \mathcal{E}]]$, we use the evaluation functor Ev of Definition 2.2.37. we have

$$\mathcal{C} \otimes \mathcal{D} \xrightarrow{G \otimes \mathcal{D}} [\mathcal{D}, \mathcal{E}] \otimes \mathcal{D} \xrightarrow{\mathrm{Ev}} \mathcal{E}.$$

This composite functor is an object in $[\mathcal{C} \otimes \mathcal{D}, \mathcal{E}]$.

The counit of the adjunction of Proposition 3.2.23 is the evaluation map

$$\mathrm{Ev} : [\mathcal{D}, \mathcal{E}] \otimes \mathcal{D} \to \mathcal{E},$$

and the unit is the functor $\mathcal{C} \to [\mathcal{D}, \mathcal{C} \otimes \mathcal{D}]$ given by $A \mapsto A \otimes (-)$.

Recall the 2-category $\mathcal{V}\mathcal{CAT}$ (Example 2.7.2(iii) and Proposition 3.1.20) whose objects, morphisms and 2-morphisms are $\mathcal{V}$-categories as in Definition 3.1.1, $\mathcal{V}$-functors as in Definition 3.1.13 and $\mathcal{V}$-natural transformations.

We learned the proof of the following from Emily Riehl.

Proposition 3.2.24. Equivalence of functor categories. *Let $\mathcal{C}$ and $\mathcal{D}$ be small $\mathcal{V}$-categories which are equivalent as in Proposition 3.1.20. Then the functor categories $[\mathcal{C}, \mathcal{E}]$ and $[\mathcal{D}, \mathcal{E}]$ are $\mathcal{V}$-equivalent as in Definition 3.1.18.*

Proof We will use Proposition 2.7.9. $\mathcal{C}$ and $\mathcal{D}$ are equivalent objects in the 2-category $\mathcal{V}\mathcal{C}at$ of Proposition 3.1.20. The equivalence between them is preserved by the 2-functor

$$[-, \mathcal{E}] : \mathcal{V}\mathcal{C}at^{op} \to \mathcal{V}\mathcal{C}at. \qquad \square$$

Here are the enriched analogs of Proposition 2.4.21 (Yoneda reduction) and Proposition 2.4.24 (Yoneda coreduction).

Proposition 3.2.25. The enriched Yoneda reduction and coreduction. *Let $\mathcal{D}$ be a small $\mathcal{V}$-category and $F : \mathcal{D} \to \mathcal{E}$ a $\mathcal{V}$-functor. Then for each object D of $\mathcal{D}$,*

$$\int^{D' \in ob\mathcal{D}} (F_{D'})^{\mathcal{D}(D,D')} \cong F_D$$

when $\mathcal{E}$ is complete and cotensored over $\mathcal{V}$ as in Definition 3.1.31, and

$$\int_{D \in ob\mathcal{D}} \mathcal{D}(D, D') \otimes F_D \cong F_{D'}$$

when $\mathcal{E}$ is cocomplete and tensored over $\mathcal{V}$. Equivalently, interchanging D and D' gives

$$\int_{D' \in ob\mathcal{D}} \mathcal{D}(D', D) \otimes F_{D'} \cong F_D.$$

Now assume that $\mathcal{E}$ is bicomplete as in Definition 2.3.25 and bitensored over $\mathcal{V}$ as in Definition 3.1.31.

Remark 3.2.26. The case of a bicomplete category $\mathcal{E}$ bitensored over $\mathcal{V}$. *This is the enriched version of Remark 2.4.25. From Proposition 3.2.25 we see that for suitable $\mathcal{E}$,*

$$F(D) \cong \int^{D' \in ob\mathcal{D}} (F_{D'})^{\mathcal{D}(D,D')} \cong \int_{D' \in ob\mathcal{D}} \mathcal{D}(D', D) \otimes F_{D'}.$$

Again, note the reversal of the morphism object in $\mathcal{D}$. The same variance considerations apply here as in the unenriched case.

For $\mathcal{E}$ as above, for each pair of objects D and D' in $\mathcal{D}$ there is the **structure map** as in (3.1.38)

$$\epsilon^F_{D,D'} : \mathcal{D}(D, D') \otimes F_D \to F_{D'}. \tag{3.2.27}$$

and its right adjoint the **costructure map** as in (3.1.39)

$$\eta^F_{D,D'} : F_D \to (F_{D'})^{\mathcal{D}(D,D')} \tag{3.2.28}$$

(We will sometimes omit the superscript F.) The adjunction is that of (2.6.39) for $\mathcal{C} = \mathcal{V}$, $X = F_D$, $Y = \mathcal{D}(D,D')$ and $Z = F_{D'}$. The structure map factors uniquely through $\mathcal{D}(D,D') \otimes_{\mathcal{D}(D,D)} F_D$, the coequalizer of

$$\begin{tikzcd}
\mathcal{D}(D,D') \otimes \mathcal{D}(D,D) \otimes F_D \arrow[d, shift right, "d_{D,D,D'} \otimes F(D) = \mu_R \otimes F_D"'] \arrow[d, shift left, "\mathcal{D}(D,D') \otimes \mu_L = \mathcal{D}(D,D') \otimes \epsilon^F_{D,D}"] \\
\mathcal{D}(D,D') \otimes F_D \arrow[d, dashed] \\
\mathcal{D}(D,D') \otimes_{\mathcal{D}(D,D)} F_D .
\end{tikzcd} \tag{3.2.29}$$

We denote the resulting **reduced structure map** by

$$\tilde{\epsilon}^F_{D,D'} : \mathcal{D}(D,D') \otimes_{\mathcal{D}(D,D)} F_D \to F_{D'} . \tag{3.2.30}$$

Remark 3.2.31. Notation to be changed later. *This notation differs from that of (7.2.36) and Definition 7.2.42 below, where the source category is assumed to have a monoidal structure. See Remark 7.2.37 below.*

We have the following analog of Proposition 3.1.11.

Proposition 3.2.32. Reduced structure map and endomorphisms. *The reduced structure map of (3.2.30) is an isomorphism when $D' = D$ and when $\mathcal{D}(D,D')$ is isomorphic to $\mathcal{D}(D,D)$.*

Proof We use the same argument as that of Proposition 3.1.11. When $D' = D$, (3.2.29) fits into the larger diagram

$$\begin{tikzcd}
 & \mathcal{D}(D,D) \otimes \mathcal{D}(D,D) \otimes F_D \arrow[d, shift right, "d_{D,D,D} \otimes F(D)"'] \arrow[d, shift left, "\mathcal{D}(D,D) \otimes \epsilon^F_{D,D}"] & \\
\mathbf{1} \otimes F(D) \arrow[r, "1_D \otimes F_D"] & \mathcal{D}(D,D) \otimes F_D \arrow[d, dashed] & \\
 & \mathcal{D}(D,D) \otimes_{\mathcal{D}(D,D)} F_D \arrow[r, "\tilde{\epsilon}^F_{D,D}"] & F_D ,
\end{tikzcd}$$

in which the composite map $\mathbf{1} \otimes F(D) \to F(D)$ is isomorphic to the identity via the left unitor (see Definition 2.6.1) in $\mathcal{V}$. This means that the coequalizer has to be $F(D)$. In view of this, an isomorphism between $\mathcal{D}(D,D')$ and $\mathcal{D}(D,D)$ induces one between the colimits

$$\mathcal{D}(D,D') \otimes_{\mathcal{D}(D,D)} F_D$$

and

$$\mathcal{D}(D,D) \otimes_{\mathcal{D}(D,D)} F_D \cong F_D. \qquad \square$$

Any object X in the enriched functor category $[\mathcal{D},\mathcal{V}]$ can be described as a coend which is a reflexive coequalizer as in Definition 2.3.60.

Proposition 3.2.33. The tautological presentation and copresentation in $[\mathcal{D},\mathcal{E}]$. *Let $\mathcal{D}$ be a small $\mathcal{V}$-category and $\mathcal{E}$ a cocomplete $\mathcal{V}$-category that is tensored over $\mathcal{V}$ as in Definition 3.1.31. Then for each object (i.e., functor $\mathcal{D} \to \mathcal{E}$) X in $[\mathcal{D},\mathcal{E}]$,*

$$X \cong \int_{D\in\mathcal{D}} よ^D \otimes X_D, \tag{3.2.34}$$

and the indicated coequalizer is reflexive. In particular for each object D' in $\mathcal{D}$, $X_{D'}$ is the reflexive coequalizer

$$X_{D'} \cong \int_{D\in\mathcal{D}} (よ^D)_{D'} \otimes X_D \cong \int_{D\in\mathcal{D}} \mathcal{D}(D,D') \otimes X_D.$$

This is **the tautological presentation of** X*.*

When $\mathcal{E}$ is cotensored over $\mathcal{V}$, E is an object in $\mathcal{E}$ and D is an object in $\mathcal{D}$, let

$$E^{(よ_D)} \in [\mathcal{D},\mathcal{E}]$$

be given by $D' \mapsto E^{\mathcal{D}(D',D)}$. (Note that this expression is covariant in D'.) Then if in addition $\mathcal{E}$ is complete, we have

$$X \cong \int^{D\in\mathcal{D}} (X_D)^{よ_D} \quad \textit{with} \quad X_{D'} \cong \int^{D\in\mathcal{D}} (X_D)^{\mathcal{D}(D',D)},$$

where the equalizers are coreflexive. This is **the tautological copresentation of** X*.*

Proof We will prove the statements about coends only. The coend on the right of(3.2.34) is an $\mathcal{E}$-valued functor on $\mathcal{D}$. Evaluating on an object D' gives

$$\int_{D\in ob\mathcal{D}} \mathcal{D}(D,D') \otimes X_D,$$

which is $X_{D'}$ by the Yoneda coreduction of Proposition 3.2.25, so the coend is X.

To show that this coequalizer is reflexive, we need to define the section

$$s : \bigvee_{D\in ob\mathcal{D}} よ^D \otimes X_D \to \bigvee_{D,D'\in ob\mathcal{D}} よ^{D'} \otimes \mathcal{D}(D,D') \otimes X_D$$

Its restriction to the Dth summand is the map to $よ^D \otimes \mathcal{D}(D,D) \otimes X_D$ induced by the identity morphism (as in Definition 3.1.1) $\mathbf{1} \to \mathcal{D}(D,D)$. $\square$

3.2D Enriched Kan extensions

Next we give the enriched analog of the formulas (2.5.11) and (2.5.12) for left and right Kan extensions as coends and ends. In this setting the small categories $\mathcal{C}$ and $\mathcal{D}$ of (2.5.7) are $\mathcal{V}$-categories, as is the target category $\mathcal{E}$, which need not be small. The functors and natural transformations are now $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations as in Definition 3.1.13. The cocompleteness/completeness requirement on $\mathcal{E}$ is strengthened by the additional requirement that it be tensored/cotensored over $\mathcal{V}$ as in Definition 3.1.31. The resulting coends/ends are enriched over $\mathcal{V}$ as in Definition 3.2.12. The following can be found in [Kel82, (4.25) and (4.24)].

Proposition 3.2.35. Enriched Kan extensions. *Let* $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ *be a symmetric monoidal category, and suppose we have a diagram similar to that of (2.5.1), namely*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{E} \\ & \mathcal{D}, \arrow[ur, dashed] & \end{tikzcd} \tag{3.2.36}$$

in which $\mathcal{C}$*,* $\mathcal{D}$ *and* $\mathcal{E}$ *are* $\mathcal{V}$*-categories with* $\mathcal{C}$ *and* $\mathcal{D}$ *small, and* F *and* K *are* $\mathcal{V}$*-functors. Then the left and right enriched Kan extensions of* $F : \mathcal{C} \to \mathcal{E}$ *along* $K : \mathcal{C} \to \mathcal{D}$ *are given by*

$$(Lan_K F)_d \cong \int_{\mathcal{C}} \mathcal{D}(K(c), d) \otimes F_c$$

$$\text{and} \quad (Ran_K F)_d \cong \int^{\mathcal{C}} F_c^{\mathcal{D}(d, K(c))},$$

when the target category $\mathcal{E}$ *is cocomplete and tensored over* $\mathcal{V}$ *in the first case, and complete and cotensored over* $\mathcal{V}$ *in the second case.*

The following is the enriched analog of Proposition 2.5.4 and also follows immediately from the definitions.

Proposition 3.2.37. Enriched Kan extensions as adjoints to precomposition. *The left (right) enriched Kan extension* Lan_K *(*Ran_K*) is equivalent to a functor* $K_! : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$ *(*$K_* : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$*) which is the left (right) adjoint of the precomposition functor* $K^* : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$.

The enriched analog of Proposition 2.6.11 is

Proposition 3.2.38. A coend reduction for enriched small monoidal categories. *Let* $(\mathcal{D}, \oplus, \mathbf{0})$ *be a small monoidal* $\mathcal{V}$*-category. Then for any two objects* X *and* Y *of* $\mathcal{D}$*,*

$$\int_{W \in ob\mathcal{D}} \mathcal{D}(W \oplus X, Y) \otimes \mathcal{D}(\mathbf{0}, W) \cong \mathcal{D}(X, Y).$$

Proposition 3.2.39. An adjunction for functor categories. *Let* $(\mathcal{V}, \otimes, \mathbf{1})$ *be a closed symmetric monoidal category, let* $(\mathcal{D}, \oplus, \mathbf{0})$ *be a (not necessarily closed) symmetric monoidal category enriched over* $\mathcal{V}$*, and let* $\mathcal{S} = [\mathcal{D}, \mathcal{V}]$ *be the category of enriched functors (see Definition 3.1.13) from* $\mathcal{D}$ *to* $\mathcal{V}$ *as in Definition 3.2.18. Then for each object* D *of* $\mathcal{D}$ *the functor* $\mathcal{V} \to \mathcal{S}$ *given by* $K \mapsto K \otimes よ^D$ *(where* $よ^D$ *is the Yoneda functor of Definition 3.1.68 and the tensor product is that of Proposition 2.6.25) is the left adjoint of the evaluation functor* $\mathrm{Ev}_D : \mathcal{S} \to \mathcal{V}$ *given by* $E \mapsto E_D$*.*

In other words, $- \otimes よ^D \dashv \mathrm{Ev}_D$*, meaning there is a natural isomorphism*

$$\mathcal{S}(K \otimes よ^D, E) \cong \mathcal{V}(K, E_D).$$

Proof We have

$$\begin{aligned}
\mathcal{S}(K \otimes よ^D, E) &= \int^{B \in \mathrm{ob}\mathcal{D}} \mathcal{V}((K \otimes よ^D)_B, E_B) \qquad \text{by Definition 3.2.18} \\
&= \int^{B \in \mathrm{ob}\mathcal{D}} \mathcal{V}(\mathcal{D}(D, B) \otimes K, E_B) \qquad \text{by the definition of } よ^D \\
&\cong \int^{B \in \mathrm{ob}\mathcal{D}} \mathcal{V}(\mathcal{D}(D, B), \mathcal{V}(K, E_B)) \\
&\qquad\qquad \text{because } \mathcal{V} \text{ is closed symmetric monoidal} \\
&\cong \mathcal{V}(K, E_D) \qquad\qquad \text{by Proposition 3.2.25.}
\end{aligned}$$

□

3.3 The Day convolution

The Day convolution [Day70] is the formal tool that makes it possible to give the categories of smashable spectra as in Definition 7.2.33, which include the category $\mathcal{S}p^G$ of orthogonal G-spectra (to be defined below in Chapter 9) a closed symmetric monoidal structure. Very briefly, let $\mathcal{V}$ be a closed cocomplete symmetric monoidal category and $\mathcal{D}$ a $\mathcal{V}$-category which is also symmetric monoidal. The Day convolution is a binary operation on the functor category $[\mathcal{D}, \mathcal{V}]$ that makes it a closed cocomplete symmetric monoidal category as well. We will see in Chapter 9 that $\mathcal{S}p_G$ fits this description, with $\mathcal{V} = \mathcal{T}_G$.

First we explain the use of the word "convolution." Classically suppose f and g are suitable real valued functions on $\mathbf{R}^n$. Their convolution $f * g$ is a third such function defined by

$$(f * g)(x) = \int^{\mathbf{R}^n} f(t)g(x - t)dt.$$

Here we are integrating over $\mathbf{R}^n$ in the sense of calculus rather than computing an end in the sense of category theory. More generally the domain $\mathbf{R}^n$ could be replaced by a Lie group and the range $\mathbf{R}$ could be replaced by a suitable ring.

We want to replace the functions f and g by functors F and G from a symmetric monoidal category $\mathcal{D}$ to a closed cocomplete symmetric monoidal category $\mathcal{V}$. First we illustrate with an elementary example.

Example 3.3.1. The Cartesian product of graded sets. *Let*

$$A = \{A_n \colon n \geqslant 0\} \qquad \textit{and} \qquad B = \{B_n \colon n \geqslant 0\}$$

be graded sets. Their Cartesian product $A \times B$ is defined by

$$(A \times B)_n = \coprod_{i+j=n} A_i \times B_j.$$

We reinterpret this as follows. Let $\mathcal{N}$ be the discrete category (Definition 2.1.7) associated with the natural numbers $\mathbf{N}$. It is symmetric monoidal under addition with 0 as unit. The graded sets A and B can be regarded as functors $\mathcal{N} \to \mathcal{S}et$, and we indicate the value of such a functor F on the object n by F_n rather than $F(n)$. Then we can intrepret $A \times B$ as a coend by

$$(A \times B)_n = \coprod_{i,j} A_i \times B_j \times \mathcal{N}(n, i+j) = \int^{\mathcal{N}\times\mathcal{N}} \mathcal{N}(i+j, n) \times A_i \times B_j.$$

Note that $\mathcal{N}$ is a symmetric monoidal category enriched over the closed symmetric monoidal category $\mathcal{S}et$, the functor category $[\mathcal{N}, \mathcal{S}et]$ is that of graded sets, and the graded Cartesian product is a closed symmetric monoidal structure on it. It is a special case of the Day convolution.

For a more interesting example, see Theorem 7.2.60 below.

Now we give the formal definition.

Definition 3.3.2. The Day convolution. *Let $\mathcal{D} = (\mathcal{D}_0, \oplus, \mathbf{0})$ be a small symmetric monoidal $\mathcal{V}$-category, where $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$ is a cocomplete closed symmetric monoidal category, and let $X, Y \in [\mathcal{D}, \mathcal{V}]$ be $\mathcal{V}$-functors (Definition 3.2.18). Then we define $X \boxtimes Y$ to be the left Kan extension (see §2.5) $Lan_{\oplus}(- \otimes -)$ of $\otimes(X \times Y)$ along $\oplus$,*

$$\begin{tikzcd}
\mathcal{D}\times\mathcal{D} \arrow[r, "X\times Y"] \arrow[dr, "\oplus"'] & \mathcal{V}\times\mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \\
& \mathcal{D} \arrow[ur, dashed, "X\boxtimes Y"'] &
\end{tikzcd}$$

In particular for each object D in $\mathcal{D}$, we have

$$(X \boxtimes Y)_D = \int^{\mathcal{D}\times\mathcal{D}} \mathcal{D}(A \oplus B, D) \otimes X_A \otimes Y_B. \tag{3.3.3}$$

The formula (3.3.3) is derived from (2.5.11), which expresses a left Kan extension as a coend. We can use it to describe the structure map

$$\epsilon^{X\boxtimes Y}_{D,D'} : \mathcal{D}(D,D')\otimes(X\boxtimes Y)_D \to (X\boxtimes Y)_{D'}$$

of (3.2.27) as follows. The source is

$$\begin{aligned}\mathcal{D}(D,D')\otimes(X\boxtimes Y)_D &\cong \mathcal{D}(D,D')\otimes\int_{\mathcal{D}\times\mathcal{D}}\mathcal{D}(A\oplus B,D)\otimes X_A\otimes Y_B\\ &\cong \int_{\mathcal{D}\times\mathcal{D}}\mathcal{D}(D,D')\otimes\mathcal{D}(A\oplus B,D)\otimes X_A\otimes Y_B\end{aligned}$$

and the map is

$$\epsilon^{X\boxtimes Y}_{D,D'} = \int_{\mathcal{D}\times\mathcal{D}} d_{A\oplus B,D,D'}\otimes X_A\otimes Y_B, \tag{3.3.4}$$

where

$$d_{A\oplus B,D,D'} : \mathcal{D}(D,D')\otimes\mathcal{D}(A\oplus B,D)\to\mathcal{D}(A\oplus B,D')$$

is the composition morphism in $\mathcal{D}$.

Day Convolution Theorem 3.3.5. *The binary operation of Definition 3.3.2 gives the functor category $[\mathcal{D},\mathcal{V}]$ a closed symmetric monodial structure in which the unit element is the $\mathcal{V}$-functor $I = よ^{\mathbf{0}}$ (see Yoneda Lemma 2.2.10) given by $I_D = \mathcal{D}(\mathbf{0},D)$. The internal Hom functor (Definition 2.6.33) $\underline{[\mathcal{D},\mathcal{V}]}(X,-)$ is the right adjoint of the functor $(-)\boxtimes X$.*

Proof The symmetries and associativities of $\mathcal{D}$ and $\mathcal{V}$ lead to natural isomorphisms between $X\boxtimes Y$ and $Y\boxtimes X$ and between $(X\boxtimes Y)\boxtimes Z$ and $X\otimes(Y\boxtimes Z)$.

We need a calculation to show that the unit I has the desired property. Using (3.3.3), we get

$$\begin{aligned}(I\boxtimes X)_D &= \int_{\mathcal{D}\times\mathcal{D}}\mathcal{D}(A\oplus B,D)\otimes I_A\otimes X_B\\ &= \int_{\mathcal{D}\times\mathcal{D}}\mathcal{D}(A\oplus B,D)\otimes\mathcal{D}(\mathbf{0},A)\otimes X_B\\ &= \int_{B\in\mathcal{D}}\left(\int_{A\in\mathcal{D}}\mathcal{D}(A\oplus B,D)\otimes\mathcal{D}(\mathbf{0},A)\right)\otimes X_B\\ &\qquad\text{by the enriched analog of Proposition 2.4.17}\\ &\cong \int_{\mathcal{D}}\mathcal{D}(D,B)\otimes X_B \qquad \text{by Proposition 3.2.38}\\ &\cong X_D \qquad \text{by Proposition 3.2.25.}\end{aligned}$$

Hence $I\boxtimes X$ is naturally isomorphic to X, as is $X\boxtimes I$ by symmetry or by a similar calculation.

The internal Hom, being the right adjoint of the functor $(-)\boxtimes X$, exists because $\mathcal{V}$ and hence $[\mathcal{D},\mathcal{V}]$ are cocomplete. □

Let $\mathcal{D}$ be the walking arrow category $J = (0 \to 1)$ of § 2.6F. It is enriched over $\mathcal{S}et$ and therefore over any bicomplete closed symmetric monoidal category $(\mathcal{C}, \otimes, *)$. The functor category $[J, \mathcal{C}]$ is the arrow category $\mathcal{C}_1$. The Yoneda functors $よ^0$ and $よ^1$ are respectively the identity morphism on $*$ and the map $\varnothing \to *$. Let $f : X_0 \to X_1$ and $g : Y_0 \to Y_1$ be objects in $\mathcal{C}^J$.

The small category J has two symmetric monoidal structures, which we denote by $\cup$ and $\cap$. For $\cup$, the unit is 0 and $1 \cup 1 = 1$. For $\cap$, the unit is 1 and $0 \cap 0 = 0$.

Theorem 3.3.6. Two monodial structures on the arrow category $\mathcal{C}_1 = [J, \mathcal{C}]$. *The symmetric monoidal structures $\cup$ and $\cap$ on J define above lead to two closed symmetric monoidal structures on $\mathcal{C}_1$ for a cocomplete closed symmetric monoidal category $(\mathcal{C}, \otimes, *)$ via the Day Convolution Theorem 3.3.5. They coincide respectively with the structures $\otimes$ and $\square$ of Definition 2.6.55.*

Proof For the moment we will denote these two monoidal structures on the arrow category also by $\cup$ and $\cap$.

To find the domain of $f \cup g$ using (3.3.3), we need

$$J(j' \cup j'', 0) \otimes f_{j'} \otimes g_{j''} = \begin{cases} X_0 \otimes Y_0 & \text{for } j' = j'' = 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

For the codomain, we need

$$J(j' \cup j'', 1) \otimes f_{j'} \otimes g_{j''} = X_{j'} \otimes Y_{j''},$$

since $J(j' \cup j'', 1) = *$, the terminal object in $\mathcal{S}et$, in all cases.

With these in hand, (3.3.3) gives

$$\begin{aligned} (f \cup g)_0 &\cong \int^{J \times J} J(j' \cup j'', 0) \otimes X_{j'} \otimes Y_{j''} \\ &\cong X_0 \otimes Y_0 \qquad \text{by Proposition 2.4.18(i)} \\ \text{and} \qquad (f \cup g)_1 &\cong \int^{J \times J} J(j' \cup j'', 1) \otimes X_{j'} \otimes Y_{j''} \\ &\cong \int^{J \times J} X_{j'} \otimes Y_{j''} \cong X_1 \otimes Y_1 \qquad \text{by Proposition 2.4.18(iii),} \end{aligned}$$

so $f \cup g$ is the evident morphism

$$f \otimes g : X_0 \otimes Y_0 \to X_1 \otimes Y_1.$$

The codomain of $f \cap g$ is

$$\begin{aligned} \int^{J \times J} J(j' \cap j'', 1) \otimes f_{j'} \otimes g_{j''} &= \int^{J \times J} f_{j'} \otimes g_{j''} \\ &= X_1 \otimes Y_1 \end{aligned}$$

as before.

To find the domain of $f \cap g$ using (3.3.3), we need

$$J(j' \cap j'', 0) \otimes f_{j'} \otimes g_{j''} = \begin{cases} \varnothing & \text{for } j' = j'' = 1 \\ X_{j'} \otimes Y_{j''} & \text{otherwise} \end{cases}$$

This means we can use Proposition 2.4.18(ii) to evaluate the relevant double coend, namely

$$\begin{aligned} (f \cap g)_0 &= \int_{J \times J} J(j' \cap j'', 0) \otimes f_{j'} \otimes g_{j''} \\ &= (X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_0 \otimes Y_0) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \\ &= (X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \end{aligned}$$

It follows that $f \cap g$ is the map

$$f \,\square\, g : (X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \to X_1 \otimes Y_1. \qquad \square$$

Proposition 3.3.7. The components of the internal Hom functor in $[\mathcal{D}, \mathcal{V}]$. *Let $\mathcal{D}$ and $\mathcal{V}$ be as in Definition 3.3.2.*

We will abbreviate the internal Hom functor $\underline{[\mathcal{D}, \mathcal{V}]}(-,-)$ by $F(-,-)$.

(i) **Relation to the categorical Hom.**

$$F(X, Y)_D \cong [\mathcal{D}, \mathcal{V}](よ^D \otimes X, Y).$$

In particular,

$$F(X, Y)_0 \cong [\mathcal{D}, \mathcal{V}](X, Y),$$

and

$$F(よ^A, Y)_D \cong [\mathcal{D}, \mathcal{V}](よ^{A+D}, Y) \cong Y_{A+D}. \tag{3.3.8}$$

(ii) **End formulation for complete $\mathcal{V}$.** *If in addition $\mathcal{V}$ is complete, then*

$$F(X, Y)_D \cong \int^{C \in \mathcal{D}} \mathcal{V}(X_C, Y_{C+D}).$$

Proof (i) The adjunction that defines the internal Hom (see Definition 2.6.33) is

$$[\mathcal{D}, \mathcal{V}](W, F(X, Y)) \cong [\mathcal{D}, \mathcal{V}](W \otimes X, Y). \tag{3.3.9}$$

By setting $W = よ^D$ as in Definition 3.1.68 for an object D in $\mathcal{D}$, we can make the right hand side equal to $(F(X, Y))_D$ since

$$[\mathcal{D}, \mathcal{V}](よ^D, -) = (-)_D$$

meaning that

$$[\mathcal{D}, \mathcal{V}](よ^D, F(X, Y)) = F(X, Y)_D.$$

Hence for $W = よ^D$, (3.3.9) reads

$$(F(X,Y))_D \cong [\mathcal{D},\mathcal{V}](よ^D \otimes X, Y)$$

(ii)

$$\begin{aligned} F(X,Y)_D &\cong [\mathcal{D},\mathcal{V}](よ^D \otimes X, Y) && \text{by (i)} \\ &\cong [\mathcal{D},\mathcal{V}](X, F(よ^D, Y)) && \text{by (3.3.9)} \\ &\cong \int^{C\in\mathcal{D}} \mathcal{V}(X_C, F(よ^D, Y)_C) && \text{by Definition 3.2.18} \\ &\cong \int^{C\in\mathcal{D}} \mathcal{V}(X_C, Y_{C+D}) && \text{by (3.3.8).} \end{aligned}$$ □

We will now descibe the structure map of (3.2.27) for $F(X,Y)$,

$$\epsilon^{F(X,Y)}_{D,D'} : \mathcal{D}(D,D') \otimes F(X,Y)_D \to F(X,Y)_{D'}.$$

In terms of the isomorphism of Proposition 3.3.7(ii), it is the composite

$$\begin{tikzcd} \mathcal{D}(D,D') \otimes \int^{C\in\mathcal{D}} \mathcal{V}(X_C, Y_{C+D}) \arrow[d, "\cong"] \\ \int^{C\in\mathcal{D}} \mathcal{V}(\mathbf{1}, \mathcal{D}(D,D')) \otimes \mathcal{V}(X_C, Y_{C+D}) \arrow[d, "\int^{C\in\mathcal{D}} \Pi_{\mathbf{1},X_C,\mathcal{D}(D,D'),Y_{C+D}}"] \\ \int^{C\in\mathcal{D}} \mathcal{V}(X_C, \mathcal{D}(D,D') \otimes Y_{C+D}) \arrow[d, "\int^{C\in\mathcal{D}} \mathcal{V}(X_C, \alpha_{C,D,D'} \otimes Y_{C+D})"] \\ \int^{C\in\mathcal{D}} \mathcal{V}(X_C, \mathcal{D}(C\oplus D, C\oplus D') \otimes Y_{C+D}) \arrow[d, "\int^{C\in\mathcal{D}} \mathcal{V}(X_C, \epsilon^Y_{C+D,C+D'})"] \\ \int^{C\in\mathcal{D}} \mathcal{V}(X_C, Y_{C+D'}), \end{tikzcd} \tag{3.3.10}$$

where $\Pi_{\mathbf{1},X_C,\mathcal{D}(D,D'),Y_{C+D}}$ is the Cartesian product morphism of Definition 2.6.50, and $\alpha_{C,D,D'} : \mathcal{D}(D,D') \to \mathcal{D}(C\oplus D, C\oplus D')$ is the addition morphism of Definition 2.6.6.

An argument similar to that of Proposition 3.2.33 shows

Proposition 3.3.11. Reflexivity of the Day convolution. *The coequalizer of (3.3.3) is reflexive.*

Proposition 3.3.12. The Day convolution with a tensored Yoneda functor. *For objects X of $\mathcal{V}$, D and D' of $\mathcal{D}$ and E of $[\mathcal{D},\mathcal{V}]$ as above, we have*

$$(E \boxtimes F^D(X))_{D'} \cong (E \boxtimes よ^D)_{D'} \otimes X,$$

where $F^D : \mathcal{V} \to [\mathcal{D},\mathcal{V}]$ is the tensored Yoneda functor of Definition 3.1.68, namely

$$F^D(X)_{D'} := \mathcal{D}(D,D') \otimes X.$$

Proof We have

$$\begin{aligned}
(E \boxtimes F^D(X))_{D'} &= \int_{\mathcal{D}\times\mathcal{D}} \mathcal{D}(A \oplus B, D') \otimes E_A \otimes F^D(X)_B \\
&\qquad \text{by (3.3.3)} \\
&= \int_{\mathcal{D}\times\mathcal{D}} \mathcal{D}(A \oplus B, D') \otimes E_A \otimes \mathcal{D}(D,B) \otimes X \\
&= \left(\int_{\mathcal{D}\times\mathcal{D}} \mathcal{D}(A \oplus B, D') \otimes E_A \otimes \mathcal{D}(D,B)\right) \otimes X \\
&\qquad \text{since tensor products preserve colimits} \\
&= \left(E \boxtimes よ^D\right)_{D'} \otimes X.
\end{aligned}$$

□

Note that the penultimate equality in the above proof is analogous to factoring a constant out of an integral!

Remark 3.3.13. Notation for the product in the Day convolution. *It is common practice to use the same symbol for the product operations in the closed symmetric monoidal category $\mathcal{V}$ and in the functor category $[\mathcal{D},\mathcal{V}]$. The isomorphism of Proposition 3.3.12 means that we could denote $\boxtimes$ by $\otimes$ without risk of ambiguity. We will do this below in Theorem 7.2.60 and Definition 9.1.21, where we use the symbol $\wedge$ to denote the smash product of two spaces, that of a spectrum with a space, and that of two spectra. In that setting the tensored Yoneda functor for the unit object (the trivial vector space) in $\mathcal{D} = \mathscr{J}_G$ is the functor sending a space to its suspension spectrum.*

Recall that in functor categories such as $[\mathcal{D},\mathcal{V}]$ we have for each object D in $\mathcal{D}$ we have the Yoneda functor (Yoneda Lemma 2.2.10) $よ^D$, which is the functor defined by $よ^D_{D'} = \mathcal{D}(D,D')$ for each object D' in $\mathcal{D}$.

Proposition 3.3.14. The Day convolution of two Yoneda functors. *Let D_1 and D_2 be objects in $\mathcal{D}$. Then in the $\mathcal{V}$-functor category $[\mathcal{D},\mathcal{V}]$,*

$$よ^{D_1} \otimes よ^{D_2} = よ^{D_1 \oplus D_2}.$$

Proof We will use (3.3.3) to calculate $よ^{D_1} \otimes よ^{D_2}$. For each D we have

$$(よ^{D_1} \otimes よ^{D_2})_D \cong \int_{\mathcal{D}\times\mathcal{D}} \mathcal{D}(A \oplus B, D) \otimes よ^{D_1}_A \otimes よ^{D_2}_B$$

$$
\begin{aligned}
&\cong \int_{\mathcal{D}\times\mathcal{D}} \mathcal{D}(A \oplus B, D) \otimes \mathcal{D}(D_1, A) \otimes \mathcal{D}(D_2, B) \\
&\cong \int_{\mathcal{D}\times\mathcal{D}} \mathcal{D}(D_1, A) \otimes \mathcal{D}(D_2, B) \otimes \mathcal{D}(A \oplus B, D) \\
&\cong \int_{\mathcal{C}} \mathcal{C}((D_1, D_2), (A, B)) \otimes F((A, B)) \\
&\qquad \text{where } \mathcal{C} := \mathcal{D} \times \mathcal{D} \text{ and } F((A, B)) := \mathcal{D}(A \oplus B, D) \\
&\cong F((D_1, D_2)) \qquad \text{by Proposition 3.2.25} \\
&\cong \mathcal{D}(D_1 \oplus D_2, D) = (よ^{D_1 \oplus D_2})_D. \qquad \square
\end{aligned}
$$

The following is proved by Mandell *et al* in [MMSS01, 22.1] in the case of topological categories.

Proposition 3.3.15. Lax symmetric monoidal functors and commutative algebras. *The category of (commutative) monoids in $[\mathcal{D}, \mathcal{V}]$ is isomorphic to that of lax (symmetric) monoidal functors $\mathcal{D} \to \mathcal{V}$ (Definition 2.6.19).*

Proof Let $R : \mathcal{D} \to \mathcal{V}$ be lax (symmetric) monoidal. Then, in the notation of Definition 2.6.19, we have a unit map $\iota : \mathbf{1} \to R(\mathbf{0})$ and a natural transformation μ from $R(-) \otimes R(-)$ to $R(- \oplus -)$. By the definition of the tensored Yoneda functor $F^{\mathbf{0}}$ and the Yoneda functor $\mathbf{1} = よ^{\mathbf{0}}$ of Yoneda Lemma 2.2.10, the maps ι and μ determine and are determined by the maps $\eta : \mathbf{1} \to R$ and $m : R \otimes R \to R$ of Definition 2.6.58 that give R the structure of a (commutative) monoid. $\square$

3.4 Simplicial sets and simplicial spaces

The category of simplicial sets is a convenient combinatorial substitute for that of topological spaces and a widely used tool in homotopy theory. A thorough modern account can be found in [GJ99].

3.4A The category of finite ordered sets

Let $\mathbf{\Delta}$ be the category of finite ordered sets $[n] = \{0, 1, \ldots, n\}$ and order preserving maps. It is an easy exercise to show that any such map can be written as a composite of the following ones:

- the **face maps** $d_i : [n-1] \to [n]$ for $0 \leqslant i \leqslant n$, where d_i is the order preserving monomorphism that does not have i in its image and
- the **degeneracy maps** $s_i : [n+1] \to [n]$ for $0 \leqslant i \leqslant n$, where s_i is the order preserving epimorphism sending i and $i+1$ to i.

These satisfy the **simplicial identities**:

(i) $d_i d_j = d_{j-1} d_i$ for $i < j$
(ii) $d_i s_j = s_{j-1} d_i$ for $i < j$
(iii) $d_i s_j = id$ for $i = j$ and for $i = j + 1$
(iv) $d_i s_j = s_j d_{i-1}$ for $i > j + 1$
(v) $s_i s_j = s_j s_{i-1}$ for $i > j$.

Definition 3.4.1. *A* **simplicial set** X *is a functor* $\mathbf{\Delta}^{op} \to \mathcal{S}et$. *It is common to denote its value on* $[n]$ *by* X_n *and call it the* **set of** n**-simplices** *of* X. *A simplicial set* X *thus consists of a collection of sets* X_n *for* $n \geqslant 0$, *along with face maps* $d_i : X_n \to X_{n-1}$ *and degeneracy maps* $s_i : X_n \to X_{n+1}$ *for* $0 \leqslant i \leqslant n$ *satisfying the identities (i)–(v) above. A simplex is* **nondegenerate** *if it is not in the image of any degeneracy map* s_i. *The category* $\mathcal{S}et_{\mathbf{\Delta}}$ *of simplicial sets is the category of such functors with natural transformations as morphisms.*

More generally a **simplicial object** X **in a category** $\mathcal{C}$ *is a functor* $X : \mathbf{\Delta}^{op} \to \mathcal{C}$. *It is common to write it as* $X_\bullet$ *to emphasize its simplicial nature. We denote the category of simplicial objects in* $\mathcal{C}$ *by* $\mathcal{C}_{\mathbf{\Delta}}$.

Similarly a **cosimplicial object** Y **in a category** $\mathcal{C}$, *sometimes denoted by* $Y^\bullet$, *is a* $\mathcal{C}$ *valued functor on* $\mathbf{\Delta}$ *whose value on* $[n]$ *is denoted by* Y^n. *It consists of a collection of objects* Y^n *in* $\mathcal{C}$ *for* $n \geqslant 0$, *along with coface maps* $d^i : Y^{n-1} \to Y^n$ *and codegeneracy maps* $s^i : Y^{n+1} \to Y^n$ *for* $0 \leqslant i \leqslant n$ *satisfying identities dual to (i)–(v) above. We denote the category of cosimplicial objects in* $\mathcal{C}$ *by* $\mathcal{C}^{\mathbf{\Delta}}$. *In particular, a* **cosimplicial space** *is an object in the category* $\mathcal{T}op^{\mathbf{\Delta}}$ *of functors* $\mathbf{\Delta} \to \mathcal{T}op$.

For an object C *in* $\mathcal{C}$, *we denote by* $cs_*(C)$ *the* **constant simplicial object at** C, *the functor* $\mathbf{\Delta}^{op} \to \mathcal{C}$ *sending each object to* C *and each morphism to* 1_C. *The* **constant cosimplicial object at** C, $cc_*(X)$ *is similarly defined.*

Simplicial sets are ubiquitous in homotopy theory, but cosimplicial sets are rarely considered. Cosimplicial spaces are more common.

Definition 3.4.2. *The cosimplicial space* $\Delta^\bullet$, **the cosimplicial standard simplex**, *is the functor* $[n] \mapsto \Delta^n$, *where the* **standard** n**-simplex** Δ^n *is the space*

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbf{R}^{n+1} : t_i \geqslant 0 \text{ and } \sum_i t_i = 1 \right\}.$$

It is homeomorphic to the n*-disk* D^n. *Its* **boundary** $\partial\Delta^n$ *is the set of points with at least one coordinate equal to 0; it is homeomorphic to* S^{n-1}. *The* i**th face** Δ^n_i *for* $0 \leqslant i \leqslant n$ *is the set of points with* $t_i = 0$*; it is homeomorphic to* D^{n-1}. *The* i**th horn** Λ^n_i *is the complement of the interior of the* i*th face in the boundary, the set of points with at least one vanishing coordinate and with* $t_i > 0$. *It is also homeomorphic to* D^{n-1}. *It is an* **inner horn** *if* $0 < i < n$*; otherwise it is an* **outer horn**.

The cosimplicial standard simplicial set $\Delta[\bullet]$ *(called the cosimplicial standard simplex in [Hir03, Definition 15.1.15]) is the functor* $[n] \mapsto \Delta[n]$*, where the simplicial set* $\Delta[n]$ *(also called the* **standard** n**-simplex***) is given by*

$$\Delta[n]_k = \mathbf{\Delta}([k], [n]).$$

The singular chain complex for Y is obtained from the free abelian groups on these sets by defining a boundary operator in terms of the face maps d_i.

Definition 3.4.3. The geometric realization $|X|$ **(or** $\mathcal{R}e(X)$**) of a simplicial set** X *is the coend (Definition 2.4.5)*

$$|X| := \int_{\mathbf{\Delta}} X_n \times \Delta^n.$$

This means the topological space $|X|$ *is the quotient of the union of all of the simplices of* X*,*

$$\coprod_n X_n \times \Delta^n,$$

obtained by gluing them together appropriately. Equivalently it is the quotient of a similar disjoint union using only the nondegenerate simplices of X*. In particular the space* Δ^n *is* $|\Delta[n]|$ *for the simplicial set* $\Delta[n]$ *of Definition 3.4.2.*

The **geometric realization** $|X|$ **of a simplicial space** X *is similarly defined as a quotient of the union of the spaces* $X_n \times \Delta^n$*, whose topologies are determined by those of the spaces* X_n *as well the spaces* Δ^n*.*

Remark 3.4.4. *Following common practice, we are using the term "standard* n*-simplex" for both the topological space* Δ^n *and the simplicial set* $\Delta[n]$ *of Definition 3.4.2 in hopes that the distinction between the two will be clear from the context. Note that* $|\Delta[n]| \cong \Delta^n$*, so* $|\Delta[\bullet]| \cong \Delta^\bullet$*.*

Remark 3.4.5. The realization of a bisimiplicial set. *It follows from the definitions that the coend*

$$\int_{\mathbf{\Delta}} X_n \times \Delta[n]$$

is the simplicial set X *itself. Now suppose that* X *is a* **bisimplicial set***, meaning a simplicial object in the category of simplicial sets or equivalently set valued functor on* $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op}$*. Then in the coend above, each* X_n *is itself a simplicial set, and the coend is another simplicial set* $|X|$*. Hirschhorn [Hir03, Definition 15.11.1] calls this the* **realization** *of the bisimplicial set* X*. In [Hir03, Theorem 15.11.6] he shows that it is naturally isomorphic to the diagonal simplicial set*

$$\mathbf{\Delta}^{op} \xrightarrow{diag} \mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \xrightarrow{X} \mathcal{S}et. \tag{3.4.6}$$

Definition 3.4.7. The singular functor. *For a topological space Y the simplicial set $\mathcal{S}ing(Y)$ (the* **singular complex** *of Y) is given by letting $\mathcal{S}ing(Y)_n$ be the set of* **all** *continuous maps $\Delta^n \to Y$. The face and degeneracy operators are defined in terms of the coface and codegeneracy operators on $\mathbf{\Delta}$.*

The following is proved by May in [May67, 14.1].

Proposition 3.4.8. $|X|$ as a CW complex. *The geometric realization $|X|$ of a simplicial set X is a CW complex with one n-cell for each nondegenerate n-simplex of X.*

Similarly we have a map

$$\coprod_n X_n \to \int_{\mathbf{\Delta}} X_n,$$

which is the set $\pi_0|X|$ of path connected components of $|X|$. Thus collapsing each Δ^n to a point in Definition 3.4.3 gives a map

$$|X| = \int_{\mathbf{\Delta}} \Delta^n \times X_n \xrightarrow{\quad \epsilon \quad} \int_{\mathbf{\Delta}} X_n = \pi_0|X|. \tag{3.4.9}$$

A simplicial space X, i.e., a functor $X : \mathbf{\Delta}^{op} \to \mathcal{T}op$, has a geometric realization $|X|$ defined as in Definition 3.4.3, but with the not necessarily discrete topology of X_n taken into account.

For a simplicial set X, $|X^{[n]}|$ is the n-skeleton of the CW complex $|X|$.

The following was proved by Kan in [Kan58a].

Proposition 3.4.10. The equivalence of $\mathcal{S}et_{\mathbf{\Delta}}$ and $\mathcal{T}op$ and of their pointed analogs. *As a functor from $\mathcal{S}et_{\mathbf{\Delta}}$ to $\mathcal{T}op$, geometric realization of Definition 3.4.3 is the left adjoint of $\mathcal{S}ing$, the singular functor of Definition 3.4.7. The adjunction*

$$|\cdot| : \mathcal{S}et_{\mathbf{\Delta}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{T}op : \mathcal{S}ing$$

and its pointed analog are equivalences of categories.

In particular for an arbitrary space X one has a weak homotopy equivalence $|\mathcal{S}ing(X)| \to X$ whose source is a CW complex. For this reason, e.g., in [BK72] (the "yellow monster"), the terms "space" and "simplicial set" are sometimes used interchangeably.

Definition 3.4.11. Topological and simplicial categories.

(i) *When $\mathcal{V} = (\mathcal{T}op, \times, *)$, we say that a $\mathcal{V}$-category is a* **topological category***. We denote the category of topological categories by $\mathcal{CAT}_{\mathcal{T}op}$ and that of small topological categories by $\mathcal{C}at_{\mathcal{T}op}$.*

(ii) *When $\mathcal{V} = (\mathcal{T}, \wedge, S^0)$, we say that a $\mathcal{V}$-category is a* **pointed topological category***. We denote the category of pointed topological categories by $\mathcal{CAT}_{\mathcal{T}}$ and that of small pointed topological categories by $\mathcal{C}at_{\mathcal{T}}$..*

(iii) When $\mathcal{V} = (\mathcal{S}et_{\mathbf{\Delta}}, \times, *)$, we say that a $\mathcal{V}$-category is a **simplicial category**. *We denote the category of simplicial categories by $\mathcal{CAT}_{\mathbf{\Delta}}$ and that of small simplicial categories by $\mathcal{C}at_{\mathbf{\Delta}}$.*

(iv) When $\mathcal{V} = (\mathcal{S}et_{\mathbf{\Delta}*}, \wedge, S^0)$, we say that a $\mathcal{V}$-category is a **pointed simplicial category**. *We denote the category of simplicial categories by $\mathcal{CAT}_{\mathbf{\Delta}*}$ and that of small pointed simplicial categories by $\mathcal{C}at_{\mathbf{\Delta}*}$.*

We will see below in Corollary 5.6.16 that every topological model category is also a simplicial one.

The adjunction

$$|\cdot| : \mathcal{S}et_{\mathbf{\Delta}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{T}op : \mathcal{S}ing$$

leads to

$$|\cdot| : \mathcal{CAT}_{\mathbf{\Delta}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{CAT}_{\mathcal{T}op} : \mathcal{S}ing$$

(see Definition 3.4.11) in the obvious way. Given a simplicial category $\mathcal{C}$, we define the topological category $|\mathcal{C}|$ to have the same objects as $\mathcal{C}$ with morphism spaces

$$|\mathcal{C}|(X, Y) = |\mathcal{C}(X, Y)|,$$

and given a topological category $\mathcal{D}$, we define the simplicial category $\mathcal{S}ing(\mathcal{D})$ to have the same objects as $\mathcal{D}$ with simplicial morphisms sets

$$\mathcal{S}ing(\mathcal{D})(X, Y) = \mathcal{S}ing(\mathcal{D}(X, Y)).$$

3.4B The nerve of a small category

Definition 3.4.12. The nerve and classifying space of a small (topological) category. *For a small category J, the nerve $N(J)$ is the simplicial set given by*

$$N(J)_n = \mathcal{C}at([n], J)$$

where $[n]$ here denotes the linearly ordered set $\{0, \ldots, n\}$ regarded as a category. The **classifying space** *BJ is the geometric realization of the nerve, $|N(J)|$.*

For a small topological category D, the similarly defined nerve $N(D)$ is a simplicial space whose geometric realization (see Definition 3.4.3) is the classifying space BD.

In other words, $N(J)_n$ is the set of diagrams in J of the form

$$j_0 \to j_1 \to \cdots \to j_{n-1} \to j_n. \tag{3.4.13}$$

Of the $n+1$ face maps $N(J)_n \to N(J)_{n-1}$, $n-1$ are obtained by composing each of the $n-1$ pairs of adjacent arrows above, and the other two are obtained by ignoring the maps from j_0 and to j_n. Equivalently, assuming that J has an initial and a terminal object, we could compose each of the $n+1$ pairs of adjacent morphisms in the diagram

$$\varnothing \to j_0 \to j_1 \to \cdots \to j_{n-1} \to j_n \to *.$$

The $n+1$ degeneracy maps $N(J)_n \to N(J)_{n+1}$ are obtained by inserting the identity map on j_i in (3.4.13) for each i.

Remark 3.4.14. Two conventions in defining the nerve. *Some authors have the arrows above going the opposite way, lowering instead of raising indices, in the set of diagrams constituting $N(J)_n$. This means their $N(J)$ is our $N(J^{op})$. The distinction is meaningless when the two small categories are isomorphic, as is the case for the one object category associated with a group G. We are following the conventions of [Hir03]. The opposite convention is used in [BK72], in which the authors never use the word "nerve," but speak only of the "space associated with a small category."*

We leave the following as an exercise for the reader.

Proposition 3.4.15. Some easy classifying spaces.

(i) *For small categories J and K, $B(J \times K) = BJ \times BK$.*
(ii) *For $[n]$ as in Definition 3.4.12, $B[n] = \Delta^n$, the standard n-simplex of Definition 3.4.2.*
(iii) *The classifying space of the one object category $\mathcal{B}G$ associated with a topological group or monoid G has the homotopy type of the usual classifying space of G. For topological groups G_1 and G_2,*

$$B(G_1 \times G_2) \cong BG_1 \times BG_2. \tag{3.4.16}$$

(iv) *Let $\mathcal{B}_{G/e}G$ be the topological category with object set G and a single morphism $\gamma_1 \to \gamma_2$ between each pair of objects. Its classifying space is a contractible free G-space EG.*

Remark 3.4.17. Variants of EG and BG. *The space EG above is not the only contractible free G-space, but one of many. In practice any such space can be used to construct a classifying space BG, whose homotopy type is independent of this choice. For example if $\widehat{G}$ is any group containing G, then the space $E\widehat{G}$ defined above is also a contractible free G-space and its G-orbit space $E\widehat{G}/G$ could serve as a classifying space for G.*

Another contractible free G-space $\mathscr{E}G$ is constructed by Milnor in [Mil56] as the colimit of iterated topological joins of G with itself. As explained by Graeme Segal in [Seg68, §3], Milnor's classifying space $\mathscr{B}G = \mathscr{E}G/G$ is the classifying space (in the sense of Definition 3.4.12) of a certain subcategory of $\mathcal{B}G \times N$ for N as in Definition 2.3.63.

We will see another instance of a contractible free $O(k)$-space for the orthogonal group $O(k)$ in Example 8.3.7(iv) below.

For the remainder of this book, BG **and** *EG* **for a topological group** *G* **will be understood to be the spaces of Proposition 3.4.15(iii) and (iv)** *unless otherwise specified.*

Proposition 3.4.18. The nerve of a connected category. *A small category J is connected as in Definition 2.1.55 iff its classifying space BJ as in Definition 3.4.12 is path connected.*

Definition 3.4.19. Contractible small categories. *A small category J is* **contractible** *if its classifying space BJ as in Definition 3.4.12 is contractible.*

Proposition 3.4.20. Some contractible small categories. *If a small category J has an initial object or a terminal object, then it is contractible.*

Definition 3.4.21. *The* **barycentric subdivision** $\operatorname{sd}\Delta[n]$ **of the n-simplex.** *Let $[n]$ be the set $\{0, 1, 2, \dots, n\}$ as before, and let $\mathscr{P}([n])$ be the category of its subsets and inclusion maps. The simplicial set $\operatorname{sd}\Delta[n]$ (for $\Delta[n]$ as in Definition 3.4.3) is the nerve $B\mathscr{P}([n])$. Its nondegenerate k-simplices are sequences of proper inclusions*

$$v_0 \subset v_1 \subset \cdots \subset v_k$$

of subsets of $[n]$ called **flags**. *In particular its vertices are subsets v of $[n]$. The geometric realization $|\operatorname{sd}\Delta[n]|$ can be mapped homeomorphically to the standard n-simplex Δ^n of Definition 3.4.2 by*

$$v \mapsto (t_0, , , , t_n) \qquad \text{where } t_i = \begin{cases} 1/|v| & \text{for } i \in v \\ 0 & \text{otherwise.} \end{cases}$$

The **barycentric subdivision** $\operatorname{sd} X$ *of a simplicial set X is*

$$\operatorname{sd} X = \operatorname*{colim}_{\Delta[n]\to X} \operatorname{sd}\Delta[n],$$

where the colimit is over all maps of simplicial sets $\Delta[n] \to X$ for all n, where $\Delta[n]$ is the simplicial set of Definition 3.4.3 whose geometric realization is the standard n-simplex Δ^n of Definition 3.4.2.

More information on the homeomorphism $|\operatorname{sd}\Delta[n]| \to |\Delta[n]| = \Delta^n$ above can be found in [GJ99, Lemma III.4.1]. The subdivision $\operatorname{sd} X$ is obtained from X by subdividing each of its nondegenerate simplices.

Remark 3.4.22. Eilenberg-Mac Lane spaces. *The classifying space construction defines a fcuntro from groups to spaces. For an abelian group A, the mutiplicationmap $A \times A \to A$ is a group homomorphism, so we get a map $BA \times BA \to BA$, which can be shown to make BA itself into an abelian topological group. Hence we could take its classifying space and get another abelian* *topological group, and so on. The* n*th iteration* $B^n A$ *is the Eilenberg-Mac Lane space* $K(A, n)$.

The following is [Hir03, Definition 15.1.16].

Definition 3.4.23. *For a simplicial set* K*, the* **category** ΔK **of simplices of** K *is the category* $(\Delta\downarrow K)$ *of Definition 2.1.51. Note here that* $\Delta : \mathbf{\Delta} \to \mathcal{S}et_{\mathbf{\Delta}}$ *(Definition 3.4.2) is a functor to the category of simplicial sets* $\mathcal{S}et_{\mathbf{\Delta}}$*, while* K *is an object in it.*

The category $\Delta^{op}K$ *is* $(\Delta K)^{op}$.

3.5 The homotopy extension property, h-cofibrations and nondegenerate base points

In this section we recall some definitions useful for studying topological categories.

3.5A h-cofibrations

Definition 3.5.1. Mapping cylinders. *Given an object in* $\mathcal{T}op$*, i.e., a topological space* X*, the corresponding* **cylinder** *is the Cartesian product* $X \times I$*, where* I *denotes the unit interval* $[0, 1]$*. For a pointed space* (X, x_0)*, the* **reduced cylinder** *is*

$$X \rtimes I = X \wedge I_+ = X \times I/\{x_0\} \times I,$$

with the base point being the image of $\{x_0\} \times I$ *as in Definition 2.1.49.*

For a morphism (continuous map) $f : X \to Y$ *in* $\mathcal{T}op$*, the* **mapping cylinder** *is the space*

$$M_f = (X \times I) \coprod Y/(x, 1) \sim f(x); \tag{3.5.2}$$

one end of the cylinder $X \times I$ *is "glued onto"* Y *using the map* f*. Equivalently it is the pushout of the diagram*

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "f"'] & X \times I \arrow[d, "j"] \\ Y \arrow[r, "j_1"'] & M_f, \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \tag{3.5.3}$$

where $i_1 : X \to X \times I$ *sends* $x \in X$ *to* $(x, 1)$.

For a pointed map $f : (X, x_0) \to (Y, y_0)$ *the* **reduced mapping cylinder** *is the space*

$$M'_f = M_f/\{x_0\} \times I; \tag{3.5.4}$$

we collapse the copy of the unit interval in M_f associated with the base point $x_0 \in X$ (whose far end is identified with the base point $y_0 \in Y$ since the map f is pointed) to form the base point of M'_f. Equivalently it is the pushout of the diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"'] & X \times I \arrow[r] & X \rtimes I \arrow[d, "j'"] \\
Y \arrow[rr, "j'_1"] & & M'_f \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}.$$

The following is elementary.

Proposition 3.5.5. The target space as deformation retract. *The diagram of (3.5.3) can be enlarged to*

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"'] & X \times I \arrow[d, "j"] \arrow[ddr, "fp_1"] & \\
Y \arrow[r, "j_1"] \arrow[drr, "1_Y"'] & M_f \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "r"] & \\
& & Y,
\end{tikzcd}$$

where p_1 denotes projection onto the first factor, making Y a deformation retract of M_f. The maps j_1 and r are homotopy equivalences. Similar statements hold in the pointed case.

We will see a construction dual to Definition 3.5.1 in Definition 4.2.5 below.

Definition 3.5.6. *Let $j : S^0 \to I_+$ (where the target is the unit interval I with disjoint base point) be the map sending the nonbase point to 0. A map of pointed spaces $i : A \to X$ is an h-***cofibration** *(or* **Hurewicz cofibration***) if it is a closed embedding and the pair (X, A) has the* **homotopy extension property***: for any pointed map $f : X \to Y$ and pointed homotopy $h : I \ltimes A \to Y$ (where $X \ltimes Y$ is as in Definition 2.1.49) with $fi = h(j \wedge A)$*

$$\begin{tikzcd}
A \arrow[r, "A \wedge j"] \arrow[d, "i"'] & A \rtimes I \arrow[d, "i \rtimes I"'] \arrow[ddr, "h"] & \\
X \arrow[r, "X \wedge j"] \arrow[drr, "f"'] & X \rtimes I \arrow[dr, dashed, "\tilde{h}"] & \\
& & Y
\end{tikzcd} \tag{3.5.7}$$

there is a map $\tilde{h} : X \rtimes I \to Y$ making the full diagram commute.

Equivalently a pointed map $i : A \to X$ is an h-cofibration iff the indicated *lifting exists in all commutative diagrams of the form*

$$\begin{tikzcd}
A \arrow[r, "h'"] \arrow[d, "i"'] & Y^{I_+} \arrow[d, "e_0"] \\
X \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{h}'"] & Y,
\end{tikzcd} \tag{3.5.8}$$

where Y^{I_+} is the space of pointed maps $I_+ \to Y$, i.e., paths in Y with no conditions on their endpoints, and e_0 is evaluation at 0. The maps h' and $\widetilde{h}'$ above are the right adjoints to the maps h and $\widetilde{h}$ of (3.5.7).

We can make sense of the diagram of (3.5.8) in any topological category $\mathcal{C}$ that is bitensored (see Definition 3.1.31) over $\mathcal{T}op$ or $\mathcal{T}$ **and define h-cofibrations there accordingly.** See Definition 5.6.7 below.

Proposition 3.5.9. Mapping cylinders and h-cofibrations. *In the category $\mathcal{T}$, a map $i : A \to X$ is an h-cofibration iff its mapping cylinder M_i' as in Definition 3.5.1 is a retract of the reduced cylinder $X \rtimes I$. This means that there is a map*

$$\widetilde{h} : X \rtimes I \to M_i' \tag{3.5.10}$$

such that $\widetilde{h}r$ is the identity map on M_i', where $r : M_i' \to X \rtimes I$ is the map given by the pushout property.

Equivalently it suffices to test the condition of (3.5.7) for the case where $Y = M_i'$.

Proof Since M_i' is the pushout of the two maps out of A in (3.5.7), there is a unique map $r : M_i' \to X \rtimes I$ with $rh' = i \rtimes I$ and $rf' = X \wedge j$ in the following diagram, in which the upper left quadrilateral is a pushout.

$$\begin{tikzcd}
A \arrow[rrr, "A \wedge j"] \arrow[dd, "i"'] & & & A \rtimes I \arrow[dl, "h'"'] \arrow[dd, "i \rtimes I"] \arrow[ddddrr, bend left, "h"] & & \\
& & M_i' \arrow[dr, "r"] & & & \\
X \arrow[urr, "f'"] \arrow[rrr, "X \wedge j"'] \arrow[ddrrrrr, bend right, "f"'] & & & X \rtimes I \arrow[dr, dashed, "\widetilde{h}"] & & \\
& & & & M_i' \arrow[dr] & \\
& & & & & Y
\end{tikzcd}$$

If i is an h-cofibration, then for $Y = M_i'$ there is a map $\widetilde{h}$ making the diagram commute, which means that $\widetilde{h}r$ is the identity map on M_i'. Conversely if such a map $\widetilde{h}$ exists, the commutativity of the outer quadrilateral above and the puhout property of M_i' means there is a unique map $M_i' \to Y$ determined by h and f. □

The following is an easy consequence of the condition of (3.5.8).

Proposition 3.5.11. Sequential colimits preserve h-cofibrations. *The class of h-cofibrations is stable under composition, and the formation of coproducts and cobase change. Given a sequence*

$$X_1 \xrightarrow{f_1} \cdots \to X_i \xrightarrow{f_i} X_{i+1} \to \cdots$$

in which each f_i is an h-cofibration, the map

$$X_j \to \operatorname*{colim}_i X_i$$

is an h-cofibration for each $j > 0$.

Definition 3.5.12. Deformation retracts. *A pair (X, A) is an* **NDR-pair** *(short for* **neighborhood deformation retract pair***) if there is a continuous map $u : X \to I$ such that $u^{-1}(0) = A$ and a homotopy $h : X \times I \to X$ such that $h(x, 0) = x$ for all $x \in X$, $h(a, t) = a$ for all $t \in I$ when $a \in A$, and $h(x, 1) \in A$ if $u(x) < 1$. (X, A) is a* **DR-pair** *if $u(x) < 1$ for all $x \in X$, in which case A is a* **deformation retract** *of X.*

The statement and proof of the following can be found [May99a, §6.4], where an h-cofibration is called a cofibration.

Theorem 3.5.13. Properties of h-cofibrations. *Let A be a closed subspace of X. Then the following are equivalent:*

(i) (X, A) is an NDR-pair as in Definition 3.5.12.
(ii) $(X \times I, X \times \{0\} \cup A \times I)$ is a DR-pair.
(iii) $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.
(iv) The inclusion $i : A \to X$ is an h-cofibration.

A proof of the following can be found in [Bre93, §VII.1] and in [May99a, Chapter 6].

Proposition 3.5.14. Some h-cofibrations. *Let the pointed space X be obtained from A by attaching cells. Then the inclusion map $i : A \to X$ is an h-cofibration. If $f : X \to Y$ is a map in $\mathcal{T}$, then the inclusion map $X \to M_f'$ to the reduced mapping cyclinder (Definition 3.5.1) is an h-cofibration.*

A based inclusion $i : A \to X$ is closed (meaning its image a closed subset of X) iff its reduced mapping cylinder M_i' is a retract of the reduced cylinder of X.

Note that an h-cofibration in the functor category $\mathcal{T}^J$ for small J is more than an objectwise h-cofibration because the choice of $\widetilde{h}$ must be natural in the objects of J.

The next four results are taken from [LMSM86, pages 488-489]. The following is straightforward.

Proposition 3.5.15. Mapping cylinders and pullbacks. *Let $f : X \to Y$ be a morphism in $\mathcal{T}^J$ for a small category J. Define the reduced mapping cylinder M'_f (Definition 3.5.1) and reduced cylinder $Y \rtimes I$ objectwise. Then the diagram*

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d, "f"'] & M'_f \arrow[d, "i"] \\ Y \arrow[r, "Y \wedge j"] & Y \rtimes I \end{tikzcd}$$

is a pullback diagram, where j is as in Definition 3.5.6, $i_0(x) = (x, 0)$ and $i(x, t) = (f(x), t)$.

Proposition 3.5.16. Retractions and closed inclusions. *Let*

$$i : A \to X \qquad \text{and} \qquad r : X \to A$$

be morphisms in $\mathcal{T}^J$ such that $ri = 1_A$. Then the diagram

$$\begin{tikzcd} A \arrow[r, "i"] & X \arrow[r, shift left, "ir"] \arrow[r, shift right, "1_X"'] & X \end{tikzcd}$$

is an equalizer and i is a closed inclusion.

Proposition 3.5.17. Pullbacks and closed inclusions. *Let*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "i"] \\ Y \arrow[r, "g"] & W \end{tikzcd}$$

be a pullback diagram in $\mathcal{T}^J$. Then if i is a closed inclusion, so is f.

Proof If i is a closed inclusion, then it is the equalizer of a pair of maps $i_1, i_2 : Z \to W \cup_Z W$. Since the diagram is a pullback, this implies that i' is the equalizer of $i_1 g$ and $i_2 g$ and hence a closed inclusion. $\square$

Lemma 3.5.18. Every h-cofibration $f : X \to Y$ in $\mathcal{T}^J$ is an objectwise closed inclusion.

Proof In Proposition 3.5.17, let $W = M'_f$ and $Z = Y \rtimes I$. $\square$

Proposition 3.5.19. Left adjoints preserve h-cofibrations. *Any topological functor F which is a continuous left adjoint preserves the class of h-cofibrations.*

Proof Let

$$F : \mathcal{C} \underset{}{\overset{}{\rightleftarrows}} \mathcal{D} : G$$

be the adjunction for the closed topological categories (as in Definition 2.6.42) $\mathcal{C}$ and $\mathcal{D}$, let $i : A \to X$ be an h-cofibration in $\mathcal{C}$ and let Y be an object in $\mathcal{D}$. The continuity of G implies that $G(Y^{I_+}) = (GY)^{I_+}$. Hence the desired lifting in the diagram

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "Fi"'] & Y^{I_+} \arrow[d] \\ FX \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

is adjoint to the one in

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & (GY)^{I_+} \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & GY. \end{tikzcd}$$

□

Now suppose that a closed topological category $\mathcal{C}$ as in Definition 2.6.42 has a symmetric monoidal structure $\otimes$ which is compatible with the smash product of pointed spaces, in the sense that for pointed spaces K and L, and objects $X, Y \in \mathcal{C}$ there is a natural isomorphism

$$(X \otimes K) \otimes (Y \otimes L) \approx (X \otimes Y) \otimes (K \wedge L)$$

compatible with the enrichment and the symmetric monoidal structures. Then given $i : A \to X$ we may form

$$i^{\otimes n} : A^{\otimes n} \to X^{\otimes n}$$

and regard it as a map in the category $\mathcal{C}^{\mathcal{B}\Sigma_n}$ of objects in $\mathcal{C}$ equipped with a Σ_n-action.

Proposition 3.5.20. Smashing preserves h-cofibrations. *If $i : A \to X$ is an h-cofibration in a pointed closed topological category $\mathcal{C}$ as in Definition 2.6.42, then for any pointed topological space K, the map*

$$i \wedge K : A \wedge K \to X \wedge K$$

is also an h-cofibration.

Proof In the diagram of (3.5.8) we replace Y by Y^K and use the fact that

$$(Y^K)^{I_+} \approx Y^{K \rtimes I} \approx (Y^{I_+})^K$$

by Proposition 3.1.37. Thus the diagram is

$$\begin{tikzcd}
A \arrow[r, "h'"] \arrow[d, "i"'] & (Y^{I_+})^K \arrow[d, "e_0"] \\
X \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{h}'"] & Y^K,
\end{tikzcd}$$

which is adjoint to

$$\begin{tikzcd}
A \wedge K \arrow[r, "h'"] \arrow[d, "i \wedge K"'] & Y^{I_+} \arrow[d, "e_0"] \\
X \wedge K \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{h}'"] & Y.
\end{tikzcd}$$

This makes $i \wedge K$ an h-cofibration. $\square$

Theorem 3.5.21. Monoidal powers preserve h-cofibrations. *If $i : A \to X$ is an h-cofibration in a pointed closed topological category $\mathcal{C}$, then $i^{\wedge n}$ is an h-cofibration in the closed topological category $\mathcal{C}^{\mathcal{B}\Sigma_n}$.*

Remark 3.5.22. *In the category of equivariant orthogonal spectra, a version of this result appears in [MMSS01, Lemma 15.8], where the reader is referred to [EKMM97, Lemma XII.2.3]. Our proof is independent of theirs.*

Proof Suppose we can show that the diagonal inclusion

$$M'_{i^{\wedge n}} \to (M'_i)^{\wedge n} \tag{3.5.23}$$

(where M'_f is the reduced mapping cylinder of Definition 3.5.1) is the inclusion of a Σ_n-equivariant retract with retraction map r_n. Then we can construct a Σ_n-equivariant retraction of

$$M'_{i^{\wedge n}} \to X^{\wedge n} \rtimes I$$

(where $I = [0, 1]$ as usual) as the composition

$$X^{\wedge n} \wedge I \xrightarrow{X^{\wedge n} \wedge \mathrm{diag}} X^{\wedge n} \rtimes I^n \approx (X \rtimes I)^{\wedge n} \xrightarrow{\widetilde{h}^{\wedge n}} (M'_i)^{\wedge n} \xrightarrow{r_n} M'_{i^{\wedge n}}$$

where $\widetilde{h}$ is the retraction of (3.5.10) and r_n is retraction of (3.5.23). Then we can apply Proposition 3.5.9 to the map $i^{\wedge n}$ to conclude that it is an h-cofibration.

For the desired retraction r_n of the embedding of (3.5.23), the key construction is the symmetric retraction of the unit n-cube onto its diagonal given by

$$(x_1, \ldots x_n) \mapsto (x_0, \ldots x_0) \qquad \text{where } x_0 = \min(x_i). \tag{3.5.24}$$

Start with the diagram in $\mathcal{C}^n$ in which each component is the pushout square

$$\begin{tikzcd}
A \rtimes \{0\} \arrow[r] \arrow[d, "i"'] & A \rtimes I \arrow[d] \\
X \arrow[r] & M'_i
\end{tikzcd}$$

We have the target exponent filtration of Definition 2.9.34 in which

$$\mathrm{fil}_0 = X^{\wedge n} \qquad \text{and} \qquad \mathrm{fil}_n = (M'_i)^{\wedge n},$$

and the spaces fil_k for $0 < k < n$ interpolate between those two. We will use only the nth stage of it. The diagram of (2.9.40) reads

$$\begin{tikzcd}
\partial_A (A \rtimes I)^{\wedge n} \arrow[r] \arrow[d] & (A \rtimes I)^{\wedge n} \arrow[d] \\
\mathrm{fil}_{n-1} \arrow[r] & (M'_i)^{\wedge n}
\end{tikzcd} \tag{3.5.25}$$

and all maps are Σ_n-equivariant. Using the retraction $M'_i \to X$ and the inclusion $X^{\wedge n} \to M'_{i^{\wedge n}}$ we get

$$\mathrm{fil}_{n-1} \to X^{\wedge n} \to M'_{i^{\wedge n}},$$

To extend it to fil_n, note that the top row of (3.5.25) can be identified with the tensor product of the identity map of $A^{\wedge n}$ with

$$\partial_{\{0\}} I^n_+ \to I^n_+.$$

The domain here is the union of a disjoint base point with the subspace of I^n consisting of all points in which at least one coordinate is 0. This identification is compatible with the action of the symmetric group. The desired extension is then constructed using the Σ_n-equivariant retraction of I^n to the diagonal given by (3.5.24), which takes $\partial_{\{0\}} I^n$ to the point $(0, \dots 0)$. $\square$

Working fiberwise one concludes

Proposition 3.5.26. Indexed monoidal products preserve h-cofibrations. *Suppose that $(\mathcal{M}, \wedge, S)$ is a symmetric monoidal category which is also a closed topological category as in Definition 2.6.42, and $p : I \to J$ is a covering category as in Definition 2.8.1. The indexed monoidal product*

$$p^{\wedge}_* : \mathcal{M}^I \to \mathcal{M}^J$$

preserves the class of h-cofibrations.

3.5B Nondegenerate base points

Definition 3.5.27. *A* **nondegenerate point** $x \in X$ *for a topological space* X *is one for which the inclusion map is an* h*-cofibration as in Definition 3.5.6. A functor* $X : J \to \mathcal{T}$ *is* **nondegenerately based** *if the space* X_j *has a nondegenerate base point for each object* j *of* J*. A pointed topological category is* **nondegenerately based** *if each of its pointed morphism spaces is.*

Note that this use of the word "degenerate" has nothing to do with degeneracies in connection with simplicial sets in §3.4.

Degenerate points are rare, so we offer a textbook example; see Example 3.5.31 for another one.

Example 3.5.28. A space with degenerate base point. *Let* $X \subset I^2$ *be the* **comb space***,*

$$X = (0 \times I) \cup \left(\bigcup_{n>0} \{1/n\} \times I \right) \cup (I \times 0),$$

and let $x = (0,1) \in X$*. The pair* $(X, \{x\})$ *does not have the homotopy extension property (see Definition 3.5.6), so the base point* x *is degenerate. Let* $Y = X \times \{0\} \cup x \times I \subset X \times I$ *and let* $f : X \to Y$ *be the inclusion. It does not extend to* $X \times I$ *because its subspace* Y *is not a retract.*

There is an easy way to deal with degenerate base points when they occur.

Definition 3.5.29. Adding a whisker. *Given a space* X *with degenerate base point* x_0*, we can replace the bad pair* (X, x_0) *with a good pair* $(\widetilde{X}, x_1)$ *constructed as follows. The space* $\widetilde{X}$ *is the union of* X *with an interval* I *attached to* X *at the point* x_0*, and* x_1 *is the other end of the interval. The map* $(\widetilde{X}, x_1) \to (X, x_0)$ *collapses* I *to* x_0*, and* $x_1 \in \widetilde{X}$ *is a nondegenerate base point.*

The construction of Definition 3.5.29 is functorial in (X, x_0) and is used below in Example 5.1.12.

Proposition 3.5.30. The suspension of a weak equivalence. *Let* $f : X \to Y$ *be a weak equivalence of spaces with nondegenerate base point. Then its suspension* $\Sigma f = S^1 \wedge f$ *is also a weak equivalence.*

Proof The nondegeneracy of the base points insures that the map from the unreduced suspension, the double cone on X, to the reduced suspension $\Sigma X = S^1 \wedge X$ (sending the line through x_0 to a point) is a homotopy equivalence, and similarly for Y. A pointed weak equivalence $X \to Y$ is easily seen to induce a homology equivalence, an isomorphism in π_1 and therefore a weak equivalence on unreduced suspensions. □

I am grateful to Greg Arone, Tyler Lawson and others for the following counterexample illustrating the necessity of a nondegenerate base point.

Example 3.5.31. A weak equivalence not preserved by suspension. *Let* $\mathbf{N}$ *denote the natural numbers with the discrete topology, and let*

$$X = \{0\} \cup \{1/n \colon n = 1, 2, 3, \dots\} \subset I$$

(topologized as a subspace of I*), with both having 0 as base point. The base point of* X *is degenerate. The comb space of Example 3.5.28 can be mapped to the unreduced cone on this* X *by sending the bottom interval* $I \times \{0\}$ *to the cone point. That map is a homotopy equivalence.*

Define $f : \mathbf{N} \to X$ *by*

$$f(n) = \begin{cases} 0 & \text{for } n = 0 \\ 1/n & \text{for } n > 0. \end{cases}$$

It is a continuous bijection (so the same is true for all of its suspensions) but not a homeomorphism since its set theoretic inverse is not continuous. The discrete space $\mathbf{N}$ *has more open sets than* X*. Moreover* X *is compact, since any open set containing 0 must have a finite complement. It is also a closed subset of the unit interval.*

The map f *is a weak equivalence since it induces a bijection of sets on* π_0*, and the higher homotopy groups of both spaces are trivial. However its single suspension is a map from an infinite wedge of circles to the* **Hawaiian earring** H *(see [EK00b]) which does not induce an isomorphism on* π_1*.* H *is a subset of the plane* $\mathbf{R}^2$*, namely the union of the circles through the origin having centers at* $(1/n, 0)$ *for all positive integers* n*. It is locally path connected, but not semi-locally simply connected.*

For $k > 1$*,* $\pi_k \Sigma^k \mathbf{N}$ *is a countable direct sum of copies of the integers and hence countable, while* $\pi_k \Sigma^k X$ *is the limit of an inverse system of finitely generated free abelian groups, and is uncountable. Moreover the space* $\Sigma^k X$*, the subject of [BM62] and [EK00a], is known to have infinitely many nontrivial rational homology groups, quite unlike* $\Sigma^k \mathbf{N}$*.*

We will consider the Hawaiian earring again in Example 4.2.4(ii),

4

Quillen's theory of model categories

> In order to compute this cohomology for commutative rings, the author was led to consider the simplicial objects over [an abelian category] $\mathcal{A}$ as forming the objects of a homotopy theory analogous to the homotopy theory of algebraic topology, then using the analogy as a source of intuition for simplicial objects. This was suggested by the theorem of Kan [Kan58b] that the homotopy theory of simplicial groups is equivalent to the homotopy theory of connected pointed spaces. The analogy turned out to be very fruitful, but there were a large number of arguments that were formally similar to well known ones in algebraic topology, so it was decided to define the notion of a homotopy theory in sufficient generality to cover in a uniform way the different homotopy theories encountered.
>
> *Daniel Quillen [Qui67, pages i–ii]*

Quillen invented model categories in [Qui67] for the reasons stated above. His ideas have become increasingly prominent in the subject in the past twenty-five years. (Of the more than 700 citations of [Qui67] listed in MathSciNet, only 18 appeared before 1994.) The most thorough accounts are the books by Hovey [Hov99] and Hirschhorn [Hir03]. A lighter and very helpful introduction is [DS95]. We also recommend [MP12, Part 4].

Our aim in this chapter and the two that follow it is not to produce another self contained account of the subject, but merely to tell our readers what they need to know to follow the arguments to be presented later in the book. We will refer the sources above for most of the proofs.

In §4.1 we will give the axioms of a model category and related definitions including those of fibrant and cofibrant objects (**??**) and fibrant and cofibrant replacement, Definition 4.1.20.

In §4.2 we introduce the three examples orginally cited by Quillen, namely topological spaces, chain complexes of R-modules and simplicial sets. The first of these is the most familiar and the most important for our purposes. In studying it we use the previously defined notions of mapping cylinders and reduced mapping cylinders (Definition 3.5.1), h-cofibrations and the homotopy extension property (Definition 3.5.6), nondegenerate base points (Definition 3.5.27) and adding a whisker, Definition 3.5.29. In studying the model category structure on simplicial sets we define Kan fibrations in Definition 4.2.17.

The word "homotopy" does not appear in the definition of a model category despite that fact that homotopy theory is its motivation. The various notions of homotopy that can be defined in a model category are the subject of §4.3. The Quillen homotopy category $\operatorname{Ho}\mathcal{M}$ of a model category $\mathcal{M}$ is introduced in Definition 4.3.16.

A functor on a model category $\mathcal{M}$ is homotopical if it factors through the homotopy category $\operatorname{Ho}\mathcal{M}$. Unfortunately not all the functors we encounter have this property. How to deal with them is the subject of §4.4. They often behave well on fibrant or cofibrant objects even if they do not behave well in general. It is often useful to replace them by derived (Definition 4.4.5) or total derived (Definition 4.4.7) versions whose existence is the subject of Proposition 4.4.6 and Proposition 4.4.8.

Functors between model categories are the subject of § 4.5. They tend to come in adjoint pairs going in opposite directions called Quillen pairs or Quillen adjunctions Definition 4.5.1. The left adjoint (also called a **left Quillen functor**) preserves cofibrations and trivial cofibrations, while the right adjoint (**right Quillen functor**) preserves fibrations and trivial fibrations. Thus a left Quillen functor preserves trivial cofibrations between cofibrant objects. Ken Brown's Lemma 5.1.7 says that this implies that it preserves **all** weak equivalences between cofibrant objects.

A Quillen pair $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$, is a **Quillen equivalence** (Definition 4.5.14) if for all cofibrant X in $\mathcal{M}$ and all fibrant Y in $\mathcal{N}$, a map $f : FX \to Y$ is a weak equivalence in $\mathcal{N}$ iff the corresponding map $X \to UY$ is a weak equivalence in $\mathcal{M}$. Theorem 4.5.17 says that a Quillen equivalence between model categories induces a categorical equivalence between the corresponding homotopy categories.

In §4.6 we study model categoric generalizations of the classical suspension and loop functors. We follow Quillen's treatment of this topic [Qui67, §I.2] very closely. Suspensions and loop objects and functors in a general model category are spelled out inDefinition 4.6.17. Stable model categories are defined in Definition 5.7.1. In Definition 5.7.3, we say that a model category is **exactly stable** if it has desuspension and delooping functors with certain properties. This notion is new as far as we know.

In § 4.7 we study fiber sequences and cofiber sequences, again following Quillen [Qui67, §I.3]. Such sequences are defined in Definition 4.7.6. For example a cofiber sequence starts with a cofibration $f : A \to B$ where A is cofibrant in a pointed model category. This leads to a second cofibration $g : B \to C$, where C is the cofiber of f and a map $m' : C \to C \vee \Sigma A$ with certain properties. Composing m' with projection onto ΣA obtained by collapsing C to the initial/terminal object. Then it turns out (Proposition 4.7.9) that ΣA is the cofiber of g and the evident map $h : C \to \Sigma A$ is also a cofibration. Its cofiber is ΣB, and we can repeat this process *ad infinitum.* Thus we get a diagram

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{\Sigma f} \cdots.$$

For any fibrant object Y, Proposition 4.7.11 gives an exact sequence

$$\pi(A, Y) \xleftarrow{f^*} \pi(B, Y) \xleftarrow{g^*} \pi(C, Y) \xleftarrow{h^*} \pi(\Sigma A, Y) \xleftarrow{(\Sigma f)^*} \cdots,$$

where $\pi(-,-)$ is defined in Definition 4.3.11. Theorem 5.7.6 says that when the model category is exactly stable as in Definition 5.7.3, this exact sequence can be extended to the left indefinitely. There is a dual notion of a fiber sequence starting with a fibration to a fibrant object, and we get a similar exact sequence by considering homotopy classes of maps to it from a cofibrant object.

In §4.8 we review Quillen's small object argument. This is the most technically challenging part of the theory. It is used to construct the factorizations required in a model category. It involves set theoretic and cardinality arguments that most homotopy theorists prefer not to think about.

4.1 Basic definitions

4.1A The definition of a model category

Definition 4.1.1. *A* **model category** $\mathcal{M}$ *is a category with three classes of morphisms called weak equivalences ($\mathscr{W}$), fibrations ($\mathscr{F}$) and cofibrations ($\mathscr{C}$), each closed under composition and containing all isomorphisms. A trivial fibration (cofibration) is one which is also a weak equivalence. These are required to satisfy the following five axioms.*

MC1 Bicompleteness axiom. $\mathcal{M}$ *has all small limits and colimits.*

MC2 Two-out-of-three axiom. *Let f and g be morphisms in $\mathcal{M}$ such that gf is defined. Then if two of f, g and gf are weak equivalences, so is the third. (Note that weak equivalences, unlike homotopy equivalences in topology, are* **not** *required to have inverses.)*

MC3 Retract axiom. *If f is a retract (Definition 2.1.56) of g and g is a weak equivalence, fibration or cofibration, then so is f.*

MC4 Lifting axiom. *Given a commutative diagram as in (2.3.11),*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y, \end{tikzcd}$$

a morphism h exists with $hi = f$ and $ph = g$, that is $i \boxslash p$ as in Definition 2.3.10, when either

(i) i is a cofibration ($i \in \mathscr{C}$) and p is a trivial fibration ($p \in \mathscr{W} \cap \mathscr{F}$) or
(ii) i is a trivial cofibration ($i \in \mathscr{W} \cap \mathscr{C}$) and p is a fibration ($p \in \mathscr{F}$).

MC5 Factorization axiom.
$\mathcal{M}$ has two functorial factorizations (as in Definition 2.2.9) F_0 and F_1 such that for any morphism $f : X \to Y$ we get commutative diagram

$$\begin{tikzcd}
& \widetilde{Y} \arrow[dr, "\delta_0 F_0(f)"] & \\
X \arrow[ur, "\delta_2 F_0(f)"] \arrow[rr, "f"] \arrow[dr, "\delta_2 F_1(f)"'] & & Y \\
& \widehat{X} \arrow[ur, "\delta_0 F_1(f)"'] &
\end{tikzcd}$$

where

- $\delta_2 F_0(f)$ *is a cofibration,*
- $\delta_0 F_0(f)$ *is a trivial fibration,*
- $\delta_2 F_1(f)$ *is a trivial cofibration and*
- $\delta_0 F_1(f)$ *is a fibration.*

Note that each intermediate object is denoted by the same letter as the original object (X or Y) it is weakly equivalent to.

When these axioms are satisfied, we say that $(\mathcal{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ **defines a model category**.

MC1 implies that $\mathcal{M}$ has an initial object $\varnothing$ and a terminal object $*$, the colimit and limit respectively of the empty diagram. When they are the same, we will say the model category is **pointed** in Definition 4.1.26 below.

We will see below in Proposition 5.1.2 that in every model category the weak equivalences satisfy a stronger condition than **MC2** called the 2-of-6 property of Definition 5.1.1.

The lifting axiom **MC4** can be reformulated as follows. The model category $\mathcal{M}$ has morphism classes $\mathscr{W}$ (weak equivalences), $\mathscr{C}$ (cofibrations) and $\mathscr{F}$ (fibrations). Hence the class of trivial cofibrations (trivial fibrations) is by definition $\mathscr{W} \cap \mathscr{C}$ ($\mathscr{W} \cap \mathscr{F}$). Then, using the notation of Definition 2.3.10,

$$(\mathscr{W} \cap \mathscr{C}) \boxslash \mathscr{F} \qquad \text{and} \qquad \mathscr{C} \boxslash (\mathscr{W} \cap \mathscr{F}).$$

The factorization axiom **MC5** says there are weak factorization systems (as in Definition 2.3.19)

$$(\mathscr{W} \cap \mathscr{C}, \mathscr{F}) \qquad \text{and} \qquad (\mathscr{C}, \mathscr{W} \cap \mathscr{F}). \tag{4.1.2}$$

MC5 also implies that every weak equivalence is the composite of a trivial cofibration followed by a trivial fibration.

It is known [JT07, Proposition 7.8] that model categories can be characterized as follows.

Proposition 4.1.3. Model categories and morphism classes. *Let $\mathcal{M}$ be a bicomplete category with morphism classes $\mathscr{W}$, $\mathscr{C}$ and $\mathscr{F}$ such that*

- *$\mathscr{W}$ satisfies the 2-of-3 property and*
- *$(\mathscr{W}\cap\mathscr{C},\mathscr{F})$ and $(\mathscr{C},\mathscr{W}\cap\mathscr{F})$ are weak factorization systems.*

Then $(\mathcal{M},\mathscr{W},\mathscr{C},\mathscr{F})$ defines a model category.

Definition 4.1.4. *An object X in a model category is* **contractible** *if the unique map $X\to *$ is a weak equivalence.*

Remark 4.1.5. The word "contractible." *Surprisingly, this term is* **not used** *in the model category literature, except in reference to certain topological spaces or simplicial sets associated with a model category. We are introducing it here for its convenience in Example 4.1.14, Proposition 4.5.9 and Lemma 5.6.20 below.*

We note that this notion of contractibility is self dual only when the model category is pointed as in Definition 4.1.26 below. In the category $\mathcal{T}op$ of topological spaces without base point, a contractible space is not one that admits a weak equivalence from the empty set.

Remark 4.1.6. The original model category axioms. *These axioms are slightly stronger than those originally given by Quillen in [Qui67]. His* **MC1** *required only* **finite** *limits and colimits, and he did not require the factorizations of* **MC5** *to be functorial. Experience has shown that the strengthened axioms are more convenient and are satisfied in nearly every interesting example.*

There is a weaker notion of a **homotopical category**, in which one has weak equivalences satisfying a stronger form of **MC2**. Logically, they should be studied before model categories, but historically they were introduced decades later. We will treat them in §5.1 below.

Remark 4.1.7. The hard part. *In order to use model category theory, one must show that the category one is interested in really has a model structure. Often the hardest part of this is verifying* **MC5***, which can involve delicate set theoretic arguments.* **We will discuss this further in § 4.8.**

Proposition 4.1.8. Any two of the three morphisms classes (fibrations, cofibrations and weak equivalences) determines the third.

Proof If we know the fibrations and cofibrations, then the trivial fibrations (trivial cofibrations) are those morphisms haveing the right (left) lifing property with respect to all cofibrations (fibrations).

If we know the fibrations and the weak equivalences, then the cofibrations (trivial cofibrations) are those morphisms having the left lifting property with respect to all trivial fibrations (all fibrations).

A dual argument works for cofibrations and weak equivalences. □

Remark 4.1.9. Changing model structures: the seesaw effect. *We will sometimes want to consider more than one model structure on the same underlying category. We may wish to alter the model structure by keeping one of the three morphism classes fixed and expanding another one. This invariably means shrinking the third class.*

For example we may want to expand the class of weak equivalences and keep the same class of cofibrations. This process is called **Bousfield localization** *and is the subject of Chapter 6 below. This means more of the cofibrations will be trivial. Since fibrations are required to have the right lifting property with respect to trivial cofibrations, there will be fewer of them, and fibrant replacement will be more interesting. On the other hand, the class of trivial fibrations, being those morphisms with the right lifting property with respect to all cofibrations, will remain the same.*

If we expand the classes of fibrations and weak equivalences, we will have both fewer cofibrations and fewer trivial cofibrations, because the lifting properties they must satisfy will be more demanding. We will see an instance of this in Remark 5.4.23 below.

Definition 4.1.10. Injective morphisms and objects. *For a class $\mathscr{C}$ of morphisms in a model category $\mathcal{M}$, a* **$\mathscr{C}$-injective morphism** *g is one that has the right lifting property with respect to each map in $\mathscr{C}$. A* **$\mathscr{C}$-injective object** *X is one for which the morphism $X \to *$ is $\mathscr{C}$-injective. A map is a* **$\mathscr{C}$-cofibration** *if it has the left lifting property with respect to every $\mathscr{C}$-injective map.*

In the notation of Definition 2.3.10, the classes of $\mathscr{C}$-injective morphisms and $\mathscr{C}$-cofibrations are $\mathcal{C}^{\boxslash}$ and ${}^{\boxslash}(\mathcal{C}^{\boxslash})$ respectively.

Example 4.1.11. Fibrations as injective morphisms. *Let $\mathscr{C}$ be the class of all cofibrations (trivial cofibrations) in $\mathcal{M}$. Then*

- *the $\mathscr{C}$-injective morphisms are the trivial fibrations (fibrations),*
- *the $\mathscr{C}$-injective objects are the contractible (meaning weakly equivalent to $*$) fibrant objects (all fibrant objects) and*
- *the $\mathscr{C}$-cofibrations are the cofibrations (trivial cofibrations).*

Proposition 4.1.12. Pushouts (pullbacks) of cofibrations (fibrations). *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, phantom, very near start, "(\lrcorner)"] & Y \end{tikzcd}$$

be a pullback (pushout) diagram in a model category $\mathcal{M}$. If p is a fibration (i is a cofibration), so is the map i (p). If p is a trivial fibration (i is a trivial cofibration), so is the map i (p).

Proof We will prove the pullback form of the statements, leaving the dual pushback form to the reader. To show that i is a fibration, suppose $j : C \to D$ is a trivial cofibration and we have a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "j"'] & A \arrow[r] \arrow[d, "i"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "p"] \\ D \arrow[r] \arrow[ur, dashed, "h_2"] \arrow[urr, dashed, "h_1"'] & B \arrow[r] & Y. \end{tikzcd}$$

Then the lifting h_1 exists because p is a fibration, and the lifting h_2 exists because the right square is a pullback. The existence of h_2 for any trivial cofibration j means that i is a fibration as claimed.

Similarly, suppose p is a trivial fibration and j is any cofibration. Then the liftings exist as before, making i a trivial fibration. □

The above generalizes as follows.

Proposition 4.1.13. Limits (colimits) preserve fibrations and trivial fibrations (cofibrations and trivial cofibrations). *Let F and F' be functors from a small category J to a model category $\mathcal{M}$, and let $\theta : F \Rightarrow F'$ be a natural transformation. Then if the map $\theta_j : F(j) \to F'(j)$ is a fibration (cofibration) for each object j of J, then the induced map $\lim_J F \to \lim_J F'$ ($\operatorname{colim}_J F \to \operatorname{colim}_J F'$) is a fibration (cofibration).*

In particular any limit (colimit) of fibrant (cofibrant) objects is fibrant (cofibrant).

Proof We will prove the statement for colimits, leaving the dual statement for limits to the reader. Let $\mathcal{M}^J$ denote the category of functors from J to $\mathcal{M}$. It has a model structure in which a morphism is a weak equivalence or a fibration if its value on each object in J is one. (There is a different model structure in which a morphism is a weak equivalence or a cofibration if its value on each object in J is one. It is needed for the dual case. Both will be studied further in §5.4 below.) Thus F and F' can be regarded as objects in $\mathcal{M}^J$ and θ as a morphism between them. Recall (Proposition 2.3.24) that the colimit functor $\mathcal{M}^J \to \mathcal{M}$ is the left adjoint of the diagonal functor $\Delta : \mathcal{M} \to \mathcal{M}^J$.

Let $p : X \to Y$ be a trivial fibration in $\mathcal{M}$, making $\Delta(p)$ a trivial fibration in $\mathcal{M}^J$. Consider the following adjoint pair of diagrams in $\mathcal{M}$ and $\mathcal{M}^J$.

$$\begin{tikzcd} \operatorname*{colim}_J F \arrow[r] \arrow[d, "\operatorname*{colim}_J \theta"'] & X \arrow[d, "p"] \\ \operatorname*{colim}_J F' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r] \arrow[d, "\theta"'] & \Delta(X) \arrow[d, "\Delta(p)"] \\ F' \arrow[r] \arrow[ur, dashed] & \Delta(Y). \end{tikzcd}$$

The hypothesis that each θ_j is a cofibration is equivalent to the existence of a lifting in the diagram in on the right. That lifting is adjoint to one on the left, which makes $\operatorname*{colim}_J \theta$ a cofibration as claimed.

Similarly let $p : X \to Y$ be a fibration in $\mathcal{M}$, making $\Delta(p)$ a fibration in $\mathcal{M}^J$, and consider the same diagrams as before. The hypothesis that each θ_j is a trivial cofibration is equivalent to the existence of a lifting in the diagram in on the right. That lifting is adjoint to one on the left, which makes $\operatorname*{colim}_J \theta$ a trivial cofibration as claimed. □

The following may seem pedantic, but it is surprisingly useful.

Example 4.1.14. Overcategories and undercategories of a model category. *Let $\mathcal{M}$ be a model category and let A be an object in it. Then we have the undercategory $(A{\downarrow}\mathcal{M})$ and the overcategory $(\mathcal{M}{\downarrow}A)$ as in Definition 2.1.51. In both cases there is a forgetful functor U to $\mathcal{M}$ obtained by ignoring the structure map ω or υ. It is known that both categories admit model structures in which a morphism $X \to Y$ (meaning a triangle as in (2.1.52) or (2.1.53)) is a weak equivalence, fibration or cofibration if its image under the forgetful map is one. A proof can be found in [Hir15].*

It follows that the initial object of $(A{\downarrow}\mathcal{M})$ is $1_A : A \to A$ while the terminal object is $A \to *$. An object $\upsilon_X : A \to X$ is cofibrant if υ_X is a cofibration in $\mathcal{M}$, fibrant if X is fibrant in $\mathcal{M}$ and contractible if X is contractible in $\mathcal{M}$.

Dually, the terminal object of $(\mathcal{M}{\downarrow}A)$ is $1_A : A \to A$ while the initial object is $\varnothing \to A$. An object $\omega_X : X \to A$ is fibrant if ω_X is a fibration in $\mathcal{M}$, contractible if ω_X is a weak equivalence, and cofibrant if X is cofibrant in $\mathcal{M}$.

4.1B Some toy examples

In the next section we will discuss Quillen's three classical examples of model categories, namely topological spaces, chain complexes of R-modules and simplicial sets. Some drier examples are the following.

Definition 4.1.15. The dual of a model category. *Let $\mathcal{M}$ be a model category. Then the opposite category $\mathcal{M}^{op}$ has a model structure in which weak equivalences are dual to those of $\mathcal{M}$ and fibrations (cofibrations) are dual to the cofibrations (fibrations) of $\mathcal{M}$.*

Definition 4.1.16. The product of a set of model categories. *For model categories $\mathcal{M}$ and $\mathcal{N}$ we can define a model category structure on $\mathcal{M} \times \mathcal{N}$ (see Definition 2.1.5) as follows. A morphism (f, g) is a weak equivalence fibration or cofibration if both f and g are. This definition can be extended to any set of model categories.*

We learned the following from Tom Goodwillie. Also see [Rie14, Example 11.2.5] and [AC14].

Example 4.1.17. Model structures on $\mathcal{S}et$. *There are nine model structures on $\mathcal{S}et$, with morphisms as in the following table. In it we say a map is* **empty** *if its domain is empty; otherwise it is* **nonempty**. *The* **empty isomorphism** *is the map from the empty set to itself.*

Cofibrations	*Weak equivalences*	*Fibrations*
All maps	*Isomorphisms*	*All maps*
Isomorphisms	*All maps*	*All maps*
All maps	*All maps*	*Isomorphisms*
Injections	*All maps*	*Surjections*
Surjections	*All maps*	*Injections*
Split injections	*All maps*	*Surjections and empty maps*
All nonempty maps and the empty isomorphism	*All maps*	*Isomorphisms and empty maps*
All maps	*All nonempty maps and the empty isomorphism*	*Isomorphisms and empty maps*
Injections	*All nonempty maps and the empty isomorphism*	*Surjections and empty maps*

Now consider the inclusion functor $F : \mathcal{S}et \to \mathcal{T}op$ that gives each set the discrete topology. The standard model structure (as opposed to the toy ones in the next example) on $\mathcal{T}op$ is given below in Definition 4.2.1. The only model structure above for which F preserves weak equivalences is the first one, and for that structure it preserves neither fibrations nor cofibrations. Hence none of the model structures on $\mathcal{S}et$ above is compatible with the standard one on $\mathcal{T}op$. Functors between model categories will be discussed further in §4.5 below.

Example 4.1.18. Three toy model structures on a bicomplete cat-

egory. *In each of the following let $\mathcal{M}$ be a bicomplete category (Definition 2.3.25), so* **MC1** *is satisfied. Let one of the three classes of morphisms (weak equivalences, cofibrations and fibrations) be the isomorphisms in $\mathcal{M}$ and let the other two classes consist of all morphisms. In each case we get a model structure on $\mathcal{M}$ for which there are obvious factorizations. For example if the cofibrations are isomorphisms, then $\alpha(f)$ and $\gamma(f)$ are each the identity on the domain of f, while $\beta(f)$ and $\delta(f)$ are each f itself.*

The structure in which all weak equivalences are isomorphisms is called the **minimal model structure***, and the other two are called* **maximal model structures***. These adjectives refer to the class of weak equivalences.*

4.1C Fibrant and cofibrant objects

Definition 4.1.19. Fibrant and cofibrant objects. *An object X is* **cofibrant** *if the morphism $\varnothing \to X$ is a cofibration, and* **fibrant** *if the morphism $X \to *$ is a fibration. An object X is* **cofibrant-fibrant** *if it is both cofibrant and fibrant. We use the words* **fibrancy** *and* **cofibrancy** *for the quality of being fibrant or cofibrant.*

A **cofibrant (fibrant) approximation** *of an object X is a weak equivalence $X_c \to X$ ($X \to X_f$) where X_c is cofibrant (X_f is fibrant). A* **fibrant cofibrant (cofibrant fibrant) approximation** *of X is a cofibrant (fibrant) approximation in which the weak equivalence is a trivial fibration (trivial cofibration).*

Hence in a fibrant cofibrant approximation X_c to X, the word "fibrant" does not refer to X_c (which is cofibrant but not necessarily cofibrant), but to the fact that the weak equivalence $X_c \to X$ is a trivial fibration.

Hirschhorn [Hir03] denotes such weak equivalences by $\widetilde{X} \to X$ and $X \to \widehat{X}$. One has the following canonical examples.

Definition 4.1.20. Fibrant and cofibrant replacement. *Let $\epsilon_X : QX \to X$ be the functorial (in X) trivial fibration obtained by applying the first factorization of* **MC5** *to the morphism $\varnothing \to X$, giving us*

$$\varnothing \longrightarrow QX \xrightarrow{\epsilon_X} X.$$

The object QX is a called the **cofibrant replacement** *of X. It is a fibrant cofibrant approximation. The pair (Q, ϵ) is an augmented functor as in Definition 2.2.8.*

Dually, by applying the second factorization of **MC5** *to the morphism $X \to *$ we get a trivial cofibration and hence a weak equivalence $\eta_X : X \to RX$ to the* **fibrant replacement** *of X, which is a cofibrant fibrant approximation. The pair (R, η) is a coaugmented functor as in Definition 2.2.8.*

Proposition 4.1.21. Universal properties of QX and RX. *Any morphism $f : X \to Y$ to a fibrant object Y factors through RX. Dually, any morphism $g : W \to X$ from a cofibrant object W factors though QX.*

Proof For the first statement, consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ RX \arrow[r] \arrow[ur, dashed, "\hat{f}"] & * \end{tikzcd}$$

The left vertical map is a trivial cofibration and the right one is a fibration, so the factorization, i.e., the indicated lifting exists.

The argument for the second statement is similar. □

Proposition 4.1.22. Fibrant (cofibrant) objects as retracts. *Let X be a fibrant (cofibrant) object in a model category. Then for any trivial cofibration $i : X \to X'$ (any trivial fibration $p : X' \to X$), X is a retract of X', meaning there is a morphism $j : X' \to X$ ($q : X \to X'$) with $ji = 1_X$ ($pq = 1_X$).*

Proof We will prove the statement about fibrant objects. Consider the lifting diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d] \\ X' \arrow[r] \arrow[ur, dashed, "j"] & * \end{tikzcd}$$

The map i is a trivial cofibration by assumption, and the other vertical map is a fibration since X is fibrant. Therefore the desired map j exists by **MC4**. □

Applying the functor R to the map $\epsilon_X : QX \to X$ and vice versa leads to a diagram

$$\begin{tikzcd} QX \arrow[r, "\epsilon_X", "\text{fibration}"'] \arrow[d, "\eta_{QX}"', "\text{cofibration}"] & X \arrow[d, "\eta_X", "\text{cofibration}"'] & QX \arrow[l, "\epsilon_X"', "\text{fibration}"] \arrow[d, "Q\eta_X", "\text{cofibration}"'] \\ RQX \arrow[r, "\text{fibration}", "R\epsilon_X"'] \arrow[rr, dashed, bend right] & RX & QRX \arrow[l, "\text{fibration}"', "\epsilon_{RX}"] \end{tikzcd} \tag{4.1.23}$$

where each morphism is a weak equivalence, each horizontal map is a fibration and each vertical map is cofibration. The objects RQX and QRX are both fibrant and cofibrant. The indicated factorization of $R\epsilon_X$ exists by Proposition 4.1.21 because it is a map to RX from a cofibrant object.

Remark 4.1.24. It's awkward. *Functorial factorization is useful for theoretical purposes, but difficult to use in practice beyond toy examples like those* *of Example 4.1.18. It is described for the category of topological spaces below in §4.2B.*

There are other functorial cofibrant and fibrant approximations besides the ones associated with the two functorial factorizations. They are discussed by Hirschhorn in [Hir03, Chapter 8]. Even they do not have the properties one might want. For example, one rarely has a functorial cofibrant approximation which does not alter objects that are cofibrant to begin with.

The following is Hirschhorn's [Hir03, Definition 8.1.15].

Definition 4.1.25. Functorial cofibrant (fibrant) approximations.

(i) A **functorial cofibrant (fibrant) approximation** *on $\mathcal{M}$ is an augmented functor (Q, ϵ) (coaugmented functor (R, η)) on $\mathcal{M}$ as in Definition 2.2.8 such that $\epsilon_X : QX \to X$ ($\eta_X : X \to RX$) is a cofibrant (fibrant) approximation to X for every object X of $\mathcal{M}$.*

(ii) A **functorial fibrant cofibrant (cofibrant fibrant) approximation** *on $\mathcal{M}$ is a functorial cofibrant (fibrant) approximation such that ϵ_X is a trivial fibration (η_X is a trivial cofibration) for every object X of $\mathcal{M}$.*

In [Hir03, §8.1] Hirschhorn shows that any two functorial cofibrant (or fibrant) approximations are equivalent in a certain sense. He also proves a similar result for fibrant and cofibrant approximations to maps. In [Hir03, §14.6] he considers categories of approximations (either fibrant or cofibrant) to an object, map or subcategory of a model category $\mathcal{M}$ and shows that in most cases they have contractible classifying spaces.

4.1D Pointed model categories

Definition 4.1.26. *A model category $\mathcal{M}$ is* **pointed** *if the map $\varnothing \to *$ (from the initial object to the terminal one) is an isomorphism, and in that case $*$ is called the* **null object***. This means that for any objects A and B there is a unique morphism $A \to B$ factoring through the initial/terminal object $*$. We will denote it by 0 and refer to it as the* **trivial map***. We will denote the product and coproduct operations in $\mathcal{M}$ by $\wedge$ and $\vee$, the* **smash product** *and* **wedge***.*

For an arbitrary model category $\mathcal{M}$, **the associated pointed model category** *$\mathcal{M}_*$ is the category $(*\downarrow\mathcal{M})$ (see Definition 2.1.51) under the terminal object $*$, meaning the category whose objects are maps $m : * \to M$, often written as (M, m), where M is an object of $\mathcal{M}$. A morphism $f : (M, m) \to (N, n)$ is a morphism $f : M \to N$ in $\mathcal{M}$ with $n = f(m)$.*

It needs to be proved that $\mathcal{M}_*$ as defined in Definition 4.1.26 is actually a model category. This is done in [Hov99, Proposition 1.1.8]. There is a functor $\mathcal{M} \to \mathcal{M}_*$ given by $M \mapsto M_+ := M \coprod *$ (adding a disjoint base point), which is the left adjoint of the forgetful functor $U : \mathcal{M}_* \to \mathcal{M}$. If $\mathcal{M}$ is already pointed, then these define an equivalence of categories. A morphism f in $\mathcal{M}_*$ is a cofibration, fibration or weak equivalence iff Uf is one in $\mathcal{M}$.

4.1E Kernel, cokernels, fibers and cofibers

Definition 4.1.27. *The* **kernel** *or* **fiber***, and* **cokernel** *or* **cofiber** *of a morphism $f : X \to Y$ in a pointed model category are the pullback and pushout objects in the diagram*

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & X \arrow[r] \arrow[d, "f"'] & * \arrow[d] \\ * \arrow[r] & Y \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & \operatorname{coker} f \end{tikzcd}$$

The kernel and cokernel are the equalizer and coequalizer (as in Definition 2.3.27) of f with the trivial map $X \to * \to Y$. The term fiber (cofiber) above has the expected meaning only when f is a fibration (cofibration). For example the kernel of $* \to Y$ is $*$, and has nothing to do with Y.

Definition 4.1.28. Homotopy fibers and cofibers. *Let CX, the* **reduced cone object of** X *and PY, the* **reduced path object of** Y*, be obtained by factoring the unique maps $X \to *$ and $* \to Y$ as*

$$X \xrightarrow{i_X} CX \xrightarrow{\simeq} * \qquad \text{and} \qquad * \xrightarrow{\simeq} PY \xrightarrow{p_Y} Y$$

where i_X is cofibration, p_Y is a fibration, and the other two maps are weak equivalences.

Then the **homotopy fiber** *and* **homotopy cofiber** *of f are the pullback and pushout objects in the diagram*

$$\begin{tikzcd} F_f \arrow[r, "p_f"] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & X \arrow[r, "i_X"] \arrow[d, "f"'] & CX \arrow[d] \\ PY \arrow[r, "p_Y"] & Y \arrow[r, "i_f"] \arrow[ur, phantom, "\lrcorner" very near end] & C_f . \end{tikzcd}$$

The weak equivalences $* \to PY$ and $CX \to *$ in the factorizations induce maps

$$\eta : \ker f \to F_f \qquad \text{and} \qquad \epsilon : C_f \to \operatorname{coker} f. \tag{4.1.29}$$

We will see in §5.8 (specifically Example 5.8.5(iii)) that η (ϵ) is a weak equivalence when f is a fibration (cofibration). In Corollary 5.6.9 we will see that he map ϵ is also a weak equivalence when f is an h-cofibration. The homotopy fiber F_f (homotopy cofiber C_f) is the homotopy equalizer (coequalizer) of f and the trivial map.

Definition 4.1.30. Reduced path space and reduced cone. *Let $\mathcal{M} = \mathcal{T}$, the category of pointed topological spaces to be studied in §4.2A. Then for a pointed space X, the* **reduced path space** *PX can be taken to be the space X^I of paths ω in X with $\omega(0) = x_0$ (the base point), with the map $PX \to X$ being $\omega \mapsto \omega(1)$. The base points of I and PX are 0 and the constant x_0-valued path. The* **reduced cone** *CX is $X \wedge I$, where the base point of I is 1.*

We can use PX and CX to define the suspension and loop objects as the pullback and pushout objects in the diagram

$$
\begin{tikzcd}
& PX \arrow[dr] & & CX \arrow[dr] & \\
\Omega X \arrow[ur] \arrow[dr] & & X \arrow[ur] \arrow[dr] & & \Sigma X. \\
& PX \arrow[ur] & & CX \arrow[ur] &
\end{tikzcd}
\tag{4.1.31}
$$

We will take this up again in Definition 4.6.17 and §4.7.

4.2 Three classical examples of model categories

Following [Qui67, §I.1], we will describe model structures on

(i) $\mathcal{T}op$, the category of compactly generated weak Hausdorff spaces and its pointed analog $\mathcal{T}$,
(ii) $\mathcal{C}h_R$, the category of nonnegatively graded (or bounded below) chain complexes of R-modules for an arbitrary ring R and
(iii) the category $\mathcal{S}et_{\boldsymbol{\Delta}}$ of simplicial sets.

4.2A The model structure on topological spaces

The details of (i) can be found in [DS95, §8].

Definition 4.2.1. Continuous fibrations and cofibrations. *A continuous map $f : X \to Y$ map is a* **weak equivalence** *if it induces isomorphisms in homotopy groups (and in path component sets) with respect to every base point. It is a* **Serre fibration** *if it has the right lifting property with respect to the inclusion*

$$j_n : I^n \times \{0\} \to I^n \times I$$

for any $n \geqslant 0$. It is a **cofibration** *if it has the left lifting property with respect to every trivial Serre fibration.*

These choices of weak equivalences, fibrations and cofibrations give a model structure on $\mathcal{T}op$. Replacing each by their pointed analogs defines a model structure on $\mathcal{T} = \mathcal{T}op_*$ (see Definition 4.1.26), the category of pointed compactly generated weak Hausdorff spaces. In both cases all spaces are fibrant. The cofibrant spaces are retracts of generalized CW complexes, meaning spaces obtained from a discrete space by attaching cells, not necessarily in dimensional order.

Every cofibration $f : X \to Y$ is the inclusion into a retract of a generalized relative CW complex, meaning that Y is the retract of a space obtained from X by attaching cells, again not necessarily in dimensional order.

A proof of the following can be found in [Hov99, pages 54–57].

Proposition 4.2.2. Detecting trivial Serre fibrations. *A continuous map $f : X \to Y$ map is a trivial Serre fibration (meaning it is both a Serre fibration and a weak equivalence) if it has the right lifting property with respect to the inclusion of the boundary*

$$i_n : S^{n-1} \to D^n$$

for any $n > 0$.

Definition 4.2.3. *A continuous map $f : X \to Y$ is a* **Hurewicz fibration** *if it has the right lifting property with respect to the inclusion*

$$j : X \times \{0\} \to X \times I$$

for any space X.

It is known that a map is a trivial Hurewicz fibration iff it has the right lifting property with respect to all h-cofibrations as in Definition 3.5.6. The map e_0 of (3.5.8) is a trivial Hurewicz fibration.

Example 4.2.4. Some spaces that are not cofibrant.

(i) Let C be the Cantor set, regarded as a subset of the unit interval I. A cofibrant approximation is the map $p : C' \to C$, where C' denotes the same set with the discrete topology. We can use Proposition 4.2.8 to show that p is a trivial Serre fibration. Suppose we have a commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d, "i_n"'] & C' \arrow[d, "p"] \\ D^n \arrow[r, "\beta"] \arrow[ur, dashed] & C. \end{tikzcd}$$

Since C is totally disconnected, the map β must send all of D^n to a single point in C. Therefore α sends all of S^{n-1} to the corresponding point in C' and the lifting exists uniquely.

(ii) The map $f : \mathbf{N} \to X$ of Example 3.5.31 is also a cofibrant approximation to the noncofibrant space X. A similar argument to the above shows that f is a trivial Serre fibration. The map Σf is a continuous bijection from a countable wedge of circles (which is cofibrant) to the Hawaiian earring

ΣX. The two spaces have distinct fundamental groups, one countable and one uncountable, so Σf is not a weak equivalence

Consider the lifting diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d, "i_n"'] & \Sigma\mathbf{N} \arrow[d, "\Sigma f"] \\ D^n \arrow[r, "\beta"] \arrow[ur, dashed, "h"] & \Sigma X. \end{tikzcd}$$

Since Σf is a bijection, there is a unique lifting h, but it cannot be continuous in general. If it were, then Σf would be a trivial Serre fibration, contradicting the fact that it is not a weak equivalence.

We do not have a description of a cofibrant approximation of ΣX or its suspensions.

There is another model structure on $\mathcal{T}op$ and a close relative of $\mathcal{T}$ due to Strøm [Str72] in which the weak equivalences are actual homotopy equivalences. It has recently been generalized by Barthel and Riehl in [BR13]. In the pointed case one must assume that the base points are **nondegenerate** as defined in Definition 3.5.27. Strøm calls such spaces **well pointed**. His model structure is sometimes called the **h-model structure** (for Hurewicz) while the one discussed above is sometimes called the **q-model structure**, for Quillen. We will use the terms "h-cofibration" and "Hurewicz fibration" for the maps of Definition 3.5.6 and Definition 4.2.3, and the unadorned "cofibration" and "fibration" for those of Definition 4.2.1. In all four cases we will use the adjective "trivial" when the map is also an equivalence in the sense of Hurewicz or Quillen as appropriate.

The map $i_n : S^{n-1} \to D^n$ for $n \geqslant 0$, the inclusion of the boundary, is both a cofibration and an h-cofibration. We will see later that all Quillen cofibrations are generated by the i_n through operations described below in Definition 4.8.13. These operations also preserve h-cofibrations. This means that **all Quillen cofibrations are h-cofibrations, but not all h-cofibrations are Quillen cofibrations**. For example the inclusion of a point into the Cantor set is an h-cofibration but not a Quillen cofibration. See Proposition 5.6.10 below for an alternate proof that all Quillen cofibrations are h-cofibrations.

The following construction is dual to the mapping cylinder of Definition 3.5.1, as one sees by comparing the diagrams (4.2.6) and (3.5.3). We will see the two of them again as examples of homotopy limits and colimits in Example 5.8.5(ii) below.

Definition 4.2.5. The mapping path space N_f *for a map* $f : X \to Y$ *is* *the pullback in the diagram*

$$\begin{tikzcd} N_f \arrow[r, "\chi_f"] \arrow[d, "\phi_f"'] \arrow[dr, phantom, very near start, "\ulcorner"] & Y^I \arrow[d, "p_0"] \\ X \arrow[r, "f"] & Y, \end{tikzcd} \tag{4.2.6}$$

where Y^I is the path space of Y, meaning the space of maps $I \to Y$, and p_0 is a evaluation at 0. Thus

$$N_f = X \times_Y Y^I = \left\{(x, \omega) \in X \times Y^I : f(x) = \omega(0)\right\}.$$

Define maps

$$\begin{array}{ccccc} X & \xrightarrow{\quad j \quad} & N_f & \xrightarrow{\quad \pi \quad} & Y \\ x & \longmapsto & (x, \omega_{f(x)}) & & \\ & & (x, \omega) & \longmapsto & \omega(1), \end{array} \tag{4.2.7}$$

where ω_y is the constant y-valued path in Y.

We will see this space again as a homotopy limit below in Example 5.8.5(ii).

Proposition 4.2.8. The map π of (4.2.7) is a Serre fibration.

Proof We need to show that there is always a lifting in the diagram

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "\alpha"] \arrow[d, "j_n"'] & N_f \arrow[d, "\pi"] \\ I^n \times I \arrow[r, "\beta"'] \arrow[ur, dashed, "h"] & Y. \end{tikzcd}$$

for j_n as in Definition 4.2.1. Since the diagram commutes, $\beta(x, 0) = \pi\alpha(x)$ for each $x \in I^n$. The restriction of β to $\{x\} \times I$ is a path in Y starting at $\pi\alpha(x)$. We can define the second coordinate (in Y^I) of the same restriction of h in terms of this path. □

4.2B Quillen's factorizations of continuous maps

Now we will outline Quillen's method of factoring a map $f : X \to Y$ in $\mathcal{T}op$. The following is [Qui67, Lemma II.3.3]. The method he used is the **small object argument**, which we will describe more formally below in §4.8.

Theorem 4.2.9. The first factorization. *Any morphism $f : X \to Y$ in $\mathcal{T}op$ can be factored as a composite*

$$X \xrightarrow{\quad i \quad} Z \xrightarrow{\quad p \quad} Y,$$

where i is a cofibration and p is a trivial fibration.

Proof Consider the diagram

$$\begin{tikzcd} X = Z^{-1} \arrow[r, "\ell_0"] \arrow[dr, "f=p_{-1}"'] & Z^0 \arrow[r, "\ell_1"] \arrow[d, "p_0"] & Z^1 \arrow[r, "\ell_2"] \arrow[dl, "p_1"] & \cdots \\ & Y & & \end{tikzcd}$$

constructed inductively as follows. To get from Z^n to Z^{n+1}, consider the set L_n of diagrams of the form

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "i_k"'] & Z^n \arrow[d, "p_n"] \\ D^k \arrow[r, "\beta"] & Y, \end{tikzcd} \tag{4.2.10}$$

for all $k > 0$,where i_k is the inclusion of the boundary. Thus L_n is the set of all maps of spheres into Z^n with null homotopies in Y. Then Z^{n+1} will be the space obtained form Z^n by attaching cells using all such maps α. It is the pushout in the diagram

$$\begin{tikzcd} \coprod\limits_{L_n} S^{k-1} \arrow[r, "\coprod \alpha"] \arrow[d, "\coprod i_k"'] & Z^n \arrow[d, "\ell_n"] \\ \coprod\limits_{L_n} D^k \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & Z^{n+1}. \end{tikzcd} \tag{4.2.11}$$

Thus ℓ_n is a cofibration because Z^{n+1} is obtained from Z^n by attaching a set of cells indexed by the set L_n.

The maps $p_n : Z^n \to Y$ give us a map

$$p : Z = \operatorname*{colim}_n Z^n \to Y, \tag{4.2.12}$$

and composing the cofibrations ℓ_n gives us a cofibration $i : X \to Z$ with $f = pi$ as desired.

We need to show that our map p is a trivial Serre fibration. This means we need a lifting for any diagram of the form

$$\begin{tikzcd} S^{m-1} \arrow[r, "\alpha"] \arrow[d, "i_m"'] & Z \arrow[d, "p"] \\ D^m \arrow[r, "\beta"] \arrow[ur, dashed] & Y. \end{tikzcd} \tag{4.2.13}$$

The compactness of S^{m-1} implies that α factors through some Z^n, so the diagram above can be replaced by

$$\begin{tikzcd}
S^{m-1} \arrow[r, "\alpha"] \arrow[d, "i_m"'] & Z_n \arrow[r] \arrow[d, "\ell_{n+1}"'] & Z \arrow[d, "p"] \\
D^m \arrow[r, "\beta"] & Z^{n+1} \arrow[r, "p_{n+1}"'] \arrow[ur] & Y
\end{tikzcd}$$

The diagonal arrow on the right exists due to the way Z is defined in (4.2.12), and it gives us the lifting needed in (4.2.13). $\square$

Note that, while the space Z constructed above gives the desired factorization, it is not an object that one would like to deal with in practice.

Following this proof, Quillen remarked (paraphrasing)

> The argument used above relied primarily on the fact that
>
> $$\mathcal{T}op(S^k, \operatorname*{colim}_n Z^n) \cong \operatorname*{colim}_n \mathcal{T}op(S^k, Z^n)$$
>
> and may be used to prove factorization whenever the fibrations (or trivial fibrations) are characterized by the right lifting property with respect to a set of maps $\{A_i \to B_i\}$ where each A_i is "sequentially small" in the sense that $\mathcal{T}op(A_i, -)$ commutes with sequential colimits. We will have further occasions to use this argument and will refer to it as the **small object argument**.

We will refer to (4.2.11) as **Quillen's diagram**. We will see similar diagrams below in (11.1.36) and (11.4.12). The small object argument is the subject of §4.8 below.

Corollary 4.2.14. The second factorization. *Any morphism $f : X \to Y$ in $\mathcal{T}op$ can be factored as a composite*

$$X \xrightarrow{\ i\ } Z \xrightarrow{\ p\ } Y,$$

where i is a trivial cofibration and p is a fibration.

Proof We could follow Quillen's suggestion and mimic the proof of Theorem 4.2.9, replacing the inclusion $i_k : S^{k-1} \to D^k$ in (4.2.10) by the inclusion $j_k : I^k \to I^k \times I$, but he used a different approach.

Let Y^I denote the space of paths in Y, and let

$$X \times_Y Y^I = \left\{ (x, \omega) \in X \times Y^I : \omega(0) = f(x) \right\}.$$

Then define maps

$$g : X \to X \times_Y Y^I \qquad \text{by} \qquad x \mapsto (x, \omega_{f(x)})$$

where ω_y is the constant path at $y \in Y$, and

$$p_1 : X \times_Y Y^I \to Y \qquad \text{by} \qquad (x, \omega) \mapsto \omega(1)$$

Then $f = p_1 g$, and it is easy to see that g is a weak equivalence and p_1 is a fibration.

Now use Theorem 4.2.9 to factor the map g. The resulting cofibration is trivial since g is a weak equivalence. □

Both Theorem 4.2.9 and Corollary 4.2.14 have pointed analogs which we leave to the reader.

Proposition 4.2.15. Fibrations are surjective and cofibrations are injective.

(i) Any Serre fibration as in Definition 4.2.1 with nonempty domain and path connected codomain is surjective.

(ii) Any cofibration $f : A \to B$ sends distinct points in A to distinct points in B.

Proof (i) Let $p : X \to Y$ be such a fibration and consider the lifting diagram

$$\begin{tikzcd} * \arrow[r, "\alpha"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ I \arrow[r, "\beta"'] \arrow[ur, dashed, "h"] & Y, \end{tikzcd}$$

where the image of α is a point $x \in X$, and β is a path from $p(x)$ to some other point $y \in Y$. Since Y is path connected, any point y can be reached by such a path, and is therefore in the image of p.

(ii) Let $f : A \to B$ be a cofibration. Then consider the lifting diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & I \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & *. \end{tikzcd}$$

The map p is a trivial Serre fibration, so the lifting h exists. Given two distinct points in A, we can choose a map α sending them to distinct points in I since A is weak Hausdorff. It follows that hf and therefore f also have distinct values on them. □

4.2C The model structure on chain complexes

The details of (ii), the model structure on $\mathcal{C}h_R$, can be found in [DS95, §7].

Definition 4.2.16. Fibrations and cofibrations of nonnegatively graded chain complexes. *A morphism in $\mathcal{C}h_R$ (a chain map) is a* **weak equivalence** *if it induces an isomorphism in homology. It is a* **fibration** *if it is surjective in all degrees. It is a* **cofibration** *if it is a monomorphism with projective cokernel in each degree.*

These choices of weak equivalences, fibrations and cofibrations give a model structure on $\mathcal{C}h_R$. The cofibrant objects are chain complexes of projective modules, and all objects are fibrant.

4.2D The model structure on the category of simplicial sets

We are now ready to define the model structure on $\mathcal{S}et_{\mathbf{\Delta}}$.

Definition 4.2.17. Fibrations and cofibrations of simplicial sets. *A morphism of simplicial sets $f : X \to Y$ is a* **weak equivalence** *if its geometric realization $|f|$ is a weak equivalence of topological spaces. (Since $|f|$ is a weak equivalence of CW complexes, it is an actual homotopy equivalence.) It is a* **cofibration** *if each map $f_n : X_n \to Y_n$ is one to one. It is a* **Kan fibration** *if $|f|$ has the right lifting property with respect to each inclusion $\Lambda_i^n \to \Delta^n$. It is* **anodyne** *if it has the left lifting property with respect to all Kan fibrations.*

Exercise. Define a Kan fibration directly in terms of simplicial sets, without referring to the geometric realization. Show that a map is a cofibration as defined above iff it has the left lifting property with respect to each inclusion $\partial\Delta^n \to \Delta^n$. Show that simplicial set X is a Kan complex as in Definition 4.2.18 iff the map $X \to *$ is a Kan fibration.

Definition 4.2.18. *A* **Kan complex** *is a simplicial set in which a map from* **each horn** *Λ_i^n extends to a map from Δ^n. The extension is not required to be unique.*

Kan complexes are the fibrant objects in the Quillen model structure on $\mathcal{S}et_{\mathbf{\Delta}}$ to be discussed below in §4.2D.

Remark 4.2.19. Variants of Kan fibrations *are defined by Joyal in [Joy02, Definition 2.1] by requiring $|f|$ to have the right lifting property with respect to* **some but not necessarily all** *of the horn inclusions $\Lambda_i^n \to \Delta^n$. A* **left fibration** *is a map that has it for $0 \leqslant i < n$, an* **inner fibration** *or* **mid fibration** *is a map that has it for $0 < i < n$, and a* **right fibration** *is a map that has it for $0 < i \leqslant n$. Left, inner and right anodyne maps are defined similarly. See [Lur09, Definition 2.0.0.3].*

These choices of weak equivalences, cofibrations and fibrations give a model structure on $\mathcal{S}et_{\mathbf{\Delta}}$, sometimes called the **Quillen model structure**, also known as the **Kan model structure**. All objects are cofibrant, and the fibrant objects are the Kan complexes, meaning simplicial sets X for which every map from the simplicial set corresponding to the horn Λ_i^n extends to $\mathbf{\Delta}([\cdot],[n])$, the simplicial set corresponding to Δ^n.

The following was stated by Quillen in [Qui67] and a proof can be found in [Hov99, Theorem 3.6.7].

Proposition 4.2.20. The Quillen equivalence of $\mathcal{S}et_{\Delta}$ and $\mathcal{T}op$ and of their pointed analogs. *The equivalence of categories of Proposition 3.4.10 is a Quillen equivalence (see Definition 4.5.14 below) of model categories.*

4.3 Homotopy in a model category

So far we have said nothing about homotopy. Classical homotopy theory begins with the definition of a homotopy between two continuous maps. Recall the following, where the diagrams mimic those of [Qui67, I.1.3].

Example 4.3.1. Two ways to define homotopy in $\mathcal{T}op$. *Given two continuous maps of topological spaces $f_0, f_1 : A \to B$, there are two equivalent ways to say when they are homotopic:*

(i) Use f_0 and f_1 to define a map $A \times \{0,1\} \to B$ and try to extend it to all of $I \times A$.

$$\begin{tikzcd} A \coprod A \arrow[r, "f_0 \coprod f_1"] \arrow[d, "\nabla"'] \arrow[dr, "\partial_0 \coprod \partial_1"] & B \\ A & A \times I \arrow[l, "\sigma"] \arrow[u, dashed, "h"'] \end{tikzcd} \tag{4.3.2}$$

Here the cofibration $\partial_0 \coprod \partial_1$ is the product of A with the inclusion of $\{0,1\}$ into the unit interval I, the composite of $\sigma(\partial_0 \coprod \partial_1)$ with the inclusion of either summand is the identity on A, and σ is a trivial fibration. In other words, $\sigma(\partial_0 \coprod \partial_1) : A \coprod A \to A$ is the fold map ∇. The map h is usually called a **homotopy between** f_0 **and** f_1. *In order to distinguish it from what comes next, we will call it a* **left homotopy**.

(ii) Use f_0 and f_1 to define a map $A \to B \times B$ and try to lift it to the path space B^I along the map $d_0 \times d_1$ sending a path to its two endpoints.

$$\begin{tikzcd} B^I \arrow[dr, "d_0 \times d_1"] & B \arrow[l, "s"'] \arrow[d, "\Delta"] \\ A \arrow[u, dashed, "k"] \arrow[r, "f_0 \times f_1"'] & B \times B \end{tikzcd} \tag{4.3.3}$$

Here the trivial cofibration s sends each $y \in B$ to the constant path at y, the composite of $(d_0 \times d_1)s$ with the projection onto either factor summand is the identity on B, and s is a trivial cofibration. In other words,

$$(d_0 \times d_1)s : B \to B \times B$$

is the diagonal map Δ. The map k is a **right homotopy between** f_0 **and** f_1.

Example 4.3.4. Two ways to define homotopy in $\mathcal{T}$. *Recall that the product and coproduct operations in $\mathcal{T}$ (Definition 2.1.48) are the smash product $\wedge$ and the wedge $\vee$. Hence we replace the diagram of (4.3.2) by*

$$\begin{tikzcd} A \vee A \arrow[rr, "f_0 \vee f_1"] \arrow[d, "\nabla"'] \arrow[drr, "\partial_0 \vee \partial_1"] & & B \\ A & & A \rtimes I, \arrow[ll, "\sigma"'] \arrow[u, dashed, "h"'] \end{tikzcd}$$

where $A \rtimes I$ is the **reduced cylinder**, *namely $A \times I/\{a_0\} \times I$ where $a_0 \in A$ is the base point. The map h is required to be based point preserving, so we get a pointed left homotopy.*

Dually, we replace (4.3.3) by

$$\begin{tikzcd} B^{I_+} \arrow[drr, "d_0 \wedge d_1"] & & B \arrow[ll, "s"'] \arrow[d, "\Delta"] \\ A \arrow[u, dashed, "k"] \arrow[rr, "f_0 \wedge f_1"] & & B \wedge B. \end{tikzcd}$$

Here B^{I_+}, the space of base point preserving maps $I_+ \to B$, is the same as unbased path space B^I. Its base point is the constant path at the base point $b_0 \in B$. That path is required to be the image of a_0 under the right homotopy k.

Remark 4.3.5. Homotopy in a topological model category. *In a (pointed) topological model category $\mathcal{M}$, meaning one that is enriched, bitensored (see Definition 3.1.31) over $\mathcal{T}op$ ($\mathcal{T}$), the objects $A \times I$ and B^I ($A \rtimes I$ and B^{I_+}) are defined and one can consider morphisms h and k as above.* **Nearly all of the model categories we will study in this book are topological.** *Topological model categories will be formally introduced in Definition 5.6.3 below.*

In a general model category we can mimic the diagrams (4.3.2) and (4.3.3), replacing $A \times I$ and B^I by objects $Cyl(A)$ and $Path(B)$ having similar properties. The two hypothetical maps are called **left and right homotopies**, and their existences are not equivalent in general.

More formally we have the following.

Definition 4.3.6. Left and right homotopies. *Let $f_0, f_1 : A \to B$ be two morphisms in a model category $\mathcal{M}$. A* **left homotopy** *between them is a map* *h making the following diagram commute.*

$$\begin{tikzcd}
A\coprod A \arrow[r, "f_0\coprod f_1"] \arrow[d, "\nabla"'] \arrow[dr, "\partial_0\coprod\partial_1"] & B \\
A & \widetilde{A} \arrow[l, "\sigma"'] \arrow[u, dashed, "h"']
\end{tikzcd}$$

where σ is a weak equivalence. When it exists we write $f_0 \overset{\ell}{\simeq} f_1$.

A **right homotopy** *between them is a map k making the following diagram commute.*

$$\begin{tikzcd}
\widetilde{B} \arrow[dr, "d_0\times d_1"] & B \arrow[l, "s"'] \arrow[d, "\Delta"] \\
A \arrow[u, dashed, "k"] \arrow[r, "f_0\times f_1"'] & B\times B
\end{tikzcd}$$

where s is a weak equivalence. When it exists we write $f_0 \overset{r}{\simeq} f_1$.

When both left and right homotopies exist, we write $f_0 \simeq f_1$, and say that f_0 and f_1 are **homotopic***.*

Two objects X and Y are **homotopy equivalent** *if there are maps $f : X \to Y$ and $g : Y \to X$ with $gf \simeq 1_X$ and $fg \simeq 1_Y$. The maps f and g then are* **homotopy equivalences***.*

Definition 4.3.7. Cylinder and path objects. *Let A and B be objects in a model category $\mathcal{M}$. A* **cylinder object** *for A is a factorization*

$$A\coprod A \xrightarrow{\partial_0\coprod\partial_1} Cyl(A) \xrightarrow{\sigma} A$$

of the fold map $\nabla : A\coprod A \to A$, where $\partial_0\coprod\partial_1$ is a cofibration and σ is a weak equivalence. The **functorial cylinder object** *for A is the one where the factorization above is the functorial one in which σ is a trivial fibration.*

Dually, a **path object** *for B is a factorization*

$$B \xrightarrow{s} Path(B) \xrightarrow{d_0\times d_1} B\times B$$

of the diagonal map $B \to B\times B$, where s is a weak equivalence and $d_0\times d_1$ is a fibration. The **functorial path object** *is similarly defined.*

The existence of functorial cylinder and path objects is proved by Hirschhorn in [Hir03, Lemma 7.3.3].

Remarks 4.3.8. Properties of cylinder and path objects.

(i) **Functoriality.** *In [Qui67, page 1.6] Quillen noted that his cylinder and path objects, which he denotes by $A\times I$ and B^I, are neither functorial* *nor the product or power of an object I. The notation was chosen only for convenience.*

(ii) **Duality.** *The notions of left and right homotopy are dual, meaning that a right homotopy in a model category $\mathcal{M}$ is the same thing as a left homotopy in $\mathcal{M}^{op}$. The same goes for cylinder and path objects. Hence statements about left homotopies and cylinder objects are equivalent to dual statements about right homotopies and path objects.*

(iii) **The topological case.** *As in Remark 4.3.5, when $\mathcal{M}$ is a (pointed) topological model category (as in Definition 5.6.3 below), we* **can** *define*

$$Cyl(A) = A \times I \qquad \text{and} \qquad Path(B) = B^I$$
$$\big(Cyl(A) = A \rtimes I \qquad \text{and} \qquad Path(B) = B^{I_+}\big)\,,$$

Quillen's caution of (i) notwithstanding. These definitions are easier than the functorial factorizations described in §4.2B.

The following three results and two definitions are originally due to Quillen [Qui67, §I.1], and are stated and proved as [Hov99, 1.2.5–8].

Proposition 4.3.9. Properties of left and right homotopy. *Let $\mathcal{M}$ be a model category in which we have morphisms*

$$X \xrightarrow{\ a\ } A \underset{f_1}{\overset{f_0}{\rightrightarrows}} B \xrightarrow{\ b\ } Y.$$

(i) Suppose $f_0 \overset{\ell}{\simeq} f_1$ as in Definition 4.3.6. Then $bf_0 \overset{\ell}{\simeq} bf_1$. Dually, if $f_0 \overset{r}{\simeq} f_1$, then $f_0a \overset{r}{\simeq} f_1a$.

(ii) If B is fibrant and $f_0 \overset{\ell}{\simeq} f_1$, then $f_0a \overset{\ell}{\simeq} f_1a$. Dually, if A is cofibrant and $f_0 \overset{r}{\simeq} f_1$, then $bf_0 \overset{r}{\simeq} bf_1$

(iii) If A is cofibrant (B is fibrant) then left (right) homotopy is an equivalence relation on $\mathcal{M}(A, B)$.

(iv) If A is cofibrant and b is a trivial fibration or weak equivalence of fibrant objects, then b induces an isomorphism

$$\mathcal{M}(A, B)/\overset{\ell}{\simeq} \xrightarrow{\ \cong\ } \mathcal{M}(A, Y)/\overset{\ell}{\simeq}\,.$$

Dually, if B is fibrant and a is a trivial cofibration or weak equivalence of cofibrant objects, then a induces an isomorphism

$$\mathcal{M}(A, B)/\overset{r}{\simeq} \xrightarrow{\ \cong\ } \mathcal{M}(X, B)/\overset{r}{\simeq}\,.$$

(v) If A is cofibrant, then $f_0 \overset{\ell}{\simeq} f_1$ implies $f_0 \overset{r}{\simeq} f_1$. Furthermore if B' is any path object for B, then there is a right homotopy $k : A \to B'$ from f_0 to f_1. Dually, if B is fibrant, then $f_0 \overset{r}{\simeq} f_1$ implies $f_0 \overset{\ell}{\simeq} f_1$ and for any cylinder object A' for A, there is a left homotopy $h : A' \to B$ from f_0 to f_1.

Corollary 4.3.10. Maps from a cofibrant object to a fibrant one. *With notation is in Proposition 4.3.9, suppose that A is cofibrant and B is fibrant. Then left and right homotopy coincide and each is an equivalence relation in $\mathcal{M}(A, B)$. Moreover, if $f_0 \simeq f_1$, for any cylinder object A' for A (path object B' for B), there is a left homotopy $h : A' \to B$ (right homotopy $k : A \to B'$) between f_0 and f_1.*

Definition 4.3.11. The sets $\pi^\ell(A, B)$, $\pi^r(A, B)$ and $\pi(A, B)$. *Let A be a cofibrant object in a model category $\mathcal{M}$. For another object B in $\mathcal{M}$, $\pi^\ell(A, B)$ denotes the set of left homotopy classes (see Proposition 4.3.9(iii)) of morphisms $A \to B$. Dually, for arbitrary A and fibrant B, $\pi^r(A, B)$ denotes the set of right homotopy classes of morphisms $A \to B$. When A is cofibrant and B is fibrant, $\pi(A, B)$ (or $\pi_0(A, B)$) denotes the set of homotopy classes of morphisms $A \to B$.*

Proposition 4.3.12. Homotopy as an equivalence relation. *Given a model category $\mathcal{M}$, in the full subcategory $\mathcal{M}_{cf}$ of cofibrant-fibrant objects (see Definition 4.1.19) homotopy is an equivalence relation among morphisms compatible with composition, and a map is a weak equivalence iff it is a homotopy equivalence as in Definition 4.3.6.*

There is a Whitehead theorem saying that a weak equivalence of cofibrant-fibrant objects (Definition 4.1.19) is a homotopy equivalence. For more details see[Hir03, Chapter 7] or [Hov99, §1.2]. Proposition 4.3.12 enables us to make the following.

Definition 4.3.13. The classical homotopy category $\pi\mathcal{M}_{\mathbf{cf}}$ of a model category *$\mathcal{M}$ is the category whose objects are cofibrant-fibrant objects (as in Definition 4.1.19) of $\mathcal{M}$ and whose morphisms are homotopy classes of morphisms in $\mathcal{M}$. For objects X and Y in $\mathcal{M}_{cf}$ we will sometimes denote the morphisms set $\pi\mathcal{M}_{cf}(X, Y)$ by $[X, Y]$.*

Remark 4.3.14. The use of square brackets. *We also used square brackets in Definition 3.2.18 in connection with enriched functors, with $[\mathcal{D}, \mathcal{C}]$ for categories $\mathcal{D}$ and $\mathcal{C}$ denoting the category whose objects are certain functors $\mathcal{D} \to \mathcal{C}$. Hopefully the distinction between the two usages will be clear from the context.*

A related notion is the localization of $\mathcal{M}$ with respect to its weak equivalences. For this we need the following.

Definition 4.3.15. Localization of a category. *If $\mathcal{C}$ is a category and $\mathscr{W}$ is a class of maps in $\mathcal{C}$, then a localization of $\mathcal{C}$ with respect to $\mathscr{W}$ is a category $L_{\mathscr{W}}\mathcal{C}$ and a functor $\gamma : \mathcal{C} \to L_{\mathscr{W}}\mathcal{C}$ such that*

(i) if $w \in \mathscr{W}$, then $\gamma(w)$ is an isomorphism, and

(ii) if $\mathcal{D}$ is a category and $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(w)$ is an isomorphism for every $w \in \mathscr{W}$, then there is a unique functor $\delta : L_{\mathscr{W}}\mathcal{C} \to \mathcal{D}$ with $\delta\gamma = F$.

A refinement of this notion for model categories will be given below in Definition 4.5.13.

It is easy to show that if such a localization exists, then it is unique up to unique isomorphism. Its existence is discussed in [Hov99, §1.2], [Hir03, §8.3], and originally by Quillen in [Qui67, §I.1]. It exists for any model category $\mathcal{M}$ [Hir03, Theorem 8.3.5]) and is known [Hir03, Theorem 8.3.6] to be equivalent to $\pi\mathcal{M}_{\mathrm{cf}}$ as in Definition 4.3.16.

Definition 4.3.16. The Quillen homotopy category $\mathrm{Ho}\,\mathcal{M}$ **of a model category** $\mathcal{M}$ *is its localization with respect to its class of weak equivalences.*

More explicitly, morphisms $A \to B$ in $\mathrm{Ho}\,\mathcal{M}$ are equivalence classes of "zig zag" diagrams of the form

$$A \leftarrow \bullet \to \bullet \leftarrow \cdots \to B \tag{4.3.17}$$

where each arrow pointing to the left (the wrong way) is a weak equivalence. If $\mathcal{M}$ is not small, the collection of such equivalence classes could be a proper class, which means that $\mathrm{Ho}\,\mathcal{M}$ could fail to be locally small.

It is known [DHKS04, 7.7] that zig zag diagrams of (4.3.17) can be assumed to have the three arrow form

$$A \leftarrow \bullet \to \bullet \leftarrow B.$$

Example 4.3.18. Homotopy categories for maximal and minimal model structures. *Suppose a bicomplete category $\mathcal{M}$ has a maximal model structure as in Example 4.1.18, i.e., one in which every morphism is a weak equivalence. Then in its homotopy category, all objects are isomorphic, so* $\mathrm{Ho}\,\mathcal{M}$ *is equivalent to the trivial category. At the other extreme, when the model structure is minimal, the homotopy category is $\mathcal{M}$ itself.*

The following was proved by Quillen as [Qui67, Corollary I.1.1].

Proposition 4.3.19. Morphisms in $\mathrm{Ho}\,\mathcal{M}$**.** *Let A be a cofibrant object and B a fibrant object in a model category $\mathcal{M}$. Then in the homotopy category* $\mathrm{Ho}\,\mathcal{M}$ *of Definition 4.3.16, the morphism set $[\gamma A, \gamma B] = \mathrm{Ho}\,\mathcal{M}(\gamma A, \gamma B)$ is naturally isomorphic to the set $\pi(A, B)$ of homotopy classes of morphisms $A \to B$ of Definition 4.3.11.*

4.4 Nonhomotopical and derived functors

Homotopy theorists like to work with functors like π_* and H_* that depend only on the homotopy type of the space involved. In terms of a model category $\mathcal{M}$, this means a functor that factors through the homotopy category $\operatorname{Ho}\mathcal{M}$. Unfortunately we sometimes have to deal with functors not having this property.

The following is taken from [DS95, §10]; also see [Lur09, A.2.4].

Example 4.4.1. Pushouts need not preserve weak equivalences. *Let J denote the category $\{a \leftarrow b \rightarrow c\}$, $\mathcal{T}op$ the category of compactly generated weak Hausdorff spaces, and $\mathcal{T}op^J$ the category of functors $J \to \mathcal{T}op$, i.e., pushout diagrams in $\mathcal{T}op$. Then we have the functor colim : $\mathcal{T}op^J \to \mathcal{T}op$ which assigns to each diagram its pushout. It is left adjoint to the diagonal functor $\Delta : \mathcal{T}op \to \mathcal{T}op^J$ which assigns to each space X the constant X-valued diagram. A morphism in $\mathcal{T}op^J$ is the obvious sort of commutative diagram.*

Now consider the morphism

$$\begin{tikzcd} D^n \arrow[d] & S^{n-1} \arrow[l] \arrow[r] \arrow[d] & D^n \arrow[d] \\ * & S^{n-1} \arrow[l] \arrow[r] & * \end{tikzcd} \tag{4.4.2}$$

in which each vertical map, and hence the morphism in $\mathcal{T}op^J$, is a weak equivalence. However the pushout of the top row (where the two maps are inclusion of the boundary) is S^n, while that of the bottom row is a point. Thus the pushout functor fails to preserve this weak equivalence in $\mathcal{T}op^J$.

It turns out there is a model structure on $\mathcal{T}op^J$ in which the top row of (4.4.2) is cofibrant but the bottom row is not, and the pushout functor **does** preserve weak equivalences between cofibrant objects. This will be discussed further in Example 5.4.14 below. Let $f : X \to Y$ be a morphism in $\mathcal{T}op^J$. It consists of three maps, $f_a : X_a \to Y_a$, $f_b : X_b \to Y_b$ and $f_c : X_c \to Y_c$.

We define the model structure on $\mathcal{T}op^J$ by saying that f is a weak equivalence/fibration if each of the three maps is, but the definition of a cofibration is more complicated. Let $\partial_b f = X_b$ and define $\partial_a f$ to be the pushout of

$$\begin{tikzcd} X_b \arrow[r] \arrow[d, "f_a"'] & X_a \arrow[d] \\ Y_b \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \partial_a f \end{tikzcd}$$

with a similar definition for $\partial_c f$. For each index we get a map

$$i_*(f) : \partial_*(f) \to Y_*.$$

For the indices a and c these are the **corner maps** of Definition 2.3.9. We say that f is a cofibration if each of these three maps is. It is a routine exercise [DS95, 10.6] to verify that this defines a model structure on $\mathcal{T}op^J$.

Proposition 4.4.3. Cofibrant objects in $\mathcal{T}op^J$. *An object X in $\mathcal{T}op^J$ is cofibrant iff X_b is a CW complex and the two maps from it are cofibrations.*

Proof By Definition 4.1.19, an object X in $\mathcal{T}op^J$ is cofibrant if the map to it from the initial object (the constant $\varnothing$-valued diagram) is a cofibration. Thus we have to consider the morphism represented by the diagram

$$\begin{tikzcd} \varnothing \arrow[d] & \varnothing \arrow[l] \arrow[r] \arrow[d] & \varnothing \arrow[d] \\ X_a & X_b \arrow[l] \arrow[r] & X_c . \end{tikzcd}$$

The first requirement for the map being a cofibration in $\mathcal{T}op^J$ is that the map $\varnothing \to X_b$ be a cofibration in $\mathcal{T}op$, which means that X_b is cofibrant. Next observe that the two pushouts are each X_b. Hence the corner maps, which are also required to be cofibrations, are the maps in the bottom row as claimed. □

In (4.4.2), the top row is cofibrant but the bottom row is not.

Remark 4.4.4. The projective model structure. *We will see much more of the ideas in this example in what follows. The model structure on $\mathcal{T}op^J$ above is an instance of the* **projective model structure** *on the category of J-diagrams (for an arbitrary small category J) in a suitable model category to be spelled out in Definition 5.4.2 and Definition 5.4.8 below.*

More generally we can ask to what extent a functor $F : \mathcal{M} \to \mathcal{N}$ from between model categories can be factored through the homotopy category $\mathrm{Ho}(\mathcal{M})$ of Definition 4.3.16. The following definitions and results are standard in model category theory and have been lifted from [Hir03, §8.4].

Definition 4.4.5. Derived functors. *Let $\mathcal{M}$ be a model category equipped with a functor F to an arbitary category $\mathcal{D}$. Consider the diagram*

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & & \mathcal{D} \\ & \mathrm{Ho}\,\mathcal{M} \arrow[ur, dashed] & \end{tikzcd}$$

A left (right) derived functor LF (RF) of F is a right (left) Kan extension (see §2.5) of F along γ. If LF (RF) exists, it comes equipped with a natural transformation

$$\epsilon : LF \cdot \gamma \Rightarrow F \qquad\qquad (\eta : F \Rightarrow RF \cdot \gamma)\,.$$

Note the reversal of handedness above and in Definition 4.4.7 below; it is not a typo. Recall from §2.5B that right (left) Kan extensions are known to exist when the source category is small and the target category is complete (cocomplete), but not in general. Derived functors will be studied in the more genral setting of homotopical categories below in §5.1B.

The following is proved as [Hir03, Proposition 8.4.4].

Proposition 4.4.6. Existence of derived functors. *Let F be as in Definition 4.4.5. If it takes trivial cofibrations (trivial fibrations) between cofibrant (fibrant) objects to isomorphisms, then LF (RF) exists.*

Definition 4.4.7. Total derived functors. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and $F : \mathcal{M} \to \mathcal{N}$ a functor. Consider the diagram*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[dr, "\gamma"'] & \mathcal{N} \arrow[r, "\delta"] & \operatorname{Ho}\mathcal{N} \\ & \operatorname{Ho}\mathcal{M} \arrow[ur, dashed] & \end{tikzcd}$$

A total left (right) derived functor $\mathbf{L}F$ ($\mathbf{R}F$) of F is a right (left) Kan extension of δF along γ. If $\mathbf{L}F$ ($\mathbf{R}F$) exists, it comes equipped with a natural transformation $\epsilon : \mathbf{L}F \cdot \gamma \Rightarrow \delta F$ ($\eta : \delta F \Rightarrow \mathbf{R}F \cdot \gamma$). Equivalently a total left (right) derived functor is such a natural transformation.

The following is a special case of Proposition 4.4.6.

Proposition 4.4.8. Existence of total derived functors. *Let F be as in Definition 4.4.7. If it takes trivial cofibrations (trivial fibrations) between cofibrant (fibrant) objects to weak equivalences, then $\mathbf{L}F$ ($\mathbf{R}F$) exists.*

Remark 4.4.9. Deriving left and right Quillen functors. *Functors F satisfying the hypotheses above are known as left (right) Quillen functors (Definition 4.5.1) and are the subject of §4.5 below. It is known that they preserve all weak equivalences between cofibrant (fibrant) objects, as explained in Remark 4.5.5. This means that the restriction of δF to the full subcategory $\mathcal{M}_c$ of cofibrant objects (the full subcategory $\mathcal{M}_f$ of fibrant objects) in $\mathcal{M}$ converts weak equivalences to isomorphisms and therefore extends* **uniquely** *to a functor $\operatorname{Ho} F$ from $\operatorname{Ho}\mathcal{M}_c = L_{\mathscr{W}}\mathcal{M}_c$ ($\operatorname{Ho}\mathcal{M}_f = L_{\mathscr{W}}\mathcal{M}_f$). Meanwhile the functorial cofibrant (fibrant) approximation functor Q (R) (see Definition 4.1.20) induces a functor $\operatorname{Ho} Q : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{M}_c$ ($\operatorname{Ho} R : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{M}_f$), so we have*

$$\mathbf{L}F = \operatorname{Ho} F \operatorname{Ho} Q \qquad (\mathbf{R}F = \operatorname{Ho} F \operatorname{Ho} R).$$

Hovey [Hov99, Definition 1.3.6] uses this as the **definition** *of $\mathbf{L}F$ and $\mathbf{R}F$. It is equivalent to our Kan extension definition.*

Example 4.4.10. Derived functors in homological algebra. *This notion of a derived functor of Definition 4.4.7 is related to the one in homological algebra in the following way. For a ring R let $\mathcal{C}h_R$ denote the category of non-negatively graded chain complexes of left R-modules. It has a model structure given in Definition 4.2.16 in which the cofibrant objects are chain complexes of projective R-modules. For an R-module N, let $K(N,0)$ denote the chain complex which is N concentrated in degree 0. It has a cofibrant approximation $P \to K(N,0)$ where P is a projective resolution of N.*

For a right R-module M, the functor $M \otimes (-)$ defines a functor $F : \mathcal{C}h_R \to \mathcal{C}h_{\mathbf{Z}}$. It has a total left derived functor $\mathbf{L}F : \operatorname{Ho}\mathcal{C}h_R \to \operatorname{Ho}\mathcal{C}h\mathbf{Z}$. Then it follows from the above that there is a natural isomorphism

$$H_i\mathbf{L}F(K(N,0)) \cong \operatorname{Tor}_i^R(M,N) \text{ for all} i \geqslant 0.$$

If $\mathbf{L}F$ as in Definition 4.4.7 exists, one could ask for a lifting of

$$\mathbf{L}F\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{N}$$

to $\mathcal{N}$ that is (unlike F) **homotopical**.

The following definition is due to Shulman [Shu06, Definition 2.5]. It is repeated by Riehl in [Rie14, Definition 2.1.18], where she calls them simply "derived functors."

Definition 4.4.11. Point set derived functors. *A point set left (right) derived functor $\mathbf{L}F : \mathcal{M} \to \mathcal{N}$ ($\mathbf{R}F : \mathcal{M} \to \mathcal{N}$) of a functor between model categories $F : \mathcal{M} \to \mathcal{N}$ is a homotopical functor together with a natural transformation $\lambda : \mathbf{L}F \Rightarrow F$ ($\mu : F \Rightarrow \mathbf{R}F$) such that $\delta\lambda : \delta\mathbf{L}F \Rightarrow \delta F$ ($\delta\mu : \delta F \Rightarrow \delta\mathbf{R}F$) is a total left (right) derived functor of F as in Definition 4.4.7.*

Hence in the left derived case we have a diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, shift left, "F"] \arrow[r, shift right, "\mathbf{L}F"'] \arrow[d, "\gamma"] & \mathcal{N} \arrow[d, "\delta"] \\
\operatorname{Ho}\mathcal{M} \arrow[r, "\mathbf{L}F"] & \operatorname{Ho}\mathcal{N}
\end{tikzcd}$$

in which both $(\mathbf{L}F)\gamma$ and $\delta(\mathbf{L}F)$ are homotopical functors $\mathcal{M} \to \operatorname{Ho}\mathcal{N}$ which support natural transformations to δF, which need not be homotopical.

4.5 Quillen functors and Quillen equivalences

The following definitions and results are standard in model category theory. Unless otherwise stated, proofs can be found in [Qui67, §I.4], [Hov99, §1.3] and [Hir03, §8.5].

Definition 4.5.1. Quillen pairs. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories with a pair of adjoint functors*

$$F : \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{N} : U.$$

and a natural isomorphism $\varphi : \mathcal{N}(FX, Y) \xrightarrow{\cong} \mathcal{M}(X, UY)$ for objects X in $\mathcal{M}$ and Y in $\mathcal{N}$. We say

(i) F is a **left Quillen functor***,*
(ii) U is a **right Quillen functor***, and*
(iii) (F, U) is a **Quillen pair***, or (F, U, φ) is a* **Quillen adjunction** *and φ is the* **adjunction isomorphism***,*

if

(a) the left adjoint F preserves both cofibrations and trivial cofibrations, and
(b) the right adjoint U preserves both fibrations and trivial fibrations.

When $\mathcal{N} = \mathcal{M}$, we say that F and U are **left and right Quillen endofunctors** *which together comprise a* **Quillen endopair***.*

Remark 4.5.2. Fibrancy and cofibrancy. *The left adjoint functor F need* **not** *preserve fibrations, so it need not send fibrant objects to fibrant objects. Similary the image of a cofibrant object under the right adjoint U need not be cofibrant.*

The following is new as far as we know, and will be used in Definition 5.7.3 below.

Definition 4.5.3. *A Quillen pair (F, U) is* **invertible** *if there is an adjunction*

$$U^{-1} : \mathcal{M} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{N} : F^{-1},$$

with F^{-1} (U^{-1}) commuting with F (U) up to natural isomorphism, and with natural transformations

$$f : FF^{-1} \Rightarrow 1_{\mathcal{N}} \qquad \text{and} \qquad u : 1_{\mathcal{M}} \Rightarrow UU^{-1}$$

that induce weak equivalences on cofibrant and fibrant objects respectively.

The following is a consequence of Proposition 2.3.36.

Proposition 4.5.4. Left (right) Quillen functors preserve colimits (limits).

The effect of right Quillen functors on homotopy Cartesian squares of fibrations as in Definition 5.8.37 is the subject of Proposition 5.8.45 below.

Remark 4.5.5. Quillen functors and Ken Brown's Lemma. *A left (right) Quillen functor preserves trivial cofibrations (fibrations) in general by definition, and hence trivial cofibrations (fibrations) between cofibrant (fibrant) objects in particular. This is known (see Ken Brown's Lemma 5.1.7 below) to imply that such functors preserve* **all** *weak equivalences between cofibrant (fibrant) objects.*

Example 4.5.6. Some Quillen pairs.

(i) **The diagonal product and coproduct diagonal adjunctions.** *Let $\mathcal{M}$ be a model category and let S be a set. Then the product $\mathcal{M}^S$ has a model category structure described in Definition 4.1.16. The product functor $\prod : \mathcal{M}^S \to \mathcal{M}$ is defined because model categories have small limits by definition. It preserves fibrations and trivial fibrations. Its left adjoint is the diagonal functor $\Delta : \mathcal{M} \to \mathcal{M}^S$, which preserves both fibrations and cofibrations as well as weak equivalences. Hence $(\Delta, \prod)$ is a Quillen pair, the* **diagonal product adjunction**. *Similarly there is a coproduct functor $\coprod : \mathcal{M}^S \to \mathcal{M}$ which preserves cofibrations and trivial cofibrations. Its right adjoint is Δ, so $(\coprod, \Delta)$ is also a Quillen pair, the* **coproduct diagonal adjunction**.

(ii) **The disjoint base point conjunction**. *Let $\mathcal{M}$ be a model category and let $\mathcal{M}_*$ be the category of pointed objects in $\mathcal{M}$ (Definition 4.1.26). Then we have the disjoint base point functor $F : \mathcal{M} \to \mathcal{M}_*$ and the forgetful functor $U : \mathcal{M}_* \to \mathcal{M}$ with $F \dashv U$. Then U preserves fibrations, cofibrations and weak equivalences. In particular it is a right Quillen functor, so (F, U) is a Quillen pair.*

(iii) **The undercategory and overcategory adjunctions.** *In the undercategory $(A{\downarrow}\mathcal{M})$ of Example 4.1.14, the forgetful functor U has a left adjoint $F : \mathcal{M} \to (A{\downarrow}\mathcal{M})$ that sends an object X to the cofibration $A \to A \amalg X$, and (F, U) is a Quillen pair. In the overcategory $(\mathcal{M}{\downarrow}A)$, the right adjoint G of the forgetful functor sends an object X to the fibration $A \times X \to A$, and (U, G) is a Quillen pair.*

(iv) **The inclusion projection adjunction**. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories. Let $\mathcal{M} \times \mathcal{N}$ have the product model structure of Definition 4.1.16. Define functors*

$$\begin{tikzcd} M \arrow[r, mapsto] & (M, \varnothing) \\ I : \mathcal{M} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{N} : P_1 \arrow[l, shift left] \\ M & (M, N) \arrow[l, mapsto] \end{tikzcd}$$

These two functors are easily seen to be adjoint. Both preserve weak equivalences, cofibrations and fibrations, so I is a left Quillen functor and P_1 is a right one. Hence (I, P_1) is a Quillen pair. **This would still be the case were we to alter the model structure on $\mathcal{N}$ in some way.** *Hence the* *existence of a Quillen adjunction between two model categories does* **not** *mean that the model structure of one determines that of the other.*

As an extreme case of this, $\mathcal{M}$ could be the trivial model category with just two objects, $\varnothing$ and $*$, and a single nonidentity morphism that is defined to be a cofibration, with the other two morphisms being trivial fibrations. Then such a Quillen adjunction exists for **any** *model category $\mathcal{N}$.*

Remark 4.5.7. Fibrations defined by a right adjoint functor. *In a Quillen pair (F, U) as in Definition 4.5.1, we require the functor U to preserve fibrations and trivial fibrations. This is not the same as* **defining** *a morphism in $\mathcal{N}$ to be a fibration or trivial fibration if its image under U is one. In the Crans-Kan Transfer Theorem 5.2.27 below, we start with an adjunction (F, U) and a model structure on $\mathcal{M}$. We then define a model structure on $\mathcal{N}$ by requiring a morphism in it to be a fibration or a weak equivalence if its image under U is one. This leads to (F, U) being a Quillen pair. As indicated in Example 4.5.6(iv), $\mathcal{N}$ could have other model structures for which U is a right Quillen functor.*

The first of the examples listed above enables us to prove the following.

Proposition 4.5.8. Products and coproducts of weak equivalences. *A product (coproduct) of weak equivalences between fibrant (cofibrant) objects is a weak equivalence.*

Proof We will prove the statement about coproducts, making use of the Quillen adjunction $(\coprod, \Delta)$ of Example 4.5.6(i). The functor $\coprod$ is a left Quillen functor, so it preserves trivial cofibrations, and in particular trivial cofibrations between cofibrant objects. By Ken Brown's Lemma 5.1.7 below, a functor which does this preserves **all** weak equivalences between cofibrant objects, not just trivial cofibrations. The result follows. $\square$

Proposition 4.5.9. Pushouts and pullbacks of weak equivalences. *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "(\lrcorner)", very near start] & Z \end{tikzcd}$$

be a pullback (pushout) diagram in a model category $\mathcal{M}$. If the two morphisms to Z (from A) are trivial fibrations (trivial cofibrations), so are the other two.

Proof We will prove the statement about pullbacks and leave the dual statemen about pushouts to the reader. We first consider the case where $Z = *$, the terminal object in $\mathcal{M}$. Then the pullback diagram is

$$\begin{tikzcd} X \times B \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, "r_2"] \\ X \arrow[r, "r_1"] & *, \end{tikzcd}$$

where the maps r_1 and r_2 are trivial fibrations, and the maps p_1 and p_2 are projections onto the factors. The two latter maps are fibrations by Proposition 4.1.12. The product $X \times B$ is fibrant since X and B are. Then the composite morphism

$$r_1 p_1 = r_1 \times r_2 = r_2 p_2$$

is a weak equivalence by Proposition 4.5.8 because it is the product of two weak equivalences of fibrant objects.

In the general case the diagram is

$$\begin{tikzcd} X \underset{Z}{\times} B \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, "r_2"] \\ X \arrow[r, "r_1"] & Z. \end{tikzcd}$$

Thus each object in the diagram is equipped with a map to Z, so we can treat it is a diagram in the overcategory $(\mathcal{M}\downarrow Z)$ as in Example 4.1.14. Then the object Z, more precisely the morphism $1_Z : Z \to Z$, is the terminal object in the category, the objects B and X are contractible as in Definition 4.1.4, and the pullback $X \underset{Z}{\times} B$ is the categorical product of B and X. Hence we have reduced the general case to the special case above. □

Definition 4.5.10. Parametrized fibrancy and cofibrancy. *Let A be an object in a model category $\mathcal{M}$. A* **parametrized cofibrant object** *X, or* **cofibrant object parametrized under** *A in $\mathcal{M}$ is an object with a cofibration $\upsilon_X : A \to X$, i.e., a cofibrant object in the undercategory $(A\downarrow\mathcal{M})$. A* **parametrized morphism***, or* **morphism parametrized under** *A between such objects is a morphism in $(A\downarrow\mathcal{M})$. Parametrized fibrant objects and parametrized morphisms (over A) between are similarly defined in terms of the overcategory $(\mathcal{M}\downarrow A)$.*

Example 4.5.11. The disjoint base point functor as a pseudoendofunctor on $\mathcal{M}od$. *The disjoint base point functor $F : \mathcal{M} \mapsto \mathcal{M}_*$ leads to a pseudo-2-functor (Definition 2.7.10) from the 2-category $\mathcal{M}od$ of model categories (Example 2.7.2 (v)) to itself that is not a 2-functor.*

Proposition 4.5.12. Properties of Quillen pairs. *For model categories $\mathcal{M}$ and $\mathcal{N}$ with a pair of adjoint functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$, the following are equivalent:*

(i) (F, U) *is a Quillen pair.*

(ii) *The left adjoint* F *preserves both cofibrations and trivial cofibrations.*

(iii) *The right adjoint* U *preserves both fibrations and trivial fibrations.*

(iv) *The left adjoint* F *preserves cofibrations and the right adjoint* U *preserves fibrations.*

(v) *The left adjoint* F *preserves trivial cofibrations and the right adjoint* U *preserves trivial fibrations.*

(vi) *The left adjoint* F *preserves cofibrations between cofibrant objects and all trivial cofibrations.*

(vii) *The right adjoint* U *preserves fibrations between fibrant objects and all trivial fibrations.*

The last two clauses above are due to Dugger [Dug01].

The following is Hirschhorn's [Hir03, Definition 3.1.1] and is a refinement of Definition 4.3.15. It is related to Bousfield localaization, as we will see in Theorem 6.2.6 below.

Definition 4.5.13. Left and right localization. *Let* $\mathcal{M}$ *be a model category with a morphism class (possibly a set)* $\mathscr{E}$*. A* **left (right) localization with respect to** $\mathscr{E}$ *is a model category* $L_{\mathscr{E}}\mathcal{M}$ *(*$R_{\mathscr{E}}\mathcal{M}$*) receiving a left (right) Quillen functor* j *from* $\mathcal{M}$ *such that*

(i) *the total left (right) derived functor of* j *sends images of* $\mathscr{E}$ *in* $\mathrm{Ho}\,(\mathcal{M})$ *to isomorphisms in* $\mathrm{Ho}\,(L_{\mathscr{E}}\mathcal{M})$ *(*$\mathrm{Ho}\,(R_{\mathscr{E}}\mathcal{M})$*), and*

(ii) *if* $\mathcal{N}$ *is another model category and* $\phi : \mathcal{M} \to \mathcal{N}$ *is a left (right) Quillen functor whose total left (right) derived functor takes the images in* $\mathrm{Ho}\,(\mathcal{M})$ *of the elements of* $\mathscr{E}$ *into isomorphisms in* $\mathrm{Ho}\,(\mathcal{N})$*, then there is a unique left (right) Quillen functor* $\delta : L_{\mathscr{E}}\mathcal{M} \to \mathcal{N}$ *(*$\delta : R_{\mathscr{E}}\mathcal{M} \to \mathcal{N}$*) such that* $\delta j = \phi$*.*

Definition 4.5.14. *A Quillen adjunction* (F, U, φ) *(or Quillen pair* (F, U)*) as in Definition 4.5.1 is a* **Quillen equivalence** *if for all cofibrant* X *in* $\mathcal{M}$ *and all fibrant* Y *in* $\mathcal{N}$*, a map* $f : FX \to Y$ *is a weak equivalence in* $\mathcal{N}$ *iff* $\varphi(f) : X \to UY$ *is a weak equivalence in* $\mathcal{M}$*. In that case we say that* F *is a* **left Quillen equivalence** *and* U *is a* **right Quillen equivalence** *and* (F, U) *is a* **pair of Quillen equivalences.** *We will sometimes write* $\mathcal{M} \simeq \mathcal{N}$ *to indicate the existence of a Quillen equivalence.*

Remark 4.5.15. Quillen equivalence and categorical equivalence. *A Quillen equivalence induces an equivalence (as in Definition 2.2.4) of homotopy categories, but need* **not** *be an equivalence of the model categories themselves. One can show that when there is a Quillen adjunction (or any adjunction) for which a map* $FX \to Y$ *is an isomorphism in* $\mathcal{N}$ *iff the adjoint map* $X \to UY$ *an isomorphism in* $\mathcal{M}$*, the categories* $\mathcal{M}$ *and* $\mathcal{N}$ *are equivalent. However the requirements of a Quillen equivalence are weaker in two respects:*

(i) we only consider morphisms in which X is cofibrant in $\mathcal{M}$ and Y is fibrant fibrant in $\mathcal{N}$, and

(ii) the logical equivalence in the definition concerns weak equivalence rather than isomorphism.

The following is an immediate consequence of the above definition.

Proposition 4.5.16. Comparing two model structures on the same underlying category. *Suppose (F, U, φ) is a Quillen adjunction as above in which the underlying categories are the same and F and U are each the identity functor. Suppose further that $\mathcal{M}$ and $\mathcal{N}$ have the same weak equivalences. Then (F, U, φ) is a Quillen equivalence.*

The following is proved by Hirschhorn as [Hir03, Theorem 8.5.23] and by Dwyer and Spalinski as [DS95, Theorem 9.7].

Theorem 4.5.17. Quillen equivalences and homotopy categories. *If (F, U, φ) is a Quillen equivalence in Definition 4.5.14, then the total derived functors*

$$\mathbf{L}F : \mathrm{Ho}\,\mathcal{M} \rightleftarrows \mathrm{Ho}\,\mathcal{N} : \mathbf{R}U$$

are equivalences of the homotopy categories $\mathrm{Ho}\,\mathcal{M}$ and $\mathrm{Ho}\,\mathcal{N}$.

The following is implied by [Hov99, Proposition 1.3.13], as explained there by Hovey.

Proposition 4.5.18. Quillen equivalences, units and counits. *Suppose (F, U, φ) as above is a Quillen adjunction as in Definition 4.5.1. Then the following are equivalent.*

(i) (F, U, φ) is a Quillen equivalence as in Definition 4.5.14.

(ii) The composite

$$X \xrightarrow{\eta_X} UFX \xrightarrow{Ur_{FX}} URFX \tag{4.5.19}$$

(where η is the counit of the adjunction and $r_Y : Y \to RY$ is functorial fibrant replacement in $\mathcal{N}$) is a weak equivalence for all cofibrant X in $\mathcal{M}$, and the composite

$$FQUY \xrightarrow{Fq_{UY}} FUY \xrightarrow{\epsilon_Y} Y \tag{4.5.20}$$

(where $q_X : QX \to X$ is functorial cofibrant replacement in $\mathcal{M}$ and ϵ is the unit of the adjunction) is a weak equivalence for all fibrant Y in $\mathcal{N}$.

(iii) The total derived functors $\mathbf{L}F : \mathrm{Ho}\,\mathcal{M} \rightleftarrows \mathrm{Ho}\,\mathcal{N} : \mathbf{R}U$ are equivalences of the homotopy categories $\mathrm{Ho}\,\mathcal{M}$ and $\mathrm{Ho}\,\mathcal{N}$.

The following is proved by Hovey as [Hov99, Corollary I.3.14].

Corollary 4.5.21. A single functor can determine a Quillen equivalence. *Suppose (F, U, φ) and (F, U', φ') are Quillen adjunctions from $\mathcal{M}$ to $\mathcal{N}$. Then (F, U, φ) is a Quillen equivalence if and only if (F, U', φ') is one. Dually, if (F', U, φ'') is another Quillen adjunction, then (F, U, φ) is a Quillen equivalence if and only if (F', U, φ'') is one.*

Hence it makes sense to say that a left (right) Quillen functor F (U) is Quillen equivalence without mentioning its adjoint or an adjunction isomorphism.

Proposition 4.5.22. Composing Quillen adjunctions. *Suppose we have model categories $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$ with functors*

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, shift left, "F_1"] \arrow[r, phantom, "\perp"] & \mathcal{M}_1 \arrow[l, shift left, "U_1"] \arrow[r, shift left, "F_2"] \arrow[r, phantom, "\perp"] & \mathcal{M}_2 \arrow[l, shift left, "U_2"] \end{tikzcd}$$

where (F_1, U_1) and (F_2, U_2) are Quillen pairs as in Definition 4.5.1. Then so is (F_2F_1, U_1U_2).

Proof Since F_1 and F_2 both preserve cofibrations, so does their composite F_2F_1. Dually, U_1U_2 preserves fibrations since U_1 and U_2 do. It follows that (F_2F_1, U_1U_2) is a Quillen pair. □

4.6 The suspension and loop functors

We follow the original treatment of this topic given in [Qui67, §I.2], to which we refer the reader for more details. After continuing the discussion of homotopy begun in §4.3, we define the suspension and loop functor in Definition 4.6.17 and state their basic properties in Theorem 4.6.24, which is [Qui67, Theorem I.2.2].

A newer treatment can be found in [Hov99, Chapter 6]. It relies on a structure Hovey calls a **framing**, developed in [Hov99, Chapter 5], building on results of [DK80]. For any model category $\mathcal{M}$, the homotopy category $\operatorname{Ho}\mathcal{M}$ is a module (as in Definition 2.6.22) over $\operatorname{Ho}\mathcal{S}et_{\boldsymbol{\Delta}}$, the homotopy category of simplicial sets as in Definition 4.2.17. This is explained in §4.7 below.

Using the definitions and notation of §4.3, we need to define a higher homotopy between two (left or right) homotopies between two maps $f_0, f_1 : A \to B$. As before we start by illustrating the idea in the case of $\mathcal{T}op$.

Example 4.6.1. A higher left homotopy. *Let $h, h' : A \times I \to B$ be two left homotopies (see Example 4.3.1(i) and Definition 4.3.6) between f_0 and f_1. The pushout of $I \leftarrow S^0 \to I$ is a circle S^1, so h and h' give us a map*

$h \coprod h' : A \times S^1 \to B$. *Then a* **higher left homotopy between** h **and** h' *is a map* $H : A \times D^2 \to B$ *making the following diagram commute.*

$$\begin{tikzcd} A \times S^1 \arrow[r, "h \coprod h'"] \arrow[d, "\sigma \coprod \sigma'"'] \arrow[dr, "A \times i_2"] & B \\ A & A \times D^2 \arrow[l, "\tau"] \arrow[u, "H"'] \end{tikzcd}$$

where $i_2 : S^1 \to D^2$ *is the usual inclusion and the maps* σ *and* σ' *are the homotopy equivalences associated with* h *and* h'.

There is a dual notion of **higher right homotopy between given right homotopies** k **and** k' *in which* $A \times D^2$ *(a solid cylinder) is replaced by* B^{D^2}, *the space of disk-like surfaces in* B.

This suggests the following analog of Definition 4.3.7, which we have not seen in the literature.

Definition 4.6.2. Solid cylinder and surface objects. *Let* A *and* B *be objects in a model category* $\mathcal{M}$. *A* **solid cylinder object** *for* A *is a factorization*

$$Cyl(A) \coprod_{A \coprod A} Cyl(A) \xrightarrow{\iota_2} Sol(A) \xrightarrow{\tau} A$$

of the secondary fold map $\sigma \coprod \sigma : Cyl(A) \coprod_{A \coprod A} Cyl(A) \to A$, *where* ι_2 *is a cofibration and* τ *is a weak equivalence.*

Dually, a **surface object** *for* B *is a factorization*

$$B \xrightarrow{s_2} Surf(B) \xrightarrow{t_2} Path(B) \underset{B \times B}{\times} Path(B)$$

of the secondary diagonal map $s \times s : B \to Path(B) \underset{B \times B}{\times} Path(B)$, *where* s_2 *is a weak equivalence and* t_2 *is a fibration.*

The following is [Qui67, Definition I.2.1].

Definition 4.6.3. Higher left and right homotopies. *Let* $f_0, f_1 : A \to B$ *be two morphisms in a model category* $\mathcal{M}$, *and let* h *and* h' *be left homotopies between them. Then a* **higher left homotopy between** h **and** h' *is a map* $H : Sol(A) \to B$ *making the following diagram commute.*

$$\begin{tikzcd} Cyl(A) \coprod_{A \coprod A} Cyl'(A) \arrow[r, "h \coprod h'"] \arrow[d, "\sigma \coprod \sigma'"'] \arrow[dr, "\iota_2"] & B \\ A & Sol(A) \arrow[l, "\tau"] \arrow[u, "H"'] \end{tikzcd}$$

where $Cyl'(A)$ is another cylinder object for A, and the maps σ and σ' are the homotopy equivalences associated with h and h'.

A **higher right homotopy** *$K : A \to Sur(B)$ between right homotopies k and k' is similarly defined.*

Quillen showed that these higher homotopies lead to sets

$$\pi_1^{\ell}(A, B;\, f_0, f_1) \qquad \text{and} \qquad \pi_1^{r}(A, B;\, f_0, f_1) \tag{4.6.4}$$

of higher homotopy classes of left and right homotopies between f_0 and f_1.

Next we consider the relation between left and right homotopies.

Definition 4.6.5. Corresponding left and right homotopies. *As in Definition 4.3.6, let $f_0, f_1 : A \to B$ be two morphisms in a model category $\mathcal{M}$ with left and right homotopies h and k, where $\widetilde{A} = Cyl(A)$ and $\widetilde{B} = Path(B)$ as in Definition 4.3.7. A* **correspondence** *between h and k is a map $H : Cyl(A) \to Path(B)$ making the following diagram commute.*

$$\begin{tikzcd}
 & Cyl(A) \arrow[r, "\sigma"] \arrow[dl, "h"'] \arrow[d, "H"] & A \arrow[r, "\partial_1"] \arrow[d, "f_1"] & Cyl(A) \arrow[d, "H"] & A \arrow[l, "\partial_0"'] \arrow[dl, "k"] \\
B & Path(B) \arrow[l, "d_0"] \arrow[r, "d_1"] & B \arrow[r, "s"] & Path(B) &
\end{tikzcd}$$

If such an H exists, we say that the left homotopy h **corresponds** *to the right homotopy k.*

Again we start with an example in $\mathcal{T}op$. The map H below (Quillen's notation) is unrelated to the map he called H in Definition 4.6.3.

Example 4.6.6. Some correspondences. *Returning to Example 4.3.1, suppose we have a map $H : A \times I \to B^I$ (where $I = [0,1]$ as usual), making the following diagram commute.*

$$\begin{tikzcd}
 & A \times I \arrow[r, "\sigma"] \arrow[dl, "h"'] \arrow[d, "H"] & A \arrow[r, "\partial_1"] \arrow[d, "f_1"] & A \times I \arrow[d, "H"] & A \arrow[l, "\partial_0"'] \arrow[dl, "k"] \\
B & B^I \arrow[l, "d_0"'] \arrow[r, "d_1"] & B \arrow[r, "s"] & B^I &
\end{tikzcd} \tag{4.6.7}$$

The map H is adjoint to a map $I^2 \to B^A$, that is a family of maps $A \to B$ parametrized by the unit square, which we will also denote by H. Its restrictions *to the vertices and edges of the square are indicated in the drawing below.*

$$\begin{tikzcd}
f_1 \arrow[r, dash, "f_1\sigma"] \arrow[d, dash, "k"'] \arrow[dr, phantom, "H"] & f_1 \arrow[d, dash, "sf_1"] \\
f_0 \arrow[r, dash, "h"'] & f_1
\end{tikzcd} \tag{4.6.8}$$

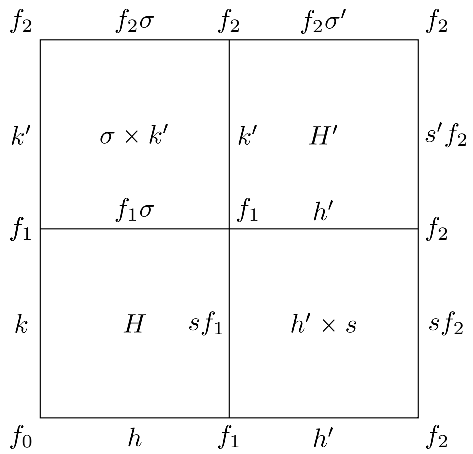
Now suppose we have a third map $f_2 : A \to B$ that is homotopic to f_1 with left and right homotopies h' and k', and a map $H' : A \times I' \to B^{I'}$ (where $I' = [1, 2]$) making the following diagram commute.

$$\begin{tikzcd}
& A \times I' \arrow[r, "\sigma'"] \arrow[dl, "h'"'] \arrow[d, "H'"] & A \arrow[r, "\partial_2"] \arrow[d, "f_2"] & A \times I' \arrow[d, "H'"] & A \arrow[l, "\partial_1"'] \arrow[dl, "k'"] \\
B & B^{I'} \arrow[l, "d_1"'] \arrow[r, "d_2"] & B \arrow[r, "s'"] & B^{I'} &
\end{tikzcd} \tag{4.6.9}$$

The corresponding map on the square $(I')^2$ has the form

$$\begin{tikzcd}
f_2 \arrow[r, dash, "f_2\sigma'"] \arrow[d, dash, "k'"'] \arrow[dr, phantom, "H'"] & f_2 \arrow[d, dash, "s'f_2"] \\
f_1 \arrow[r, dash, "h'"'] & f_2
\end{tikzcd} \tag{4.6.10}$$

Combining (4.6.8) and (4.6.10) we can form a map $H'' : (I'')^2 \to B^A$, *where* $I'' = [0,2]$*, as follows.*



The analog of (4.6.7) and (4.6.9) is

$$\begin{tikzcd}
 & A \times I'' \arrow[r, "\sigma''"] \arrow[dl, "h'\cdot h"'] \arrow[d, "H''"] & A \arrow[r, "\partial_2"] \arrow[d, "f_2"] & A \times I'' \arrow[d, "H''"] & A \arrow[l, "\partial_1"'] \arrow[dl, "k'\cdot k"] \\
B & B^{I''} \arrow[l, "d_1"'] \arrow[r, "d_2"] & B \arrow[r, "s''"] & B^{I''} &
\end{tikzcd}$$

where $I'' = [0,2]$ *and the* **composite left and right homotopies** $h' \cdot h$ **and** $k' \cdot k$ *are defined by*

$$(h' \cdot h)(a,t) = \begin{cases} h(a,t) & \text{for } 0 \leqslant t \leqslant 1 \\ h'(a,t-1) & \text{for } 1 \leqslant t \leqslant 2 \end{cases} \tag{4.6.11}$$

and

$$(k' \cdot k)(a)(t) = \begin{cases} k(a)(t) & \text{for } 0 \leqslant t \leqslant 1 \\ k'(a)(t-1) & \text{for } 1 \leqslant t \leqslant 2. \end{cases} \tag{4.6.12}$$

In particular we could have $f_2 = f_0$ *with* h' *and* k' *the inverse homotopies* h^{-1} *and* k^{-1} *of* h *and* k *given by*

$$h^{-1}(a,t) := h(a,1-t) \quad \text{and} \quad k^{-1}(a)(t) := k(a)(1-t). \tag{4.6.13}$$

Proposition 4.6.14. The bijection of left and right homotopy sets. *With notation as in Definition 4.6.5, for each left homotopy* h *there is a corresponding right homotopy* k *and* vice versa*. The sets* $\pi_1^\ell(A,B;\, f_0,f_1)$ *and* $\pi_1^r(A,B;\, f_0,f_1)$ *of (4.6.4) are naturally isomorphic, and we denote them by* $\pi_1(A,B;\, f_0,f_1)$*.*

Proposition 4.6.15. The composition of left and right homotopies

of maps in $\mathcal{T}op$ defined in (4.6.11) and (4.6.12) can be defined in a general model category $\mathcal{M}$. Hence we have maps

$$\pi_1^\ell(A, B;\, f_1, f_2) \times \pi_1^\ell(A, B;\, f_0, f_1) \to \pi_1^\ell(A, B;\, f_0, f_2)$$

and similarly for right homotopies. This composition is compatible with the bijection of Proposition 4.6.14.

Finally, consider the category $\mathcal{M}(A, B)$ whose objects are morphisms $A \to B$ and whose morphisms are homotopies, either left or right, with composition of morphisms being composition as in (4.6.11) and (4.6.12). It is a groupoid in which the inverse of a morphism is defined as in (4.6.13).

Definition 4.6.16. Quillen's fundamental group $\pi_1(A, B)$. *Let $\mathcal{M}$ be a pointed model category as in Definition 4.1.26. For cofibrant A and fibrant B we will abbreviate the group $\pi_1(A, B;\, 0, 0)$ of Proposition 4.6.14 by $\pi_1(A, B)$.*

This group is not to be confused with the set $\pi(A, B)$ (for cofibrant A and fibrant B) of Definition 4.3.11.

Now we are ready to discuss the suspension and loop functors. The reader is invited to compare this definition with the one suggested in (4.1.31).

Definition 4.6.17. The suspension and loop objects and functors. *For a cofibrant object A in a pointed model category $\mathcal{M}$, the* **suspension object** *ΣA is the cokernel (Definition 4.1.27) of the map*

$$A \vee A \to Cyl(A), \tag{4.6.18}$$

where $Cyl(A)$ is the functorial cylinder object of A as in Definition 4.3.7.

Dually, for a fibrant object B, the **loop object** *ΩB is the kernel of the map*

$$Path(B) \to B \wedge B, \tag{4.6.19}$$

where $Path(B)$ is the functorial path object of B.

Both of these definitions are natural, so we can regard Σ and Ω as functors.

Remark 4.6.20. The functoriality of suspension and loop objects. *The maps of (4.6.18) and (4.6.19) can be regarded components of natural transformations between the evident functors, so Σ and Ω themselves are functors. We will see in Corollary 4.7.2 below that these define functors $\Sigma : \mathcal{M}_c \to \mathcal{M}_c$ and $\Omega : \mathcal{M}_f \to \mathcal{M}_f$, where $\mathcal{M}_c$ and $\mathcal{M}_f$ denote the full subcategories of cofibrant and fibrant objects of $\mathcal{M}$.*

For an adjunction relating these two functors in the case of a topological model category, see Example 5.6.12 below.

Remark 4.6.21. *In the case $\mathcal{M} = \mathcal{T}$, ΣA as in Definition 4.6.17 is the usual reduced suspension and ΩB is the usual loop space.*

Recall that a morphism $f : X \to Y$ in a pointed model category has a homotopy fiber F_f and cofiber C_f as in Definition 4.1.28. In the case of $\mathcal{T}$ (or more generally a pointed topological model category as in Definition 5.6.3 below),

$$F_f = \{(x, \omega) \in X \wedge PY : f(x) = \omega(1)\}$$

and

$$C_f = CX \vee_X Y,$$

the mapping cone of f, which is a quotient of the union of Y and $X \times I$.

Definition 4.6.22. The pinch map. *Let $\mathcal{M}$ be a pointed topological model category. As noted in Remarks 4.3.8(iii), the cylinder object for a cofibrant object A can defined as $Cyl(A) = A \rtimes I$. This makes suspension ΣA the cokernel of the map*

$$A \wedge (\partial I)_+ \to Cyl(A), \qquad \text{where } \partial I = \{0,\, 1\}\,.$$

Let

$$\partial' I = \{0,\, 1/2,\, 1\}\,,$$

which has ∂I as a subspace. Then the cokernel of $A \wedge (\partial' I)_+ \to Cyl(A)$ is isomorphic to $\Sigma A \vee \Sigma A$. Using this isomorphism, we define the **pinch map**

$$\Sigma A \xrightarrow{\Diamond\!\!\Diamond_A} \Sigma A \vee \Sigma A$$

to be the map of cokernels of the horizontal maps in the diagram

$$\begin{tikzcd}
A \wedge (\partial I)_+ \arrow[r] \arrow[d] & Cyl(A) \arrow[d, equal] \\
A \wedge (\partial' I)_+ \arrow[r] & Cyl(A).
\end{tikzcd}$$

We leave the following as an exercise for the reader.

Proposition 4.6.23. The pinch map and homotopy groups. *For a cofibrant object A in a pointed topological model category, the pinch map $\Diamond\!\!\Diamond_A : \Sigma A \to \Sigma A \vee \Sigma A$ of Definition 4.6.22 induces a natural (in X) group structure in the set $\pi^\ell(\Sigma A, X)$ of Definition 4.3.11 (which is $\pi(\Sigma A, X)$ when X is fibrant) which is abelian if A itself is a suspension.*

The following was stated and proved by Quillen as [Qui67, Theorem I.2.2].

Theorem 4.6.24. Total derived suspension and loop functors. *In a pointed model category $\mathcal{M}$, $\pi_1(A, B)$ (see Definition 4.6.16) for cofibrant A and fibrant B gives a group valued functor on $\operatorname{Ho}\mathcal{M}^{op} \times \operatorname{Ho}\mathcal{M}$. There are also functors $\mathbf{L}\Sigma$, $\mathbf{R}\Omega : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{M}$ and canonical isomorphisms*

$$\pi(\Sigma A, B) \cong \pi_1(A, B) \cong \pi(A, \Omega B),$$

the isomorphisms being those of Proposition 4.3.19.

Remarks 4.6.25. Total derived functors.

(i) *As the notation indicates, the adjoint functors* $\mathbf{L}\Sigma$ *and* $\mathbf{R}\Omega$ *are the total left and right derived functors (Definition 4.4.7) of the functors* Σ *and* Ω *of Definition 4.6.17.*

(ii) *The functors* Σ *and* Ω *can be iterated. For any* X*,* $\mathbf{L}\Sigma^n X$ *(*$\mathbf{R}\Omega^n X$*) is a cogroup (group) object in* $\operatorname{Ho}\mathcal{M}$ *for* $n \geqslant 1$ *which is abelian for* $n \geqslant 2$*.*

4.7 Fiber and cofiber sequences

Again we follow the treatment of this topic by Quillen in [Qui67, §I.3]. He called them fibration and cofibration sequences. **Assume throughout this section that $\mathcal{M}$ is a pointed model category** as in Definition 4.1.26. We will summarize Quillen's development of fiber sequences, leaving most of the dual theory of cofiber sequences as exercises for the reader. Fiber and cofiber sequences are defined in Definition 4.7.6.

Lemma 4.7.1. The fiber of a fibrant fibration is fibrant, the first Dr. Seuss lemma. *Let* $f : X \to Y$ *be a fibration in a pointed model category* $\mathcal{M}$*. Then its kernel as in Definition 4.1.27 is fibrant, and both* X *and the homotopy fiber* F_f *as in Definition 4.1.28 are fibrant if* Y *is.*

Dually, if f *is a cofibration, then its cokernel is cofibrant, and both* Y *and the homotopy cofiber* C_f *are cofibrant if* X *is.*

The second Dr. Seuss lemma is Lemma 5.6.20 below.

Proof We will only prove the fibration statement.

To show that $\ker f$ is fibrant, let $j : A \to B$ be a trivial cofibration and consider the following commutative diagram, in which α is arbitrary.

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "j"'] & \ker f \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed, "h'"] \arrow[urr, dashed, "h"'] & * \arrow[r] & Y
\end{tikzcd}$$

The lifting $h : B \to X$ exists because f is a fibration. Once it has been chosen, the lifting $h' : Y \to F$ exists uniquely because the right square is a pullback diagram. We can find such an h' for any trivial cofibration j, so the map $\ker f \to *$ is a fibration and $\ker f$ is fibrant.

For the homotopy fiber F_f, consider a similar commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "j"'] & F_f \arrow[r, "p_f"] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "f"] \\
B \arrow[r, "\beta"'] \arrow[ur, dashed, "h'"] & PY \arrow[r, "p_Y"] \arrow[ur, dashed, "h"'] & Y
\end{tikzcd}$$

As above, we conclude that the map $F_f \to PY$ is a fibration. Since p_Y is a fibration, so is the map fp_f. It follows that X and F_f are fibrant if Y is. □

Corollary 4.7.2. The loop (suspension) functor of Definition 4.6.17 sends fibrant (cofibrant) objects to fibrant (cofibrant) objects.

Proof If A is cofibrant, so are the objects $A \vee A$ and $Cyl(A)$. Thus we can apply Lemma 4.7.1 to the cokernel of (4.6.18) and conclude that ΣA is cofibrant. The object ΩB for fibrant B is fibrant by a similar argument. □

We will illustrate Quillen's construction in the pointed model category $\mathcal{T}$ (pointed topological spaces), in which the functorial path object for X is X^I as in Definition 4.1.30.

Example 4.7.3. The homotopy action of ΩY on F. *For a sequence in $\mathcal{T}$*

$$F \longrightarrow X \xrightarrow{f} Y$$

in which f is a fibration with kernel F, consider the diagram

$$\begin{tikzcd}
F \times_X X^{I_+} \times_X F \arrow[r, "p_2"] \arrow[d, "\pi"'] & X^{I_+} \arrow[d, "(d_0, p^{I_+})"] \\
F \times \Omega Y \arrow[r, "i \times j"] & X \times_Y Y^{I_+},
\end{tikzcd} \tag{4.7.4}$$

where $Y^{I_+} = \mathcal{T}(I_+, Y)$ and $X^{I_+} = \mathcal{T}(I_+, X)$ are the spaces of paths in Y and X. The space in the upper left is that of paths in X with endpoints in $F = f^{-1}(y_0) \subseteq X$ (where $y_0 \in Y$ is the base point), so the projection p_2 is the inclusion map into the full path space of X. The map π sends such a path ω to the ordered pair consisting of its starting point in F and its image in Y^{I_+}, which is necessarily closed at the base point $y_0 \in Y$ (since the endpoints of ω lie in $F = f^{-1}(y_0)$) and thus a point in ΩY. The map on the right sends a path ω in X to the ordered pair consisting of its starting point $d_0(\omega) \in X$ and the path $f\omega \in Y^{I_+}$. In the bottom map, $j : \Omega Y \to Y^I$ is the inclusion of the loop space into the path space Y^{I_+}.

It is easy to show that diagram is a pullback diagram, both vertical maps are weak equivalences and the spaces on the right are weakly equivalent to X. **Since π is an equivalence, it has an inverse in $\mathrm{Ho}\,\mathcal{T}$, where we define $m = p_3\pi^{-1} : F \wedge \Omega Y \to F$.**

The following was proved by Quillen as [Qui67, Proposition I.3.1]. It is also proved with a different method by Hovey as [Hov99, Theorem 6.2.1].

Proposition 4.7.5. The homotopy action of ΩY on F. *In a pointed model category $\mathcal{M}$, let $f : X \to Y$ be a fibration with kernel F with Y fibrant as in Lemma 4.7.1. Then the analog of (4.7.4),*

$$\begin{tikzcd} F \wedge_X Path(X) \wedge_X F \arrow[r, "p_2"] \arrow[d, "\pi"'] & Path(X) \arrow[d, "(d_0, Path(f))"] \\ F \wedge \Omega Y \arrow[r, "p \wedge j"] & X \wedge_Y Path(Y), \end{tikzcd}$$

leads to a right action $m : F \wedge \Omega Y \to F$ in $\operatorname{Ho}\mathcal{M}$ of the group object ΩY (see Remarks 4.6.25 (ii)) on F which is independent of the choices of path objects.

Definition 4.7.6. Fiber and cofiber sequences. *A* **fiber sequence** *in $\operatorname{Ho}\mathcal{M}$ for a pointed model category $\mathcal{M}$ is a diagram*

$$W \to X \to Y \qquad \text{with a right action } W \wedge \Omega Y \to W$$

which is isomorphic to a diagram

$$F \xrightarrow{p} X \xrightarrow{f} Y \qquad \text{with a right action } F \wedge \Omega Y \xrightarrow{m} F \tag{4.7.7}$$

where f is a fibration, Y is fibrant, and F is the **fiber of** *f, namely its kernel as in Definition 4.1.27. The right action m is as in Proposition 4.7.5.*

Dually, a **cofiber sequence** *is a diagram isomorphic to*

$$X \xrightarrow{f} Y \xrightarrow{i} C \qquad \text{with a right coaction } C \xrightarrow{m'} C \vee \Sigma X \tag{4.7.8}$$

where X is cofibrant, f is a cofibration, and C is the **cofiber of** *f, namely its cokernel as in Definition 4.1.27. The construction of the right coaction m' is dual to that of the right action m above.*

The next two results are [Qui67, Propositions I.3.3 and I.3.4].

Proposition 4.7.9. Extending fiber and cofiber sequences. *If (4.7.7) is a fiber sequence, so is*

$$\Omega Y \xrightarrow{\partial} F \xrightarrow{p} X \qquad \text{with a right action } \Omega Y \wedge \Omega X \xrightarrow{n} \Omega Y$$

where ∂ is the composite

$$\Omega Y \xrightarrow{(0, \Omega Y)} F \wedge \Omega Y \xrightarrow{m} F$$

and the map induced by the action n,

$$\pi(A, \Omega Y) \times \pi(A, \Omega X) \xrightarrow{n_*} \pi(A, \Omega Y)$$

(see Definition 4.3.11 for the definition of the set $\pi(-,-)$) for cofibrant A is given by

$$(\lambda, \mu) \mapsto ((\Omega p)_* \mu)^{-1} \cdot \lambda.$$

Dually, if (4.7.8) is a cofiber sequence, then so is

$$Y \xrightarrow{i} C \xrightarrow{\delta} \Sigma X \qquad \text{with a right coaction } \Sigma X \xrightarrow{n'} \Sigma X \vee \Sigma Y,$$

where δ is the composite

$$C \xrightarrow{m'} C \vee \Sigma X \xrightarrow{(*, \Sigma X)} \Sigma X$$

and the coaction

$$\pi(\Sigma X, B) \times \pi(\Sigma Y, B) \xrightarrow{(n')^*} \pi(\Sigma X, B)$$

for fibrant B is given by

$$(\lambda', \mu') \mapsto ((\Sigma u)^* \mu')^{-1} \cdot \lambda'.$$

Corollary 4.7.10. Long fiber and cofiber sequences. *Let $f : X \to Y$ be a morphism in a pointed topological model category.*

(i) If Y is fibrant, f is a fibration, and F_f is as in Definition 4.1.28, then we have an infinite sequence of morphisms, a **long fiber sequence***,*

$$\cdots \longrightarrow \Omega^2 Y \xrightarrow{-\Omega\partial_f} \Omega F_f \xrightarrow{-\Omega p_f} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\partial_f} F_f \xrightarrow{p_f} X \xrightarrow{f} Y,$$

in which any two adjacent morphisms form a fiber sequence.

(ii) Dually if X is cofibrant, f is a cofibration, and C_f is as in Definition 4.1.28, then we have an infinite sequence of morphisms, a **long cofiber sequence***,*

$$X \xrightarrow{f} Y \xrightarrow{i_f} C_f \xrightarrow{\delta_f} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i_f} \Sigma C_f \xrightarrow{-\Sigma\delta_f} \Sigma^2 X \longrightarrow \cdots,$$

in which any two adjacent morphisms form a cofiber sequence.

Classically, the above are known as **Puppe sequences**, named after Dieter Puppe (1930–2005). We will refer to the two exact sequences below as **the Puppe exact seqences**.

Proposition 4.7.11. Exact sequences for fiber and cofiber sequences.

(i) Given the long fiber sequence of Corollary 4.7.10(i), for each cofibrant A the sequence

$$\cdots \longrightarrow \pi(A, \Omega^q F) \xrightarrow{(\Omega^q p_f)_*} \pi(A, \Omega^q X) \xrightarrow{(\Omega^q f)_*} \pi(A, \Omega^q Y) \xrightarrow{(\Omega^{q-1}\partial_f)_*} \cdots$$

$$\cdots \xrightarrow{(\partial_f)_*} \pi(A, F) \xrightarrow{(p_f)_*} \pi(A, X) \xrightarrow{f_*} \pi(A, Y)$$

(see Definition 4.3.11 for the meaning of $\pi(-,-)$, and Theorem 4.6.24

for the group structures on $\pi(-,\Omega-)$ and $\pi(\Sigma-,-)$) is exact in the following sense:

(a) *The image of $(p_f)_*$ in $\pi(A,X)$ is the preimage of the trivial element in $\pi(A,Y)$.*
(b) *The composite $(p_f\partial_f)_*$ is trivial and $(p_f)_*(\alpha_1) = (p_f)_*(\alpha_2)$ iff $\alpha_2 = \alpha_1 \cdot \lambda$ for some $\lambda \in \pi(A,\Omega Y)$*
(c) *The composite $\partial_{f*}(\Omega f)_*$ is trivial and $(\partial_f)_*(\lambda_1) = \partial_{f*}(\lambda_2)$ iff $\lambda_2 = (\Omega f)_*\mu \cdot \lambda_1$ for some $\mu \in \pi(A,\Omega X)$.*
(d) *The sequence of group homomorphisms from $\pi(A,\Omega X)$ to the left is exact in the usual sense.*

(ii) Dually, given the long cofiber sequence of Corollary 4.7.10(ii), for each fibrant B the sequence

$$\cdots \longrightarrow \pi(\Sigma^q C,B) \xrightarrow{(\Sigma^q i_f)^*} \pi(\Sigma^q Y,B) \xrightarrow{(\Sigma^q f)^*} \pi(\Sigma^q X,B) \xrightarrow{(\Sigma^{q-1}\delta_f)^*} \cdots$$
$$\cdots \xrightarrow{(\delta_f)^*} \pi(C,B) \xrightarrow{(i_f)^*} \pi(Y,B) \xrightarrow{f^*} \pi(X,B)$$

is exact in the sense of (i)(a)–(i)(d) above with $(\partial_f)_*$ and (p_{f*}) replaced by $(\delta_f)^*$ and $(i_f)^*$, and the action m_* of (i)(b) and n_* of (i)(c) replaced by $(m')^*$ and $(n')^*$.

In §5.7A below, we will see that these exact sequences can be extended indefinitely to the right in the case of spectra.

4.8 The small object argument

The small object argument is a method introduced by Quillen in [Qui67] (and here in the proof of Theorem 4.2.9) and later improved by Bousfield in [Bou77] to construct the factorizations needed for a model structure. It is also needed to construct localization functors, the subject of Chapter 6 below.

We will state the theorem first and then give the relevant definitions. Proofs can be found in [Hov99, 2.1.4], [Hir03, 10.5.16], [Lur09, A.1.2.5] and [MP12, §15.1]. See [Gar09] for further discussion.

Theorem 4.8.1. The small object argument. *Let $\mathcal{C}$ be a cocomplete (meaning that all small colimits exist) category with a class of morphisms $\mathcal{I}$ having small domains relative to $\mathcal{I}$, with smallness as in Definition 4.8.18 below. Then there is a functorial factorization of an arbitrary morphism f as $f = f''f'$, where f' is in the saturated class generated by $\mathcal{I}$ (see Definition 4.8.13 below) and f'' is in $\mathcal{I}^{\boxslash}$, meaning that it has the right lifting property (Definition 2.3.10) with respect to $\mathcal{I}$.*

Definition 4.8.2. *A class of morphisms in a cocomplete category $\mathcal{C}$* **permits the small object argument** *if it satisfies the hypothesis of Theorem 4.8.1.*

First we need to define small objects. The set theoretic notions (ordinals and cardinals) relevant to the definition Definition 4.8.8 below are discussed in [Hov99, §2.1.1], [SS00, §2] and in [Hir03, Chapter 10]. We recall them briefly.

Two sets A and B **have the same cardinality** if there is a bijection between them. The the cardinalilty of B **exceeds** that of A if there is a bijection between A and a subset of B but not one between A and B itself.

An **ordinal** λ is the well ordered set of all smaller ordinals. It can also be regarded as a category in which there is a unique morphism $\alpha \to \beta$ whenever $\alpha \leqslant \beta$. Every ordinal λ has a successor $\lambda + 1$. A **limit ordinal** is one that is neither zero nor a successor. The smallest infinite ordinal, often denoted by ω, is the first limit ordinal.

Example 4.8.3. The first few ordinals. *Since an ordinal is the well ordered set of all smaller ordinals, the smallest ordinal, commonly denoted by 0, is the empty set $\varnothing$. The next few are*

$$\begin{aligned} 1 &:= \{\varnothing\} = \{0\}, \\ 2 &:= \{\varnothing, \{\varnothing\}\} = \{0, 1\} \\ 3 &:= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{0, 1, 2\} \end{aligned}$$

and so on, with $n + 1$ being the successor of n. The smallest infinite ordinal and its successors are

$$\begin{aligned} \omega &:= \{0, 1, 2, 3, \dots\} \quad \text{(the set of all nonegative integers)} \\ \omega + 1 &:= \omega \cup \{\omega\} \\ \omega + 2 &:= (\omega + 1) \cup \{\omega + 1\} = \omega \cup \{\omega, \omega + 1\} \\ &\vdots \end{aligned}$$

The next limit ordinal is

$$\begin{aligned} 2\omega &:= \omega \cup \{\omega, \omega + 1, \omega + 2, \dots\} \\ &:= \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}. \end{aligned}$$

An ordinal is a **cardinal** if its cardinality exceeds that of all smaller ordinals. Thus all finite ordinals are cardinals, as is ω. The ordinals $\omega + 1$ and 2ω are not cardinals because there is a bijection between ω and each of them.

Definition 4.8.4. Transfinite composition. *Let $\mathcal{C}$ be a cocomplete category and λ an ordinal. A* **λ-sequence** *is a colimit preserving functor $X : \lambda \to \mathcal{C}$, i.e., is a diagram*

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_\beta \longrightarrow \cdots$$

For any limit ordinal $\gamma < \lambda$, the map

$$\operatorname*{colim}_{\beta<\gamma} X_\beta \to X_\gamma$$

is an isomorphism. The **composition of the λ-sequence** *is the map*

$$X_0 \to \operatorname*{colim}_{\beta<\lambda} X_\beta. \tag{4.8.5}$$

If $\mathcal{I}$ is a collection of morphisms that includes the maps $X_\beta \to X_{\beta+1}$ for all β with $\beta+1 < \lambda$, we say that the map (4.8.5) is a **transfinite composition of maps in $\mathcal{I}$**.

Example 4.8.6. Pedestrian cases of transfinite composition.

(i) For $\lambda = 0$, a λ-sequence is an object X_0 in $\mathcal{C}$, and the map of (4.8.5) is its identity morphism.

(ii) For $\lambda = \aleph_0$, a λ-sequence is an ordinary one, and the map of (4.8.5) is evident map from X_0 to the sequential colimit.

The **cardinality** $|A|$ of a set A is the smallest ordinal for which there is a bijection $|A| \to A$. A **cardinal** κ is an ordinal for which $|\kappa| = \kappa$.

Definition 4.8.7. Ordinals filtered by a cardinal. *Let γ be a cardinal. An ordinal α is* **γ-filtered** *if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| < \gamma$, then the supremum of A is less that α, $\sup A < \alpha$.*

The above is a lower bound on the limit ordinal α. Any limit ordinal is filtered by a finite cardinal.

Definition 4.8.8. Small objects. *Let $\mathcal{C}$ be a cocomplete category, $\mathcal{D}$ a subcategory of $\mathcal{C}$, and let κ be a cardinal. Then an object A in $\mathcal{C}$ is* **κ-small relative to $\mathcal{D}$** *if for all κ-filtered ordinals λ and all λ-sequences as in Definition 4.8.4 such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta+1 < \lambda$, the map of sets*

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. It is **small relative to $\mathcal{D}$** *if it is κ-small relative to $\mathcal{D}$ for some κ. It is* **small** *if it is small relative to $\mathcal{C}$ itself.*

The object A is **finite (relative to $\mathcal{D}$)** *if it is small relative to $\mathcal{D}$ for a finite cardinal κ. In this case, maps from A commute with colimits of arbitrary λ-sequences, as long as λ is a limit ordinal.*

In the case of a finite cardinal κ, the above notions are the same as those in Definition 2.3.63 and Definition 2.3.69.

The following properties of small objects are proved as [Hir03, 10.4.8 and 10.4.9].

Proposition 4.8.9. Colimits preserve smallness. *Let $\mathcal{C}$ be a cocomplete category with a subcategory $\mathcal{D}$. Then any colimit of objects that are small relative to $\mathcal{D}$ is also small relative to $\mathcal{D}$.*

Proposition 4.8.10. Smallness and factorization. *Let $\mathcal{C}$ and $\mathcal{D}$ be as in Definition 4.8.8, and let $\mathcal{I}$ be a set of maps in $\mathcal{C}$ for which each domain and codomain is small relative to $\mathcal{D}$. Then if X is small relative to $\mathcal{D}$ and the map $X \to Y$ is a transfinite composition of pushouts of elements of $\mathcal{I}$, then Y is small relative to $\mathcal{D}$. In particular if $f : X \to Z$ is a morphism in $\mathcal{C}$ with X small relative to $\mathcal{D}$, and $f' : X \to Y$ is the map given by Theorem 4.8.1, then Y is also small relative to $\mathcal{D}$.*

Definition 4.8.11. Combinatorial model categories. *A model category $\mathcal{M}$ is* **combinatorial** *if it is cofibrantly generated (see §5.2 below) and* **locally presentable**, *meaning that each of its objects is a colimit of small objects in a set W. An object A is* **small** *if there is a regular cardinal κ such that for every small category T with morphism set of size $< \kappa$ and every functor $X : T \to \mathcal{M}$, there is an isomorphism*

$$\operatorname*{colim}_{t\in T} \mathcal{M}(A, X_t) \to \mathcal{M}(A, \operatorname*{colim}_{t\in T} X_t).$$

In $\mathcal{S}et_{\mathbf{\Delta}}$ (but not in $\mathcal{T}op$ or $\mathcal{T}$) we know that finite complexes are small and that every object is a colimit of finite complexes, so it is combinatorial.

Definition 4.8.12. Accessible categories. *An object X in a category $\mathcal{C}$ is* **κ-compact** *for a cardinal number κ if the functor $\mathcal{C}(X, -)$ preserves κ-directed colimits. A category $\mathcal{C}$ is* **accessible** *if there is a κ such that $\mathcal{C}$ is closed under κ-directed colimits and each object in it is such a colimit of objects in a set K of κ-compact objects.*

A category if **locally presentable** *if it is accessible and cocomplete.*

Next we need to discuss saturation.

Definition 4.8.13. *Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{I}$ be a class of morphisms in it. The* **regular class** *$\mathcal{R}eg(\mathcal{I})$ generated by $\mathcal{I}$ is the smallest class containing $\mathcal{I}$ and all isomorphisms that is closed under coproducts, pushouts, and transfinite compositions as in Definition 4.8.4. A class of morphisms closed under these operations is said to be* **regular**. *The pushout operation refers to pushout diagrams of the form*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d] \\ X \arrow[r, "j"'] & Y \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \qquad (4.8.14)$$

where i is in $\mathcal{I}$ and f is arbitrary. Then the map j is in the regular class generated by $\mathcal{I}$.

The **saturated class** $\mathcal{S}at(\mathcal{I})$ *generated by* $\mathcal{I}$ *is the smallest class containing* $\mathcal{I}$ *and all isomorphisms that is closed the operations above and* **under retracts**. *A class of morphisms closed under these operations is said to be* **saturated**.

Lurie uses the term **weakly saturated** for saturated as above in [Lur09, Definition A.1.2.2], as does Riehl in [Rie14, §11.1]. She uses **saturated** in connection with homotopical categories (to be studied below in §5.1) in [Rie14, Remark 2.1.8], following [DHKS04]. May and Ponto use the term **left saturated** in [MP12, Definition 14.1.7.] for the notion of Definition 4.8.13. They define a dual notion that involves pullbacks and transfinite sequential limits instead of pushouts and transfinite compositions, which are transfinite sequential colimits by definition. See [CF00] for more discussion.

The following is implied by the relevant definitions. See [MP12, Proposition 14.1.8] or [Hir03, Proposition 10.3.2] for a proof.

Proposition 4.8.15. Morphisms defined by a left lifting property. *Let* $\mathcal{R}$ *be a class of morphisms in a cocomplete category* $\mathcal{C}$. *Then the class* ${}^{\boxslash}\mathcal{R}$ *(the class of morphisms having the left lifting property with respect to* $\mathcal{R}$ *as in Definition 2.3.10) is saturated as in Definition 4.8.13.*

Proposition 4.8.16. Removing redundant maps. *Suppose the category* $\mathcal{C}$ *in Definition 4.8.13 is cocomplete with coproduct* $\amalg$. *Suppose the morphism class* $\mathcal{I}$ *contains morphisms* $f : A \to B$, $g : C \to D$ *and* $f \amalg g : A \amalg C \to B \amalg D$. *Let* $\mathcal{I}'$ *be the same class with* $f \amalg g$ *removed. Then* $\mathcal{I}'$ *generates the same saturated class as* $\mathcal{I}$.

Proof Consider the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & A \amalg C \arrow[d, "f \amalg C"] & \\
B \arrow[r] \arrow[ur, phantom, "\lrcorner"] & B \amalg C \arrow[d, "B \amalg g"'] & C \arrow[l] \arrow[d, "g"] \\
& B \amalg D \arrow[ur, phantom, "\llcorner"] & D \arrow[l]
\end{tikzcd}$$

in which both squares are pushouts. Then all vertical maps are in the saturated class of $\mathcal{I}'$, including the composite

$$(B \amalg g)(f \amalg C) = f \amalg g. \qquad \square$$

The following is [Hir03, Proposition 10.3.4].

Proposition 4.8.17. Saturated classes in a model category. *In any model category the classes of cofibrations and of trivial cofibrations are each saturated.*

A closely related definition is the following generalization of the notion of a cell complex.

Definition 4.8.18. *Let $\mathcal{C}$ be a category with pushouts and an initial object $*$, and let $\mathcal{I}$ be a class of morphisms of $\mathcal{C}$. Then a morphism $f : W \to X$ in $\mathcal{C}$ is* **relative $\mathcal{I}$-cellular** *or a* **relative $\mathcal{I}$-cell complex***, if it is transfinite composition of pushouts of maps in $\mathcal{I}$. An object X in $\mathcal{C}$ is an* **$\mathcal{I}$-cell complex** *if the map $* \to X$ is in that class.*

An object in $\mathcal{C}$ is **small relative to** *$\mathcal{I}$ or $\mathcal{I}$-small if it is small relative to the category of $\mathcal{I}$-cell complexes, as in Definition 4.8.8.*

The class of relative $\mathcal{I}$-cell complexes is smaller than the saturated class (Definition 4.8.13) generated by $\mathcal{I}$ because it need not be closed under retracts.

Proposition 4.8.19. Smallness with respect to a smaller saturated class. *Let $\mathcal{C}$ be a category with pushouts and let $\mathcal{J}$ be a class of morphisms in it, each of which is in the* **saturated class** *generated by $\mathcal{I}$. If an object A is $\mathcal{I}$-small as in Definition 4.8.18, then it is also $\mathcal{J}$-small.*

Proof Since A is $\mathcal{I}$-small, the map

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism when each map $X_\beta \to X_{\beta+1}$ is in the saturated class generated by $\mathcal{I}$. Our assumption about $\mathcal{J}$ implies that the saturated class generated by it is contained in the one generated by $\mathcal{I}$, so the result follows. □

Example 4.8.20. CW complexes as $\mathcal{I}$-cell complexes. *Let $\mathcal{C} = \mathcal{T}op$ and let*

$$\mathcal{I} = \{i_n : n \geqslant 0\} \qquad \text{where } i_n \text{ is the map } S^{n-1} = \partial D^n \to D^n$$

as in (5.2.10) below. Then the saturated class (Definition 4.8.13) generated by $\mathcal{I}$ consists of all composites of maps $j : X \to Y$ where Y is obtained from X by attaching an n-cell for some n. Hence the pushout diagram of (4.8.14) reads

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, "i_n"'] & X \arrow[d, "j"] \\ D^n \arrow[r, "g"'] & Y. \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Then a CW complex is an $\mathcal{I}$-cell complex, but $\mathcal{I}$-cell complexes are more general. In a CW complex we start with the initial object (the empty set), attach some 0-cells (points) to get a discrete set (the 0-skeleton), then attach some 1-cells to get the 1-skeleton, and so on. In an $\mathcal{I}$-cell complex we start with the initial object and add cells **in any order** *regardless of dimension.*

5

Model category theory since Quillen

> By definition a model category is just an ordinary category with three specified classes of morphisms, called *fibrations*, *cofibrations* and *weak equivalences*, which satisfy a few simple axioms that are deliberately reminiscent of properties of topological spaces. Surprisingly enough, these axioms give a reasonably general context in which it is possible to set up the basic machinery of homotopy theory. The machinery can then be used immediately in a large number of different settings, as long as the axioms are checked in each case. Although many of these settings are geometric (spaces (§8), fibrewise spaces (3.11), G-spaces [DK85], spectra [BF78], diagrams of spaces [DK84]...), some of them are not (chain complexes (§7), simplicial commutative rings [Qui70], simplicial groups [Qui69b]...). Certainly each setting has its own technical and computational peculiarities, but the advantage of an abstract approach is that they can all be studied with the same tools and described in the same language.
>
> *William Dwyer and Jan Spalinksi, [DS95, Introduction]*

In this chapter we discuss some tools in model category theory developed since Quillen's work that we will need later. The best such tool, Bousfield localization, is the subject of Chapter 6.

All of the model categories of interest in this book are pointed and topological, meaning that they are enriched (Definition 3.1.1) and bitensored (Definition 3.1.31) over $\mathcal{T}$, the category of pointed topological spaces. How this enrichment interacts with the model structure is the subject of §5.6.

We prefer topological model categories to simplicial ones (meaning ones enriched over simplicial sets) because equivariant homotopy theory does not play nicely with simplicial sets. Being in the topological world enables us to speak of maps or functors inducing weak equivalences between mapping spaces. This means we do **not** need to rely on the theory of framings developed by Hovey in [Hov99, Chapter 5], in which he shows that the homotopy category of an arbitrary model category looks like that of a simplicial model category. In other words it is a module (in the sense of Definition 2.6.22) over $\mathrm{Ho}\,\mathcal{S}et_{\boldsymbol{\Delta}}$. **Thus when Hovey would speak of an isomorphism between simplicial mapping sets that he can define in the homotopy category**

of a model category $\mathcal{M}$, we can speak instead of weak equivalences between mapping spaces in the topological model category $\mathcal{M}$ itself.

Homotopical categories are treated in §5.1. They were first introduced by Kan et al in [DHKS04]. Here we have weak equivalences but no defined class of fibrations or cofibrations, and no requirement of completeness or cocompleteness. These weak equivalences are required to satisfy a 2-of-6 property (spelled out in Definition 5.1.1) that implies the two-out-of-three axiom of Definition 4.1.1.

Every category has a homotopical structure in which the weak equivalences are the isomorphisms. We can use a functor $F : \mathcal{C} \to \mathcal{D}$ to define a homotopical structure on $\mathcal{C}$ in which weak equivalences are morphisms mapping to isomorphisms in $\mathcal{D}$; see Proposition 5.1.5. In that case we say that F is a **homotopy functor**. For example, weak equivalences in $\mathcal{T}$ and $\mathcal{T}op$ are defined to be maps inducing isomorphisms of homotopy groups. A functor between homotopical categories that preserves weak equivalences is said to be **homotopical**.

A homotopical category $\mathcal{M}$ has **a homotopy category** $\mathrm{Ho}\,\mathcal{M}$ similar to that of a model category as in Definition 4.3.16. Given a functor F between homotopical categories that is not homotopical, in favorable cases we can define **derived functors** LF and RF as in Definition 4.4.5, and **total derived functors** $\mathbf{L}F$ and $\mathbf{R}F$ as in Definition 4.4.7. See Definition 5.1.11 and Theorem 5.1.13.

In §5.1C we discuss precofibrations (called **flat maps** in [HHR16, Definition B.9]), flat functors and flat objects. A **precofibration** (Definition 5.1.15) is a map that plays nicely with pushouts. A **flat functor** (Definition 5.1.20) between cocomplete homotopical categories is one that is homotopical and preserves colimits. When we have a monoidal structure $\wedge$, we define a **flat object** X to be one for which the functor $X \wedge (-)$ is flat. Being able to identify flat objects in a model category can be very helpful, as we shall see below in Chapter 9.

Cofibrantly and compactly generated model categories are the subject of §5.2. These are model categories in which we specify sets of morphisms $\mathcal{I}$ and $\mathcal{J}$ (**cofibrant generating sets**) which in some sense generate all of the cofibrations and trivial cofibrations respectively, and therefore, along with the homotopical structure, determine the entire model structure. Experience has shown this to be a convenient way to describe a model structure. **We will use it in all model categories in this book from now on.**

In theory, one could dualize this notion and define **fibrantly generated model categories** in which fibrations and trivial fibrations are generated by specified sets, but we know of no work where this is done. **The study of model categories is self dual in theory but not in practice.**

From a technical standpoint, the advantage of cofibrant generation is the following. If one want to show that a morphism has the right lifting property with respect to all (trivial) cofibrations, it is enough to show that it has with respect to all morphisms in the set $\mathcal{I}$ ($\mathcal{J}$). Generating sets for $\mathcal{T}op$ and $\mathcal{T}$ are given in Example 5.2.9.

The notion of cofibrant generation leads to the following question. Given a bicomplete homotopical category $\mathcal{M}$, when does a pair of morphism sets $\mathcal{I}$ and $\mathcal{J}$ lead to a model structure? This question is answered by the Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24.

Suppose we have a cofibrantly generated model category $\mathcal{M}$ and an adjunction

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left, "U"] \end{tikzcd}$$

with $\mathcal{N}$ bicomplete. It is a **transfer adjunction** if it meets certain technical conditions given in Definition 5.2.25. The Crans-Kan Transfer Theorem 5.2.27 says there is a cofibrantly generated model structure on $\mathcal{N}$ in which a map in $\mathcal{N}$ is a weak equivalence if its image under U is one, and with cofibrant generating sets $F\mathcal{I}$ and $F\mathcal{J}$. **We will use this several times later in the book to construct new model categories.**

One example is Theorem 5.2.34, where we describe a way to enlarge the class of cofibrations (and therefore shrink the class of fibrations) in a model category without changing the class of weak equivalences. We will use this in the category $\mathcal{S}p^G$ of orthogonal G-spectra as indicated in Model structure conditions 9.0.4(i).

Proper model categories. A model category is left (right) proper if the pushout (pullback) of any weak equivalence along a cofibration (fibration) is again a weak equivalence, as explained in Definition 5.3.1. In §5.3 we study proper model categories, that is ones that are both left and right proper. Many model categories of interest, including $\mathcal{T}op$, $\mathcal{T}$, $\mathcal{S}et_{\boldsymbol{\Delta}}$ and its pointed analog, are proper.

Recall that in $\mathcal{T}$ a morphism is a weak equivalence if it induces an isomrphism in all homotopy groups. This is generalized in Theorem 5.6.21 as follows. Let $\mathcal{M}$ be a left proper cofibrantly generated (pointed) topological model category. Then a morphism $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence iff

$$f_* : \mathcal{M}(K, X) \to \mathcal{M}(K, Y)$$

is a weak equivalence of (pointed) topological spaces for each K that is a domain or codomain of a generating cofibration. In the case $\mathcal{M} = \mathcal{T}$, such Ks are spheres and disks, so we get our condition about homotopy groups.

Functor categories. In §5.4 we study the category $\mathcal{M}^J$ of functors from a small category J (the **indexing category**) to a cofibrantly generated model category $\mathcal{M}$. We need this because we will see Chapter 7, specifically in §7.2, that various categories of spectra can be described in such terms. $\mathcal{M}^J$ has a model structure, called the **projective** one, in which a morphism (meaning natural transformation between functors) $f : X \to Y$ is a weak equivalence or a fibration if its jth component is one for each object j in J. We can then define cofibrations in terms of left lifting properties.

This theory works best when $\mathcal{M}$ is closed symmetric monoidal, and J along with the category of $\mathcal{M}$-valued functors on it are enriched over $\mathcal{M}$. Strictly speaking we cannot discuss this until we have developed the notion of a monoidal model category in §5.5, culminating with Definition 5.5.9. The monoidal and model structures need to mesh in a certain way. We also need to know what it means for a one model category to be enriched over a symmetric monoidal one, which is the subject of §5.6. Thus the formal theory of enriched functor categories will have to wait until §5.6B, when we will have the needed technical tools.

Even when J and $\mathcal{M}$ are ordinary categories, they are enriched over $\mathcal{S}et$, and $\mathcal{M}$ is bitensored over it (as in Definition 3.1.31) since it is bicomplete as in Definition 2.3.25. Thus for each object j in J, the Yoneda functor よ^j (which sends each j' in J to the set $J(j, j')$) is $\mathcal{S}et$-valued. Furthermore its product with other objects and morphisms in $\mathcal{M}$ is defined. We will see in Theorem 5.4.10 that cofibrant generating sets for $\mathcal{M}^J$ can be formed by taking the set of products of these Yoneda functors よ^j with maps the cofibrant generating sets $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{M}$. Its enriched analog is Theorem 5.6.26.

Now suppose that J has a full subcategory K with inclusion functor $\alpha : K \to J$. The latter induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$. It has a left adjoint $\alpha_!$ which sends a functor $K \to \mathcal{M}$ to its left Kan extension along α. In Theorem 5.4.21 we show that $(\alpha_!, \alpha^*)$ is a transfer adjunction. We can use it to transfer the projective model structure on $\mathcal{M}^K$ to get a new model structure on $\mathcal{M}^J$. In it a morphism is a weak equivalence or a fibration if its kth component is one for each object k in the subcategory $K \subseteq J$. These conditions are weaker than those for the projective model structure, which involve **all** components of the morphism. Hence we have more weak equivalences and fibrations and therefore **fewer cofibrations** than before. For this reason we call the new model structure on $\mathcal{M}^J$ a **confinement** of the projective one. In the extreme case when the subcategory K is empty, all maps in $\mathcal{M}^J$ are fibrations and weak equivalences, and the only cofibrations are isomorphisms.

We give an enriched version of Theorem 5.4.21 in Theorem 5.6.38. The relation between confinement and enlargement is studied in §5.4D. Its application to G-spectra is the subject of Example 5.4.33.

Monoidal model categories, which we call **Quillen rings**, are the subject of §5.5. Let $(\mathcal{M}, \wedge, S)$ be a model category with a monoidal structure. This choice of notation reflects a bias toward pointed topological model categories equipped with a smash product. A monoidal structure in a cocomplete category enables one to define the pushout product $f \square g$ of morphisms f and g as in Definition 2.6.12. In the **pushout product axiom** of Definition 5.5.9, we require it to be a cofibration when f and g are cofibrations.

In many examples, such as the usual smash product in $\mathcal{T}$, the unit object S is cofibrant, **but we do not require this**. Indeed, for reasons explained below in Remark 7.0.7(ii), we will need a model structure on the category of spectra in which the unit object S^{-0} (the sphere spectrum) is **not** cofibrant. Instead of requiring S to be cofibrant, we require that the smash product of its cofibrant approximation $QS \to S$ with any cofibrant object be a weak equivalence. This is the **unit axiom** of Definition 5.5.9.

A third requirement, the **monoid axiom** of Definition 5.5.22, was formulated by Schwede and Shipley in [SS00, Definition 3.3]. It says that the smash product of any object with a trivial cofibration is a weak equivalence, and likewise for any morphism that is obtained as a transfinite composition of pushouts of such maps. They use it to construct MSs on certain categories associated with a monoidal model category in [SS00, Theorem 4.1], which is stated below as Theorem 5.5.25.

Given a Quillen ring $(\mathcal{M}, \wedge, S)$, we define modules and algebras over it in Definition 5.5.17. Suppose we have such a module $\mathcal{N}$ with a cofibrant object C and fibrant object X. In **??** we show that the morphism object $\mathcal{N}(C, X)$ is fibrant and that certain maps between such objects are fibrations.

Enriched model categories are introduced in §5.6. In an ordinary model category $\mathcal{N}$ one is concerned with lifting diagrams (2.3.11), namely

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y. \end{tikzcd}$$

In the enriched setting we do not have individual morphisms that can commute in this way. We need to reformulate the above in terms of maps between morphism objects in $\mathcal{N}$. This is done in Definition 5.6.3, which is a restatement of Definition 5.5.17. We are primarily interested **pointed topological model categories**, meaning model categories enriched over variants of $\mathcal{T}$. Certain notions associated with $\mathcal{T}$ can be extended to them.

For an object X in such a category $\mathcal{M}$, we can define $\pi_0 X$ to be the pointed set of path connected components in the pointed space $\mathcal{M}(S^0, X)$. In Definition 5.6.7 we define h-cofibrations in $\mathcal{M}$. In Proposition 5.6.11 we show that π_0 of a sequential colimit of h-cofibrations behaves as expected. In Example 5.6.12 we discuss suspension and loop functors in such categories.

In Corollary 5.6.16 we show that topological model categories are special cases of simplicial ones. This is convenient because most of the literature on model categories concerns the simplicial case.

In Lemma 5.6.17, Theorem 5.6.21 and Corollary 5.6.24, we indicate useful technical tools for identifying weak equivalences in terms of maps from cofibrant to fibrant objects.

In §5.6C we will show that modifying the model structure on a Quillen ring in certain ways, without altering the monoidal structure, leads to another Quillen ring which satisfies the monoid axiom if the original one did.

- Theorem 5.6.34 says this about enlarging a model structure as in Theorem 5.2.34.
- Theorem 5.6.39 says this about confining a model structure as in Theorem 5.4.21.

Theorem 5.6.35 is a similar statement about the monoidal structure on a functor category $[\mathscr{J}, \mathcal{M}]$ given by the Day convolutions, when $\mathcal{M}$ is a Quillen ring and $\mathscr{J}$ is a symmetric monoidal category enriched over it.

In Chapter 9 we will see that there are eight different model structures on the category of orthogonal G-spectra. The one monoidal structure provided by the Day convolution is compatible with each of them, so we get eight different Quillen rings.

Homotopy limits and colimits are the subject of §5.8. As we saw in §4.4, an objectwise weak equivalence of diagrams, that is of $\mathcal{M}$-valued functors on a (small) indexing category J, may not lead to a weak equivalence of limits or colimits. Example 2.3.66 and Example 2.3.65 show that sequential limits and colimits can behave in unexpected ways. A related difficulty is the failure of the homotopy category $\mathrm{Ho}\,\mathcal{C}$ of a model category $\mathcal{C}$ to have limits and colimits.

The construction of homotopy limits and colimits is designed to address these problems. After defining them, there are two questions one can ask:

- Under what circumstances are they weakly equivalent to the corresponding ordinary limits and colimits? We will see that there is a canonical map (5.8.2) from the limit to the homotopy limit, and dually for colimits (5.8.3). Theorem 5.8.16 below is a partial answer to this question.
- When are they homotopy invariant, meaning when does an objectwise weak equivalence of diagrams induce a weak equivalence of their homotopy limits or colimits?

In §5.8A we give the original definition of Bousfield and Kan [BK72, Chapter XI] in Definition 5.8.1. We list some standard examples in Example 5.8.5. Their homotopy invariance properties are discussed in § 5.8B. The effect of changing the indexing category on a homotopy limit or colimit is the subject of §5.8D.

In §5.8E we specialize to the case where diagram is indexed by a generalized direct $\mathcal{M}$-category as in Definition 5.6.31. This includes the case of telescopes, which are homotopy sequential colimits. They figure prominently in our study of spectra starting in Chapter 7.

In §5.8F we discuss homotopy limits in right proper model categories as in Definition 5.3.1. We define homotopy pullbacks in Definition 5.8.30, homotopy fiber squares in Definition 5.8.38, and homotopy fibers in Definition 5.8.42.

5.1 Homotopical categories

In [DHKS04] (the "blue beast"), Dan Kan and three of his former students, Bill Dwyer, Phil Hirschhorn and Jeff Smith, initiated the study of homotopical categories. Roughly speaking, these are model categories for which fibrations and cofibrations have not yet been defined. The authors wanted to see how much of the theory could be deduced from having only defined weak equivalences. Summaries of their work can be found in [Shu06, §2-4] and [Rie14, Chapter 2]. It is relevant for us because in the categories of spectra (Chapter 7) and G-spectra (Chapter 9) we know what the weak equivalences are (see Definition 7.0.9), but there is more than one plausible way to define a model structure.

The material in the first two subsections below is similar to that of [HHR16, §B.1], while that of §5.1C matches [HHR16, §B.2].

5.1A Basic definitions

Definition 5.1.1. *A* **homotopical category** *is a category $\mathcal{M}$ equipped with a* **wide** *subcategory $\mathscr{W}$ ("wide" meaning every object of $\mathcal{M}$ is in $\mathscr{W}$) whose morphisms (the weak equivalences) satisfy the* **2-of-6 property***: given a diagram of the form*

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$$

with gf and hg in $\mathscr{W}$, the morphisms f, g, h and hgf are also in it. It is a **minimal homotopical category** *if in addition the only morphisms in $\mathscr{W}$ are isomorphisms.*

A **homotopy functor** *$F : \mathcal{M} \to \mathcal{C}$ is one that sends weak equivalences to isomorphisms. A* **homotopical functor** *between homotopical categories is one that preserves weak equivalences. A* **homotopical equivalence** *between homotopical categories is a pair of homotopical functors as in Definition 2.2.4. We will refer to $\mathscr{W}$, the collection of weak equivalences, as a* **homotopical structure** *on $\mathcal{M}$.*

The definition of a homotopical category above is taken from [Rie14, 2.1.1]. In the original definition of [DHKS04, 7.5], repeated as [Shu06, 2.1], $\mathscr{W}$ is a class of morphisms satisfying the 2-of-6 property and **containing all identity morphisms**. The two definitions are equivalent.

We are using the symbol $\mathcal{M}$ to suggest that a homotopical category is somewhat like a model category, but note that the definition has no requirement of completeness or cocompleteness.

The following is proved in [DHKS04, Proposition 9.2].

Proposition 5.1.2. Every model category is homotopical.

Definition 5.1.3. Functors from a small category to a homotopical one. *Let J be a small category and $\mathcal{M}$ a homotopical category. Given a functor $X : J \to \mathcal{M}$, we denote its value on an object j of J by X_j, and similarly for natural transformations between such functors. We define a the* **strict homotopical structure** *on the functor category $\mathcal{M}^J$ by saying that morphism $f : X \to Y$ weak weak equivalence if f_j is one for each j in J.*

The following is an immdediate consequence of the definitions above.

Proposition 5.1.4. Homotopical equivalences between functor categories. *For a homotopical category $\mathcal{M}$, an equivalence of small categories $J \to K$ induces a homotopical equivalence $\mathcal{M}^K \to \mathcal{M}^J$.*

An isomorphism f is necessarily a weak equivalence, as we see from the diagram

$$\bullet \xrightarrow{f} \bullet \xrightarrow{f^{-1}} \bullet \xrightarrow{f} \bullet.$$

Hence every category has a **minimal homotopical structure** in which the weak equivalences are the isomorphisms.

The 2-of-6 condition implies the 2-of-3 property required of weak equivalences in a model category, as we see from the diagrams

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet = \bullet, \qquad \bullet \xrightarrow{f} \bullet = \bullet \xrightarrow{g} \bullet \qquad \text{and} \qquad \bullet = \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet.$$

It is known that every model category satisfies this condition and is therefore underlain by a homotopical category. This is proved in [DHKS04, 9.3] as follows. They modify the model category axioms of §4.1 by strengthening the 2-of-3 condition of **MC2** to the 2-of-6 condition, and dropping the requirement of **MC3** that retractions preserve weak equivalences. Then they show that their modified axioms are equivalent to those of §4.1.

Proposition 5.1.5. Homotopical structures defined by functors. *For any category $\mathcal{M}$ with a functor F to a homotopical category $\mathcal{C}$, the class of morphisms in $\mathcal{M}$ mapping to weak equivalences in $\mathcal{C}$ defines a homotopical structure on $\mathcal{M}$, which we will say is* **defined by** *F.*

In particular for any category $\mathcal{C}$, the class of morphisms in $\mathcal{M}$ mapping to isomorphisms in $\mathcal{C}$ under F defines a homotopical structure on $\mathcal{M}$.

Proof The 2-of-6 condition for weak equivalences in $\mathcal{C}$ implies it for maps in $\mathcal{M}$ that map to weak equivalences in $\mathcal{C}$. □

Indeed this is how weak equivalences are defined in the three classical model categories of §4.2. Stable equivalences of spectra are by definition maps inducing isomorphisms of stable homotopy groups or equivalences of morphism spaces into Ω-spectra.

The following is a tool for producing a homotopical structure on a category $\mathcal{M}$ via a functor to another homotopical category $\mathcal{C}$. If $\mathcal{M}$ had a homotopical structure to begin with, it could acquire another one. This will be particularly helpful in §9.3 below.

Proposition 5.1.6. Homotopical structures for equivalent categories. *Suppose we have functors $F : \mathcal{M} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{M}$ inducing an equivalence of categories as in Definition 2.2.4 with $\mathcal{C}$ homotopical. Then $\mathcal{M}$ has a homotopical structure defined by F as in Proposition 5.1.5, and the categorical equivalence is homotopical as in Definition 5.1.1.*

Proof We need to show that both functors preserve weak equivalences. The homotopical structure on $\mathcal{M}$ is defined so that F is homotopical. To show that G is homotopical, let $g : X \to Y$ be a weak equivalence in $\mathcal{C}$. Then Gg is by definition a weak equivalence in $\mathcal{M}$ iff FGg is one in $\mathcal{C}$. Since the categories are equivalent, there is a natural equivalence $\epsilon : FG \Rightarrow 1_{\mathcal{C}}$. This means we have the following diagram in $\mathcal{C}$.

$$\begin{tikzcd} FG(X) \arrow[r, "\epsilon_X", "\cong"'] \arrow[d, "FG(g)"'] & X \arrow[d, "g"] \\ FG(Y) \arrow[r, "\epsilon_Y", "\cong"'] & Y \end{tikzcd}$$

This means

$$FG(g) = (\epsilon_Y)^{-1} g \epsilon_X,$$

making it the composite of three weak equivalences and hence a weak equivalence itself. □

The following is proved as [DHKS04, 14.5], and in slightly different language, as [Hov99, Lemma 1.1.12] and [Hir03, Corollary 7.7.2].

Ken Brown's Lemma 5.1.7. *Let $F : \mathcal{M} \to \mathcal{C}$ be a functor from a model category $\mathcal{M}$ to a homotopical category $\mathcal{C}$, e.g., to another model category. If it sends trivial cofibrations between cofibrant objects (trivial fibrations between fibrant objects) to weak equivalences, then it sends* **all weak equivalences** *between cofibrant (fibrant) objects to weak equivalences.*

It would be nice to generalize this to the case where $\mathcal{M}$ is a homotopical category and F sends "trivial precofibrations" (weak equivalences which are precofibrations as in Definition 5.1.15 below) between precofibrant objects in to weak equivalences in $\mathcal{C}$, but we do not know how to prove it. The proof of Ken Brown's Lemma 5.1.7 uses factorization in the model category $\mathcal{M}$, which we do not have in a homotopical category.

5.1B Deformations and derived functors

A homotopical category $\mathcal{M}$ has **a homotopy category** $\operatorname{Ho}\mathcal{M}$ with a localization functor $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ as in Definition 4.3.15, subject to the set theoretic difficulties cited there. For the minimal homotopical structure (in which weak equivalences are isomorphisms), $\operatorname{Ho}\mathcal{M}$ is $\mathcal{M}$ itself. A homotopy functor $\mathcal{M} \to \mathcal{C}$ factors uniquely through $\operatorname{Ho}\mathcal{M}$; see Corollary 5.1.10 below. A homotopical functor $F : \mathcal{M} \to \mathcal{N}$ induces a functor between their homotopy categories.

Proposition 5.1.8. Homotopy functors and the homotopy category. *The transformation $\mathcal{M}(X,-) \to \operatorname{Ho}\mathcal{M}(X,-)$ induced by γ is the universal natural transformation from $\mathcal{M}(X,-)$ to a homotopy functor.*

Proof The assertion is that if $F : \mathcal{M} \to \mathcal{S}et$ is a homotopy functor and $\mathcal{M}(X,-) \Rightarrow F$ a natural transformation, then there is a unique dotted arrow making the diagram

$$\begin{tikzcd} \mathcal{M}(X,-) \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\gamma"'] & F \\ \operatorname{Ho}\mathcal{M}(\gamma X, \gamma(-)) \arrow[ur, Rightarrow, dashed] & \end{tikzcd} \tag{5.1.9}$$

commute. Before describing the proof we make an observation about the property characterizing the functor $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$. For homotopy functors F and G on $\mathcal{M}$, this property supplies unique factorizations $F = \widetilde{F}\circ\gamma$ and $G = \widetilde{G}\circ\gamma$. It also implies that composition with γ gives a bijection between the set of natural transformations $\widetilde{G} \to \widetilde{F}$ and $G \to F$.

With this in mind we now turn to the proof of the proposition. By the Yoneda Lemma 2.2.10, the horizontal arrow in (5.1.9) is given by an element of $F(X)$. By the observation above, the set of natural transformations

$$\operatorname{Ho}\mathcal{M}(\gamma X, \gamma(-)) \to F$$

is in bijection with the set of natural transformations

$$\operatorname{Ho}\mathcal{M}(\gamma X, -) \to \widetilde{F}$$

which, again by Yoneda, is in one to one correspondence with the elements of $\widetilde{F}(\gamma X) = F(X)$. The map between these sets corresponding to the two ways of going around (5.1.9) is the identity. $\square$

Corollary 5.1.10. Homotopy functors on $\mathcal{M}$ factor through its homotopy category. *Suppose that $\mathcal{M}$ is a homotopical category, and that $X \in \mathcal{M}$ has the property that $\mathcal{M}(X,-)$ is a homotopy functor. Then the natural transformation $\mathcal{M}(X,-) \to \operatorname{Ho}\mathcal{M}(X,-)$ is a bijection.*

Now let $F : \mathcal{M} \to \mathcal{N}$ be a functor that is not necessarily homotopical. In favorable cases we can define **derived functors** LF and RF as in Definition 4.4.5, and **total derived functors** $\mathbf{L}F$ and $\mathbf{R}F$ as in Definition 4.4.7. However our previously stated existence results for them, Proposition 4.4.6 and Proposition 4.4.8, do not apply here because they are stated in terms of fibrations and cofibrations. They say that a left/right derived (or total left/right derived) functor exists if the original functor behaves well on cofibrant/fibrant objects.

Recall that each object X in a model category has a cofibrant replacement $X^c \to X$ and a fibrant replacement $X \to X^f$ with both maps being weak equivalences that are functorial in X. In the homotopical setting we seek similar functors from or to a subcategory of "good" objects on which the functor we are trying to derive behaves well. **For the rest of this subsection we shall only concern ourselves with left derived functors and related notions, leaving the formulation of their right analogs as exercises for the reader.**

With this in mind we have the following.

Definition 5.1.11. *A* **left deformation** *on a homotopical category $\mathcal{M}$ is a functor $Q : \mathcal{M} \to \mathcal{M}$ (denoted by R in the right case) together with a natural transformation $q : Q \Rightarrow 1$ inducing a weak equivalence on each object. We will abusively say that objects in the image of Q are* **cofibrant**, *even though $\mathcal{M}$ does not have a model structure. A* **left deformation retract** *$\mathcal{M}_Q \subseteq \mathcal{M}$ is the full subcategory of objects in the image of Q. It is a* **left F-deformation retract** *if the restriction to $\mathcal{M}_Q$ of a functor F defined on $\mathcal{M}$ is homotopical.*

A **left deformation of a functor** *$F : \mathcal{M} \to \mathcal{N}$ of homotopical categories is a left deformation on $\mathcal{M}$ such that F is homotopical on an associated subcategory M_Q of cofibrant objects. When F admits a left deformation, we say that F is* **left deformable**.

Example 5.1.12. Adding a whisker as a left deformation. *Suppose F is a functor on $\mathcal{T}$ that is homotopical on spaces with nondegenerate base point. Then the functor $X \mapsto \widetilde{X}$ of Definition 3.5.29 is a left deformation of F.*

The functor Q is always homotopical. When $\mathcal{M}$ is a model category, cofibrant replacement is a left deformation for any left Quillen functor (Definition 4.5.1) F. The notation is meant to suggest that Q is a generalization of cofibrant replacement.

A proof of the following can be found in [DHKS04, 41.2-5] and [Rie14, 2.2.8].

Theorem 5.1.13. Existence of a left derived functor. *If a functor $F : \mathcal{M} \to \mathcal{N}$ between homotopical categories has a left deformation Q, then FQ is a left derived functor of F.*

5.1C Precofibrations, flat functors and flat objects

The precofibrations that we will define here (called **flat maps** in [HHR16, Definition B.9]) will be used in our study of G-spectra in Chapter 9, in particular in our teatment of indexed smash products in §9.6B.

We give the following model category definition here for the reader's convenience. Model categories with this property are the subject of §5.3 and §5.8F below.

Definition 5.1.14. Proper model categories. *A model category is* **left proper** *if the pushout of any weak equivalence along a cofibration is again a weak equivalence. In other words, given a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

where h is a weak equivalence and f is a cofibration, k is also a weak equivalence. There is a dual notion of **right proper** *that involves fibrations and pullbacks, and a model category with both properties is said to be simply* **proper***.*

In general a bicomplete homotopical category $\mathcal{M}$ may have more than one model structure. The classes of cofibrations and fibrations are not determined by the homotopical structure. However there is a property shared by cofibrations in many model categories, such as left proper ones, which can be described in terms of the homotopical structure alone, assuming cocompleteness. We call such maps **precofibrations**. This suggests a class of preferred objects (precofibrant objects) analogous to cofibrant objects in a model category.

Definition 5.1.15. Precofibrations and related notions.

(i) A **precofibration** *$f : A \to B$ in a cocomplete homotopical category is a morphism with the property that for every map $A \to C$ and every weak equivalence $h : C \to C'$, the induced map of pushouts, i.e., the map k in*

$$\begin{tikzcd} C \arrow[d] \arrow[rr, bend left, "h", "\simeq"'] & A \arrow[l, "g"'] \arrow[r, "g'"] \arrow[d, "f"] & C' \arrow[d] \\ B \cup_A C & B \arrow[l] \arrow[r] & B \cup_A C', \arrow[from=2-1, to=2-3, bend right, "k"'] \end{tikzcd} \tag{5.1.16}$$

is also a weak equivalence. A **trivial precofibration** *is a precofibration that is also a weak equivalence.*

(ii) An object in such a category is **precofibrant** *if the map to it from the initial object is a precofibration. This means its coproduct with any weak equivalence is again a weak equivalence.*

(iii) Such a category **has enough precofibrants** *if each object in it admits a weak equivalence from a precofibrant object.*

Remark 5.1.17. Cocompleteness is more than we need *for this definition to make sense. The only colimits we need are the pushouts in (i) and the initial object in (ii).*

If f is a cofibration in a model category, then so are the other two vertical maps in (5.1.16) by Proposition 4.1.12.

Remark 5.1.18. Precofibrant objects *are discussed by Michael Batanin and Clemens Berger in [BB17]. They use the term "h-cofibration," short for homotopical cofibration and not to be confused with h-cofibration as in Definition 3.5.6, for precofibration in [BB17, Definition 1.1].*

[BB17, Lemma 1.4] says (i) An object Z is precofibrant if and only if $Z \vee (-)$ preserves weak equivalences.

(ii) The class of weak equivalences is closed under finite coproducts if and only if all objects of the model category are precofibrant.

(iii) The class of weak equivalences is closed under arbitrary coproducts whenever all objects are precofibrant and weak equivalences are closed under filtered colimits along coproduct injections.

The following is an immediate consequence of the two preceding definitions.

Proposition 5.1.19. *A model category is left proper as in Definition 5.1.14 iff* **its cofibrations are precofibrations***, **its cofibrant objects are precofibrant***, and* **it has enough precofibrants**.

Definition 5.1.20. *A* **flat functor** $F : \mathcal{M} \to \mathcal{N}$ *between cocomplete homotopical categories is a functor that is homotopical and preserves colimits. A* **flat object** X *in a cocomplete monoidal homotopical category $(\mathcal{M}, \wedge, S)$ is one for which the endofunctor $X \wedge (-)$ is flat.*

In homological algebra one defines Tor in terms of projective resolutions. It can also be defined in terms of flat resolutions, where an R-module M is flat if the functor $M \otimes_R (-)$ preserves exactness, or equivalently it preserves monomorphisms.

The following is an exercise for the reader.

Proposition 5.1.21. Properties of precofibrations.

(i) Limits and colimits of precofibrations are precofibrations.

(ii) Composites (including transfinite ones) of precofibrations are precofibrations.

(iii) Any cobase change (see §2.3A) of a precofibration is a precofibration.

(iv) If a retract of a weak equivalence is a weak equivalence, then a retract of a precofibration is precofibration. In particular, this is true when $\mathcal{M}$ is a model category.

Proposition 5.1.22. Precofibrations and pushouts. *Suppose that*

$$\begin{tikzcd} X_1 \arrow[d, "\simeq"'] & A_1 \arrow[l, "\flat"'] \arrow[r, "\flat"] \arrow[d, "\simeq"] & Y_1 \arrow[d, "\simeq"] \\ X_2 & A_2 \arrow[l] \arrow[r, "\flat"] & Y_2 \end{tikzcd} \tag{5.1.23}$$

is a diagram in which the morphisms denoted by the musical flat symbol $\flat$ are precofibrations. If the vertical maps are weak equivalences, then so is the map

$$X_1 \underset{A_1}{\cup} Y_1 \to X_2 \underset{A_2}{\cup} Y_2$$

of pushouts.

Proof First suppose that $A_1 = A_2 = A$. Then

$$X_1 \underset{A}{\cup} Y_1 \to X_1 \underset{A}{\cup} Y_2$$

is a weak equivalence since $A \to X_1$ is a precofibration. The map $X_1 \to X_1 \underset{A}{\cup} Y_2$ is a precofibration, since it is a cobase change of $A \xrightarrow{\flat} Y_2$ along $A \to X_1$. But this implies that

$$X_1 \underset{A}{\cup} Y_2 \to X_2 \underset{X_1}{\cup} \left(X_1 \underset{A}{\cup} Y_2\right) = X_2 \underset{A}{\cup} Y_2.$$

is a weak equivalence. Putting these together gives the result in this case.

For the general case, consider the following diagram

$$\begin{tikzcd} X_1 \arrow[d] \arrow[dd, bend right=60, "\simeq"'] & A_1 \arrow[l, "\flat"'] \arrow[r, "\flat"] \arrow[d, "\simeq"] & Y_1 \arrow[d] \arrow[dd, bend left=60, "\simeq"] \\ X_1 \underset{A_1}{\cup} A_2 \arrow[d] & A_2 \arrow[l] \arrow[r] \arrow[d, equal] & A_2 \underset{A_1}{\cup} Y_1 \arrow[d] \\ X_2 & A_2 \arrow[l] \arrow[r, "\flat"] & Y_2 \,. \end{tikzcd}$$

The fact that the maps $A_1 \to X_1$ and $A_1 \to Y_1$ are precofibrations implies that the upper vertical maps (hence all the vertical maps) are weak equivalences, and that the maps in the middle row are precofibrations. It also implies that

$$A_1 \to X_1 \underset{A_1}{\cup} Y_1$$

is a precofibration. Since $A_1 \to A_2$ is a weak equivalence, this means that

$$X_1 \underset{A_1}{\cup} Y_1 \to A_2 \underset{A_1}{\cup} \left(X_1 \underset{A_1}{\cup} Y_1 \right)$$

is a weak equivalence. But this is the map from the pushout of the top row of (5.1.23) to the pushout of the middle row. By the case in which $A_1 = A_2$, the map from the pushout of the middle row to the pushout of the bottom row is a also a weak equivalence. This completes the proof. □

Proposition 5.1.24. Precofibrations in factorizations. *If $\mathcal{M}$ has the property that every map can be factored into a precofibration followed by a weak equivalence, then Proposition 5.1.22 holds with the assumption that only one of the maps in the top row of (5.1.23) is a precofibration.*

Proof Suppose that the map $A_1 \to X_1$ is a precofibration, and factor $A_1 \to Y_1$ into a precofibration $A_1 \to Y_1'$ followed by a weak equivalence $Y_1' \to Y_1$ Now consider the diagram

$$\begin{tikzcd}
X_1 \arrow[d, equal] & A_1 \arrow[l, "\flat"'] \arrow[r, "\flat"] \arrow[d, equal] & Y_1' \arrow[d, "\sim"] \\
X_1 \arrow[d, "\sim"'] & A_1 \arrow[l, "\flat"'] \arrow[r] \arrow[d, "\sim"] & Y_1 \arrow[d, "\sim"] \\
X_2 & A_2 \arrow[l] \arrow[r, "\flat"'] & Y_2 .
\end{tikzcd}$$

By Proposition 5.1.22, the map from the pushout of the top row to the pushout of the middle row is a weak equivalence, as is the map from the pushout of the top row to the pushout of the bottom row. The map from the pushout of the middle row to the pushout of the bottom row is then a weak equivalence by the two out of three property of weak equivalences. □

Remark 5.1.25. Precofibrations in the category of G-spectra. *In the category $\mathcal{S}p^G$ equipped with the stable weak equivalences (Definition 9.0.2 and Proposition 9.1.4), the h-cofibrations (Definition 5.6.7) will turn out to be precofibrations; see Theorem 7.3.13 below. The mapping cylinder construction then factors every map into a precofibration followed by a weak equivalence, so Proposition 5.1.24 applies.*

We will use the following below in Proposition 9.6.6.

Proposition 5.1.26. Smashing with weak equivalences of flat objects. *Suppose that every object Z in a symmetric monoidal homotopical category $(\mathcal{M}, \wedge, S)$ admits a weak equivalence $\widetilde{Z} \to Z$ from a flat object $\widetilde{Z}$ as in Definition 5.1.20. If $X \to Y$ is a weak equivalence of flat objects, then $X \wedge Z \to Y \wedge Z$ is a weak equivalence for any Z.*

Proof This follows from the diagram

$$\begin{tikzcd}
X \wedge \widetilde{Z} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & X \wedge Z \arrow[d] \\
Y \wedge \widetilde{Z} \arrow[r, "\simeq"] & Y \wedge Z.
\end{tikzcd}$$

□

5.2 Cofibrantly and compactly generated model categories

5.2A Generating sets of cofibrations and trivial cofibrations

In any model category the fibrations are those morphisms having the right lifting property with respect to all trivial cofibrations, and the trivial fibrations are those morphisms have the right lifting property with respect to all cofibrations. The cofibrations are determined by the fibrations in a similar manner. Every weak equivalence can be factored as a trivial cofibration followed by a trivial fibration. This means that the model structure is determined by any two of the following three collections of morphisms:

- weak equivalences ($\mathscr{W}$)
- cofibrations ($\mathscr{C}$) and trivial cofibrations ($\mathscr{W} \cap \mathscr{C}$)
- fibrations ($\mathscr{F}$) and trivial fibrations ($\mathscr{W} \cap \mathscr{F}$).

It is often convenient to define a model structure by identifying the weak equivalences and a minimal set $\mathcal{I}$ ($\mathcal{J}$) of (trivial) cofibrations with domains that are small in the sense of Definition 4.8.8, which generate all the others in the sense of Definition 4.8.13.

As far as we know, the following first appeared in [DHK97, Chapter 2]. That work was the precursor to [DHKS04].

Definition 5.2.1. *A model category $\mathcal{M}$ is* **cofibrantly generated** *if there are sets of morphisms $\mathcal{I}$, the* **set of generating cofibrations** *and $\mathcal{J}$, the* **set of generating trivial cofibrations***, each permitting the small object argument (Definition 4.8.2), such that*

(i) the class $\mathscr{F}$ of fibrations is $\mathcal{J}^{\boxslash}$, i.e., a map is a fibration iff it has the right lifting property (Definition 2.3.10) with respect to each morphism in $\mathcal{J}$ and
(ii) the class $\mathscr{W} \cap \mathscr{F}$ of trivial fibrations is $\mathcal{I}^{\boxslash}$.

We will refer to $\mathcal{I}$ and $\mathcal{J}$ as **cofibrant generating sets** *of $\mathcal{M}$. We will sometimes say that $\mathcal{M}$ is* **cofibrantly generated by** $(\mathcal{I}, \mathcal{J})$*. As in (2.3.12), we will denote by cofib($\mathcal{I}$) (cofib($\mathcal{J}$)) the classes ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ (${}^{\boxslash}(\mathcal{J}^{\boxslash})$), that of cofibrations (trivial cofibrations) in $\mathcal{M}$.*

The class $\mathscr{C}$ ($\mathscr{W} \cap \mathscr{C}$) of (trivial) cofibrations, that is $\mathrm{cofib}(\mathcal{I})$ ($\mathrm{cofib}(\mathcal{J})$), is easily seen to contain the regular and saturated classes (as in Definition 4.8.13) generated by $\mathcal{I}$ ($\mathcal{J}$). Morphisms in the regular class generated by $\mathcal{I}$ ($\mathcal{J}$) are called **regular $\mathcal{I}$-cofibrations ($\mathcal{J}$-cofibrations)** in [DHK97, 7.2(iii)] and elsewhere. See [Hir03, Proposition 11.2.1] for a proof of the following, which says that all cofibrations are in the saturated class.

It also makes it much easier to determine whether a given morphism is a (trivial) fibration.

Proposition 5.2.2. The set $\mathcal{I}$ ($\mathcal{J}$) generates all (trivial) cofibrations. *In a cofibrantly generated model category the class $\mathscr{C}$ ($\mathscr{W} \cap \mathscr{C}$) of (trivial) cofibrations is the saturated class (Definition 4.8.13) generated by $\mathcal{I}$ ($\mathcal{J}$).*

A similar definition could be made in terms of fibrations, but this comes up in practice far less frequently. It is discussed briefly in [DHK97, §7.6] and used in [Isa04] and in [BHK+15].

Remark 5.2.3. Notation for the sets of generating cofibrations and generating trivial cofibrations. *It is common in the literature to denote these sets by I and J. We prefer to use the symbols $\mathcal{I}$ and $\mathcal{J}$ (note the different font) so we can reserve I for the unit interval $[0,1]$ and J for a generic small category.*

Remark 5.2.4. One generating (trivial) cofibration is enough. *We could replace the typically infinite sets $\mathcal{I}$ and $\mathcal{J}$ of Definition 5.2.1 by singletons consisting in each case of the coproduct of all the maps in the original set. This would lead to the same structure since any retract of a (trivial) cofibration is a (trivial) cofibration. However it is usually more convenient to deal with the infinitely many maps in $\mathcal{I}$ and $\mathcal{J}$ one at a time.*

The following is an exercise for the reader.

Proposition 5.2.5. The product of two cofibrantly generated model categories. *Let $\mathcal{M}$ and $\mathcal{M}'$ be cofibrantly generated model categories with pairs of cofibrant generating sets $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{I}', \mathcal{J}')$. Then $\mathcal{M} \times \mathcal{M}'$ (see Definition 4.1.16) is a model category cofibrantly generated by the pair*

$$\left((\mathcal{I} \times *) \cup (* \times \mathcal{I}'), (\mathcal{J} \times *) \cup (* \times \mathcal{J}')\right),$$

where $*$ denotes the identity map in the terminal object in either category. A similar statement holds for any such product of cofibrantly generated model categories.

In other words, the generating sets for the product are the unions of those for the two factors.

The following is introduced by Mandell *et al* in [MMSS01, Definition 5.9].

Definition 5.2.6. *An object in a topological model category is* **compact** *if for any sequence of h-cofibrations (Definition 3.5.6)*

$$\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots ,$$

the map $\operatorname{colim} \mathcal{M}(A, X_n) \to \mathcal{M}(A, \operatorname{colim} X_n)$ *is a homeomorphism. A* **compactly generated model category** $\mathcal{M}$ *is a cofibrantly generated topological model category (Definition 5.2.1) in which each object A appearing as a domain or codomain in $\mathcal{I}$ and $\mathcal{J}$ is compact.*

This use of the term "compactly generated" is **not** the same as that of Definition 2.1.48 in connection with topological spaces.

This notion of compactness is a form of relative finiteness as in Definition 2.3.69. It is also a form of relative smallness (with respect to a finite cardinal and the subcategory of h-cofibrations) as in Definition 4.8.8.

In $\mathcal{T}op$ this definition of compactness is equivalent to the usual one.

Most model categories one encounters in practice are cofibrantly generated, and when they are topological they are compactly generated.

The following is a variant of [Hov01b, Definition 4.1], which Hovey attributes to Voevodsky.

Definition 5.2.7. *An object in a topological model category $\mathcal{M}$ is* **finitely presented** *if for any sequential diagram*

$$X_0 \to \cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots ,$$

the map $\operatorname{colim} \mathcal{M}(A, X_n) \to \mathcal{M}(A, \operatorname{colim} X_n)$ *is a homeomorphism.*

A cofibrantly generated topological model category $\mathcal{M}$ (Definition 5.2.1) with cofibrant genrating sets $\mathcal{I}$ and $\mathcal{J}$ is

(i) **finitely generated** *if each object A appearing as a domain or codomain in $\mathcal{I}$ and $\mathcal{J}$ is finitely presented, and*

(ii) **almost finitely generated** *if the domains and codomains of the generating cofibrations are finitely presented, and if there is a set of trivial cofibrations $\mathcal{J}'$ with finitely presented domains and codomains such that a map f whose codomain is fibrant is a fibration if and only if f has the right lifting property with respect to $\mathcal{J}'$.*

A more general notion of compactness that we will need in Definition 6.3.1 below is the following.

Definition 5.2.8. Compact objects relative to a set of morphisms $\mathcal{I}$. *An object W is* **compact** *relative to $\mathcal{I}$ if there is a cardinal γ such that for any relative $\mathcal{I}$-cell complex $X \to Y$ (meaning Y is obtained from X by attaching a sequence of "cells" via pushouts along morphisms in $\mathcal{I}$ as in (5.2.12) below),* *any map $W \to Y$ lifts to an object obtained from X by attaching at most γ cells.*

Example 5.2.9. Cofibrantly generated model structures on topological spaces, $\mathcal{T}op$, and pointed topological spaces, $\mathcal{T}$. *In $\mathcal{T}op$ (see §4.2A), let the set of generating cofibrations be*

$$\mathcal{I} = \{i_n : n \geqslant 0\} \qquad \text{where } i_n \text{ is the map } S^{n-1} = \partial D^n \to D^n \tag{5.2.10}$$

(with $D^0 = *$ and $\partial D^0 = \varnothing$), and let the set of generating trivial cofibrations be

$$\mathcal{J} = \{j_n : n \geqslant 0\} \qquad \text{where } j_n \text{ is the map } (\{0\} \to [0,1]) \times I^n. \tag{5.2.11}$$

Pushing out along one of the former, that is forming the pushout of a diagram of the form

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & X \\ D^n & \end{tikzcd} \tag{5.2.12}$$

is the same thing as attaching an n-cell to X with attaching map f. Thus one can produce all cofibrant objects (CW complexes) by starting with the terminal object (a point) and repeatedly (perhaps transfinitely) pushing out along the maps of (5.2.10). All cofibrations from an arbitrary space X can be obtained by repeatedly attaching cells, not necessarily in order of dimension. Such maps $X \to Y$ are called **relative CW complexes**. *This is a special case of Definition 4.8.18.*

The analogous sets for $\mathcal{T}$ are

$$\mathcal{I}_+ = \{i_{n+} : n \geqslant 0\} \qquad \text{and} \qquad \mathcal{J}_+ = \{j_{n+} : n \geqslant 0\} \tag{5.2.13}$$

where i_{n+} is the map $S^{n-1}_+ \to D^n_+$ and j_{n+} is the map $I^n_+ \to I^{n+1}_+$. The analog of (5.2.12) is the diagram

$$\begin{tikzcd} S^{n-1}_+ \arrow[r, "f_+"] \arrow[d] & X \\ D^n_+ & \end{tikzcd}$$

with $f_+|S^n = f$, for which the pushout is the same as that of (5.2.12). Thus the cofibrant objects are pointed CW complexes.

As in Definition 4.2.1, a **Serre fibration** *is a map with the right lifting property for all maps in (5.2.11) or (5.2.13).*

Example 5.2.14. A cofibrantly generated model structure on simplicial sets. *In $\mathcal{S}et_{\boldsymbol{\Delta}}$ with the Quillen model structure of Definition 4.2.17, a set of generating cofibrations is*

$$\mathcal{I}_{\Delta} = \left\{ i_n^{\Delta} \colon n \geqslant 0 \right\} \qquad \text{where } i_n^D \text{ is the map } \partial\Delta^n \to \Delta^n, \tag{5.2.15}$$

the inclusion of the boundary of the standard n-simplex as in Definition 3.4.2. Let the set of generating trivial cofibrations be

$$\mathcal{J}_{\Delta} = \left\{ j_{n,i}^{\Delta} \colon 0 \leqslant i \leqslant n \right\} \qquad \text{where } j_{n,i}^{\Delta} \text{ is the map } \Lambda_i^n \to \Delta^n, \tag{5.2.16}$$

where Λ_i^n is the ith horn as in Definition 3.4.2.

Remark 5.2.17. The Strøm model structure on $\mathcal{T}op$, *which was introduced in [Str72] and discussed in §4.2, in which the weak equivalences are actual homotopy equivalences, is known* **not** *to be cofibrantly generated. See [Rap10, Remark 4.7] and [BR13] for further discussion.*

Given a bicomplete homotopical category $\mathcal{M}$ (Definition 5.1.1 below), one can ask when two classes of morphisms $\mathcal{I}$ and $\mathcal{J}$ could serve as the generating sets of cofibrations and trivial cofibrations for a cofibrantly generated model structure on $\mathcal{M}$. The domains of both $\mathcal{I}$ and $\mathcal{J}$ must be small as in Definition 4.8.8. The following is proved as [Hir03, 11.2.9].

Proposition 5.2.18. The generating trivial cofibrations can be assumed to be relative $\mathcal{I}$-cell complexes. *Let $\mathcal{M}$ be a cofibrantly generated model category with a generating set $\mathcal{I}$ of cofibrations. If $\mathcal{J}$ is a generating set of trivial cofibrations, then there is a bijection of it with a set $\tilde{\mathcal{J}}$ having the same domains as $\mathcal{J}$ in which each map is a relative $\mathcal{I}$-cell complex as in Definition 4.8.18.*

This is proved by using the small object argument Theorem 4.8.1 based on $\mathcal{I}$ to factor the maps in $\mathcal{J}$.

5.2B Transferring a model structure from one category to another

The following definition and proposition are taken from [HKRS17, §2.1].

Definition 5.2.19. Right and left induced model structures. *Let $(\mathcal{M}, \mathscr{W}, \mathscr{C}, \mathscr{F})$ define a model category as in Definition 4.1.1, and suppose there are adjoint functors*

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "V"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "R"] \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "U"] \end{tikzcd}$$

where the categories $\mathcal{K}$ and $\mathcal{C}$ are bicomplete. If they exist,

- *the* **right induced model structure on** $\mathcal{C}$ *is given by*
$$(\mathcal{C},\, U^{-1}\mathscr{W},\, {}^{\boxslash}(U^{-1}(\mathscr{F}\cap\mathscr{W})),\, U^{-1}\mathscr{F})$$
and
- *the* **left induced model structure on** $\mathcal{K}$ *is given by*
$$(\mathcal{K},\, V^{-1}\mathscr{W},\, V^{-1}\mathscr{C},\, (V^{-1}(\mathscr{C}\cap\mathscr{W}))^{\boxslash}).$$

We say that U **makes fibrations and weak equivalences in** $\mathcal{C}$*, they are* **created by** U *and that they are* **lifted along the right adjoint** U*. Similarly* V **makes cofibrations and weak equivalences in** $\mathcal{K}$*, they are* **created by** V *and they are* **lifted along the left adjoint** V*.*

The Crans-Kan Transfer Theorem 5.2.27 is a classical example of a right induced model structure. Left induced model structures are harder to come by, but we will see eight of them in Corollary 9.3.17 below.

If the right induced model structure exists on $\mathcal{C}$, then both of its weak factorization systems are right induced from the weak factorization systems on $\mathcal{M}$ of (4.1.2), i.e., the right classes are created by U. Similarly if the left induced model structure exists on $\mathcal{K}$, then both of its weak factorization systems are left induced from the ones on $\mathcal{M}$, i.e., left classes are created by V.

The following is proved by Kathryn Hess, Magdalena Kedziorek, Emily Riehl and Brooke Shipley as [HKRS17, Proposition 2.1.4].

Proposition 5.2.20. The acyclicity condition. *Suppose we have a model category* $\mathcal{M}$ *and adjunctions as in Definition 5.2.19, namely*

$$\begin{tikzcd} \mathcal{K} \arrow[r, shift left, "V"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "R"] \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{C}, \arrow[l, shift left, "U"] \end{tikzcd}$$

and that the right (left) induced (from those of (4.1.2)) factorization systems exist in $\mathcal{C}$ *(*$\mathcal{K}$*). Then*

- *the right induced model structure exists on* $\mathcal{C}$ *iff*
$${}^{\boxslash}(U^{-1}\mathscr{F}) \subseteq U^{-1}\mathscr{W}$$
and
- *the left induced model structure exists on* $\mathcal{K}$ *iff*
$$(V^{-1}\mathscr{C})^{\boxslash} \subseteq V^{-1}\mathscr{W}.$$

Corollary 5.2.21. A left induced acyclicity condition. *With hypotheses as in Proposition 5.2.20, suppose in addition that each generating cofibration in* $\mathcal{M}$ *is isomorphic to one in the image of* V*. Then* $(V^{-1}\mathscr{C})^{\boxslash} \subseteq V^{-1}\mathscr{W}$*.*

Proof Suppose that $p : X \to Y$ is a morphism $\mathcal{K}$ lying in $(V^{-1}\mathscr{C})^{\boxslash}$. This means that for any morphism $i : A \to B$ for which $V(i)$ is a cofibration in $\mathcal{M}$, there is a lifting h in

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, "h"] & Y \end{tikzcd}$$

We need to show that $V(p)$ is a weak equivalence in $\mathcal{M}$. It would be a trivial fibration and hence a weak equivalence if we knew that it had the right lifting property with respect to each generating cofibration in $\mathcal{M}$. Since these are all, up to isomorphism, in the image of V by assumption, we have the desired right liftings. □

Corollary 5.2.22. Factorization systems in retract categories. *With hypotheses as in Proposition 5.2.20, suppose in addition that $\mathcal{C}$ ($\mathcal{K}$) is a full reflective (coreflective) subcategory of $\mathcal{M}$ as in Definition 2.2.49, meaning that the composite functor LU (RV) is identity functor on $\mathcal{C}$ ($\mathcal{K}$). Then the right (left) induced factorization systems exist in $\mathcal{C}$ ($\mathcal{K}$).*

Proof In the left case, let $f : X \to Y$ be a morphism in $\mathcal{K}$, and let $V(f) = pi$ be either of the factorizations of its image in the model category $\mathcal{M}$. Then in $\mathcal{K}$ we have

$$f = RV(f) = R(pi) = R(p)R(i),$$

giving the desired factorization in $\mathcal{K}$. □

Corollary 5.2.23. A left induced model structure. *With hypotheses as in Proposition 5.2.20, suppose in addition that RV is the identity functor on $\mathcal{K}$ and VR is naturally equivalent to the identity functor on $\mathcal{M}$, making $\mathcal{K}$ and $\mathcal{M}$ equivalent as categories. Then there is a left induced model structure on $\mathcal{K}$ such that the left adjunction in Proposition 5.2.20 is a Quillen equivalence.*

Proof Since RV is the identity functor, $\mathcal{K}$ has the desired factorization systems by Corollary 5.2.22. Since VR is naturally equivalent to the identity functor, the set of generating cofibrations of $\mathcal{M}$ is isomorphic to a set in the image of V. Thus we have the left acyclicity condition by Corollary 5.2.21. □

In other words, the right (left) induced model structure exists in $\mathcal{C}$ ($\mathcal{K}$) iff the maps one would expect to be trivial cofibrations (trivial fibrations) really are weak equivalences. In the right induced case one is asking for certain "cofibrations" in $\mathcal{C}$ to behave nicely under the right adjoint U, and similarly in the left induced case. In general there is no expectation that a right adjoint should play nicely with cofibrations. This makes the condition difficult to verify.

The authors of [HKRS17] discuss ways of verifying their acyclicity condition for a class of model categories they call **accessible**. These are not to be confused with accessible categories as in Definition 4.8.12. Their accessible model categories include cofibrantly generated ones, which we will now discuss.

The next result is proved by Hirschhorn in [Hir03, Theorem 11.3.1], where he attributes it to Dan Kan. It is also is also proved as [DHK97, 8.1] and as [Hov99, Theorem 2.1.19]. We will use both it and the Crans-Kan Transfer Theorem 5.2.27 below repeatedly in this book.

Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24. *Let $\mathcal{M}$ be a bicomplete homotopical category (Definition 5.1.1 below), for which $\mathscr{W}$ is the class of weak equivalences, with morphisms sets $\mathcal{I}$ and $\mathcal{J}$ such that:*

(i) Both $\mathcal{I}$ and $\mathcal{J}$ permit the small object argument as in Definition 4.8.2, meaning that both have small domains relative to themselves (Definition 4.8.18).

(ii) Every $\mathcal{J}$-cofibration is an $\mathcal{I}$-cofibration and a weak equivalence, that is

$$\mathcal{S}at(\mathcal{J}) \subseteq \mathcal{S}at(\mathcal{I}) \cap \mathscr{W},$$

for $\mathcal{S}at(\mathcal{I})$ and $\mathcal{S}at(\mathcal{J})$ as in Definition 4.8.13.

(iii) Every morphism with the right lifting property with respect to $\mathcal{I}$ also has it with respect to $\mathcal{J}$ and is a weak equivalence, that is $\mathcal{I}^{\boxslash} \subseteq \mathcal{J}^{\boxslash} \cap \mathscr{W}$.

(iv) One of the following two conditions holds:

(a) a weak equivalence that is an $\mathcal{I}$-cofibration is also a $\mathcal{J}$-cofibration, that is $\mathcal{S}at(\mathcal{I}) \cap \mathscr{W} \subseteq \mathscr{C}(\mathcal{J})$, or

(b) a weak equivalence having the right lifting property with respect to $\mathcal{J}$ also has it with respect to $\mathcal{I}$, that is $\mathcal{J}^{\boxslash} \cap \mathscr{W} \subseteq \mathcal{I}^{\boxslash}$.

Then $\mathcal{M}$ has a cofibrantly generated model category structure with the specified weak equivalences for which $\mathcal{I}$ and $\mathcal{J}$ are the generating sets of cofibrations and trivial cofibrations. In particular both conditions of (iv) hold.

There is an alternative formulation due to May and Ponto [MP12, Theorem 15.2.3] which requires, in addition to the smallness condition of (i), that $\mathcal{J} \subseteq \mathscr{W}$, the **acyclicity condition** and $\mathscr{W} \cap \mathcal{J}^{\boxslash} = \mathcal{I}^{\boxslash}$, the **compatibility condition**.

Definition 5.2.25. Transfer adjunctions. *Let $\mathcal{M}$ be a cofibrantly generated model category, let $\mathcal{N}$ be a bicomplete category and let*

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left, "U"] \end{tikzcd}$$

be a pair of adjoint functors (§2.2). For cofibrant generating sets $\mathcal{I}$ and $\mathcal{J}$ be of $\mathcal{M}$, let $F\mathcal{I} = \{Fi\colon i \in \mathcal{I}\}$ and $F\mathcal{J} = \{Fj\colon j \in \mathcal{J}\}$. Then the above is a **transfer adjunction**, *and (F, U) is a* **transfer pair**, *if* *(i) both $F\mathcal{I}$ and $F\mathcal{J}$ permit the small object argument (see Definition 4.8.2) in $\mathcal{N}$ and*

(ii) U takes relative $F\mathcal{J}$-cell complexes (Definition 4.8.18) in $\mathcal{N}$ to weak equivalences in $\mathcal{M}$.

One might call the above a **right** transfer adjunction, but we will make no use of the dual notion.

The next result is an exercise for the reader.

Proposition 5.2.26. The product of transfer adjunctions is a transfer adjunction. *Suppose we have transfer adjunctions*

$$\begin{tikzcd} \mathcal{M}_i \arrow[r, shift left, "F_i"] \arrow[r, phantom, "\perp"] & \mathcal{N}_i \arrow[l, shift left, "U_i"] \end{tikzcd} \qquad \text{for } i = 1, 2$$

as in Definition 5.2.25. Then the product of Proposition 2.2.18,

$$\begin{tikzcd} \mathcal{M}_1 \times \mathcal{M}_2 \arrow[r, shift left, "F_1 \times F_2"] \arrow[r, phantom, "\perp"] & \mathcal{N}_2 \times \mathcal{N}_2, \arrow[l, shift left, "U_1 \times U_2"] \end{tikzcd}$$

is also a transfer adjunction.

The following is an example of a right induced model structure in the sense of Definition 5.2.19 that will be used repeatedly in this book. It is proved by Hirschhorn in [Hir03, Theorem 11.3.2], where he attributes it to Dan Kan. It is very similar to [Cra95, Theorem 3.3] (which is cited as its source in [DHK97, Lemma 9.1]), [Bla96, Theorem 4.14] and [SS00, Lemma 2.3], where some other references are given.

Crans-Kan Transfer Theorem 5.2.27. *Let*

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left, "U"] \end{tikzcd}$$

be as in Definition 5.2.25. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure***), for which $f\mathcal{I}$ and $F\mathcal{J}$ are cofibrant generating sets, and the weak equivalences and fibrations are respectively the maps taken by U to weak equivalences and fibrations in $\mathcal{M}$. Furthermore, with respect to this model structure, (F, U) is a Quillen pair as in Definition 4.5.1.*

This can be proved by showing that the indicated structure in $\mathcal{N}$ satisfies the conditions of the Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24. A more direct argument is given by David Blanc in [Bla96, Theorem 4.14]. We say that U **makes weak equivalences in** $\mathcal{N}$ or that they are **lifted along the right adjoint** U.

As noted in Remark 4.5.7, $\mathcal{N}$ could have other model structures for which (F, U) is again a Quillen pair.

Remark 5.2.28. The hard part of using the Crans-Kan Transfer Theorem 5.2.27. *In practice the second of Crans and Kan's two conditions, which says that the right adjoint U takes trivial cofibrations in $\mathcal{N}$ to weak equivalences in $\mathcal{M}$, is the harder one to verify. It is an instance of the acyclicity condition of Proposition 5.2.20.*

Corollary 5.2.29. Fibrations in the transferred model structure. *In the situation of the Crans-Kan Transfer Theorem 5.2.27, the fibrations in $\mathcal{N}$ are those maps whose images under U are fibrations in $\mathcal{M}$, i.e., U* **makes fibrations in** *$\mathcal{N}$.*

Proof This is a special case of Proposition 2.3.13. A map $p : X \to Y$ in $\mathcal{N}$ is a fibration iff it has the right lifting property with respect to $Fj : FA \to FB$ for each map $j : A \to B$ in $\mathcal{J}$. In other words there is always a lifting in the following diagram in $\mathcal{N}$

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "Fj"'] & X \arrow[d, "p"] \\ FB \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Since $F \dashv U$, this is equivalent to the existence of a lifting in the corresponding diagram in $\mathcal{M}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & UX \arrow[d, "Up"] \\ B \arrow[r] \arrow[ur, dashed] & UY. \end{tikzcd}$$

This lifting exists for each $j \in \mathcal{J}$ iff Up is a fibration in $\mathcal{M}$. □

Remark 5.2.30. Model structures on categories of algebras. *In many applications of the Crans-Kan Transfer Theorem 5.2.27, the category $\mathcal{N}$ is the category of objects in $\mathcal{M}$ with some additional structure, U is the forgetful functor and its left adjoint F sends an object in $\mathcal{M}$ to the appropriate sort of free object generated by it. $\mathcal{N}$ could be the category $\mathcal{M}^T$ of T-algebras for a monad (T, η, μ) (see Definition 2.2.40) on $\mathcal{M}$ where the functor T preserves cofibrations and trivial cofibrations, making (F, U) a Quillen pair as explained in Proposition 4.5.12. Condition (ii) in the Crans-Kan Transfer Theorem 5.2.27 means that if we have a pushout diagram in $\mathcal{N}$,*

$$\begin{tikzcd} FA \arrow[r, "Fj"] \arrow[d] & FB \arrow[d] \\ X \arrow[r, "f"] \arrow[ur, phantom, "\lrcorner" very near end] & Y, \end{tikzcd}$$

where $j : A \to B$ is a generating trivial cofibration in $\mathcal{M}$ (which means that Fj is a trivial cofibration in $\mathcal{N}$), then f is a weak equivalence.

Corollary 5.2.31. The case where $\mathcal{M}$ is bireflective in $\mathcal{N}$. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$ of cofibrations and trivial cofibrations, let $\mathcal{N}$ be a bicomplete category and let*

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "A"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left, "B"] \end{tikzcd}$$

be a pair of adjoint functors making $\mathcal{M}$ a bireflective subcategory of $\mathcal{N}$ as in Definition 2.2.50. Then (A, B) is a transfer pair as in Definition 5.2.25, so $\mathcal{N}$ has a transferred model structure, making (A, B) a Quillen pair. If in addition AB is naturally isomorphic to the identity functor on $\mathcal{N}$, then the adjunction is a Quillen equivalence.

Proof For Definition 5.2.25(i), $A\mathcal{I}$ and $A\mathcal{J}$ permit the small object argument because $BA\mathcal{I} = \mathcal{I}$ and $BA\mathcal{J} = \mathcal{J}$ do. Permitting the small object argument has to do with colimits, and these are preserved by B (see Proposition 2.3.36) since it is a left adjoint. For (ii), B takes relative $A\mathcal{J}$-cell complexes (Definition 4.8.18) to $BA\mathcal{J}$-cell complexes, which are $\mathcal{J}$-cell complexes and hence weak equivalences in $\mathcal{M}$.

Now suppose that AB is is naturally isomorphic to the identity functor on $\mathcal{N}$, X is cofibrant in $\mathcal{M}$, Y is fibrant in $\mathcal{N}$, and $f : AX \to Y$ is a weak equivalence in $\mathcal{N}$. □

In the §5.4 we will study the functor category $\mathcal{M}^J$ for a cofibrantly generated model category $\mathcal{M}$ and a small category J. We will see that it has a cofibrantly generated model structure defined in terms of the one on $\mathcal{M}$. The same goes for $\mathcal{M}^K$ for a full subcategory K of J. In §5.4C we will see that $\mathcal{M}^K$ is a bireflective subcategory of $\mathcal{M}^J$, so the two are related by an adjunction as above. In this case the right category as well as the left one comes equipped with a model structure, and Corollary 5.2.31 gives us a way to construct a new model structure on it. The new structure has fewer cofibrations than the old one, and we will refer to the former as a **confinement** of the latter.

We will use the following in §5.4D below to show that the composite adjunctions in (5.4.27) are transfer adjunctions.

Proposition 5.2.32. Certain composites of transfer adjunctions are transfer adjunctions. *Suppose we have cofibrantly generated model categories $\mathcal{M}$ and $\mathcal{M}'$ with cofibrant generating sets $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{I}', \mathcal{J}')$ respectively, a bicomplete category $\mathcal{N}$, and transfer adjunctions as in Definition 5.2.25,*

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "F_1"] \arrow[r, phantom, "\perp"] & \mathcal{M}' \arrow[l, shift left, "U_1"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{M}' \arrow[r, shift left, "F_2"] \arrow[r, phantom, "\perp"] & \mathcal{N}. \arrow[l, shift left, "U_2"] \end{tikzcd} \tag{5.2.33}$$

Assume additionally that

(a) *the morphism sets $F_2F_1\mathcal{I}$ and $F_2F_1\mathcal{J}$ (as well as $F_2\mathcal{I}'$ and $F_2\mathcal{J}'$) permit the small object argument in $\mathcal{N}$;*

(b) *U_1 sends weak equivalences in the given model structure on $\mathcal{M}'$ to weak equivalences in $\mathcal{M}$; and either*

(c′) *induced trivial cofibrations in $\mathcal{M}'$ are trivial cofibrations in its given model structure or*

(c″) *$\mathcal{M}'$ is a bireflective subcategory of $\mathcal{N}$.*

Then the composite adjunction of Proposition 2.2.19,

$$\mathcal{M} \underset{U_1U_2}{\overset{F_2F_1}{\underset{\perp}{\rightleftarrows}}} \mathcal{N}$$

is also a transfer adjunction.

Note that the requirement that the adjunction on the left of (5.2.33) be a transfer adjunction does not involve the given model structure on $\mathcal{M}'$. The Crans-Kan Transfer Theorem 5.2.27 produces a right induced model structure on $\mathcal{M}'$ which may differ from the given one. The proposition leads to model structures on $\mathcal{N}$ induced from the given ones on $\mathcal{M}$ and $\mathcal{M}'$. The variants (c′) and (c″) are satisfied in the clockwise and counterclockwise ways of going around from the lower left to the upper right in the diagrams of (5.4.27), (5.4.32) and (5.4.34) below.

Unlike earlier statements about composites of adjunctions (Proposition 2.2.19 and Proposition 4.5.22), this one requires additional hypotheses for the following reason. Requiring the second adjunction of (5.2.33) to be a transfer adjunction with respect to the given model structure on $\mathcal{M}'$ is not the same as requiring it to be one with respect to the model structure on $\mathcal{M}'$ induced by the first adjunction. Even if we knew it was such a transfer, showing that the composite adjunction is one would still be more than a formality.

Proof The smallness condition needed for the composite adjunction is assumption (a).

For the second condition of Definition 5.2.25 in the composite adjunction, we need to show that U_1U_2 sends a relative $F_2F_1\mathcal{J}$-cell complex (Definition 4.8.18) f to a weak equivalence in $\mathcal{M}$.

Assumption (c′) implies that such a map are also a relative $F_2\mathcal{J}'$-cell complexes. Hence its image under U_2 is a weak equivalence since (F_2, U_2) is a transfer pair. Its image under U_1U_2 is then a weak equivalence by (b).

Assumption (c″) implies that $U_2F_2F_1\mathcal{J}$ is isomorphic to $F_1\mathcal{J}$, so $U_2F_2F_1\mathcal{J}$ is a set of weak equivalences in $\mathcal{M}$. This means that U_1U_2 sends relative $F_2F_1\mathcal{J}$-complexes in $\mathcal{N}$ to weak equivalences in $\mathcal{M}'$ as desired. □

The following example of right induction (as in Definition 5.2.19) appears not to be in the literature. It will be useful for us in Chapter 9. It gives us a way to add more cofibrations without altering the weak equivalences in a cofibrantly generated model category $\mathcal{M}$. We will refer to the given model structure on $\mathcal{M}$ as the **orginal model structure** and call the new one the **enlarged model structure** or the **model structure enlarged by** F. The word "enlarged" here refers to the class of cofibrations, not that of weak equivalences, which is unchanged, or that of fibrations, which becomes smaller. It will be used to construct the model structure we need on the category of G-spectra starting in (9.2.7) below.

Theorem 5.2.34. Enlarging the class of cofibrations in a cofibrantly generated model category. *Let $\mathcal{M}$ and $\mathcal{M}'$ be cofibrantly generated model categories with pairs of cofibrant generating sets $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{I}', \mathcal{J}')$. Suppose further that there is an adjunction (which need* **not** *be a Quillen adjunction)*

$$\begin{tikzcd} \mathcal{M}' \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "U"] \end{tikzcd} \tag{5.2.35}$$

such that both $F\mathcal{I}'$ and $F\mathcal{J}'$ permit the small object argument in $\mathcal{M}$, U sends relative $F\mathcal{J}'$-complexes to weak equivalences in $\mathcal{M}'$, and U preserves weak equivalences. Thus (F, U) is a transfer pair, a condition which does not involve the model structure on $\mathcal{M}$, with the additional requirment that U preserves weak equivalences.

Consider the following composite adjunction, which we will refer to as the **enlarging adjunction***.*

$$\begin{tikzcd} (X, X') \arrow[r, mapsto] & (X, FX') \arrow[r, mapsto] & X \amalg FX' \\ \mathcal{M} \times \mathcal{M}' \arrow[r, shift left, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left, "\mathcal{M}\times U"] \arrow[r, shift left, "\amalg"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "\Delta"] \\ (Y, UY) & (Y, Y) \arrow[l, mapsto] & Y, \arrow[l, mapsto] \end{tikzcd} \tag{5.2.36}$$

where the adjunction on the right is the coproduct diagonal adjunction of Example 4.5.6(i).

Then the composite adjunction of (5.2.36) is a transfer adjunction as in Definition 5.2.25, and there is a cofibrantly generated model structure on $\mathcal{M}$ for which the above is a Quillen adjunction. It has the same weak equivalences as the original one but more cofibrations and hence fewer fibrations. It has cofibrant generating sets $\mathcal{I} \cup F\mathcal{I}'$ and $\mathcal{J} \cup F\mathcal{J}'$.

The same holds if we replace the adjunction of (5.2.35) with a set of adjunctions having similar properties, all having the same $\mathcal{M}$ on the right but possibly different $\mathcal{M}'$s on the left, and modify (5.2.36) accordingly.

Remark 5.2.37. Not a Quillen adjunction. *To repeat, the adjunction of (5.2.35) need* **not** *be a Quillen adjunction. The theorem is of interest only in the case when it is not. If it were, the new model structure on $\mathcal{M}$ would* *coincide with the old one since it would have the same weak equivalences and cofibrations.*

Since there are more enlarged cofibrations in $\mathcal{M}$ than original ones, there are fewer enlarged fibrations. A morphism in $\mathcal{M}$ is an enlarged fibration iff it is an original one **and** its image under U is a fibration in $\mathcal{M}'$.

Proof The pair $(\amalg(\mathcal{M} \times F), (\mathcal{M} \times U)\Delta)$ is a transfer pair as in Definition 5.2.25, so $\mathcal{M}$ has a model structure for which a set of generating cofibrations is

$$\left\{i_1 \amalg F i_2 \colon i_1 \in \mathcal{I},\, i_2 \in \mathcal{I}'\right\}.$$

For the statement about enlarged weak equivalences, note that the right adjoint sends a morphism $f : Y \to Z$ in $\mathcal{M}$ to (f, Uf) in $\mathcal{M} \times \mathcal{M}'$. If f is a weak equivalence in the original module structure in $\mathcal{M}$, then Uf is a weak equivalence in $\mathcal{M}'$ by assumption. It follows that (f, Uf) is a weak equivalence in $\mathcal{M} \times \mathcal{M}'$, so f is also weak equivalence in the enlarged model structure. For the converse, an enlarged weak equivalence g maps to (g, Ug) so g must be an original weak equivalence.

The statement about generating sets follows from Proposition 5.2.5. $\square$

5.3 Proper model categories

Now, as promised in §5.1C, we take up the study of proper model categories as in Definition 5.1.14, which we repeat here.

Definition 5.3.1. Proper model categories again. *A model category is* **left proper** *if the pushout of any weak equivalence along a cofibration is again a weak equivalence. In other words, given a pushout diagram*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] \arrow[ur, phantom, "\lrcorner", very near end] & D \end{tikzcd} \tag{5.3.2}$$

where h is a weak equivalence and f is a cofibration, k is also a weak equivalence. There is a dual notion of **right proper** *that involves fibrations and pullbacks, and a model category with both properties is said to be simply* **proper**.

When (5.3.2) is a pushout diagram, g is a cofibration whenever f is, without any assumption on h or left properness. When it is a pullback diagram, f is a fibration whenever g is one.

The right proper version of the following is proved by Bousfield in [Bou01, Lemma 9.4].

Proposition 5.3.3. Weaker conditions for right and left properness. *A model category is left proper as in Definition 5.3.1 if given a pushout diagram (5.3.2) where h is a weak equivalence, f is a cofibration and A is cofibrant, k is also a weak equivalence. It is right proper if given a pullback diagram similar to (5.3.2) where k is a weak equivalence, g is a fibration and D is fibrant, h is also a weak equivalence.*

The following is proved by Hirschhorn as [Hir03, Corollary 13.1.3].

Proposition 5.3.4. A model category is left (right) proper if all objects in it are cofibrant (fibrant).

Corollary 5.3.5. *The categories $\mathcal{T}$ and $\mathcal{T}op$ are right proper, and $\mathcal{S}et_{\boldsymbol{\Delta}}$ and its pointed analog are left proper.*

The following is proved by Hirschhorn as [Hir03, Theorems 13.1.10 and 13.1.13].

Theorem 5.3.6. The categories $\mathcal{T}op$, $\mathcal{T}$, $\mathcal{S}et_{\boldsymbol{\Delta}}$ and its pointed analog are all proper.

5.4 The category of functors from a small category to a cofibrantly generated model category

5.4A The projective and injective model structures on $\mathcal{M}^J$

Let $\mathcal{M}$ be a model category and let J be a small category. We now consider the functor category $\mathcal{M}^J$, the category of J-shaped diagrams in $\mathcal{M}$. For such a diagram X we will denote its value on an object j of J by X_j, and similarly for morphisms $f : X \to Y$ of diagrams.

Functoriality means that each morphism $\theta : j \to j'$ in J induces a morphism

$$X_\theta : X_j \to X_{j'} \qquad \text{in } \mathcal{M}.$$

Collectively these define a **structure map**

$$\epsilon^X_{j,j'} : J(j,j') \times X_j \to X_{j'} \tag{5.4.1}$$

with suitable properties.

In order to define a model structure on $\mathcal{M}^J$, we make the following.

Definition 5.4.2. *A morphism $f : X \to Y$ in $\mathcal{M}^J$ is*

(i) A **projective weak equivalence** *or* **strict weak equivalence** *if each f_j is a weak equivalence;*
(ii) A **projective fibration** *or* **strict fibration** *if each f_j is a fibration;*

(iii) A **projective cofibration** *if it has the left lifting property with respect to strict trivial fibrations.*

This is the **projective model structure** *on $\mathcal{M}^J$. In the* **injective model structure** *a map f is a injective weak equivalence or cofibration if each f_j is one, and injective fibrations are defined in terms of right lifting properties.*

The above use the word "injective" is unrelated to that of Definition 4.1.10, and that of "projective " is unrelated to its use in homological algebra. We will make little use of the injective model structure in this book. This is another instance of the lack of symmetry in the subject.

Definition 5.4.3. Cofibrant (fibrant) diagrams. *For model category $\mathcal{M}$ and a small category J, a* **projectively cofibrant (injectively fibrant) diagram** *is an object in $\mathcal{M}^J$ which is cofibrant (fibrant) in the projective (injective) model structure.*

Proposition 5.4.4. Cofibrations and cofibrant objects in $\mathcal{M}^J$. *Let $\mathcal{M}^J$ have the projective model structure as in Definition 5.4.2.*

(i) For a cofibration $i : A \to B$ in $\mathcal{M}^J$, each map $i_j : A_j \to B_j$ is a cofibration.
(ii) Each object X_j of a cofibrant diagram X in $\mathcal{M}^J$ is cofibrant in $\mathcal{M}$.

Proof (i) The map $i : A \to B$ is a cofibration if it has the left lifting property with respect to each trivial fibration $p : X \to Y$. Since trivial fibrations in $\mathcal{M}^J$ are defined objectwise, this means that for each object j in J there is a lifting in the diagram

$$\begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"'] & X_j \arrow[d, "p_j"] \\ B_j \arrow[r] \arrow[ur, dashed] & Y_j. \end{tikzcd}$$

To show that i_j is a cofibration, we need to know that **every** trivial fibration $q : W \to Z$ in $\mathcal{M}$ is the jth component of one in $\mathcal{M}^J$. For this we can use the constant diagram functor $\Delta : \mathcal{M} \to \mathcal{M}^J$ of §2.3C. For any j, q is the jth component of the trivial fibration $\Delta(q)$.

(ii) A diagram A in $\mathcal{M}^J$ is cofibrant iff the map to it from constant $*$-valued diagram is a cofibration. Cofibrations in $\mathcal{M}^J$ are defined in terms of the left lifting property with respect to trivial fibrations, which are defined objectwise. Thus the map $* \to X_j$ must be a cofibration for each object j in J, so A_j must be cofibrant in $\mathcal{M}$ as claimed. □

However, the structure maps of (5.4.1) for a cofibrant diagram (in the projective model structure on $\mathcal{M}^J$) need not be cofibrations, as the following illustrates.

Example 5.4.5. A cofibrant diagram whose structure maps are not all cofibrations. *Let K be a cofibrant object other than $*$ in a pointed model category $\mathcal{M}$, and let j be an object in a small category J suchthat $J(j', j)$ is empty for all objects j' distcint from j. Let X be a functor in $\mathcal{M}^J$ defined by*

$$X_{j'} = \begin{cases} K & \text{for } j' = j \\ * & \text{otherwise} \end{cases}$$

The left lifting property that the map $* \to X$ needs to have to be cofibration need only be checked at the object j since the functor is trivial everwhere else. Hence X is cofibrant in $\mathcal{M}^J$ because K is cofibrant in $\mathcal{M}$. On the other hand, for any object $j' \neq j$ for which the morphism set $J(j, j')$ is nonempty, the structure map

$$J(j, j') \times X_j = J(j, j') \times K \xrightarrow{\epsilon^X_{j,j'}} X_{j'} = *$$

is not a cofibration.

Remark 5.4.6. The insuffiency of objectwise cofibrancy. *The necessary cofibrancy condition of Proposition 5.4.4 is* **not** *sufficient in general for the following reason. Let $p : X \to Y$ be a trivial cofibration in $\mathcal{M}^J$ and let A be a cofibrant object in $\mathcal{M}^J$. Hence for each object j in J we need a lifting h_j in the diagram*

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X_j \arrow[d, "p_j"] \\ A_j \arrow[r, "f_j"'] \arrow[ur, dashed, "h_j"] & Y_j, \end{tikzcd}$$

where p_j is a trivial fibration in $\mathcal{M}$. In addition we must be able to assemble such diagrams in $\mathcal{M}$ to a similar one in $\mathcal{M}^J$. This means that the liftings h_j, like the maps f_j and p_j, must be compatible with the structure maps.

Similarly if $i : A \to B$ is a cofibration in $\mathcal{M}^J$, then each map i_j is necessarily a cofibration, but this condition is not sufficient.

Strict maps are designated by the words "level" in [MMSS01], [MM02] and [GM11], and "objectwise" in [Hir03]. We will use the words strict and objectwise interchangeably. In the simplicial case (meaning when $\mathcal{M}$ is $\mathcal{S}et_{\mathbf{\Delta}}$), the projective structure is also known as the **Bousfield-Kan model structure** since it was introduced in [BK72, XI.8]. The injective or **Heller model structure** was introduced in [Hel88]. Both were shown by their original authors to be proper and cofibrantly generated, and both are combinatorial, as in Definition 4.8.11. The two are known to be Quillen equivalent, and in the Bousfield-Kan (Heller) model structure every object is cofibrant (fibrant) when the same is true in $\mathcal{M}$.

Example 5.4.7. The case where J is a finite groupoid. *Groupoids were studied in §2.1E. There we saw (Proposition 2.1.33) that a finite groupoid is made up of connected components, also known as orbits, each determined up to isomorphism by its isotropy group (see Definition 2.1.29(iv)) and its cardinality. Important examples for us are groupoids $\mathcal{B}_T G$ associated with a finite G-set T for a finite group G; see Definition 2.1.31.*

For a model category $\mathcal{M}$ and a groupoid J, an object X in the functor category $\mathcal{M}^J$ is a collection of objects X_j in $\mathcal{M}$ for each object j in J. When j and j' are in the same connectected component of j, then the functor X gives is a family of isomorphisms $X_j \to X_{j'}$ related by a free transitive action of the group $G_j = J(j,j)$ by precomposition and by a similar action of the isomorphic group $G_{j'}$ by postcomposition. In particular, the Yoneda functor $よ^j$ assigns the value $\varnothing$ to each j' not in the connected component of j, and a free transitive G_j-set to each j' in its connected component.

Cofibrations in $\mathcal{M}^J$ with its projective model structure (Definition 5.4.2) have the following description. Choose an object t in each connected component of J. Then a cofibration $i : X \to Y$ is determined by an arbitrary collection of cofibrations $i_t : X_t \to Y_t$ for each such t. The maps i_j for other $j \in J$ are uniquely determined by this data. The same goes for fibrations and weak equivalences in $\mathcal{M}^J$. It follows that **the projective and injective model structures are the same**.

We will discuss this further in §5.5E below.

5.4B Cofibrant generation in functor categories

Definition 5.4.8. Generating cofibrations and trivial cofibrations in $\mathcal{M}^J$. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. For each object j in J as above, let $よ^j$ in $\mathcal{S}et^J$ be the Yoneda functor of Yoneda Lemma 2.2.10, namely the functor defined by $よ^j(k) = J(j,k)$. Define morphism sets in $\mathcal{M}^J$ by*

$$\begin{aligned} & F^J\mathcal{I} := \left\{ よ^j \otimes f \colon f \in \mathcal{I}, j \in J \right\} \\ \text{and} \qquad & F^J\mathcal{J} := \left\{ よ^j \otimes f \colon f \in \mathcal{J}, j \in J \right\} \end{aligned}$$

(the meaning of $\otimes$ will be explained in Remark 5.4.9 below) where for a morphism $f : A \to B$ in $\mathcal{M}$

$$(よ^j \otimes f)_k = \coprod_{J(j,k)} f,$$

the disjoint union indexed by the set $J(j,k)$ of copies of f. (We are using the fact that the cocomplete category $\mathcal{M}$ is tensored (see Definition 3.1.31) over

$\mathcal{S}et$.) A morphism $\lambda : k \to \ell$ in J induces the morphism

$$\lambda_* : \coprod_{J(j,k)} f \to \coprod_{J(j,\ell)} f$$

that sends the copy of f in the source corresponding to $\kappa \in J(j,k)$ to the one in the target corresponding to $\lambda\kappa \in J(j,\ell)$.

Remark 5.4.9. Left and right tensor products. *Here we are tensoring objects and morphisms in $\mathcal{M}$ on the left with the $\mathcal{S}et$-valued Yoneda functor $よ^j$, using the fact that $\mathcal{M}$ is tensored over $\mathcal{S}et$ as in Definition 3.1.31. In §5.6 below we will consider enriched functors $J \to \mathcal{N}$ where J and $\mathcal{N}$ are both enriched over a symmetric monoidal model category $\mathcal{M}$, to be defined in Definition 5.5.9. With additional assumptions on J, such functors are known as* **spectra** *(see §7.2), which are the subject of stable homotopy theory. Given such a functor X and an object M in $\mathcal{M}$, we will write their tensor product* **on the other side** *as $M \wedge X$ rather than $X \otimes M$. See Proposition 7.2.49 below.*

The following is proved as [Hir03, Theorem 11.6.1], where the term **level model structure** is used.

Theorem 5.4.10. The projective model structure on $\mathcal{M}^J$ for a small category J and cofibrantly generated model category $\mathcal{M}$. *The projective model structure on $\mathcal{M}^J$ of Definition 5.4.2 is proper (Definition 5.3.1 below) if $\mathcal{M}$ is, and cofibrantly generated with generating sets $F^J\mathcal{I}$ and $F^J\mathcal{J}$ of Definition 5.4.8. Its weak equivalences (fibrations) are strict weak equivalences (fibrations). Its cofibrations are retracts of transfinite compositions of pushouts of elements of $F^J\mathcal{I}$.*

Corollary 5.4.11. Some cofibrant objects in $\mathcal{M}^J$. *For any cofibrant object K in $\mathcal{M}$ and any object j in J, the object $よ^j \otimes K$ is cofibrant in $\mathcal{M}^J$.*

Using Theorem 5.3.6 we have

Corollary 5.4.12. The projective model structure on $\mathcal{M}^J$ for a small category J is proper when $\mathcal{M}$ is $\mathcal{T}op$, $\mathcal{T}$, $\mathcal{S}et_{\Delta}$ or its pointed analog.

The enriched analogs of Theorem 5.4.10 and Corollary 5.4.11 are Theorem 5.6.26 and Corollary 5.6.27 below.

Here is a sketch of Hirschhorn's proof. He compares $\mathcal{M}^J$ with $M^{|J|}$, where $|J|$ is the discrete category associated with J as in Definition 2.1.7, that is the category having the same objects as J but only identity morphisms. The category $M^{|J|}$ is simply the product of copies of $\mathcal{M}$ indexed by the object set of J. The model structure of $M^{|J|}$ is straightforward; a morphism in it is a fibration, cofibration or weak equivalence iff it is a strict fibration, cofibration or weak equivalence. There is a forgetful functor $U = u^* : \mathcal{M}^J \to M^{|J|}$

induced by the inclusion functor $u : |J| \to J$. It sends a functor X in $\mathcal{M}^J$ to its collection of values X_j on the objects j of J.

In [Hir03, Definition 11.5.27] he constructs a left adjoint functor

$$F^J = u_! : \mathcal{M}^{|J|} \to M^J$$

to U and shows that it satisfies the hypotheses of the Crans-Kan Transfer Theorem 5.2.27. For an object X in $\mathcal{M}^{|J|}$, for each object k in J we have

$$(u_!X)_k = \coprod_{j \in J} J(j,k) \times X_j, \qquad \text{so} \qquad u_!X = \coprod_{j \in J} よ^j \times X_j.$$

The resulting generating sets of morphisms in $\mathcal{M}^J$ are those of Definition 5.4.8. Given an object X in $\mathcal{M}^{|J|}$, meaning a collection of objects in $\mathcal{M}$ indexed by the set of objects in J, consider the diagram

$$\begin{tikzcd} {|J|} \arrow[rr, "X"] \arrow[dr, "u"'] & & \mathcal{M} \\ & J \arrow[ur, "Lan_u X"'] & \end{tikzcd}$$

where as usual $Lan_u X$ denotes the left Kan extension of X along u. Then $u_!X = Lan_u X$, the left adjoint of the precomposition functor $U = u^*$; see Proposition 2.5.4. We will refer to the Quillen adjunction

$$\begin{tikzcd} \mathcal{M}^{|J|} \arrow[r, shift left, "F^J = u_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J \arrow[l, shift left, "U = u^*"] \end{tikzcd} \tag{5.4.13}$$

as the **Hirschhorn adjunction**. For each object j in J, the composition of U with the projection functor $p_j : \mathcal{M}^{|J|} \to \mathcal{M}$ (the precomposition functor induced by the inclusion of the one object discrete category into $|J|$ corresponding to j) is the evaluation map Ev_j of Definition 2.2.37.

Theorem 5.4.10 can be generalized in two ways:

(i) The **injective model structure** on $\mathcal{M}^J$ for a combinatorial model category $\mathcal{M}$ (defined below in Definition 4.8.11) has weak equivalences and cofibrations (instead of fibrations) defined strictly. This is discussed in [Lur09, §A.2.8 and §A.3.3]. The two structures are Quillen equivalent since they have the same weak equivalences.

(ii) We can assume that both J and $\mathcal{M}$ are enriched over a closed symmetric monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, \mathbf{1})$. This is discussed in [BDS16, §6.2]. We will take it up in §5.6 below. In particular the enriched analogs of Theorem 5.4.10 and Theorem 5.4.21 are Theorem 5.6.26 and Theorem 5.6.38.

Example 5.4.14. The projective model structure on the category of pushout diagrams. *Let J be the pushout category $\{a \leftarrow b \to c\}$ as in*

Example 4.4.1, and let $\mathcal{M}$ be as in Theorem 5.4.10. Then $\mathcal{M}^J$ is the category of pushout diagrams in $\mathcal{M}$. We denote a typical object X in this category by

$$X_a \xleftarrow{\alpha_X} X_b \xrightarrow{\beta_X} X_c,$$

and a typical morphism $g : X \to Y$ by

$$\begin{tikzcd} X_a \arrow[d, "g_a"] & X_b \arrow[l, "\alpha_X"'] \arrow[r, "\beta_X"] \arrow[d, "g_b"] & X_c \arrow[d, "g_c"] \\ Y_a & Y_b \arrow[l, "\alpha_Y"'] \arrow[r, "\beta_Y"] & Y_c. \end{tikzcd}$$

Then g is a weak equivalence or fibration if each of g_a, g_b and g_c is one.

The three Yoneda functors $J \to \mathcal{S}et$ are given by

$$よ^a(j) = \begin{cases} * & \text{for } j = a \\ \varnothing & \text{for } j \neq a \end{cases}$$

$$よ^b(j) = * \quad \text{for all } j$$

$$よ^c(j) = \begin{cases} * & \text{for } j = c \\ \varnothing & \text{for } j \neq c \end{cases}$$

It follows that for each $g : A \to B$ in $\mathcal{I}$ or $\mathcal{J}$, we get three generating cofibrations or trivial cofibrations in $\mathcal{M}^J$, namely

$$\begin{tikzcd} A \arrow[d, "g"] & \varnothing \arrow[l] \arrow[r] \arrow[d] & \varnothing, \arrow[d] \\ B & \varnothing \arrow[l] \arrow[r] & \varnothing \end{tikzcd} \quad \begin{tikzcd} A \arrow[d, "g"] \arrow[r, equal] & A \arrow[d, "g"] \arrow[r, equal] & A \arrow[d, "g"] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \varnothing \arrow[d] & \varnothing \arrow[l] \arrow[r] \arrow[d] & A \arrow[d, "g"] \\ \varnothing & \varnothing \arrow[l] \arrow[r] & B. \end{tikzcd}$$

Note that if A and B are cofibrant (as is the case with $\mathcal{T}op$), then in each of the above diagrams the horizontal maps are cofibrations. Thus we can conclude that if the domains of $\mathcal{I}$ and $\mathcal{J}$ are cofibrant, then a cofibrant object X in $\mathcal{M}^J$ must not only be objectwise cofibrant, **its maps must be cofibrations as well**. *The bottom row of (4.4.2) does not meet this requirement, so it is not a cofibrant object in $\mathcal{T}op^J$, as noted in Example 4.4.1.*

Example 5.4.15. The projective model structure on $\mathcal{M}^G$. *Let G be a group, and let $\mathcal{M}$ be as in Theorem 5.4.10. Then $\mathcal{M}^G$, the category of objects in $\mathcal{M}$ with G-action, is the category of $\mathcal{M}$-valued functors on the one object category J associated with G, as explained in Example 2.3.35 (iii). Therefore it has a projective model structure with generating sets as in Definition 5.4.8. The Yoneda functor (Yoneda Lemma 2.2.10) $よ^j$ for the unique object j of J is the free G-set G. It follows that the generating sets for the projective model structure on $\mathcal{M}^G$ are $G \otimes \mathcal{I}$ and $G \otimes \mathcal{J}$, where $\otimes$ denotes the categorical product in $\mathcal{M}$. We call this the* **underlying model structure on $\mathcal{M}^G$** *since a morphism in it is a weak equivalence if the underlying morphism in $\mathcal{M}$ is one. There are other model structures on $\mathcal{M}^G$ that we will study below in §8.6.*

Example 5.4.16. The walking arrow category. *Let J be the walking arrow category* **2** *of Definition 2.1.6, let $K = |J|$ be the discrete category (as in Definition 2.1.7) with two objects, and let $\alpha : K \to J$ be the functor that is an isomorphism on object sets. Then $\mathcal{M}^K \cong \mathcal{M} \times \mathcal{M}$ and $\mathcal{M}^J \cong \mathcal{M}_1$, the arrow category of Definition 2.1.51(v), whose objects are morphisms in $\mathcal{M}$. Then the functor α^* sends a object $f : X \to Y$ in $\mathcal{M}^J$ (morphism in $\mathcal{M}$) to the object (X, Y) in $\mathcal{M} \times \mathcal{M}$. The left Kan extension $\alpha_!$ sends (X, Y) to the morphism $X \to X \amalg Y$ and the right Kan extension α_* sends it to $X \times Y \to Y$.*

An object (X, Y) in $\mathcal{M}^K$ is cofibrant iff X and Y are each cofibrant in $\mathcal{M}$. An object $f : X \to Y$ in $\mathcal{M}^K$ is cofibrant iff X and Y are each cofibrant in $\mathcal{M}$ **and** *f is a cofibration. Thus the left adjoint $\alpha_!$ sends (X, Y) to the map $X \to X \amalg Y$, which is a cofibration, so $\alpha_!$ preserves cofibrant objects. The right adjoint α_* sends (X, Y) to the map $X \times Y \to Y$, which need not be a cofibration, so α_* does not preserve cofibrant objects.*

Objects (X, Y) in $\mathcal{M}^K$ and $f : X \to Y$ in $\mathcal{M}^K$ are fibrant iff X and Y are each fibrant in $\mathcal{M}$. In the latter case there is no condition on f since we are using the projective model structure. The left adjoint $\alpha_!$ may not preserve fibrant objects since the coproduct of fibrant objects need not be fibrant. The right adjoint α_* does preserve fibrant objects since the product of fibrant objects is fibrant.

In the following $\mathcal{M}$ is assumed to be pointed for convenience, but this hypothesis is not essential.

Proposition 5.4.17. The projective model structure for the coproduct of two indexing categories. *Let $\mathcal{M}$ be a pointed cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$, and let J and K be small categories. Consider the projective models structures as in Definition 5.4.2 on $\mathcal{M}^J$, $\mathcal{M}^K$ and $\mathcal{M}^{J \amalg K}$ for $J \amalg K$ as in Definition 2.1.5. The last of these is isomorphic as a category to the product (as in Definition 4.1.16 and Proposition 5.2.5) of the first two. Then the projective model structure on $\mathcal{M}^{J \amalg K}$ is isomorphic to the product (as in Proposition 5.2.5) of those on $\mathcal{M}^J$ and $\mathcal{M}^K$.*

Proof In $\mathcal{M}^{J \amalg K}$ a map $f : X \to Y$ is a weak equivalence or a fibration iff f_j is one for each $j \in J$ and f_k is one for each $k \in K$.

For $j \in J$, the Yoneda functor $よ^j$ is defined by

$$(よ^j)_\ell = (J \amalg K)(j, \ell) = \begin{cases} J(j, \ell) & \text{for } \ell \in J \\ \varnothing & \text{for } \ell \in K, \end{cases}$$

and $よ^k$ for $k \in K$ is similarly defined. It follows that for $i \in \mathcal{I}$ and $j \in J$,

$$\left(よ^j \otimes i\right)_\ell = \begin{cases} J(j, \ell) \otimes i & \text{for } \ell \in J \\ * & \text{for } \ell \in K, \end{cases}$$

and similarly for $よ^k \otimes i$ for $k \in K$.

It follows that the cofibrant generating sets for the projective model structure on $\mathcal{M}^{J \amalg K}$ are the same as those for the product of the projective model structures on $\mathcal{M}^J$ and $\mathcal{M}^K$. A projective weak equivalence on $\mathcal{M}^{J \amalg K}$ is the same as the product of projective weak equivalences on $\mathcal{M}^J$ and $\mathcal{M}^K$. □

The next two statements, Proposition 5.4.18 and Corollary 5.4.20, have dual analogs involving the injective model structure which we leave to the reader.

Proposition 5.4.18. Quillen adjunctions between projective model structures. *Let $\mathcal{M}$ be a model category and let $\alpha : K \to J$ be a functor between small categories K and J. Let the functor categories $\mathcal{M}^K$ and $\mathcal{M}^J$ have the projective model structures of Theorem 5.4.10. Then the functors $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$ and $\alpha_! : \mathcal{M}^K \to \mathcal{M}^J$ given respectively by precomposition with α and left Kan extension, form a Quillen pair $(\alpha_!, \alpha^*)$ as in Definition 4.5.1 between the projective model structures on $\mathcal{M}^K$ and $\mathcal{M}^J$ of Definition 5.4.2.*

We will give an enriched analog of the above in Proposition 5.6.29.

Proof The functor $\alpha_!$ is the left adjoint of α^* by Proposition 3.2.37. To show that we have a Quillen adjunction it suffices by Proposition 4.5.12 (iii) to show that the right adjoint α^* preserves fibrations and trivial fibrations. This follows immediately from Definition 5.4.2. □

Remark 5.4.19. The right Kan extension. *In the situation of Proposition 5.4.18, the functor α^* also has a right adjoint α_* given by right Kan extension. A similar argument shows that (α^*, α_*) is a Quillen pair relating $\mathcal{M}^K$ and $\mathcal{M}^J$ with their injective model structures. However it is* **not** *a Quillen pair with respect to the projective model structures, as Example 5.4.16 illustrates.*

Corollary 5.4.20. The colimit as a left Quillen functor. *For a model category $\mathcal{M}$ and a small category J, the functors*

$$\operatorname*{colim}_J : \mathcal{M}^J \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{M} : \Delta$$

form a Quillen pair relating the model categories $\mathcal{M}^J$ (with the projective model structure) and $\mathcal{M}$.

Dually, there is a Quillen adjunction

$$\Delta \dashv \lim_J$$

involving the injective model structure on $\mathcal{M}^J$.

Proof We apply Proposition 5.4.18 to the functor $\alpha : J \to *$, where $*$ is the trivial category and α sends each object of J to the single object of $*$. This means that $\mathcal{M}^* = \mathcal{M}$ and $\alpha^* : \mathcal{M} \to \mathcal{M}^J$ is the constant diagram or diagonal functor Δ of §2.3C. Its left adjoint $\alpha_! : \mathcal{M}^J \to \mathcal{M}$ is the colimit functor by Proposition 2.3.24. □

5.4C Confinement or right induction from subcategories of J

The following will be needed in Chapter 7 and Chapter 9 below to construct positive model structures on categories of spectra. We will sometimes refer to the new model structure on the functor category $\mathcal{M}^J$ as a **confinement** of the original one, since it has fewer cofibrations than before.

Theorem 5.4.21. Confined model structures on $\mathcal{M}^J$ right induced from subcategories of J. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$, and let $\alpha : K \to J$ be a fully faithful functor (as in Definition 2.1.12) between small categories. In particular, K could be a full subcategory of J. Then*

(i) The functors $(\alpha_!, \alpha^*)$ form a transfer pair as in Definition 5.2.25, so the projective model structure on $\mathcal{M}^K$ induces a model structure on $\mathcal{M}^J$ as in the Crans-Kan Transfer Theorem 5.2.27, and

(ii) the sets

$$\bigcup_{k \in \mathrm{ob}K} \text{よ}^{\alpha(k)}\mathcal{I} \qquad \textit{and} \qquad \bigcup_{k \in \mathrm{ob}K} \text{よ}^{\alpha(k)}\mathcal{J}$$

are cofibrant generating sets for the induced model structure on $\mathcal{M}^J$.

(iii) Furthermore, with respect to this model structure on $\mathcal{M}^J$, $(\alpha_!, \alpha^*)$ is a Quillen equivalence as in Definition 4.5.14.

An enriched analog of this is Theorem 5.6.38 below.

Proof (i) Recall that Proposition 2.5.15 deals with left Kan extensions along fully faithful functors such as α. It says that $\mathcal{M}^K$ is a retract of $\mathcal{M}^J$, so Corollary 5.2.31 implies that $\alpha_!, \alpha^*$ is a transfer adjunction.

(ii) By the Crans-Kan Transfer Theorem 5.2.27 it suffices to show that the indicated generating sets are the images under $\alpha_!$ of the generating sets of Theorem 5.4.10 for the projective model structure on $\mathcal{M}^K$, namely

$$\bigcup_{k \in \mathrm{ob}K} \text{よ}^{k}\mathcal{I} \qquad \text{and} \qquad \bigcup_{k \in \mathrm{ob}K} \text{よ}^{k}\mathcal{J}.$$

For each object k of K we have the diagram

$$\begin{tikzcd} K \arrow[rr, "\text{よ}^k"] \arrow[dr, "\alpha"'] & & \mathcal{M} \\ & J. \arrow[ur, dashed, "\alpha_! \text{よ}^k"'] & \end{tikzcd}$$

Recall that $\alpha_!$ is the left adjoint of the precomposition functor α^* by definition. Thus for $k \in K$ and any $Y \in \mathcal{M}^J$ we have

$$\mathcal{M}^J(\alpha_! \text{よ}^k, Y) \cong \mathcal{M}^K(\text{よ}^k, \alpha^* Y) \cong (\alpha^* Y)_k \cong Y_{\alpha(k)} \cong \mathcal{M}^J(\text{よ}^{\alpha(k)}, Y),$$

so $\alpha_! \text{よ}^k \cong \text{よ}^{\alpha(k)}$ as desired.

(iii) The adjunction is a Quillen equivalence by Corollary 5.2.31. □

Remark 5.4.22. The positive stable model structure in [MMSS01, §14]. *That reference is concerned with the case where J is the natural numbers and nondecreasing maps, and K is the set of positive integers. In [MMSS01, Theorem 14.1] they say that a cofibration in the positive model structure must be a homeomophism in degree 0. Their situation is simpler than ours since in their case there are no maps from objects in K to objects in J not in K. The case of the ordinary Mandell-May category $\mathscr{J}$ is similar, but the equivariant Mandell-May category $\mathscr{J}_G$ is not.*

Remark 5.4.23. Comparing the confined and projective model structures on $\mathcal{M}^J$. *In the model structure on $\mathcal{M}^J$ induced from that on $\mathcal{M}^K$, a map f is a fibration or a weak equivalence iff f_j is one for* **each object** j **of** J **in the image of** α*. In the projective model structure on $\mathcal{M}^J$ a map f is a fibration or a weak equivalence iff f_j is one for* **all objects** j **of** J*. This weaker condition in the induced case means there are more fibrations and weak equivalences than in the projective model structure.*

Hence there are fewer cofibrations and trivial cofibrations because they are required to have the left lifting property with respect to more morphisms. This also follows from the fact that the cofibrant generating sets in the induced structure are smaller than those in the projective one.

Theorem 5.4.21 gives us a distinct model structure on the diagram category $\mathcal{M}^J$ for each full subcategory of the small category J. In the extreme case when the subcategory of J is empty, all maps in $\mathcal{M}^J$ are fibrations and weak equivalences, and the only cofibrations, trivial or otherwise, are isomorphisms. This is one of the two maximal model structures of Example 4.1.18.

Proposition 5.4.24. Cofibrant approximations in the confined model structure. *With notation as in Theorem 5.4.21, let A be a projectively cofibrant object in $\mathcal{M}^J$. For a fully faithful functor $\alpha : K \to J$, define a functor $Q_\alpha A$ in $\mathcal{M}^J$ by the coend (see Definition 2.4.5)*

$$(Q_\alpha A)_j = \int_{k \in K} J(\alpha(k), j) \times A_{\alpha(k)}$$

with the evident structure maps. Equivalently,

$$Q_\alpha A = \alpha_! \alpha^*(A).$$

Let $q_\alpha : Q_\alpha A \to A$ be the counit ϵ_A of the adjunction $\alpha_! \dashv \alpha^*$ as in Definition 2.2.20. Then it is a cofibrant approximation to A in the confined model structure on $\mathcal{M}^J$.

Proof First we need to show that q_α is a weak equivalence in the confined model structure. By definition this holds if

$$\alpha^* q_\alpha = \alpha^* \alpha_! \alpha^* : \alpha^* Q_\alpha A \to \alpha^* A$$

is a weak equivalence in the projective model structure for $\mathcal{M}^K$. By Proposition 2.5.15, the functor $\alpha^*\alpha_!$ is naturally equivalent to the identity functor, so $\alpha^* Q_\alpha A$ is naturally isomorphic to, and hence weakly equivalent to $\alpha^* A$.

Next we need to show that $Q_\alpha A = \alpha_! \alpha^* A$ is cofibrant in the confined model structure on $\mathcal{M}^J$. Since the left adjoint $\alpha_!$ sends projectively cofibrant objects in $\mathcal{M}^K$ to confined cofibrant objects in $\mathcal{M}^J$, it suffices to show that $\alpha^* A$ is cofibrant.

For this we will show that (α^*, α_*) is a Quillen adjunction between the projective model structures on $\mathcal{M}^J$ and $\mathcal{M}^K$, by showing that α_* preserves fibrations and trivial fibrations. For a functor X in $\mathcal{M}^K$, we know by (2.5.12) that for each j in J,

$$\alpha_*(X)_j = \int^{k \in K} X_k^{J(j,\alpha(k))}. \tag{5.4.25}$$

For a fibration or trivial fibration $p : X \to Y$ in $\mathcal{M}^K$, each induced map $p_k : X_k \to Y_k$ is a fibration or trivial fibration in $\mathcal{M}$, as is the map $p_k^{J(j,\alpha(k))}$, being the product fibrations or trivial fibrations. It follows that he same is true of the map of ends $\alpha_*(p)$. Thus α_* is a right Quillen functor, so α^* is a left Quillen functor. This means that $\alpha^* A$ and hence $Q_\alpha A = \alpha_! \alpha^* A$ are cofibrant as required. □

5.4D The relation between confinement and enlargement of model structures

Now we will discuss the relation between confinement as in Theorem 5.4.21 and enlargement as in Theorem 5.2.34.

Theorem 5.4.26. Confinement and enlargement. *Let $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{M}$, and $\mathcal{M}'$ be cofibrantly generated model categories. Suppose we have a commutative diagram*

$$\begin{tikzcd}
\mathcal{M}\times\mathcal{M}' \arrow[r, shift left, "\mathcal{M}\coprod F"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "B\times B'"] \arrow[d, phantom, "\dashv"] & \mathcal{M} \arrow[l, shift left, "\mathcal{M}\times U"] \arrow[d, shift left, "B"] \arrow[d, phantom, "\dashv"] \\
\mathcal{L}\times\mathcal{L}' \arrow[u, shift left, "A\times A'"] \arrow[r, shift left, "\mathcal{L}\coprod E"] \arrow[r, phantom, "\perp"] & \mathcal{L} \arrow[l, shift left, "\mathcal{L}\times V"] \arrow[u, shift left, "A"]
\end{tikzcd} \tag{5.4.27}$$

where (F, U), (E, V), (A, B) and (A', B') are transfer adjunctions (as in Definition 5.2.25) in which each right adjoint preserves weak equivalences in the given model structure on the category on the right, $\mathcal{L}$ ($\mathcal{L}'$) is a bireflective subcategory of $\mathcal{M}$ ($\mathcal{M}'$) as in Definition 2.2.50, and the functors A and A'

preserve trivial cofibrations. Suppose that the images of the cofibrant generating sets of $\mathcal{L} \times \mathcal{L}'$ *in* $\mathcal{M}$ *permit the small object argument there.*

Then the two composite adjunctions between the lower left and upper right corners in the diagram are also transfer adjunctions. Starting with the given model structures in $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{L}$ *and* $\mathcal{L}'$, *we get three more model structures on* $\mathcal{M}$ *transferred from those on the three other categories in the diagram. The top horizontal adjunction gives the enlarged model structure, the right vertical adjunction the confined model structure, and the composite adjunction gives what could be called the* **enlarged confined** *or the* **confined enlarged** *model structure. Only the last of these depends on the choice of* $\mathcal{L}'$ *and its adjunctions.*

Note that the top adjunction in (5.4.27) is the enlarging adjunction of Theorem 5.2.34, while the one on the right is a generalization of the one in Theorem 5.4.21. Given these two adjunctions, one might want to choose $\mathcal{L}'$ and its adjunctions so as to make the diagram a pullback in some sense, like the one in (5.4.34) below. We leave such a formulation to the future.

Note also that the adjunctions in the diagram are not assumed to be Quillen adjunctions. The transfer adjunction hypothesis depends on the model structure on the domain of the left adjoint, making no reference to the one on the codomain.

Proof It is easy to check that the horizontal adjunctions are transfer adjunctions. It follows from Proposition 5.2.26 that the left vertical adjunction is is one as well.

In the left vertical adjunction, the lower category is a bireflective subcategory (as in Definition 2.2.50) of the upper one by Proposition 2.2.51.

We will show that the hypotheses of Proposition 5.2.32 are met for each of the composite adjunctions. The first of them, the smallness condition, is met by assumption. For the second, note that $B \times B'$ and $\mathcal{L} \times V$ preserve given weak equivalences because B, B' and V each do. The functor $A \times A'$ preserves trivial cofibrations, so (c′) is satisfied by the clockwise composite of left adjoints. In the counterclockwise case (c″) is satisfied. □

Example 5.4.28. A trivial choice of $\mathcal{L}'$ leading to an uninteresting enlarged confined model structure on $\mathcal{M}$. *Assuming that* $\mathcal{M}$ *and* $\mathcal{M}'$ *in Theorem 5.4.26 are pointed,* $\mathcal{L}'$ *could be the trivial pointed category with a single object and a single morphism. Then the adjunctions of* (A', B') *and* (E, V) *would lead to model structures on* $\mathcal{L}$ *and* $\mathcal{M}'$ *in which all morphisms are trivial fibrations and all cofibrations are isomorphisms. Thus the resulting class of cofibrations in* $\mathcal{M}$ *would be minimal (isomorphisms only) while those of weak equivalences and fibrations would be maximal, meaning all maps.*

Corollary 5.4.29. Four model structures on $\mathcal{M}$. *With notation as in Theorem 5.4.26, denote the enlarged, confined and enlarged confined model* *structures on $\mathcal{M}$ by $\mathcal{M}_{enla}$, $\mathcal{M}_{conf}$, and $\mathcal{M}_{enco}$. Then the diagram of (5.4.27) can be expanded to*

$$\begin{tikzcd}
\mathcal{M}_{enla} \arrow[rrr, shift left] \arrow[dr, shift left] \arrow[dddd, shift left] & & & \mathcal{M} \arrow[lll, shift left, "\top"] \arrow[dl, shift left, "\perp"] \arrow[dddd, shift left, "\dashv"] \\
& \mathcal{M}\times\mathcal{M}' \arrow[r, shift left, "\mathcal{M}\coprod F"] \arrow[ul, shift left, "\top"] \arrow[dd, shift left, "B\times B'"] & \mathcal{M} \arrow[l, shift left, "\mathcal{M}\times U"] \arrow[ur, shift left] \arrow[dd, shift left, "B"] \arrow[l, phantom, "\perp"] & \\
& & & \\
& \mathcal{L}\times\mathcal{L}' \arrow[r, shift left, "\mathcal{L}\coprod E"] \arrow[uu, shift left, "A\times A'", "\dashv"'] \arrow[dl, shift left] & \mathcal{L} \arrow[l, shift left, "\mathcal{L}\times V"] \arrow[uu, shift left, "A", "\dashv"'] \arrow[dr, shift left] \arrow[l, phantom, "\perp"] & \\
\mathcal{M}_{enco} \arrow[rrr, shift left] \arrow[ur, shift left, "\top"] \arrow[uuuu, shift left, "\dashv"] & & & \mathcal{M}_{conf} \arrow[lll, shift left, "\top"] \arrow[ul, shift left, "\perp"] \arrow[uuuu, shift left]
\end{tikzcd} \tag{5.4.30}$$

where the horizontal and vertical arrows on the outer square and the upper right diagonal arrows are all identity functors, and the other diagonal left (right) adjoints are such that the diagram of left (right) adjoints commutes. The diagonal and outer adjunctions are Quillen pairs, but the inner ones are not.

Note that none of the left adjoints in the inner square of (5.4.30) is the composite of other left adjoints in the diagram that are left Quillen functors, so Proposition 4.5.22 does not imply that any of the inner left adjoints are left Quillen functors. The same goes for right adjoints.

Note also the reversal of direction of the outer horizontal left adjoints from the inner ones. Categorically the outer functors are both left and right adjoints since they are all identity functors, but only the ones indicated as left (right) adjoints preserve cofibrations (fibrations).

Corollary 5.4.31. Confinement and enlargement for functor categories. *Let $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$ be cofibrantly generated model categories and let $\alpha : K \to J$ be a fully faithful functor between small categories. Suppose we have a commutative diagram of the form*

$$\begin{tikzcd}
\mathcal{M}^J\times\mathcal{M}' \arrow[r, shift left, "\mathcal{M}^J\coprod F"] \arrow[d, shift left, "\alpha^*\times B"] & \mathcal{M}^J \arrow[l, shift left, "\mathcal{M}^J\times U"] \arrow[l, phantom, "\perp"] \arrow[d, shift left, "\alpha^*"] \\
\mathcal{M}^K\times\mathcal{M}'' \arrow[r, shift left, "\mathcal{M}^K\coprod E"] \arrow[u, shift left, "\alpha_!\times A", "\dashv"'] & \mathcal{M}^K \arrow[l, shift left, "\mathcal{M}^K\times V"] \arrow[l, phantom, "\perp"] \arrow[u, shift left, "\alpha_!", "\dashv"']
\end{tikzcd} \tag{5.4.32}$$

where (F, U), (E, V) and (A, B) are transfer adjunctions (as in Definition 5.2.25) *in which U and V preserve weak equivalences, and $\mathcal{M}''$ is a bireflective subcategory of $\mathcal{M}'$ as in Definition 2.2.50.*

Then the two composite adjunctions in the diagram are also transfer adjunctions. Starting with the projective model structures in $\mathcal{M}^J$ and $\mathcal{M}^K$, and the given ones on $\mathcal{M}'$ and $\mathcal{M}''$, we get three more model structures on $\mathcal{M}^J$ transferred from those on the three other categories in the diagram. The top horizontal adjunction gives the enlarged model structure, the right vertical adjunction the confined model structure, and the composite adjunction gives what could be called the **enlarged confined** *or the* **confined enlarged** *model structure. The last of these depends on the choice of $\mathcal{M}''$ and its adjunctions.*

Proof Note that $\mathcal{M}^K$ is a bireflective subcategory of $\mathcal{M}^J$ by Proposition 2.5.15, and $(\alpha_!, \alpha^*)$ is a transfer adjunction by Theorem 5.4.21. This makes the present situation a special case of Theorem 5.4.26. □

We do not have an abstract description of the optimal choice of $\mathcal{L}'$ in terms of the other categories in Theorem 5.4.26, but we do have a good one in the following case.

Example 5.4.33. Application to G-spectra. *Let*

$$\begin{aligned} \mathcal{M} &= \mathcal{T}^G, \qquad J = \mathscr{J}_G \qquad \text{and} \qquad K = \mathscr{J}_G^+; \\ \mathcal{M}^J &= [\mathscr{J}_G, \mathcal{T}^G] = \mathcal{S}p^G \\ \mathcal{M}^K &= [\mathscr{J}_G^+, \mathcal{T}^G] =: \mathcal{S}p_+^G \\ \mathcal{M}' &= \prod_{H\subset G} [\mathscr{J}_H, \mathcal{T}^H] = \prod_{H\subset G} \mathcal{S}p^H \\ \mathcal{M}'' &= \prod_{H\subset G} [\mathscr{J}_H^+, \mathcal{T}^H] =: \prod_{H\subset G} \mathcal{S}p_+^H, \end{aligned}$$

Here $\mathcal{S}p_+$ denotes the category of positively indexed spectra. (See Chapter 9 below for more discussion of these categories.) Hence (5.4.27) reads

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\prod\limits_{H\subseteq G} \mathcal{S}p^H \arrow[r, shift left, "\bigvee G\underset{H}{\ltimes}(-)"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\prod \alpha^*"] \arrow[d, phantom, "\dashv"] & \mathcal{S}p^G \arrow[l, shift left, "\prod i_H^G"] \arrow[d, shift left, "\alpha^*"] \arrow[d, phantom, "\dashv"] \\
\prod\limits_{H\subseteq G} \mathcal{S}p_+^H \arrow[r, shift left, "\bigvee G\underset{H}{\ltimes}(-)"] \arrow[r, phantom, "\perp"] \arrow[u, shift left, "\prod \alpha_!"] & \mathcal{S}p_+^G. \arrow[l, shift left, "\prod i_H^G"] \arrow[u, shift left, "\alpha_!"]
\end{tikzcd} \tag{5.4.34}$$

Here each category in the lower row is a bireflective subcategory of the one above it. Each right adjoint functor preserves weak equivalences.

The resulting four model structures on $\mathcal{S}p^G$ are the ones on the left in

Figure 7.1 and the four unstable ones listed in Theorem 9.2.13, where cofibrant generating sets are indicated for each of them.

5.5 Monoidal model categories

A monoidal model category is more than simply a model category as in Definition 4.1.1 equipped with a closed monoidal structure as in def-closed-SMC. We need the monoidal structure to mesh with the model structure in certain ways.

A closed symmetric monoidal structure is a special case of a two variable adjunction as in Definition 2.6.26. Just as Quillen adjunctions as in Definition 4.5.1 are the right adjunctions to use in model category theory, we will define a **two variable Quillen adjunction** in Definition 5.5.3. It is a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$, for model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, in which the pushout product of cofibrations in $\mathcal{C}$ and $\mathcal{D}$ is a cofibration in $\mathcal{E}$, which is trivial if either of its factors is. Such adjunctions and related constructions are the subject of §5.5A.

This sets the stage for the definition of a monoidal model category, which we call a **Quillen ring**, and related notions in §5.5B. A Quillen ring is required to satisfy the **pushout product and unit axioms** stated in Definition 5.5.9. The unit axiom is only needed in the case where the unit object fails to be cofibrant. While it is cofibrant in the most familiar examples (see Proposition 5.5.13), we will need a model structure on the category of spectra in which it is not. See Remark 5.5.11.

In §5.5C we will consider additional axioms that a Quillen ring $\mathcal{M}$ might satisfy. They are designed to insure that certain additional categories associated with $\mathcal{M}$ are again model categories and possibly Quillen rings.

The first of these is the **Schwede-Shipley monoid axiom** of Definition 5.5.22. It says that certain morphisms in $\mathcal{M}$ that one would expect to be weak equivalences really are weak equivalences. In Theorem 5.5.25 we see that the categories of suitably defined modules or algebras over a monoid R in $\mathcal{M}$ have their own model structures. Schwede and Shipley prove this in each case by showing that a certain adjunction is a transfer adjunction as in Definition 5.2.25, so the Crans-Kan Transfer Theorem 5.2.27 can be used to give a model structure on the new category.

Next we have the **White commutative monoid axiom** of Definition 5.5.26. It is relevant to the existence of a model structure on the category of commutative monoids in a Quillen ring $\mathcal{M}$; see Theorem 5.5.27. The proof of this theorem is not difficult, but verifying its hypotheses in the case of orthogonal G-spectra is. We take up this problem in Chapter 10.

In §5.5D we study the arrow category of a compactly generated Quillen ring. The results there will be needed in §10.3.

5.5A Quillen bifunctors

The definitions in this subsection are taken from [Hov99, §4.2], where Hovey attributes them to Jeff Smith.

Before defining monoidal model categories (in Definition 5.5.9 below), we will state a result about monoidal categories that might be model categories. In §9.2 we will use the Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24 to show that the category of G-spectra, which is known to be closed symmetric monoidal, has a cofibrantly generated model structure defined in terms of certain generating sets $\mathcal{I}$ and $\mathcal{J}$.

Recall that a set of morphisms $\mathcal{I}$ in a category $\mathcal{C}$ generates a set of cofibrations cofib$(\mathcal{I})$ defined in terms of lifting properties in Definition 5.2.1. The following is proved in [Hov99, Lemma 4.2.4].

Lemma 5.5.1. Pushout corners of cofibrations. *Suppose*

$$\otimes : \mathcal{C}' \times \mathcal{C}'' \to \mathcal{C}$$

is a two variable adjunction (see Definition 2.6.26), and that $\mathcal{I}'$, $\mathcal{I}''$ and $\mathcal{I}$ are sets of maps in the three categories. Suppose as well that $\mathcal{I}' \,\square\, \mathcal{I}'' \subseteq \mathcal{I}$; see Definition 2.6.12. Then

$$\mathit{cofib}(\mathcal{I}') \,\square\, \mathit{cofib}(\mathcal{I}'') \subseteq \mathit{cofib}(\mathcal{I}).$$

The special case where the three categories and the three collections of maps are the same yields the following.

Corollary 5.5.2. Pushout corner maps in closed monoidal categories. *Let $\mathcal{I}$ be a set of maps in a closed monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$ (Definition 2.6.33). Then*

$$\mathit{cofib}(\mathcal{I}) \,\square\, \mathit{cofib}(\mathcal{I}) \subseteq \mathit{cofib}(\mathcal{I}).$$

Now we have a variation on Definition 2.6.26, which is taken from [Hov99, Definition 4.2.1].

Definition 5.5.3. *For model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, a two variable adjunction $(\wedge, \mathrm{Hom}_\ell, \mathrm{Hom}_r, \varphi_\ell, \varphi_r)$ as in Definition 2.6.26 is a* **two variable Quillen adjunction** *if for each pair of cofibrations $f : C_1 \to C_2$ in $\mathcal{C}$ and $g : D_1 \to D_2$ in $\mathcal{D}$, the pushout corner map (see Definition 2.6.12) $f \,\square\, g$ is a cofibration in $\mathcal{E}$ which is trivial if either f or g is trivial. The functor $\wedge$ here is said to be a* **Quillen bifunctor***.*

The pushout corner condition above is simplified by the following, which is proved by Hovey as [Hov99, Corollary 4.2.5].

Proposition 5.5.4. Cofibrant generating sets and Quillen bifunctors. *Suppose we have model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ with a two variable adjunction as in Definition 2.6.26. Suppose further that $\mathcal{C}$ and $\mathcal{D}$ are cofibrantly generated with generating sets $(\mathcal{I},\mathcal{J})$ and $(\mathcal{I}',\mathcal{J}')$ respectively. Then $\wedge$ is a Quillen bifunctor if and only if $\mathcal{I}\,\square\,\mathcal{I}'$ consists of cofibrations and both $\mathcal{I}\,\square\,\mathcal{J}'$ in $\mathcal{E}$ and $\mathcal{J}\,\square\,\mathcal{I}'$ consist of trivial cofibrations in $\mathcal{E}$.*

The following is [Hov99, Lemma 4.2.2] and its proof is an exercise for the reader. The maps $\mathcal{D}_{\diamondsuit}(i,p)$ and $\mathcal{C}_{\diamondsuit}(j,p)$ are instances of the lifting test map of Definition 2.3.14.

Lemma 5.5.5. Quillen bifunctors and lifting test maps. *For model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, let*

$$(\wedge, \mathrm{Hom}_\ell, \mathrm{Hom}_r, \varphi_\ell, \varphi_r)$$

be a two variable adjunction as in Definition 2.6.26. Then the following are equivalent:

(i) $\wedge$ is a Quillen bifunctor.

(ii) Given a cofibration $i : A \to B$ in $\mathcal{C}$ and a fibration $p : X \to Y$ in $\mathcal{E}$, the induced map

$$\mathcal{D}_{\diamondsuit}(i,p) : \mathrm{Hom}_\ell(B,X) \to \mathrm{Hom}_\ell(B,Y) \underset{\mathrm{Hom}_\ell(A,Y)}{\times} \mathrm{Hom}_\ell(A,X)$$

is a fibration in $\mathcal{D}$ which is trivial if either i or p is trivial.

(iii) Given a cofibration $j : A' \to B'$ in $\mathcal{D}$ and a fibration $p : X \to Y$ in $\mathcal{E}$, the induced map

$$\mathcal{C}_{\diamondsuit}(j,p) : \mathrm{Hom}_r(B',X) \to \mathrm{Hom}_r(B',Y) \underset{\mathrm{Hom}_r(A',Y)}{\times} \mathrm{Hom}_r(A',X)$$

is a fibration in $\mathcal{C}$ which is trivial if either j or p is trivial.

Corollary 5.5.6. Special cases of lifting test maps. *For model categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, let*

$$(\wedge, \mathrm{Hom}_\ell, \mathrm{Hom}_r, \varphi_\ell, \varphi_r)$$

be a two variable Quillen adjunction as in Definition 5.5.3. Then

(i) Given a cofibrant object B in $\mathcal{C}$ and a fibration $p : X \to Y$ in $\mathcal{E}$, the induced map

$$p_* : \mathrm{Hom}_\ell(B,X) \to \mathrm{Hom}_\ell(B,Y)$$

is a fibration which is trivial when p is trivial.

(ii) Given a cofibrant object B' in $\mathcal{D}$ and a fibration $p : X \to Y$ in $\mathcal{E}$, the induced map

$$p_* : \mathrm{Hom}_r(B',X) \to \mathrm{Hom}_\ell(B',Y)$$

is a fibration which is trivial when p is trivial.

(iii) Given a cofibration $i : A \to B$ in $\mathcal{C}$ and a fibrant object X in $\mathcal{E}$, the induced map

$$i^* : \mathrm{Hom}_\ell(B, X) \to \mathrm{Hom}_\ell(A, X)$$

is a fibration in $\mathcal{D}$ which is trivial if i is trivial.

(iv) Given a cofibration $j : A' \to B'$ in $\mathcal{D}$ and a fibrant object X in $\mathcal{E}$, the induced map

$$j^* : \mathrm{Hom}_r(B', X) \to \mathrm{Hom}_r(A', X)$$

is a fibration in $\mathcal{D}$ which is trivial if j is trivial.

Proof The first two statements are the second and third parts of Lemma 5.5.5 for $A = \varnothing$ and $A' = \varnothing$ by Proposition 2.3.18(i). The second two statements are the second and third parts of Lemma 5.5.5 for $Y = *$ by Proposition 2.3.18(ii). $\square$

Corollary 5.5.7. Cofibrant generating sets in two variable adjunctions. *Suppose $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a two variable adjunction between model categories. Suppose also that $\mathcal{C}$ and $\mathcal{D}$ are cofibrantly generated, with generating cofibrations $\mathcal{I}$ and $\mathcal{I}'$ respectively, and generating trivial cofibrations $\mathcal{J}$ and $\mathcal{J}'$ respectively. Then $\otimes$ is a Quillen bifunctor if and only if $\mathcal{I} \square \mathcal{I}'$ consists of cofibrations and both $\mathcal{I} \square \mathcal{J}'$ and $\mathcal{J} \square \mathcal{I}'$ consist of trivial cofibrations in $\mathcal{E}$.*

Proposition 5.5.8. Quillen adjunctions associated with a Quillen bifunctor. *Suppose we have a two variable Quillen adjunction as in Definition 5.5.3.*

(i) The ordinary adjunctions of (2.6.27) and (2.6.28),

$$(C \wedge -) : \mathcal{D} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E} : \mathrm{Hom}_\ell(C, -)$$

and

$$(- \wedge D) : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E} : \mathrm{Hom}_r(D, -),$$

are Quillen adjunctions as in Definition 4.5.1 when C and D are cofibrant objects in $\mathcal{C}$ and $\mathcal{D}$ respectively, and thus fibrant objects in $\mathcal{C}^{op}$ and $\mathcal{D}^{op}$.

(ii) The equivalent ordinary adjunctions of Proposition 2.6.31,

$$\mathrm{Hom}_\ell^{op}(E, -) : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D}^{op} : \mathrm{Hom}_r(E, -)$$

and

$$\mathrm{Hom}_r^{op}(-, E) : \mathcal{D} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}^{op} : \mathrm{Hom}_\ell(-, E),$$

are Quillen adjunctions when E is a fibrant object in $\mathcal{E}$.

Proof By Proposition 4.5.12 it suffices to show that the left (right) adjoints preserve cofibrations (fibrations) and trivial cofibrations (trivial fibrations).

For the first adjunction of (i), let $d : D_1 \to D_2$ be a cofibration in $\mathcal{D}$. Then, as explained in Example 2.6.14, $C \wedge d$ is the map $(\varnothing_{\mathcal{C}} \to C) \square d$, where $\varnothing_{\mathcal{C}}$ is in the initial object in $\mathcal{C}$. Therefore it is a cofibration which is trivial when g is trivial. The argument for the second adjunction is similar.

For the first adjunction of (ii), each fibration in $\mathcal{D}^{op}$ corresponds to a cofibration $d : D_1 \to D_2$ in $\mathcal{D}$. Thus we need to show that the map

$$d^* : \mathrm{Hom}_r(D_2, E) \to \mathrm{Hom}_r(D_1, E)$$

is a fibration in $\mathcal{C}$. This follows from Corollary 5.5.6(iv). Similarly, the second adjunction of Proposition 2.6.31 is a Quillen adjunction by Corollary 5.5.6(iii). □

5.5B Quillen rings

Definition 5.5.9. *A* **(symmetric) monoidal model category** *or* **(commutative) Quillen ring** *is a closed (symmetric) monoidal category* $(\mathcal{M}, \wedge, S)$ *with a model structure satisfying the following two axioms.*

(i) **Pushout product axiom.** *The operation* $\wedge$ *is a Quillen bifunctor as in Definition 5.5.3. This means that* $f \square g$ *(see Definition 2.6.12) is a cofibration whenever* f *and* g *are, and it is a trivial cofibration if in addition either* f *or* g *is one.*

(ii) **Unit axiom**. *Let* $q : QS \to S$ *be the cofibrant replacement (see Definition 4.1.20) of the unit object* S*. Then smashing the source and target of* q *on either side with a cofibrant object* K *gives a weak equivalence.*

From now on, all Quillen rings are assumed to be commutative unless otherwise stated.

Remark 5.5.10. *This is a followup to Remark 2.6.24. The terms* **Quillen ring** *and* **Quillen module** *(see Definition 5.5.17 below) are not common in the literature. Their use was suggested by Vigleik Angeltveit in [Ang08, §3].*

Remark 5.5.11. *Applying the pushout product axiom to the maps* $* \to A$ *and* $* \to B$ *for cofibrant* A *and* B *leads to the conclusion that* $A \wedge B$ *is also cofibrant.*

The unit axiom is redundant if the unit object S *is cofibrant, but in general it is needed to ensure that the homotopy category has a monoidal structure with unit* S*. See [Hov99, Theorem 4.3.2]. We will see below in Corollary 7.4.49 that in our model structure of choice for the category* $\mathcal{S}p^G$ *of equivariant* G*-spectra, the sphere spectrum* S^{-0} *(its unit object) is* **not** *cofibrant.*

In [Lur09, Definition A.3.1.2], Lurie's definition of a monoidal model category, he assumes that the unit object is cofibrant.

Example 5.5.12. $\mathcal{S}et$ **as a Quillen ring.** *Let the symmetric monoidal category* $(\mathcal{S}et, \times, *)$ *have the model structure in which weak equivalences are isomorphisms and all maps are fibrations and cofibrations; see Example 4.1.17. It is a Quillen ring. While every category is enriched over* $\mathcal{S}et$*, and every bicomplete category is a* $\mathcal{S}et$*-module as in Definition 2.6.42, we will see in Remark 5.5.20 below a similar statement about model categories is not true.*

The following is proved by Hovey in [Hov99, Propositions 4.2.8 and 4.2.11, and Corollaries 4.2.10 and 4.2.12]. Recall (Definition 2.1.48) that for us, topological spaces are assumed to be compactly generated weak Hausdorff. Thus our category $\mathcal{T}op$ is not the same as Hovey's **Top**, the category of **all** topological spaces.

Proposition 5.5.13. Some Quillen rings. *The following model categories are Quillen rings as in Definition 5.5.9:*

$(\mathcal{S}et_{\boldsymbol{\Delta}}, \times, *)$ *(simplicial sets under Cartesian product),*
$(\mathcal{S}et_{\boldsymbol{\Delta}*}, \wedge, S^0)$ *(pointed simplicial sets under smash product),*
$(\mathcal{T}op, \times, *)$ *(topological under Cartesian product) and*
$(\mathcal{T}, \wedge, S^0)$ *(pointed topological spaces under smash product).*

Proposition 5.5.4 implies the following.

Proposition 5.5.14. The pushout product axiom in the cofibrantly generated case. *Let* $(\mathcal{M}, \wedge, S)$ *be a closed monoidal model category that is cofibrantly generated with generating sets* $\mathcal{I}$ *and* $\mathcal{J}$*. Then the pushout product axiom of Definition 5.5.9(i) holds iff each morphism in* $\mathcal{I} \square \mathcal{I}$ *is a cofibration, and each one in* $\mathcal{J} \square \mathcal{I}$*,* $\mathcal{I} \square \mathcal{J}$ *and* $\mathcal{J} \square \mathcal{J}$ *is a trivial cofibration.*

The following is the application of Lemma 5.5.5 to the case where all three categories are the same, and we denote it by $\mathcal{N}$. In that case the functor $\wedge : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ is a closed monoidal structure.

Proposition 5.5.15. The pushout product and the lifting test map. *Let* $(\mathcal{N}, \wedge, S)$ *be a model category with a closed symmetric monoidal structure. Then the following are equivalent:*

(i) $\wedge$ *satisfies the pushout product axiom, meaning it is a Quillen bifunctor.*
(ii) Given a cofibration $i : A \to B$ *and a fibration* $p : X \to Y$ *in* $\mathcal{M}$*, the induced map*

$$\mathcal{N}_{\diamondsuit}(i,p) : \mathcal{N}(B,X) \to \mathcal{N}(B,Y) \underset{\mathcal{N}(A,Y)}{\times} \mathcal{N}(A,X)$$

is a fibration in $\mathcal{N}$ *which is trivial if either* i *or* p *is trivial.*

Definition 5.5.16. *Given Quillen rings* $(\mathcal{M}, \wedge, S)$ *and* $(\mathcal{N}, \otimes, T)$*, a* **strong (weak) monoidal Quillen adjunction** *between them is a Quillen adjunction* (F, U, φ) *(Definition 4.5.1) such that the left adjoint* F *is strong (oplax* *monoidal) as in Definition 2.6.19 and the map $F(q) : F(QS) \to F(S)$ (where $q : QS \to S$ is functorial cofibrant replacement in $\mathcal{M}$) is a weak equivalence.*

The condition on $F(q)$ above is of course redundant when the unit object S is cofibrant. Hovey [Hov99, Definition 4.2.16] requires the left adjoint F to be strong monoidal. Schwede-Shipley [SS03a, Definition 3.6] only require it to be oplax monoidal, but they also require that for the oplax functor F, the map $F(X \wedge Y) \to F(X) \otimes F(Y)$ is a weak equivalence for cofibrant X and Y. More precisely, they require the right adjoint U to be lax monoidal, which by Proposition 2.6.21 is equivalent to requiring F to be oplax monoidal.

It can be shown that (symmetric) monoidal model categories and strong monoidal Quillen pairs form a 2-category as in §2.7; see [Hov99, page 113].

The following is the model category analog of Definition 2.6.42. We will approach this definition from a different perspective below in Definition 5.6.3.

Definition 5.5.17. Quillen modules. *Given a Quillen ring $(\mathcal{M}, \wedge, S)$ as in Definition 5.5.9, an* $\mathcal{M}$**-model category** *or* **Quillen** $\mathcal{M}$**-module** *is a closed $\mathcal{M}$-module category $\mathcal{N}$ (Definition 2.6.42) with a model structure such that the functor*

$$\mathcal{M} \times \mathcal{N} \xrightarrow{\ \wedge\ } \mathcal{N} \tag{5.5.18}$$

is a Quillen bifunctor (and hence part of a two variable Quillen adjunction as explained in Definition 5.5.3) and, when S is not cofibrant in $\mathcal{M}$ and $q : QS \to S$ is its functorial cofibrant replacement, the map the

$$q \wedge X : QS \wedge X \to S \wedge X$$

is a weak equivalence for all cofibrant X in $\mathcal{N}$. When $\mathcal{M} = \mathcal{T}$, we say that $\mathcal{N}$ is a **pointed topological model category**.

An $\mathcal{M}$**-model Quillen functor** *between two such model categories is an $\mathcal{M}$-module functor that is also a Quillen functor. An* $\mathcal{M}$**-model Quillen adjunction** *is similarly defined. An* $\mathcal{M}$**-model natural transformation** *between two such functors is any natural transformation between them.*

A Quillen $\mathcal{M}$-module $\mathcal{N}$ is a **Quillen** $\mathcal{M}$**-algebra** *if it is also a Quillen ring.*

The Quillen ring $\mathcal{M}$ is **topological (simplicial)** *if it is a Quillen $\mathcal{T}op$-algebra (Quillen $\mathcal{S}et_\Delta$-algebra).*

Proposition 5.5.19. An adjunction isomorphism. *For $\mathcal{M}$ and $\mathcal{N}$ as in Definition 5.5.17, let M be an object in $\mathcal{M}$ and let X and Y be objects in $\mathcal{N}$. Then there is a natural isomorphism*

$$\mathcal{N}(M \wedge X, Y) \cong \mathcal{M}(M, \mathcal{N}(X, Y)).$$

Proof This is the adjunction φ_r of Definition 2.6.26 for the case

$$(\mathcal{C}, \mathcal{D}, \mathcal{E}) = (\mathcal{M}, \mathcal{N}, \mathcal{N}) \text{ with objects } (C, D, E) = (M, X, Y). \qquad \square$$

Remark 5.5.20. Not all model categories are Quillen $\mathcal{S}et$-modules. *Let $\mathcal{M} = \mathcal{S}et$ as in Example 5.5.12, and let $\mathcal{N} = \mathcal{T}op$, so the functor of (5.5.18) sends the object (A, X) in $\mathcal{S}et \times \mathcal{T}op$ to the Cartesian product $A \times X$ with the discrete topology on A. It is* **not** *a Quillen bifunctor because the pushout product $f \,\square\, g$ of a cofibration (meaning any map) f in $\mathcal{S}et$ with a cofibration g in $\mathcal{T}op$ will be another cofibration in $\mathcal{T}op$ only when f is one to one.*

Thus, even though all bicomplete categories are closed $\mathcal{S}et$-modules as in Definition 2.6.42, there appears to be no Quillen ring over which all model categories are Quillen modules. On the other hand it is known [Hov99, Chapter 5, specifically Theorem 5.5.3] that the homotopy category (Definition 4.3.16) $\mathrm{Ho}\,\mathcal{M}$ of an arbitary model category $\mathcal{M}$ is enriched over $\mathrm{Ho}\,\mathcal{S}et_\Delta$, the homotopy category for the model structure of Definition 4.2.17 on the category $\mathcal{S}et_\Delta$ of simplicial sets.

We will use the following below in the proof of Theorem 7.3.29.

Proposition 5.5.21. The tensor cotensor and smash Hom adjunctions, and lifting test maps. *For $\mathcal{M}$ and $\mathcal{N}$ as in Definition 5.5.17,*

(i) for each cofibrant object A of $\mathcal{M}$ there is a Quillen adjunction

$$A \wedge (-) : \mathcal{N} \underset{\perp}{\overset{\longrightarrow}{\longleftarrow}} \mathcal{N} : (-)^A$$

and for each cofibrant object C of $\mathcal{N}$ there is a Quillen adjunction

$$(-) \wedge C : \mathcal{M} \underset{\perp}{\overset{\longrightarrow}{\longleftarrow}} \mathcal{N} : \mathcal{N}(C, -);$$

(ii) for each fibrant object X in $\mathcal{N}$ there are Quillen adjunctions

$$(\mathcal{N}(-, X))^{op} : \mathcal{N} \underset{\perp}{\overset{\longrightarrow}{\longleftarrow}} \mathcal{M}^{op} : X^{(-)}$$

and

$$(X^{-})^{op} : \mathcal{M} \underset{\perp}{\overset{\longrightarrow}{\longleftarrow}} \mathcal{N}^{op} : \mathcal{N}(-, X);$$

(iii) given a cofibration $i : A \to B$ in $\mathcal{M}$ and a fibration $p : X \to Y$ in $\mathcal{N}$, the induced map

$$\mathcal{N}_\Diamond(i, p) : X^B \to Y^B \underset{Y^A}{\times} X^A$$

is a fibration in $\mathcal{N}$ which is trivial if either i or p is;

(iv) given a cofibration $j : C \to D$ in $\mathcal{N}$ and a fibration $p : X \to Y$ in $\mathcal{N}$, the induced map

$$\mathcal{M}_\Diamond(j, p) : \mathcal{N}(D, X) \to \mathcal{N}(D, Y) \underset{\mathcal{N}(C,Y)}{\times} \mathcal{N}(C, X)$$

is a fibration in $\mathcal{M}$ which is trivial if either j or p is.

Proof The first two statements follow from Proposition 5.5.8 and the second two follow from Lemma 5.5.5. □

As in the case of Definition 2.6.42, the collection of $\mathcal{M}$-model categories, Quillen adjunctions and natural transformations form a 2-category.

5.5C The monoid axiom and friends

The following is [SS00, Definition 3.3].

Definition 5.5.22. *A Quillen ring $(\mathcal{M}, \wedge, S)$ as in Definition 5.5.9 satisfies the* **monoid axiom** *if every morphism that is obtained as a transfinite composition of pushouts of smash products of any object with trivial cofibrations is a weak equivalence. Equivalently, the regular class (as in Definition 4.8.13)*

$$\mathcal{R}eg((\mathscr{W} \cap \mathscr{C}) \wedge \mathcal{M})$$

generated by the class of trivial cofibrations, $\mathscr{W} \cap \mathscr{C}$, (this is the notation of Definition 4.1.1) smashed with objects in $\mathcal{M}$, is made up of weak equivalences.

In the definition above, the pushouts of cofibrations may be smashed with **differing** objects before being transfinitely composed. This condition is therefore stronger than requiring that the smash product of any trivial cofibration with any object be a weak equivalence.

The following is [SS00, Lemma 3.5].

Lemma 5.5.23. The monoid axiom in the cofibrantly generated case. *The monoid axiom of Definition 5.5.22 holds if every morphism that is obtained as a transfinite composition of pushouts of smash products of any object with maps in $\mathcal{J}$ is a weak equivalence*

The following is implied by [Hov98, Lemma 2.3], as explained there by Hovey.

Theorem 5.5.24. The symmetric monoidal categories $(\mathcal{T}op, \times, *)$ and $(\mathcal{T}, \wedge, S^0)$ both satisfy the monoid axiom.

The following is proved by Schwede and Shipley as [SS00, Theorem 4.1]. The idea of the proof is to show that the adjunction of (2.6.67) is a transfer adjunction as in Definition 5.2.25, so the Crans-Kan Transfer Theorem 5.2.27 applies to it. This is the point of the monoid axiom.

Theorem 5.5.25. Some model categories associated with a Quillen ring. *Let $(\mathcal{M}, \wedge, S)$ be a cofibrantly generated Quillen ring as in Definition 5.5.9 satisfying the monoid axiom of Definition 5.5.22, in which every object is small relative to $\mathcal{M}$.*

(i) Let R be a monoid in $\mathcal{M}$ as in Definition 2.6.58. Then the category of left R-modules is a cofibrantly generated model category.

(ii) Let R be a commutative monoid in $\mathcal{M}$. Then the category of R-modules is a cofibrantly generated, monoidal model category satisfying the monoid axiom.

(iii) Let R be a commutative monoid in $\mathcal{M}$. Then the category of R-algebras as in Definition 2.6.63 is a cofibrantly generated model category. Every cofibration of R-algebras whose source is cofibrant as an R-module is also a cofibration of R-modules. In particular, if the unit S of the smash product is cofibrant in $\mathcal{M}$, then every cofibrant R-algebra is also cofibrant as an R-module.

The following is defined by David White in [Whi17, Definitions 3.1 and 3.4].

Definition 5.5.26. *A Quillen ring $(\mathcal{M}, \wedge, S)$ satisfies the* **White commutative monoid axiom** *if whenever h is a trivial cofibration, so is $h^{\square n}/\Sigma_n$ for all $n > 0$. Here $h^{\square n}$ denotes the n-fold pushout power in $\mathcal{M}$, defined in Definition 2.6.12(iii).*

It satisfies the **strong commutative monoid axiom** *if in addition $h^{\square n}/\Sigma_n$ is a cofibration whenever h is one. In particular this means that the n-fold symmetric power of Definition 2.6.63,*

$$\mathrm{Sym}^n(X) = X^{\wedge n}/\Sigma_n.$$

is cofibrant when X is, making Sym^n a left Quillen functor.

The following, with a slightly weaker smallness hypothesis, is proved by White as [Whi17, Theorem 3.2]. Here the idea of the proof is to show that the adjunction of (2.6.68) is a transfer adjunction as in Definition 5.2.25, so the Crans-Kan Transfer Theorem 5.2.27 applies to it. The commutative monoid axiom is designed for this purpose.

Theorem 5.5.27. The category of commutative monoids in a Quillen ring. *Let $(\mathcal{M}, \wedge, S)$ be a cofibrantly generated Quillen ring satisfying the commutative monoid axiom of Definition 5.5.26 and the monoid axiom of Definition 5.5.22, and assume that the domains of the generating cofibrations and trivial cofibrations are small as in Definition 4.8.8. Let R be a commutative monoid in $\mathcal{M}$, and assume that the functor Sym_R of Definition 2.6.63 commutes with filtered colimits. Then the category* $\mathbf{Comm}_R\mathcal{M}$ *of commutative R-algebras is a cofibrantly generated model category in which a map is a weak equivalence or fibration if and only if it is one in $\mathcal{M}$. In particular, when $R = S$, this gives a model structure on the category* $\mathbf{Comm}\,\mathcal{M}$ *commutative monoids in $\mathcal{M}$.*

Remark 5.5.28. *The existence of a model structure on* $\mathbf{Comm}\,\mathcal{M}$ *of commutative algebras in a Quillen ring $(\mathcal{M}, \wedge, S)$ implies that the fibrant replacement S_f (see Definition 4.1.19) is a commutative algebra, but there are cases where* *it is known not be one. For example, in the original category of spectra (see Chapter 7) the fibrant replacement of the sphere spectrum would have to be a commutative ring object. This would mean that its 0th space, the infinite loop space*

$$\underset{n}{hocolim}\, \Omega^n S^n$$

would have to be a commutative ring object in the category $\mathcal{T}$ of pointed topological spaces. However such objects are known to be products of Eilenberg-Mac Lane spaces, which the above space clearly is not. See [SS00, Remark 4.5] and [MMSS01, §14] for more discussion. Another aspect of this difficulty is described in Example 10.5.2 below.

5.5D The arrow category of a compactly generated Quillen ring

Next we will discuss the notions of §2.6F in the case of a compactly generated (see Definition 5.2.6) Quillen ring. Given such a category $(\mathcal{M}, \wedge, S)$ with generating sets $\mathcal{I}$ and $\mathcal{J}$, we denote its arrow category by $\mathcal{M}_1$. As we saw in §2.6F, it has a closed symmetric monoidal structure defined in terms of the pushout product operation $\square$ based on the formation of pushout corner maps. Since $\mathcal{M}_1 = \mathcal{M}^J$ for $J = (0 \to 1)$, the two object category with a single nonidentity morphism, the results of §5.4 apply here.

The following is a special case of [Hov99, Theorem 5.1.3], in which J could be a category of the form $(0 \to 1 \to 2 \to \cdots)$ where the chain of morphisms is of arbitrary length. See also [Hov14, Theorem 3.1]. we will use it below in §10.3.

Proposition 5.5.29. The arrow category of a compactly generated Quillen ring. *Let $(\mathcal{M}, \wedge, S)$ be a Quillen ring with generating sets $\mathcal{I}$ and $\mathcal{J}$ and let $\mathcal{M}_1$ be its arrow category.*

Under the projective (in the sense of Definition 5.4.2), a morphism

$$(X_0 \to X_1) \to (Y_0 \to Y_1) \tag{5.5.30}$$

is a weak equivalence or fibration iff each of $X_i \to Y_i$ is. It is a cofibration iff both $X_0 \to Y_0$ and the corner map

$$X_1 \underset{X_0}{\cup} Y_0 \to Y_1 \tag{5.5.31}$$

are cofibrations. An object $X_0 \to X_1$ is cofibrant if X_0 is cofibrant and $X_0 \to X_1$ is a cofibration.

The model structure on $\mathcal{M}_1$ is cofibrantly generated. The generating (trivial) cofibrations in $\mathcal{M}_1$ are of two types. Type I are the maps

$$(K \to K) \to (L \to L),$$

where the inner arrows are identity maps, and type II are the maps

$$(* \to K) \to (* \to L)$$

were $K \to L$ is running through the set $\mathcal{I}$ (respectively $\mathcal{J}$).

$\mathcal{M}_1$ is a Quillen ring under the monoidal structure given by the pushout product operation.

Proof We will derive this description of $\mathcal{M}_1 = \mathcal{M}^J$ for the case for $J = (0 \to 1)$ from Theorem 5.4.10. Since J has two objects, there are two Yoneda functors, $よ^0$ and $よ^1$. The sets $J(0,0)$, $J(0,1)$ and $J(1,1)$ each have one element, and $J(1,0)$ is empty. This means that for a morphism $f : K \to L$ in $\mathcal{M}$, $よ^0 \otimes f$ is

$$(K \to K) \to (L \to L)$$

and $よ^1 \otimes f$ is $(* \to K) \to (* \to L)$. This accounts for the two types of morphisms in $F^J\mathcal{I}$ and $F^J\mathcal{J}$.

Hirschhorn's left adjoint functor of (5.4.13) $F^J : \mathcal{M}^{|J|} \to \mathcal{M}^J$ in this case is given by

$$(X_0, X_1) \mapsto (X_0 \to X_0 \amalg X_1).$$

From this it is easy to verify the adjunction isomorphism

$$\begin{aligned}\mathcal{M}^{|J|}((X_0, X_1), U(Y_0 \xrightarrow{g} Y_1)) &\cong \mathcal{M}^J(F^J(X_0, X_1), (Y_0 \xrightarrow{g} Y_1)) \\ &\cong \mathcal{M}(X_0, Y_0) \times \mathcal{M}(X_1, Y_1).\end{aligned}$$

We refer the reader to [Hov99, Theorem 5.1.3] for the rest of the proof that $\mathcal{M}_1$ is cofibrantly generated. The statement about the monoid structure is proved in [Hov14, Theorem 3.1 (5)]. □

The following is proved by Hovey in [Hov14, Theorem 3.1 and Proposition 3.2]. We will use it below in Proposition 10.3.5.

Proposition 5.5.32. *Let $(\mathcal{M}, \wedge, S)$ be a cofibrantly generated Quillen ring with generating sets $\mathcal{I}$ and $\mathcal{J}$. Equipped with the structure of Proposition 5.5.29, $\mathcal{M}_1$ is a cofibrantly generated Quillen ring which satisfies the monoid axiom if $\mathcal{M}$ does.*

Hovey's proof relies on results of [SS00]. He mentions their Lemma 2.3, and their Theorems 3.1 and 3.3. As far as we can tell these should be Lemma 3.5, and Theorems 4.1 and 4.3 respectively.

5.5E Indexed products in an enriched monoidal model category

In this subsection we will study the indexed monoidal products of §2.9 in a monoidal model category as in Definition 5.5.9. It has two monoidal structures relating to wedges and smash products, and the results of §2.9 apply to both. Recall the notion of a finite covering category $p : \widetilde{K} \to K$ of Definition 2.8.1, exemplified by the functor between groupoids induced by a map of finite G-sets as explained in Example 2.9.1.

We will assume throughout this subsection that the categories $\widetilde{K}$ and K are finite groupoids, meaning each has finitely many objects and morphisms, with each morphism being invertible. Each has a finite number of connected components (see Definition 2.1.21), each of which is characterized up to isomorphism by its isotropy group (see Definition 2.1.29(iv)) and number of objects; see Remark 2.1.34.

Example 5.5.33. Group induction. *Let G be a finite group with a subgroup H, let $K = \mathcal{B}G$ and $\widetilde{K} = \mathcal{B}_{G/H}G$, so the finite covering category $p : \widetilde{K} \to K$ is induced by the map of G-sets $G/H \to G/G$. The category $\mathcal{M}^K$ is that of objects in $\mathcal{M}$ with G-action. By the case of Proposition 2.1.38 where $T = H/H$, there is a categorical equivalence $j : \mathcal{B}H \to \widetilde{K}$, so by Corollary 2.1.40, $\mathcal{M}^{\widetilde{K}}$ is equivalent to $\mathcal{M}^{\mathcal{B}H}$, the category of H-objects in $\mathcal{M}$. The composite functor*

$$\mathcal{M}^{\mathcal{B}H} \xleftarrow[\simeq]{j^*} \mathcal{M}^{\mathcal{B}_{G/H}G} \xleftarrow{p^*} \mathcal{M}^{\mathcal{B}G}$$

is the forgetful functor i_H^G (as in Definition 2.2.25) that sends an object in $\mathcal{M}$ with an action of G to the same object with the action restricted to H. It has a left adjoint that sends an H-object X in $\mathcal{M}$ to the appropriate indexed coproduct. In the pointed topological case it is

$$X \mapsto G \underset{H}{\ltimes} X,$$

the group induction functor of Definition 2.2.25. The space on the right is the orbit space under the diagonal action of H on $G \ltimes X$.

Let $(\mathcal{N}, \wedge, S)$ be a compactly generated (Definition 5.2.6) Quillen ring with generating sets $\mathcal{I}$ and $\mathcal{J}$, enriched (as in Definition 5.6.3) over a concrete compactly generated closed symmetric monoidal model category $(\mathcal{M}, \otimes, \mathbf{1})$.

By Proposition 5.4.17, a coproduct decomposition of $\widetilde{K}$ or K into connected components (as in Definition 2.1.5 and Definition 2.1.21) induces a product decomposition of the projective model structure on $\mathcal{N}^{\widetilde{K}}$ or $\mathcal{N}^K$.

The finite covering (Definition 2.8.1) $p : \widetilde{K} \to K$ above induces functors

$$p_*^{\wedge} : \mathcal{N}^{\widetilde{K}} \to \mathcal{N}^K \qquad \text{and} \qquad p_*^{\vee} : \mathcal{N}^{\widetilde{K}} \to \mathcal{N}^K, \tag{5.5.34}$$

the indexed smash product and indexed wedge respectively, as in Definition 2.9.6. Here the superscripts on p refer to the smash product and wedge operations. When K has one object $*$, it is isomorphic to the category $\mathcal{B}G$, where G is the automorphism group of $*$. In that case for each $k \in \widetilde{K}$, the functor p induces a monomorphism $G_k \to G$, where G_k is the automorphism group of k. For an element X of $\mathcal{N}^{\widetilde{K}}$, we denote $p_*^{\wedge} X$ by $X^{\wedge \widetilde{K}}$. We will make use of the maps $p_*^{\wedge}$ and $p_*^{\vee}$ in §9.3B and §10.2 below.

Given a map $f : A \to B$ in $\mathcal{N}^{\widetilde{K}}$, that is a suitable collection of maps

$$f_k : A_k \to B_k \qquad \text{for } k \in \widetilde{K},$$

we get an indexed corner map

$$\underset{k \in \widetilde{K}}{\square} f_k : \partial_A B^{\wedge \widetilde{K}} \to B^{\wedge \widetilde{K}} \tag{5.5.35}$$

as in Definition 2.9.29, namely the pushout product (as in Definition 2.6.12) of the maps f_k. Under the projective model structure on $\mathcal{N}^{\widetilde{K}}$, a generating cofibration consists of a collection

$$\left(\text{よ}^k \wedge i_k = G_k \underset{H_k}{\ltimes} i_k \right) \tag{5.5.36}$$

where $i_k \in \mathcal{I}$, the set of generating cofibrations for $\mathcal{N}$, and we have one k from each connected component (as in Definition 2.1.21) in $\widetilde{K}$.

Let $\mathcal{M}$ be a pointed model category and let S be a finite set. The coproduct diagonal adjunction of Example 4.5.6(i) can be used to show that a wedge of cofibrations indexed by S is again a cofibration. We can regard S as a discrete category (Definition 2.1.7) and consider the functor category $\mathcal{M}^S$. Its objects are simply collections of objects in $\mathcal{M}$ indexed by S. The projective and injective model structures on $\mathcal{M}^S$ (Definition 5.4.2) coincide with the product model sructure of Definition 4.1.16. It follows that a discretely indexed wedge of cofibrations is again a cofibration.

The first step in dealing with the nondiscrete case is to consider the category $\mathcal{M}^{\mathcal{B}_{\Sigma_n}\Sigma_n}$. In the language of §2.1E, the groupoid $\mathcal{B}_{\Sigma_n}\Sigma_n$ is 1-connected and is the universal cover (as in Definition 2.1.27) of $\mathcal{B}\Sigma_n$ under the evident covering $p : \mathcal{B}_{\Sigma_n}\Sigma_n \to \mathcal{B}\Sigma_n$. We have an adjunction of functor categories

$$\begin{tikzcd} \mathcal{M}^{\mathcal{B}_{\Sigma_n}\Sigma_n} \arrow[r, shift left=1.5ex, "p_*^{\vee}"] \arrow[r, phantom, "\perp"] & \mathcal{M}^{\mathcal{B}\Sigma_n} \arrow[l, shift left=1.5ex, "p^*"] \end{tikzcd} \tag{5.5.37}$$

as in Example 2.1.41 with $G = \Sigma_n$ and $H = e$. The category on the left is equivalent to $\mathcal{M}$. An object in the category on the right is an object of $\mathcal{M}$ with an action of the group Σ_n. The right adjoint p^* is equivalent to the forgetful functor to $\mathcal{M}$, and the left adjoint sends an object in $\mathcal{M}$ to its n-fold wedge with the symmetric group Σ_n permuting its summands.

If we endow the two categories of (5.5.37) with their projective model structures, then the adjunction is **not** a Quillen adjunction. Projective cofibrations on the right have domains and codomains with trivial group action, so the image under the left adjoint of a cofibration in $\mathcal{M}$ is not a projective cofibration. We can rectify this difficulty by enlarging the collection of cofibrations in $\mathcal{M}^{\mathcal{B}\Sigma_n}$ by applying Theorem 5.2.34 to (5.5.37).

Similarly, given finite groups $H \subseteq G$, we have a groupoid covering (as in Definition 2.1.23) $p : \mathcal{B}_{G/H}G \to \mathcal{B}G$ and an adjunction

$$\begin{tikzcd} \mathcal{M}^{\mathcal{B}_{G/H}G} \arrow[r, shift left, "p_*^\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M}^{\mathcal{B}G} \arrow[l, shift left, "p^*"] \end{tikzcd}$$

with the category on the left being equivalent to $\mathcal{M}^{\mathcal{B}H}$. Starting with the projective model structures on the two categories and applying Theorem 5.2.34 to this adjunction, we get an enlarged model structure on $\mathcal{M}^{\mathcal{B}G}$ in which for each cofibration $i : A \to B$ in $\mathcal{M}$, the indexed wedge

$$G \underset{H}{\ltimes} A \xrightarrow{G \underset{H}{\ltimes} i} G \underset{H}{\ltimes} B$$

is a cofibration in $\mathcal{M}^{\mathcal{B}G}$.

More generally, given a finite G-set T, we get an adjunction

$$\begin{tikzcd} \mathcal{M}^{\mathcal{B}_TG} \arrow[r, shift left, "p_*^\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M}^{\mathcal{B}G}. \arrow[l, shift left, "p^*"] \end{tikzcd} \tag{5.5.38}$$

Applying Theorem 5.2.34 to it gives us an enlarged model structure on $\mathcal{M}^{\mathcal{B}G}$ in which a T-indexed wedge of cofibrations in $\mathcal{M}$ is again a cofibration. Since an ordinary wedge of cofibrations is a cofibration (this being discrete case discussed above), as long as T contains a copy of G/H for each subgroup H up to conjugacy, **any** indexed wedge of cofibrations in $\mathcal{M}$ is again a cofibration in the enlarged model structure. Two examples of finite G-sets with this property are

$$\coprod_{H \subseteq G} G/H \qquad \text{and} \qquad \mathscr{P}(G),$$

where the disjoint union on the left is over all subgroups H of G, and $\mathscr{P}(G)$ denotes the power set of G, on which G acts by left multiplication, the subject of Example 8.1.2 below. The power set has a subset isomorphic to the disjoint union. These considerations lead to the following, which will be used below in the proof of Theorem 9.8.4. See Remark 8.6.19 below for an explanation of the word "equifibrant."

Proposition 5.5.39. The equifibrant model structure on $\mathcal{M}^{\mathcal{B}G}$. *For a finite group G, let T be a finite G-set containing a subset isomorphic to G/H for each subgroup $H \subseteq G$, and let $p : \mathcal{B}_TG \to \mathcal{B}G$ be the evident covering. For a pointed model category $\mathcal{M}$, let $\mathcal{M}^{\mathcal{B}G}$ have the model structure given by the application of Theorem 5.2.34 to the adjunction of (5.5.38) starting with the projective model structure on both categories. We call this model structure, which is independent of the choice of such a T, the* **equifibrant** *model structure.*

Let S be another finite G-set, $r : \mathcal{B}_S G \to \mathcal{B}G$ the evident covering, and $i : A \to B$ a projective cofibration in $\mathcal{M}^{\mathcal{B}_S G}$. Then the indexed wedge $r_*^{\vee} i$ is a cofibration in the equifibrant model structure on $\mathcal{M}^{\mathcal{B}G}$.

Similar considerations lead to an equifibrant model structure on $\mathcal{M}^{\mathcal{B}_T G}$ for any finite G-set T. Any map of G-sets $\widetilde{T} \to T$ leads to a groupoid covering $r : \mathcal{B}_{\widetilde{T}} G \to \mathcal{B}_T G$ and therefore an adjunction

$$\begin{tikzcd} \mathcal{M}^{\mathcal{B}_{\widetilde{T}} G} \arrow[r, shift left, "r_*^{\vee}"] \arrow[r, phantom, "\perp"] & \mathcal{M}^{\mathcal{B}_T G}. \arrow[l, shift left, "r^*"] \end{tikzcd} \tag{5.5.40}$$

similar to that of (5.5.38). We can use Theorem 5.2.34 to enlarge the model structure on the right so that any $\widetilde{T}$-indexed wedge of cofibrations is again a cofibration. The finite G-set T is isomorphic to a disoint union of orbits G/H for various H. We can require that the map $\widetilde{T} \to T$ be such that the preimage of each summand G/H contains a copy of G/K for each $K \subseteq H$.

Corollary 5.5.41. The equifibrant model structure on $\mathcal{M}^{\mathcal{B}_T G}$. *For a finite group G and finite G-set T, let $\widetilde{T} \to T$ be a map of finite G-sets such that the preimage of each summand of T of the form G/H contains a subset isomorphic to G/K for each subgroup $K \subseteq H$, and let $p : \mathcal{B}_{\widetilde{T}} G \to \mathcal{B}_T G$ be the evident covering. For a pointed model category $\mathcal{M}$, let $\mathcal{M}^{\mathcal{B}_T G}$ have the model structure given by the application of Theorem 5.2.34 to the adjunction of (5.5.40) starting with the projective model structure on both categories. This model structure is independent of the choice of such a $\widetilde{T}$.*

Let S be another finite G-set over T, $r : \mathcal{B}_S G \to \mathcal{B}_T G$ the evident covering, and $i : A \to B$ a projective cofibration in $\mathcal{M}^{\mathcal{B}_S G}$. Then the indexed wedge $r_*^{\vee} i$ is a cofibration in the equifibrant model structure on $\mathcal{M}^{\mathcal{B}_T G}$.

5.6 Enriched model categories

5.6A Motivation

The category $\mathcal{S}p^G$ of G-spectra and equivariant maps, to be defined below in Definition 9.0.2, is enriched over the closed symmetric monoidal model category of pointed G-spaces $\mathcal{T}^G$, so it is convenient to have a notion of an enriched model category.

We suppose that $\mathcal{N}$ is an $\mathcal{M}$-category (Definition 3.1.1) for a closed symmetric monoidal category $\mathcal{M}$, and that the underlying categories $\mathcal{N}_0$ and $\mathcal{M}_0$ are both model categories. **We will assume that $\mathcal{M}_0$ is concrete as explained in Remark 3.1.27.** This means that objects in $\mathcal{M}$, such as $\mathcal{N}(X, Y)$, are sets with additional structure (such as a topology and possibly a base point and/or a G-action) which is preserved by all maps in sight. The enrichment of the model structure on $\mathcal{N}_0$ needs to be compatible with the one on $\mathcal{M}_0$.

In order to motivate the definition, consider the following.

In an ordinary model category $\mathcal{N}$ one is concerned with lifting diagrams (2.3.11), namely

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y. \end{tikzcd}$$

From this we can derive a diagram of morphism sets as in (2.3.15),

$$\begin{tikzcd} h \arrow[rrrr, dashed, mapsto] \arrow[dddd, dashed, mapsto] & & & & g \arrow[dddd, mapsto] \\ & \mathcal{N}(B,X) \arrow[r, "p_*"] \arrow[d, "i^*"'] & \mathcal{N}(B,Y) \arrow[d, "i^*"] & & \\ & \mathcal{N}(A,X) \arrow[r, "p_*"] & \mathcal{N}(A,Y) & & \\ & & & & \\ f \arrow[rrrr, mapsto] & & & & pf = gi, \end{tikzcd} \tag{5.6.1}$$

which in general is **not** a pullback diagram. The existence of h thus depends on the surjectivity of the pullback corner map, also known as the lifting test map of Definition 2.3.14,

$$\mathcal{N}_{\diamond}(i,p) : \mathcal{N}(B,X) \to \mathcal{N}(A,X) \underset{\mathcal{N}(A,Y)}{\times} \mathcal{N}(B,Y). \tag{5.6.2}$$

Now suppose instead that $\mathcal{N}$ is a category enriched over a Quillen ring $\mathcal{M}$. This means that the objects $\mathcal{N}(-,-)$ above are not morphism sets but objects in $\mathcal{M}$, and we could hope that $\mathcal{N}_{\diamond}(i,p)$ is a fibration.

This suggests the following, which is a restatement of Definition 5.5.17 and can be found in [GM11, Definition 4.19]. In the simplicial case, where $\mathcal{M} = \mathcal{S}et_{\mathbf{\Delta}}$, this definition (but not the terminology) coincides with [Hir03, Definition 9.1.6] and [Qui67, Definition II.2.2]. In the latter, conditions (i) and (ii) below are denoted by SM0 and SM7 respectively, while Hirschhorn denotes them by M6 and M7. In the topological case, where $\mathcal{M} = \mathcal{T}op$, this definition coincides with [EKMM97, Definition VII.4.2]. In the pointed topological case, where $\mathcal{M} = \mathcal{T}$, it can found in [MMSS01, Definition 5.12].

We are mainly interested in the pointed topological case, where $\mathcal{M}$ is some variant of $\mathcal{T}$ and is in any case enriched over it. Most of the model categories we will study later in this book have such a structure.

Definition 5.6.3. Quillen $\mathcal{M}$-modules. *Let $\mathcal{M}$ be a Quillen ring as in*

Definition 5.5.9. A **Quillen $\mathcal{M}$-module** *is an $\mathcal{M}$-category (Definition 3.1.1) $\mathcal{N}$ underlain by a model category $\mathcal{N}_0$ such that, with notation as above,*

(i) $\mathcal{N}$ is bitensored over $\mathcal{M}$, i.e., it is equipped with a two variable adjunction (with $\mathcal{V}$ and $\mathcal{C}$ replaced by $\mathcal{M}$ and $\mathcal{N}$) as in Proposition 3.1.47.

(ii) When i is a cofibration and p is a fibration in $\mathcal{N}_0$, the map of (5.6.2) in $\mathcal{M}$ is a fibration which is a weak equivalence if either i or p is one in $\mathcal{N}_0$. In this case we say that i **has the homotopy left lifting property with respect to** *p, and p* **has the homotopy right lifting property with respect to** *i.*

When $\mathcal{M} = (\mathcal{T}op, \times, *)$ ($\mathcal{M} = (\mathcal{T}, \wedge, S^0)$), we say that $\mathcal{N}$ is a **(pointed) topological model category**.

When $\mathcal{M} = \mathcal{S}et_{\mathbf{\Delta}}$ ($\mathcal{M} = \mathcal{S}et_{\mathbf{\Delta}*}$), we say that $\mathcal{N}$ is a **(pointed) simplicial model category**.

A Quillen $\mathcal{M}$-module $\mathcal{N}$ is a **Quillen $\mathcal{M}$-algebra** *if it is also a Quillen ring as in Definition 5.5.9.*

Definition 5.6.4. The set $\pi_0 X$. *Let X be an object in a pointed topological (simplicial) model category $\mathcal{M}$. Then $\pi_0 X$ is the pointed set of path connected components in the pointed space (simplicial set) $\mathcal{M}(S^0, X)$, the base point of the set being the component of the map that factors through the initial object $*$ of $\mathcal{M}$. Given two objects X and Y in $\mathcal{M}$, we will abbreviate $\pi_0\mathcal{M}(X, Y)$ by $\pi_0(X, Y)$, which coincides with the set $\pi(X, Y)$ of Definition 4.3.11 when X is cofibrant and Y is fibrant.*

This set has a natural group structure when there is a map $X \vee X \to X$ with suitable properties, such as the pinch map of Definition 4.6.22 when $X = \Sigma A$ for some cofibrant A. See Proposition 4.6.23.

We will make use of the following in §7.3B.

Definition 5.6.5. Homotopy invariants. *Suppose that there is a set of compact cofibrant objects $\{A_\alpha\}$ in $\mathcal{M}$ such that map $g : X \to Y$ in $\mathcal{M}$ is a weak equivalence iff the induced map*

$$\pi_0\mathcal{M}(A_\alpha, X) \xrightarrow{g*} \pi_0\mathcal{M}(A_\alpha, Y)$$

is an isomorphism for each α. We will abbreviate $\pi_0\mathcal{M}(A_\alpha, -)$ by $\pi_\alpha(-)$. Then the collection of functors π_α is a **complete set of homotopy invariants for $\mathcal{M}$.**

It is possible to convert the collection of $\mathcal{S}et$-valued functors $\{\pi_\alpha\}$ into a single functor into a suitable bicomplete category $\mathcal{C}$. We leave the details to the reader.

Example 5.6.6. The category $\mathcal{T}^G$ of pointed G-spaces *and equivariant maps for a finite group G has a model structure (see Theorem 8.6.2 below) in which a map $f : X \to Y$ is a weak equivalence iff for each subgroup $H \subseteq G$, the fixed point map $f^H : X^H \to Y^H$ is an ordinary weak equivalence. This means it has a complete set of homotopy invariants as in Definition 5.6.5 for which the set of objects A_α is*

$$\left\{ G \underset{H}{\ltimes} S^n : n \geqslant 0, H \subseteq G \right\}.$$

For trivial G, this set is that of spheres of all nonnegative dimensions, and the weak equivalence condition of Definition 5.6.5 coincides with that of Definition 4.2.1.

As indicated following Definition 3.5.6, we have the following.

Definition 5.6.7. *As in Definition 3.5.6, let $j : S^0 \to I_+$ (where the target is the unit interval I with disjoint base point) be the map sending the nonbase point to 0. A map $i : X \to Y$ in a pointed topological model category $\mathcal{M}$ is an* h**-cofibration** *if the indicated lifting exists in all commutative diagrams of the form*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "i"'] & Y^{I_+} \arrow[d, "e_0"] \\ Y \arrow[r, "\beta"'] \arrow[ur, dashed, "h"] & Y^{S^0}, \end{tikzcd}$$

where Y is fibrant and e_0 is the map of cotensor products induced by the inclusion $i_0 : S^0 \to I_+$ that sends the nonbase point to 0.

Equivalently there is a map $\widetilde{h} : Y \rtimes I \to Y$, the left adjoint of h, making the following diagram commute,

$$\begin{tikzcd} X \arrow[r, "X \wedge j"] \arrow[d, "i"'] & X \rtimes I \arrow[d, "i \rtimes I"'] \arrow[ddr, "\widetilde{\alpha}"] & \\ Y \arrow[r, "Y \wedge j"] \arrow[drr, "\widetilde{\beta}"'] & Y \rtimes I \arrow[dr, dashed, "\widetilde{h}"] & \\ & & Y, \end{tikzcd}$$

where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the left adoints of α and β.

The following can be proved using the argument of [Hat02, Proposition 0.17].

Lemma 5.6.8. The case where X is contractible. *For an h-cofibration $i : X \to Y$ as in Definition 5.6.7, let $q : Y \to \operatorname{coker} f$ be the map of Definition 4.1.27. If X is contractible, q is a weak equivalence.*

Corollary 5.6.9. Cofibers and quotients. *For an h-cofibration $f : X \to Y$, let C_f be the mapping cone or cofiber of f as in Definition 4.1.27. Then the map $\epsilon : C_f \to \operatorname{coker} f \cong Y/X$ of (4.1.29) is a weak equivalence.*

Proof The subspace CX of C_f is contractible, its inclusion into C_f is an h-cofibration, and $C_f \cong Y/X$, so the result follows from Lemma 5.6.8. □

Proposition 5.6.10. *In a pointed topological model category $\mathcal{M}$,* **every cofibration is an h-cofibration***.*

Proof The map i_0 in Definition 5.6.7 is a trivial cofibration. This means that Corollary 5.5.6(iii) (or (iv)) implies that the map e_0 is a trivial fibration. It follows that the lifting h exists for any cofibration i. □

Proposition 5.6.11. *For a pointed topological model category $\mathcal{M}$, let*

$$X_0 \to X_1 \to X_2 \to \cdots$$

be a diagram in which each map is an h-cofibration as in Definition 5.6.7. Then for π_0 as in Definition 5.6.4,

$$\pi_0 \operatorname*{colim}_n X_n \cong \operatorname*{colim}_n \pi_0 X_n.$$

Proof Every h-cofibration is a closed inclusion by definition. The compactness of the unit interval implies that any pointed path in $\operatorname*{colim}_n X_n$ in the image of one in some X_n, so the corresponding element in $\pi_0 \operatorname*{colim}_n X_n$ is in the image of $\pi_0 X_n$. □

Example 5.6.12. The suspension and loop functors in pointed topological model categories. *Let $\mathcal{M} = (\mathcal{T}, \wedge, S^0)$ as in Definition 5.6.3. Then the isomorphisms of (3.1.48), with $\mathcal{C} = \mathcal{N}$ and $K = S^1$, read*

$$\Omega\mathcal{N}(X, Y) \xleftarrow[\cong]{\phi_r} \mathcal{N}(\Sigma X, Y) \xrightarrow[\cong]{\phi_\ell} \mathcal{N}(X, \Omega Y), \tag{5.6.13}$$

where $\Sigma X = S^1 \wedge X$ and $\Omega Y = Y^{S^1}$. For $\mathcal{N} = \mathcal{T}$, this is the usual adjunction between the suspension and loop functors related to Example 2.2.30 (ii).

We learned the following from Phil Hirschhorn.

Example 5.6.14. Mapping to a nonfibrant object is not a right Quillen functor. *Let $\mathcal{M}$ be a pointed topological model category with an object X that is not fibrant. Then the functor $X^{(-)} : \mathcal{T}^{op} \to \mathcal{M}$ sending a space K to X^K is not a right Quillen functor. Consider the cofibration $i : * \to S^0$ in $\mathcal{T}$, which corresponds to a fibration in $\mathcal{T}^{op}$. The induced map*

$$X = X^{S^0} \xrightarrow{i^*} X^* = *$$

is not a fibration because X is not fibrant.

We can apply Proposition 2.6.45 to this situation as follows.

Proposition 5.6.15. Changing Quillen rings. *Let $\mathcal{M}$ be a commutative Quillen ring as in Definition 5.5.9 and let $\mathcal{N}$ be a $\mathcal{M}$-enriched model category as in Definition 5.6.3 underlain by an ordinary model category $\mathcal{N}_0$. Suppose there is another commutative Quillen ring $\mathcal{M}'$ with a strong monoidal Quillen adjunction as in Definition 5.5.16,*

$$F : \mathcal{M}' \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{M} : G.$$

Then there is a Quillen $\mathcal{M}'$-module $\mathcal{N}'$ underlain by $\mathcal{N}_0$.

Proof The two variable adjunction of Definition 5.6.3(i) exists by Proposition 2.6.45. Right adjoints preserve limits and hence pullbacks by Proposition 2.3.36. Hence we can apply the right adjoint functor G to the map $\mathcal{N}_\diamond(i,p)$ of (5.6.2) and get a similar map in $\mathcal{M}'$.

Since G is a right Quillen functor, it preserves fibrations and

$\mathcal{N}'_\diamond(i,p) = G(\mathcal{N}_\diamond(i,p))$ is a fibration which is trivial if either i or p is a weak equivalence, as required by Definition 5.6.3(ii). □

By applying Proposition 5.6.15 to the Quillen equivalence of Proposition 4.2.20, we get the following.

Corollary 5.6.16. Topological model categories are simplicial model categories. *If $\mathcal{N}$ is a (pointed) topological model category, there is a (pointed) simplicial category $\mathcal{N}'$ having the same objects as $\mathcal{N}$ with morphism objects*

$$\mathcal{N}'(X,Y) := \mathcal{S}ing(\mathcal{N}(X,Y)),$$

where $\mathcal{S}ing$ denotes the singular functor of Definition 3.4.7.

This means that any theorem about simplicial model categories is also true for topological model categories. Most of the literature on model categories concerns the simplicial case.

The following lemma will be of great help in Chapter 6. It is proved (in the simplicial setting) as [Hir03, Theorem 9.7.4]. The simplicial statement implies the topological one by Corollary 5.6.16. See Theorem 5.6.21 and Lemma 5.8.51 below for similar statements.

Lemma 5.6.17. Detecting weak equivalences in a topological Quillen module with fibrant and cofibrant approximations. *Let $\mathcal{M}$ and $\mathcal{N}$ be as in Definition 5.6.3 with $\mathcal{M}$ topological. If $f : X \to Y$ is a map in a topological model category $\mathcal{N}$, then the following are equivalent:*

(i) The map f is a weak equivalence.

(ii) For **some** *fibrant approximation $\hat{f} : \hat{X} \to \hat{Y}$ to f (Definition 4.1.19) and every cofibrant object W the map $\hat{f}_* : \mathcal{N}(W,\hat{X}) \to \mathcal{N}(W,\hat{Y})$ of topological spaces is a weak equivalence.*

(iii) *For* **every** *fibrant approximation* $\widehat{f} : \widehat{X} \to \widehat{Y}$ *to* f *and every cofibrant object* W *the map* $\widehat{f}_* : \mathcal{N}(W, \widehat{X}) \to \mathcal{N}(W, \widehat{Y})$ *is a weak equivalence.*
(iv) *For* **some** *cofibrant approximation* $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ *to* f *and every fibrant object* Z *the map* $\widetilde{f}^* : \mathcal{N}(\widetilde{Y}, Z) \to \mathcal{N}(\widetilde{X}, Z)$ *is a weak equivalence.*
(v) *For* **every** *cofibrant approximation* $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ *to* f *and every fibrant object* Z *the map* $\widetilde{f}^* : \mathcal{N}(\widetilde{Y}, Z) \to \mathcal{N}(\widetilde{X}, Z)$ *is a weak equivalence.*

The simplicial form of the following is proved by Hirschhorn as [Hir03, Corollary 9.7.5].

Corollary 5.6.18. Detecting weak equivalences which are maps between fibrant or cofibrant objects. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be as in Definition 5.6.3 with* $\mathcal{M}$ *topological. (In particular* $\mathcal{N}$ *could be* $\mathcal{M}$ *itself.) Let* $f : X \to Y$ *be a map in* $\mathcal{N}$.

(i) *If* X *and* Y *are fibrant, then* f *is a weak equivalence if and only if for every cofibrant object* W *in* $\mathcal{N}$ *the map* $f_* : \mathcal{N}(W, X) \to \mathcal{N}(W, Y)$ *is a weak equivalence of pointed topological spaces.*
(ii) *If* X *and* Y *are cofibrant, then* f *is a weak equivalence if and only if for every fibrant object* Z *in* $\mathcal{N}$ *the map* $f^* : \mathcal{N}(Y, Z) \to \mathcal{N}(X, Z)$ *is a weak equivalence of pointed topological spaces.*

Corollary 5.6.19. Homotopy invariance of morphism objects. *The functor*

$$\mathcal{N}(-\,,-) : \mathcal{N}^{op} \times \mathcal{N} \to \mathcal{M}$$

is homotopical when the first variable is cofibrant in $\mathcal{N}$ *(and thus fibrant in* $\mathcal{N}^{op}$*) and the second variable is fibrant.*

The following is comparable to Lemma 4.7.1.

Lemma 5.6.20. The fiber of a trivial fibrant fibration is contractible, the second Dr. Seuss lemma. *Let* $\mathcal{M}$ *be a pointed topological model category as in Definition 5.6.3 and let* $p : X \to Y$ *be a fibration between fibrant objects. Then* p *is a trivial fibration iff its fiber* F *as in Lemma 4.7.1 is contractible as in Definition 4.1.4.*

The first Dr. Seuss lemma is Lemma 4.7.1.

Proof The contractibility of F is equivalent to that of $\mathcal{M}(W, F)$ for all cofibrant W by Lemma 5.6.17. Consider the long exact sequence of Proposition 4.7.11,

$$\begin{tikzcd}
\pi(W,\Omega X) \arrow[r, "(\Omega p)_*"] & \pi(W,\Omega Y) \arrow[r, "\partial_*"] & \pi(W,F) \arrow[r, "i_*"] & \pi(W,X) \arrow[r, "p_*"] & \pi(W,Y) \\
\pi(\Sigma W, X) \arrow[u, "\phi_\ell", "\cong"'] \arrow[r, "p_*"] & \pi(\Sigma W, Y). \arrow[u, "\phi_\ell", "\cong"'] & & &
\end{tikzcd}$$

The vertical isomorphisms ϕ_ℓ are the maps of (5.6.13). Recall (Corollary 4.7.2) that the suspension of a cofibrant object is cofibrant.

Hence if p is a weak equivalence, both maps labeled p_* are isomorphisms and hence $\pi(W, F)$ has one element. This set is $\pi_0\mathcal{M}(W, F)$, and we can say the same about $\pi_n\mathcal{M}(W, F) \cong \pi(\Sigma^n W, F)$ for each $n \geqslant 0$. It follows (using Lemma 5.6.17 again) that $\mathcal{M}(W, F)$ and hence F itself are contractible.

Conversely, if F is contractible, the map $p_* : \pi(W, X) \to \pi(W, Y)$ is an isomorphism for all cofibrant W, so p is a weak equivalence. □

The following is proved in simplicial form by Hovey in [Hov01b, Proposition 3.2]. It is similar to Lemma 5.6.17 with the additional assumption that $\mathcal{M}$ is cofibrantly generated.

Theorem 5.6.21. Detecting weak equivalences. *Let $\mathcal{M}$ be a left proper cofibrantly generated (pointed) topological model category as in Definition 5.2.1 with generating set of cofibrations $\mathcal{I}$. Then a morphism $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence iff*

$$f_* : \mathcal{M}(K, X) \to \mathcal{M}(K, Y) \tag{5.6.22}$$

is a weak equivalence of (pointed) topological spaces for each K that is a domain or codomain of a morphism in $\mathcal{I}$. If each such K is cofibrant, then the left properness condition is not needed.

Remark 5.6.23. Warning about a reference to [Hir03]. *Hovey's proof of the simplicial form of Theorem 5.6.21 (and also his proof of [Hov01b, Corollary 3.5]) makes use of Lemma 5.6.17, which he refers to as "Theorem 18.8.7" of [Hir03], but Hirschhorn's book (which was in preprint form at the time) has since been revised, and the result in question is now [Hir03, Theorem 9.7.4]. Hovey also cites Hirschhorn's "Lemma 11.3.2," which is now (thanks, Phil!) [Hir03, Proposition 13.5.6].*

Theorem 5.6.21 in the pointed case says that a morphism $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence iff certain maps f_* are weak equivalences in $\mathcal{T}$. Since such weak equivalences are characterized by their behavior on homotopy groups, we can refined this statement further as follows.

For each $i \geqslant 0$ have

$$\begin{aligned} \mathcal{M}(S^i \wedge K, X) &\cong \mathcal{T}(S^i, \mathcal{M}(K, X)) \quad \text{by Proposition 5.5.19 below} \\ &= \Omega^i \mathcal{M}(K, X), \\ \text{so } \pi_0\mathcal{M}(S^i \wedge K, X) &\cong \pi_0\Omega^i\mathcal{M}(K, X) = \pi_i\mathcal{M}(K, X). \end{aligned}$$

If $\mathcal{M}$ is compactly generated as in Definition 5.2.6, then each K in (5.6.22) is compact, as is $S^i \wedge K$. Thus we have the following.

Corollary 5.6.24. Detecting weak equivalences with π_0. *Let $\mathcal{M}$ be a* *left proper compactly generated (pointed) topological model category. Then a morphism $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence iff the induced map*

$$\pi_0 f_* : \pi_0\mathcal{M}(L, X) \to \pi_0\mathcal{M}(L, Y)$$

is an isomorphism for each compact object L in $\mathcal{M}$. If the domains and codomains of $\mathcal{I}$ are all cofibrant as well as compact, then f is a weak equivalence iff $\pi_0 f_*$ is an isomorphism for all compact cofibrant L.

5.6B Functors into an enriched model category

We will now discuss the enriched analog of Theorem 5.4.10, Theorem 5.6.26 below, which was originally proved by Brun-Dundas-Stolz' as [BDS16, Theorem 6.2.7]. It concerns functors into an enriched model category. Let $\mathcal{N}$ and $\mathcal{M}$ be as in Definition 5.6.3. Suppose there is a small indexing category $\mathcal{D}$ enriched over $\mathcal{M}$. For each object d of $\mathcal{D}$, let End_d be its endomorphism category, meaning the one object full subcategory of $\mathcal{D}$ as before. We denote the functor category by $[\mathcal{D}, \mathcal{N}]$ as in Definition 3.2.18. Let $\star$ denote the trivial $\mathcal{M}$-category, meaning the category with one object $*$ whose endomorphism object is the initial object $\mathbf{1}$ of $\mathcal{M}$.

Then for each functor X in $[\mathcal{D}, \mathcal{N}]$ and each object d in $\mathcal{D}$ we have a diagram

$$\begin{tikzcd}
\star \arrow[r] \arrow[dr, "\mathrm{Ev}_d X"'] & \mathrm{End}_d \arrow[r] \arrow[d, "\mathrm{Ev}'_d X"] & \mathcal{D} \arrow[dl, "X"] \\
& \mathcal{N} &
\end{tikzcd}$$

where the horizontal maps are the obvious inclusions, the map $\mathrm{Ev}_d X$ is evaluation of X at d and $\mathrm{Ev}'_d X$ is the restriction of X to End_d. This leads to a diagram of categories and functors.

$$\begin{tikzcd}
\mathcal{N} & {[\star, \mathcal{N}]} \arrow[l, "\cong"'] & {[\mathrm{End}_d, \mathcal{N}]} \arrow[l] & {[\mathcal{D}, \mathcal{N}]} \arrow[l, "\mathrm{Ev}'_d"] \arrow[ll, bend right, "\mathrm{Ev}_d"']
\end{tikzcd}$$

An object K in $[\mathrm{End}_d, \mathcal{N}]$ is an object in $\mathcal{N}$ equipped with an action of the endomorphism monoid $\mathrm{End}_d(d, d) = \mathcal{D}(d, d)$. When the functors above have left adjoints, we get

$$\begin{tikzcd}
\mathcal{N} \arrow[r, "\cong"] & {[\star, \mathcal{N}]} \arrow[r, "\mathcal{D}(d,d)\otimes(-)"'] \arrow[rr, bend left, "F_d"] & {[\mathrm{End}_d, \mathcal{N}]} \arrow[r, "G_d"'] & {[\mathcal{D}, \mathcal{N}]}
\end{tikzcd}$$

where F_d and G_d are the enriched tensored Yoneda and corestriction functors; see Yoneda Lemma 2.2.10 and Definition 3.1.68. For an object K in $\mathcal{N}$,

$\mathcal{D}(d,d) \otimes K$ is the corresponding object with free action of $\mathcal{D}(d,d)$, and the functor $F_d K$ is given by $F_d K(d') = \mathcal{D}(d,d') \otimes K$.

The functor G_d on K endowed with a $\mathcal{D}(d,d)$-action is given by

$$G_d K(d') = \mathcal{D}(d,d') \otimes_{\mathcal{D}(d,d)} K.$$

This object in $\mathcal{N}$ is the coequalizer of the maps

$$\mathcal{D}(d,d') \otimes \mathcal{D}(d,d) \otimes K \rightrightarrows \mathcal{D}(d,d') \otimes K$$

given by the right action of $\mathcal{D}(d,d)$ on $\mathcal{D}(d,d')$ and its left action on K.

Now assume that for each object d in $\mathcal{D}$, the enriched functor category $[\mathrm{End}_d, \mathcal{N}]$ is underlain by a cofibrantly generated model category $\mathcal{N}_d$ with generating sets $\mathcal{I}_d$ and $\mathcal{J}_d$ and weak equivalences $\mathscr{W}_d$. Assume also that the left adjoints G_d above exist, and use them to define sets of maps $G\mathcal{I}$, $G\mathcal{J}$ and $\mathscr{W}$ in $[\mathcal{D}, \mathcal{N}]_0$ by

$$G\mathcal{I} := \bigcup_{d \in \mathcal{D}} G_d \mathcal{I}_d, \qquad G\mathcal{J} := \bigcup_{d \in \mathcal{D}} G_d \mathcal{J}_d \tag{5.6.25}$$

and

$$\mathscr{W} := \left\{ f \in [\mathcal{D}, \mathcal{N}]_0 \colon \ \mathrm{Ev}'_d f \in \mathscr{W}_d \, \forall \, d \in \mathcal{D} \right\}.$$

Then [BDS16, Theorem 6.2.7], the enriched analog of Theorem 5.4.10, is the following.

Theorem 5.6.26. A cofibrantly generated model structure on the category underlying $[\mathcal{D}, \mathcal{N}]$. *With notation and assumptions as above, assume further that each functor category $[End_d, \mathcal{N}]$ is bitensored over $\mathcal{M}$, the domains of $\mathcal{GI}$ and $\mathcal{GJ}$ (see (5.6.25)) are small relative to $\mathcal{GI}$-cell and $\mathcal{GJ}$-cell respectively, and that $\mathcal{GJ}$-cell is contained in $\mathscr{W}$. Then the underlying category $[\mathcal{D}, \mathcal{N}]_0$ is a cofibrantly generated model category where a map $f : X \to Y$ is a fibration (weak equivalence) iff for each $d \in \mathcal{D}$, $\mathrm{Ev}'_d f$ is a fibration (weak equivalence) in the model structure $\mathcal{N}_d$ on $[End_d, \mathcal{N}]_0$. The set of generating (trivial) cofibrations is ($\mathcal{GJ}$) $\mathcal{GI}$.*

Like Theorem 5.4.10, this is proved by showing that the data satisfy the conditions of the Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24. The following is the analog of Corollary 5.4.11.

Corollary 5.6.27. Some cofibrant objects in $[\mathcal{D}, \mathcal{N}]$. *For any cofibrant object K in $\mathcal{M}$ and any object D in $\mathcal{D}$, the object $よ^D \otimes K$ is cofibrant in $[\mathcal{D}, \mathcal{N}]$.*

Proposition 5.6.28. The enriched Yoneda adjunction as a Quillen pair. *Let $(\mathcal{M}, \wedge, S)$ be a Quillen ring as in Definition 5.5.9. Suppose there is a small indexing category $\mathcal{D}$ enriched over $\mathcal{M}$. For each object D in the small category $\mathcal{D}$, let $\mathrm{Ev}_D : [\mathcal{D}, \mathcal{M}] \to \mathcal{M}$ be the enriched evaluation functor* *that sends an enriched functor $X : \mathcal{D} \to \mathcal{M}$ to X_D, its value in $\mathcal{M}$. Let $よ^D : \mathcal{D} \to \mathcal{M}$ be the enriched Yoneda functor that sends an object E in $\mathcal{D}$ to the morphism object $\mathcal{D}(D,E)$ in $\mathcal{M}$, and let $F^D : \mathcal{M} \to [\mathcal{D},\mathcal{M}]$ be given by $M \mapsto よ^D \wedge M$, the enriched tensored Yoneda functor as in Definition 3.1.68. Then we have Quillen adjunction*

$$\begin{tikzcd} M \arrow[r, mapsto] & よ^D \wedge M \\ F^D : \mathcal{M} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & {[\mathcal{D},\mathcal{M}]} : \mathrm{Ev}_D \arrow[l, shift left] \\ X_D & X. \arrow[l, mapsto] \end{tikzcd}$$

In particular, if M is a cofibrant object in $\mathcal{M}$, then $よ^D \wedge M$ is cofibrant in $[\mathcal{D},\mathcal{M}]$, and if X if fibrant in $[\mathcal{D},\mathcal{M}]$, then X_D is fibrant in $\mathcal{M}$.

Proof The adjunction $F^D \dashv \mathrm{Ev}_D$ is that of Proposition 3.1.71. We need to show that it is a Quillen adjunction, which we will do by showing that Ev_D is a right Quillen functor, meaning a functor that preserves fibrations and trivial fibrations. This is immediate because by definition, a morphism in $[\mathcal{D},\mathcal{M}]$ is a (trivial) fibration if its evaluation at each D is one. □

The following is an enriched analog of Proposition 5.4.18, and it can proved the same way.

Proposition 5.6.29. Quillen adjunctions between enriched projective model structures. *Let $\mathcal{M}$ be a model category and let $\alpha : K \to J$ be an $\mathcal{M}$-functor between small categories K and J enriched over $\mathcal{M}$. Then the functors*

$$U = \alpha^* : \mathcal{M}^J \to \mathcal{M}^K$$

given by precomposition with α and $F = \alpha_! : \mathcal{M}^K \to \mathcal{M}^J$ given by left Kan extension, form a Quillen pair (F,U) as in Definition 4.5.1 between the projective model structures on $\mathcal{M}^K$ and $\mathcal{M}^J$ of Definition 5.4.2.

Proposition 5.6.30. Projective model structures for similar indexing categories. *With notation as in Proposition 5.6.29, suppose in addition that the functor $\alpha : K \to J$ induces an isomorphism of object sets. Then (F,U) is a Quillen equivalence as in Definition 4.5.14.*

The following is an $\mathcal{M}$-category (for a Quillen ring $\mathcal{M}$) analog of Definition 2.1.58.

Definition 5.6.31. *Let $\mathcal{M} = (\mathcal{M}_0, \wedge, S)$ be a Quillen ring as in Definition 5.5.9. A* **generalized direct (inverse) $\mathcal{M}$-category** *$\mathcal{D}$ is a small $\mathcal{M}$-category as in Definition 3.1.1 equipped with a function that assigns a nonnegative integer, the* **degree** *(denoted by $|\alpha|$ for an object α), to each object of $\mathcal{D}$, satisfying that the following.*

- *Every degree preserving morphism in $\mathcal{D}_0$, the ordinary category underlying $\mathcal{D}$, is an isomorphism.*
- *Every morphism in $\mathcal{D}_0$ that is not an isomorphism raises (lowers) degree.*
- *Let α and β be objects of $\mathcal{D}$. If $|\alpha| > |\beta|$ ($|\alpha| < |\beta|$), then $\mathcal{D}(\alpha, \beta)$ is the initial (terminal) object in $\mathcal{M}$.*
- *For $|\alpha| \leqslant |\beta|$ ($|\alpha| \geqslant |\beta|$), $\mathcal{D}(\alpha, \beta)$ is cofibrant (fibrant) in $\mathcal{M}$.*
- *For $|\alpha| = |\alpha'| \leqslant |\beta|$ ($|\alpha| \geqslant |\beta'| \leqslant |\beta|$), the reduced composition morphism*

$$\mathcal{D}(\alpha', \beta) \underset{\mathcal{D}(\alpha',\alpha')}{\wedge} \mathcal{D}(\alpha, \alpha') \to \mathcal{D}(\alpha, \beta)$$
$$\left(\mathcal{D}(\beta', \beta) \underset{\mathcal{D}(\beta',\beta')}{\wedge} \mathcal{D}(\alpha, \beta') \to \mathcal{D}(\alpha, \beta)\right)$$

of Proposition 3.1.10 is an isomorphism.

Example 5.6.32. Some generalized direct $\mathcal{M}$-categories. *We will study several such categories later in this book.*

(i) *The category $\mathscr{J}_K^{\mathbf{N}}$ of Definition 7.1.13 below used to define presymmetric spectra.*
(ii) *The categories $\mathscr{J}_K^{\mathbf{N}}$, $\mathscr{J}_K^{\mathbf{\Sigma}}$ and $\mathscr{J}_K^{\mathbf{O}}$ of Definition 7.2.4 below used to define presymmetric, symmetric and orthogonal spectra in Definition 7.2.33.*
(iii) *The $\mathscr{J}_K^{\mathbf{O}}$-algebras $\mathscr{J}_L^{\mathbf{F}}$ of Definition 7.2.19 below. They are generalizations of (ii) that will provide a framework for the study of (iv) below.*
(iv) *The Mandell-May category $\mathscr{J}_G$ and of Definition 8.9.24 below. It is enriched over $\mathcal{T}_G$, the category of pointed G-spaces (for a finite group G) with the Bredon model structure of Definition 8.6.1. The functor category $[\mathscr{J}_G, \mathcal{T}_G]$ is that of orthogonal G-spectra as in Definition 9.0.2 below.*

Remark 5.6.33. Altering the degree function. *The degree of each object in direct category is a nonnegative integer, but no use is made of addition of degrees in Definition 5.6.31. Therefore if we were to alter the degree function linearly, replacing $|\alpha|$ by $m|\alpha|+b$ for fixed integers $m > 0$ and $b \geqslant 0$, we would not change the structure of the category.*

5.6C Monoidal structures in enriched model categories

Here we will show that when we modify the model structure on a Quillen ring (Definition 5.5.9), the same monoidal structure makes the new model category a Quillen ring again under appropriate circumstances. The model category constructions we will study are

(i) enlargement as in Theorem 5.2.34, the subject of Theorem 5.6.34 below,
(ii) functor categories of Definition 5.4.2, the subject of Theorem 5.6.35, and
(iii) induction from a subcategory of the indexing category as in Theorem 5.4.21, also known as confinement, the subject of Theorem 5.6.39.

Theorem 5.6.34. The monoidal structure for an enlarged monoidal model category. *With notation as in Theorem 5.2.34, which concerns an adjunction*

$$\begin{tikzcd} \mathcal{M}' \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M}, \arrow[l, shift left, "U"] \end{tikzcd}$$

assume in addition that $(\mathcal{M}', \wedge, S')$ and $(\mathcal{M}, \wedge, S)$ are both algebras (as in Definition 5.5.17) over a Quillen ring $(\mathcal{L}, \wedge, S_0)$. We also assume that the right adjoint U is symmetric monoidal as in Definition 2.6.20. Then with respect to the enlarged model structure, the original monoidal structure on $\mathcal{M}$ satisfies

(i) the pushout product axiom of Definition 5.5.9(i),

(ii) the unit axiom Definition 5.5.9(ii), and

(iii) (if it holds for $\mathcal{M}$ with its original model structure and for $\mathcal{M}'$) the monoid axiom of Definition 5.5.22.

The example we have in mind is $\mathcal{L} = \mathcal{T}$, $\mathcal{M} = \mathcal{S}p^G$ (the category of orthogonal G-spectra for a finite group G, the subject of Chapter 9 below) and $\mathcal{M}' = \mathcal{S}p^H$ for $H \subseteq G$, with (F, U) being the change of group adjunction of (9.1.18).

Proof (i) We need to show that for enlarged cofibrations i_1 and i_2, $i_1 \,\square\, i_2$ is also an enlarged cofibration which is trivial if either i_1 or i_2 is trivial. Recall that $\mathcal{M}$ and $\mathcal{M}'$ are cofibrantly generated model categories with pairs of cofibrant generating sets $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{I}', \mathcal{J}')$. To show that $i_1 \,\square\, i_2$ is a cofibration, it suffices to consider the case where they are both generating cofibrations in the enlarged model structure, meaning that both are elements of $\mathcal{I} \cup F\mathcal{I}'$. In the case where one or both is a trivial cofibration, it suffices to treat the case where one or both is an element of $\mathcal{J} \cup F\mathcal{J}'$. In each of these situations the pushout product axiom in $\mathcal{M} \times \mathcal{M}'$ implies that the pushout product in the enlarged model structure of $\mathcal{M}$ has the desired property.

(ii) The assumption that U is symmetric monoidal implies that S' is isomorphic to $U(S)$, making (S, S') isomorphic to $(S, U(S))$ in $\mathcal{M} \times \mathcal{M}'$. Thus we can take it to be the unit in $\mathcal{M} \times \mathcal{M}'$. Since U preserves trivial fibrations (such as the cofibrant approximation $q : QS \to S$), $U(q)$ is a cofibrant approximation in $\mathcal{M}'$. Thus

$$(q, U(q)) : (QS, QU(S)) \to (S, U(S)).$$

is a cofibrant approximation to the unit in $\mathcal{M} \times \mathcal{M}'$.

Now let K be a cofibrant object in the enlarged model structure on $\mathcal{M}$. We need to show that $q \wedge K$ is a weak equivalence in the enlarged model structure on $\mathcal{M}$, which means showing that $(q \wedge K, U(q) \wedge U(K))$ is a weak equivalence in $\mathcal{M} \times \mathcal{M}'$. This is the case since $\mathcal{M}$ and $\mathcal{M}'$ both satisfy the unit axiom.

(iii) Since the enlarged model structure on $\mathcal{M}$ is cofibrantly generated, by Lemma 5.5.23 it suffices to that every morphism that is obtained as a transfinite composition of pushouts of smash products of any object with maps in its generating set $\mathcal{J} \cup F\mathcal{J}'$ of trivial cofibrations in $\mathcal{M}$ is a weak equivalence. This means showing the ta image of such a map under U is a weak equivalence in $\mathcal{M} \times \mathcal{M}'$. Again this is the case bacause both $\mathcal{M}$ with its original model structure $\mathcal{M}'$ satisfy the monoid axiom. □

Theorem 5.6.35. Quillen rings and the Day convolution. *Let $(\mathcal{M}, \wedge, S)$ be a Quillen ring as in Definition 5.5.9, and let $(\mathscr{J}, \oplus, 0)$ be a symmetric monoidal category enriched over $\mathcal{M}$ in which all morphism objects are cofibrant with $\mathscr{J}(0,0) = S$. Let the functor category $[\mathscr{J}, \mathcal{M}]$ have the projective model structure of Definition 5.4.2. Then the closed symmetric monoidal structure on $\mathcal{N}$ given by the Day Convolution Theorem 3.3.5, which we will also denote by $\wedge$, makes it a Quillen ring with unit $よ^0$. In other words, the monoidal structure satisfies*

(i) the pushout product axiom,
(ii) the unit axiom, and
(iii) (if $\mathcal{M}$ satisfies it) the monoid axiom.

Proof We will abbreviate the functor category $[\mathscr{J}, \mathcal{M}]$ by $\mathcal{N}$.

(i) We use Proposition 5.5.15, which says that it is equivalent to the map $\mathcal{N}_\diamondsuit(i,p)$ being a fibration which is trivial if either the cofibration $i : A \to B$ or the fibration $p : X \to Y$ is. In the projective model structure, a map $f : W \to Z$ is a (trivial) fibration iff for each object j in $\mathscr{J}$, $f_j : W_j \to Z_j$ is a (trivial) fibration in $\mathcal{M}$. The domain of $\mathcal{N}_\diamondsuit(i,p)$ is $\mathcal{N}(B,X)$, whose jth component is the end

$$\mathcal{N}(B,X)_j \cong \int^{k \in \mathscr{J}} \mathcal{M}(B_k, X_{j+k})$$

by Proposition 3.3.7(ii).

We now examine the codomain $\diamondsuit(i,p)$ (see Definition 2.3.14) of $\mathcal{N}_\diamondsuit(i,p)$. For each pair of objects j and k in $\mathscr{J}$, we have the diagram

$$\begin{tikzcd}
\mathcal{M}(B_k, X_{j+k}) \arrow[r, "(p_{j+k})_*"] \arrow[d, "(i_k)^*"'] & \mathcal{M}(B_k, Y_{j+k}) \arrow[d, "(i_k)^*"] \\
\mathcal{M}(A_k, X_{j+k}) \arrow[r, "(p_{j+k})_*"] & \mathcal{M}(A_k, Y_{j+k})
\end{tikzcd}$$

with the pullback corner map

$$\mathcal{M}_\diamondsuit(i_j, p_{j+k}) : \mathcal{M}(B_k, X_{j+k}) \to \diamondsuit(i_k, p_{j+k}). \tag{5.6.36}$$

It follows that

$$\diamondsuit(i,p)_j = \int^{k \in \mathscr{J}} \diamondsuit(i_k, p_{j+k}).$$

The pushout product axiom for $\mathcal{M}$ implies that the map of (5.6.36) is a fibration which is trivial if either i_k or p_{j+k} is, so it is trivial if either the cofibration $i : A \to B$ of the fibration $p : X \to Y$ is trivial.It follows that for each j and k in $\mathscr{J}$, the map of (5.6.36)is a fibration that is trivial if either i or p is. (By Proposition 5.5.15 this is equivalent to the pushout axiom for $\mathcal{M}$.) This means that the same is true of the map $\mathcal{N}_\diamondsuit(i,p)$. The pushout product axiom for $([\mathscr{J}, \mathcal{M}], \wedge, よ^0)$ follows.

(ii) Let $q : QS \to S$ be a cofibrant approximation in $\mathcal{M}$. Then

$$よ^0 \wedge q : よ^0 \wedge QS \to よ^0 \wedge S \cong よ^0 \tag{5.6.37}$$

is a cofibrant approximation in $\mathcal{N}$. The unit axiom requires that for any cofibrant object X in $\mathcal{N}$, the map

$$X \wedge よ^0 \wedge q \cong X \wedge q$$

is a weak equivalence in $\mathcal{N}$. This will be true if for each j in $\mathscr{J}$, the map $X_j \wedge q$ is a weak equivalence in $\mathcal{M}$. Now the cofibrancy of X implies that of X_j is cofibrant by Proposition 5.4.4(ii), so $X_j \wedge q$ is a weak equivalence by the unit axiom for $\mathcal{M}$.

(iii) Now suppose that $\mathcal{M}$ satisfies the monoid axiom of Definition 5.5.22. We need to show that if $i : A \to B$ is a trivial cofibration in $\mathcal{N}$, then $i \wedge X$ is a weak equivalence for any object X in $\mathcal{N}$. Once we have done so, pushouts and transfinite compositions in $\mathcal{N}$ can be computed objectwise, so the monoidal axiom for $\mathcal{N}$ will follow from the same property in $\mathcal{M}$.

If $i : A \to B$ is a trivial projective cofibration in $\mathcal{N}$, then $i_{j'} : A_{j'} \to B_{j'}$ is trivial cofibration in $\mathcal{M}$ for each object j' in $\mathscr{J}$. The monoid axiom for $\mathcal{M}$ implies that

$$i_{j'} \wedge K_{j''} : A_{j'} \wedge K_{j''} \to B_{j'} \wedge K_{j''}$$

is a weak equivalence for all $j', j'' \in \mathscr{J}$. For each $k \in \mathscr{J}$ we have

$$(A \wedge K)_k \cong \int_{(j',j'') \in \mathscr{J} \times \mathscr{J}} \mathscr{J}(j' + j'', k) \wedge A_{j'} \wedge K_k$$

by (3.3.3), and there is is a similar formula for $(B \wedge K)_k$. It follows that for each $j', j'', k \in \mathscr{J}$, the map

$$\mathscr{J}(j'+j'',k) \wedge i_{j'} \wedge K_k : \mathscr{J}(j'+j'',k) \wedge A_{j'} \wedge K_k \to \mathscr{J}(j'+j'',k) \wedge B_{j'} \wedge K_k$$

is a weak equivalence, so the maps $(i \wedge K)_k$ for each k, and therefore $i \wedge K$ itself are weak equivalences. $\square$

The following enriched analog of Theorem 5.4.21 and Proposition 5.4.24 can be proved in the same way. For $\mathscr{J}$ and $\mathcal{M}$ as in Theorem 5.6.35 with an ideal $\mathscr{K} \subseteq \mathscr{J}$ as in Definition 2.6.9, we have the confined model structure on $[\mathscr{J}, \mathcal{M}]$ of Theorem 5.4.21. We want to study it as a monoidal model category.

Theorem 5.6.38. An induced model structure on $[\mathscr{J}, \mathcal{M}]$. *Let $\mathcal{M}$ be a cofibrantly generated Quillen ring with generating sets $\mathcal{I}$ and $\mathcal{J}$, and let $\alpha : \mathscr{K} \to \mathscr{J}$ be a fully faithful $\mathcal{M}$-functor between small categories $\mathscr{K}$ and $\mathscr{J}$ enriched over $\mathcal{M}$. In particular, $\mathscr{K}$ could be a full subcategory of $\mathscr{J}$. Then*

(i) the projective model structure on $[\mathscr{K}, \mathcal{M}]$ induces a model structure on $[\mathscr{J}, \mathcal{M}]$ as in the Crans-Kan Transfer Theorem 5.2.27,

(ii) the sets

$$\mathcal{I}_{\mathscr{K}} = \bigcup_{k \in \mathrm{ob}\,\mathscr{K}} \text{よ}^{\alpha(k)}\mathcal{I} \qquad \text{and} \qquad \mathcal{J}_{\mathscr{K}} = \bigcup_{k \in \mathrm{ob}\,\mathscr{K}} \text{よ}^{\alpha(k)}\mathcal{J}$$

are cofibrant generating sets for the induced model structure on $[\mathscr{J}, \mathcal{M}]$, and

(iii) For a projectively cofibrant object A in $[\mathscr{J}, \mathcal{M}]$, let

$$Q_\alpha A = \alpha_! \alpha^*(A).$$

and let $q_\alpha : Q_\alpha A \to A$ be the counit ϵ_A of the enriched adjunction $\alpha_! \dashv \alpha^*$ as in Definition 2.2.20. Then it is a cofibrant approximation to A in the induced model structure on $[\mathscr{J}, \mathcal{M}]$.

Theorem 5.6.39. The confined model structure as a closed symmetric monoidal category. *Let $\mathscr{J}$ and $\mathcal{M}$ be as in Theorem 5.6.35, and let $\mathscr{K} \subset \mathscr{J}$ be an ideal as in Definition 2.6.9. Assume further that $\mathcal{M}$ is cofibrantly generated with generating sets $\mathcal{I}$ and $\mathcal{J}$. Then under the confined model structure on $[\mathscr{J}, \mathcal{M}]$ of Theorem 5.6.38 and the closed symmetric monoidal structure on $[\mathscr{J}, \mathcal{M}]$ given by the Day Convolution Theorem 3.3.5, $[\mathscr{J}, \mathcal{M}]$ is a Quillen ring with unit よ^0. In other words, the monoidal structure satisfies*

(i) the pushout product axiom,
(ii) the unit axiom, and
(iii) (if $\mathcal{M}$ satisfies it) the monoid axiom.

Proof (i) Recall that the confined model structure on $[\mathscr{J}, \mathcal{M}]$ has more fibrations and weak equivalences and fewer cofibrations than the projective one. Its cofibrant generating sets are $\mathcal{I}_{\mathscr{K}}$ and $\mathcal{J}_{\mathscr{K}}$ as in Theorem 5.6.38(ii). Here we omit α from the notation since we are assuming that $\mathscr{K}$ is a subcategory of $\mathscr{J}$ rather than a category mapping to it via a functor α.

By Proposition 5.5.14 the pushout product axiom reduces to showing that each map in the set $\mathcal{I}_{\mathscr{K}} \square \mathcal{I}_{\mathscr{K}}$ is a cofibration, and replacing either or both factors by $\mathcal{J}_{\mathscr{K}}$ yields a set of trivial cofibrations. Let $i_1 : A_1 \to B_1$ and $i_2 : A_2 \to B_2$

be two generating cofibrations of $\mathcal{M}$, and let k_1 and K_2 be two objects of $\mathscr{K}$. Then

$$(よ^{k_1} i_1) \,\square\, (よ^{k_2} i_2) \cong よ^{k_1+k_2}(i_1 \,\square\, i_2),$$

which is a cofibration in $[\mathscr{J}, \mathcal{M}]$ because $i_1 \,\square\, i_2$ is a cofibration in $\mathcal{M}$. The argument for trivial cofibrations is similar.

(ii) The domain of the map of (5.6.37) that we used in the proof of Theorem 5.6.35 need not be cofibrant if the object 0 in $\mathscr{J}$ is not in $\mathscr{K}$. We will use the cofibrant approximation $Q_\alpha よ^0$ of $よ^0$ given by Theorem 5.6.38(iii). Thus we have

$$Q_\alpha よ^0 \wedge QS \xrightarrow{q_\alpha \wedge QS} よ^0 \wedge QS \xrightarrow{よ^0 \wedge q} よ^0 \wedge S \cong よ^0.$$

The middle object is projectively cofibrant. The unit axiom for $\mathcal{M}$ implies that the map $q_\alpha \wedge QS$ is a weak equivalence in the induced model structure. The composition above is therefore a cofibrant approximation to the unit $よ^0$ (which need not be projectively cofibrant) in the induced model structure.

We can now proceed roughly as in the proof of the unit axiom in Theorem 5.6.35. It requires that for any induced cofibrant object A in $[\mathscr{J}, \mathcal{M}]$, the map $A \wedge q_\alpha \wedge q$ is a weak equivalence in $[\mathscr{J}, \mathcal{M}]$ with the induced model structure. This will be true if for each j and j' in $\mathscr{J}$ with $j + j'$ in $\mathscr{K}$, the map $A_j \wedge (q_\alpha)_{j'} \wedge q$ is a weak equivalence in $\mathcal{M}$. The domain of that map is nontrivial only when j and j' are both in $\mathscr{K}$, so those are the only cases we need to consider.

Now the cofibrancy of A implies that of A_j by Proposition 5.4.4(ii), and that of the domain of $(q_\alpha)_{j'}$ is implied by the cofibrancy of morphism objects in $\mathscr{J}$. Therefore $A_j \wedge (q_\alpha)_{j'} \wedge q$ is a weak equivalence by the unit axiom for $\mathcal{M}$.

(iii) Assuming that $\mathcal{M}$ satisfies the monoid axiom, we need to show that the smash product of any generating trivial induced cofibration i of $[\mathscr{J}, \mathcal{M}]$ with another object K in $[\mathscr{J}, \mathcal{M}]$ is an induced weak equivalence. This will imply the monoid axiom for $[\mathscr{J}, \mathcal{M}]$ since pushouts and transfinite compositions in $[\mathscr{J}, \mathcal{M}]$ can be computed objectwise.

By Theorem 5.6.38(ii), each generating trivial cofibration has the form $よ^{\alpha(k)} \wedge j$ for an object k in $\mathscr{K}$ and a generating trivial cofibration $i : A \to B$ for $\mathcal{M}$. The jth component of this map is $\mathscr{J}(\alpha(k), j) \wedge i$. This is a trivial cofibration since i is one and $\mathscr{J}(\alpha(k), j)$ is cofibrant. We can show that for each object k' in $\mathscr{K}$, the $\alpha(k')$th component of $よ^{\alpha(k)} \wedge i \wedge K$ is a weak equivalence by an argument similar used in the proof of Theorem 5.6.35(iii). $\square$

The following is a consequence of the previous four theorems and should be compared to Corollary 5.4.29.

Corollary 5.6.40. Four monoidal model structures on $[\mathscr{J}, \mathcal{M}]$. *Let $\mathcal{M}$,*

M', F and U be as in Theorem 5.6.34. Let $\mathscr{J}$ be as in Theorem 5.6.35, and let $\mathscr{K} \subset \mathscr{J}$ be an ideal as in Definition 2.6.9. Similarly let $(\mathscr{J}', \oplus, 0)$ be a symmetric monoidal category enriched over $\mathcal{M}'$ with similar properties and an ideal $\mathscr{K}' \subset \mathscr{J}'$. For brevity, let

$$\begin{aligned} \mathcal{N} &:= [\mathscr{J}, \mathcal{M}] & \qquad \mathcal{N}' &:= [\mathscr{J}', \mathcal{M}'] \\ \mathcal{P} &:= [\mathscr{K}, \mathcal{M}] & \qquad \mathcal{P}' &:= [\mathscr{K}', \mathcal{M}'] \end{aligned}$$

Then we have a diagram similar to (5.4.30),

$$\begin{tikzcd}
\mathcal{N}_{enla} \arrow[rrr, shift left] \arrow[dr, shift left] \arrow[ddd, shift left] & & & \mathcal{N} \arrow[lll, shift left, "\top"] \arrow[dl, shift left, "\perp"] \arrow[ddd, shift left] \\
& \mathcal{N} \times \mathcal{N}' \arrow[ul, shift left, "\top"] \arrow[r, shift left, "\mathcal{N} \coprod F"] \arrow[d, shift left, "B \times B'"] & \mathcal{N} \arrow[l, shift left, "\mathcal{N} \times U"] \arrow[l, phantom, "\perp"] \arrow[ur, shift left] \arrow[d, shift left, "B"] & \\
& \mathcal{P} \times \mathcal{P}' \arrow[u, shift left, "A \times A'"] \arrow[u, phantom, "\dashv"] \arrow[r, shift left, "\mathcal{P} \coprod E"] \arrow[dl, shift left] & \mathcal{P} \arrow[l, shift left, "\mathcal{P} \times V"] \arrow[l, phantom, "\perp"] \arrow[u, shift left, "A"] \arrow[u, phantom, "\dashv"] \arrow[dr, shift left] & \\
\mathcal{N}_{enco} \arrow[uuu, shift left, "\dashv"] \arrow[ur, shift left, "\top"] \arrow[rrr, shift left] & & & \mathcal{N}_{conf} \arrow[lll, shift left, "\top"] \arrow[ul, shift left, "\perp"] \arrow[uuu, shift left, "\dashv"]
\end{tikzcd} \tag{5.6.41}$$

in which $\mathcal{N} = [\mathscr{J}, \mathcal{M}]$ has the monoidal structure given by the Day convolution of Definition 3.3.2. Under the four model structures on $\mathcal{N}$, it is a Quillen ring that satisfies the monoid axiom whem $\mathcal{M}$ and $\mathcal{M}'$ do.

Proof. The diagram is obtained from (5.4.30) via the substitutions

$$\begin{aligned} \mathcal{M} &\mapsto \mathcal{N}, & \qquad & & \mathcal{M}' &\mapsto \mathcal{N}', \\ \mathcal{L} &\mapsto \mathcal{P}, & \qquad &\text{and} & \mathcal{L}' &\mapsto \mathcal{P}'. \end{aligned}$$ □

Remark 5.6.42. The case of orthogonal G-spectra. *Our application of this diagram is the case where $\mathcal{N} = [\mathscr{J}, \mathcal{M}]$ is the category of orthogonal G-spectra $[\mathscr{J}_G, \mathcal{T}^G]$ for a finite group G. It is illustrated in Figure 7.1, where the four model structures on the left are instances of the four outer ones in (5.6.41) as follows.*

$$\mathcal{N}_{enla} \longmapsto \boxed{\textit{equifibrant model structure}} \qquad \mathcal{N} \longmapsto \boxed{\textit{projective model structure}}$$

$$\mathcal{N}_{enco} \longmapsto \boxed{\textit{positive equifibrant model structure}} \qquad \mathcal{N}_{conf} \longmapsto \boxed{\textit{positive model structure}}$$

The four model structures on the right in Figure 7.1 are obtained by stabilizing the ones on the left, and stabilization is a form of Bousfield localization, the subject of Chapter 6.

5.7 Stable and exactly stable model categories

The following is due to [Hov99, Definition 7.1.1].

Definition 5.7.1. *A pointed model category $\mathcal{S}p$ is* **stable** *if the functors $\mathbf{L}\Sigma, \mathbf{R}\Omega : \operatorname{Ho}\mathcal{S}p \to \operatorname{Ho}\mathcal{S}p$ of Theorem 4.6.24 are equivalences on the homotopy category of $\mathcal{S}p$.*

Stable model categories are studied in great depth by Schwede and Shipley in [SS03b]. We will denote such a category by $\mathcal{S}p$ rather than $\mathcal{M}$ to remind the reader that we are typically dealing with spectra rather than spaces.

The following notion of **exact stability** is new as far as we know. We will see in Theorem 5.7.6 and in §5.7B that it is helpful in establishing certain exact sequences of homotopy groups. In §7.3E and §7.4E, we will show that the categories of spectra we are interested in are exactly stable.

It may be the case that all stable model categories are exactly stable, and this might follow from the results of [SS03b]. We leave this question for the interested reader.

Before giving the definition, we need the following observation. For a fibrant replacement functor R with coaugmentation r in any pointed topological model category, there is a natural transformation $\mu : R\Omega \Rightarrow \Omega R$. For each object X, the map μ_X is the image of the identity morphism on ΩX under the composite map

$$\begin{tikzcd}
\mathcal{S}p(\Omega X, \Omega X) \arrow[r, "\cong"] & \Omega\mathcal{S}p(\Omega X, X) \arrow[d, "\Omega(R_{\Omega X, X})"'] & \\
& \Omega\mathcal{S}p(R\Omega X, RX) \arrow[r, "\cong"] & \mathcal{S}p(R\Omega X, \Omega RX)
\end{tikzcd} \tag{5.7.2}$$

$$1_{\Omega X} \longmapsto \mu_X,$$

where the vertical arrow is induced by the endofunctor R as in (3.1.14), and the horizontal isomorphisms are those of (5.6.13).

Definition 5.7.3. *A model category $\mathcal{S}p$ is* **exactly stable** *if it is pointed topological and the following conditions are met.*

(i) For each $k > 0$, the Quillen pair (Σ^k, Ω^k) is invertible as in Definition 4.5.3.

We will refer to Σ^{-1} (Ω^{-1}) *as* **desuspension** (**delooping**), *and we denote* $\Omega^k\Omega^{-k}$ *by* Θ^k. *The accompanying natural transformations are denoted by*

$$\sigma^k : \Sigma^k\Sigma^{-k} \Rightarrow 1_{\mathcal{S}p} \qquad \text{and} \qquad \theta^k : 1_{\mathcal{S}p} \Rightarrow \Theta^k.$$

The superscripts on σ, θ *and* Θ *are not exponents. Instead we have*

$$\begin{aligned} \sigma^k_X &\cong \sigma_X(\Sigma\sigma_{\Sigma^{-1}X})\cdots(\Sigma^{k-1}\sigma_{\Sigma^{1-k}X}) \\ \text{and} \qquad \theta^k_X &\cong (\Omega^{k-1}\theta_{\Omega^{1-k}X})(\Omega^{k-2}\theta_{\Omega^{2-k}X})\cdots\theta_X, \end{aligned}$$

and these maps are weak equivalences on cofibrant and fibrant objects respectively.

(ii) For each object X *and each* $k > 0$, *the unit and counit maps for the Quillen adjunction* $\Sigma^k \dashv \Omega^k$,

$$\eta^k_X : X \to \Omega^k\Sigma^k X \qquad \text{and} \qquad \epsilon^k_X : \Sigma^k\Omega^k X \to X,$$

are weak equivalences.

(iii) For each object X, *the morphism* μ_X *of (5.7.2) is a weak equivalences.*

Such a category has an endofunctor Θ^∞ *with coaugmentation* θ^∞. *For each object* X, $\Theta^\infty X$ *is the homotopy colimit or telescope (see Example 5.8.5(iv) and Lemma 5.8.20 below) of the diagram*

$$X \xrightarrow{\theta_X} \Theta X \xrightarrow{\theta_{\Theta X}} \Theta^2 X \xrightarrow{\theta_{\Theta^2 X}} \cdots, \tag{5.7.4}$$

and θ^∞_X *is the evident map* $X \to \Theta^\infty X$.

Proposition 5.7.5. Exact stability implies stability.

Proof The existence of functors in the homotopy category induced by Σ^{-1} and Ω^{-1} means that $\mathbf{L}\Sigma$ and $\mathbf{R}\Omega$ are equivalences. $\square$

5.7A Extending the Puppe exact sequences

Now we show that the exact sequences of Proposition 4.7.11 can be extended indefinitely to the right in the stable setting, as promised at the end of §4.7.

Theorem 5.7.6. Exact sequences in the exactly stable case. *Let* $\mathcal{S}p$ *be a model category that is exactly stable as in Definition 5.7.3. Then the long exact sequences of Proposition 4.7.11 can be extended indefinitely to the right with all terms having natural abelian group structures. In the case of Proposition 4.7.11(i) we have*

$$\cdots \longrightarrow \pi(A, \Omega^{-q}F) \xrightarrow{(\Omega^{-q}p_f)_*} \pi(A, \Omega^{-q}X) \xrightarrow{(\Omega^{-q}f)_*} \pi(A, \Omega^{-q}Y) \xrightarrow{(\Omega^{-q-1}\partial_f)_*} \cdots$$

(where $\pi(-,-)$ *is as in Definition 4.3.11) and for Proposition 4.7.11(ii) we have*

$$\cdots \longrightarrow \pi(\Sigma^{-q}C, B) \xrightarrow{(\Sigma^{-q}i_f)^*} \pi(\Sigma^{-q}Y, B) \xrightarrow{(\Sigma^{-q}f)^*} \pi(\Sigma^{-q}X, B) \xrightarrow{(\Sigma^{-q-1}\delta_f)^*} \cdots$$

Proof For the fiber sequence of (4.7.7) we have a diagram of fibrant objects

$$\begin{tikzcd} F \arrow[r, "p_f"] \arrow[d, "\omega_F"', "\simeq"] & X \arrow[r, "f"] \arrow[d, "\omega_X"', "\simeq"] & Y \arrow[d, "\omega_Y"', "\simeq"] \\ \Omega\Omega^{-1}F \arrow[r, "\Omega\Omega^{-1}p_f"] & \Omega\Omega^{-1}X \arrow[r, "\Omega\Omega^{-1}f"] & \Omega\Omega^{-1}Y \arrow[r, "\Omega^{-1}\partial_f"] & \Omega^{-1}F \arrow[r, "\Omega^{-1}p_f"] & \Omega^{-1}X \arrow[r, "\Omega^{-1}f"] & \Omega^{-1}Y, \end{tikzcd} \tag{5.7.7}$$

where the maps $\omega_{(-)}$ are induced by the natural transformation ω of Definition 5.7.3(i), and the bottom row is an extended fiber sequence. Applying the functor $\pi(A, -)$ for cofibrant A enables us the extend the exact sequence of Proposition 4.7.11(i) three more terms to the right, and this procedure can be repeated any number of times.

A dual procedure can be applied to extend the exact sequence of Proposition 4.7.11(ii). □

5.7B The Adams exact sequence

Now we will show that when the pointed topological model category $\mathcal{S}p$ is exactly stable, applying the functor $\pi(A, R(-))$ to a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C$$

as in (4.7.8) leads to an additional long exact sequence comparable to the first one in Theorem 5.7.6, the **Adams exact sequence**. This name is new to our knowledge, but the sequence has been around for half a century, having first appeared in [Ada64, Lecture 3].

The following was proved by Adams for original spectra in [Ada64, Lemma 3.5] and again in [Ada74b, Proposition III.3.10], and by Gaunce Lewis (1949–2006) and May for G-spectra in [LMSM86, Lemma III.2.1]. None of those books was written in the language of model categories, so there was no mention of fibrant replacement.

Lemma 5.7.8. Maps into a stable homotopy cofiber sequence. *Let $\mathcal{S}p$ be an exactly stable model category. For a cofibrant object A and a morphism $f : X \to Y$ in $\mathcal{S}p$ with homotopy cofiber C_f as in Definition 4.1.28, the sequence*

$$\pi(A, RX) \xrightarrow{R(f)_*} \pi(A, RY) \xrightarrow{R(i_f)_*} \pi(A, RC_f),$$

is exact, meaning that the image of $R(f)_*$ is the preimage of the trivial element in $\pi(A, RC_f)$, where R is a fibrant replacement functor.

Proof Let $\alpha : A \to RY$ be a morphism for which $R(i_f)\alpha$ is homotopic to the trivial map. Then consider the homotopy commutative diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, dashed, "\Sigma^{-1}\gamma"'] & A \arrow[r, "i_A"] \arrow[d, "\alpha"] & CA \arrow[r, "\delta_A"] \arrow[d, dashed, "\beta"] & \Sigma A \arrow[d, dashed, "\gamma"] \\ RX \arrow[r, "R(f)"'] & RY \arrow[r, "R(i_f)"'] & RC_f \arrow[r, "R(\delta_f)"'] & R\Sigma X, \end{tikzcd}$$

in which the first row is a cofiber sequence and the bottom row is the fibrant replacement of one. The map β is an extension of $R(i_f)\alpha$ over the cone CA, which exists because the latter map is homotopic to the trivial map. For the map γ, note that $R(\delta_f)\beta i_A$ is homotopic to $R(\delta_f)R(i_f)\alpha$, which is trivial. It follows that there is an extension γ of $R(\delta_f)\beta$ to ΣA.

The map γ is adjoint to the vertical map in the following diagram.

$$\begin{tikzcd} & & A \arrow[d] \\ RX \arrow[r, "R(\eta_X)", "\simeq"'] & R\Omega\Sigma X \arrow[r, "\mu_{\Sigma X}", "\simeq"'] & \Omega R\Sigma X. \end{tikzcd}$$

Here the map η_X is the weak equivalence of Definition 5.7.3(ii) and $\mu_{\Sigma X}$ is that of Definition 5.7.3(iii). The objects in the bottom row are each fibrant, so the functor $\pi(A, -)$ converts the maps to isomorphisms. Hence we get the desired element in $\pi(A, RX)$. □

We can use exact stability to give the set $\pi(A, RX)$ for cofibrant A an abelian group structure which is natural in X. For each $k > 0$, we have maps

$$\begin{tikzcd} \pi(A, RX) \arrow[r, "(R\eta_X^k)_*", "\cong"'] & \pi(A, R\Omega^k\Sigma^k X) & \\ & \pi(A, \Omega^k R\Sigma^k X) \arrow[u, "\cong"'] \arrow[r, "\cong"] & \pi(\Sigma^k A, R\Sigma^k X) \end{tikzcd} \tag{5.7.9}$$

where $\eta_X^k : X \to \Omega^k\Sigma^k X$ is the weak equivalence of Definition 5.7.3(ii). Since η_X^k is a weak equivalence, $\mathcal{S}p(A, R\eta_X^k)$ is one by Lemma 5.6.17, so $\pi(A, R\eta_X^k)$ is an isomorphism. The vertical map is based on the composite weak equivalence of fibrant objects

$$\Omega^k R\Sigma^k X \xrightarrow{\Omega^{k-1}\mu_{\Sigma^k X}} \Omega^{k-1}R\Omega\Sigma^k X \xrightarrow{\Omega^{k-2}\mu_{\Omega\Sigma^k X}} \cdots \longrightarrow R\Omega^k\Sigma^k X,$$

for $\mu_{(-)}$ as in (5.7.2). Such a morphism is converted to an isomorphism by the functor $\pi(A, -)$. The map on the right of (5.7.9) is induced by an adjunction isomorphism. In the case of original spectra, this isomorphism for $k = 1$ was the subject of [Ada74b, Theorem III.3.7].

It follows that $\pi(A, RX)$ is isomorphic to $\pi(\Sigma^2 A, R\Sigma^2 X)$, which is an abelian group. Hence application of the functor $\pi(A, R(-))$ to the extended cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{i_f} C_f \xrightarrow{\delta_f} \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma i_f} \Sigma C_f \xrightarrow{\Sigma \delta_f} \cdots \tag{5.7.10}$$

leads to a long exact sequence of abelian groups going indefinitely to the right. We can extend it to the left with the dual of (5.7.7), namely the following diagram of cofibrant objects

$$\begin{tikzcd}
 & & & X \arrow[r, "f"] & Y \arrow[r, "i_f"] & C_f \\
\Sigma^{-1}X \arrow[r, "\Sigma^{-1}f"] & \Sigma^{-1}Y \arrow[r, "\Sigma^{-1}i_f"] & \Sigma^{-1}C_f \arrow[r, "\Sigma^{-1}\delta_f"] & \Sigma\Sigma^{-1}X \arrow[u, "\sigma_X", "\simeq"'] \arrow[r] & \Sigma\Sigma^{-1}Y \arrow[u, "\sigma_Y", "\simeq"'] \arrow[r] & \Sigma\Sigma^{-1}XC_f \arrow[u, "\sigma_{C_f}", "\simeq"']
\end{tikzcd}$$

where the maps σ_{-} are induced by the natural transformation σ of Definition 5.7.3(i).

Replacing X by $\Sigma^{-k}X$ in (5.7.9) leads to another isomorphism

$$\pi(A, R\Sigma^{-k}X) \xrightarrow{\cong} \pi(\Sigma^k A, R\Sigma^k\Sigma^{-k}X) \xrightarrow[\cong]{R(\sigma_X^k)_*} \pi(\Sigma^k A, RX),$$

which leads to the following.

Theorem 5.7.11. The Adams exact sequence. *Let $\mathcal{S}p$ be an exactly stable pointed topological model category. For a morphism $f : X \to Y$ in $\mathcal{S}p$ with homotopy cofiber C_f as in Definition 4.1.28, there is a long exact sequence similar to that of Proposition 4.7.11(i),*

$$\cdots \longrightarrow \pi(\Sigma^q A, RX) \xrightarrow{(Rf)_*} \pi(\Sigma^q A, RY) \xrightarrow{(Ri_f)_*} \pi(\Sigma^q A, RC_f) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi(A, RX) \xrightarrow{(Rf)_*} \pi(A, RY) \xrightarrow{(Ri_f)_*} \pi(A, RC_f) \xrightarrow{(R\delta_f)_*} \cdots$$

$$\cdots \longrightarrow \pi(A, R\Sigma^q X) \xrightarrow{(R\Sigma^q f)_*} \pi(A, R\Sigma^q Y) \xrightarrow{(R\Sigma^q i_f)_*} \pi(A, R\Sigma^q C_f) \xrightarrow{(R\Sigma^q \delta_f)_*} \cdots$$

where the homomorphisms in the second and third rows are induced by the maps in (5.7.10), and the last homomorphism in the top row is the composite

$$\pi(\Sigma^q A, RC_f) \xrightarrow{(R\delta_f)_*} \pi(\Sigma^q A, R\Sigma X) \xrightarrow{\cong} \pi(\Sigma^{q-1} A, RX),$$

with the isomorphism being induced by a weak equivalence similar to that of (5.7.9).

5.8 Homotopy limits and colimits

The original source for this material is [BK72, Chapters XI and XII]. More recent treatments can be found in [Hir03, Chapters 18 and 19], [Dug17], [Rie14, Chapters 5 and 6] and [Shu06]. Homotopy limits and colimits in a combinatorial model category (Definition 4.8.11 below) are described concisely in [Lur09, A.2.8].

Very briefly, given a topological model category $\mathcal{M}$ as in Definition 5.6.3 and a small category J, one has functors

$$\operatorname{colim}, \lim : \mathcal{M}^J \to \mathcal{M}$$

as in Definition 2.3.22, which are not homotopical in general. Here we are using the projective (injective) model structure on $\mathcal{M}^J$ (as in Definition 5.4.2) in the colimit (limit) case.

Since the two functors are not homotopical, they do not induce functors between the corresponding homotopy categories. Example 4.4.1 is an elementary case of this difficulty. As explained in §4.4, derived functors (when they exist) provide a way around this problem. **The homotopy colimit (homotopy limit) is a point set left (right) derived functor (as in Definition 4.4.11) of colim (lim).**

Curiously the definition of homotopy limits and colimits does **not** require a model structure on the category in question. They were originally defined for simplicial sets, and the definition can easily be modified to work for any topological category. On the other hand, the problems they are designed to address, illustrated in Example 4.4.1 and Example 7.2.70 below, are model theoretic, as are the theorems about them, such as Theorem 5.8.8, Theorem 5.8.9, Theorem 5.8.10 and Theorem 5.8.16 below.

5.8A The Bousfield-Kan definition

We begin with the definitions of homotopy limits (**homotopy inverse limits** in their terminology) and homotopy colimits (**homotopy direct limits**) of [BK72, Chapter XI], repeated with minor differences and more modern notation in [Hir03, Chapter 18]. These concern functors from a small category J to the category $\mathcal{S}et_{\mathbf{\Delta}}$ of simplicial sets. We refer the reader to [Hir03, Chapter 19] for the case where $\mathcal{S}et_{\mathbf{\Delta}}$ is replaced by a more general model category.

We will state the definitions for a topological model category as in Definition 5.6.3. We will often refer the reader to [Hir03, Chapter 18], which concerns homotopy limits and colimits in simplicial model categories. Since all topological model categories are simplicial as well by Corollary 5.6.16, Hirschhorn's results apply here.

Let J be a small category. Let $\mathcal{M}$ be a topological model category and let $X : J \to \mathcal{M}$ be a functor. We denote its image on an object j in J by X_j, and we denote the category of such functors by $\mathcal{M}^J$, in which morphisms are natural transformations of functors. Morphism objects in both $\mathcal{M}$ and $\mathcal{M}^J$ are topological spaces since $\mathcal{M}$ is topological. Given two functors (J-diagrams) X and Y, the morphism object on $\mathcal{M}^J$, meaning the set of natural transformations between the two functors, is the end (Definition 2.4.5)

$$\mathcal{M}^J(X,Y) \cong \int^J \mathcal{M}(X_j, Y_j)$$

by Proposition 2.4.19 and Definition 3.2.18.

For an object c in a category $\mathcal{C}$, recall the over and under categories $\mathcal{C}{\downarrow}c$ and $c{\downarrow}\mathcal{C}$ of Definition 2.1.51. Note that $(c{\downarrow}\mathcal{C})^{op} = \mathcal{C}^{op}{\downarrow}c$. The small category J has a nerve $N(J)$, the simplicial set given in Definition 3.4.12, as does $J{\downarrow}j$ for an object j of J. An n-simplex in $N(J{\downarrow}j)$ is a diagram of the form

$$j_0 \longrightarrow j_1 \longrightarrow \cdots \longrightarrow j_n \longrightarrow j,$$

and we get a map $f_j : N(J{\downarrow}j) \to N(J)$ by dropping that last morphism in the diagram. The simplicial set $N(J{\downarrow}j)$ has a base point, the vertex $*$ corresponding to the identity morphism $j \to j$. A morphism $\beta : j \to j'$ in J induces a map of simplicial sets $J{\downarrow}\beta : N(J{\downarrow}j) \to N(J{\downarrow}j')$, and hence a functor $J{\downarrow}- : J \to \mathcal{S}et_{\boldsymbol{\Delta}}$, i.e., an J-shaped diagram of simplicial sets sending j to the nerve $N(J{\downarrow}j)$. The maps f_j lead to an isomorphism $\lim_J N(J{\downarrow}-) \to N(J)$. The identity map on $N(J{\downarrow}j)$ is homotopic to the composite

$$N(J{\downarrow}j) \to * \to N(J{\downarrow}j).$$

The following is the Bousfield-Kan definition of [BK72, XI.3.2], which they stated for diagrams of simplicial sets, modified to work for diagrams in a topological model category.

Definition 5.8.1. The homotopy limit and colimit of a diagram in a topological model category. *Let $\mathcal{M}$ be a topological model category as in Definition 5.6.3. Note that such a model category is bitensored over $\mathcal{T}op$ as in Definition 3.1.31. Let $X : J \to \mathcal{M}$ be a J-diagram in $\mathcal{M}$. Its homotopy limit and colimit are defined by*

$$\begin{aligned} \operatorname*{holim}_J X &:= \int^J X_j^{B(J\downarrow j)} \\ \text{and} \qquad \operatorname*{hocolim}_J X &:= \int_J B(J^{op}{\downarrow}j) \times X_j. \end{aligned}$$

The homotopy limit and colimit are both natural in X. The functor

$$\operatorname{holim} : \mathcal{M}^J \to \mathcal{M}$$

is the right adjoint of the functor $W \mapsto (J{\downarrow}-) \times W$, which assigns to each object W in $\mathcal{M}$ the functor sending each object j in J to the object $N(J{\downarrow}j) \times W$. The corresponding functor for hocolim is the left adjoint of $W \mapsto (J^{op}{\downarrow}-) \times W$.

The equivalence $B(J{\downarrow}j) \to *$ leads to a map

$$\int^J \mathrm{Hom}(*, X_j) \cong \int^J X_j = \lim_J X \xrightarrow{\quad\eta\quad} \operatorname{holim}_J X \tag{5.8.2}$$

which need **not** be a weak equivalence. Dually, there is a map

$$\epsilon : \operatorname{hocolim}_J X \to \operatorname{colim}_J X. \tag{5.8.3}$$

See Theorem 5.8.16 below for conditions guaranteeing that η and ϵ are weak equivalences.

[BK72, XI.3.5] offers the following case where η fails to be a weak equivalence.

Example 5.8.4. Bousfield-Kan's toy counterexample. *Let J be the small category $a \rightrightarrows b$ and let X be an object in $(\mathcal{S}et_{\boldsymbol{\Delta}})^J$ with $X_a = *$ and X_b fibrant. Then $\lim_J X$ is either $*$ or the empty set, depending on whether the two maps $* \to X_b$ are the same. The classifying spaces BJ and $B(J{\downarrow}b)$ are equivalent to S^1, while $B(J{\downarrow}a)$ is contractible. It follows that $\operatorname{holim}_J X$ is equivalent to the loop space ΩX_b and therefore not equivalent in general to $\lim_J X$.*

Each of the examples below save the first one is taken directly from [BK72, Chapter XI]. In each case, **there are similar examples for diagrams in $\mathcal{T}op$, $\mathcal{T}$, $\mathcal{T}op^G$, $\mathcal{T}^G$ for any group** G, and more generally for any topological model category or simplicial model category as in as in Definition 5.6.3. Recall (Corollary 5.6.16) that all topological model categories are simplicial. We leave these formulations to the reader. Simplicial versions of these examples are discussed in more detail in [Rie14, §6.4 and §6.5].

Example 5.8.5. Some Bousfield-Kan homotopy limits and colimits. *The examples here are described as diagrams of simplicial sets, following [BK72].* **They could be replaced by diagrams of topological spaces or of objects in a topological model category.**

(i) **Fixed point sets and orbit spaces of group actions.** *Let G be a group and J the corresponding one object category $\mathcal{B}G$ of Definition 2.1.31 ; we will denote its single object by $*$. Then a J-diagram X is an action of G on simplicial set which we also denote by X, for which*

$$\lim_J X = X^G \qquad \text{and} \qquad \operatorname{colim}_J X = X_G = X/G,$$

the fixed point set and orbit space of the action, as noted in Example 2.3.35(iii). Then BJ is the classifying space BG of G while $B(J{\downarrow}*)$, which has a vertex *for each element of G, is the* **contractible free G-space** *EG of Proposition 3.4.15(iv). It follows that*

$$holim\, X = \mathrm{Hom}(N(J{\downarrow}*), X)^G =: X^{hG},$$

the **homotopy fixed point set** *of X, and*

$$hocolim\, X = N(J^{op}{\downarrow}*) \underset{G}{\times} X =: X_{hG},$$

the **homotopy orbit set** *of the simplicial set X, also known as the* **Borel construction**.

When G acts trivially on X, we have $lim\, X = X^G = X$ and

$$holim\, X = X^{hG} = \mathrm{Hom}(N(J), X).$$

Thus the map $\eta : X^G \to X^{hG}$ of (5.8.2) is induced by the equivariant map $EG \to *$. In general these two objects are quite different. The Sullivan conjecture [Sul71], proved by Miller in [Mil84], says that the two are equivalent when G is finite and X is a finite complex.

These notions will be critical in what follows. They will be repeated as formal definitions in §8.3A below.

(ii) **Mapping path spaces and mapping cylinders.** *Let J be the category $a \to b$, so a simplicial functor X on J is simply a map $f : X_a \to X_b$ and $\lim_J X = X_a$. A similar statement is true whenever J has an initial object. The classifying spaces of J and $J{\downarrow}b$ are each unit intervals, while that of $J{\downarrow}a$ is a point. It follows that*

$$\underset{J}{holim}\, X = N_f = \left\{(x, \omega) \in X_a \times X_b^I : \omega(0) = f(x)\right\},$$

(where I denotes the unit interval $[0,1]$ and X_b^I denotes the path space of X_b) the **mapping path space of** *f as in Definition 4.2.5. The map $\eta : \lim_J X = X_a \to \underset{J}{holim}\, X$ of (5.8.2) sends $x \in X_a$ to $(x, \omega_{f(x)})$ where ω_y is the constant y-valued path in X_b. We also have a map $j : \underset{J}{holim}\, X \to X_a$ given by $(x, \omega) \mapsto x$. Since $j\eta = 1_{X_a}$, X_a is a retract of $\underset{J}{holim}\, X$.*

It is known to be an equivalence when X_a and X_b are fibrant. In the pointed case the base point is the pair (x_0, ω_0), where $x_0 \in X_a$ is the base point and ω_0 is the constant path at the base point $y_0 \in X_b$.

Dually, $\underset{J}{colim}\, X = X_b$ (and similarly whenever J has a terminal object), $\underset{J}{hocolim}\, X$ is the mapping cylinder

$$M_f = ((X_a \times I) \amalg X_b) / ((x, 1) \sim f(x)),$$

and the map $\epsilon : M_f \to X_b$ is an equivalence for cofibrant X_a and X_b. In *the pointed case the homotopy colimit is the reduced mapping cylinder M'_f of Definition 3.5.1. The map $\epsilon : M_f \to X_b$ is given by $(x,t) \mapsto f(x)$ and $y \mapsto y$. We also have an inclusion $i : X_b \to M_f$ with $\epsilon i = 1_{X_b}$, so X_b is a retract of M_f.*

(iii) **Homotopy pullbacks and pushouts, including homotopy fibers and cofibers.** *Let J be the category $a' \to b \leftarrow a''$, so a simplicial functor X on J is a pullback diagram*

$$X_{a'} \xrightarrow{f'} X_b \xleftarrow{f''} X_{a''}$$

and

$$\lim_J X = \{(x',x'') \in X_{a'} \times X_{a''} : f'(x') = f''(x'')\} := X_{a'} \underset{X_b}{\times} X_{a''},$$

The classifying spaces of J and $J{\downarrow}b$ are unit intervals while those of $J{\downarrow}a'$ and $J{\downarrow}a'$ are points. It follows that

$$\begin{aligned} \operatorname{holim}_J X &= \{(x',x'',\omega) \in X_{a'} \times X_{a''} \times X_b^I \\ &\qquad : f'(x') = \omega(0),\ f''(x'') = \omega(1)\} \\ &=: X_{a'} \underset{X_b}{\times}{}^h X_{a''}, \end{aligned}$$

where X_b^I denotes the path space of X_b. This is the **homotopy pullback**. *It is the ordinary pullback in*

$$\begin{tikzcd} X_{a'} \underset{X_b}{\times}{}^h X_{a''} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_b^I \arrow[d] \\ X_{a'} \times X_{a''} \arrow[r, "(f',f'')"] & X_b \times X_b \end{tikzcd} \qquad \begin{tikzcd} \omega \arrow[d, mapsto] \\ (\omega(0), \omega(1)). \end{tikzcd} \tag{5.8.6}$$

Meanwhile the ordinary limit is the pullback of the diagram

$$\begin{tikzcd} X_{a'} \underset{X_b}{\times} X_{a''} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_b \arrow[d, "\Delta"] \\ X_{a'} \times X_{a''} \arrow[r, "(f',f'')"] & X_b \times X_b, \end{tikzcd}$$

which we can map to the diagram of (5.8.6) by using the constant path map $X_b \to X_b^I$.

The map $\eta : \lim_J X \to \operatorname{holim}_J X$ of (5.8.2) is known to be an equivalence when either f' or f'' is a fibration, but it need not be one in general.

- *When $X_{a'} = X_{a''} = *$ mapping to distinct points in a connected fibrant X_b, the ordinary limit is empty while the homotopy limit is the space of paths in X_b connecting the two image points. This is Example 5.8.4.*

- *When $X_{a'} = *$ and $X_{a''}$ is arbitrary, the homotopy limit is known as the* **homotopy fiber** *of the map $X_{a''} \to X_b$, while the ordinary limit is the preimage of under this map of the image of $X_{a'}$.*

Replacing J by J^{op} yields a pushout diagram

$$X_{a'} \xleftarrow{f'} X_b \xrightarrow{f''} X_{a''}$$

The **homotopy pushout** $\operatorname*{hocolim}_{J^{op}} X$ *is also known as the* **double mapping cylinder**

$$\begin{aligned} &cyl(X_{a'}, X_b, X_{a''}) \\ &\quad := ((X_b \times I) \amalg X_{a'} \amalg X_{a''}) / ((y,0) \sim f'(y),\ (y,1) \sim f''(y)). \end{aligned}$$

The map ϵ from it to the usual pushout is an equivalence if either map is a cofibration. When $X_{a'} = X_{a''} = *$, the ordinary pushout is also $*$ while the homotopy pushout is the unreduced suspension of X_b. In this case the maps are cofibrations only when $X_b = *$. When $X_{a'} = *$ and $X_{a''}$ is arbitrary, the ordinary pushout is the quotient $X_{a''}/X_b$ while the homotopy pushout is the **mapping cone** $C_{f''}$ *of $f'' : X_b \to X_{a''}$, also known as the* **homotopy cofiber***, namely the ordinary pushout of $CX_b \leftarrow X_b \to X_{a''}$, where CX_b is the cone on X_b.*

(iv) **Towers and telescopes.** *Recall the sequential limit category N^{op} of Definition 2.3.63,*

$$0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots$$

and let X be the diagram (a tower)

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \longleftarrow \cdots .$$

Then the ordinary inverse limit (a sequential limit as in Definition 2.3.63) is

$$\lim_{N^{op}} X = \left\{ (x_0, x_1, \dots) \in \prod_{j \geqslant 0} X_j \colon f_j(x_j) = x_{j-1} \text{ for } j > 0 \right\}.$$

The homotopy limit is

$$\operatorname*{holim}_{N^{op}} X = \left\{ (\omega_0, \omega_1, \dots) \in \prod_{j \geqslant 0} X_j^I \colon f_j(\omega_j(1)) = \omega_{j-1}(0) \text{ for } j > 0 \right\},$$

where each $\omega_j : I \to X_j$ is a path. The two are known to be weakly equivalent when each X_j is fibrant and each map f_j is a fibration; see Theorem 5.8.16 below.

Dually, replacing N^{op} by N, the sequential colimit category of Definition 2.3.63, yields a telescope diagram

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots .$$

The colimit (a sequential colimit as in Definition 2.3.63) is

$$\operatorname*{colim}_N X = \left(\coprod_{j\geqslant 0} X_j\right) / x_j \sim f_j(x_j)$$

and the homotopy colimit is the **telescope**

$$\operatorname*{hocolim}_N X = \left(\coprod_{j\geqslant 0} X_j \times I\right) / (x_j, 1) \sim (f_j(x_j), 0).$$

The two are known to be weakly equivalent when each X_j is cofibrant and each f_j is a cofibration; see Theorem 5.8.16 below.

Remark 5.8.7. Telescopes as ordinary sequential colimits. *For*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots .$$

as in Example 5.8.5(iv), let

$$\widetilde{X}_n = \left(\coprod_{0\leqslant j<n} X_j \times I\right) \amalg X_n / (x_j, 1) \sim (f_j(x_j), 0).$$

Then we have inclusions $\widetilde{f}_j : \widetilde{X}_j \to \widetilde{X}_{j+1}$, and maps $p_n : \widetilde{X}_n \to X_n$. In particular $\widetilde{X}_0 = X_0$, $\widetilde{X}_1$ is the mapping cylinder M_{f_0} as in Example 5.8.5(ii), and $\widetilde{X}_n$ is a quotient of the union of the first n mapping cylinders. This leads to a diagram

$$\begin{tikzcd}
\widetilde{X}_0 \arrow[r, "\widetilde{f}_0"] \arrow[d, "p_0"', "="] & \widetilde{X}_1 \arrow[r, "\widetilde{f}_1"] \arrow[d, "p_1"] & \widetilde{X}_2 \arrow[r, "\widetilde{f}_2"] \arrow[d, "p_2"] & \cdots \\
X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] & X_2 \arrow[r, "f_2"] & \cdots
\end{tikzcd}$$

and a map $p : \operatorname*{colim}_N \widetilde{X}_n \to \operatorname*{colim}_N X_n$. The ordinary colimit of the top row is the homotopy colimit of the bottom row. In particular, $\widetilde{X}_{n+1}$ is the pushout in the diagram

$$\begin{tikzcd}
x \arrow[r, mapsto] & (x,1) \\
\widetilde{X}_n \arrow[r] \arrow[d, "f_n p_n"'] & \widetilde{X}_n \times I \arrow[d] \\
X_{n+1} \arrow[r] & \widetilde{X}_{n+1}.
\end{tikzcd}$$

See Lemma 5.8.20 for further discussion.

5.8B Homotopy invariance

The following results are taken from [Hir03, Theorems 18.5.1–18.5.3], where they are stated and proved for simplicial model categories. We are stating them for topological model categories, which are simplicial by Corollary 5.6.16.

In each of the following three theorems, $\mathcal{M}$ is a topological model category, J is a small category, and X and Y are objects in the functor category $\mathcal{M}^J$, meaning J-diagrams in $\mathcal{M}$. Thus a morphism $f : X \to Y$ in $\mathcal{M}^J$ is a natural transformation of functors whose value on an object j of J is denoted by $f_j : X_j \to Y_j$.

Theorem 5.8.8. Homotopy limits (colimits) preserve fibrations (cofibrations). *Let $f : X \to Y$ be a morphism in $\mathcal{M}^J$. Then if each $f_j : X_j \to Y_j$ is a fibration (cofibration), then the induced map of homotopy limits (homotopy colimits) is fibration (cofibration).*

Theorem 5.8.9. Homotopy limits (colimits) of fibrant (cofibrant) objects are fibrant (cofibrant). *Let X be an object in $\mathcal{M}^J$. Then if each X_j is a fibrant (cofibrant), then the homotopy limit (homotopy colimit) is fibrant (cofibrant).*

Theorem 5.8.10. Homotopy limits (colimits) preserve weak equivalences of fibrant (cofibrant) objects. *Let $f : X \to Y$ be a morphism in $\mathcal{M}^J$. Then if each $f_j : X_j \to Y_j$ is a weak equivalence of fibrant (cofibrant) objects, then the induced map of homotopy limits (homotopy colimits) is weak equivalence.*

In the case $\mathcal{M} = \mathcal{T}op$ with its usual model structure as in Definition 4.2.1, Dugger-Isaksen [DI04, Theorem A.7] show that one can remove the cofibrancy hypothesis in Theorem 5.8.10. They show that **any** objectwise weak equivalence of diagrams in $\mathcal{T}op$ induces a weak equivalence of homotopy colimits. They do this by comparing this model structure with that of Strøm [Str72], in which weak equivalences are homotopy equivalences. They show that both structures lead to the same homotopy colimits up to weak equivalence in the usual sense. In the Strøm model structure all objects are cofibrant, so no cofibrant replacement is needed.

Since all objects in $\mathcal{T}op$ are fibrant, any objectwise weak equivalence of diagrams in $\mathcal{T}op$ induces a weak equivalence of homotopy limits as well.

5.8C Homotopy colimits via the two sided bar construction

The following construction has the homotopy colimit as a special case. More information can be found in [Rie14, Chapters 4 and 5]. We will use it in Theorem 8.8.3 below.

Definition 5.8.11. The two sided categorical bar construction. *Let J be a small (topological) category with (continuous) functors $T : J^{op} \to \mathcal{T}op$ and $S : J \to \mathcal{T}op$. Then $B(T, J, S) = |B_\bullet(T, J, S)|$ (the geomtric realization of a simplicial space) where*

$$B_n(T, J, S) = \left\{(t, \underline{j}, s) \colon \underline{j} \in N(J)_n, t \in T(j_n), s \in S(j_0)\right\},$$

in which $\underline{j}$ is a diagram in J of the form

$$j_0 \to j_1 \to \cdots \to j_n.$$

$B_n(T, J, S)$ is topologized as a subspace of the product

$$\left(\coprod_{j \in Ob\,(J)} T(j)\right) \times \left(\coprod_{j', j'' \in Ob\,(J)} J(j', j'')\right)^n \times \left(\coprod_{j \in Ob\,(J)} S(j)\right).$$

5.8D Change of indexing category

Here we will deal with questions analogous to those of § 2.3H. As we did there, we will confine the discussion to homotopy colimits, leaving the dual statements about homotopy limits to the reader. They can be found in [Dug17, §6.10].

Suppose we have small categories J and K, a topological model category $\mathcal{M}$, and functors

$$J \xrightarrow{S} K \xrightarrow{X} \mathcal{M}.$$

This leads to a morphism

$$\phi_S : \operatorname*{hocolim}_J XS \to \operatorname*{hocolim}_K X \tag{5.8.12}$$

and we want to know when it is a weak equivalence.

Here is the homotopy analog of Definition 2.3.80.

Definition 5.8.13. Homotopy final functors. *A functor $S : J \to K$ is* **homotopy final** *(or* **homotopy terminal***, or* **homotopy left cofinal***) if for each $k \in K$ the undercategory $(k{\downarrow}S)$ as in Definition 2.1.51 is non-empty and contractible as in Definition 3.4.19. When S is the inclusion of a subcategory J of K, we say that J* **is homotopy final in** *K.*

Note that this contractibility requirement on $(k{\downarrow}\alpha)$ is stronger than the connectivity requirement of Definition 2.3.80.

The following analog of Corollary 2.3.83 is proved by Dugger as [Dug17, Lemma 6.8]. He states it for the case $\mathcal{M} = \mathcal{T}op$, but his proof works in the generality stated here.

Lemma 5.8.14. Homotopy colimits indexed by categories with terminal objects. *Suppose the small category K has a terminal object k as in Example 2.1.16(ii) and $\mathcal{M}$ is a topological model category. Let S be the inclusion functor of the trivial category into K corresponding to k. Then for any functor $X : K \to \mathcal{M}$, the composite*

$$\underset{K}{hocolim}\, X \xrightarrow{\ \epsilon\ } \underset{K}{colim}\, X \xrightarrow{\ \phi_S\ } X(k),$$

for ϵ as in (5.8.3) and ϕ_S as in (2.3.79), is a weak equivalence.

For Dugger this is a step toward proving the following analog of Theorem 2.3.82, which is his [Dug17, Theorem 6.7].

Theorem 5.8.15. Homotopy colimit maps induced by homotopy final functors. *Let $\mathcal{M}$ be a topological model category and let $S : J \to K$ be a homotopy final functor as in Definition 5.8.13. Then for any functor $X : K \to \mathcal{M}$, the induced map ϕ_α of (5.8.12) is a weak equivalence.*

5.8E Homotopy colimits indexed by generalized direct $\mathcal{M}$-categories.

Recall that a direct category as in Definition 2.1.58 is a small category in which each object X has a nonnegative integer $|X|$ (its degree) assigned to it, and there are no morphisms that lower degree. Examples include the indexing categories relevant to mapping cylinders (see Example 5.8.5(ii)), pushouts (Example 5.8.5(iii)) and sequential colimits as in Example 5.8.5(iv). Their duals are indexed by inverse categories. If $\mathcal{D}$ is a direct category and $\mathcal{M}$ is a model category, then the functor category $\mathcal{M}^{\mathcal{D}}$ has the projective model structure of Definition 5.4.2.

In the three examples cited above, J is a direct category, and a cofibrant diagram is one in which each of the objects is cofibrant and each of the maps is a cofibration. For more complicated $\mathcal{D}$ the description of cofibrant objects in $\mathcal{M}^{\mathcal{D}}$ may not be so simple.

Projectively cofibrant (injectively fibrant) functors from generalized direct (inverse) $\mathcal{M}$-categories (as in Definition 5.6.31) are of interest in light of the following result, which is a special case of Hirschhorn's [Hir03, Theorem 19.9.1]. He speaks there of Reedy categories, of which direct and inverse categories are special cases.

Theorem 5.8.16. Equivalence of certain categorical and homotopy

limits/colimits. *Let $\mathcal{M}$ be a topological model category and $\mathcal{D}$ a direct (inverse) category as in Definition 2.1.58. Then for a projectively cofibrant (injectively fibrant) diagram in $\mathcal{M}^{\mathcal{D}}$, the natural map*

$$\epsilon : \operatorname*{hocolim}_{\mathcal{D}} X \to \operatorname*{colim}_{\mathcal{D}} X \qquad \left(\eta : \lim_{\mathcal{D}} X \to \operatorname*{holim}_{\mathcal{D}} X\right)$$

of (5.8.3) (of (5.8.2)) is a weak equivalence. In particular this holds for a sequential colimit (limit) in which the objects are all cofibrant (fibrant) and the maps are all cofibrations (fibrations).

Corollary 5.8.17. The case of telescopes, pushouts, coequalizers and their duals. *Let $\mathcal{M}$ be a topological model category and let N be the sequential colimit category*

$$0 \to 1 \to 2 \to 3 \to \cdots ,$$

so an object X in $\mathcal{M}^N$ is a diagram

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots . \tag{5.8.18}$$

Its homotopy colimit is the telescope of Example 5.8.5(iv). If each object X_j is cofibrant and each map f_j is a cofibration, then the map

$$\epsilon : \operatorname*{hocolim}_{N} X \to \operatorname*{colim}_{N} X$$

of (5.8.3) from the homotopy colimit to the categorical colimit is a weak equivalence.

Similar statements hold for pushouts and coequalizers (e.g. cofibers).

For the dual diagrams, meaning towers, pullbacks and equalizers, the map η from the ordinary limit to the homotopy limit is a weak equivalence when the objects in the diagram are fibrant and the maps are fibrations.

The following results will be useful in our study of spectra in Chapter 7 and later chapters. They are elementary and surely known to the experts, but we have not seen them explicitly stated in the literature.

The following is needed in the proof of Lemma 7.3.22 below.

Lemma 5.8.19. Telescopes and towers of weak equivalences and of isomorphisms. *Let $\mathcal{M}$ and N be as in Corollary 5.8.17.*

(i) If each map f_n in (5.8.18) is a weak equivalence, then the evident map from X_0 to the homotopy colimit is also a weak equivalence.

(ii) If each map f_n in (5.8.18) is an isomorphism, then X_0 is a retract of the homotopy colimit, meaning there is a map $r_X : \operatorname{hocolim} X \to X_0$ such that the composite

$$X_0 \longrightarrow \operatorname{hocolim} X \xrightarrow{r_X} X_0$$

is the identity.

Dually, if each map in a sequential limit X is a weak equivalence, then so is the evident map from the homotopy limit to X_0. If each map is an isomorphism, then X_0 is a retract of the homotopy limit.

Proof We will only prove the statements about telescopes.

(i) Consider the coend of Definition 5.8.1 that defines the homotopy colimit. In this case each of the classifying spaces $B(N^{op}\downarrow n)$ is contractible. Thus the coend is the quotient of a countable coproduct of objects each equivalent to X_0 obtained by identifying them with each other via the maps f_n. The result follows.

(ii) Since each X_j is isomorphic to X_0, the coend of Definition 5.8.1 is isomorphic to

$$X_0 \times \int_J B(J^{op}\downarrow j),$$

and the desired map r_X is projection onto the first factor. □

The following is needed in the proof of Proposition 7.1.19, Lemma 7.3.22 and Corollary 7.3.24 below.

Lemma 5.8.20. Telescopes as sequential colimits. *Let $\mathcal{M}$ be a topological model category.*

(i) Telescopes, that is homotopy colimits over the sequential colimit category N of Definition 2.3.63, preserve finite limits. In particular they preserve pullbacks.
(ii) A functor that preserves sequential colimits also preserves homotopy sequential colimits.
(iii) A homotopy sequential colimit is also an ordinary sequential colimit in which each map is an h-cofibration as in Definition 5.6.7.
(iv) If A is compact as in Definition 5.2.6, then the map

$$hocolim\, \mathcal{M}(A, X_n) \to \mathcal{M}(A, hocolim\, X_n)$$

is a homeomorphism.
(v) For π_0 as in Definition 5.6.4,

$$\pi_0(hocolim\, X_n) \cong colim\, \pi_0(X_n).$$

(vi) Let $X' \to X$ denote the following map of sequential diagrams in a topological model category $\mathcal{M}$.

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X_0 \arrow[r, "f_0"] \arrow[d, "f_0"] & X_1 \arrow[r, "f_1"] \arrow[d, "f_1"] & \cdots \\ X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] & X_2 \arrow[r, "f_2"] & \cdots \end{tikzcd} \tag{5.8.21}$$

Then the induced map $hocolim\, X' \to hocolim\, X$ is a weak equivalence.

In view of this result and Definition 5.2.6, we make the following.

Definition 5.8.22. Compact Quillen functors. *A right Quillen functor is compact if it commutes with homotopy sequential colimits. A left Quillen functor is compact if its right adjoint is. A Quillen pair is compact if both of its functors are.*

In a Quillen ring $(\mathcal{M}, \wedge, S)$, for an object K that is compact as in Definition 5.2.6, the Quillen adjoint functors $K \wedge (-)$ and $\mathcal{M}(K, -)$ are compact as above.

Proof of Lemma 5.8.20 For each $n \geqslant 0$, let $[n]$ be the full subcategory of N whose objects are the natural numbers $\leqslant n$. Then there are functors $[n] \to [m]$ for $n \geqslant m$ and $N \to [m]$ given by $i \mapsto \min(m, i)$. For simplicity we denote the restriction of a functor $X : N \to \mathcal{M}$ to the subcategory $[n]$ by X as well. This leads to a diagram

$$\operatorname*{hocolim}_{[0]} X \to \operatorname*{hocolim}_{[1]} X \to \operatorname*{hocolim}_{[2]} X \to \cdots \to \operatorname*{hocolim}_{N} X, \tag{5.8.23}$$

so we can define a functor $\widetilde{X} : N \to \mathcal{M}$ by $\widetilde{X}_n = \operatorname*{hocolim}_{[n]} X$. Then we have

$$\operatorname*{hocolim}_{N} X = \operatorname*{colim}_{N} \widetilde{X}, \tag{5.8.24}$$

and (i) and (ii) follow. Since the maps in the diagram $\widetilde{X}$ are standard inclusion maps associated with mapping cylinders, they are h-cofibrations as claimed in (iii).

For (iv), each of the maps (but the last) in the diagram (5.8.23) is an h-cofibration (Definition 5.6.7), so the result follows.

For (v), $\widetilde{X}_n$ is weakly equivalent to X_n. The former should be thought of as a telescope with $n+1$ lenses joined by n cylinders. One has an inclusion map $i_n : X_n \to \widetilde{X}_n$, and a projection map $p_n : \widetilde{X}_n \to X_n$. Both are weak equivalences with

$$f_n = p_{n+1} \widetilde{f}_n i_n \qquad \text{for } n \geqslant 0.$$

It follows that $\pi_0 \widetilde{X}_n \cong \pi_0 X_n$, and

$$\begin{aligned} \pi_0 \text{hocolim}\, X_n &\cong \pi_0 \text{colim}\, \widetilde{X}_n \text{ by (5.8.24)} \\ &\cong \text{colim}\, \pi_0 \widetilde{X}_n \text{ by Proposition 5.6.11 since each map} \\ &\qquad \text{in the diagram is an } h\text{-cofibration} \\ &\cong \text{colim}\, \pi_0 X_n \text{ since } X_n \text{ and } \widetilde{X}_n \text{ are weakly equivalent.} \end{aligned}$$

For (vi), the map of homotopy colimits associated with (5.8.21) is the map of ordinary colimits associated with the middle rows of

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, equal] & X_0 \arrow[r, "f_0"] \arrow[d, "\simeq"', "i_0"] & X_1 \arrow[r, "f_1"] \arrow[d, "\simeq"', "i_1"] & \cdots \\
\varnothing \arrow[r] \arrow[d] & \tilde{X}_0 \arrow[r, "\tilde{f}_0"] \arrow[d, "\tilde{f}_0"] & \tilde{X}_1 \arrow[r, "\tilde{f}_1"] \arrow[d, "\tilde{f}_1"] & \cdots \\
\tilde{X}_0 \arrow[r, "\tilde{f}_0"] \arrow[d, equal] & \tilde{X}_1 \arrow[r, "\tilde{f}_1"] \arrow[d, "\simeq"', "p_1"] & \tilde{X}_2 \arrow[r, "\tilde{f}_2"] \arrow[d, "\simeq"', "p_2"] & \cdots \\
X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] & X_2 \arrow[r, "f_2"] & \cdots
\end{tikzcd}$$

The induced map of colimits is a weak equivalence. □

Proposition 5.8.25. Telescopes and homotopy Cartesian squares. *Let J be as in Corollary 5.8.17 and let $\mathcal{M}$ be a topological model category in which sequential colimits preserve finite products. Let $p : X \to Y$ be a morphism in $\mathcal{M}^J$, namely a diagram*

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "p_0"'] & X_1 \arrow[r, "f_1"] \arrow[d, "p_1"'] & X_2 \arrow[r, "f_2"] \arrow[d, "p_2"'] & X_3 \arrow[r, "f_3"] \arrow[d, "p_3"'] & \cdots \\
Y_0 \arrow[r, "g_0"] & Y_1 \arrow[r, "g_1"] & Y_2 \arrow[r, "g_2"] & Y_3 \arrow[r, "g_3"] & \cdots
\end{tikzcd}$$

Suppose further that for each $n > 0$, the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "p_0"] \arrow[d, "f_{n-1}f_{n-2}\cdots f_0"'] & Y_0 \arrow[d, "g_{n-1}g_{n-2}\cdots g_0"] \\
X_n \arrow[r, "p_n"] & Y_n
\end{tikzcd} \tag{5.8.26}$$

is homotopy Cartesian as in Definition 5.8.37. Then the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "p_0"] \arrow[d, "f_\infty"'] & Y_0 \arrow[d, "g_\infty"] \\
\operatorname*{hocolim}_J X \arrow[r, "p_\infty"] & \operatorname*{hocolim}_J Y,
\end{tikzcd} \tag{5.8.27}$$

where the maps f_∞, g_∞ and p_∞ are the homotopy colimits of the corresponding maps in (5.8.26), is also homotopy Cartesian.

Proof The pullback in (5.8.26) is $W_n := Y_0 \underset{Y_n}{\times} X_n$, and the pullback corner map to it from X_0 is a weak equivalence since the diagram is homotopy Cartesian. Thus we get a diagram

$$X_0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow W_3 \longrightarrow \cdots$$

in which each map is a weak equivalence. Its homotopy colimit, which we denote by W_∞, is the pullback object of (5.8.27) by Lemma 5.8.20(i). By Lemma 5.8.19, the map $X_0 \to W_\infty$ is also a weak equivalence, which means that (5.8.27) is homotopy Cartesian as claimed. □

We will use following definition in our study of spectra in Chapter 7.

Definition 5.8.28. *A model category is* **telescopically closed** *if every homotopy sequential colimit (telescope) of fibrant objects is fibrant.*

Model categories in which all objects are fibrant, such as $\mathcal{T}$, are, in particular, telescopically closed.

Hovey defines (with attribution to Voevodsky) a condition in [Hov01b, Definition 4.1] guaranteeing that every sequential colimit of fibrant objects is fibrant. It is that of Definition 5.2.7(ii). The fibrancy consequence is the subject of [Hov01b, Lemma 4.3]

5.8F Homotopy limits in right proper model categories

Right proper model categories as in Definition 5.3.1 are a convenient setting to generalize the notions of homotopy pullback and homotopy fiber in $\mathcal{T}op$ and its variants (all of which are proper) introduced in Example 5.8.5(iii). As before we start with a pullback diagram

$$X' \xrightarrow{f'} Y \xleftarrow{f''} X'' \tag{5.8.29}$$

The homotopy pullback is defined by replacing the maps f' and f'' by fibrations using a functorial factorization F_1 as in **MC5** and then taking the ordinary pullback, as explained in [Hir03, §13.3.1].

Definition 5.8.30. Homotopy pullbacks in a right proper model category. *Let F be a functorial factorization as in Definition 2.2.9, similar to F_1 of* **MC5** *in that it factors every map $f : X \to Y$ in a right proper model category $\mathcal{M}$ as*

$$X \xrightarrow{j_f} F(f) \xrightarrow{p_f} Y,$$

where j_f is a trivial cofibration and p_f is a fibration. Then the **homotopy pullback** *of (5.8.29) is the ordinary pullback of*

$$F(f') \xrightarrow{p_{f'}} Y \xleftarrow{p_{f''}} F(f'').$$

The diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "j_{f'}"'] & Y \arrow[d, equal] & X'' \arrow[l, "f''"'] \arrow[d, "j_{f''}"] \\
F(f') \arrow[r, "p_{f'}"] & Y & F(f''). \arrow[l, "p_{f''}"']
\end{tikzcd}$$

leads to a natural map from the ordinary pullback to the homotopy pullback,

$$j_{f',f''} : X' \underset{Y}{\times} X'' \to F(f') \underset{Y}{\times} F(f'') \tag{5.8.31}$$

Strictly speaking, this map depends on the choice of functorial factorization F, but we will see below in Proposition 5.8.34 any two such differ by a weak equivalence.

Remark 5.8.32. Warning. *The homotopy pullback is* **not** *to be confused with the homotopy limit (Definition 5.8.1) of a pullback diagram discussed in Example 5.8.5(iii). It is known (Proposition 5.8.36 below) that the two are weakly equivalent when X', X'' and Y are each fibrant.*

For the following see [Hir03, Proposition 13.3.4 and 13.3.9].

Proposition 5.8.33. Homotopy invariance of the homotopy pullback. *In a right proper model category $\mathcal{M}$, suppose we have a commutative diagram*

$$\begin{tikzcd}
X'_0 \arrow[r, "f'_0"] \arrow[d, "\simeq"'] & Y_0 \arrow[d, "\simeq"] & X''_0 \arrow[l, "f''_0"'] \arrow[d, "\simeq"] \\
X'_1 \arrow[r, "f'_1"] & Y_1 & X''_1 \arrow[l, "f''_1"']
\end{tikzcd}$$

in which the vertical maps are weak equivalences. Then the induced map of homotopy pullbacks is also a weak equivalence. If in addition at least one map in each row is a fibration, then the induced map of ordinary pullbacks is also a weak equivalence.

It turns out that resulting object is also independent (up to weak equivalence) of the choice of functorial factorization F. The following are proved by Hirschhorn as [Hir03, Proposition 13.3.7 and Corollary 13.3.8].

Proposition 5.8.34. Flexibility of the homotopy pullback. *If in a right proper model category*

$$X' \xrightarrow{j'} W' \xrightarrow{p'} Y \quad \text{and} \quad X'' \xrightarrow{j''} W'' \xrightarrow{p''} Y$$

are factorizations of f' and f'' in which j' and j'' are weak equivalences, and p' and p'' are fibrations, then the homotopy pullback of (5.8.29) is naturally weakly equivalent to each of $W' \underset{Y}{\times} W''$, $X' \underset{Y}{\times} W''$ and $W' \underset{Y}{\times} X''$.

Corollary 5.8.35. Pullbacks involving a fibration. *If either f' or f'' in (5.8.29) is a fibration, then the map $j_{f',f''}$ of (5.8.31) from the ordinary pullback to the homotopy pullback is a weak equivalence.*

The following is proved by Hirschhorn as [Hir03, Proposition 19.5.3].

Proposition 5.8.36. Homotopy pullbacks and homotopy limits. *Let $\mathcal{M}$ be a right proper topological model category with a pullback diagram*

$$X' \xrightarrow{f'} Y \xleftarrow{f''} X''$$

in which all three objects are fibrant. Then its homotopy pullback as in Definition 5.8.30 is naturally weakly equivalent to its homotopy limit $X' \times_Y^h X''$ as in Definition 5.8.1 and Example 5.8.5(iii).

Definition 5.8.37. Homotopy Cartesian squares. *Let*

$$\begin{tikzcd} A \arrow[r, "g''"] \arrow[d, "g'"'] & X'' \arrow[d, "f''"] \\ X' \arrow[r, "f'"] & Y \end{tikzcd}$$

be a commutative diagram in a model category. The diagram with A removed is a pullback diagram $\mathbf{X}$ and there is a canonical map $\alpha : A \to \lim \mathbf{X}$, the pullback corner map of Definition 2.3.9. The diagram above is **homotopy Cartesian** *if α is a weak equivalence. Dually, it is* **homotopy coCartesian** *if the map to Y from the evident pushout is a weak equivalence.*

The following definition should be compared to the above.

Definition 5.8.38. Homotopy fiber squares. *Let*

$$\begin{tikzcd} A \arrow[r, "g''"] \arrow[d, "g'"'] & X'' \arrow[d, "f''"] \\ X' \arrow[r, "f'"] & Y \end{tikzcd} \tag{5.8.39}$$

be a commutative diagram in a right proper model category. It is a **homotopy fiber square** *is the map from A to the homotopy pullback of (5.8.29) is a weak equivalence.*

Proposition 5.8.40. Equivalence of homotopy fiber squares and homotopy Cartesian squares. *Let $\mathcal{M}$ be a right proper model category and suppose that either f' or f'' in (5.8.39) is a fibration. Then (5.8.39) is a homotopy fiber square iff it is a homotopy Cartesian square.*

Proof Corollary 5.8.35 says that in this case the map $j_{f',f''}$ from the categorical pullback to the homotopy one is a weak equivalence. This means that one of the maps to them from from A is a weak equivalence iff the other one is. □

Remark 5.8.41. Comparison with the Bousfield-Friedlander definition of [BF78, A.2] and Definition 5.8.37. *The former involves a factorization of only one of the maps f' and f'', meaning they require the map from*

A to either $X' \underset{Y}{\times} W''$ or $W' \underset{Y}{\times} X''$ to be a weak equivalence. This is equivalent to Definition 5.8.38 by Proposition 5.8.34.

Definition 5.8.42. *Consider the diagram (5.8.29) with $X'' = *$ in a right proper model category $\mathcal{M}$. Then* **a point in** *Y is a map $f'' : * \to Y$ and the* **fiber of f' at that point** *is the pullback of that diagram.*

The homotopy pullback of the diagram of Definition 5.8.38 need not be fibrant, but the following object always is.

Definition 5.8.43. *Consider the diagram (5.8.29) with $X'' = *$ in a right proper model category $\mathcal{M}$. Then the* **homotopy fiber** *of f' at that point is the homotopy pullback of the diagram*

$$F(f') \xrightarrow{p_{f'}} Y \xleftarrow{f''} *.$$

Homotopy fibers will be used in the study of localizing subcategories in §6.3C below. These in turn figure in the slice filtration of §11.1.

The following is [Hir03, Proposition 13.4.6].

Proposition 5.8.44. The homotopy fiber of a fibration. *When the map f' is a fibration, then the fiber of Definition 5.8.42 is weakly equivalent to the homotopy fiber of Definition 5.8.43.*

The next two results concern homotopy Cartesian squares as in Definition 5.8.37. They will be used in the proof of Theorem 7.3.29.

Proposition 5.8.45. Right Quillen functors preserve homotopy Cartesian squares of fibrations. *Let $\mathcal{M}$ be model category in which*

$$\begin{tikzcd} X_0 \arrow[r, "p_0"] \arrow[d, "f_0"'] & Y_0 \arrow[d, "g_0"] \\ X_1 \arrow[r, "p_1"] & Y_1 \end{tikzcd} \tag{5.8.46}$$

is a homotopy Cartesian square (as in Definition 5.8.37) where p_0 and p_1 are fibrations, and let $U : \mathcal{M} \to \mathcal{N}$ be a right Quillen functor. Then

$$\begin{tikzcd} UX_0 \arrow[r, "Up_0"] \arrow[d, "Uf_0"'] & UY_0 \arrow[d, "Ug_0"] \\ UX_1 \arrow[r, "Up_1"] & UY_1 \end{tikzcd} \tag{5.8.47}$$

is a homotopy Cartesian square in $\mathcal{N}$ in which Up_0 and Up_1 are fibrations.

Proof Let P_0 denote the pullback in (5.8.46), so the pullback corner map $X_0 \to P_0$ is a weak equivalence. The map $P_0 \to Y_0$ is a fibration because p_1 is one. This means that the pullback corner map is a weak equivalence of fibrant objects parametrized over Y_0, as in Definition 4.5.10. It follows from **??** that right Quillen functors preserve such weak equivalences. We also know that right Quillen functors preserve limits (Proposition 4.5.4), so UP_0 is the pullback of (5.8.47). The map to it from UX_0 is a weak equivalence, so the result follows. □

The following is a homotopy analog of Proposition 2.3.6. It will be used in the proof of Theorem 7.3.29 below.

Proposition 5.8.48. Composing homotopy Cartesian squares. *Suppose we have homotopy Cartesian squares*

$$\begin{tikzcd} X_0 \arrow[r, "p_0"] \arrow[d, "f_0"'] & Y_0 \arrow[d, "g_0"] \\ X_1 \arrow[r, "p_1"] & Y_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_1 \arrow[r, "p_1"] \arrow[d, "f_1"'] & Y_1 \arrow[d, "g_1"] \\ X_2 \arrow[r, "p_2"] & Y_2 \end{tikzcd} \tag{5.8.49}$$

in a right proper model category $\mathcal{M}$, in which each p_i is a fibration. Then

$$\begin{tikzcd} X_0 \arrow[r, "p_0"] \arrow[d, "f_1 f_0"'] & Y_0 \arrow[d, "g_1 g_0"] \\ X_2 \arrow[r, "p_2"] & Y_2 \end{tikzcd} \tag{5.8.50}$$

is also homotopy Cartesian.

Proof Let P_0 and P_1 denote the pullbacks of the two squares in (5.8.49). Then we have a diagram

$$\begin{tikzcd} X_0 \arrow[r, "a_0"] \arrow[dr, "f_0"'] & P_0 \arrow[r, "a_0'"] \arrow[d, "b_0"] & P_0' \arrow[r, "p_0'"] \arrow[d, "b_0'"] & Y_0 \arrow[d, "g_0"] \\ & X_1 \arrow[r, "a_1"] \arrow[dr, "f_1"'] & P_1 \arrow[r, "p_1'"] \arrow[d, "b_1"] & Y_1 \arrow[d, "g_1"] \\ & & X_2 \arrow[r, "p_2"] & Y_2 \end{tikzcd}$$

in which p_0' and p_1' are fibrations, a_0 and a_1 are weak equivalences and each square is a pullback. Then P_0' is the pullback of (5.8.50) by Proposition 2.3.6, so it suffices to show that the map a_0' is a weak equivalence.

To this end we factor the vertical map b_0' functorially as a trivial cofibration followed by a fibration and choose $P_{1/2}$ to be the pullback of the lower square in the following diagram.

$$\begin{tikzcd}
P_0 \arrow[r, "a_0'"] \arrow[d, "b_{0,0}"'] & P_0' \arrow[d, "b_{0,0}'"] \\
P_{1/2} \arrow[r, "a_{1/2}"] \arrow[d, "b_{0,1}"'] & P_{1/2}' \arrow[d, "b_{0,1}'"] \\
X_1 \arrow[r, "a_1"] & P_1
\end{tikzcd}$$

The left column need not be the functorial factorization of b_0. Since $b_{0,1}'$ is a fibration, $b_{0,1}$ is also one. Recall that a_1 is a weak equivalence by hypothesis, so $a_{1/2}$ is a one since $\mathcal{M}$ is right proper.

Then it follows from Proposition 2.3.6 that P_0 is the pullback of the upper square. Since $a_{1/2}$ and $b_{0,0}'$ are weak equivalences, Proposition 4.5.9 tells us that a_0' is also one. □

The following lemma will be used in the proof of Theorem 7.4.43 below, in which the map corresponding to i is a trivial fibration between cofibrant objects.

Lemma 5.8.51. A homotopy Cartesian diagram and a weak equivalence of morphism objects. *Let $\mathcal{M}$ be a right proper Quillen ring as in Definition 5.5.9, and let $\mathcal{N}$ be a Quillen $\mathcal{M}$-module as in Definition 5.6.3. Let $i : A \to B$ be a weak equivalence between cofibrant objects in $\mathcal{N}$ and let $p : X \to Y$ be a fibration between fibrant objects in $\mathcal{N}$. Then the following diagram in $\mathcal{M}$ is homotopy Cartesian as in Definition 5.8.37.*

$$\begin{tikzcd}
\mathcal{N}(B,X) \arrow[r, "p_*"] \arrow[d, "i^*"'] & \mathcal{N}(B,Y) \arrow[d, "i^*"] \\
\mathcal{N}(A,X) \arrow[r, "p_*"] & \mathcal{N}(A,Y)
\end{tikzcd} \tag{5.8.52}$$

Moreover both vertical maps are weak equivalences.

Proof Suppose first that i is a trivial cofibration. Then the diagram coincides with that of (5.6.1) and the map of (5.6.2) is a weak equivalence. This means our diagram is homotopy Cartesian in that case.

It follows from Proposition 5.5.8 that both maps labelled p^* are fibrations since A and B are both cofibrant, so $\mathcal{N}(A, -)$ and $\mathcal{N}(B, -)$ are right Quillen functors, which we are applying to the fibration p.

Since X and Y are fibrant, the functors

$$\mathcal{N}(-, X), \mathcal{N}(-, Y) : \mathcal{N}^{op} \to \mathcal{M}$$

are right Quillen functors by Proposition 5.5.8. Hence they preserve weak equivalences between fibrant objects in $\mathcal{N}^{op}$. A weak equivalence between cofibrant objects in $\mathcal{N}$ is opposite to such a map between fibrant objects in $\mathcal{N}^{op}$.

Thus the maps i^* and p_* in (5.8.52) are weak equivalences and fibrations respectively. Since $\mathcal{M}$ is right proper, the pullback of a weak equivalence along a fibration is again a weak equivalence. Hence $\mathcal{N}(B, X)$ is weakly equivalent to both the pullback and $\mathcal{N}(A, X)$, so the diagram is homotopy Cartesian. $\square$

6

Bousfield localization

The problem that led to Bousfield's model category structure was that of constructing a *localization functor* for a homology theory. That is, given a homology theory h_*, the problem was to define for each space X a local space $L_{h_*}X$ and a natural homology equivalence $X \to L_{h_*}X$. There had been a number of partial solutions to this problem (perhaps the most complete being that of Bousfield and Kan [BK72]), but each of these was valid only for some special class of spaces, and only for certain homology theories. In [Bou75], Bousfield constructed a functorial h_*-localization for an arbitrary homology theory h_* and for every simplicial set. In Bousfield's model category structure, a fibrant approximation to a space (i.e.„ a weak equivalence from a space to a fibrant space) was exactly a localization of that space with respect to h_*.

Phillp Hirschhorn, [Hir03, page ix]

Bousfield localization, first introduced by Bousfield in [Bou75], is one of the most useful constructions in model category theory. Briefly, one starts with a model category $\mathcal{M}$ and enlarges the class of weak equivalences to form a new model category $\mathcal{M}'$ with the same underlying category as $\mathcal{M}$. The cofibrations and trivial fibrations of $\mathcal{M}'$ are the same as those of $\mathcal{M}$. Recall that trivial fibrations are by definition maps having the right lifting property with respect to cofibrations, and fibrations are by definition maps having the right lifting property with respect to trivial cofibrations. Since there are more weak equivalences in $\mathcal{M}'$ than in $\mathcal{M}$, there are **more trivial cofibrations** and hence **fewer fibrations**. There are also fewer fibrant objects, and fibrant replacement tends to be more interesting (or drastic) in $\mathcal{M}'$ than in $\mathcal{M}$. As indicated in Remark 4.1.7, the hard part of showing that the new model structure exists is verifying that it satisfies the factorization axiom **MC5**.

The definition of Bousfield localization does not require that the model category in question be cofibrantly generated. We have previously discussed two other ways of modifying a cofibrantly generated model structure without changing the underlying category. The first is the enlargement procedure of Theorem 5.2.34, which leaves the class $\mathscr{W}$ of weak equivalences unchanged while enlarging the class $\mathscr{C}$ of cofibrations. The second applies to a functor category $\mathcal{M}^J$. We can replace its projective model structure by the one induced from the similar one on $\mathcal{M}^K$ for a subcategory K of J as in Theorem 5.4.21. It has both more weak equivalences and fibrations than the projective model structure, as explained in Remark 5.4.23.

In each of these three cases there is a Quillen adjunction in which both functors are the identity. The left adjoint has the original model category as its domain in the first two cases and as its codomain in the third case.

Table 6.1 indicates how the classes of weak equivalences $\mathscr{W}$, cofibrations $\mathscr{C}$ and fibrations $\mathscr{F}$ in the new model structure on $\mathcal{M}$ or $\mathcal{M}^J$ compare with those in the original one.

Table 6.1 *Three methods of altering a cofibrantly generated model structure. Compare with Figure 7.1 and Theorem 9.2.13.*

Construction	$\mathscr{W}$	$\mathscr{C}$	$\mathscr{F}$	Identity functor from new to original model category
Enlargement as in Theorem 5.2.34, e.g. equifibrant enlargement.	Same	More	Less	Right Quillen
Confinement as in Theorem 5.4.21, e.g. positivization.	More	Less	More	Left Quillen
Bousfield localization as in this chapter, e.g. stabilization.	More	Same	Less	Right Quillen

In § 6.1 we give three examples, each dating from the 1970s and due to Bousfield. They are localization of spaces with respect to a generalized homology theory, the same for spectra, and the passage from strict equivalences of spectra to stable ones. A fourth example, in which the nth Postnikov section of a space is its fibrant replacement, is given below in Example 6.2.13 and Example 6.2.14.

In § 6.2 we discuss more general approaches to Bousfield localization, of which there are two. Roughly speaking, they amount to redefining the class of weak equivalences and redefining the fibrant replacement functor. For the former one specifies a set or class of morphisms in $\mathcal{M}$ that one wants to be weak equivalences in $\mathcal{M}'$. This could be anything from a single morphism f to the class of morphisms that induce isomorphisms after applying some functor, such as a homology theory. In most cases the new class of weak equivalences is bigger than the union of the original ones with the specified set of additional morphisms. If you invite a few new morphisms to the party, they will bring all of their friends. See Definition 6.2.1 for details.

We assume throughout that our original model category $\mathcal{M}$ is enriched over another model category, possibly itself, so that we can speak of weak equivalences of morphism objects. We also need to assume that $\mathcal{M}$ is proper as in Definition 5.3.1. Given a morphism set $\mathcal{S}$, we say that an object Z is **$\mathcal{S}$-local** if each map $f : X \to Y$ in $\mathcal{S}$ induces a weak equivalence $f^* : \mathcal{M}(Y, Z) \to \mathcal{M}(Y, Z)$. These will turn out to be the fibrant objects in the new model structure. Then we say a map $g : A \to B$ is an **$\mathcal{S}$-local equivalence** the induced map

$$g^* : \mathcal{M}(B, Z) \to \mathcal{M}(A, Z)$$

is a weak equivalence for each $\mathcal{S}$-local object Z. Such maps will be the weak equivalences in the new model structure. This process is known as **left Bousfield localization**.

There is of course a dual notion in which we enlarge the class of weak equivalences but retain the same class of fibrations. This leads to fewer cofibrations and cofibrant objects, and a more interesting cofibrant replacement functor. **We will make no use of this notion in this book.**

The second major approach to left Bousfield localization is indicating what the new fibrant replacement functor Υ should be. It is the subject of §6.2B. In Definition 6.2.15 we say that for a homotopy idempotent functor $\Upsilon : \mathcal{M} \to \mathcal{M}$, a map $g : X \to Y$ is **Υ-equivalence** (**Υ-fibration**) if Υg is a weak equivalence (fibration). Υ-cofibrations are defined to be ordinary cofibrations.

Theorem 6.2.16 is about how these two approaches interact. It spells out properties that such a Υ must have in relation to a morphism class $\mathscr{C}$ in order to yield the same class of weak equivalences in a potential new model structure. The assumptions are that Υ-local objects detect $\mathscr{C}$-local equivalences the same way that $\mathscr{C}$-local objects do, and that each $\mathscr{C}$-local object Z is also a Υ-local object. The conclusions are that Υ-equivalences are $\mathscr{C}$-local equivalences, that the map $X \to \Upsilon X$ is always a $\mathscr{C}$-local equivalence and that fibrant approximation in $\mathcal{M}'$ is related to that in $\mathcal{M}$ in a certain way.

§6.3 is a collection of results about when Bousfield localization is possible. Hirschhorn's Theorem 6.3.4 says that a model category $\mathcal{M}$ satisfying some conditions spelled out in Definition 6.3.2 can be localized with respect to any morphism set $\mathcal{S}$.

Now suppose the morphism set $\mathcal{S}$ consists of cofibrations and that the topological model category $\mathcal{M}$ is cofibrantly generated with generating sets $\mathcal{I}$ and $\mathcal{J}$. In Definition 6.3.8 we define an enlargement $\overline{\Lambda(S)}$ of $\mathcal{J}$ related to $\mathcal{S}$ that will, under favorable hypotheses, serve as the generating set of trivial cofibrations in the localized model structure. One is tempted to think this is always the case, but there is a simple counterexample due to Bousfield (Example 6.3.11) that shows otherwise.

6.1 It's all about fibrant replacement

Now we give three striking examples of fibrant replacement. They involve Bousfield localization [Bou75] and Ω-spectra [BF78].

6.1A The Bousfield localization of a space with respect to a homology theory E_* is its fibrant replacement

We now turn to Bousfield localization of spaces, the subject of [Bou75]. It is a special case of a more general procedure, localization of a topological model category with respect to a class of morphisms, the subject of Definition 6.2.5 below.

Here we have a homology theory E_* defined on $\mathcal{T}$, and the class of morphisms is that of E_*-equivalences. We say that a pointed space W is **E_*-local** if any E_*-equivalence $f : X \to Y$ (meaning a map for which $E_*(f)$ is an isomorphism) induces an isomorphism $f^* : [Y, W] \to [X, W]$. An **E_*-localization of** X is a map $\lambda : X \to L_E X$ which is an E_*-equivalence to an E_*-local space. It follows that any map from X to an E_*-local space factors uniquely through $L_E X$ and that λ factors uniqely through any E_*-equivalence out of X. If $L_E X$ exists, it is unique up to weak equivalence.

The idea is to construct a new model structure on the category $\mathcal{T}$ in which fibrant replacement is E_*-localization. This is done by defining cofibrations to be the usual ones and expanding the class of weak equivalences to that of E_*-equivalences. Fibrations and trivial fibrations are then defined by their lifting properties. This means there are more trivial cofibrations and hence fewer fibrations than in the standard model structure. We can use Lemma 5.6.17 to show that a model structure with these cofibrations and weak equivalences has E_*-localization as its fibrant replacement functor. Any E_*-equivalence has a cofibrant approximation, so every fibrant object is E_*-local.

As indicated in Remark 4.1.7, the difficulty lies in proving that there really is a model structure with the desired cofibrations and weak equivalences satisfying **MC5**. The factorization F_0 of **MC5** is easy: the one given by the standard model structure on $\mathcal{T}$ will do since it consists of a standard cofibration, which is also an E_*-cofibration by definition, followed by a standard trivial fibration. The latter is defined by the same lifting property as an E_*-trivial fibration and is automatically an E_*-equivalence. In particular cofibrant replacement is the same in the E_*-model structure as in the standard one.

The factorization F_1 of **MC5** is another matter. It gives fibrant replacement of the source when the target is a point. It is the subject of [Bou75, Theorem 11.1], whose proof involves a cardinality argument.

6.1B Bousfield localization of spectra

Bousfield proves an analogous localization theorem for spectra in [Bou79], but his proof does not use model categories. Instead he completes a program suggested by Frank Adams.

Suppose we want to localize a spectrum A with respect to a homology theory represented by a spectrum E. Consider the category of maps $K \to A$ where $E_*(K) = 0$. Let ${}_EA \to A$ be the "direct limit" (or colimit in more modern terminology) of all such maps. Then its cofiber (Definition 4.7.6) A_E is the desired localization of A.

Adams suggested this in a lecture at the University of Chicago in the early 1970s. Bousfield, who was in the audience, asked him how he knew the collection $\{K \to A\}$ was a set. Adams did not have an answer.

In [Bou79, Lemmas 1.12 and 1.13] Bousfield showed that it suffices to consider E_*-acyclic CW spectra K with cardinality (of the set of cells in K) bounded by that of the union of the groups $\pi_k E$ for all k. For example, this union is countable when $\pi_k E$ is countable for each k. This collection of E_*-acyclic CW spectra **is** a set, so we can form the direct limit as Adams suggested.

6.1C Fibrant spectra are Ω-spectra

A model structure on the category of spectra was first defined by Bousfield-Friedlander in [BF78]. In it the fibrant replacement of an arbitrary spectrum is the Ω-spectrum equivalent to it. We will discuss this in much more detail below in Chapter 7.

6.2 Bousfield localization in more general model categories.

The technique introduced by Bousfield in [Bou75] and outlined in §6.1A is quite useful and and can be used in other settings. Starting with a model category $\mathcal{M}$, we want to introduce a new model category $\mathcal{M}'$ with the same underlying category as $\mathcal{M}$. We do this by enlarging the set of weak equivalences, keeping the collection of cofibrations as they were, and modifying the collection of fibrations accordingly. Assuming this can be done, note that since $\mathcal{M}'$ has more weak equivalences than $\mathcal{M}$, it has more trivial cofibrations and hence **fewer fibrations and fewer fibrant objects.** Hence fibrant replacement in $\mathcal{M}'$ is more drastic than it is in $\mathcal{M}$, producing objects with stronger properties. The resulting Bousfield localization functor is fibrant replacement. **The term "Bousfield localization" is also used for the passage from $\mathcal{M}$ to $\mathcal{M}'$.**

As noted in Remark 4.1.7, the hardest part of showing that the new collections of weak equivalences, fibration and cofibrations constitute a model structure is the verification of the factorization axiom, **MC5** of Definition 4.1.1. It can involve delicate set theoretic arguments.

One way to enlarge the class of weak equivalences in a model category $\mathcal{M}$ is to consider a functor F from $\mathcal{M}$ to either an ordinary category $\mathcal{C}$ or another model category $\mathcal{N}$, possibly $\mathcal{M}$ itself. It must convert weak equivalences to either isomorphisms in $\mathcal{C}$ or to weak equivalences in $\mathcal{N}$. Then our new notion of a map f being a weak equivalence is that $F(f)$ is either an isomorphism in $\mathcal{C}$ or a weak equivalence in $\mathcal{N}$. One case of this will be studied in §6.2B.

Another way enlarge the class of weak equivalences in $\mathcal{M}$ is to specify a class $\mathscr{E}$ of morphisms (which could consist of a single map) that one would like to be weak equivalences in the new model structure and then enlarge it appropriately. This will be done in Definition 6.2.1 below.

6.2A Localization with respect to a collection of morphisms

We need to work in a model category that is enriched over another model category (possibly itself) so that we can speak of weak equivalences, rather than isomorphisms, of morphism objects. We will assume that the model category $\mathcal{M}$ is topological, i.e., that the morphism sets $\mathcal{M}(X, Y)$ come equipped with natural topologies. The following terminology is taken from [Hir03, Definitions 3.1.4 and 3.1.1].

Definition 6.2.1. $\mathscr{E}$-local notions. *Let $\mathscr{E}$ be a class (possibly a set) of morphisms in a topological model category $\mathcal{M}$. A* **$\mathscr{E}$-local object** *Z is a fibrant object for which the morphism*

$$f^* : \mathcal{M}(B, Z) \to \mathcal{M}(A, Z)$$

is a weak equivalence for any morphism $f : A \to B$ in $\mathscr{E}$. A morphism $g : X \to Y$ is a **$\mathscr{E}$-local equivalence** *if for every $\mathscr{E}$-local object Z the map*

$$g^* : \mathcal{M}(Y, Z) \to \mathcal{M}(X, Z)$$

is a weak equivalence. A **$\mathscr{E}$-trivial cofibration** *is a cofibration which is also a $\mathscr{E}$-local equivalence. A morphism is a* **$\mathscr{E}$-local fibration** *(or a* **$\mathscr{E}$-fibration***) if its has the right lifting property with respect to all $\mathscr{E}$-trivial cofibrations. An object Z is* **$\mathscr{E}$-fibrant** *if the map $Z \to *$ is a $\mathscr{E}$-fibration.*

If $\mathscr{E}$ consists of a single morphism f, we will speak of **f-local objects**, **f-local equivalences***, and so on. If that morphism is from the initial object to some object A, the terms* **A-local** *and* **A-null** *are sometimes used for f-local objects.*

These notions can be dualized by saying that a cofibrant object W is **$\mathscr{E}$-colocal** if the map $f_* : \mathcal{M}(W, A) \to \mathcal{M}(W, B)$ is a weak equivalence for each

f in $\mathscr{E}$, and so on. These lead to a notion of right Bousfield localization dual to left Bousfield localization as in Definition 6.2.5.

Remark 6.2.2. The nontopological case. *Hirschhorn's definitions do not require $\mathcal{M}$ to be a topological model category. The conditions on f^* and g^* above are stated in terms of the homotopy function complexes* map$(-,-)$ *of [Hir03, Chapter 17] instead the topological spaces $\mathcal{M}(-,-)$.*

We will see in Proposition 6.2.12 below that under additional hypotheses, an object is $\mathscr{E}$-local iff it is $\mathscr{E}$-fibrant.

Proposition 6.2.3. A weak equivalence is a $\mathscr{E}$-local equivalence, and so on. *Let $\mathcal{M}$ and $\mathscr{E}$ be as in Definition 6.2.1. Then*

(i) every weak equivalence is a $\mathscr{E}$-local equivalence,
(ii) every $\mathscr{E}$-local fibration is a fibration,
(iii) every $\mathscr{E}$-fibrant object is fibrant and
(iv) a morphism is a $\mathscr{E}$-local trivial fibration iff it is a trivial fibration.

Proof The first statement is part of [Hir03, Proposition 3.1.5].

For (ii), a $\mathscr{E}$-local fibration has the right lifting property with respect to every cofibration which is $\mathscr{E}$-local equivalence and hence, by (i), with respect to every trivial cofibration. Therefore it is a fibration.

For (iii), if $Y \to *$ is a $\mathscr{E}$-fibration, it is a fibration by (ii), so Y is fibrant.

For (iv), both model structures have the same set of cofibrations and hence the same set of trivial fibrations. □

Proposition 6.2.4. Expanding the class of weak equivalences. *Let $\mathscr{E}$ be a class of morphisms in a topological model category $\mathcal{M}$. Suppose there is a model category $\mathcal{M}'$ having the same underlying category and the same class of cofibrations as $\mathcal{M}$, but with $\mathscr{E}$-local equivalences as weak equivalences. Then its fibrations and fibrant objects are the $\mathscr{E}$-fibrations and $\mathscr{E}$-fibrant objects of $\mathcal{M}$ as in Definition 6.2.1.*

Definition 6.2.5. *The* **left Bousfield localization** *of a model category $\mathcal{M}$ with respect to a class of morphisms $\mathscr{E}$ is the model category $\mathcal{M}'$ (if it exists) of Proposition 6.2.4. We denote by $L_{\mathscr{E}} : \mathcal{M} \to \mathcal{M}'$ the functor underlain by the identity functor, we denote $\mathcal{M}'$ by $L_{\mathscr{E}}\mathcal{M}$, and we denote the fibrant replacement functor in $\mathcal{M}'$ by $\Upsilon_{\mathscr{E}}$.*

When $\mathscr{E} = \{f\}$ we denote these functors by L_f and Υ_f. When it is a set $\mathcal{S}$ we denote them by $L_{\mathcal{S}}$ and $\Upsilon_{\mathcal{S}}$.

The use of the symbol $L_{\mathscr{E}}$ in both the above and Definition 4.5.13 is justified by the following, which is Hirschhorn's [Hir03, Theorem 3.3.19].

Theorem 6.2.6. Left Bousfield localization as left localization. *The left Bousfield localization $L_{\mathscr{E}}\mathcal{M}$ of Definition 6.2.5 is a left localization as in*

Definition 4.5.13 where the left Quillen functor $j : \mathcal{M} \to L_{\mathscr{E}}\mathcal{M}$ is the identity functor.

Corollary 6.2.7. Detecting weak equivalences in $L_{\mathscr{E}}\mathcal{M}$. *Let $\mathcal{M}$ be a model category with a morphism class $\mathscr{E}$ such that the left Bousfield localization $L_{\mathscr{E}}\mathcal{M}$ exists. Let $\mathcal{C}$ be a bicomplete category and suppose that $F : \mathcal{M} \to \mathcal{C}$ is a functor sending weak equivalences and morphisms in $\mathscr{E}$ to isomorphisms. Then F sends weak equivalences in $L_{\mathscr{E}}\mathcal{M}$ to isomorphisms in $\mathcal{C}$.*

Proof Recall (Example 4.1.18) that $\mathcal{C}$ has a model structure in which weak equivalences are isomorphisms, and all morphisms are both fibrations and cofibrations. Its homotopy category is isomorphic to $\mathcal{C}$ itself.

It follows that F is a left (and a right) Quillen functor. By Theorem 6.2.6, the total left derived functor

$$\mathbf{L}F : \mathrm{Ho}\,(\mathcal{M}) \to \mathrm{Ho}\,(\mathcal{C}) \cong \mathcal{C}$$

factors uniquely through $\mathrm{Ho}\,(L_{\mathscr{E}}\mathcal{M})$, where the images of weak equivalences in $L_{\mathscr{E}}\mathcal{M}$ are isomorphisms. The result follows. □

As in the case of Definition 6.2.1, Hirschhorn makes these definitions for an arbitrary model category, replacing the mapping space $\mathcal{M}(X,Y)$ by its simplicial analog.

The case of left localization for $\mathcal{M} = \mathcal{T}$ (in which all objects are fibrant) where $\mathscr{E}$ consists of a single morphism f was studied extensively by Dror Farjoun in [Far96a].

Note that the definition of $\mathscr{E}$-local equivalence of Definition 6.2.1 differs slightly from that of [Bou75]. We are now requiring g to induce a weak equivalence of mapping spaces $\mathcal{M}(-,K)$ while Bousfield only requires an isomorphism of sets of homotopy classes of maps $[-,K]$, the set of path connected components of $\mathcal{M}(-,K)$. For the case $\mathcal{M} = \mathcal{T}$ and $\mathscr{E} = \{f\}$, it is shown in [Far96b, Corollary 1.3] that the two requirements are equivalent.

When $\mathscr{E}$ is a set $\mathcal{S}$ (rather than a proper class) of morphisms

$$\{f_\alpha : X_\alpha \to Y_\alpha\},$$

we could replace $\mathcal{S}$ by the single morphism

$$\textstyle\coprod_\alpha X_\alpha \xrightarrow{\coprod_\alpha f_\alpha} \coprod_\alpha Y_\alpha.$$

However it is usually not convenient to do so, because it is easier to deal with the f_α one at a time as in Remark 5.2.4. On the other hand, considering a proper class $\mathscr{E}$ of morphisms, as Hirschhorn does in [Hir03, Chapter 3], is *a priori* more general. For example, when localizing with respect to a homology theory E_*, one wants to consider the class of all E_*-equivalences.

Proposition 6.2.8. Left Bousfield localization as a left Quillen functor. *If the functor $L_{\mathscr{E}}$ of Definition 6.2.5 exists, there is a Quillen pair (Definition 4.5.1)*

$$L_{\mathscr{E}} : \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} L_{\mathscr{E}}\mathcal{M} : U,$$

where the functors $L_{\mathscr{E}}$ and U are each the identity functor on the underlying category.

Proof By assumption, the model category $L_{\mathscr{E}}\mathcal{M}$ has the same underlying category and cofibrations as $\mathcal{M}$. It follows that $L_{\mathscr{E}}\mathcal{M}$ has the same trivial fibrations as $\mathcal{M}$ since they are defined in term of the right lifting property with respect to cofibrations. Every weak equivalence in $\mathcal{M}$ is also one in $L_{\mathscr{E}}\mathcal{M}$, so every trivial cofibration in $\mathcal{M}$ is also one in $L_{\mathscr{E}}\mathcal{M}$. Similarly every fibration in $L_{\mathscr{E}}\mathcal{M}$ is also one in $\mathcal{M}$. This means the identity functor is left Quillen as functor from $\mathcal{M}$ to $L_{\mathscr{E}}\mathcal{M}$ and right Quillen as functor going the other way. The result follows. □

Remark 6.2.9. Left Bousfield localization is not a Quillen equivalence. *The adjunction of Proposition 6.2.8 is generally not a Quillen equivalence. Let $X \to Y$ be a morphism in $\mathcal{M}$ which is a $\mathscr{E}$-local equivalence but not a weak equivalence. Then the same is true of the composite*

$$QX \to X \to Y \to R_{\mathscr{E}}Y, \tag{6.2.10}$$

where Q is a cofibrant replacement functor in $\mathcal{M}$ and $R_{\mathscr{E}}$ is a fibrant replacement functor in $L_{\mathscr{E}}\mathcal{M}'$. Since the functors $L_{\mathcal{C}}$ and U of Proposition 6.2.8 are each the identity functor, the composite morphism of (6.2.10) can be regarded as a morphism in either category. Being an weak equivalence in $L_{\mathscr{E}}\mathcal{M}$ does not make it one in $\mathcal{M}$.

The following is proved by Hirschhorn as [Hir03, Theorem 3.1.6].

Theorem 6.2.11. Quillen pairs and $\mathscr{E}$-local objects and equivalences. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let*

$$F : \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{N} : U.$$

be a Quillen pair as in Definition 4.5.1.

(i) Let $\mathscr{E}$ be a class of morphisms in $\mathcal{M}$. Then the following are equivalent.

- *(a) The total left derived functor $\mathbf{L}F : \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ (Definition 4.4.7) of F takes the images in $\mathrm{Ho}\,\mathcal{M}$ of the elements of $\mathscr{E}$ into isomorphisms in $\mathrm{Ho}\,\mathcal{N}$.*
- *(b) The functor F takes every cofibrant approximation (Definition 4.1.19) to a morphism in $\mathscr{E}$ into a weak equivalence in $\mathcal{N}$.*
- *(c) The functor U takes every fibrant object of $\mathcal{N}$ into a $\mathscr{E}$-local object of $\mathcal{M}$.*

(d) *The functor F takes every $\mathscr{E}$-local equivalence between cofibrant objects into a weak equivalence in $\mathcal{N}$.*

(ii) *Dually, let $\mathscr{D}$ be a class of morphisms in $\mathcal{N}$. Then the following are equivalent.*

(a) *The total right derived functor $\mathbf{R}U : \mathrm{Ho}\,\mathcal{N} \to \mathrm{Ho}\,\mathcal{M}$ of U takes the images in $\mathrm{Ho}\,\mathcal{N}$ of the elements of $\mathscr{D}$ into isomorphisms in $\mathrm{Ho}\,\mathcal{M}$.*

(b) *The functor U takes every fibrant approximation to a morphism in $\mathscr{D}$ into a weak equivalence in $\mathcal{M}$.*

(c) *The functor F takes every cofibrant object of $\mathcal{M}$ into a $\mathscr{D}$-local object of $\mathcal{N}$.*

(d) *The functor U takes every $\mathscr{D}$-local equivalence between fibrant objects into a weak equivalence in $\mathcal{M}$.*

Proposition 6.2.12. $\mathscr{E}$-fibrant means $\mathscr{E}$-local. *Using the terminology of Definition 6.2.1, if the functor $L_{\mathscr{E}}$ exists, an object X is $\mathscr{E}$-fibrant iff it is $\mathscr{E}$-local in the original model category $\mathcal{M}$.*

Proof If X is $\mathscr{E}$-local, then it is fibrant by definition. Let $f : A \to B$ be a morphism in $\mathscr{E}$. Then the diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & *. \end{tikzcd}$$

in $\mathcal{M}_0$ (the ordinary category underlying the topological category $\mathcal{M}$) with a arbitrary leads to

$$\begin{tikzcd} h \in \mathcal{M}_0(B,X) \arrow[d, mapsto] & \mathcal{M}(B,X) \arrow[r, "p_*"] \arrow[d, "f^*"'] & \mathcal{M}(B,*) \arrow[r, equal] \arrow[d] & * \\ a \in \mathcal{M}_0(A,X) & \mathcal{M}(A,X) \arrow[r, "p_*"] & \mathcal{M}(A,*) \arrow[r, equal] & * \end{tikzcd}$$

in which the map f^* is a weak equivalence since X is $\mathscr{E}$-local. Since the two objects on the right are each $*$, f^* is also the pullback corner map. This means the lifting h exists, so p is an $\mathscr{E}$-fibration and X is $\mathscr{E}$-fibrant.

Conversely, suppose that X is $\mathscr{E}$-fibrant, which means that p is an $\mathscr{E}$-fibration. Then it is also a fibration, so X is fibrant. We need to show that it is $\mathscr{E}$-local. If $L_{\mathscr{E}}$ exists, Proposition 6.2.8 says there is a Quillen pair as in Theorem 6.2.11 in which $\mathcal{N} = L_{\mathscr{E}}\mathcal{M}$ and both functors are the identity. Then the image under U of a fibrant object in $L_{\mathscr{E}}\mathcal{M}$, i.e., an $\mathscr{E}$-fibrant object, is $\mathscr{E}$-local by Theorem 6.2.11(i)(c). □

Example 6.2.13. n-connected maps as weak equivalences. *Consider the category $\mathcal{T}$ of pointed topological spaces with its usual model structure.*

For a fixed integer $n \geqslant 0$, we can expand the collection of weak equivalences to that of maps $f : X \to Y$ inducing an isomorphism in π_i for $i \leqslant n$. This means that any n-connected space is weaky equivalent to a point. While there is no homology theory E_* for which such maps are the E_*-equivalences, we can say what fibrant replacement means. The local objects are those spaces W for which $\pi_i W = 0$ for $i > n$. There is a functorial way to kill the homotopy groups of a space X above dimension n by attaching cells of dimensions above $n+1$, leading to the nth Postnikov section $P^n X$. The resulting map $X \to P^n X$ is fibrant replacement in the new model category structure. See Example 6.2.14 below for another approach to $P^n X$.

Example 6.2.14. Postnikov sections as localizations. *Let f be $S^{n+1} \to *$ or $* \to S^{n+1}$. In either case X is f-local iff $\Omega^{n+1} X$ is contractible, which is equivalent to the requirement that $\pi_k X = 0$ for $k > n$. It follows that $L_f X \simeq P^n X$, the nth Postnikov section of X, meaning the space obtained from X by killing all homotopy groups above dimension n. It was of course originally constructed without reference to model categories by attaching cells to X. We will use this example in constructing the slice filtration of Chapter 11. It was discussed previously in Example 6.2.13 and will appear again in Example 6.3.11.*

6.2B Localization via an idempotent functor

An alternative to localizing with respect to a collection of morphisms $\mathscr{E}$ (which are to be weak equivalences in the new model structure) is to localize with respect to an idempotent coaugmented functor Υ which is to be fibrant replacement.

The following definition is due to Bousfield [Bou01].

Definition 6.2.15. Υ-structures on a model category. *Let $\mathcal{M}$ be a model category with a coaugmented endofunctor $\Upsilon : \mathcal{M} \to \mathcal{M}$ as in Definition 2.2.8 with coaugmentation η. We say Υ is* **homotopy idempotent** *if the maps $\Upsilon \eta_X$ and $\eta_{\Upsilon X}$ from ΥX to $\Upsilon^2 X$ are weak equivalences for each object X in $\mathcal{M}$. We say that an object X is* **Υ-local** *if η_X is a weak equivalence. We say that a morphism $g : X \to Y$ in $\mathcal{M}$ is*

- *a* **Υ-equivalence** *if Υg is a weak equivalence,*
- *a* **Υ-cofibration** *if it is a cofibration and*
- *a* **Υ-fibration** *if it has the right lifting property with repect to every Υ-trivial cofibration, meaning every cofibration that is a Υ-equivalence.*

Note how Definition 6.2.15 compares with Definition 6.2.1. In the latter a map $g : X \to Y$ is an $\mathscr{E}$-local equivalence if the induced map $g^* : \mathcal{M}(Y, Z) \to \mathcal{M}(X, Z)$ is a weak equivalence for each $\mathscr{E}$-local object Z. In the former the condition is that Υg is a weak equivalence. This raises two questions:

- Given such a functor Υ, can we find a morphism class $\mathscr{E}$ such that g is a Υ-equivalence iff it is an $\mathscr{E}$-equivalence? One candidate is the class of all Υ-equivalences. It would be better to have a minimal morphism set $\mathcal{S}$ with a similar relation to Υ.
- Conversely, given a morphism class $\mathscr{E}$, can we find an idempotent functor Υ such that g is a $\mathscr{E}$-equivalence iff is a Υ-equivalence? One candidate might be a fibrant replacement functor for $L_{\mathscr{E}}\mathcal{M}$. It would be better to have a more explicitly described functor.

The following will be used in the proof of Theorem 7.3.23 below.

Theorem 6.2.16. The relation between a morphism class and a corresponding homotopy idempotent functor. *Let $\mathcal{M}$ be a pointed topological Quillen ring as in Definition 5.5.9, and let $\mathcal{N}$ be a Quillen $\mathcal{M}$-module as in Definition 5.5.17 with a class of morphisms $\mathscr{E}$. Let $\Upsilon : \mathcal{N} \to \mathcal{N}$ be a coaugmented functor, with coaugmentation η as in Definition 2.2.8, having the following properties:*

- *It is homotopy idempotent as in Definition 6.2.15.*
- *Every $\mathscr{E}$-local object is Υ-local.*

Then

(i) If $g : X \to Y$ in $\mathcal{N}$ is a Υ-equivalence as in Definition 6.2.15, then it is a $\mathscr{E}$-local equivalence as in Definition 6.2.1.
(ii) The map $\eta_X : X \to \Upsilon X$ is an $\mathscr{E}$-local equivalence for all X.
(iii) For a fibrant approximation functor R in $\mathcal{N}$ as in Definition 4.1.25 with coaugmentation η', for all X in $\mathcal{N}$ the composite morphism

$$X \xrightarrow{\eta'_X} RX \xrightarrow{\eta_{RX}} \Upsilon RX$$

is an $\mathscr{E}$-local equivalence to an $\mathscr{E}$-local object. In particular, if every object in $\mathcal{N}$ is fibrant, then the map η_X has this property.
(iv) A map $g : X \to Y$ is a $\mathscr{E}$-local equivalence iff Rg is a Υ-equivalence.

Proof (i) If $g : X \to Y$ is a Υ-equivalence, then the induced map

$$g^* : \mathcal{N}(Y, Z) \to \mathcal{N}(X, Z)$$

is a weak equivalence for each Υ-object X, and hence for each $\mathscr{E}$-local object Z by 6.2.16. This makes g an $\mathscr{E}$-local equivalence by definition.

(ii) Since Υ is homotopy idempotent by hypothesis, the map $\Upsilon\eta_X$ is a weak equivalence, which makes η_X an $\mathscr{E}$-local equivalence by (i).

(iii) The map $\eta'_X : X \to RX$ (the canonical map from an object to its functorial fibrant approximation) is a trivial cofibration and hence a weak equivalence, so its composite with the $\mathscr{E}$-local equivalence $\eta_{RX} : RX \to \Upsilon RX$ is again an $\mathscr{E}$-local equivalence.

(iv) If g is an $\mathscr{E}$-local equivalence, then by (iii), the same is true of ΥRg. Since the latter is a map of $\mathscr{E}$-local objects, it is a weak equivalence.

For the converse, it follows from (i) that if ΥRg is a weak equivalence, then Rg is an $\mathscr{E}$-local equivalence. Since the map $X \to RX$ is weak equivalence, g is an $\mathscr{E}$-local equivalence. $\square$

6.2C The relation between localization, confinement and enlargement

Here we will continue the discussion of §5.4D. Recall the situation of Theorem 5.4.26. We have four cofibrantly generated model categories $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{M}$, and $\mathcal{M}'$. They fit into the diagram of (5.4.27), namely

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left=1.5, "\mathcal{M} \coprod F"] \arrow[r, phantom, "\perp"] \arrow[d, shift right=1.5, leftarrow, "A \times A'"'] \arrow[d, phantom, "\dashv"] \arrow[d, shift left=1.5, "B \times B'"] & \mathcal{M} \arrow[l, shift left=1.5, "\mathcal{M} \times U"] \arrow[d, shift right=1.5, leftarrow, "A"'] \arrow[d, phantom, "\dashv"] \arrow[d, shift left=1.5, "B"] \\
\mathcal{L} \times \mathcal{L}' \arrow[r, shift left=1.5, "\mathcal{L} \coprod E"] \arrow[r, phantom, "\perp"] & \mathcal{L}, \arrow[l, shift left=1.5, "\mathcal{L} \times V"]
\end{tikzcd}$$

where the adjunctions are not Quillen pairs. As explained in Corollary 5.4.29, this leads to four different model structures on $\mathcal{M}$ and the following diagram.

$$\begin{tikzcd}
\mathcal{M}_{enla} \arrow[r, shift left=1.5] \arrow[r, phantom, "\top"] \arrow[d, shift right=1.5, leftarrow] \arrow[d, phantom, "\dashv"] \arrow[d, shift left=1.5] & \mathcal{M} \arrow[l, shift left=1.5] \arrow[d, shift right=1.5, leftarrow] \arrow[d, phantom, "\dashv"] \arrow[d, shift left=1.5] \\
\mathcal{M}_{enco} \arrow[r, shift left=1.5] \arrow[r, phantom, "\top"] & \mathcal{M}_{conf} \arrow[l, shift left=1.5]
\end{tikzcd}$$

Here each underlying category is $\mathcal{M}$, each functor is the identity and each adjunction is a Quillen adjunction.

Now suppose we have a morphism class $\mathscr{E}$ in $\mathcal{M}$, and that each of the four model structures above can be left Bousfield localized with respect to $\mathscr{E}$. (In the next section we will discuss when such localization is possible.) Then we get four more model structures on $\mathcal{M}$ as in the following diagram.

$$\begin{tikzcd}
L_{\mathscr{E}}\mathcal{M}_{enla} \arrow[rrr, shift left] \arrow[rrr, phantom, "\top"] \arrow[dr, shift left] \arrow[dr, phantom, "\top"] \arrow[ddd, shift left] \arrow[ddd, phantom, "\dashv"] & & & L_{\mathscr{E}}\mathcal{M} \arrow[lll, shift left] \arrow[dl, shift left] \arrow[dl, phantom, "\perp"] \arrow[ddd, shift left] \arrow[ddd, phantom, "\dashv"] \\
& \mathcal{M}_{enla} \arrow[r, shift left] \arrow[r, phantom, "\top"] \arrow[ul, shift left] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \mathcal{M} \arrow[l, shift left] \arrow[ur, shift left] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \\
& \mathcal{M}_{enco} \arrow[r, shift left] \arrow[r, phantom, "\top"] \arrow[u, shift left] \arrow[dl, shift left] \arrow[dl, phantom, "\top"] & \mathcal{M}_{conf} \arrow[l, shift left] \arrow[u, shift left] \arrow[dr, shift left] \arrow[dr, phantom, "\perp"] & \\
L_{\mathscr{E}}\mathcal{M}_{enco} \arrow[rrr, shift left] \arrow[rrr, phantom, "\top"] \arrow[ur, shift left] \arrow[uuu, shift left] & & & L_{\mathscr{E}}\mathcal{M}_{conf} \arrow[lll, shift left] \arrow[ul, shift left] \arrow[uuu, shift left]
\end{tikzcd} \tag{6.2.17}$$

Again each arrow denotes the identity functor and each adjunction is Quillen. Each diagonal adjunction is a case of the one in Proposition 6.2.8.

We will see an instance of this diagram for the category of orthogonal G-spectra in Figure 7.1 below. In that case cofibrant generating sets for the eight model structures are given in Theorem 9.2.13.

6.2D Localizations and Quillen rings

Given a Quillen ring $(\mathcal{M}, \otimes, \mathbf{1})$ as in Definition 5.5.9 and a morphism class $\mathscr{E}$ in $\mathcal{M}$, what can we say about the monoidal structure on $L_{\mathscr{E}}\mathcal{M}$? What conditions on $\mathcal{M}$ are needed to insure that it satisfies the various axioms of §5.5? These questions are addressed by David White in [Whi].

It is convenient to use the following, which is [Whi, Definition 4.4].

Definition 6.2.18. *For a Quillen ring $\mathcal{M}$ and morphism class $\mathscr{E}$, the functor $L_{\mathscr{E}}$ is a* **monoidal Bousfield localization** *if $L_{\mathscr{E}}\mathcal{M}$ satisfies the pushout product and unit axioms of Definition 5.5.9, and its cofibrant objects are flat as in Definition 5.1.20.*

He proves the following as [Whi, Theorems B].

Theorem 6.2.19. *Let M be a cofibrantly generated Quillen ring in which cofibrant objects are flat. Then $L_{\mathscr{E}}$ is a monoidal Bousfield localization if and only if every map of the form $f \otimes K$, where f is in $\mathscr{E}$ and K is cofibrant, is an $\mathscr{E}$-local equivalence. If the domains of the generating cofibrations $\mathcal{I}$ are cofibrant, it suffices to consider $f \otimes K$ for K in the set of domains and codomains of the morphisms in $\mathcal{I}$.*

6.3 When is left Bousfield localization possible?

For which model categories $\mathcal{M}$ and morphism sets $\mathcal{S}$ does the notion of $\mathcal{S}$-local equivalence lead to a new model structure and hence a Bousfield localization (fibrant replacement) functor $U_{\mathcal{S}}$? Hirschhorn [Hir03, Theorem 4.1.1] showed this can be done whenever $\mathcal{M}$ is left proper as in Definition 5.3.1 and **cellular** as in Definition 6.3.1 below. Barwick in [Bar10, Theorem 4.7] and Lurie in [Lur09, A.3.7.3] give proofs that this can be done whenever $\mathcal{M}$ is left proper and **combinatorial** as in Definition 4.8.11, the result being originally due to Jeff Smith. In both cases the new model category, denoted by $L_f\mathcal{M}$, which has the same underlying category as $\mathcal{M}$, is again left proper and cellular/combinatorial.

6.3A Cellular, Hirschhorn, combinatorial and accessible model categories

The terms **cellular** [Hir03, Definition 12.1.1] and **combinatorial** [Bar10, Definition 1.21] both refer to a cofibrantly generated model category $\mathcal{M}$ (§5.2) with generating sets $\mathcal{I}$ of cofibrations and $\mathcal{J}$ of trivial cofibrations. Combinatorial and accessible model categories were defined in Definition 4.8.11 and Definition 4.8.12.

Definition 6.3.1. *A model category is* **cellular** *if it is a cofibrantly generated with generating sets $\mathcal{I}$ and $\mathcal{J}$ in which*

(i) the domain and codomain of each morphism in $\mathcal{I}$ is compact relative to $\mathcal{I}$ as in Definition 5.2.8,
(ii) the domain of each morphism in $\mathcal{J}$ is small relative to $\mathcal{I}$ as in Definition 4.8.18 and
(iii) the cofibrations are effective monomorphisms as in Definition 2.1.10.

The categories $\mathcal{T}op$, $\mathcal{T}$ and $\mathcal{S}et_{\Delta}$ with their standard model structures are each cellular.

Definition 6.3.2. *A* **Hirschhorn category** *is a cofibrantly generated model category that is left proper (Definition 5.3.1) and cellular (Definition 6.3.1).*

We will need the following to define the stable model structure in Definition 7.3.6 below.

Proposition 6.3.3. Functors into a Hirschhorn category. *Let $\mathcal{M}$ be a Hirschhorn category. Then for a small category J, the functor category $\mathcal{M}^J$ with its projective model structure (as in Definition 5.4.2) is also a Hirschhorn category.*

Proof $\mathcal{M}^J$ is cofibrantly generated by Theorem 5.6.26.

To see that it is left proper, suppose we have a diagram such as (5.3.2) in

$\mathcal{M}^J$. Then for each object j in J we have a similar diagram in $\mathcal{M}$. The map f_j is a cofibration since f is one, and h_j is a weak equivalence because h is one. The left properness of $\mathcal{M}$ means that k_j is a weak equivalence, so k is a weak equivalence in $\mathcal{M}^J$.

The conditions required for $\mathcal{M}^J$ to be cellular can be verified objectwise. □

The following is proved by Hirschhorn as [Hir03, Theorem 4.1.1] and restated as [Hov01b, Theorem 2.2].

Theorem 6.3.4. Hirschhorn categories are localizable. *Let $\mathcal{M}$ be a Hirschhorn category (Definition 6.3.2) with a set (not a proper class) of morphisms $\mathcal{S}$. Then the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ of Definition 6.2.1 exists and is again a Hirschhorn category.*

Remark 6.3.5. The cofibrant generating sets for $L_{\mathcal{S}}\mathcal{M}$. *Since the two model categories $\mathcal{M}$ and $L_{\mathcal{S}}\mathcal{M}$ have the same cofibrations, $\mathcal{I}$ serves as a generating set for the latter as well as for the former. The original generating set $\mathcal{J}$ of trivial cofibrations in $\mathcal{M}$ needs to be enlarged since $L_{\mathcal{S}}\mathcal{M}$ has more weak equivalences. Hirschhorn defines a generating set $\mathcal{J}_{\mathcal{S}}$ to be a set of representatives of the isomorphism classes of inclusions of subcomplexes that are $\mathcal{S}$-local equivalences of $\mathcal{I}$-cell complexes (as in Definition 4.8.18) of suitable cardinality. The cardinality condition ensures that it is a set rather than a proper class. This is enough to prove the theorem, but in practice one wants a more economical set with a more explicit description. In Theorem 7.3.36 and Theorem 7.4.52 below we will go to some trouble to give such descriptions in cases of interest.*

The following is proved by Hovey as [Hov01b, Proposition 2.3].

Proposition 6.3.6. Localization of Quillen equivalences. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Hirschhorn categories with morphism sets $\mathcal{S}$ and $\mathcal{S}'$ respectively and a Quillen equivalence (Definition 4.5.14)*

$$F : \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{M}' : U.$$

Suppose further that $F(Qf)$ (where Q denotes functorial cofibrant replacment in $\mathcal{M}$) is an $\mathcal{S}'$-local equivalence for all $f \in S$. Then F induces a Quillen equivalence (in the sense of Corollary 4.5.21) $L_{\mathcal{S}}\mathcal{M} \to L_{\mathcal{S}'}\mathcal{M}'$ iff for each $\mathcal{S}$-local object X in $\mathcal{M}$ there is an $\mathcal{S}'$-local object Y in $\mathcal{M}'$ with X weakly equivalent to UY in $\mathcal{M}$.

For the results of Barwick [Bar10, Theorem 4.7] and Lurie [Lur09, A.3.7.3] we need Definition 4.8.11 and Definition 4.8.12. Presentable and accessible ∞-categories are the subject of [Lur09, Chapter 5]. The former can be localized in the sense of Bousfield; this is studied in [Lur09, §5.5].

The following recognition result is due to Jeff Smith, who first announced it at the Barcelona Conference on Algebraic Topology of 1998. It is proved as

[Bek00, Theorem 1.7 and Propositions 1.15 and 1.19] and [Bar10, Proposition 2.2]. It states that under certain conditions, a combinatorial model category can be defined by specifying its weak equivalences and generating cofibrations, without having to specify a set of generating trivial cofibrations.

Theorem 6.3.7. Smith's recognition principle. *Suppose $\mathcal{C}$ is a complete locally presentable category with an accessible subcategory $\mathscr{W}$ and set of morphisms $\mathcal{I}$ such that*

(i) $\mathscr{W}$ satisfies the two-out-of-three axiom.
(ii) The set $\mathcal{I}^{\boxslash}$ is contained in $\mathscr{W}$.
(iii) The intersection $(W \cap^{\boxslash} (\mathcal{I}^{\boxslash}))$ is closed under pushouts and transfinite composition.

Then $\mathcal{C}$ is a combinatorial model category with weak equivalences $\mathscr{W}$, cofibrations ${}^{\boxslash}(\mathcal{I}^{\boxslash})$, and fibrations $(\mathscr{W} \cap^{\boxslash} (\mathcal{I}^{\boxslash}))^{\boxslash}$.

6.3B $\mathcal{S}$-horns and related notions

The following is a modification of [Hir03, Definitions 4.2.1 and 4.2.2].

Definition 6.3.8. $\mathcal{S}$-horns. *Let $\mathcal{M}$ be a topological model category as in Definition 5.6.3 with a set of cofibrations $\mathcal{S}$. The* **full set of $\mathcal{S}$-horns** *is the set*

$$\begin{aligned}\Lambda(\mathcal{S}) &= \{f \,\square\, i_n \colon f \in \mathcal{S},\ n \geqslant 0\} \\ &= \left\{A \times D^n \cup_{A \times S^{n-1}} B \times S^{n-1} \to B \times D^n \colon (f : A \to B) \in \mathcal{S},\ n \geqslant 0\right\},\end{aligned}$$

where $i_n : S^{n-1} \to D^n$ as in Example 5.2.9, and $\square$ denotes the pushout corner as in Definition 2.6.12. In the pointed case we have

$$\begin{aligned}\Lambda(\mathcal{S}) &= \{f \,\square\, i_{n+} \colon f \in \mathcal{S},\ n \geqslant 0\} \\ &= \left\{A \wedge D^n_+ \cup_{A \wedge S^{n-1}_+} B \wedge S^{n-1}_+ \to B \wedge D^n_+ \colon (f : A \to B) \in \mathcal{S},\ n \geqslant 0\right\}.\end{aligned}$$

(Compare with Definition 2.6.15.)

If $\mathcal{M}$ is also cofibrantly generated with generating trivial cofibrations $\mathcal{J}$, the **augmented set of $\mathcal{S}$-horns** *is the set*

$$\overline{\Lambda(\mathcal{S})} = \mathcal{J} \cup \Lambda(\mathcal{S}).$$

Similar morphism sets will appear below as generating sets of trivial cofibrations in categories of spectra in the corner map theorems, Theorem 7.3.36, Theorem 7.4.52 and its special case Theorem 9.2.11.

Proposition 6.3.9. $\mathcal{S}$-horns, $\mathcal{S}$-local equivalences and $\mathcal{S}$-fibrant objects. *Let $\mathcal{M}$ be a topological Hirschhorn category (Definition 6.3.2) with a cofibration set $\mathcal{S}$. Then each map in $\overline{\Lambda(\mathcal{S})}$ as in Definition 6.3.8 is an $\mathcal{S}$-local* *equivalence as in Definition 6.2.1. An object X in $\mathcal{M}$ is $\mathcal{S}$-fibrant (Definition 6.2.1) iff the map $X \to *$ is $\overline{\Lambda(\mathcal{S})}$-injective as in Definition 4.1.10.*

Proof The assertion that each map in $\overline{\Lambda(\mathcal{S})}$ as in Definition 6.3.8 is an $\mathcal{S}$-local equivalence is proved by Hirschhorn as [Hir03, Proposition 4.2.3], and he proves in [Hir03, Proposition 4.2.4] that X is $\mathcal{S}$-local (Definition 6.2.1) iff the map $X \to *$ is $\overline{\Lambda(\mathcal{S})}$-injective. If $X \to *$ is $\overline{\Lambda(\mathcal{S})}$-injective it is also $\mathcal{J}$-injective and hence a fibration in $\mathcal{M}$. This means that in addition to being $\mathcal{S}$-local, X is fibrant and hence $\mathcal{S}$-fibrant by Proposition 6.2.12. □

The Bousfield localization of a cofibrantly generated model category is the subject of [Hir03, Chapter 3]. Given such a category $\mathcal{M}$ with generating sets $\mathcal{I}$ and $\mathcal{J}$, its localization at a map f has generating sets $\mathcal{I}$ and some superset $\mathcal{J}_f$ of $\mathcal{J}$. There have to be more trivial cofibrations in $L_f\mathcal{M}$ because there are more weak equivalences than in $\mathcal{M}$. While we know that such a $\mathcal{J}_f$ exists, we know of no general explicit description of it.

Suppose $\mathcal{M} = \mathcal{T}op$ (or $\mathcal{T}$) with generating sets $\mathcal{I}$ and $\mathcal{J}$ ($\mathcal{I}_+$ and $\mathcal{J}_+$) as in Example 5.2.9, and $\mathcal{S}$ consists of a single cofibration f between cofibrant objects, i.e., CW complexes. We are interested in the left localization (as in Definition 6.2.1) $L_f\mathcal{T}op$ ($L_f\mathcal{T}$) as a cofibrantly generated model category. Since its cofibrations are the same as those of $\mathcal{T}op$ ($\mathcal{T}$), they are generated by $\mathcal{I}$ ($\mathcal{I}_+$). One might think that its trivial cofibrations are generated by $\overline{\Lambda(f)}$ as in Definition 6.3.8, but we will see in Example 6.3.11 below that this is not always the case.

The following is proved by Hirschhorn as [Hir03, Propositions 1.4.4–1.4.7].

Proposition 6.3.10. Properties of $\overline{\Lambda(f)}$. *Let $f : A \to B$ be cofibration in $\mathcal{T}op$ ($\mathcal{T}$) with A a (pointed) CW complex, and let $\overline{\Lambda(f)}$ be as in Definition 6.3.8. Then*

(i) A map $p : X \to Y$ is $\overline{\Lambda(f)}$-injective iff it is a fibration having the homotopy right lifting property (Definition 5.6.3 (i)) with respect to f.
(ii) Every relative $\overline{\Lambda(f)}$-complex as in Definition 4.8.18 and every trivial cofibration is a $\overline{\Lambda(f)}$-cofibration.
(iii) A space X is $\overline{\Lambda(f)}$-injective iff it is f-local as in Definition 6.2.1.

The following can also be found in [Hir03, Example 2.1.6].

Example 6.3.11. Bousfield's counterexample. *Let $f : S^m \to D^{m+1}$ be the inclusion i_{m+1} of the boundary, which is a cofibration between cofibrant objects in $\mathcal{T}op$. This is equivalent (up to change of superscript) to one of the maps in Example 6.2.14, so the resulting functor L_f is P^{m-1}, the $(m-1)$th Postnikov section. We also know that*

$$i_n \,\square\, f = i_n \,\square\, i_{m+1} = i_{m+n+1},$$

so

$$\overline{\Lambda(f)} = \mathcal{J} \cup (\mathcal{I} \,\Box\, f) = \mathcal{J} \cup \{i_{m+n+1} : n \geqslant 0\}\,.$$

Then the map $p : PK(\mathbf{Z}, m) \to K(\mathbf{Z}, m)$ (where $PK(\mathbf{Z}, m)$ is the space of pointed paths in $K(\mathbf{Z}, m)$) has the right lifting properties needed to be $\overline{\Lambda(f)}$-injective. The cofibration $i : * \to S^m$ does not have the left lifting property with respect to p, so it is not a $\overline{\Lambda(f)}$-cofibration. On the other hand it is an f-local equivalence since $P^{m-1}S^m$ is contractible. Therefore it is a trivial cofibration in $L_f\mathcal{T}op$. This means that $\overline{\Lambda(f)}$ is **not** *a generating set for such maps.*

6.3C Localizing subcategories of a topological model category.

In this subsection we describe a type of Bousfield localization in a model category $\mathcal{M}$ based on a subcategory with certain properties similar to those of the subcategory of spaces with given connectivity; see Example 6.2.14. We will use the same method in Chapter 11 to define the all important slice filtration on the category of G-spectra.

Let $\mathcal{M}$ be a topological model category. The following properties of a full subcategory τ of $\mathcal{M}$ imitate those of n-connected spaces. Those of its complement $\tau^\perp$ imitate those of the subcategory of spaces having trivial homotopy groups above a given dimension.

Definition 6.3.12. Localizing subcategories.
A subcategory τ of $\mathcal{M}$ is **localizing** *if*

(i) Any object weakly equivalent to one in τ is also in τ.
(ii) If $W \to X \to Y$ is a cofiber sequence (Definition 4.7.6) with W in τ, then X is in τ iff Y is in τ.
(iii) Any coproduct (finite or infinite) of objects in τ is in τ.

The **complement** *$\tau^\perp$ of τ is the subcategory of objects Y such that the space $\mathcal{M}(X, Y)$ is contractible for all X in τ.*

Remark 6.3.13. *These conditions imply that τ is closed under retracts and filtered homotopy colimits. If $W = A \vee B$ is in τ, we can show that A is by the following variant of the Eilenberg swindle. Let $W_i = A_i \vee B_i$ with $A_i = A$ and $B_i = B$ for $i \geqslant 0$. Then consider the map*

$$\bigvee_{i \geqslant 0} W_i \to \bigvee_{i \geqslant 0} W_i$$

then maps A_i isomorphically to A_{i+1} and B_i isomorphically to B_i. Since τ is closed under arbitrary wedges, it contains the source and target of this map. Therefore its cofiber, which is $A_0 = A$ is also in τ.

The second condition says that τ is closed under cofibers and extensions. It *does* **not** *say that W is in τ if X and Y are, i.e., in the stable case the fiber of a map between two objects in τ need not be in τ.*

Definition 6.3.14. *A localizing subcategory τ as in Definition 6.3.12 is* **generated by a set of objects** $T = \{T_\alpha\}$ *if it is the smallest subcategory of $\mathcal{M}$ containing the objects of T and closed under weak equivalence, cofibers, extensions and arbitrary wedges.*

Each of the localizing categories we shall consider below are generated by a set T as above.

Remark 6.3.15. *The set T in Definition 6.3.14 could be replaced by the single object*

$$\bigvee_\alpha T_\alpha.$$

since τ is closed under retracts. We will denote this object abusively by T.

Example 6.3.16. *The category of n-connected spaces in $\mathcal{T}$ is generated by the object S^{n+1}.*

As in Example 6.2.14 we can ask for a functor $P^\tau : \mathcal{M} \to \tau^\perp$, the analog of nth Postnikov section P^n, and define $P_\tau X$ to be the fiber of the map $X \to P^\tau X$, the analog of the n-connected cover P_{n+1}. When $\mathcal{M}$ is left proper and cellular, we have the machinery of Bousfield localization available by Hirschhorn's theorem [Hir03, Theorem 4.1.1].

Theorem 6.3.17. The functors P^τ and P_τ. *Let $\mathcal{M}$ be a topological model category which is Hirschhorn as in Definition 6.3.2. Let τ be a localizing subcategory of $\mathcal{M}$ (Definition 6.3.12) generated by a cofibrant object T (Definition 6.3.14), and let $f : T \to *$ be the unique map. Then the Dror Farjoun localization functor L_f of §6.2, which we denote by P^τ, is the left adjoint of the inclusion functor $\tau^\perp \to \mathcal{M}$ (see Definition 6.3.12). We will denote the fiber of the map $X \to P^\tau X$ by $P_\tau X$.*

PART TWO

SETTING UP EQUIVARIANT STABLE HOMOTOPY THEORY

7

Spectra and stable homotopy theory

First I have to explain the meaning of the word "stable" in algebraic topology. We say that some phenomenon is **stable**, if it can occur in any dimension, or in any sufficiently large dimension, and if it occurs in essentially the same way independent of dimension, provided perhaps that the dimension is sufficiently large…

The construction of our category is in several steps. In particular, we will distinguish between "functions", "maps" and "morphisms"…

J. Frank Adams, [Ada74b, pages 123 and 140]

This chapter is about a model category theoretic framework for the passage from unstable to stable homotopy theory. We have come along way since the work of Adams quoted above.

7.0A The original definition of spectra

Originally (in [Lim59], [Spa59], [Whi62], [Ada74b] and [BF78]) a **spectrum** X was defined to be a sequence of pointed spaces or simplicial sets X_n for $n \geqslant 0$, along with **structure maps**

$$\epsilon_n^X : \Sigma X_n \to X_{n+1}. \tag{7.0.1}$$

A **map of spectra** $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ that play nicely with the structure maps. One knows that ϵ_n^X is adjoint to a **costructure map**

$$\eta_n^X : X_n \to \Omega X_{n+1}. \tag{7.0.2}$$

Experience has shown that spectra for which this map is a weak equivalence for all n are convenient to work with. They are known as **Ω-spectra**. Some authors, starting with Gaunce Lewis, Peter May, Mark Steinberger (1950–2018), and Jim McClure [LMSM86], refer to spectra defined without requiring η_n^X

to be a weak equivalence as " prespectra", reserving the term "spectrum" for what we are calling an Ω-spectrum.

There are two notions of weak equivalence one might consider here. A map $f : X \to Y$ is a **strict equivalence** if each f_n is a weak equivalence. This means that $\pi_*(f_n)$ is an isomorphism for each n. It is a **stable equivalence** if it induces an isomorphism of **stable homotopy groups** defined by

$$\pi_k X = \operatorname*{colim}_n \pi_{n+k} X_n. \tag{7.0.3}$$

This condition is far looser and can be met even if none of the f_n is a weak equivalence. It turns out that for an Ω-spectrum X,

$$\pi_k X \cong \pi_{n+k} X_n \qquad \text{for all } n.$$

The notion of spectrum is generalized as follows in Definition 7.1.1. Replace $\mathcal{T}$ or $\mathcal{S}et_{\mathbf{\Delta}*}$ by a pointed model category $\mathcal{M}$, and replace Σ with a left Quillen endofunctor T with right adjoint Ω_T. The resulting category is denoted by $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$, in which an object X is a sequence of objects X_n in $\mathcal{M}$ equipped with structure maps $\epsilon_n^X : TX_n \to X_{n+1}$. We call X a **Hovey spectrum** since the category in this level of generality was first studied by Hovey in [Hov01b].

The special case of this when the left Quillen functor T (which is suspension, $X \mapsto S^1 \wedge X$ in the original case) is given by $X \mapsto K \wedge X$ for a fixed cofibrant object K is called a **presymmetric spectrum** in Definition 7.1.13. (We know of no interesting example where T does not have this form.) These objects were studied (without being called presymmetric spectra) by Bjørn Dundas, Oliver Röndigs, and Paul Østvær in [DRØ03]. Such a spectrum is the same thing as an $\mathcal{M}$-valued functor on a certain **indexing category** $\mathscr{J}_K^{\mathbf{N}}$ enriched over $\mathcal{M}$; see Theorem 7.2.32 and Definition 7.2.4. Functors on a larger category, that of all finite simplicial sets, were studied earlier by Manos Lydakis in [Lyd98]. In each case, we are dealing with enriched functors, so we can use the enriched category theory of Chapter 3 to study them. In particular this applies to spectra as originally defined, which we refer to as **the original case**.

The indexing category $\mathscr{J}_{S^1}^{\mathbf{N}}$ (see Definition 7.2.4), which we abbreviate here by $\mathscr{J}^{\mathbf{N}}$, is monoidal (as in Definition 2.6.1) under addition, but surprisingly this monoidal structure is **not** symmetric. (How could addition fail to be commutative?) This lack of symmetry will be explained in Remark 7.2.11. It means that the category of $\mathcal{M}$-valued functors on it, the category of presymmetric spectra, does not have a symmetric monoidal structure (or even one without symmetry) implied by functoriality. **In hindsight, this is the reason for the longstanding difficulty in defining the smash product of spectra.** To get a feel for how hard it was, try reading the 32 pages of [Ada74b, III.4].

Had $\mathscr{J}^{\mathbf{N}}$ been symmetric monoidal, we could have used the Day Convolution Theorem 3.3.5 to show that the functor category $[\mathscr{J}^{\mathbf{N}}, \mathcal{T}]$, that is the original category of spectra, has a closed symmetric monoidal structure. In other words, it would have a smash product with all of the nice features one could hope for, not just up to homotopy, but on the nose! Its definition would be categorical rather than homotopy theoretic. **The failure of spectra to have a nice smash product was a major headache for many years.**

7.0B New ways to define spectra

Roughly speaking, $\mathscr{J}^{\mathbf{N}}$ does not have enough morphisms to be symmetric monoidal. One way to fix this is to replace it by a category $\mathscr{J}^{\mathbf{\Sigma}}$ (see Definition 7.2.4) having the same objects (natural numbers) but bigger morphism spaces. For $m \leqslant n$, $\mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m}, \mathbf{n})$ is not just one copy of S^{n-m}, but a wedge of them indexed by the set of one to one maps from the m-element set $\mathbf{m}$ to the n-element set $\mathbf{n}$. The number of such maps is $n!/(n-m)!$. The composition map

$$j_{m,n,p} : \mathscr{J}^{\mathbf{\Sigma}}(\mathbf{n}, \mathbf{p}) \wedge \mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m}, \mathbf{n}) \to \mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m}, \mathbf{p}) \tag{7.0.4}$$

is determined by the composition of such one to one maps. In particular $\mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m}, \mathbf{m}) \cong \Sigma_{m+}$, the symmetric group on m letters with a disjoint base point, also known as the wedge of $m!$ copies of S^0. **The category $\mathscr{J}^{\mathbf{\Sigma}}$ is symmetric monoidal.**

A $\mathcal{T}$-valued functor on $\mathscr{J}^{\mathbf{\Sigma}}$ is called a **symmetric spectrum**, and we denote the category of such spectra by $\mathcal{S}p^{\mathbf{\Sigma}}$. These were first studied by Mark Hovey, Brooke Shipley and Jeff Smith in [HSS00], with $\mathcal{T}$ replaced by the category of pointed simplicial sets. The mth component X_m of a symmetric spectrum X comes equipped with an action of Σ_m. The hypotheses of the Day Convolution Theorem 3.3.5 are met by $\mathcal{S}p^{\mathbf{\Sigma}}$, so **it is closed symmetric monoidal**. It has a smash product with all the nice properties one could hope for! We will define other categories of spectra with similar properties in Definition 7.2.33 below, where we call them .

This construction of a closed symmetric monoidal category of spectra in [HSS00] was **not the first**. That distinction belongs to the category of S-modules defined a few years earlier to Tony Elmendorf, Igor Kriz, Mike Mandell and Peter May in [EKMM97]. Their construction is more complicated and we will not use it here.

Remark 7.0.5. The difference between presymmetric, symmetric and smashable spectra. *In the category $\mathscr{J}_K^{\mathbf{N}}$ (see Definition 7.2.4), the morphism object $\mathscr{J}_K^{\mathbf{N}}(\mathbf{n}, \mathbf{n}+\mathbf{k}) = K^{\wedge k}$ depends only on k. This means that an $\mathcal{M}$-valued functor X on it (a presymmetric spectrum) consists of a sequence* *of objects X_n in $\mathcal{M}$, and for each n and k a structure map*

$$\epsilon^X_{n,k} : \mathscr{J}^{\mathbf{N}}_K(\mathbf{n}, \mathbf{n}+\mathbf{k}) \wedge X_n = K^{\wedge k} \wedge X_n \to X_{n+k}. \tag{7.0.6}$$

The domain of this map is the image of X_n under the endofunctor $K^{\wedge k} \wedge (-)$ on $\mathcal{M}$, and this functor is independent of n.

This independence of n does not hold in the other indexing categories we will consider in Definition 7.2.4 and Definition 7.2.19. This means that for $\mathcal{M}$-valued functors on those categories (symmetric and smashable spectra), the domain of the analog of (7.0.6) cannot be described in terms of an endofunctor independent of n, so they are not Hovey spectra.

7.0C Model structures for spectra

So far this introductory discussion, like [HHR16, Appendix A], has been categorical rather then homotopy theoretic. We will now turn to homotopy theory, which means talking about model structures on various categories of spectra.

One can define the **strict or projective model structure** on the category of spectra by saying that $f : X \to Y$ is a strict fibration or weak equivalence if the same is true of each f_n. Then one defines cofibrations to be maps with suitable left lifting properties. Once we have identified spectra as $\mathcal{T}$-valued functors on a certain indexing category $\mathscr{J}$ this strict model structure becomes the projective model structure of Definition 5.4.2.

We will see that this strict model structure is inadequate for our purposes for three different reasons. It needs to be **stabilized**, **positivized** and, in the equivariant case, **enlarged** so as to be equifibrant. We will explain each of these terms in due course. See Remark 8.6.19 for an explanation of the word "equifibrant."

Remark 7.0.7. Three reasons to modify the projective model structure.

(i) *The first defect of the strict model structure is that its notion of weak equivalence is too restrictive. Stable homotopy theorists are hard wired to expand the collection of weak equivalences to include all stable equivalences, that is to all maps inducing isomorphisms of stable homotpy groups.* **This is a situation crying out for left Bousfield localization**, *the subject of Chapter 6. The term* **stabilization** *refers to this instance of it. It works very nicely, and it turns out that the stably fibrant objects are precisely the Ω-spectra, a most pleasant state of affairs. A fibrant replacement R (originally denoted by Q in [BF78]) can be defined to be the well known functor*

$$(RX)_n = \operatorname*{hocolim}_k \Omega^k X_{n+k}.$$

It is sometimes referred to as **spectrification***; this term's first use may be in [LMSM86, page 4].*

To our knowledge, stabilization was first described as a form of Bousfield localization by Hovey in [Hov01b]. This chapter is heavily influenced by his point of view, as well as that of [MMSS01]. Curiously, Bousfield localization in this context was not mentioned by Bousfield himself in [BF78].

(ii) **Positivization** *has to do with defining a model structure on the category of commutative ring spectra, which we also refer to as commutative algebras in the category of spectra. We will take this up in detail in Chapter 10, but we can illustrate the basic difficulty here. A commutative ring object R in a closed symmetric monoidal category, such as a category of smashable spectra $\mathcal{S}p$, is one having a map $R \wedge R \to R$ with suitable properties. One gets a category* $\mathbf{Comm}\,\mathcal{S}p$ *of such objects as in Definition 2.6.58. Since $\mathcal{S}p$ is cocomplete, we have the free commutative algebra functor* $\mathrm{Sym} : \mathcal{S}p \to \mathbf{Comm}\,\mathcal{S}p$ *of (2.6.65),*

$$X \mapsto \mathrm{Sym}\,(X) := \bigvee_{n \geqslant 0} \mathrm{Sym}^n X,$$

where Sym^n *is the nth symmetric product functor,*

$$X \mapsto (X^{\wedge n})_{\Sigma_n}.$$

The functor Sym *is left adjoint to the forgetful functor*

$$U : \mathbf{Comm}\,\mathcal{S}p \to \mathcal{S}p.$$

We would like to have a model structure on the category $\mathbf{Comm}\,\mathcal{S}p$ *for which the pair of functors* (Sym, U) *is a Quillen pair with the stable model structure on $\mathcal{S}p$. This means that the functors should satisfy the hypotheses of the Crans-Kan Transfer Theorem 5.2.27, in particular that* $U\mathrm{Sym}$ *should preserve stably trivial cofibrations between cofibrant objects in $\mathcal{S}p$.*

Now consider the map

$$s_1 : S^{-1} \wedge S^1 \to S^{-0}, \tag{7.0.8}$$

which we will generalize in (7.3.3) below. In the original case its mth component is

$$\begin{cases} * \to S^0 & \text{for } m = 0 \\ 1_{S^m} : S^m \to S^m & \text{for } m > 0, \end{cases}$$

and in the orthogonal case it is

$$\mathscr{J}(1, \mathbf{m}) \wedge S^1 = \mathscr{J}(1, \mathbf{m}) \wedge \mathscr{J}(0, 1) \xrightarrow{j_{0,1,m}} \mathscr{J}(0, m) = S^m,$$

the domain being a point when $m = 0$. In any case it is a stably trivial cofibration between cofibrant objects. However the spectra $\mathrm{Sym}^n(S^{-0}) \cong S^{-0}$ *and* $\mathrm{Sym}^n(S^{-1} \wedge S^1)$ *for $n > 1$ are* **wildly different***, as will be* *explained in Example 10.5.2 below. This means that* Sym^n *cannot be a left Quillen functor,* **unless we alter the stable model structure on $\mathcal{S}p$.**

We will spell out exactly how this is done in Definition 7.4.36 below. In the symmetric and orthogonal cases it means weakening the condition for a map $f : X \to Y$ *to be a fibration or a weak equivalence. We require that* f_m *be one* **only for** $m > 0$*; we no longer care about* f_0*. This means there are more fibrations and weak equivalences, and hence fewer cofibrations and cofibrant objects than before. For a map* $i : A \to B$ *to be a positive cofibration,* **the map $i_0 : A_0 \to B_0$ must be a pointed homeomorphism***, so in a cofibrant object* K*,* K_0 *must be a point.*

In particular, **the sphere spectrum S^{-0} is no longer cofibrant***, but* S^{-1} *and* $S^{-1} \wedge S^1$ *still are. The map of (7.0.8) is a cofibrant approximation for* S^{-0}*, but it is no longer required to be preserved as a weak equivalence by the functor* Sym^n*. This means the bad behavior of the map* $\mathrm{Sym}^n(e_1)$ *is no longer a concern.*

(iii) Finally we have **equifibrant enlargement** *in the case of orthogonal* G*-spectra. We will discuss it in §9.2 below.*

These three modifications (stabilization, positivization and equifibrant enlargement) can be done independently of each other in any combination in the categories where they are applicable, and they commute with each other. They are indicated in Figure 7.1 by horizontal, vertical and diagonal arrows respectively. Thus we get two model structures (projective and stable) on the original category of spectra, four on symmetric and orthogonal spectra (those of Definition 7.4.36 below) and eight on orthogonal G-spectra. Each of these model structures is cofibrantly generated, and we will identify their generating sets in Theorem 9.2.13 below. **The model structure we will use in subsequent chapters is the positive stable equifibrant one on the bottom right**.

The fact that these three constructions commute with each other follows from the commutativity of the diagram of (6.2.17). It can also be inferred after the fact from the cofibrant generating sets of the eight resulting model structures.

In the original case, X is called an **Ω-spectrum** if the map costructure map of η_n^X (7.0.2) is a weak equivalence for all n. Hovey's generalization is the notion of a Ω_T-spectrum (which he calls a U-spectrum) in Definition 7.1.6. In the original category of spectra, which we denote here by $\mathcal{S}p$ (and by $\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)$ in Definition 7.1.1), two notions of weak equivalence are studied in [BF78]. In the language of §5.1, the category of spectra has two different homotopical structures.

Definition 7.0.9. Homotopical structures on the category of spectra.

(i) A map $f : X \to Y$ *is a* **strict equivalence** *if* $f_n : X_n \to Y_n$ *is a weak*

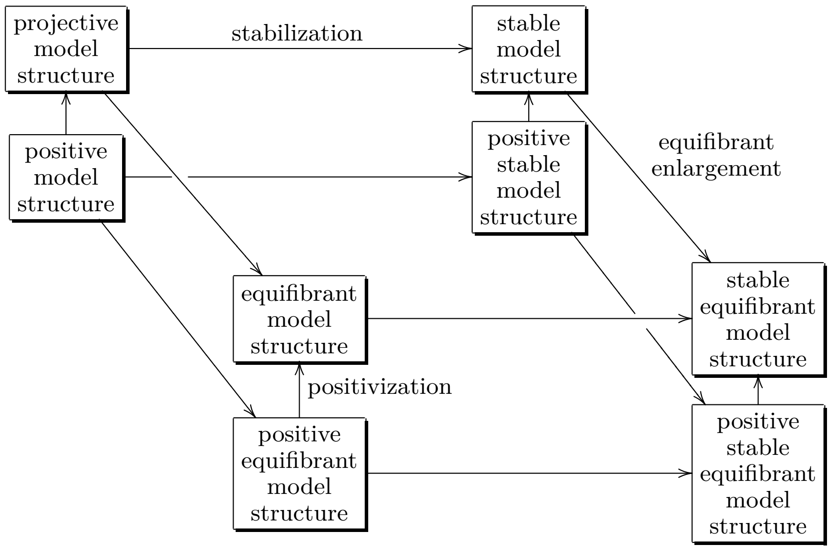


Figure 7.1 Model categories run amok: eight model structures on the category of orthogonal G-spectra. This diagram is a special case of (6.2.17). Each functor is the identity and each arrow indicates the direction of the left Quillen functor. See Table 6.1 for the constructions and Theorem 9.2.13 for cofibrant generating sets.

equivalence for each n. In the corresponding model structure, all spectra are fibrant.

(ii) *It is a* **stable equivalence** *if certain weaker conditions are met. The maps η_n^X can be iterated as in the diagram*

$$X_n \xrightarrow{\eta_n^X} \Omega X_{n+1} \xrightarrow{\Omega\eta_{n+1}^X} \Omega^2 X_{n+2} \xrightarrow{\Omega^2\eta_{n+2}^X} \cdots, \tag{7.0.10}$$

which is the nth component of (5.7.4). This enables us to define the **stable homotopy groups** *of the spectrum X by*

$$\pi_k X := \operatorname*{colim}_n \pi_k \Omega^n X_n \cong \operatorname*{colim}_n \pi_{k+n} X_n. \tag{7.0.11}$$

We can make sense of this definition even for $k < 0$ by defining $\pi_i X_n$ to be trivial for $i < 0$. The groups in the sequential colimit on the right are then defined for all n and could be nontrivial for $k + n > 0$. A more formal definition will be given below in Definition 7.3.14.

A stable equivalence is a map inducing an isomorphism of stable homotopy groups. This notion leads to a model structure in which the fibrant objects are the Ω-spectra.

7.0D The stable model structure

The goal of §7.3 is to describe the stable model structure as a left Bousfield localization of the strict one with respect to a certain set of maps $\mathcal{S}$. Alternatively it could be described in terms of the homotopy idempotent functor Θ^∞ of Definition 5.7.3 using Theorem 6.2.16.

What should the morphism set $\mathcal{S}$ be? To see what $\mathcal{S}$ should be, suppose X is an Ω-spectrum, meaning that the map $\eta_n^X : X_n \to \Omega X_{n+1}$ is a weak equivalence for each $n \geqslant 0$. By Theorem 5.6.21, this is equivalent to requiring the map

$$(\eta_n^X)_* : \mathcal{T}(A, X_n) \to \mathcal{T}(A, \Omega X_{n+1}) \tag{7.0.12}$$

to be a weak equivalence for certain pointed spaces A.

In this chapter we will see that

(i) There are spectra S^{-n} (the **Yoneda spectra** of Definition 7.1.30 below) having the property that for any spectrum Y,

$$\mathcal{S}p(S^{-n}, Y) = Y_n. \tag{7.0.13}$$

Note here that $\mathcal{S}p$ is enriched over $\mathcal{T}$, so its morphism objects are pointed spaces.

More explicitly, the nth Yoneda spectrum S^{-n} is defined by

$$(S^{-n})_k = \begin{cases} * & \text{for } k < n \\ S^{k-n} & \text{otherwise} \end{cases} \tag{7.0.14}$$

with structure map $\epsilon_k^{S^{-n}} : S^1 \wedge S^{k-n} \to S^{k+1-n}$ being the evident homeomorphism for $k \geqslant n$.

To see that (7.0.13) holds, note that a map $f : S^{-n} \to Y$ is a compatible collection of maps

$$f_k : (S_k^{-n}) = S^{k-n} \to Y_n \qquad \text{for } k \geqslant 0.$$

The nth component is a pointed map $S^0 \to Y_n$, and the space of such maps is Y_n it self. The higher components of f must be compatible with the structure maps. This means that for $k \geqslant n$, the diagram

$$\begin{tikzcd} \Sigma(S^{-n})_k \arrow[r, "\Sigma f_k"] \arrow[d, "\epsilon_k^{S^{-n}}"'] & \Sigma Y_k \arrow[d, "\epsilon_k^Y"] \\ (S^{-n})_{k+1} \arrow[r, "f_{k+1}"] & Y_{k+1} \end{tikzcd}$$

must commute. Since the structure map on the left is a homeomorphism by (7.0.14), we have

$$f_{k+1} \cong \epsilon_k^Y(\Sigma f_k).$$

This means that for each $k > n$, f_k is determined by f_n, and that any map of pointed spaces $f_n : S^0 \to Y_n$ can be uniquely extended for a map of spectra $f : S^{-n} \to Y$.

(ii) There are maps of spectra

$$s_n^A : \Sigma A \wedge S^{-n-1} \to A \wedge S^{-n}, \tag{7.0.15}$$

(the **stabilizing maps** of Definition 7.3.1 below; the smash product of a space with a spectrum is defined in Proposition 7.1.14) such that the map of (7.0.12) is a weak equivalence iff the map

$$(s_n^A)^* : \mathcal{S}p(A \wedge S^{-n}, X) \to \mathcal{S}p(\Sigma A \wedge S^{-n-1}, X)$$

is one. Explicitly, the mth component of the map of (7.0.15) is the evident isomorphism for $m \neq n$ (when the kth components of the two spectra are isomorphic) and the trivial one for $k = n$, when the domain is a point but the codomain need not be. When $A = S^0$, this is the map s_n of (1.4.12).

Hence, in the language of Definition 6.2.1, a spectrum X is an Ω-spectrum iff it is $\mathcal{S}$-local with respect to the set of morphisms

$$\mathcal{S} = \left\{ s_n^A : n \geqslant 0 \right\}$$

where A ranges over the domains and codomains of the set of generating cofibrations for $\mathcal{T}$, namely the set $\mathcal{I}_+$ of (5.2.13).

In §7.3 we give the details of stabilization of Hovey spectra as a form of left Bousfield localization. Recall that localization can be defined either in terms of a collection of morphisms that one wants to be weak equivalences in the new model structure, or in terms of a homotopy idempotent endofunctor that one wants to serve as fibrant replacement. The former is a set $\mathcal{S}$ of **stabilizing maps**, defined in Definition 7.3.1. The latter is Hovey's functor Θ^∞ defined in Definition 5.7.3(iii) and studied in §7.3A.

Our Theorem 6.2.16 gives conditions under which an endofunctor and a morphism set lead to the same Bousfield localization. Lemma 7.3.22 shows that Θ^∞ and $\mathcal{S}$ meet these conditions.

In § 7.3D we give explicit cofibrant generating sets for the stable model structure in terms of those for $\mathcal{M}$ and the set $\mathcal{S}$.

In §7.4 we give a parallel treatment of smashable spectra. The four main theorems of § 7.3 each have counterparts here that are indicated in a table at the start of the section. The punch line is Theorem 7.4.52, the smashable analog of Theorem 7.3.36.

7.0E The prickly case of symmetric spectra

Symmetric spectra are problematic in that stable homotopy groups need not be preserved by stable equivalences, as is emphasize by Hovey-Shipley-Smith in [HSS00, §3.1] where they say "**THIS WILL NOT WORK**." We are introducing them only as a logical step toward defining orthogonal spectra and orthogonal G-spectra, which we refer to collectively as **smashable spectra** in Definition 7.2.33. They are our real objects of interest, and they do not suffer from this malady. When the proof of a theorem is harder in the symmetric case, we will only give it for the smashable case.

As an illustration of this difficulty with symmetric stable equivalences, consider the stablizing map

$$s_m : S^{-m-1} \wedge S^1 \to S^{-m}$$

of (1.4.11). The Yoneda spectrum S^{-m} here has to be defined differently from the original case. We will do it for Hovey spectra in Definition 7.1.30 and for smashable spectra in Definition 7.2.52. In particular the components of the symmetric analog of the spectra in (1.4.11) are

$$(S^{-m})_k = \mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m},\mathbf{k}) \cong \begin{cases} \bigvee\limits_{k!/(k-m)!} S^{k-m} & \text{for } k \geqslant m \\ * & \text{otherwise,} \end{cases}$$

and therefore

$$(S^{-m-1} \wedge S^1)_k \cong \begin{cases} \bigvee\limits_{k!/(k-m-1)!} S^{k-m} & \text{for } k \geqslant m+1 \\ * & \text{otherwise.} \end{cases}$$

Thus unlike in the case of ordinary spectra, the kth components of S^{-m} and $S^{-m-1} \wedge S^1$ in $\mathcal{S}p^{\mathbf{\Sigma}}$ differ for all $k \geqslant m$ because they have different numbers of wedge summands.

Hence we need to be more careful about how we **define** the map s_m. Consider the diagram

$$\begin{tikzcd}
\bigvee\limits_{k!/(k-m-1)!} S^{k-m-1} \wedge S^1 \arrow[d, equal] & \\
\mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m+1},\mathbf{k}) \wedge \mathscr{J}^{\mathbf{\Sigma}}(\mathbf{0},\mathbf{1}) \arrow[d, "\mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m+1},\mathbf{k})\wedge ?"'] \arrow[dr, "(s_m)_k"] & \\
\mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m+1},\mathbf{k}) \wedge \mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m},\mathbf{m+1}) \arrow[r, "j_{m,m+1,k}"'] \arrow[d, equal] & \mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m},\mathbf{k}) \arrow[d, equal] \\
\bigvee\limits_{k!/(k-m-1)!} S^{k-m-1} \wedge \bigvee\limits_{(m+1)!} S^1 & \bigvee\limits_{k!/(k-m)!} S^{k-m}
\end{tikzcd}$$

where $j_{m,m+1,k}$ is the composition morphism of (7.0.4). Thus in order to define the kth component of s_m, we need only to choose a summand of $\mathscr{J}^{\mathbf{\Sigma}}(\mathbf{m},\mathbf{m+1})$. These summands are indexed by one to one maps of $\mathbf{m}$ into

$\mathbf{m+1}$. We choose the standard inclusion, that is the order preserving map sending $\mathbf{m}$ to the first m elements of $\mathbf{m+1}$. Formally this choice is the map

$$\alpha_{m,0,1} : \mathscr{J}^{\boldsymbol{\Sigma}}(\mathbf{0},\mathbf{1}) \to \mathscr{J}^{\boldsymbol{\Sigma}}(\mathbf{m},\mathbf{m+1})$$

of Definition 2.6.6.

In any case, we see that the stabilizing map s_m does **not induce an isomorphism of stable homotopy groups**. This means we cannot define a stable equivalence to be a map that does so.

In the language of §6.2, there are two ways to define stable equivalence in $\mathcal{S}p^{\Sigma}$:

(i) It is a weak equivalence in the localization of $\mathcal{S}p^{\Sigma}$ (with its projective model structure as in Definition 5.4.2) with respect to the set $\mathcal{S}$ of morphisms s_m of (1.4.11) for $m > 0$. See Definition 6.2.1.

(ii) It is an Υ-equivalence as in Definition 6.2.15, for a suitable homotopy idempotent functor Υ, which is stable fibrant replacement. The functor we will use is similar to Θ^∞ as in Definition 5.7.3.

For a spectrum Z in any of the categories we will consider, the map $s_m : S^1 \wedge S^{-m-1} \to S^{-m}$ (for a suitably defined S^{-m}) induces the left vertical map in

$$\begin{tikzcd}
\mathcal{S}p(S^{-m}, Z) \arrow[rr, "\cong"] \arrow[d, "s_m^*"'] & & Z_m \arrow[d, "\eta_m^Z"] \\
\mathcal{S}p(S^1 \wedge S^{-m-1}, Z) \arrow[r, "\cong"] & \mathcal{S}p(S^{-m-1}, \Omega Z) \arrow[r, "\cong"] & \Omega Z_{m+1}
\end{tikzcd} \tag{7.0.16}$$

This is a weak equivalence for each $m \geqslant 0$ iff the right vertical map is one, meaning iff Z is an Ω-spectrum. Hence, **the $\mathcal{S}$-local objects in $\mathcal{S}p$ are precisely the Ω-spectra**.

In [HSS00, §5.6], they make a distinction between **stable equivalences** as just defined, and **stable homotopy equivalences**, meaning maps inducing isomorphisms of stable homotopy groups, which not all stable equivalences do. In [HSS00, Theorem 3.1.11] they show that a stable homotopy equivalences is a stable equivalence. The proof makes use of the functor Θ^k of Definition 5.7.3(i), which he denotes by R^k.

A symmetric spectrum is said to be **semistable** ([HSS00, Definition 5.6.1]) if the map to its stable fibrant replacement (which is always a stable equivalence) is a stable homotopy equivalence. They show that $S^1 \wedge S^{-1}$ is not semistable. [HSS00, Proposition 5.6.5] says that a map between semistable spectra is a stable equivalence iff it is a stable homotopy equivalence.

For more discussion, see Stefan Schwede's untitled book project [Sch07, Examples I.2.12, I.4.31 and II.4.2]. He discusses long exact sequences of stable homotopy groups of symmetric spectra in [Sch07, §I.4.1]. In [Sch07, Proposition I.4.7], he gives the Adams exact sequence for a stable cofiber sequence of

Theorem 5.7.11 and the Puppe exact sequence for a stable fiber sequence as in Proposition 4.7.11(i). He discusses semistable spectra in [Sch07, §I.4.5].

Stabilization of orthogonal spectra (see Definition 7.2.33) is better behaved than that of symmetric spectra. The indexing category is $\mathscr{J}^{\mathbf{O}} = \mathscr{J}^{\mathbf{O}}_{S^1}$ as in Definition 7.2.4(iii). For $m \leqslant n$, the morphism space $\mathscr{J}^{\mathbf{O}}(\mathbf{m}, \mathbf{n})$ is informally a wedge of copies of S^{n-m} parametrized by the Stiefel manifold $O(n)/O(m)$ of orthogonal embeddings $\mathbf{R}^m \to \mathbf{R}^n$. In Proposition 7.2.15 we will see that the connectivity of the evident map $S^{n-m} \to \mathscr{J}^{\mathbf{O}}(\mathbf{m}, \mathbf{n})$ increases linearly with n. This means that the stabilizing maps s_m in $\mathcal{S}p^{\mathbf{O}}$, like those in $\mathcal{S}p^{\mathbf{N}}$ and unlike those in $\mathcal{S}p^{\boldsymbol{\Sigma}}$, induce isomorphisms in stable homotopy groups.

7.0F Comparing our approach with those of [Lyd98], [HSS00], [Hov01b], [DRØ03], and [Sch07]

There is substantial overlap between the concepts presented in this chapter and in the works listed above, but also some important differences. In both cases spectra are described as enriched functors from an indexing category $\mathscr{J}$ to a Quillen ring $\mathcal{M}$ (over which $\mathscr{J}$ is enriched) satisfying appropriate technical hypotheses. The differences are as follows.

(i) For them the target category is either some variant of pointed simplicial sets or a category suitable for motivic homotopy theory. For us it is some variant of pointed topological spaces. We chose spaces over simplicial sets because studying representation spheres for a finite group G is easier if we do not have to worry about triangulations. The two approaches lead to Quillen equivalent theories. In the equivariant case, this is spelled out in [DRØ03, §9.3]. In our case all objects in the target category $\mathcal{M}$ are fibrant, while in theirs all objects are cofibrant. Their maps tend to be better behaved than ours.

(ii) The differing target categories lead to different fibrant replacement functors. In both cases one has a sequential diagram involving costructure maps similar to that of Definition 5.7.3(iii). For us the fibrant replacement is its homotopy colimit or telescope, while for them it is the ordinary colimit. In both cases one needs some technical assumptions on the target category (which are met in the examples of interest) to insure that these (homotopy) colimits of fibrant objects are again fibrant.

Colimits, even sequential ones, can be problematic in homotopy theory, as we saw in Example 2.3.65. Homotopy groups do not commute with sequential colimits in general, but they do when all the maps in the diagram are closed inclusions. The same is true of the functor $\Omega_K = (-)^K$ in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)$ for cofibrant K, since we are assuming that K is compact as in Definition 5.2.6. However it is not reasonable to assume that the maps in (7.1.8) are closed inclusions, so we use the homotopy colimit or telescope instead.

We remind the reader that telescopes enjoy many of the properties of ordinary sequential colimits; see Lemma 5.8.19, Lemma 5.8.20 and Proposition 5.8.25. By Lemma 5.8.20(iii), a homotopy sequential colimit is an ordinary sequential colimit in which each map is a closed inclusion, so we avoid the pitfalls of Example 2.3.65.

(iii) Our indexing categories are leaner than theirs. For Lydakis [Lyd98], a spectrum is a homotopical functor to simplicial sets from the full category of finite simplicial sets. For such a functor X, the nth "space" is its value on the simplicial set $S^n = \partial\Delta^{n+1}$. His unit object (the sphere spectrum) is the inclusion functor.

[DRØ03, §2.5 and §2.6] define categories $K\mathrm{Sph}$ and $K\mathrm{Sph}^{\Sigma}$ which are simplicial analogs of our $\mathscr{J}_K^{\mathbf{N}}$ and $\mathscr{J}_K^{\mathbf{\Sigma}}$. Later they consider full symmetric monoidal subcategories $\mathcal{C}$ of $\mathcal{S}et_{\mathbf{\Delta}}^{fin}$ containing $K\mathrm{Sph}^{\Sigma}$.

Our indexing categories of Definition 7.2.19 are also symmetric monoidal subcategories of $\mathcal{M}$ containing $\mathscr{J}_K^{\mathbf{\Sigma}}$, but they are far from full. See Remark 7.2.25 below.

(iv) [HSS00], [DRØ03], and [Sch07] each make use of **injective** Ω-spectra, where injective here means fibrant in the injective model structure of Definition 5.4.2. They define a map $f : X \to Y$ of symmetric spectra to be a stable equivalence if the functor $\mathcal{S}p^{\Sigma}(-, Z)$ converts it to a weak equivalence for each injective Ω-spectrum Z. For us (and in [Hov01b]) a stable equivalence is defined to an $\mathcal{S}$-local equivalence, which is equivalent (see (7.0.16)) to requiring $\mathcal{S}p^{\Sigma}(f, Z)$ be a weak equivalence in $\mathcal{M}$ for each (not necessarily injective) Ω-spectrum Z.

7.1 Hovey's generalization of spectra

We now proceed to Hovey's generalization of the original definition of spectra.

Definition 7.1.1. *Let $\mathcal{M}$ be a model category with a Quillen endopair (T, Ω_T) as in Definition 4.5.1 that is compact as in Definition 5.8.22. Then* **the category of Hovey spectra** $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ *has as its objects sequences*

$$(X_0, X_1, X_2, \dots)$$

of objects in $\mathcal{M}$ equipped with structure maps $\epsilon_n^X : TX_n \to X_{n+1}$. A morphism $f : X \to Y$ in this category is a collection $\{f_n : X_n \to Y_n : n \geqslant 0\}$ of *morphisms in $\mathcal{M}$ that commute with the structure maps, as in the diagram*

$$\begin{tikzcd} TX_n \arrow[r, "\epsilon_n^X"] \arrow[d, "Tf_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ TY_n \arrow[r, "\epsilon_n^Y"] & Y_{n+1}. \end{tikzcd} \tag{7.1.2}$$

We will denote the composite map

$$T^k X_n \xrightarrow{T^{k-1}\epsilon_n^X} T^{k-1}X_{n+1} \xrightarrow{T^{k-2}\epsilon_{n+1}^X} \cdots \xrightarrow{\epsilon_{n+k-1}^X} X_{n+k} \tag{7.1.3}$$

by $\epsilon_{n,k}^X$, so $\epsilon_n^X = \epsilon_{n,1}^X$.

We will refer to the adjunction $T^\ell \dashv \Omega_T^\ell$ for $\ell > 0$ as the **Hovey adjunction**.

Remark 7.1.4. Notation for the structure and costructure map. *Our use of the symbol η_n^X for the costructure map $X_n \to \Omega X_{n+1}$ adjoint to the structure map ϵ_n^X is unusual. Hovey denotes the latter by σ (and η_n^X by $\tilde{\sigma}$) in [Hov01b, Definitions 1.1 and 3.1]. We chose our symbols to be consistent with the uses of ϵ and η to denote augmentation and coaugmentation as in Definition 2.2.8 and to denote the counit and unit of an adjunction as in Definition 2.2.20.*

He does not require his Quillen functors to be compact. Our compactness condition means that the right adjoint functor Ω_T commutes with homotopy sequential colimits.

Strictly speaking the map we are calling ϵ_n^X should be denoted by $(\epsilon_X)_n$ since it is the nth component at X of a natural transformation ϵ, but the latter notation is too cumbersome. The same goes for η_n^X. On the other hand, once we see that the category of spectra is often an enriched functor category in Theorem 7.2.32 below, then in the notation of (3.1.38), ϵ_n^X will be an abbreviation for $\epsilon_{\mathbf{n},\mathbf{n+1}}^X$.

Remark 7.1.5. Notation for the Quillen endopair. *Hovey denotes the endopair by (T, U), but we prefer to reserve the symbol U for various forgetful functors. In the original case the pair was (Σ, Ω), and this usage is compatible with Definition 4.6.17. In view of the latter, it would be confusing to use the same symbols for their generalizations here. We follow [DRØ03] in replacing U by Ω_T. We will use the symbol Ω in Definition 7.2.29 below in a way that generalizes its original usage.*

Definition 7.1.6. *A* **Hovey Ω_T-spectrum** *is a spectrum X for which X_n is fibrant and the right adjoint of ϵ_n^X,*

$$\eta_n^X : X_n \to \Omega_T X_{n+1}, \tag{7.1.7}$$

is a weak equivalence for each n. We will denote the composite map

$$X_n \xrightarrow{\eta_n^X} \Omega_T X_{n+1} \xrightarrow{\Omega_T \eta_{n+1}^X} \Omega_T^2 X_{n+2} \longrightarrow \cdots \longrightarrow \Omega_T^k X_{n+k} \tag{7.1.8}$$

by $\eta_{n,k}^X$.

In the next section we will assume additionally that the model category $\mathcal{M}$ is **stabilizable** as in Definition 7.2.1, and in particular telescopically closed as in Definition 5.8.28. We will need the telescopically closed condition on $\mathcal{M}$ in order to define our fibrant replacement functor Θ^∞ in Definition 5.7.3. For a spectrum X, the nth component of $\Theta^\infty X$ is a certain homotopy sequential colimit of fibrant objects in $\mathcal{M}$, and we need these telescopes themselves to be fibrant. In §7.3D and §7.4D we will need the domains and codomains of the cofibrant generating maps of $\mathcal{M}$ are cofibrant.

$\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)$ is the original category of spectra in which the Quillen endopair is (Σ, Ω) and a Ω_T-spectrum is an Ω-spectrum. In this case the model category $\mathcal{M} = \mathcal{T}$ is symmetric monoidal as in Definition 5.5.9, and the left Quillen functor $T = \Sigma$ is the smash product $S^1 \wedge -$.

In Definition 7.1.13 below we will say a Hovey spectrum is **presymmetric** when the left Quillen functor T of Definition 7.1.1 is given by $X \mapsto K \wedge X$ for a fixed compact (as in Definition 5.2.6) cofibrant object K in $\mathcal{M}$. The compactness of K will guarantee that of the two Quillen functors $K \wedge (-)$ and $(-)^K$ and compact as in Definition 5.8.22. In that case a spectrum can be interpreted (see Theorem 7.2.32 below) as an $\mathcal{M}$-valued functor on a certain indexing category $\mathscr{J}_K^{\mathbf{N}}$ spelled out in Definition 7.2.4 below. Such functors are the subject of §7.2A. This will mean we can use the enriched category theory of Chapter 3 to study them.

This section will end with a brief description (in Proposition 7.1.33 below) of the projective model structure on the category above. It generalizes the strict model structure in the original case, the one in which a map of spectra $f : X \to Y$ is a fibration or weak equivalence when each $f_n : X_n \to Y_n$ is one. Experience has shown that it is too rigid for the purposes of stable homotopy theory. A stable equivalence $g : X \to Y$ is **not** required to induce a weak equivalence in each degree.

From (7.1.2) we get the following diagram in $\mathcal{S}et$.

$$\begin{tikzcd} f_n \in \mathcal{M}(X_n, Y_n) \arrow[r, "T_{X_n,Y_n}"] & \mathcal{M}(TX_n, TY_n) \arrow[r, "(\epsilon_n^Y)_*"] & \mathcal{M}(TX_n, Y_{n+1}) \\ & & f_{n+1} \in \mathcal{M}(X_{n+1}, Y_{n+1}) \arrow[u, "(\epsilon_n^X)^*"'] \end{tikzcd} \tag{7.1.9}$$

The morphisms $f_n \in \mathcal{M}(X_n, Y_n)$ and $f_{n+1} \in \mathcal{M}(X_{n+1}, Y_{n+1})$ make (7.1.2) commute iff they have the same image in $\mathcal{M}(TX_n, Y_{n+1})$. It follows that the morphism set $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)(X, Y)$ is the equalizer of two maps

$$\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)(X, Y) \dashrightarrow \prod_n \mathcal{M}(X_n, Y_n) \rightrightarrows \prod_n \mathcal{M}(TX_n, Y_{n+1}) \tag{7.1.10}$$

derived from the two maps to $\mathcal{M}(TX_n, Y_{n+1})$ in (7.1.9), which leads to the end in Proposition 7.2.48 below.

Since Ω_T is the right adjoint of T, we can rewrite (7.1.10) as

$$\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)(X, Y) \dashrightarrow \prod_n \mathcal{M}(X_n, Y_n) \rightrightarrows \prod_n \mathcal{M}(X_n, \Omega_T Y_{n+1}) \tag{7.1.11}$$

where the two maps are obtained in a similar way from the adjoint of (7.1.9), which is

$$\begin{tikzcd} \mathcal{M}(X_n, Y_n) \arrow[r, "(\Omega_T)_{X_n, Y_n}"] & \mathcal{M}(\Omega_T X_n, \Omega_T Y_n) \arrow[r, "(\eta^X_{n-1})^*"] & \mathcal{M}(X_{n-1}, \Omega_T Y_n) \\ & & \mathcal{M}(X_{n-1}, Y_{n-1}). \arrow[u, "(\eta^Y_{n-1})_*"'] \end{tikzcd} \tag{7.1.12}$$

For any cofibrant object K in a Quillen ring $\mathcal{M}$, the functor $T = K \wedge -$ is a left Quillen functor. Categories $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K \wedge -)$ of spectra constructed in this way have more convenient properties those of the general Hovey spectra of Definition 7.1.1.

Since a Quillen ring $\mathcal{M}$ is by definition closed as a symmetric monoidal category, the endofunctor $T = K \wedge (-)$ always has a right adjoint

$$\Omega_K = (-)^K = \mathcal{M}(K, -),$$

as explained in Definition 5.5.17 and the definitions leading to it. With this in mind we make the following definition.

Definition 7.1.13. Presymmetric spectra. *For a Quillen ring $\mathcal{M}$, a category of Hovey spectra (Definition 7.1.1) $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ is a* **category of presymmetric spectra** *(and an object in it a* **presymmetric spectrum***) if the functor T is $\Sigma_K := K \wedge (-)$ for a compact cofibrant object K of $\mathcal{M}$. We will denote the category of such spectra (abusively) by $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)$.*

We want to define a Quillen $\mathcal{M}$-structure on $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)$. The following is an easy consequence of the definitions.

Proposition 7.1.14. Enriched presymmetric spectra. *For a Quillen ring $\mathcal{M}$ as in Definition 5.5.9, the category $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)$ as in Definition 7.1.13 can be given the structure of an $\mathcal{M}$-category as in Definition 3.1.1 that is bitensored over $\mathcal{M}$ as in Definition 3.1.31, making it a closed $\mathcal{M}$-module as in Definition 2.6.42.*

The morphism object $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)(X, Y)$ is the equalizer of (7.1.10), or equivalently that of (7.1.11) (with T being the functor $K \wedge -$ in both cases), *regarded as a diagram in $\mathcal{M}$ rather than in $\mathcal{S}et$. For an object A in $\mathcal{M}$, its tensor and cotensor products with a spectrum X are given by*

$$(A \wedge X)_n = A \wedge X_n \qquad \text{and} \qquad (X^A)_n = (X_n)^A = \mathcal{M}(A, X_n).$$

The structure map for $A \wedge X$ is the composite

$$\begin{tikzcd} K \wedge A \wedge X_n \arrow[rr, "\epsilon_n^{A \wedge X}"] \arrow[dr, "t \wedge X_n"'] & & A \wedge X_{n+1} \\ & A \wedge K \wedge X_n \arrow[ur, "A \wedge \epsilon_n^X"'] & \end{tikzcd},$$

where t swaps the factors A and K.

The structure map for X^A is the left adjoint of its costructure map, which is the composite

$$\begin{tikzcd} \mathcal{M}(A, X_n) \arrow[rr, "\eta_n^{(X^A)}"] \arrow[d, "(\eta_n^X)_*"'] & & \mathcal{M}(K, \mathcal{M}(A, X_{n+1})) \\ \mathcal{M}(A, \mathcal{M}(K, X_{n+1})) \arrow[dr, "\cong", "\varphi^{-1}_{A, X_{n+1}}"'] & & \mathcal{M}(A \wedge K, X_{n+1}) \arrow[u, "\cong", "\varphi'_{K, X_{n+1}}"'] \\ & \mathcal{M}(K \wedge A, X_{n+1}), \arrow[ur, "t^*"'] & \end{tikzcd}$$

where φ and φ' are the adjunction isomorphisms of Definition 2.2.13 for $K \wedge (-) \dashv \mathcal{M}(K, -)$ and $A \wedge (-) \dashv \mathcal{M}(A, -)$ respectively.

For spectra X and Y, there is an adjunction isomorphism

$$\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)(A \wedge X, Y) \cong \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)(X, Y^A). \tag{7.1.15}$$

Remark 7.1.16. The Hovey twist. *The following should be compared with [Hov01b, Remark 1.6] and [DRØ03, Remark 2.13].*

The categories $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K \wedge -)$ and $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ are both tensored over the Quillen ring $\mathcal{M}$. For a functor $X : \mathscr{J}_K^{\mathbf{N}} \to \mathcal{M}$ and an object M in $\mathcal{M}$, the functor $M \wedge X$ is defined by $(M \wedge X)_n = M \wedge X_n$ with structure map $\epsilon_n^{M \wedge X} : K \wedge M \wedge X_n \to M \wedge X_{n+1}$ being the composite

$$K \wedge M \wedge X_n \xrightarrow{t \wedge X_n} M \wedge K \wedge X_n \xrightarrow{M \wedge \epsilon_n^X} M \wedge X_{n+1}, \tag{7.1.17}$$

where the twist map t swaps the first two factors.

The costructure map $\eta_n^{M \wedge X}$, the right adjoint of $\epsilon_n^{M \wedge X}$, is the composite

$$M \wedge X_n \xrightarrow{\varphi(t \wedge X_n)} (M \wedge K \wedge X_n)^K \xrightarrow{(M \wedge \epsilon_n^X)^K} (M \wedge X_{n+1})^K, \tag{7.1.18}$$

where

$$\begin{tikzcd} \mathcal{M}(K \wedge M \wedge X_n, M \wedge K \wedge X_{n+1}) \arrow[d, "\varphi=\varphi_{M \wedge X_n, M \wedge K \wedge X_{n+1}}"'] \\ \mathcal{M}(M \wedge X_n, (M \wedge K \wedge X_{n+1})^K) \end{tikzcd}$$

is the adjunction isomorphism of Definition 2.2.13 for $K \wedge (-) \dashv (-)^K$.

We also have the suspension spectrum functor $\Sigma_K^\infty : \mathcal{M} \to [\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ *given by* $(\Sigma_K^\infty M)_n = K^{\wedge n} \wedge M$ *with the structure map* $\epsilon_n^{\Sigma_K^\infty M}$ *being*

$$K \wedge (\Sigma_K^\infty M)_n = K \wedge K^{\wedge n} \wedge M \xrightarrow{j_{0,n,n+1} \wedge M} K^{\wedge(n+1)} \wedge M = (\Sigma_K^\infty M)_{n+1},$$

where $j_{0,n,n+1}$ *is the usual isomorphism, which is also the composition morphism of (7.2.5) below.*

The composite functor $\Sigma_K^\infty \Sigma_K : \mathcal{M} \to [\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ *extends along* Σ_K^∞ *to an endofunctor which we also denote by* Σ_K, *so that the following diagram commutes.*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\Sigma_K"] \arrow[d, "\Sigma_K^\infty"'] & \mathcal{M} \arrow[d, "\Sigma_K^\infty"] \\ {[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]} \arrow[r, "\Sigma_K"] & {[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}].} \end{tikzcd}$$

In Definition 7.1.13 we used the symbol Σ_K *for the endofunctor* $K \wedge (-)$ *of* $\mathcal{M}$. *Here we are using it for an endofunctor of* $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ *which differs subtly from the tensor functor* $K \wedge (-)$.

For a spectrum X *we have* $(\Sigma_K X)_n = K \wedge X_n$, *and the structure map*

$$\epsilon_n^{\Sigma_K X} : K \wedge K \wedge X_n \to K \wedge X_{n+1}$$

is $K \wedge \epsilon_n^X$. *This map does not involve a twist and is therefore* **not** *the same as the map* $\epsilon_n^{K \wedge X}$ *of (7.1.17) for the case* $M = K$.

The costructure map $\eta_n^{\Sigma_K X}$, *the right adjoint of* $\epsilon_n^{\Sigma_K X}$ *is*

$$\varphi(\epsilon_n^{\Sigma_K X}) : K \wedge X_n \to (K \wedge X_{n+1})^K,$$

which is not the same as (7.1.18) for $M = K$.

Hence **the spectra** $K \wedge X$ **and** $\Sigma_K X$ **have the same components but different structure and costructure maps**. *The spectrum* $\Sigma_K X$ *is a special case of the spectrum* TX *of Definition 7.1.27 below.*

Proposition 7.1.19. Preservation of certain colimits by Ω_K**.** *Let* $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)$ *be as in Definition 7.1.13. Then the functor* Ω_K *preserves sequential colimits of h-cofibrations as in Definition 5.6.7, and homotopy sequential colimits.*

Proof Since Ω_K is $\mathcal{M}(K, -)$ and K is compact as in Definition 5.2.6, it preserves sequential colimits of h-cofibrations by the definition of a compact object. It preserves homotopy sequential colimits by Lemma 5.8.20(iii). $\square$

The following should be compared with Yoneda Lemma 2.2.10. The comparison is precise for presymmetric spectra as in Definition 7.1.13 but imperfect in general due to the nonfunctoriality of the Hovey spectra of Definition 7.1.1.

Definition 7.1.20. Evaluation and tensored Yoneda functors. *Let $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ be as in Definition 7.1.1. For $m \geqslant 0$, the m***th evaluation functor**

$$\mathrm{Ev}_m : \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T) \to \mathcal{M}$$

*is given by $X \mapsto X_m$. The m***th tensored Yoneda functor**

$$T^{-m} : \mathcal{M} \to \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$$

(denoted by F_m in [Hov01b, Definition 1.2] and [HSS00, Definition 2.2.5]) which is the left adjoint of Ev_m, is given by

$$(T^{-m}M)_n = \begin{cases} * & \text{for } 0 \leqslant n < m \\ T^{n-m}M & \text{for } n \geqslant m \end{cases} \tag{7.1.21}$$

with the obvious structure maps, where $*$ denotes the initial object of $\mathcal{M}$. In particular $T^{-0}M$ is the **Hovey suspension spectrum** *associated with M. In the presymmetric case (Definition 7.1.13) we will sometimes denote T^{-m} by K^{-m} with*

$$(K^{-m}M)_n = \begin{cases} * & \text{for } 0 \leqslant n < m \\ M \wedge K^{n-m} & \text{for } n \geqslant m. \end{cases} \tag{7.1.22}$$

We have the following analog of Proposition 5.6.28 with a similar proof.

Proposition 7.1.23. The Yoneda adjunction for Hovey spectra. *With notation as above, for each $m \geqslant 0$ the adjunction $T^{-m} \dashv \mathrm{Ev}_m$ is a Quillen adjunction. In particular, if A is a cofibrant object in $\mathcal{M}$, then $T^{-m}A$ is projectively cofibrant in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$.*

Remark 7.1.24. The right adjoint R_m of the evaluation functor Ev_m *(denoted by M_m in [Hov01b, Remark 1.4] and by R_m in [HSS00, Definition 2.2.5]) is given by*

$$(R_m M)_n = \begin{cases} \Omega_T^{m-n}M & \text{for } n \leqslant m \\ * & \text{for } n > m, \end{cases}$$

where $*$ denotes the terminal object in $\mathcal{M}$. The structure map for $n < m$,

$$T\Omega_T^{m-n}M \to \Omega_T^{m-n-1}M,$$

is the left adjoint of the identity map on $\Omega_T^{m-n}M$.

Remark 7.1.25. The functors Σ^∞ and Ω^∞. *In the original case, the functor T^{-0} sends a pointed space K to the suspension spectrum $\Sigma^\infty K$ defined by*

$$(\Sigma^\infty K)_n = K \wedge S^n,$$

so it is also denoted by Σ^∞. Its right adjoint Ev_0 is often denoted by Ω^∞. When X is an Ω-spectrum, $\Omega^\infty X$ is an infinite loop space. The equivariant generalizations of these functors will be studied in Chapter 9 below.

In [Hov01b] the functor T^{-m} is denoted by F_m. We are using this notation to suggest that it is related to the inverse of the iterate T^m. In the original case T^{-0} is the functor Σ^∞ sending a pointed space to its suspension spectrum.

Recall that T is an endofunctor on $\mathcal{M}$. Our T^{-m} is not the actual inverse (which need not exist) of T^m since the former is not an endofunctor but rather a functor from $\mathcal{M}$ to $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$. We will see in Definition 7.1.27 below that is extends along Σ^∞ to an endofunctor of $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ which we also denote by T^{-m}. It is consistent with desuspension as in Definition 5.7.3(i).

Example 7.1.26. Coevaluation in the original case. *For an original spectrum X in $\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)$, the mth tensored Yoneda functor (again for $m \geqslant 0$) is the formal mth desuspension functor defined by*

$$(\Sigma^{-m} X)_n = \begin{cases} * & \text{for } 0 \leqslant n < m \\ X_{n-m} & \text{for } n \geqslant m. \end{cases}$$

The category $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ is bicomplete (with limits and colimits being evaluated objectwise; see [Hov01b, Lemma 1.3]. This means the nth object of a limit (colimit) is the limit (colimit) of the nth objects.

The Quillen pair (T, Ω_T) on $\mathcal{M}$ extends to a similar one on $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ by applying the two functors objectwise as in Definition 7.1.1; see [Hov01b, Lemma 1.5].

Definition 7.1.27. T, T^{-1} and Ω_T as endofunctors of $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$. *Let $X = \{X_n, \epsilon_n^X \colon n \geqslant 0\}$ be a Hovey spectrum as in Definition 7.1.1. For $\ell > 0$, let $T^\ell X$ be the Hovey spectrum with $(T^\ell X)_n = T^\ell(X_n)$ and $\epsilon_n^{T^\ell X} = T^\ell(\epsilon_n^X)$, and define $\Omega_T^\ell X$ in a similar way.*

For $m \geqslant 0$, let $T^{-m} X$ be the Hovey spectrum with

$$(T^{-m} X)_n = \begin{cases} X_{n-m} & \text{for } n \geqslant m \\ * & \text{otherwise,} \end{cases}$$

and

$$\epsilon_n^{T^{-m} X} = \begin{cases} \epsilon_{n-m}^X & \text{for } n \geqslant m \\ * \to X_0 & \text{for } n = m-1 \\ * \to * & \text{otherwise.} \end{cases}$$

See Remark 7.1.16 for more discussion.

The following is an immediate consequence of Definition 7.1.20 and Definition 7.1.27.

Proposition 7.1.28. T^ℓ and T^{-m} commute. *The following diagrams commute*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "T^{-m}"] \arrow[d, "T^\ell"'] & \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T) \arrow[d, "T^\ell"] \\ \mathcal{M} \arrow[r, "T^{-m}"] & \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T) \arrow[r, "T^{-m}"] \arrow[d, "T^\ell"'] & \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T) \arrow[d, "T^\ell"] \\ \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T) \arrow[r, "T^{-m}"] & \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T) \end{tikzcd}$$

for $\ell > 0$ and $m \geqslant 0$, but neither composite is $T^{-(m-\ell)}$ for $\ell \leqslant m$ nor $T^{\ell-m}$ for $\ell > m$. On the other hand,

$$\begin{aligned} T^{\ell_1}T^{\ell_2} &= T^{\ell_1+\ell_2} && \text{for } \ell_1, \ell_2 > 0 \\ \text{and} \qquad T^{-m_1}T^{-m_2} &= T^{-(m_1+m_2)} && \text{for } m_1, m_2 \geqslant 0. \end{aligned}$$

Remark 7.1.29. Two stable stable equivalences. *While the composite functor $T^\ell T^{-m}$ is not equal to $T^{-(m-\ell)}$ or to $T^{\ell-m}$, we will see later that it is stably equivalent to it on spectra $X = T^{-0}M$ for an object M of $\mathcal{M}$. For $\ell > 0$ and $m \geqslant 0$, we have*

$$\begin{aligned} (T^\ell T^{-m}X)_n &= \begin{cases} T^{\ell+n-m}M & \text{for } n \geqslant m \\ * & \text{for } n < m \end{cases} \\ \text{and} \qquad (T^{-(m-\ell)}X)_n &= \begin{cases} T^{n-m+\ell}M & \text{for } n \geqslant m-\ell \\ * & \text{for } n < m-\ell. \end{cases} \end{aligned}$$

There is a map

$$T^\ell T^{-m}X \to \begin{cases} T^{-(m-\ell)}X & \text{for } m \geqslant \ell > 0 \\ T^{\ell-m}X & \text{for } \ell > m \geqslant 0 \end{cases}$$

whose nth component is

$$\begin{cases} 1_{T^{\ell+n-m}M} & \text{for } n \geqslant m \\ * \to T^{\ell+n-m}M & \text{for } m-\ell \leqslant n < m \\ * \to * & \text{for } n < m-\ell. \end{cases}$$

We will see in Corollary 7.3.25 below that this map is a stable equivalence. For $\ell = 1$, it is the stabilizing map s_m^M of Definition 7.3.1. For the presymmetric case, see Proposition 7.3.20.

Definition 7.1.30. Yoneda spectra. *Let $(\mathcal{M}, \wedge, S)$ be a symmetric monoidal model category, and let $\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$ be the categrory of Hovey spectra of Definition 7.1.1. The m***th Yoneda spectrum***, is $T^{-m}S$ as in (7.1.21) for $m \geqslant 0$. A* **generalized suspension spectrum** *is one of the form $T^{-m}M$ for an object M in $\mathcal{M}$ and $m \geqslant 0$.*

We have chosen the term "Yoneda spectrum" because of its connection with the Yoneda functor, which will be apparent below in Definition 7.2.52. It is unlikely that Yoneda himself ever considered such a thing, just as Galois is unlikely to have ever thought about Galois cohomology.

Remark 7.1.31. Other ways to generalize a suspension spectrum. *Sometimes it is convenient to consider a spectrum K in which*

$$K_m = \begin{cases} \Sigma^{m-k} K_k & \text{for } m \geqslant k \\ * & \text{otherwise,} \end{cases} \tag{7.1.32}$$

and it is common to call it a suspension spectrum even though it does not conform with Definition 7.1.30. A **finite spectrum** *is usually understood to be one having the stable homotopy type of the above where K_k is a finite CW complex.*

Here are two familiar examples.

(i) As Adams noted in [Ada74b, page 141], there is a difficulty constructing the stable form of the Hopf map $\eta : S^3 \to S^2$. One would like it to be the 2-component of a map $S^1 \wedge S^{-0} \to S^{-0}$, that is a map to the sphere spectrum from its suspension, and then define the higher components to be suspensions of η. The problem with this is that there is no way to define the 0 and 1-components of such a map because η does not desuspend. (A similar problem exists with an even simpler example, the degree d map of the sphere spectrum to itself.)

There are two ways out of this dilemma. The first is to change the target spectrum to its stably fibrant replacement, in which the mth space is

$$\operatorname*{hocolim}_k \Omega^k S^{k+m}.$$

Then one has the desired maps for small m. The second is to modify the domain spectrum $S^1 \wedge S^{-0}$ by replacing its 0 and 1-components (S^1 and S^2 respectively) by a point. This new spectrum supports the desired map to the sphere spectrum and has the form of (7.1.32).

(ii) Fix a prime p and let $V(0)$ (the notation of [Tod71]) denote the mod p Moore spectrum. Its mth component for $m > 0$ is the cofiber of a degree p map in S^m, but there is no such map on S^0. Hence we need to define $V(0)_0$ to be a point as in (7.1.32). Similar problems come up with higher Smith-Toda complexes and with stunted projective spaces having negative dimensional cells.

The following is [Hov01b, Theorem 1.13]. In the presymmetric case (Definition 7.1.13), Theorem 7.2.32 below implies that it is a special case of Theorem 5.6.26.

Proposition 7.1.33. The projective model structure on Hovey spectra. *Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating sets $\mathcal{I}$ and $\mathcal{J}$. In $\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$, let*

$$\begin{aligned} \mathcal{I}_T &= \bigcup_{m\geqslant 0} T^{-m}\mathcal{I} \\ \text{and} \qquad \mathcal{J}_T^{proj} &= \bigcup_{m\geqslant 0} T^{-m}\mathcal{J} \end{aligned}$$

for T^{-m} as in Definition 7.1.20.

Then there is a cofibrantly generated model structure on $\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$ in which the weak equivalences are maps $f : X \to Y$ such that $f_n : X_n \to Y_n$ is a weak equivalence in $\mathcal{M}$ for each n, and the cofibrant generating sets are $\mathcal{I}_T$ and $\mathcal{J}_T^{proj}$. We call it the **projective model structure**, *and will refer to the various special morphisms types in it as* **projective cofibrations** *and so on. If $\mathcal{M}$ is left (right) properas in Definition 5.3.1, so is $\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$.*

Remark 7.1.34. Second warning about a Hirschhorn reference. *In the sentence preceding [Hov01b, Theorem 1.13], Hovey refers to [Hir03, Chapter 11] in reprint form for information about proper model categories. It is [Hir03, Chapter 13] in the published book. Proper model categories are treated here in §5.3 and §5.8F.*

In §7.3 below we will define the stable model structure on $\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$ as a left Bousfield localization of the projective one.

Corollary 7.1.35. Projective trivial fibrations in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$. *Any map*

$$p : X \to Y \qquad \text{in } \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$$

having the right lifting property with respect to $\mathcal{I}_T$ as in Proposition 7.1.33 is a weak equivalence and hence a trivial fibration in the projective model structure of Definition 5.4.2.

The following characterization of projective (trivial) cofibrations in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$ is proved by Hovey as [Hov01b, Proposition 1.14]. The proof makes use of the right adjoint of the evaluation functor Ev_m described in Remark 7.1.24.

Proposition 7.1.36. Projective (trivial) cofibrations of Hovey spectra. *A morphism $f : X \to Y$ in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)$ is a projective (trivial) cofibration iff f_0 is a (trivial) cofibration in $\mathcal{M}$ and for each $n > 0$ the pushout corner map (Definition 2.3.9) λ_n^f for the diagram*

$$\begin{tikzcd} TX_{n-1} \arrow[r, "Tf_{n-1}"] \arrow[d, "\epsilon_{n-1}^X"'] & TY_{n-1} \arrow[d, "\epsilon_{n-1}^Y"] \\ X_n \arrow[r, "f_n"] & Y_n \end{tikzcd}$$

is one as well.

The special case of the above with $X_n = *$ for all n is the following.

Corollary 7.1.37. Cofibrant Hovey spectra. *A Hovey spectrum Y is cofibrant iff for each $n \geqslant 0$, Y_n is cofibrant and the map $\epsilon_n^Y : TY_n \to Y_{n+1}$ is a cofibration.*

Corollary 7.1.38. Cofibrant approximations of Hovey spectra. *A map $f : X \to Y$ of Hovey spectra is a cofibrant approximation as in Definition 4.1.19 iff for each $n \geqslant 0$, X_n is cofibrant, ϵ_n^X is a cofibration, and f_n is a trivial fibration.*

Proposition 7.1.39. Functorial fibrant approximation in the projective model structure. *Let R be a functorial fibrant approximation on $\mathcal{M}$ as in Definition 4.1.25. Then a functorial fibrant approximation $R^{\mathbf{N}}$ in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ with the projective model structure is given by $(R^{\mathbf{N}}X)_n = R(X_n)$.*

Remark 7.1.40. *When $\mathcal{M} = \mathcal{T}$ (or $\mathcal{M} = \mathcal{T}^G$ for a finite group G as in Definition 8.6.1 below), all objects are fibrant, so R and therefore $R^{\mathbf{N}}$ could be the identity functor.*

7.2 The functorial approach to spectra

Spectra were originally defined as sequences of spaces (or objects in a suitable model category $\mathcal{M}$) equipped with certain structure maps, as we have seen.

In the past 20 years it has become apparent that another perspective involving enriched category theory is more convenient. Pioneering papers in this direction included [HSS00], [MMSS01] and [MM02], written by subsets of Mark Hovey, Mike Mandell, Peter May, Brooke Shipley, Stefan Schwede and Jeff Smith, along with [Lyd98] and [DRØ03].

We start with

(i) a model category $\mathcal{M}$ that is stabilizable as in Definition 7.2.1 below, and
(ii) a generalized direct $\mathcal{M}$-category $\mathscr{J}$ as in Definition 5.6.31, to be named later.

Definition 7.2.1. *A model category is* **stabilizable** *if it is a Hirschhorn category (Definition 6.3.2) and a pointed topological Quillen ring as in Definition 5.6.3 that is telescopically closed as in Definition 5.8.28, and compactly generated as in Definition 5.2.6 with the domains of its cofibrant generating morphisms being cofibrant.*

These conditions are met in cases of most interest in this book, the categories of pointed topological spaces, possibly with group action.

The following is a consequence of Theorem 5.6.21.

Proposition 7.2.2. Homotopy groups for stabilizable model categories. *A* **stabilizable** *model category has a complete set of homotopy invariants as in Definition 5.6.5.*

Then **a spectrum X is an enriched $\mathcal{M}$-valued functor on $\mathscr{J}$**. We will denote its value on an object j in $\mathscr{J}$ by X_j. We will consider several different direct $\mathcal{M}$-categories $\mathscr{J}$, and hence get several types of spectra. In each case the functor category $[\mathscr{J}, \mathcal{M}]$ has a projective model structure, derived from that on $\mathcal{M}$, in which every object is fibrant.

This means the enriched category theory of Chapter 3 is applicable. Indeed that chapter was written with this application in mind. In particular, when $\mathscr{J}$ is symmetric monoidal, the Day Convolution Theorem 3.3.5 gives us a closed symmetric monoidal structure on the functor category $[\mathscr{J}, \mathcal{M}]$.

Since $\mathcal{M}$ is cofibrantly generated, the results of §5.4 are also applicable. Thus we get a cofibrantly generated model structure on $[\mathscr{J}, \mathcal{M}]$ in which a morphism $f : X \to Y$ is a weak equivalence or fibration if $f_j : X_j \to Y_j$ is one for each object j in $\mathscr{J}$. This is the **projective model structure**.

Experience has shown that in order to do stable homotopy theory, one wants a weaker notion of weak equivalence that the objectwise condition above. Classically a stable equivalence is a map inducing an isomorphism of stable homotopy groups. **The resulting stable model structure is a Bousfield localization of the projective one.** It will be discussed below in §7.3 and §7.4.

7.2A Indexing categories for spectra

Now we will define our indexing categories $\mathscr{J}$. They come in several different flavors, leading to several different types of spectra.

Remark 7.2.3. The compact cofibrant object K. *Each of our indexing categories is defined in terms of a compact cofibrant object K in a pointed topological Quillen ring $\mathcal{M}$. In every case we will consider, K is a sphere of some positive dimension. Initially the reader would do well to assume that $K = S^1$. In the equivariant case it will be S^{ρ_G} (see §8.9) instead, where ρ_G is the real regular representation of the finite group G.*

Recall that $(\mathcal{M}, \wedge, S)$ is a pointed topological Quillen ring as in Definition 5.5.9, such as some variant of $\mathcal{T}$. Each $\mathscr{J}$ is a direct $\mathcal{M}$-category as in Definition 5.6.31, and we define a spectrum X to be an $\mathcal{M}$-valued $\mathcal{M}$-functor (as in Definition 3.1.13) on $\mathscr{J}$. We will denote its value on an object j in $\mathscr{J}$ by X_j. The category of such functors is denoted by $[\mathscr{J}, \mathcal{M}]$ as in Definition 3.2.18. Later we will use notation similar to that of Definition 7.1.13.

Definition 7.2.4. Some indexing categories. *Let $\mathcal{M}$ be a stabilizable model category as in Definition 7.2.1 with a compact (as in Definition 5.2.6)* *cofibrant object K having the form $S^1 \wedge \overline{K}$. The categories $\mathscr{J}_K^{\mathbf{N}}$, $\mathscr{J}_K^{\mathbf{\Sigma}}$, $\mathscr{J}_K^{\mathbf{O}}$ and $\mathscr{J}_K^{\mathbf{U}}$ each have finite sets $\mathbf{n} = \{1, 2, \ldots n\}$ for $n \geqslant 0$ (with $\mathbf{0}$ being the empty set $\varnothing$) as objects. In each case we will write the composition morphisms as*

$$j_{m,n,p} : \mathscr{J}_K^{\mathbf{F}}(\mathbf{n}, \mathbf{p}) \wedge \mathscr{J}_K^{\mathbf{F}}(\mathbf{m}, \mathbf{n}) \to \mathscr{J}_K^{\mathbf{F}}(\mathbf{m}, \mathbf{p}). \tag{7.2.5}$$

We will refer to $\mathcal{M}$-valued functors on these categories as presymmetric, symmetric, orthogonal and unitary spectra respectively; *see Definition 7.2.33 below. When the subscript K is omitted, it is understood to be S^1 in the first three cases, and S^2 in the unitary case. Their morphism objects, which lie in $\mathcal{M}$, are as follows.*

(i) For the first category $\mathscr{J}_K^{\mathbf{N}}$,

$$\mathscr{J}_K^{\mathbf{N}}(\mathbf{m}, \mathbf{n}) = \begin{cases} * & \text{for } m > n \\ S & \text{for } m = n \\ K^{\wedge(n-m)} & \text{otherwise.} \end{cases}$$

For $m \leqslant n \leqslant p$ the composition morphism in $\mathcal{M}$,

$$j_{m,n,p} : \mathscr{J}_K^{\mathbf{N}}(\mathbf{n}, \mathbf{p}) \wedge \mathscr{J}_K^{\mathbf{N}}(\mathbf{m}, \mathbf{n}) \to \mathscr{J}_K^{\mathbf{N}}(\mathbf{m}, \mathbf{p}),$$

is the standard isomorphism

$$K^{\wedge(p-n)} \wedge K^{\wedge(n-m)} \to K^{\wedge(p-m)}.$$

In particular

$$\mathscr{J}_K^{\mathbf{N}}(\mathbf{0}, \mathbf{n}) = K^{\wedge n}$$

and

$$\mathscr{J}_K^{\mathbf{N}}(\mathbf{m}, \mathbf{m}) = S,$$

the monoidal unit object in $\mathcal{M}$.

(ii) In the second category $\mathscr{J}_K^{\mathbf{\Sigma}}$, morphism objects are defined in terms of the symmetric group on n letters Σ_n. The morphism object $\mathscr{J}_K^{\mathbf{\Sigma}}(\mathbf{m}, \mathbf{n})$ is the coproduct in $\mathcal{M}$ of objects $K^{\wedge(n-m)}$ indexed by the set of inclusions $\mathbf{m} \to \mathbf{n}$, with composition induced by that of inclusions of finite sets. In other words,

$$\mathscr{J}_K^{\mathbf{\Sigma}}(\mathbf{m}, \mathbf{n}) = \Sigma_n \underset{\Sigma_{n-m}}{\ltimes} K^{\wedge(n-m)}.$$

In particular

$$\mathscr{J}_K^{\mathbf{\Sigma}}(\mathbf{0}, \mathbf{n}) = K^{\wedge n}$$

and

$$\mathscr{J}_K^{\mathbf{\Sigma}}(\mathbf{m}, \mathbf{m}) = \Sigma_{m+} \wedge S$$

(the coproduct of $m!$ copies of the unit object S), so $\mathscr{J}_K^{\mathbf{\Sigma}}(\mathbf{m}, \mathbf{n})$ has a

left action of Σ_n *and right action of* Σ_m*. As in* Definition 2.2.32 *and* Definition 3.1.69*, we denote these by*

$$\begin{tikzcd}[column sep=small] \Sigma_{n+} \wedge \mathscr{J}_K^{\mathbf{\Sigma}}(\mathbf{m},\mathbf{n}) \arrow[dr, "\mu_L"'] & & \mathscr{J}_K^{\mathbf{\Sigma}}(\mathbf{m},\mathbf{n}) \wedge \Sigma_{m+} \arrow[dl, "\mu_R"] \\ & \mathscr{J}_K^{\mathbf{\Sigma}}(\mathbf{m},\mathbf{n}). & \end{tikzcd} \tag{7.2.6}$$

The composition map $j_{m,n,p}$ *is a fold map of degree*

$$\begin{aligned} k &= n!\binom{p-m}{p-n} \\ &= (n-m+1)\cdots(n-1)n \\ &\qquad\qquad (p-n+1)(p-n+2)\cdots(p-m), \end{aligned}$$

a product of n *(not necessarily distinct) integers satisfying*

$$n! \leqslant k \leqslant (p-n+1)(p-n+2)\cdots(p-1)p = p!/(p-n)!.$$

Let $\mathscr{J}_K^{\mathbf{\Sigma}+}$ *be the full subcategory of* $\mathscr{J}_K^{\mathbf{\Sigma}}$ *in which the objects are positively indexed, and let* $j^+ : \mathscr{J}_K^{\mathbf{\Sigma}+} \to \mathscr{J}_K^{\mathbf{\Sigma}}$ *denote the inclusion functor.*

(iii) For the third category $\mathscr{J}_K^{\mathbf{O}}$*, the object* K *must be chosen so that the action of the symmetric group* Σ_n *on* $K^{\wedge n}$ *extends to an action of the orthogonal group* $O(n)$*. (The inclusion of* Σ_n *into* $O(n)$ *is via the usual permutation matrices.) We replace inclusions* $\mathbf{m} \to \mathbf{n}$ *for* $m \leqslant n$*, used in the definition of* $\mathscr{J}_K^{\mathbf{\Sigma}}$*, by orthogonal embeddings* $\mathbf{R}^m \to \mathbf{R}^n$*. The space of such embeddings is the Stiefel manifold*

$$O(m,n) := O(n)/O(n-m). \tag{7.2.7}$$

Every such embedding determines an orthogonal complement $\mathbf{R}^{n-m} \subseteq \mathbf{R}^n$*. This defines an orthogonal* $\mathbf{R}^{n-m}$*-bundle over* $O(m,n)$*. Its Thom space is*

$$\begin{aligned} \mathit{Thom}(O(m,n);\mathbf{R}^{n-m}) &\cong O(n) \underset{O_{n-m}}{\ltimes} S^{n-m} \\ &\cong \begin{cases} O(n)_+ & \text{for } n = m \\ SO(n) \underset{SO_{n-m}}{\ltimes} S^{n-m} & \text{for } n > m, \end{cases} \end{aligned}$$

where $SO(n)$ *is the special orthogonal group.*

Composition of orthogonal embeddings

$$\mathbf{R}^m \to \mathbf{R}^n \to \mathbf{R}^p$$

for $m \leqslant n \leqslant p$ *leads to a map*

$$O(m,n) \times O(n,p) \to O(m,p)'$$

which Thomifies to

$$\begin{array}{c} \left(O(n) \underset{O(n-m)}{\ltimes} S^{n-m}\right) \wedge \left(O(p) \underset{O(p-n)}{\ltimes} S^{p-n}\right) \\ \Big\downarrow \\ O(p) \underset{O(p-m)}{\ltimes} S^{p-m}. \end{array} \tag{7.2.8}$$

If we replace the spheres above by the corresponding smash powers of K, we get a map

$$\begin{array}{c} \left(O(n) \underset{O(n-m)}{\ltimes} K^{\wedge(n-m)}\right) \wedge \left(O(p) \underset{O(p-n)}{\ltimes} K^{\wedge(p-n)}\right) \\ \Big\downarrow \\ O(p) \underset{O(p-m)}{\ltimes} K^{\wedge(p-m)}. \end{array} \tag{7.2.9}$$

The morphism objects in $\mathscr{J}_K^{\mathbf{O}}$ are

$$\mathscr{J}_K^{\mathbf{O}}(\mathbf{m},\mathbf{n}) = \begin{cases} * & \text{for } n < m \\ O(n)_+ & \text{for } n = m \\ O(n) \underset{O(n-m)}{\ltimes} K^{\wedge(n-m)} & \text{for } n > m \end{cases} \tag{7.2.10}$$

and composition morphisms

$$\begin{array}{ccc} \mathscr{J}_K^{\mathbf{O}}(\mathbf{n},\mathbf{p}) \wedge \mathscr{J}_K^{\mathbf{O}}(\mathbf{m},\mathbf{n}) & & \\ \| & & \\ \left(O(p) \underset{O(p-n)}{\ltimes} K^{\wedge(p-n)}\right) \wedge \left(O(n) \underset{O(n-m)}{\ltimes} K^{\wedge(n-m)}\right) & & \\ \Big\downarrow & & \\ O(p) \underset{O(p-m)}{\ltimes} K^{\wedge(p-m)} & = & \mathscr{J}_K^{\mathbf{O}}(\mathbf{m},\mathbf{p}) \end{array}$$

for $m \leqslant n \leqslant p$ as in (7.2.9).

(iv) For the fourth category $\mathscr{J}_K^{\mathbf{U}}$, the object K must be chosen so that the action of the symmetric group Σ_n on $K^{\wedge n}$ extends to an action of the unitary group $U(n)$. Its morphism objects are the complex analogs of the ones in the orthogonal case.

We learned the following from Peter May.

Remark 7.2.11. The category $\mathscr{J}_K^{\mathbf{N}}$ is monoidal under addition but not symmetric monoidal. *By Definition 3.1.51 the former means that the image of the Yoneda functor $よ^{\mathbf{0}} : \mathscr{J}_K^{\mathbf{N}} \to \mathcal{M}$ is a monoidal subcategory. Its* *objects are the smash powers of K. If $\mathscr{J}_K^{\mathbf{N}}$ were symmetric, its image in $\mathcal{M}$ would have a twist isomorphism*

$$\tau_{m,n} : K^{\wedge m} \wedge K^{\wedge n} \to K^{\wedge n} \wedge K^{\wedge m}$$

as in (2.6.2). There is such a morphism in $\mathcal{M}$, **but it is not in the image of the Yoneda functor $よ^{\mathbf{0}}$.** *Recall that*

$$\mathscr{J}_K^{\mathbf{N}}(\mathbf{m}+\mathbf{n}, \mathbf{m}+\mathbf{n}) = S,$$

the unit object in the symmetric monoidal model category $\mathcal{M}$. For example, when $\mathcal{M} = \mathcal{T}$, this unit is S^0. The two corresponding endomorphisms of the smash power $K^{\wedge(m+n)}$ are the identity and constant maps. The twisting isomorphism is not among them when $m, n > 0$.

On the other hand we have the following. A proof in the symmetric case is given in [DRØ03, Lemma 2.14], and a similar argument gives the orthogonal case.

Proposition 7.2.12. The categories $\mathscr{J}_K^{\mathbf{\Sigma}}$, $\mathscr{J}_K^{\mathbf{O}}$ and $\mathscr{J}_K^{\mathbf{U}}$ are symmetric monoidal under addition.

Remark 7.2.13. *Each of the categories of Definition 7.2.4 is strictly monoidal as in Definition 2.6.4 since there is precisely one object for each natural number. Therefore we need not take the care noted in Remark 2.6.8. The same goes for the categories of Definition 7.2.19 below.*

Some of the morphism objects in these categories are

$$\begin{aligned} \mathscr{J}_K^{\mathbf{O}}(\mathbf{0}, \mathbf{n}) &\cong K^{\wedge n}, \\ \mathscr{J}_K^{\mathbf{O}}(\mathbf{m}, \mathbf{m}) &\cong O(m) \ltimes S, \\ \text{and } \mathscr{J}_K^{\mathbf{O}}(\mathbf{m}, \mathbf{m}+\mathbf{1}) &\cong SO(m+1) \ltimes K, \end{aligned}$$

where $X \ltimes Y$ is as in Definition 2.1.49. For the last of these, we claim that the Stiefel manifold $O(m+1)/O(1)$ is homeomorphic to the special orthogonal group $SO(m+1)$. An orthogonal embedding $\mathbf{R}^m \to \mathbf{R}^{m+1}$ is a choice of m orthonormal vectors in $\mathbf{R}^{m+1}$, and there is a unique $(m+1)$th unit vector orthogonal to all of them and having the right orientation. The resulting real line bundle over $O(m, m+1)$ is trivial. More examples of these morphism spaces are given in Example 8.9.28 below.

The categories $\mathscr{J}_K^{\mathbf{N}}$ and $\mathscr{J}_K^{\mathbf{\Sigma}}$ are denoted by KSph and $K\mathrm{Sph}^{\Sigma}$ (with T in place of K) in [DRØ03, §2.5 and §2.6]. They prove that the latter is symmetric monoidal in [DRØ03, Lemma 2.14].

We leave the following three results as exercises for the reader.

Proposition 7.2.14. The morphism objects in each of the categories of Definition 7.2.4 are cofibrant and compact.

Proposition 7.2.15. Connectivity of orthogonal and unitary morphism objects. *The Stiefel manifold $O(m+n)/O(n)$ is $(n-1)$-connected, and the space $\mathscr{J}_{S^1}^{\mathbf{O}}(\mathbf{m}, \mathbf{m}+\mathbf{n})$ is a CW complex of the form*

$$O(m+n) \underset{O(n)}{\ltimes} S^n \simeq S^n \cup e^{2n-1} \cup \cdots .$$

More generally, if $K = S^k$ (possibly with higher dimensional cells attached), then

$$\mathscr{J}_K^{\mathbf{O}}(\mathbf{m}, \mathbf{m}+\mathbf{n}) = O(m+n) \underset{O(n)}{\ltimes} K^{\wedge n} \simeq S^{kn} \cup e^{(k+1)n-1} \cup \cdots .$$

Similarly $\mathscr{J}_{S^2}^{\mathbf{U}}(\mathbf{m}, \mathbf{m}+\mathbf{n})$ is a CW complex of the form

$$U(m+n) \underset{U(n)}{\ltimes} S^n \simeq S^{2n} \cup e^{4n-1} \cup \cdots .$$

More generally, if $K = S^{2k}$, then

$$\mathscr{J}_K^{\mathbf{U}}(\mathbf{m}, \mathbf{m}+\mathbf{n}) = U(m+n) \underset{U(n)}{\ltimes} K^{\wedge n} \simeq S^{2kn} \cup e^{2(k+1)n-1} \cup \cdots .$$

Proposition 7.2.16. The space $\mathscr{J}_{S^1}^{\mathbf{O}}(\mathbf{m}, \mathbf{n})$ as a subspace of $\Omega^m S^n$. *For an orthogonal embedding $f : \mathbf{R}^m \to \mathbf{R}^n$ for $m \leqslant n$ and a vector $a \in f(\mathbf{R}^m)^\perp$, the orthogonal complement of the image of f, define an affine map*

$$f_a : \mathbf{R}^m \to \mathbf{R}^n \qquad by \qquad f_a(x) = a + f(x).$$

The space

$$E(m,n) = \left\{(f,a) \in O(n)/O(n-m) \times \mathbf{R}^n : a \in f(\mathbf{R}^m)^\perp\right\}.$$

is the total space of the vector bundle over the Stiefel manifold $O(n)/O(n-m)$ used to define $\mathscr{J}_{S^1}^{\mathbf{O}}(\mathbf{m}, \mathbf{n})$. Thus we have a map from $E(m,n)$ to the space of affine embeddings of $\mathbf{R}^m \to \mathbf{R}^n$. Passing to one point compactifications gives us a map from $\mathscr{J}_{S^1}^{\mathbf{O}}(\mathbf{m}, \mathbf{n})$ to a set of pointed embeddings $S^m \to S^n$, which is a subspace of $\Omega^m S^n = \mathcal{T}(S^m, S^n)$. Thus we get a faithful functor

$$e : \mathscr{J}_{S^1}^{\mathbf{O}} \to \mathcal{T}. \tag{7.2.17}$$

Example 7.2.18. Complexes K whose smash powers support orthogonal and unitary group actions.

(i) The classical example is $K = S^1$ with $\mathcal{M} = \mathcal{T}$. Then $K^{\wedge n} = S^n$, on which $O(n)$ acts via its usual action α_n on $\mathbf{R}^n$, of which S^n is the one point compactification. Similarly for $K = S^2$, $U(n)$ actions on $K^{\wedge n} = S^{2n}$.

(ii) For a finite group G with real regular representation ρ_G, let $K = S^{\rho_G}$ with $\mathcal{M} = \mathcal{T}^G$. Here $\mathcal{T}^G$ denotes the category of pointed G-spaces and equivariant maps. We will discuss it as a model category in §8.6 below. Let α_n denote the standard action of $O(n)$ on $\mathbf{R}^n$ as above.

To define an action of $O(n)$ on $K^{\wedge n} = S^{n\rho_G}$, consider the group $G \times O(n)$. Then $p_1\rho$ and $p_2\alpha_n$ are representations of the group $G \times O(n)$, where

$$G \xleftarrow{p_1} G \times O(n) \xrightarrow{p_2} O(n)$$

p_1 and p_2 are the evident homomorphisms. Their degrees are $|G|$ and n respectively. Hence their tensor product $p_1\rho \otimes p_2\alpha_n$ has degree $n|G|$. Thus the action of G on $S^{n\rho_G}$ extends to an action of $G \times O(n)$ and therefore one of $O(n)$ and hence its subgroup Σ_n.

Another type of indexing category is needed in Chapter 9, where we will consider orthogonal G-spectra for a finite group G, and again in Chapter 12 where we construct the real bordism spectrum $MU_{\mathbf{R}}$. The former are orthogonal spectra for $\mathcal{M} = \mathcal{T}^G$ (the category of pointed G-spaces and equivariant maps as in Definition 8.6.1 below) and $L = S^{\rho_G}$, for which the functor on $\mathscr{J}^{\mathbf{O}}_{S^1}$ extends to a larger symmetric monoidal indexing category $\mathscr{J}_G$ (the Mandell-May category of Definition 8.9.24 below) having **more objects**, namely orthogonal representations of G. This new category is also a direct $\mathcal{M}$-category as in Definition 5.6.31. Two additional examples will be considered in §12.1.

With this in mind we make the following, which is a variant of the notion of a $\mathcal{C}$-algebra in Definition 2.6.20.

Definition 7.2.19. Spectral $\mathscr{J}^{\mathbf{O}}$-algebras. *Let $\mathcal{M}$ be as in Definition 7.2.4 with $K = S^1$, and let $\{\pi_\alpha\}$ be a complete set of homotopy invariants for $\mathcal{M}$ as in Definition 5.6.5. Let L be a compact cofibrant object in $\mathcal{M}$ of the form $S^1 \wedge \overline{L}$. A* **spectral $\mathscr{J}^{\mathbf{O}}$-algebra** *is a strict symmetric monoidal (as in Definition 2.6.4) direct $\mathcal{M}$-category (as in Definition 5.6.31) $(\mathscr{J}^{\mathbf{F}}_L, \oplus, 0)$ receiving a symmetric monoidal $\mathcal{M}$-functor*

$$i^{\mathbf{F}}_{\mathbf{O}} : \mathscr{J}^{\mathbf{O}} \to \mathscr{J}^{\mathbf{F}}_L \tag{7.2.20}$$

as in Definition 2.6.19 such that $|i^{\mathbf{F}}_{\mathbf{O}}(\mathbf{m})| = mg$ for is a fixed positive integer g. We say that $\mathscr{J}^{\mathbf{F}}_L$ has **degree** *g as a spectral $\mathscr{J}^{\mathbf{O}}$-algebra.*

We will abbreviate $i^{\mathbf{F}}_{\mathbf{O}}(\mathbf{n})$ by n whenever it appears as a subscript or as a variable in a morphism object $\mathscr{J}^{\mathbf{F}}_L(V, W)$, and we require that

$$\mathscr{J}^{\mathbf{F}}_L(0, n) \cong L^{\wedge n}.$$

In addition to the functors π_α of Definition 5.6.5, we will need functors $\pi_{\alpha,n}$ for $n \geqslant 0$ given by

$$\pi_{\alpha,n}(-) := \pi_0\mathcal{M}(A_\alpha \wedge L^{\wedge n}, -). \tag{7.2.21}$$

$\mathscr{J}^{\mathbf{F}}_L$ must satisfy the following conditions.

(i) **The compact cofibrance condition.** *Morphism objects are cofibrant and compact, each isomorphism class of objects is a singleton, and*

$$\mathscr{J}^{\mathbf{F}}_L(0, V) \wedge \mathscr{J}^{\mathbf{F}}_L(0, W) \cong \mathscr{J}^{\mathbf{F}}_L(0, V \oplus W).$$

(ii) **The direct summand condition.** *For each object V there is an object V' such that $V \oplus V' \cong \mathbf{n}$ for some $n > 0$.*

(iii) **The stable homotopy condition.** *For objects V, W and U in $\mathscr{J}_L^{\mathbf{F}}$, let $\xi_{V,W,U}$ be the composite*

$$\begin{tikzcd} \mathscr{J}_L^{\mathbf{F}}(V \oplus W, U) \wedge \mathscr{J}_L^{\mathbf{F}}(0, W) \arrow[rr, "\xi_{V,W,U}"] \arrow[dr, "\mathscr{J}_L^{\mathbf{F}}(V\oplus W,U)\wedge\omega^{\mathbf{F}}_{V,0,W}"'] & & \mathscr{J}_L^{\mathbf{F}}(V, U) \\ & \mathscr{J}_L^{\mathbf{F}}(V \oplus W, U) \wedge \mathscr{J}_L^{\mathbf{F}}(V, V \oplus W), \arrow[ur, "j_{V,V+W,U}"'] & \end{tikzcd} \tag{7.2.22}$$

where $j_{V,V+W,U}$ is the composition morphism similar to that of (7.2.5), and $\omega^{\mathbf{F}}_{V,0,W}$ is the addition morphism of Definition 2.6.6. Then for each object V in $\mathscr{J}_L^{\mathbf{F}}$, each positive integer m and each homotopy invariant π_α as in Definition 5.6.5, **the map $\pi_{\alpha,n}\xi_{V,m,n}$ of (7.2.21) is an isomorphism for large n.**

An **ideal** *$\mathscr{L}_L^{\mathbf{F}} \subseteq \mathscr{J}_L^{\mathbf{F}}$ (as in Definition 2.6.9) is a full subcategory such that for objects V in $\mathscr{L}_L^{\mathbf{F}}$ and W in $\mathscr{J}_L^{\mathbf{F}}$, $V \oplus W$ is also in $\mathscr{L}_L^{\mathbf{F}}$. It is* **positive** *if it contains the object $i_{\mathbf{O}}^{\mathbf{F}}(\mathbf{1})$ but not $i_{\mathbf{O}}^{\mathbf{F}}(\mathbf{0})$.*

For an enriched $\mathcal{M}$-valued functor X on $\mathscr{J}_L^{\mathbf{F}}$, we will denote its values and on V and $i_{\mathbf{O}}^{\mathbf{F}}(\mathbf{n})$ by X_V and X_n respectively. We will often write $V \oplus W$ as $V + W$ when it appears as an index. We will denote the enriched $\mathcal{M}$-valued Yoneda functors $よ^V$ and $よ^{i_{\mathbf{O}}^{\mathbf{F}}(\mathbf{n})}$ by S^{-V} and L^{-n} respectively.

Remark 7.2.23. The motivation for Definition 7.2.19. *The example that this definition is designed for is the Mandell-May category $\mathscr{J}_G$ for a finite group G of Definition 8.9.24 below. Its objects are finite dimensional orthogonal representations V of G, which is our reason for using that symbol here. It is a $\mathscr{J}_{S^1}^{\mathbf{O}}$-algebra in which $L = S^\rho$, where $\rho = \rho_G$, the regular representation of G. The underlying Quillen ring $\mathcal{M}$ is $\mathcal{T}^G$ with the Bredon model structure of Definition 8.6.1 below.*

It is known that every irreducible orthogonal representation of G is a direct summand of ρ. It follows that every finite dimensional orthogonal representation V of G is a summand of some multiple of ρ. This means that $\mathscr{J}_G$ satisfies the direct summand condition of (ii), hence the name.

The stable homotopy condition of (iii) excludes the difficulty with symmetric spectra of §7.0E.

Our reason for defining the positive ideals is that we will need to consider the positive Mandell-May category $\mathscr{J}_G^+ \subset \mathscr{J}_G$ of Definition 8.9.24.

The compact cofibrance condition of (i) and the direct summand condition of (ii) are satisfied by all of the examples we will consider in this book. The stable homotopy condition for $\mathscr{J}^{\mathbf{O}}$ itself is not obvious, so we prove it now. A similar condition for $\mathscr{J}_K^{\mathbf{\Sigma}}$ does not hold, as we saw in §7.0E.

Proposition 7.2.24. The stable homotopy condition of Definition 7.2.19 (iii) holds for $\mathscr{J}^{\mathbf{O}}$, *and the map $\xi_{k,m,n}$ is $(2n-2k-m-1)$-connected.*

The stable homotopy condition also holds for $\mathscr{J}^{\mathbf{U}}$, which is a spectral $\mathscr{J}^{\mathbf{O}}$-algebra of degree 2.

Proof In the orthogonal case the composite of (7.2.22) for the map $\xi_{k,m,n}$ is

$$\begin{tikzcd}
\mathscr{J}^{\mathbf{O}}(\mathbf{k}+\mathbf{m},\mathbf{n})\wedge\mathscr{J}^{\mathbf{O}}(0,\mathbf{m}) \arrow[rr, "\xi_{k,m,n}"] \arrow[dr, "\mathscr{J}^{\mathbf{O}}(\mathbf{k}+\mathbf{m},\mathbf{n})\wedge\omega^{\mathbf{F}}_{\mathbf{k},0,\mathbf{m}}"'] & & \mathscr{J}^{\mathbf{O}}(\mathbf{k},\mathbf{n}) \\
& \mathscr{J}^{\mathbf{O}}(\mathbf{k}+\mathbf{m},\mathbf{n})\wedge\mathscr{J}^{\mathbf{O}}(\mathbf{k},\mathbf{k}+\mathbf{m}). \arrow[ur, "j_{k,k+m,n}"'] &
\end{tikzcd}$$

Using Proposition 7.2.15 with $K=S^1$, we can identity the spaces in the diagram above as

$$\begin{tikzcd}
S^{m+r}\cup e^{2(m+r)-1}\cup\cdots \arrow[d, equal] & & \\
(S^r\cup e^{2r-1}\cup\cdots)\wedge S^m \arrow[rr, "\xi_{k,m,n}"] \arrow[dr] & & S^{m+r}\cup e^{2(m+r)-1}\cup\ldots \\
& (S^r\cup e^{2r-1}\cup\cdots)\wedge(S^m\cup e^{2m-1}\cup\cdots), \arrow[ur] &
\end{tikzcd}$$

where $r=n-k-m$, and the composite map is the identity below dimension

$$2n-2k-m=n+(n-2k-m),$$

which can be made to exceed n itself by any given amount for large n. This means that for any compact cofibrant object A_α in $\mathcal{M}$, the map $\pi_{\alpha,n}\xi_{k,m,n}$ of (7.2.21) is an isomorphism large n.

The unitary case is similar. □

Remark 7.2.25. Indexing categories as subcategories of the Quillen ring $\mathcal{M}$. *It is possible, but not necessary for our purposes, to embed each of our indexing categories $\mathscr{J}$ into $\mathcal{M}$ monoidally by sending each object V to $\mathscr{J}(0,V)$. See Proposition 7.2.16 for the orthogonal case, and Definition 8.9.23 below for the case of orthogonal G-spectra.*

This is the approach taken in [DRØ03], where most indexing categories are **full** *subcategories of $\mathcal{M}$. Our subcategories are not full.*

Definition 7.2.26. The indexing group $R\mathbf{F}$ *of a spectral $\mathscr{J}^{\mathbf{O}}$-algebra $\mathscr{J}^{\mathbf{F}}_L$ is the Grothendieck group of its object set, which is an abelian monoid since $\mathscr{J}^{\mathbf{F}}_L$ is symmetric monoidal. We will refer to an element of $R\mathbf{F}$ that is not the image of an object of $\mathscr{J}^{\mathbf{F}}_L$ as a* **virtual representation***.*

We will see later that spectra defined in terms of $\mathscr{J}^{\mathbf{F}}_L$ have homotopy groups graded over $R\mathbf{F}$.

The following is an easy consequence of the direct summand condition of Definition 7.2.19(ii).

Proposition 7.2.27. Virtual representations. *Each element of the indexing group $R\mathbf{F}$ can be written as $V - ni_{\mathbf{\Sigma}}^{\mathbf{F}}(\mathbf{1})$ for an object V in $\mathscr{J}_L^{\mathbf{F}}$ and an integer $n \geqslant 0$.*

Example 7.2.28. Some indexing groups.

(i) For $\mathbf{F} = \mathbf{\Sigma}$ and $\mathbf{F} = \mathbf{O}$, the indexing group $R\mathbf{F}$ is the integers.

(ii) For a finite group G, the object set for the Mandell-May category $\mathscr{J}_G$ (see Definition 8.9.24 below) is the monoid of isomorphism classes of real orthogonal representations of G. Hence the indexing group $R\mathbf{F}$ is the additive group of the real orthogonal representation ring $RO(G)$. In this case $i_{\mathbf{\Sigma}}^{\mathbf{F}}(\mathbf{1})$ is ρ_G, the real regular representation of G.

We will consider certain ideals (as in Definition 7.2.19) in certain spectral $\mathscr{J}^{\mathbf{O}}$-algebras in Chapter 9, specifically Theorem 9.2.13, where we will define the positive stable equifibrant model structure on $\mathcal{S}p^G$, the category of G-spectra. For a finite group G we have the Mandell-May category $\mathscr{J}_G$ of Definition 8.9.24 in which the objects are orthogonal finite dimensional representations V of G and the morphism objects have to do with nonequivariant (as in Definition 3.1.59) orthogonal embeddings. Hence V and W are isomorphic if $|V| = |W|$, meaning they have the same underlying degree. The ideal $\mathscr{L}_G$ of interest is that of representations V with $V^G \neq 0$. It is the principal ideal generated by the one dimensional trivial representation.

The inclusion functor $i : \mathscr{L}_J^{\mathbf{F}} \to \mathscr{J}_L^{\mathbf{F}}$ induces a precomposition functor

$$i^* : [\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}] \to [\mathscr{L}_K^{\mathbf{F}}, \mathcal{M}].$$

It has a left adjoint, namely the left Kan extension

$$i_! : [\mathscr{L}_K^{\mathbf{F}}, \mathcal{M}] \to [\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$$

by Proposition 2.5.4. This leads to an induced model structure on the category of spectra $[\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$ by Theorem 5.4.21.

We could also transfer the projective model structure from $[\mathscr{K}, \mathcal{M}]$ for **any** full subcategory $\mathscr{K}$ of $\mathscr{J}_K^{\mathbf{F}}$, including one with a single object, as explained in Remark 5.4.23. We will see later that requiring $\mathscr{K}$ to be an ideal guarantees that each induced equivalence is also a stable equivalence.

The following notions will be useful.

Definition 7.2.29. Structured spheres and loops. *Let $\mathcal{M}$ and K be as in Definition 7.2.4 and let $\mathscr{J}_L^{\mathbf{F}}$ be a spectral $\mathscr{J}^{\mathbf{O}}$-algebra as in Definition 7.2.19. For each object W in $\mathscr{J}_L^{\mathbf{F}}$, let*

$$S^W := \mathscr{J}_L^{\mathbf{F}}(0, W),$$

generalizing the sphere in the original case. Note that for $W = i_{\mathbf{O}}^{\mathbf{F}}(\mathbf{n})$, this object is $L^{\wedge n}$.

To generalize the loop space, let Ω^W denote the functor $\mathcal{M}(S^W, -)$, the right adjoint of $S^W \wedge -$. For $W = i_{\mathbf{O}}^{\mathbf{F}}(\mathbf{n})$, this functor is also denoted by Ω_L^n.

In the examples we consider in this book, $\mathcal{M}$ is some variant of $\mathcal{T}$, meaning that its objects are pointed topological spaces, possibly with some extra structure. The underlying space of the object L is always a sphere in the usual sense, as are the objects S^W defined above.

Definition 7.2.30. Homotopy groups of objects in $\mathcal{M}$. *With notation as in Definition 7.2.29, let W be an object in $\mathscr{J}_L^{\mathbf{F}}$ and X and object in the pointed topological model category $\mathcal{M}$. Then $\pi_W X$ is the set of path connected components of the pointed space $\mathcal{M}(S^W, X)$.*

Whenever S^W possesses a pinch map $S^W \to S^W \vee S^W$, the set $\pi_W X$ acquires a natural group structure in the usual way. When ever S^W is a suspension, meaning whenever the representation W has the form $1 + \overline{W}$, we have the pinch map $\otimes_{S^{\overline{W}}}$ as in Definition 4.6.22. See Remark 8.9.5 below.

We will use the notation $\mathscr{J}_L^{\mathbf{F}}$ to denote **any** spectral $\mathscr{J}^{\mathbf{O}}$-algebra . Some important examples for this book are the following.

Example 7.2.31. Mandell-May categories. *In each of the following, the morphism objects are cofibrant and compact as in Definition 5.2.6.*

(i) For a finite group G, the **Mandell-May category** *$\mathscr{J}_G$ of Definition 8.9.24 below is a $\mathscr{J}_{S^1}^{\mathbf{O}}$-algebra, where the underlying Quillen module $\mathcal{M}$ is the category of pointed topological G-spaces $\mathcal{T}^G$. The inclusion functor $i_{\mathbf{O}}^{\mathbf{F}}$ sends the nth object to $n\rho_G$ (the n-fold direct sum of the regular representation of G) with the $O(n)$-action of Example 7.2.18(ii).*

(ii) The real and complex Mandell-May categories as in Definition 12.1.2 below.

7.2B Spectra as functors

Our reason for introducing the first of the categories in Definition 7.2.4 is the following, which is comparable to [DRØ03, Proposition 2.12].

Theorem 7.2.32. Presymmetric spectra as $\mathcal{M}$-valued functors on $\mathscr{J}_K^{\mathbf{N}}$. *Let $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K \wedge -)$ be a category of presymmetric spectra as in Definition 7.1.13. Then it is isomorphic to the $\mathcal{M}$-enriched functor category $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$ as in Definition 3.2.18.*

Proof Since $\mathscr{J}_K^{\mathbf{N}}$ has one object for each natural number n, such a functor determines a sequence of objects $\{X_n\}$ in $\mathcal{M}$ with structure maps

$$\mathscr{J}_K^{\mathbf{N}}(\mathbf{m}, \mathbf{n}) \wedge X_m \to X_n \qquad \text{for all } m, n \geqslant 0.$$

On the other hand, an object $X = \{X_n\}$ in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K \wedge -)$ has structure maps

$$T^{n-m}X_m = K^{\wedge(n-m)} \wedge X_n \to X_n$$

and the two structures coincide. More details can be found in the proof of [DRØ03, Proposition 2.12]. □

Since presymmetric spectra can be thought of as $\mathcal{M}$-valued functors on $\mathscr{J}_K^{\mathbf{N}}$, we will define other kinds of spectra as $\mathcal{M}$-valued functors on the other indexing categories. Hence there is a theorem stating that our definition of symmetric spectra is the same as that of [HSS00], proved by Dundas-Röndigs-Østvær as [DRØ03, Proposition 2.15].

Definition 7.2.33. Spectra as functors. *Let $\mathcal{M}$ and K be as in Definition 7.2.4. Then $\mathcal{M}$-valued $\mathcal{M}$-enriched functors on the categories $\mathscr{J}_K^{\mathbf{N}}$, $\mathscr{J}_K^{\mathbf{\Sigma}}$, $\mathscr{J}^{\mathbf{O}}$ and a spectral $\mathscr{J}^{\mathbf{O}}$-algebra $\mathscr{J}_L^{\mathbf{F}}$ (as in Definition 7.2.19) are* **presymmetric, symmetric, orthogonal, unitary and extraorthogonal spectra** *respectively. We will sometimes denote these functor categories by*

$$\begin{aligned} \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K) &= [\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}], \\ \mathcal{S}p^{\mathbf{\Sigma}}(\mathcal{M}, K) &= [\mathscr{J}_K^{\mathbf{\Sigma}}, \mathcal{M}], \\ \mathcal{S}p^{\mathbf{O}}(\mathcal{M}, K) &= [\mathscr{J}_K^{\mathbf{O}}, \mathcal{M}], \\ \mathcal{S}p^{\mathbf{U}}(\mathcal{M}, K) &= [\mathscr{J}_K^{\mathbf{U}}, \mathcal{M}], \\ \text{and} \qquad \mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L) &= [\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]. \end{aligned}$$

We will refer to orthogonal, unitary and extraorthogonal spectra collectively as **smashable spectra***. They are ones for which the indexing category is a spectral $\mathscr{J}^{\mathbf{O}}$-algebras,*

We are using the term **smashable** because the spectra in question have a nice smash product given by the Day convolution in Theorem 7.2.60. The same goes for symmetric spectra, but we are excluding them due to the difficulties of §7.0E. Many of the statements we will make about smashable spectra also hold in the symmetric case. Exceptions include those which rely in some way on Definition 7.2.19(iii), which fails for $\mathscr{J}_K^{\mathbf{\Sigma}}$.

Remark 7.2.34. Converting an ordinary spectrum to a smashable one. *Our indexing categories are related by inclusion functors,*

$$\begin{tikzcd} & & & \mathscr{J}_K^{\mathbf{U}} \\ \mathscr{J}_K^{\mathbf{N}} \arrow[r, "i_{\mathbf{N}}^{\mathbf{\Sigma}}"] & \mathscr{J}_K^{\mathbf{\Sigma}} \arrow[r, "i_{\mathbf{\Sigma}}^{\mathbf{O}}"] & \mathscr{J}_K^{\mathbf{O}} \arrow[ur, "i_{\mathbf{O}}^{\mathbf{U}}"] \arrow[dr, "i_{\mathbf{O}}^{\mathbf{F}}"'] & \\ & & & \mathscr{J}_L^{\mathbf{F}} \end{tikzcd}$$

Each of them induces a precomposition functor between the corresponding categories of spectra. Each precomposition functor has a left adjoint given by *left Kan extension. Thus we get a diagram*

$$
\begin{tikzcd}
& & & \mathcal{S}p^{\mathbf{U}}(\mathcal{M},K) \\
\mathcal{S}p^{\mathbf{N}}(\mathcal{M},K) \arrow[r, "(i_{\mathbf{N}}^{\mathbf{\Sigma}})_!"] & \mathcal{S}p^{\mathbf{\Sigma}}(\mathcal{M},K) \arrow[r, "(i_{\mathbf{\Sigma}}^{\mathbf{O}})_!"] & \mathcal{S}p^{\mathbf{O}}(\mathcal{M},K) \arrow[ur, "(i_{\mathbf{O}}^{\mathbf{U}})_!"] \arrow[dr, "(i_{\mathbf{O}}^{\mathbf{F}})_!"'] & \\
& & & \mathcal{S}p^{\mathbf{F}}(\mathcal{M},K).
\end{tikzcd}
\tag{7.2.35}
$$

An ordinary or presymmetric spectrum is an object in the functor category on the left. It can be converted to a symmetric, orthogonal or extraorthogonal one by applying the appropriate functor.

The diagram of (7.2.35) is similar in spirit to the Main Diagram of [MMSS01, page 442]. Their prolongation functors $\mathbb{P}$ are our left Kan extensions, and their forgetful functors $\mathbb{U}$ are our precomposition functors.

Our presymmetric spectra for $\mathcal{M} = \mathcal{T}$ and $K = S^1$ coincides with the prespectra of [MMSS01, Example 4.1]. Our symmetric and orthogonal spectra in this case are the same as those of [MMSS01, Examples 4.2 and 4.4].

Such a functor X is a collection of objects X_n in $\mathcal{M}$ with structure maps

$$
\begin{aligned}
\epsilon_{n,k}^X &: \mathscr{J}_K^{\mathbf{F}}(\mathbf{n}, \mathbf{n}+\mathbf{k}) \wedge X_n \to X_{n+k}, \\
\text{or} \quad \epsilon_{V,W}^X &: \mathscr{J}_L^{\mathbf{F}}(V, V+W) \wedge X_V \to X_{V+W} \\
&\quad \text{in the extraorthogonal case.}
\end{aligned}
\tag{7.2.36}
$$

We abbreviate $\epsilon_{n,1}^X$ by ϵ_n^X as in (7.1.3), and **we will often abbreviate** $\mathscr{J}_K^{\mathbf{F}}(\mathbf{n}, \mathbf{n}+\mathbf{k})$ $(\mathscr{J}_L^{\mathbf{F}}(V, V+W))$ **by** $J_{n,k}^{\mathbf{F}}$ $(J_{V,W}^{\mathbf{F}})$. For $k = 0$ this structure map amounts to an action of X_n by the group Σ_n in the symmetric case and by $O(n)$ in the orthogonal case. In the superorthogonal case we have an action of $O(n)$ and additional structure maps related to the additional objects in the indexing category.

Remark 7.2.37. Change of notation. *This notation differs from that of (3.2.27), where we wrote $\epsilon_{V,V'}^X$ (with $V' = V \oplus W$) instead of $\epsilon_{V,W}^X$. There the source category was not assumed to have a monoidal structure. See Remark 3.2.31.*

Definition 7.2.38. The restricted and reduced structure maps for a smashable spectrum. *Let X be a smashable spectrum, meaning and $\mathcal{M}$-valued functor on a spectral $\mathscr{J}^{\mathbf{O}}$-algebra $\mathscr{J}_L^{\mathbf{F}}$ as in Definition 7.2.19. The* **restricted structure map**

$$\bar{\epsilon}_V^X := \epsilon_V^X(\omega_{V,0,1}^{\mathbf{F}} \wedge X_V) : L \wedge X_V \to X_{V+1},$$

where $\omega_{V,0,1}^{\mathbf{F}}$ is the map of Definition 2.6.6. More generally let

$$\bar{\epsilon}_{V,W}^X := \epsilon_{V,W}^X(\omega_{V,0,W}^{\mathbf{F}} \wedge X_V) : S^W \wedge X_V \to X_{V+W}, \tag{7.2.39}$$

where S^W is as in Definition 7.2.29.

The **the reduced structure map**

$$\widetilde{\epsilon}^X_{V,W} : \mathscr{J}^{\mathbf{F}}_L(V, V+W) \underset{\mathscr{J}^{\mathbf{F}}_L(V,V)}{\wedge} X_V \to X_{V+W} \tag{7.2.40}$$

is that of (3.2.30).

Remark 7.2.41. Smashable and symmetric spectra have more structure then presymmetric spectra. *For $k = 1$, our structure map is*

$$\epsilon^X_n : \mathscr{J}^{\mathbf{F}}_K(\mathbf{n}, \mathbf{n+1}) \wedge X_n \to X_{n+1},$$

and the object $\mathscr{J}^{\mathbf{F}}_K(\mathbf{n}, \mathbf{n+1})$ varies with n, unlike the presymmetric case. In the symmetric case it is the wedge of $(n+1)!$ copies of K. In the orthogonal case it is

$$O(n+1) \underset{O(1)}{\ltimes} K \cong \Sigma SO(n+1) \ltimes K$$

by (7.2.10).

Thus in the symmetric case we have $(n+1)!$ maps $K \wedge X_n \to X_{n+1}$ compatible with the actions of Σ_n and Σ_{n+1} on X_n and X_{n+1}. In the orthogonal case we have an infinite family of them parametrized by $O(n+1)$. It is this **additional structure** *that enables us to define the smash product of spectra in a more elegant way than Adams was able to in [Ada74b, Part III].*

Definition 7.2.42. The costructure map for smashable, symmetric and presymmetric spectra. *Let $J^{\mathbf{F}}_{V,W} = \mathscr{J}^{\mathbf{F}}_K(V, V+W)$ for any of the* **F** *in Definition 7.2.33. The costructure map*

$$\eta^X_{V,W} : X_V \to \mathcal{M}(J^{\mathbf{F}}_{V,W}, X_{V+W}) \tag{7.2.43}$$

is the right adjoint of the structure map $\epsilon^X_{V,W} : J^{\mathbf{F}}_{V,W} \wedge X_V \to X_{V+W}$.

For any of the spectra of Definition 7.2.33, let the **restricted costructure map**

$$\overline{\eta}^X_{V,W} : X_V \to \Omega^W X_{V+W} = \mathcal{M}(S^W, X_{V+W}) \tag{7.2.44}$$

be the adjoint of the map $\overline{\epsilon}^X_{V,W}$ of Definition 7.2.38. We denote $\overline{\eta}^X_{V,1}$ by simply $\overline{\eta}^X_V$. In particular $\overline{\eta}^X_n$ is **not** *the right adjoint of ϵ^X_n except in the presymmetric case.*

An alternate description of the restricted costructure map will be given below in Lemma 7.4.32.

Definition 7.2.45. *A* **smashable, symmetric or presymmetric Ω-spectrum** *X is one for which the map $\overline{\eta}^X_{V,W}$ of (7.2.44) is a weak equivalence for all V and W.*

The direct summand of Definition 7.2.19(ii) (which also holds for $\mathscr{J}^{\mathbf{N}}_K$ and $\mathscr{J}^{\mathbf{\Sigma}}_K$) gives us a simpler condition for a smashable, symmetric or presymmetric spectrum to be an Ω-spectrum.

Proposition 7.2.46. A recognition criterion for Ω-spectra. *A smashable, symmetric or presymmetric spectrum X is an Ω-spectrum if the map $\overline{\eta}^X_{V,1}$ is a weak equivalence for all V.*

Proof Note that $\overline{\eta}^X_{V,n}$ for $n > 1$ is the n-fold composite

$$X_V \xrightarrow{\overline{\eta}^X_V} \Omega X_{V+1} \xrightarrow{\Omega\overline{\eta}^X_{V+1}} \cdots \xrightarrow{\Omega^{n-1}\overline{\eta}^X_{V+n-1}} \Omega^n X_{V+n},$$

so it is a weak equivalence if $\overline{\eta}^X_V$ is one for all V. For given objects V and W, choose an object W' such that $S^W \wedge S^{W'} \cong K^{\wedge n}$ for some $n > 0$. Here we are using the direct summand condition of Definition 7.2.19(ii). Now consider the diagram

$$\begin{tikzcd}
& X_V \arrow[dl, "\overline{\eta}^X_{V,W}"'] \arrow[dr, "\overline{\eta}^X_{V,n}", "\simeq"'] & \\
\Omega^W X_{V+W} \arrow[rr, "\Omega^W\overline{\eta}^X_{V+W,W'}"] \arrow[dr, "\Omega^W\overline{\eta}^X_{V+W,n}"', "\simeq"] & & \Omega^n X_{V+n} \arrow[dl, "\Omega^n\overline{\eta}^X_{V+n,W}"] \\
& \Omega^W\Omega^n X_{V+W+n} \cong \Omega^n\Omega^W X_{V+W+n}. &
\end{tikzcd} \tag{7.2.47}$$

The 2-of-6 property (see Definition 5.1.1 and Proposition 5.1.2) implies that each morphism in the diagram, including $\overline{\eta}^X_{V,W}$, is a weak equivalence. □

7.2C Morphism objects of spectra, Yoneda spectra, tautological presentations, the Day convolution and function spectra

Since we are dealing with enriched functor categories as in Definition 3.2.18, we have the following.

Proposition 7.2.48. Morphism objects as enriched ends. *Let $\mathscr{J}$ be one of the categories $\mathscr{J}^{\mathbf{N}}_K$, $\mathscr{J}^{\mathbf{\Sigma}}_K$, $\mathscr{J}^{\mathbf{O}}_K$ or $\mathscr{J}^{\mathbf{U}}_K$ of Definition 7.2.4. Then each morphism object in $[\mathscr{J}, \mathcal{M}]$ is the enriched end*

$$[\mathscr{J}, \mathcal{M}](X, Y) = \int^{\mathbf{n}\in ob\,\mathscr{J}} \mathcal{M}(X_n, Y_n).$$

In the extraorthogonal case we have

$$[\mathscr{J}^{\mathbf{F}}_L, \mathcal{M}](X, Y) = \int^{V\in ob\,\mathscr{J}^{\mathbf{F}}_L} \mathcal{M}(X_V, Y_V).$$

The morphism objects described in Proposition 7.2.48 are in $\mathcal{M}$.

We can generalize the tensor and cotensors of Proposition 7.1.14 as follows.

Proposition 7.2.49. Tensors and cotensors of smashable, symmetric or presymmetric spectra. *Let X be a spectrum in any of the categories of Definition 7.2.33, which we denote here by $\mathcal{S}p$. For an object M in $\mathcal{M}$, its tensor and cotensor products with X are given by*

$$(M \wedge X)_V = M \wedge X_V \qquad \text{and} \qquad (X^M)_V = (X_V)^M = \mathcal{M}(M, X_V).$$

The structure map for $M \wedge X$ is the composite

$$\begin{tikzcd} J_{V,W} \wedge M \wedge X_V \arrow[rr, "\epsilon^{M \wedge X}_{V,W}"] \arrow[dr, "t \wedge X_V"'] & & M \wedge X_{V+W} \\ & M \wedge J_{V,W} \wedge X_V, \arrow[ur, "M \wedge \epsilon^X_{V,W}"'] & \end{tikzcd}$$

where t swaps the factors M and $J_{V,W}$ and $J_{V,W} = \mathscr{J}^{\mathbf{F}}_K(V, V+W)$.

The structure map for X^M is the left adjoint of its costructure map, which is the composite

$$\begin{tikzcd} \mathcal{M}(M, X_V) \arrow[rr, "\eta^{(X^M)}_{V,W}"] \arrow[d, "(\eta^X_{V,W})_*"'] & & \mathcal{M}(J_{V,W}, \mathcal{M}(M, X_{V+W})) \\ \mathcal{M}(M, \mathcal{M}(J_{V,W}, X_{V+W})) \arrow[dr, "\cong"'] & & \mathcal{M}(J_{V,W} \wedge M, X_{V+W}) \arrow[u, "\cong"'] \\ & \mathcal{M}(M \wedge J_{V,W}, X_{V+W}). \arrow[ur, "t^*"'] & \end{tikzcd}$$

For spectra X and Y there is an adjunction isomorphism

$$\mathcal{S}p(M \wedge X, Y) \cong \mathcal{S}p(X, Y^M). \tag{7.2.50}$$

Proposition 7.2.51. For a cofibrant object A in a Quillen ring $\mathcal{M}$, if a smashable, symmetric or presymmetric spectrum X is a Ω_K-spectrum, so is X^A as in Proposition 7.2.49.

Proof By the recognition criterion of Proposition 7.2.46, it suffices to show that for each object V in the indexing category, the map

$$(X_V)^A \xrightarrow{\overline{\eta}^{(X^A)}_V} \Omega_K((X_{V+1})^A)$$

is a weak equivalence. Note that by definition

$$\begin{aligned} \Omega_K((X_{V+1})^A) &= \mathcal{M}(K, (X_{V+1})^A) = \mathcal{M}(K, \mathcal{M}(A, X_{V+1})) \\ &\cong \mathcal{M}(A \wedge K, X_{V+1}) \cong \mathcal{M}(A, \mathcal{M}(K, X_{V+1})) \\ &= \mathcal{M}(A, \Omega_K X_{V+1}) = (\Omega_K X_{V+1})^A, \end{aligned}$$

$$\text{so} \qquad \overline{\eta}^{(X^A)}_V \cong (\overline{\eta}^X_V)^A.$$

By Corollary 5.6.19 the functor $(-)^A$ is homotopical on fibrant objects in $\mathcal{M}$. Since $\overline{\eta}_V^X$ is a weak equivalence of fibrant objects, the same is true of its image under $(-)^A$. This makes $\overline{\eta}_V^{(X^A)}$ a weak equivalence as required. □

The following is a generalization of Definition 7.1.30.

Definition 7.2.52. More Yoneda spectra. *Let $\mathscr{J}$ be one of the categories $\mathscr{J}_K^{\mathbf{N}}$, $\mathscr{J}_K^{\mathbf{\Sigma}}$, $\mathscr{J}_K^{\mathbf{O}}$ or $\mathscr{J}_K^{\mathbf{U}}$ of Definition 7.2.4. For $m \geqslant 0$ the Yoneda spectrum S^{-m} in $[\mathscr{J}, \mathcal{M}]$ is given by*

$$\left(S^{-m}\right)_n = \mathscr{J}(\mathbf{m}, \mathbf{n}).$$

In particular for each such $\mathscr{J}$,

$$\left(S^{-0}\right)_n = K^{\wedge n}.$$

When necessary we will denote them by $S_{\mathbf{N}}^{-m}$, $S_{\mathbf{\Sigma}}^{-m}$, $S_{\mathbf{O}}^{-m}$, and $S_{\mathbf{U}}^{-m}$.

For $\mathscr{J}_L^{\mathbf{F}}$ as in Definition 7.2.19 and an object V in $\mathscr{J}_L^{\mathbf{F}}$, the Yoneda spectrum S^{-V} (or $S_{\mathbf{F}}^{-V}$) in $[\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$ is given by

$$\left(S^{-V}\right)_W = \mathscr{J}_L^{\mathbf{F}}(V, W).$$

The composition map

$$j_{U,V,W} : \mathscr{J}_L^{\mathbf{F}}(V, W) \wedge \mathscr{J}_L^{\mathbf{F}}(U, V) \to \mathscr{J}_L^{\mathbf{F}}(U, W)$$

is the Wth component of a map

$$j_{U,V} : S^{-V} \wedge \mathscr{J}_L^{\mathbf{F}}(U, V) \to S^{-U}. \tag{7.2.53}$$

We will write

$$L_{\mathbf{F}}^{-m} := S_{\mathbf{F}}^{-i_{\mathbf{O}}^{\mathbf{F}}(\mathbf{m})}. \tag{7.2.54}$$

A **generalized suspension spectrum** *is one of the form $M \wedge S_{\mathbf{F}}^{-V}$ for a cofibrant object M in $\mathcal{M}$, where $\mathbf{F}$ any one of $\mathbf{N}$, $\mathbf{\Sigma}$, $\mathbf{O}$, $\mathbf{U}$ or $\mathbf{F}$.*

Here is another analog (like Proposition 7.1.23) of Proposition 5.6.28 with a similar proof.

Proposition 7.2.55. The Yoneda adjunction for smashable, symmetric or presymmetric spectra. *With notation as above, for each V the adjunction $S^{-V} \dashv \mathrm{Ev}_V$ is a Quillen adjunction. In particular, if A is a cofibrant object in $\mathcal{M}$, then $K^{-m}A$ is projectively cofibrant in $[\mathscr{J}, \mathcal{M}]$.*

In the presymmetric case, we have

$$(A \wedge K^{-m})_n = \begin{cases} * & \text{for } n < m \\ A \wedge K^{\wedge(n-m)} & \text{otherwise,} \end{cases} \tag{7.2.56}$$

and the structure map $\epsilon_n^{A \wedge K^{-m}}$ is an isomorphism for $n \geqslant m$. In particular this spectrum is cofibrant by Corollary 7.1.37

The following is an application of Proposition 3.2.33.

Proposition 7.2.57. The tautological presentation of a smashable, symmetric or presymmetric spectrum. *Let $\mathscr{J}$ be one of the categories $\mathscr{J}_K^{\mathbf{N}}$, $\mathscr{J}_K^{\mathbf{\Sigma}}$, $\mathscr{J}_K^{\mathbf{O}}$ or $\mathscr{J}_K^{\mathbf{U}}$ of Definition 7.2.4. Then each spectrum X in $[\mathscr{J}, \mathcal{M}]$ is isomorphic to the enriched coend*

$$\int_{\mathscr{J}} X_m \wedge K^{-m}$$

and each Y in $[\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$ is isomorphic to

$$\int_{\mathscr{J}_L^{\mathbf{F}}} X_V \wedge S^{-V}.$$

In both cases we are using the fact that the functor category is tensored over $\mathcal{M}$ on the left, so

$$\left(X_m \wedge K^{-m}\right)_n = X_m \wedge \left(K^{-m}\right)_n \qquad \text{and} \qquad \left(Y_V \wedge S^{-V}\right)_W = Y_V \wedge \left(S^{-V}\right)_W .$$

Since the indexing categories other than $\mathscr{J}_K^{\mathbf{N}}$ are symmetric monoidal by Proposition 7.2.12, the Day Convolution Theorem 3.3.5 applies in those cases. Recall that the smash product $X \wedge Y$ is the left Kan extension of the composite functor $\wedge(X \times Y)$ along $\oplus$ in the diagram

$$\begin{tikzcd}
\mathscr{J} \times \mathscr{J} \arrow[r, "X \times Y"] \arrow[dr, "\oplus"'] & \mathcal{M} \times \mathcal{M} \arrow[r, "\wedge"] & \mathcal{M} \\
& \mathscr{J}. \arrow[ur, dashed, "X \wedge Y = Lan_{\oplus}(\wedge(X \times Y))"'] &
\end{tikzcd} \tag{7.2.58}$$

Equivalently its nth component is the coend

$$(X \wedge Y)_n = \int^{\mathscr{J} \times \mathscr{J}} \mathscr{J}(\mathbf{a} \oplus \mathbf{b}, \mathbf{n}) \wedge X_a \wedge Y_b. \tag{7.2.59}$$

In this setting the Day Convolution Theorem 3.3.5 reads as follows. Its application to stable homotopy theory was first observed by Jeff Smith.

Theorem 7.2.60. Day convolution for smashable or symmetric spectra. *With notation as in Definition 7.2.33, let $\mathscr{J}$ be one of the indexing categories other than $\mathscr{J}_K^{\mathbf{N}}$. Then the binary operation of (7.2.58) and (7.2.59) gives the functor category $[\mathscr{J}, \mathcal{M}]$ a closed symmetric monodial structure in which the unit element is the Yoneda spectrum K^{-0} as in Definition 7.2.52. The internal Hom functor (Definition 2.6.33) $\underline{[\mathscr{J}, \mathcal{M}]}(X, -)$ is the right adjoint of the functor $X \wedge (-)$. We will sometimes refer to $\underline{[\mathscr{J}, \mathcal{M}]}(X, Y)$ as the* **function spectrum** *$F(X, Y)$.*

Classically the existence of the function spectrum $F(X, Y)$ was proved using the Brown Representability Theorem of [Bro62], and it was only defined up to weak equivalence. Now we have an explicit description of it as a special case of Proposition 3.3.7.

Proposition 7.2.61. The function spectrum as an end. *With notation as in Theorem 7.2.60, for each object W in $\mathscr{J}$,*

$$F(X,Y)_W \cong \int^{V\in\mathscr{J}} \mathcal{M}(X_V, Y_{V+W}) \cong [\mathscr{J},\mathcal{M}](S^{-W}\wedge X, Y).$$

The structure map

$$\epsilon_{W,U}^{F(X,Y)} : \mathscr{J}(W, W\oplus U)\wedge F(X,Y)_W \to F(X,Y)_{W+U}$$

has a description similar to that of (3.3.10).

The following is a special case of Proposition 3.3.15 and is proved as [MMSS01, Proposition 22.1]. It applies to some well known Thom spectra discussed below in §9.1G and in Chapter 12.

Proposition 7.2.62. Lax symmetric monoidal functors and commutative ring spectra. *The category of lax (symmetric) monoidal functors $\mathscr{J}\to\mathcal{M}$ is the category (symmetric) monoid objects in $[\mathscr{J},\mathcal{M}]$.*

7.2D Properties of Yoneda spectra

In this subsection $\mathcal{S}p$ will denote any category of smashable or symmetric spectra $[\mathscr{J},\mathcal{M}]$ as in Definition 7.2.33. The monoidal unit in $\mathcal{M}$ will be denoted by S^0.

Let

$$\xi_{V,W} : S^W \wedge S^{-V\oplus W} \to S^{-V} \tag{7.2.63}$$

(where S^W is as in Definition 7.2.29 and $S^{-V\oplus W}$ is as in Definition 7.2.19(ii)) be the map whose Uth component is the map $\xi_{V,W,U}t$ of (7.2.22), where t swaps the two factors of the domain. In particular, $\xi_{V,0}$ is the identity map on S^{-V}. Smashing both sides of (7.2.63) on the left with S^V gives us a map

$$S^V\wedge\xi_{V,W} : S^{V\oplus W}\wedge S^{-V\oplus W}\to S^V\wedge S^{-V}. \tag{7.2.64}$$

The following is a special case of Proposition 3.2.39.

Proposition 7.2.65. S^{-V} represents the Vth object functor. *For each object V of $\mathscr{J}$, the functor $\mathcal{M}\to\mathcal{S}p$ given by $X\mapsto S^{-V}\wedge X$ is the left adjoint of the evaluation functor $\mathrm{Ev}_V : \mathcal{S}p\to\mathcal{M}$ given by $E\mapsto E_V$. Hence for every spectrum E, and every object X of $\mathcal{M}$,*

$$\mathcal{S}p(S^{-V}\wedge X, E) = \mathcal{M}(X, E_V). \tag{7.2.66}$$

For $X = S^0$ this reads

$$\mathcal{S}p(S^{-V}, E) = E_V.$$

Thus we have a Yoneda adjunction as in Remark 2.2.35,

$$S^{-V} \wedge - : \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}p : \mathrm{Ev}_V$$

In particular,

$$\begin{aligned} \mathcal{S}p(S^{-V}, E) &= E_V, && \text{the case of (7.2.66) where } X = S^0 \\ \mathcal{S}p(S^{-0} \wedge X, E) &= \mathcal{M}(X, E_0), && \text{the case } V = 0 \\ &= \mathcal{M}(X, \Omega_K^\infty E) \end{aligned}$$

where the 0th object functor Ω_K^∞ (Ω^∞ in the original case) sends a spectrum E to the object E_0. We will denote the functor $X \mapsto X \wedge S^{-0}$, by Σ^∞ as in the original case. For an object X we have

$$\mathcal{S}p(\Sigma^\infty X, E) = \mathcal{S}p(S^{-0} \wedge X, E) = \mathcal{M}(X, \Omega_T^\infty E),$$

so the functors $\Sigma^\infty : \mathcal{M} \to \mathcal{S}p$ and $\Omega_T^\infty : \mathcal{S}p \to \mathcal{M}$ are adjoint.

Corollary 7.2.67. Rigidity of Yoneda spectra. *For an object X,*

$$\mathcal{S}p(S^{-V}, S^{-W} \wedge X) \cong \mathscr{J}(W, V) \wedge X,$$

so there are no nontrivial maps $S^{-V} \to S^{-W} \wedge X$ (meaning that the morphism object is a point) when $\dim W > \dim V$. In particular there is no nontrivial map $S^{-0} \to S^{-W} \wedge X$ for $\dim W > 0$.

Now consider the spectrum $S^V \wedge S^{-V}$, which is given by

$$(S^V \wedge S^{-V})_W = (S^{-V})_W \wedge S^V = \mathscr{J}(V, W) \wedge S^V.$$

This is the source of the structure map $\epsilon_{V,W}^{S^{-0}}$ of (7.2.36) to $S^W = (S^{-0})_W$, so we have a map of spectra

$$s_V : S^V \wedge S^{-V} \to S^{-0}, \tag{7.2.68}$$

which we call the **stabilizing map**. It is the map $\xi_{0,V}$ of (7.4.5). For each $n > 0$ we have the map

$$S^{(n+1)V} \wedge S^{-(n+1)V} \xrightarrow{S^V \wedge \xi_{nV,V}} S^{nV} \wedge S^{-nV} \tag{7.2.69}$$

as in (7.2.64).

Example 7.2.70. A curious limit of spectra. *We can use the maps of (7.2.69) to form a diagram*

$$S^{-0} \longleftarrow S^V \wedge S^{-V} \longleftarrow S^{2V} \wedge S^{-2V} \longleftarrow \cdots$$

in which each map is a stable equivalence. The map to $s^{nV} \wedge S^{-nV}$ *is* $S^V \wedge \xi_{nV,V}$ *as in (7.2.64).*

The limit of the diagram may be computed objectwise. Since

$$\begin{aligned}(S^{(n+1)V} \wedge S^{-(n+1)V})_{nV} &\cong S^{(n+1)V} \wedge (S^{-(n+1)V})_{nV} \\ &\cong S^{(n+1)V} \wedge \mathscr{J}((n+1)V, nV) = *\end{aligned}$$

for each $n \geqslant 0$*, we have*

$$\lim_n S^{nV} \wedge S^{-nV} = *,$$

despite the fact that each spectrum in the diagram is equivalent to the sphere spectrum and each map is a stable equivalence. The corresponding homotopy limitis also contracible.

The reason this odd behavior is possible is that the spectra in question are **not stably fibrant**. *Recall Theorem 5.8.10 says that a homotopy limit of weak equivalences between fibrant objects is a weak equivalence, but these weak equivalences are not between fibrant objects.*

7.3 Stabilization and model structures for Hovey spectra

In this section we will discuss the passage from the projective model structure on a category of Hovey spectra to the stable one as a form of Bousfield localization. We will do the same for smashable spectra in §7.4. As explained at the beginning of Chapter 6, there are two approaches to this construction: redefining the class of weak equivalences by adding some new morphisms to it, and redefining the fibrant replacement functor.

These approaches are the subjects of §7.3A and §7.3C. In the former we specify a certain countable collection of morphisms that we call **stabilizing maps** in Definition 7.3.1. In the original case these were described informally in §7.0D. The fibrant replacement functor in the original case was described briefly in Remark 7.0.7(i). It is a special case of the functor Θ^∞ of Definition 5.7.3. The relation between stable equivalence and fibrant replacement is the subject of Theorem 7.3.23.

In §7.3D we discuss cofibrant generating sets for the stable model structure on the category of Hovey spectra. It has the same cofibrations as, but more trivial cofibrations than the projective model structure. The main result is Theorem 7.3.36, **the first corner map theorem**. We call it that because the cofibrant generating set for the stable model structure is obtained from the one for the projective model staructure by adjoining certain corner maps (as in Definition 2.3.9), the pushout products (as in Definition 2.6.12) of the stabilizing maps of Definition 7.3.1 and the generating cofibrations of the ground category $\mathcal{M}$. A similar result for symmetric spectra is [HSS00, Corollary 3.4.1].

In §7.3E we show that that the category of original spectra is exactly stable as in Definition 5.7.3. In §7.3F we will generalize this to Hovey spectra. This will enable us to apply Theorem 5.7.6 and Theorem 5.7.11 to get the expected long exact sequences of homotopy groups.

7.3A The stabilizing maps

Definition 7.3.1. The stabilizing maps s_m^M. *For $\mathcal{M}$ and T as in Definition 7.1.1, let M be an object in $\mathcal{M}$ and $m \geqslant 0$ an integer. The map in the category of spectra $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$*

$$s_m^M : T^{-(m+1)}(TM) \to T^{-m}M$$

is the left adjoint (under the adjunction $T^{-m-1} \dashv \mathrm{Ev}_{m+1}$ as in Definition 7.1.20) of the identity map on the object TM in $\mathcal{M}$. When M is the unit object $\mathbf{1}$, we denote this map simply s_m.

In [MMSS01, Definition 8.4], a similar map is defined and denoted by λ_m. The maps of Remark 7.1.29 for $\ell > 1$ are ℓ-fold composites of maps of this form.

More explicitly, since the functor $T^{-m-1} : \mathcal{M} \to \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ of Definition 7.1.20 is the left adjoint of the evaluation functor

$$\mathrm{Ev}_{m+1} : \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T) \to \mathcal{M},$$

we have

$$\begin{aligned} \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)(T^{-(m+1)}(TM), T^{-m}M) &\cong \mathcal{M}(TM, \mathrm{Ev}_{m+1}(T^{-m}M)) \\ &= \mathcal{M}(TM, (T^{-m}M)_{m+1}) \\ &= \mathcal{M}(TM, TM) \end{aligned}$$

and the morphism s_m^M on the left is the isomorphic image of 1_{TM} on the right. This means it is the counit of the adjunction

$$T^{-m+1} \dashv \mathrm{Ev}_{m+1}$$

evaluated on the object $T^{-m}M$ in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$, the map $\epsilon_{T^{-m}M}$ of Definition 2.2.20. Equivalently,

$$(s_m^M)_n = \begin{cases} * \to * & \text{for } 0 \leqslant n < m \\ * \to M & \text{for } n = m \\ 1_{T^nM} & \text{for } n > m. \end{cases} \tag{7.3.2}$$

In the presymmetric case, we denote

$$s_m^{\mathbf{1}} : K \wedge K^{-m-1} \to K^{-m} \tag{7.3.3}$$

(where K^{-m} is the Yoneda spectrum of Definition 7.2.52 and $\mathbf{1}$ is the unit object in $\mathcal{M}$) by s_m, and $s_m^M = M \wedge s_m$. The map e_1 of (7.0.8) is $s_0^{\mathbf{1}}$ in the case $K = S^1$.

The stabilizing map in the original case described in § 7.0D is a special case of this one, and the stabilizing maps of § 7.4C and (7.2.68) below are comparable to it.

Proposition 7.3.4. The stabilizing maps are projective cofibrations of Hovey spectra. *Let $\mathcal{M}$ be a stabilizable model category. Then the stabilizing maps of Definition 7.3.1 are projective cofibrations between cofibrant objects as in Proposition 7.1.33.*

Proof The cofibrancy of the spectra $T^{-m-1}TM$ and $T^{-m}M$ for cofibrant M follows easily from Corollary 7.1.37.

We will use the characterization of Proposition 7.1.36 to show that the maps are projective cofibrations.

For $n = m$, the diagram there reads

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & M, \end{tikzcd}$$

so the pushout corner map is $* \to M$, which is a cofibration since M is cofibrant.

For $n = m + 1$ we have

$$\begin{tikzcd} * \arrow[r] \arrow[d] & TM \arrow[d, "1_{TM}"] \\ TM \arrow[r, "1_{TM}"] & TM, \end{tikzcd}$$

so the pushout corner map is 1_{TM}.

For all other n each morphism the diagram is an identity map. □

Remark 7.3.5. Stabilizing maps as cofibrations. *Later in this chapter we will consider other categories of spectra in which similar stabilizing maps (see (7.2.63) below) are* **not** *projective cofibrations. We will need to use a functorial factorization*

$$\begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, "\widetilde{s}"'] & & Y \\ & \widetilde{Y}_s \arrow[ur, "\widehat{s}"'] & \end{tikzcd}$$

where $\widetilde{s}$ is a projective cofibration and $\widehat{s}$ is a projective weak equivalence. This can be obtained by a mapping cylinder construction as in Definition 3.5.1. A factorization in which $\widehat{s}$ is also a fibration exists since we are in a model category, **but it is not needed here***. Since s and $\widetilde{s}$ differ by a projective* *weak equivalence, they are interchangeable for the purpose of defining Bousfield localization. We will need to replace each map s by $\widetilde{s}$ for the purpose of describing cofibrant generating sets for the stable model structure in § 7.4D below.*

Definition 7.3.6. Stable equivalence and the stable model structure. *Let $\mathcal{M}$ be a Hirschhorn category (Definition 6.3.2) with a generating set of cofibrations $\mathcal{I}$ with cofibrant domains and a left Quillen endofunctor T (Definition 4.5.1). The* **stable model structure on** $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ *is the left Bousfield localization (Definition 6.2.1) of the projective model structure of Proposition 7.1.33 (under which $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ is also a Hirschhorn category by Proposition 6.3.3) with respect to the morphism set*

$$\mathcal{S} = \left\{ s_m^C \colon m \geqslant 0 \right\} \tag{7.3.7}$$

for s_m^C the stabilizing map of Definition 7.3.1, where C runs through the domains and codomains of $\mathcal{I}$. A **stable equivalence** *is an $\mathcal{S}$-local equivalence (see Definition 6.2.1) and a* **stable fibration** *is an $\mathcal{S}$-fibration. A* **stably fibrant spectrum** *is one that is $\mathcal{S}$-fibrant.*

The Hirschhorn categories we will consider here are stabilizable as in Definition 7.2.1, but in view of Theorem 6.3.4, the definition above makes sense without these additional assumptions.

Remark 7.3.8. The original definition of stable equivalence. *The first definition of stable model structures on the categories $\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)$ and $\mathcal{S}p^{\mathbf{N}}(\mathcal{S}et_\Delta, \Sigma)$ is that of Bousfield-Friedlander [BF78]. It is* **not** *a special case of the one above. They define stable equivalences to be maps inducing isomorphisms of stable homotopy groups (as in (7.0.11) and Definition 7.3.14), which they also define. Hovey proves these two definitions are equivalent in the original case in [Hov01b, Corollary 3.5]. (Here again he refers to "Theorem 18.8.7" of [Hir03], which is now [Hir03, Theorem 9.7.4].) Since the two model structures have the same cofibrations, namely the projective ones, it suffices to show they have the same weak equivalences. We will do this for Hovey spectra in Corollary 7.3.25.*

As we saw in § 7.0D, the $\mathcal{S}$-local objects (and hence the $\mathcal{S}$-fibrant objects by Proposition 6.2.12) in $\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)$ are the Ω-spectra. Indeed, the stabilizing maps were chosen for this very reason. This means that a map of spectra $g : X \to Y$ is an $\mathcal{S}$-local equivalence iff the map

$$g^* : \mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)(Y, Z) \to \mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)(X, Z)$$

is a weak equivalence for every Ω-spectrum Z. Hence this definition is in terms of generalized cohomology rather than stable homotopy groups.

Hovey shows that it suffices to show that the two model structures have the same fibrant objects, which are the Ω-spectra in both cases.

The following is proved by Hovey as [Hov01b, Theorem 3.4]. The original case is the statement that stably fibrant spectra are Ω-spectra, which is proved in [BF78]. A generalization will be proved as Corollary 7.4.46 below.

Theorem 7.3.9. Stably fibrant Hovey spectra are Ω_T-spectra. *Let $\mathcal{M}$ and T be as in Definition 7.3.6 with $\mathcal{M}$ topological. Then a spectrum is stably fibrant (equivalently $\mathcal{S}$-local by Proposition 6.2.12) iff it is a Ω_T-spectrum as in Definition 7.1.6. The map s_m^M of Definition 7.3.6 is a stable equivalence for each $m \geqslant 0$ and each cofibrant M in $\mathcal{M}$.*

As in Remark 6.2.2, Hovey does not assume that $\mathcal{M}$ is topological, but we will continue to do so here, and to assume that it is stabilizable as in Definition 7.2.1.

Proof By Definition 6.2.1, an $\mathcal{S}$-local spectrum Y (for $\mathcal{S}$ as in Definition 7.3.6) is one that is objectwise fibrant for which the map $(s_m^M)^*$ in the diagram

$$\begin{tikzcd} \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)(T^{-m}M,Y) \arrow[r, "(s_m^M)^*"] \arrow[d, "\cong"'] & \mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)(T^{-(m+1)}(TM),Y) \arrow[d, "\cong"] \\ & \mathcal{M}(TM, Y_{m+1}) \arrow[d, "\cong"] \\ \mathcal{M}(M, Y_m) \arrow[r, "(\eta_m^Y)_*"] & \mathcal{M}(M, \Omega_T Y_{m+1}) \end{tikzcd} \tag{7.3.10}$$

is a weak equivalence for all $m \geqslant 0$ and for any M which is a domain and codomain of the set $\mathcal{I}$ of generating cofibrations of $\mathcal{M}$. Here the two upper isomorphisms follow from the adjunctions $T^{-m} \dashv \mathrm{Ev}_m$ and $T^{-m-1} \dashv \mathrm{Ev}_{m-1}$ while the lower one follows from $T \dashv \Omega_T$. The map η_n^Y is costructure map of (7.1.7). Thus the map $(\eta_m^Y)_*$ is a weak equivalence in all such cases. By Theorem 5.6.21 this means η_m^Y is one as well. This makes Y a Ω_T-spectrum as in Definition 7.1.6.

For the converse, for a Ω_T-spectrum Y, η_m^Y is a weak equivalences, so $(s_m^M)^*$ is also one, making Y stably fibrant. $\square$

In the presymmetric case, the diagram of (7.3.10) reads

$$\begin{tikzcd} \mathcal{S}p^{\mathbf{N}}(\mathcal{M},K)(M \wedge K^{-m},Y) \arrow[r, "(s_m^M)^*"] \arrow[d, "\cong"'] & \mathcal{S}p^{\mathbf{N}}(\mathcal{M},K)(M \wedge K \wedge K^{-(m+1)},Y) \arrow[d, "\cong"] \\ & \mathcal{M}(M \wedge K, Y_{m+1}) \arrow[d, "\cong"] \\ \mathcal{M}(M, Y_m) \arrow[r, "(\eta_m^Y)_*"] & \mathcal{M}(M, Y_{m+1}^K). \end{tikzcd}$$

Theorem 7.3.11. The five lemma for Hovey spectra. *Suppose we have* *a map of cofiber sequences in* $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "i_f"] \arrow[d, "\beta"] & C_f \arrow[d, "\gamma"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "i_{f'}"] & C_{f'}. \end{tikzcd} \tag{7.3.12}$$

Then if any two of the vertical maps is a stable equivalence, so is the third.

Proof The diagram of cofiber sequences leads to a similar diagram of the second Puppe exact sequences of generalized cohomology groups of Theorem 5.7.6 for any stably fibrant spectrum B. Then the five lemma implies that if any two of α^*, β^*, and γ^* are isomorphisms, so is the third. This means the third vertical map in (7.3.12) induces an isomorphism in $\pi(-, B)$ for all stably fibrant B, making it a stable equivalence. □

Theorem 7.3.13. Stable h-cofibrations of Hovey spectra are stable precofibrations.

Proof Suppose we have an h-cofibration $f : A \to B$ with maps g, g' and h as in (5.1.16). With the mapping cylinder construction, we can assume that the maps g and g' are also h-cofibrations. We have a pushout diagram

$$\begin{tikzcd} A \vee A \arrow[r, "f \vee g"] \arrow[d, "\text{fold}"'] & B \vee C \arrow[r] \arrow[d] & C_f \vee C_g \arrow[d, equal] \\ A \arrow[r] & B \cup_A C \arrow[r] & C_f \vee C_g \end{tikzcd}$$

in which the object on the lower center is the same as the one on the lower left in (5.1.16), and each row is a cofiber sequence. There is a similar diagram with g replaced by g'.

Thus the stable equivalence h leads to diagrams

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, equal] & C \arrow[r] \arrow[d, "h"] & C_g \arrow[d, "\ell"] \\ A \arrow[r, "g'"] & C' \arrow[r] & C_{g'} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \cup_A C \arrow[r] \arrow[d, "k"] & C_f \vee C_g \arrow[d, "C_f \vee \ell"] \\ A \arrow[r] & B \cup_A C' \arrow[r] & C_f \vee C_{g'} \end{tikzcd}$$

in which the rows are cofiber sequences. Since h is a stable equivalence, the first diagram and the five lemma imply that ℓ is a stable equivalence. This means that the second diagram and the five lemma imply that k is a also a stable equivalence, making the h-cofibration f a precofibration as claimed. □

7.3B Stabilizing maps and stable homotopy groups

Now we will discuss the relation between the stabilizing maps and stable homotopy groups. The latter are defined in the original case in (7.0.11), which relates them to the ordinary homotopy groups of the spaces making up the spectrum. A key fact here is that a map in $\mathcal{T}$ is by definition a weak equivalence if it induces an isomorphism of homotopy groups. The assumption that our target category $\mathcal{M}$ is stabilizable as in Definition 7.2.1 means that it has a complete set of homotopy invariants as in Definition 5.6.5. These should be thought of as substitutes for homotopy groups.

The following definition should be compared with Definition 7.3.16 below.

Definition 7.3.14. The stable homotopy groups of a spectrum. *Let X be a spectrum as in Definition 7.2.33 and let V be an object in its indexing category $\mathscr{J}$. (Thus V is a natural number in each case but the extraorthogonal one.) Then its V***th stable homotopy group** *(also known as the V***th homotopy group***) is*

$$\pi_V X = \operatorname*{colim}_n \pi_V \Omega_L^n X_n \cong \operatorname*{colim}_n \pi_{V,n} X_n, \tag{7.3.15}$$

where $\pi_{V,n}(-) = \pi_0 \mathcal{M}(S^V \wedge L^{\wedge n}, -)$, and the colimit is the sequential one associated with the following diagram in $\mathcal{M}$.

$$X_0 \xrightarrow{\overline{\eta}_0^X} \Omega_L X_1 \xrightarrow{\Omega_L \overline{\eta}_1^X} \cdots\cdots \xrightarrow{\Omega_L^{n-1}\overline{\eta}_{n-1}^X} \Omega_L^n X_n \xrightarrow{\Omega_L^n \overline{\eta}_n^X} \cdots .$$

Here the homotopy groups of objects in $\mathcal{M}$ are as in Definition 7.2.30 and the maps $\overline{\eta}_k^X$ are as in (7.2.44).

We can extend this definition to elements V in the indexing group $R\mathbf{F}$ of Definition 7.2.26 because for each such V, $V+n$ is an object in $\mathscr{J}$ for sufficiently large n. In the second colimit of (7.3.15) we can define $\pi_W Y$ (for Y an object in $\mathcal{M}$) to be trivial when W is in $R\mathbf{F}$ but not in $\mathscr{J}$.

The set $\pi_{V,n}X$ has an (abelian) group structure when $S^V \wedge L^{\wedge n}$ is a (double) suspension. L itself is a suspension by Definition 7.2.19. The corresponding collection of functors need not be a complete set of homotopy invariants as in Definition 5.6.5, as illustrated in Example 5.6.6.

The following is a generalization of (7.0.11) and is related to Definition 7.3.14.

Definition 7.3.16. Homotopy invariants of spectra. *Let $\mathcal{S}p$ be the category of presymmetric, symmetric or orthogonal spectra, where the target category $\mathcal{M}$ has a complete set of homotopy invariants $\{\pi_\alpha\}$ as in Definition 5.6.5. Then*

$$\pi_{\alpha,k} X := \operatorname*{colim}_n \pi_\alpha \Omega_K^{n+k} X_n \cong \operatorname*{colim}_n \pi_{\alpha,n+k} X_n \tag{7.3.17}$$

for a spectrum X and each (possibly negative) integer k, where

$$\pi_{\alpha,m}(-) := \pi_0 \mathcal{M}(A_\alpha \wedge K^{\wedge m}, -), \tag{7.3.18}$$

and both $\Omega_K^m X_n$ and $K^{\wedge m}$ are understood to be a point when $m < 0$. Note *that we are using the symbol $\pi_{\alpha,m}$ to denote a $\mathcal{S}et$ valued functor both on $\mathcal{S}p$ in (7.3.17) and on $\mathcal{M}$ in (7.3.18).*

In the extraorthogonal case, we can make a similar definition with K replaced by L. In the Hovey spectrum case, we can replace Ω_K by Ω_T and $A_\alpha \wedge K^{\wedge m}$ by $T^m A_\alpha$. These set $\pi_{\alpha,m}X$ coincides with $\pi_{V,m}X$ as in Definition 7.3.14 when $A_\alpha = S^V$.

Definition 7.3.19. Stable homotopy equivalences of spectra. *Let $\mathcal{S}p$ be one of the categories of spectra of Definition 7.2.33, and suppose that the target category $\mathcal{M}$ has a complete set of homotopy invariants for $\mathcal{M}$ as in Definition 5.6.5. Then a morphism $f: X \to Y$ in $\mathcal{S}p$ is a* **stable homotopy equivalence** *if $\pi_{\alpha,k}f$ (for $\pi_{\alpha,k}$ as in (7.3.17)) is an isomorphism for each α and each integer k.*

This definition differs from that of a stable equivalence in Definition 7.3.6. **It begs the question of whether the stabilizing maps themselves are stable homotopy equivalences.** If they are, then Corollary 6.2.7 implies that **all** stable equivalences are stable homotopy equivalences. We will deal with the converse question in Theorem 7.3.23**??** for Hovey spectra, and in Theorem 7.4.29**??** for smashable spectra.

In §7.0E we saw that the stabilizing maps for symmetric spectra are **not** stable homotopy equivalences. It turns out that they are in each of the other cases. We will deal with the smoothly smashable case below in Theorem 7.4.12 after listing the relevant stabilizing maps in Definition 7.4.8. We will deal with the presymmetric case here.

Proposition 7.3.20. Presymmetric stabilizing maps. *Let $\mathcal{M}$ be a stabilizable model category as in Definition 7.2.1 with a compact cofibrant object K. Then in the category $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)$, the map $s_m : K \wedge K^{-m-1} \to K^{-m}$ of (7.3.3) is a stable homotopy equivalence for each $m \geqslant 0$.*

A similar statement for Hovey spectra can be proved in the same way. We leave the details to the reader.

Proof Recall that every stabilizable model category has a complete set of homotopy invariants (as in Definition 5.6.5) by Proposition 7.2.2.

For the map s_m, we have

$$(K \wedge K^{-m-1})_{m+p} = \begin{cases} * & \text{for } -m \leqslant p < 1 \\ K^{\wedge p} & \text{for } p \geqslant 1, \end{cases}$$

$$(K^{-m})_{m+p} = \begin{cases} * & \text{for } -m \leqslant p < 0 \\ K^{\wedge p} & \text{for } p \geqslant 0, \end{cases}$$

$$\text{and} \qquad (s_m)_{m+p} = \begin{cases} * \to * & \text{for } -m \leqslant p < 0 \\ * \to S & \text{for } p = 0 \\ 1_{K^{\wedge p}} & \text{for } p > 0. \end{cases}$$

Using (7.3.17), we see that

$$\pi_i(s_m) = \operatorname*{colim}_p \pi_{i+m+p}(s_m)_{m+p},$$

which is an identity map since $(s_m)_{m+p}$ is one for $p > 0$. This makes s_m an stable homotopy equivalence as claimed. □

7.3C Stabilization via a homotopy idempotent functor

The functors Θ and Θ^∞ for Hovey spectra are those of Definition 5.7.3.

Recall the discussion of § 7.0F. We need to use of the homotopy colimit above rather than the categorical one to insure that Θ^∞ is well behaved.

Remark 7.3.21. The functor Θ^k in the presymmetric case. *Since the right adjoint functor is $(-)^K$, which we will denote by Ω_K, in the presymmetric case, we have*

$$(\Theta^k X)_n = X_{n+k}^{(K^{\wedge k})} = \Omega_K^k X_{n+k},$$

and the structure map

$$\epsilon_n^{\Theta^k X} : K \wedge \Omega_K^k X_{n+k} \to \Omega_K^k X_{n+k+1}$$

is adjoint to

$$\Omega_K^k(\eta_{n+k}^X) : \Omega_K^k X_{n+k} \to \Omega_K^{k+1} X_{n+k+1}.$$

The compactness of K implies that Ω_K commutes with Θ^∞ by Lemma 5.8.20(iv).

Lemma 7.3.22. Properties of Θ^∞ for Hovey spectra. *With notation as in Definition 5.7.3,*

(i) The map $\theta_{\Theta^\infty X} : \Theta^\infty X \to \Theta(\Theta^\infty X)$ is a projective weak equivalence. In particular $\Theta^\infty X$ is a Ω_T-spectrum as in Definition 7.1.6. This map is the same as

$$\Theta^\infty(\theta_X) : \Theta^\infty X \to \Theta^\infty(\Theta X).$$

(ii) If Z is a Ω_T-spectrum, then the map $\theta_Z^\infty : Z \to \Theta^\infty Z$ is a projective equivalence, so Z is Θ^∞-local. In particular for any spectrum X, the map

$$\theta_{\Theta^\infty X}^\infty : \Theta^\infty X \to \Theta^\infty(\Theta^\infty X)$$

is a projective weak equivalence. Similarly the map $\theta_Z : Z \to \Theta Z$ is a projective weak equivalence.

Note that Ω_K preserves homotopy sequential colimits because K is compact as in Definition 5.2.6, by Lemma 5.8.20(ii).

Hovey proves (i) for his definition of Θ in [Hov01b, Proposition 4.6].

Proof Let $W = \Theta^\infty X$.

(i) The nth component of W is $\operatorname{hocolim}_k \Omega_T^k X_{n+k}$, so that of $\Theta(W)$ is

$$\Omega_T W_{n+1} = \Omega_T(\operatorname{hocolim}_k \Omega_T^k X_{n+k+1}) \cong \operatorname{hocolim}_k \Omega_T^{k+1} X_{n+k+1}.$$

This is **not the same** as W_n since

$$\begin{aligned} W_n &= \operatorname{hocolim}_{k\geqslant 0} \Omega_T^k X_{n+k} \\ \Omega_T W_{n+1} &= \operatorname{hocolim}_{k\geqslant 1} \Omega_T^k X_{n+k} = \operatorname{hocolim}_{k\geqslant 0} \Omega_T^{k+1} X_{n+k+1}. \end{aligned}$$

The map $W_n \to \Omega_T W_{n+1}$ is an instance of the weak equivalence of Lemma 5.8.20(vi).

Our assumption that $\mathcal{M}$ is telescopically closed as in Definition 5.8.28 means that W is projectively fibrant. To show it is a stably fibrant, it remains to show that η_n^W is a weak equivalence for each n. This follows from the isomorphism above.

(ii) A Ω_T-spectrum Z is by definition one for which η_Z is a projective weak equivalence. This implies that η_Z^∞ is one by Lemma 5.8.19(i). □

The next result is similar to one proven by Hovey as [Hov01b, Theorems 4.9 and 4.12, and Corollary 4.11]. Our assumptions about $\mathcal{M}$ enable us to give a simpler proof.

Theorem 7.3.23. Stable equivalences and Θ^∞ for Hovey spectra. *Let $\mathcal{M}$ be a stabilizable model category as in Definition 7.2.1, and let T be a left Quillen endofunctor on $\mathcal{M}$.*

(i) If $f : X \to Y$ is a map in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ such that $\Theta^\infty f$ is a projective weak equivalence, then f is a stable equivalence as in Definition 7.3.6.

(ii) The maps $\theta_X^k : X \to \Theta^k X$ for $k > 0$, and $\theta_X^\infty : X \to \Theta^\infty X$ are stable equivalences for all spectra X.

(iii) For all X in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ the map $X \to \Theta^\infty X$ is a stable equivalence into a Ω_T-spectrum as in Definition 7.1.6, and therefore a fibrant approximation for the stable model structure.

(iv) If a map $f : X \to Y$ is a stable equivalence, then $\Theta^\infty f$ is a projective equivalence.

(v) Every stable homotopy equivalence of Hovey spectra is a stable equivalence.

Proof The first four statements are essentially a special case of Theorem 6.2.16 in which the homotopy idempotent functor Υ is the present Θ^∞. From Lemma 7.3.22(i), it follows easily that $\theta_{\Theta^\infty X}^k : \Theta^\infty X \to \Theta^k(\Theta^\infty X)$ (as in Definition 5.7.3) is a weak equivalence for each k. This means that each map in the diagram defining

$$\Theta^\infty(\Theta^\infty X) = \operatorname{hocolim}_k \Theta^k(\Theta^\infty X)$$

is a weak equivalence. This makes the map $\theta^{\infty}_{\Theta^{\infty}X} : \Theta^{\infty}X \to \Theta^{\infty}(\Theta^{\infty}X)$ a weak equivalence by Lemma 5.8.19. Hence Θ^{∞} a homotopy idempotent functor.

The second hypothesis of Theorem 6.2.16 follows from the second one of Lemma 7.3.22, since every $\mathcal{S}$-local spectrum is a Ω_T-spectrum by Theorem 7.3.9.

For (v), suppose X is an Ω_T-spectrum. Then $\pi_\alpha X_n \cong \pi_{\alpha-n}X$ for each α and for each $n \geqslant 0$. Therefore a stable homotopy equivalence $f : X \to Y$ between Ω_T-spectra induces a weak equivalence $f_n : X_n \to Y_n$ for each n. It is therefore a projective weak equivalence and hence a stable equivalence. □

Corollary 7.3.24. The functor Θ^{∞} for presymmetric spectra. *The properties of Θ^{∞} stated in Lemma 7.3.22 and Theorem 7.3.23 hold for presymmetric spectra.*

Corollary 7.3.25. Stable equivalences of Hovey spectra and stable homotopy groups. *Let $\mathcal{M}$ be a stabilizable model category as in Definition 7.2.1 with a compact left Quillen functor T. Then a map $f : X \to Y$ in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ of Hovey spectra is a stable equivalence iff it is a stable homotopy equivalence, meaning that the induced map*

$$\operatorname*{colim}_n \pi_\alpha(\Omega_T^{n+k}X_n) \xrightarrow{f_*} \operatorname*{colim}_n \pi_\alpha(\Omega_T^{n+k}Y_n) \tag{7.3.26}$$

is an isomorphism for each α and for all integers (positive or negative) k, with the understanding that $\Omega_T^m X = *$ when $m < 0$.

Proof The colimits on either side of (7.3.26) should be regarded as the stable homotopy groups of X and Y. We will show that our hypothesis on f, that it induces an isomorphism on these groups, is equivalent to its fibrant replacement $\Theta^{\infty}f$ being a projective equivalence. This is equivalent to f being a stable equivalence by Theorem 7.3.23(i) and (iv).

We have

$$\begin{aligned}
(\Theta^{\infty}X)_m &= \operatorname*{hocolim}_n \Omega_T^n X_{m+n} \\
\pi_0(A_\alpha \wedge K^k, (\Theta^{\infty}X)_m) &\cong \operatorname*{colim}_n \pi_0(A_\alpha \wedge K^k, \Omega_T^n X_{m+n}) \\
&\cong \operatorname*{colim}_n \pi_0(A_\alpha \wedge K^k, \Omega_T^{n-m} X_n) \\
&\cong \operatorname*{colim}_n \pi_0(A_\alpha, \Omega_T^{n+k-m} X_n) \\
&\cong \operatorname*{colim}_n \pi_\alpha(\Omega_T^{n+k-m} X_n),
\end{aligned}$$

and similarly for Y. Hence for each α, $k \geqslant 0$ and $m \geqslant 0$, we have

$$\pi_0(A_\alpha \wedge K^k, (\Theta^{\infty}X)_m) \xrightarrow[\cong]{(\Theta^{\infty}f_m)_*} \pi_0(A_\alpha \wedge K^k, (\Theta^{\infty}Y)_m).$$

By our hypothesis on $\mathcal{M}$, this means that $\Theta^{\infty}f_m$ is a weak equivalence in $\mathcal{M}$ for each $m \geqslant 0$, so $\Theta^{\infty}f$ is a projective equivalence, and the result follows. □

7.3D Cofibrant generating sets for the stable model structure on Hovey spectra

We will now define cofibrant generating sets (as in Definition 5.2.1) for the stable model structure on $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ when $\mathcal{M}$ is stabilizable as in Definition 7.2.1. **In this subsection we will abbreviate $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ by $\mathcal{S}p$.**

The set of generating cofibrations is $\mathcal{I}_T$ of Proposition 7.1.33 since cofibrations are the same in both the projective and stable model structures. It follows that the same is true of trivial fibrations.

Proposition 7.3.27. Trivial stable fibrations in $\mathcal{S}p$. *Any map $p : X \to Y$ in $\mathcal{S}p$ having the right lifting property with respect to $\mathcal{I}_T$ as in Proposition 7.1.33 is a trivial stable fibration.*

For the set of generating trivial cofibrations we start with the set $\mathcal{J}_T^{\mathrm{proj}}$ of Proposition 7.1.33 and add some morphisms related to the stabilizing maps of Definition 7.3.1. We will rely on the fact (Proposition 7.3.4) that these maps are projective cofibrations.

For the case at hand, define

$$\mathcal{J}_T = \mathcal{J}_T^{\mathrm{proj}} \cup (\mathcal{I} \,\square\, \mathcal{S}), \tag{7.3.28}$$

where $\mathcal{I}$ is the set of generating cofibrations for $\mathcal{M}$, $\mathcal{S}$ as in Definition 7.3.6, and $\square$ is the pushout corner operation of Definition 2.6.12. Since the maps in $\mathcal{S}$ are projective cofibrations, the maps in $\mathcal{I} \,\square\, \mathcal{S}$ are by Corollary 5.5.2, so the maps in $\mathcal{J}_T$ are all projective cofibrations.

We will show in Theorem 7.3.36 below that $\mathcal{I}_T$ and $\mathcal{J}_T$ are cofibrant generating sets for the stable model structure by showing that they satisfy the four conditions of the Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24. Dwyer-Hirschhorn-Kan's fourth condition is implied by the following, which is comparable to [HSS00, Lemma 3.4.15].

Theorem 7.3.29. Trivial stable fibrations are projective weak equivalences. *Let $p : X \to Y$ be a morphism in $\mathcal{S}p$ which is a stable equivalence and has the right lifting property with respect to the set $\mathcal{J}_T$ of (7.3.28). Then p is a projective weak equivalence.*

Proof Any map p with the right lifting property with respect to $\mathcal{J}_T^{\mathrm{proj}}$ is a projective (i.e., strict) fibration, so p_m is a fibration in $\mathcal{M}$ for each m.

To analyze the right lifting property with respect to the pushout corner maps in $\mathcal{J}_T$, we use Proposition 3.1.53 with the categories $\mathcal{C}$ and $\mathcal{E}$ replaced by $\mathcal{M}$ and $\mathcal{S}p$, and the maps g, i and f replaced by p, s_m^C (see Definition 7.3.6) and f. It says that p has the right lifting property with respect to $f \,\square\, s_m^C$ (for a morphism $f : A \to B$ in $\mathcal{I}$) in $\mathcal{S}p$ iff f has the left lifting property with respect to the lifting test map $\mathcal{S}p_{\diamondsuit}(s_m^C, p)$ (Definition 2.3.14) in $\mathcal{M}$. This is the pullback corner map for the following diagram in $\mathcal{M}$.

$$\begin{tikzcd}
\mathcal{S}p(T^{-m}C, X) \arrow[r, "p_*"] \arrow[d, "(s_m^C)^*"'] & \mathcal{S}p(T^{-m}C, Y) \arrow[d, "(s_m^C)^*"] \\
\mathcal{S}p(T^{-m-1}(TC), X) \arrow[r, "p_*"] & \mathcal{S}p(T^{-m-1}(TC), Y).
\end{tikzcd} \tag{7.3.30}$$

If each such f has the left lifting property with respect to the pullback corner map of (7.3.30), the latter is a trivial fibration and hence a weak equivalence, so the diagram is homotopy Cartesian.

Using the adjunctions $T^{-m} \dashv \mathrm{Ev}_m$, $T^{-m-1} \dashv \mathrm{Ev}_{m+1}$ and $T \dashv \Omega_T$, we can embed (7.3.30) in a larger diagram

$$\begin{tikzcd}
\mathcal{M}(C, X_m) \arrow[r, "(p_m)_*"] \arrow[d, "\cong"'] & \mathcal{M}(C, Y_m) \arrow[d, "\cong"] \\
\mathcal{S}p(T^{-m}C, X) \arrow[r, "p_*"] \arrow[d, "(s_m^C)^*"'] & \mathcal{S}p(T^{-m}C, Y) \arrow[d, "(s_m^C)^*"] \\
\mathcal{S}p(T^{-m-1}(TC), X) \arrow[r, "p_*"] \arrow[d, "\cong"'] & \mathcal{S}p(T^{-m-1}(TC), Y) \arrow[d, "\cong"'] \\
\mathcal{M}(TC, X_{m+1}) \arrow[r, "(p_{m+1})_*"] \arrow[d, "\cong"'] & \mathcal{M}(TC, Y_{m+1}) \arrow[d, "\cong"] \\
\mathcal{M}(C, \Omega_T X_{m+1}) \arrow[r, "(\Omega_T^m p_{m+1})_*"] & \mathcal{M}(C, \Omega_T Y_{m+1}),
\end{tikzcd} \tag{7.3.31}$$

The outer diagram above, which is isomorphic to (7.3.30) and is therefore homotopy Cartesian, is the image under the functor $\mathcal{M}(C, -)$ of

$$\begin{tikzcd}
X_m \arrow[r, "p_m"] \arrow[d, "\eta_m^X"'] & Y_m \arrow[d, "\eta_m^Y"] \\
\Omega_T X_{m+1} \arrow[r, "\Omega_T p_{m+1}"] & \Omega_T Y_{m+1}.
\end{tikzcd} \tag{7.3.32}$$

We know that $\mathcal{M}(C, -)$ is a right Quillen functor by Proposition 5.5.21(i) and that such functors preserve homotopy Cartesian squares by Proposition 5.8.45.

We can use Theorem 5.6.21 to deduce that (7.3.32) is also homotopy Cartesian. It follows from Proposition 5.8.45 that the same is true of

$$\begin{tikzcd}
\Omega_T^i X_{m+i} \arrow[r, "\Omega_T^i p_{m+i}"] \arrow[d, "\Omega_T^i \eta_{m+i}^X"'] & \Omega_T^i Y_{m+i} \arrow[d, "\Omega_T^i \eta_{m+i}^Y"] \\
\Omega_T^{i+1} X_{m+i+1} \arrow[r, "\Omega_T^{i+1} p_{m+i+1}"] & \Omega_T^{m+i+1} Y_{m+i+1}.
\end{tikzcd}$$

Repeated use of Proposition 5.8.48 tells us that the diagram

$$\begin{tikzcd} X_m \arrow[r, "p_m"] \arrow[d] & Y_m \arrow[d] \\ \Omega_T^k X_{m+k} \arrow[r, "\Omega_T^k p_{m+k}"] & \Omega_T^k Y_{m+k} \end{tikzcd} \tag{7.3.33}$$

is homotopy Cartesian for each $k > 0$.

Proposition 5.8.25 implies that the diagram

$$\begin{tikzcd} X_m \arrow[r, "p_m"] \arrow[d] & Y_m \arrow[d] \\ (\Theta^\infty X)_m = \operatorname*{hocolim}_k \Omega_T^k X_{m+k} \arrow[r, "(\Theta^\infty p)_m"] & \operatorname*{hocolim}_k \Omega_T^k Y_{m+k} = (\Theta^\infty Y)_m \end{tikzcd} \tag{7.3.34}$$

is also homotopy Cartesian. Then our assumption that p is a stable equivalence means that $(\Theta^\infty R^{\mathbf{N}} p)_m$ (by Theorem 7.3.23(iv)) and hence $(\Theta^\infty p)_m$ and p_m are weak equivalences, making p a projective weak equivalence as claimed. □

The following is a consequence of Corollary 7.3.25.

Proposition 7.3.35. Some easy stable equivalences. *Let $f : X \to Y$ be a morphism in $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)$ of spectra such that f_n is a weak equivalence for $n \gg 0$. Then f is a stable equivalence.*

Note that each of the maps s_m^L of Definition 7.3.1 fits this description since its nth component is an isomorphism for large n.

Theorem 7.3.36. Cofibrant generating sets for the stable model structure, the corner map theorem for Hovey spectra. *When $\mathcal{M}$ is a a stabilizable model category as in Definition 7.2.1, the generated with generating sets $\mathcal{I}_T$ as in Proposition 7.1.33 and*

$$\mathcal{J}_T = \mathcal{J}_T^{proj} \cup (\mathcal{I} \,\square\, \mathcal{S})$$

as in (7.3.28).

Note the similarity between the set $\mathcal{I} \,\square\, \mathcal{S}$ and the set of $\mathcal{S}$-horns in Definition 6.3.8. An analogous result for smashable spectra will be given below in Theorem 7.4.52, and for orthogonal G-spectra in Theorem 9.2.11.

Example 7.3.37. Cofibrant generating sets in the original case. *In $\mathcal{T}$ the cofibrant generating sets are*

$$\mathcal{I} = \left\{ i_n : S_+^{n-1} \to D_+^n \right\} \qquad \text{and} \qquad \mathcal{J} = \left\{ j_n : I_+^n \to I_+^{n+1} \right\}.$$

In $\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)$ the set of stabilizing maps is

$$\mathcal{S} = \left\{ s_m : S^1 \wedge S^{-1-m} \to S^{-m} : m \geqslant 0 \right\}$$

In each degree s^m is either an identity morphism or the map from the initial object $*$ by (7.3.2). Thus we can use Example 2.6.14 to describe the pushout corner maps $s_m \square i_n$, which are

$$(i_n \square s_m)_k = \begin{cases} 1_* & \text{for } 0 \leqslant k < m \\ i_n & \text{for } k = m \\ 1_{D^n \ltimes S^{k-m}} & \text{for } k > m, \end{cases}$$

where $X \ltimes Y$ is as in Definition 2.1.49. The morphisms in $\mathcal{I}_\Sigma$ and $\mathcal{J}_\Sigma^{proj}$ are

$$(i_n \wedge S^{-m})_k = \begin{cases} 1_* & \text{for } 0 \leqslant k < m \\ \Sigma^{k-m} i_n & \text{for } k \geqslant m \end{cases}$$

and

$$(j_n \wedge S^{-m})_k = \begin{cases} 1_* & \text{for } 0 \leqslant k < m \\ \Sigma^{k-m} j_n & \text{for } k \geqslant m. \end{cases}$$

Proof of Theorem 7.3.36. We will show that the sets $\mathcal{I}_T$ and $\mathcal{J}_T$ satisfy the four conditions of the Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24. This will mean that we have a model structure on the category of Hovey spectra $\mathcal{S}p$ of Definition 7.1.1 that is cofibrantly generated with the same weak equivalences, cofibrations and hence the same trivial cofibrations as those of the stable model structure. Since any model structure is uniquely determined by such data, we have the one we are looking for.

We now deal with Dwyer-Hirschhorn-Kan's four conditions.

(i) Dwyer-Hirschhorn-Kan's first condition has to do with smallness. We need to show that the domains of $\mathcal{I}_T$ and $\mathcal{J}_T$ are small relative to $\mathcal{I}_T$ and $\mathcal{J}_T$ respectively. The key point here is that the domains of $\mathcal{I}_T$, $\mathcal{J}_T^{\text{proj}}$ and $\mathcal{S}$ are all cofibrant and the maps in them are cofibrations.

Any spectrum in which the underlying objects of $\mathcal{M}$ are cofibrant is small relative to $\mathcal{I}_T$. The domains of $\mathcal{I}_T$ and $\mathcal{J}_T$ fit this description, so they are small relative to $\mathcal{I}_T$. Moreover each of the maps in $\mathcal{J}_T$ is a cofibration and therefore in the saturated class (Definition 4.8.13) generated by $\mathcal{I}_T$ by Proposition 5.2.2. This means that any object small relative to $\mathcal{I}_T$ is also small relative to $\mathcal{J}_T$ by Proposition 4.8.19, so Dwyer-Hirschhorn-Kan's first condition is satisfied.

(ii) We need to show that the maps in $\mathcal{J}_T$ are all cofibrations and stable equivalences. We have already seen that they are cofibrations. Each map in $\mathcal{J}_T^{\text{proj}}$ is a strict weak equivalence and hence a stable equivalence. The maps in $\mathcal{S}$ are stable equivalences by definition. To show that each map in $\mathcal{I} \square \mathcal{S}$ is one, let $f : A \to B$ be a map in $\mathcal{I}$ and consider the diagram

$$\begin{tikzcd}
A \wedge T^{-m}(T^m QC) \arrow[r, "A \wedge s_m^{QC}"] \arrow[d, "f \wedge T^{-m}(T^m QC)"'] & A \wedge T^{-0}(QC)_{s_m} \arrow[d, "\beta"] \arrow[ddr, bend left, "f \wedge T^{-0}(QC)"] & \\
B \wedge T^{-m}(T^m QC) \arrow[r, "\alpha"] \arrow[drr, "B \wedge s_m^{QC}"'] & P \arrow[dr, "f \square s_m^{QC}"] & \\
 & & B \wedge T^{-0}(QC)_{s_m},
\end{tikzcd}$$

in which P is the pushout of the two maps from the upper left. The nth components of the maps $A \wedge s_m^{QC}$ (and hence α) and $B \wedge s_m^{QC}$ are identity maps for large n, so the maps are stable equivalences by Proposition 7.3.35. Hence $f \square s_m^{QC}$ is one as required by Dwyer-Hirschhorn-Kan's second condition.

(iii) We need to show that each map $f : X \to Y$ having the right lifting property with respect to $\mathcal{I}_T$ also has it with respect to $\mathcal{J}_T$ and is a stable equivalence. A map with the former property is a strict trivial fibration, therefore a projective weak equivalence and hence a stable equivalence. We have seen that each map in $\mathcal{J}_T$ is a cofibration and hence in the saturated class generated by $\mathcal{I}_T$. This means that the right lifting property with respect to to $\mathcal{I}_T$ also has it with respect to $\mathcal{J}_T$.

(iv) Theorem 7.3.29 gives the converse of the previous condition, i.e., that a stable equivalence $p : X \to Y$ with the right lifting property with respect to $\mathcal{J}_T$ (see (7.3.28)) also has it with respect to $\mathcal{I}_T$ and is therefore a trivial fibration. □

7.3E Exact sequences for original spectra

Here we and in §7.4E, will show that certain categories of spectra are exactly stable as in Definition 5.7.3. This will enable us to apply Theorem 5.7.6 and Theorem 5.7.11 to get the Puppe and Adams exacxt sequences of homotopy groups.

We begin with the original case, the category $\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)$ with its stable model structure. The kth desuspension and delooping functors for $k \geqslant 1$ are given by

$$(\Sigma^{-k} X)_n := \begin{cases} X_{n-k} & \text{for } n \geqslant k \\ * & \text{for } n < k, \end{cases} \tag{7.3.38}$$

which coincides with the definition of formal desuspension given in Example 7.1.26, and

$$(\Omega^{-k} X)_n := X_{n+k}. \tag{7.3.39}$$

For $k = 1$, this coincides with with Hovey's shift functor s_- of [Hov01b, Definition 3.7], where it is defined for Hovey spectra. In that category it has a left adjoint s_+ defined by $(s_+X)_n = X_{n-1}$, which is our formal desuspension Σ^{-1}.

The $(n+k)$th component of the map $\sigma_X^k : \Sigma^k\Sigma^{-k}X \to X$ is the structure map

$$\epsilon_{n,k}^X : \Sigma^k X_n \to X_{n+k}. \tag{7.3.40}$$

We will show that σ_X^k is a stable equivalence for all X below in Lemma 7.3.43(iv).

The nth component of the map $\omega_X^k : X \to \Omega^k\Omega^{-k}X$ is the costructure map

$$\eta_{n,k}^X : X_n \to \Omega^k X_{n+k}. \tag{7.3.41}$$

This map η_X is a special case of the map η_X of Definition 5.7.3. When X is fibrant, meaning when X is an Ω-spectrum, this map is a weak equivalence for each n. This makes ω_X^k a projective weak equivalence and therefore a stable equivalence.

The maps of (7.3.40) and (7.3.41) are not to be confused with the maps

$$\epsilon_X^k : X \to \Omega^k\Sigma^k X \qquad \text{and} \qquad \eta_X^k : \Sigma^k\Omega^k X \to X$$

adjoint to the identity maps on $\Sigma^k X$ and $\Omega^k X$ respectively, for a space or spectrum X.

We will have the following once we have proved Lemma 7.3.43 and Lemma 7.3.47. They give the first two and third conditions of Definition 5.7.3 respectively.

Theorem 7.3.42. The original category of spectra $\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, \Sigma)$ with its stable model structure is exactly stable as in Definition 5.7.3.

Hence Theorem 5.7.6 and Theorem 5.7.11 apply, and we get the usual long exact sequences of stable homotopy groups.

Lemma 7.3.43. The suspension and loop isomorphisms for original spectra. *For an original spectrum X, the following maps are stable equivalences:*

(i) The map $\eta_X^k : X \to \Omega^k\Sigma^k X$ adjoint to the identity on $\Sigma^k X$. Thus we have a **suspension isomorphism**

$$\pi_q X \cong \pi_{q+k}\Sigma^k X.$$

(ii) The map $\epsilon_X^k : \Sigma^k\Omega^k X \to X$ adjoint to the identity on $\Omega^k X$. Thus we have a **loop isomorphism**

$$\pi_q X \cong \pi_{q-k}\Omega^k X.$$

(iii) The map $\theta_X^k : X \to \Omega^k\Omega^{-k}X$.
(iv) The map $\sigma_X^k : \Sigma^k\Sigma^{-k}X \to X$.

Proof By Corollary 7.3.25, and map between original spectra is a stable equivalence iff it induces an isomorphism of stable homotopy groups, and we will use that property to recognize stable equivalences.

(i) Let F^k denote the functor $\Omega^k\Sigma^k$. For an arbitrary spectrum X, we will construct a diagram

$$\begin{tikzcd} \pi_q X \arrow[r, "(\eta^k_X)_*"] \arrow[d, equal] & \pi_q F^k X \arrow[d, equal] \arrow[dl] \\ \pi_q X \arrow[r, "(\eta^k_X)_*"'] & \pi_q F^k X \end{tikzcd} \tag{7.3.44}$$

for each $k > 0$ and each integer q. This will imply that $\pi_q\eta^k_X$ is an isomorphism, making η^k_X a stable equivalence. The groups in (7.3.44) are

$$\begin{tikzcd} \operatorname*{colim}_n \pi_{q+n}X_n \arrow[r, "(\eta^k_{X_n})_*"] \arrow[d, "(\eta^X_{n,k})_*"', "\cong"] & \operatorname*{colim}_n \pi_{q+n}F^kX_n \arrow[d, "\cong"', "(\eta^{F^kX}_{n,k})_*"] \arrow[dl, "(\Omega^k\epsilon^X_{n,k})_*"] \\ \operatorname*{colim}_n \pi_{q+n}\Omega^kX_{n+k} \arrow[r] & \operatorname*{colim}_n \pi_{q+n}\Omega^kF^kX_{n+k}, \end{tikzcd} \tag{7.3.45}$$

where each map of colimits is induced by the indicated map of nth components, the bottom one being induced by $\Omega^k\eta^k_{X_{n+k}}$.

The nth component of the left vertical map is

$$\pi_{q+n}X_n \xrightarrow{(\eta^X_{n,k})_*} \pi_{q+n}\Omega^kX_{n+k} \cong \pi_{q+n+k}X_{n+k},$$

which is a k-fold composite of maps in the diagram that defines the colimit in the first place. It follows that the map of colimits is the identity. The argument for the right vertical map is similar. The result follows.

(ii) Let G^k denote the functor $\Sigma^k\Omega^k$, so $\Omega^kG^k = F^k\Omega^k$. For an arbitrary spectrum X, we will construct a diagram

$$\begin{tikzcd} \pi_q G^k X \arrow[r, "(\epsilon^k_X)_*"] \arrow[d, equal] & \pi_q X \arrow[d, equal] \arrow[dl] \\ \pi_q G^k X \arrow[r, "(\epsilon^k_X)_*"'] & \pi_q X \end{tikzcd} \tag{7.3.46}$$

for each $k > 0$ and each integer q. This will imply that $\pi_q\epsilon^k_X$ is an isomorphism, making ϵ^k_X a stable equivalence. The groups in (7.3.46) are the outer four groups of

$$\begin{tikzcd}
\operatorname*{colim}_n \pi_{q+n} G^k X_n \arrow[rr, "(\epsilon^k_{X_n})_*"] \arrow[d, "(\eta^{G^k X}_{n,k})_*"', "\cong"] & & \operatorname*{colim}_n \pi_{q+n} X_n \arrow[dl, "(\eta^k_{X_n})_*"'] \arrow[d, "\cong"', "(\eta^X_{n,k})_*"] \\
& \operatorname*{colim}_n \pi_{q+n} F^k X_n \arrow[dl, "(F^k \eta^X_{n,k})_*"] & \\
\operatorname*{colim}_n \pi_{q+n} \Omega^k G^k X_{n+k} \arrow[rr, "(\Omega^k \epsilon^k_{X_{n+k}})_*"'] & & \operatorname*{colim}_n \pi_{q+n} \Omega^k X_{n+k},
\end{tikzcd}$$

where the maps between colimits are induced componentwise as before. The vertical maps are isomorphisms for the reasons described in the proof of (i). The result follows.

(iii) The nth component of θ^k_X is the map

$$\eta^X_{n,k} : X_n \to \Omega^k X_{n+k}.$$

We have already that the induced map of colimits

$$\operatorname*{colim}_n \pi_{n+q} X_n \to \operatorname*{colim}_n \pi_{n+q} \Omega^k X_{n+k}$$

is an isomorphism, so θ^k_X is a stable equivalence.

(iv) The $(n+k)$th component of σ^k_X is the map

$$\epsilon^X_{n,k} : \Sigma^k X_n \to X_{n+k}.$$

The the left vertical map in

$$\begin{tikzcd}
\operatorname*{colim}_n \pi_{q+n} F^k X_n \arrow[r, "\cong"] \arrow[d, "(\Omega^k \epsilon^X_{n,k})"'] & \operatorname*{colim}_n \pi_{q+n+k} \Sigma^k X_n \arrow[d, "(\epsilon^X_{n,k})_*"] \\
\operatorname*{colim}_n \pi_{q+n} \Omega^k X_{n+k} \arrow[r, "\cong"] & \operatorname*{colim}_n \pi_{q+n+k} X_{n+k},
\end{tikzcd}$$

is the vertical isomorphism of (7.3.45), so the right vertical map is also an isomorphism, making σ^k_X a stable equivalence as desired. □

Now we need to show that there is a fibrant replacement functor R for which the map μ_X of (5.7.2) is a stable equivalence. To this end we want to show that the functor Θ^∞ of Equation 5.7.2 is a fibrant replacement functor. This means showing that its coaugmentation is always a weak equivalence and that its image is always $\mathcal{S}$-fibrant, which is the same as being $\mathcal{S}$-local by Proposition 6.2.12.

Lemma 7.3.47. Fibrant replacement for original spectra commutes with the loop functor. *In the original category of spectra, $\mathcal{S}p(\mathcal{T}, \Sigma)$, the endofunctor $R = \Theta^\infty$ of Definition 5.7.3, is a fibrant replacement functor satisfying the condition of Definition 5.7.3(iii).*

Proof As noted above, we need to show that for each original spectrum X, θ_X^∞ is a stable equivalence and $\Theta^\infty X$ is $\mathcal{S}$-local.

By Theorem 7.3.9, being stably fibrant is equivalent to being an Ω-spectrum. This means that each map in (5.7.4) is a weak equivalence, so θ_X^∞ is a weak equivalence.

To show $\Theta^\infty X$ is $\mathcal{S}$-local, meaning an Ω-spectrum, we need to show that the map

$$\eta_n^{\Theta^\infty X} : (\Theta^\infty X)_n \to (\Theta\Theta^\infty X)_n$$

is a weak equivalence for each n. For this we have

$$\begin{aligned} (\Theta^\infty X)_n &= \operatorname*{hocolim}_k \Omega^k X_{n+k} \\ \text{and} \qquad (\Theta\Theta^\infty X)_n &= \Omega(\Theta^\infty X)_{n+1} \\ &= \Omega \operatorname*{hocolim}_k \Omega^k X_{n+k+1} \\ &\cong \operatorname*{hocolim}_k \Omega^{k+1} X_{n+k+1} \qquad \text{by Lemma 5.8.20 (iv)} \\ &\qquad\qquad\qquad\qquad\qquad \text{since } S^1 \text{ is compact.} \end{aligned}$$

These two homotopy colimits are not the same even though the corresponding categorical colimits are. The first telescope $(\Theta^\infty X)_n$ is a union of cylinders for $k \geqslant 0$, while the second one $(\Theta\Theta^\infty X)_n$ is the union of the same cylinders for $k \geqslant 1$. It follows that $(\Theta^\infty X)_n$ is the mapping cylinder for the composite

$$X_n \xrightarrow{\eta_n^X} \Omega X_{n+1} \xrightarrow{(\theta_{\Theta X}^\infty)_n} \operatorname*{hocolim}_m \Omega^{k+1} X_{n+k+1}.$$

Recall that for a pointed map $f : A \to B$, the map $M_f \to B$ is a weak equivalence as in Proposition 3.5.5. Our map $\eta_n^{\Theta^\infty X}$ has this form, so it is a weak equivalence as desired.

For the condition of Definition 5.7.3(iii) with $R = \Theta^\infty$, a similar calculation (not requiring the use of Proposition 3.5.5) shows that the map $\mu_X : \Omega RX \to R\Omega X$ is an isomorphism. □

7.3F Exact stability for Hovey spectra

For a category of Hovey spectra $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ as in Definition 7.1.1, we could use the same argument to prove similar statements in which the functors Σ and Ω are replaced by T and Ω_T. Then we define functors T^{-1} and Ω_T^{-1} by

$$(T^{-1}X)_n := \begin{cases} X_{n-1} & \text{for } n > 0 \\ * & \text{for } n = 0. \end{cases} \quad \text{and } (\Omega_T^{-1}X)_n := X_{n+1}. \qquad (7.3.48)$$

as in (7.3.38) and (7.3.39).

Remark 7.3.49. Hovey desuspension and the Yoneda functor. *The functor we called T^{-1} in Definition 7.1.20 is* **not** *the analog of desuspension since its source category is $\mathcal{M}$ rather than $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$. Nevertheless we will use the same symbol for our analog of desuspension. We can regard $\mathcal{M}$ as a subcategory of $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$, via the embedding functor*

$$T^{-0} : \mathcal{M} \to \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$$

in which

$$(T^{-0}M)_n = T^n M \qquad \text{for } M \in \mathcal{M}.$$

This makes the functor T^{-1} of (7.3.48) an extension of the functor of the same name in Definition 7.1.20 from the subcategory $\mathcal{M} \cong T^{-0}\mathcal{M}$ to all of $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$.

The following can be proved by the same method as that used to prove Theorem 7.3.42, replacing Σ, Ω and their inverses by T, Ω_T and their inverses. In particular this can be done in both the statement and proof of the analog of Lemma 7.3.43.

Proposition 7.3.50. Hovey desuspension and delooping. *In any category of Hovey spectra $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ as in Definition 7.1.1 there are functors T^{-1} and Ω_T^{-1} as in (7.3.48) with natural transformations*

$$\hat{\theta} : 1_{\mathcal{M}} \Rightarrow \Omega_T^k \Omega_T^{-k} =: \Theta_T^k \qquad \text{and} \qquad \hat{\sigma} : T^k T^{-k} \Rightarrow 1_{\mathcal{M}}$$

similar to those of Definition 5.7.3(i), such that for each spectrum X and each $k > 0$, the following maps are stable equivalences:

(i) $\hat{\eta}_X^k : X \to \Omega_T^k T^k X$ *adjoint to the identity on* $T^k X$*,*
(ii) $\hat{\epsilon}_X^k : T^k \Omega_T^k X \to X$ *adjoint to the identity on* $\Omega_T^k X$*,*
(iii) $\hat{\theta}_X^k : X \to \Omega_T^k \Omega_T^{-k} X$*, and*
(iv) $\hat{\sigma}_X^k : T^k T^{-k} X \to X$*.*

We use the notation $\hat{\eta}_X^k$ to avoid confusion with the map $\eta_X^k : X \to \Omega^k \Sigma^k X$ adjoint to the identity on $\Sigma^k X$, and similarly for the other three maps listed above.

In order to show that $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ is exactly stable as in Definition 5.7.3, we need an additional assumption relating the functors T and Ω_T to Σ and Ω.

The two pairs of functors coincide in the presymmetric case when $K = S^1$. We will be interested in the case where $\mathcal{M} = \mathcal{T}^G$ for a finite group G and $K = S^\rho$, where $\rho = \rho_G$ is the regular real representation of G.

In the presymmetric case we need to assume that $K \cong S^1 \wedge \overline{K}$ for some compact cofibrant $\overline{K}$. When $K = S^\rho$, $\overline{K}$ is the one point compactification of the reduced regular representation of G. See Example 8.9.8 below.

The following is the presymmetric analog of Lemma 7.3.47, and can be proved in the same way.

Lemma 7.3.51. Fibrant replacement for presymmetric spectra. *The fibrant replacement functor for $\mathcal{S}p^{\mathbf{N}}(\mathcal{T}, K)$ with its stable model structure, $R = \Theta^\infty$ as in Definition 5.7.3 commutes with Ω_T.*

Definition 7.3.52. Spectral adjunctions. *Let $\mathcal{M}$ be a stabilizable (as in Definition 7.2.1) pointed topological model category. An invertible (as in Definition 4.5.3) Quillen endopair (T, Ω_T) on $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ is* **spectral** *if there are left Quillen endofunctors $\overline{T}$ and $\overline{T}^{-1}$ (***reduced Hovey suspension and desusupension***) and adjoint right Quillen endofunctors $\overline{\Omega}_T$ and $\overline{\Omega}_T^{-1}$ (***reduced Hovey looping and delooping***) satisfying the following conditions.*

(i) There are natural isomorphisms

$$T \cong \Sigma\overline{T}, \qquad T^{-1} \cong \Sigma^{-1}\overline{T}^{-1},$$
$$\Omega_T \cong \Omega\overline{\Omega}_T \qquad \text{and} \qquad \Omega_T^{-1} \cong \Omega^{-1}\overline{\Omega}_T^{-1}.$$

Moreover the four left (right) Quillen functors, Σ, $\overline{T}$ (Ω, $\overline{\Omega}_T$), and their inverses, commute with each other up to natural isomorphism.

(ii) For each $k > 0$, there are natural transformations

$$\sigma^k : \Sigma^k\Sigma^{-k} \Rightarrow 1_{\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)}, \qquad \overline{\sigma}^k : \overline{T}^k\overline{T}^{-k} \Rightarrow 1_{\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)},$$
$$\theta^k : 1_{\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)} \Rightarrow \Omega^k\Omega^{-k} \qquad \text{and} \qquad \overline{\theta}^k : 1_{\mathcal{S}p^{\mathbf{N}}(\mathcal{M},T)} \Rightarrow \overline{\Omega}_T^k\,\overline{\Omega}_T^{-k},$$

with σ^k and $\overline{\tau}^k$ (θ^k and $\overline{\Omega}_T^k$) inducing weak equivalences on cofibrant (fibrant) objects. We define $\hat{\sigma}^k = \sigma^k\overline{\sigma}^k$ and $\hat{\theta}^k = \theta^k\overline{\theta}^k$.

Theorem 7.3.53. Conditions for exact stability in Hovey spectra. *Let $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ be a Hovey category of spectra in which the Quillen pair (T, Ω_T) is spectral as in Definition 7.3.52.*

In the presymmetric case, in which $T = K \wedge -$ and $\Omega_T = \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)(K, -)$ for a compact cofibrant object K, we assume that $K \cong S^1 \wedge \overline{K}$ for a compact cofibrant object $\overline{K}$. Then $\overline{T} = \overline{K} \wedge -$ and $\overline{\Omega}_T = \mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)(\overline{K}, -)$.

Then $\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$ is exactly stable as in Definition 5.7.3.

Proof Consider the following diagram for a Hovey spectrum X.

$$\begin{tikzcd}
\Sigma^k\Sigma^{-k}T^kT^{-k}X \arrow[r, "\sigma^k_{T^kT^{-k}X}"] \arrow[d, "\cong"'] & \overline{T}^k\overline{T}^{-k}\Sigma^k\Sigma^{-k}X \arrow[r, "\cong"] \arrow[d, "\overline{\sigma}^k_{\Sigma^k\Sigma^{-k}X}"'] & T^kT^{-k}X \arrow[d, "\hat{\sigma}^k_X"] \\
T^kT^{-k}\Sigma^k\Sigma^{-k}X \arrow[r, "\hat{\sigma}^k_{\Sigma^k\Sigma^{-k}X}"'] & \Sigma^k\Sigma^{-k}X \arrow[r, "\sigma^k_X"'] & X
\end{tikzcd}$$

The maps $\widehat{\sigma}^k_X$ and $\widehat{\sigma}^k_{\Sigma^k\Sigma^{-k}X}$ in the diagram are stable equivalences by Proposition 7.3.50(iv). It follows from the 2-of-6 property (see Definition 5.1.1) for stable equivalences that each map in the diagram, including σ^k_X, is a stable equivalence as required for exact stability. There is a similar diagram involving right Quillen functors and showing that θ^k_X is a stable equivalence. This means that the first condition of Definition 5.7.3 is satisfied.

For the second condition of Definition 5.7.3, the decompositions of the functors T and Ω_T lead to decompositions of the unit and counit of the adjunction $T^k \dashv \Omega^k_T$. In the former case we have diagram

$$\begin{tikzcd}
X \arrow[rr, "\overline{\eta}^k_X"] \arrow[d, "\eta^k_X"'] \arrow[drr, "\widehat{\eta}^k_X"] & & \overline{\Omega}^k_T \overline{T}^k X \arrow[d, "\overline{\Omega}^k_T \eta^k_{\overline{T}^k X}"] \\
\Omega^k\Sigma^k X \arrow[rr, "\Omega^k \overline{\eta}^k_{\Sigma^k X}"] & & \Omega^k_T T^k X
\end{tikzcd}$$

that commutes up to natural isomorphism, where $\overline{\eta}^k$ is the unit for the adjunction $\overline{T}^k \dashv \overline{\Omega}^k_T$. This enables us to use the 2-of-6 property to show that η^k_X is a stable equivalence as we did above. A similar argument can be made for the counit ϵ^k_X.

For the third condition of Definition 5.7.3, the fibrant replacement functor Θ^∞ commutes with $\Omega_T \cong \overline{\Omega}_T\Omega$, so it commutes with Ω. $\square$

7.4 Stabilization and model structures for smashable spectra

In §7.3 we described the projective and stable model structures on the category of Hovey spectra (Definition 7.1.1), of which presymmetric spectra (Definition 7.1.13) are a special case. Our aim in this section is to do the same for smashable spectra, meaning functors from a spectral $\mathscr{J}^{\mathbf{O}}$-algebra $\mathscr{J}^{\mathbf{F}}_L$ as in Definition 7.2.19 to a stabilizable (as in Definition 7.2.1) model category $\mathcal{M}$ (over which $\mathscr{J}^{\mathbf{F}}_L$ is enriched) equipped with a compact cofibrant object L. As in Definition 7.2.33, we denote the category of such spectra by

$$\mathcal{S}p = \mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L) := [\mathscr{J}^{\mathbf{F}}_L, \mathcal{M}].$$

The following should be compared with the discussion in [MMSS01, §14].

Definition 7.4.1. The positive model structure. *Let $\mathscr{L}^{\mathbf{F}}_L$ be a positive ideal (as in Definition 7.2.19) in $\mathscr{J}^{\mathbf{F}}_L$. The* **positive model structure** *on the category of smashable spectra $[\mathscr{J}^{\mathbf{F}}_L, \mathcal{M}]$ is the one induced up from the projective model structure on $[\mathscr{L}^{\mathbf{F}}_L, \mathcal{M}]$ as in Theorem 5.4.21.*

Theorem 5.4.21 implies the following.

Proposition 7.4.2. Cofibrant generating set for the positive model structure *are $\mathcal{I}_L^{\mathbf{F},+}$ and $\mathcal{J}_L^{\mathbf{F},+}$ as in (7.4.40) below.*

We will study the Bousfield localization of this structure with respect to a collection $\mathcal{S}$ of stabilizing maps spelled out in (7.4.9) below.

Remark 7.4.3. Why the ideal? *The reader may wonder why we are introducing the positive ideal $\mathscr{L}_L^{\mathbf{F}}$. It has to do with defining a model structure on the category of commutative ring spectra. We refer the reader to Remark 7.0.7(ii) for more information. For the time being the reader may assume the ideal is all of $\mathscr{J}_L^{\mathbf{F}}$ if they wish.*

In particular we want to prove a generalization of Theorem 7.3.36, namely Theorem 7.4.52 below. The following table indicates the parallel steps in the two proofs.

§7.3	§7.4
Theorem 7.3.23	Theorem 7.4.29
Proposition 7.3.27	Proposition 7.4.42
Theorem 7.3.29	Theorem 7.4.43 and Proposition 7.4.50
Theorem 7.3.36	Theorem 7.4.52

7.4A The stable model structure for smashable spectra

A smashable spectrum X is a collection of objects X_V in $\mathcal{M}$ for each object V in $\mathscr{J}_L^{\mathbf{F}}$. There are structure maps generalizing those of (7.2.36),

$$\epsilon_{V,W}^X : \mathscr{J}_L^{\mathbf{F}}(V, V+W) \wedge X_V \to X_{V+W} \tag{7.4.4}$$

Recall (Definition 7.3.1) that the stabilizing maps for presymmetric spectra were of the form

$$s_m : K \wedge K^{-1-m} \to K^{-m}.$$

One could define similar maps in the symmetric and orthogonal cases, with K^{-m} as in Definition 7.2.52. Since the smash product of spectra is defined in these cases, the map s_m is the same as $s_0 \wedge K^{-m}$, where $s_0 : K \wedge K^{-1} \to K^{-0}$. This latter map is more subtle than in the presymmmetric case, even when $K = S^1$, because for $n > 0$, the nth spaces of $S^1 \wedge S^{-1}$ and of S^{-0} are not the same.

We need a different collection of stabilizing maps for smashable spectra. The map $\xi_{0,W}$ of (7.2.63) has the form

$$\xi_{0,W} : S^W \wedge S^{-W} \to S^{-0}, \tag{7.4.5}$$

and in this case the first map in (7.2.22) is the identity morphism on

$$\mathscr{J}_L^{\mathbf{F}}(0, W) \wedge \mathscr{J}_L^{\mathbf{F}}(W, U).$$

When $W = i_{\mathbf{N}}^{\mathbf{F}}\mathbf{n}$, then $S^W = L^{\wedge n}$ and $S^{-W} = L^{-n}$, and we denote the map above by

$$\xi_{0,n} : L^{\wedge n} \wedge L^{-n} \to L^{-0} = S^{-0}. \tag{7.4.6}$$

Let

$$\hat{\mathcal{S}} = \left\{\xi_{V,W} \colon V, W \in ob\, \mathscr{J}_L^{\mathbf{F}}\right\}. \tag{7.4.7}$$

This is the collection of stabilizing maps we want to consider, but we can accomplish the same thing with a smaller set (see (7.4.9) below) with the help of the direct summand condition of Definition 7.2.19(ii).

Definition 7.4.8. *The* **stable model structure on** $[\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$ *is the left Bousfield localization (Definition 6.2.1) of the projective model structure of Proposition 7.1.33 with respect to the morphism set*

$$\mathcal{S} = \left\{\xi_{V,n} \colon L^{\wedge n} \wedge S^{-V} \wedge L^{-n} \to S^{-V}\right\}, \tag{7.4.9}$$

where V ranges over all objects of $\mathscr{J}_L^{\mathbf{F}}$ and n ranges over all integers $n > 0$. (We exclude the case $n = 0$ because $\xi_{V,0}$ is the identity map on S^{-V} and hence uninteresting.) A **stable equivalence** *is an $\mathcal{S}$-local equivalence (see Definition 6.2.1) and a* **stable fibration** *is an $\mathcal{S}$-fibration. A* **stably fibrant spectrum** *is one that is $\mathcal{S}$-fibrant.*

Remark 7.4.10. Theorem 7.3.11 and Theorem 7.3.13 hold for smashable spectra *with the same proofs as before. We leave the details to the reader.*

Proposition 7.4.11. Getting from $\mathcal{S}$ to $\hat{\mathcal{S}}$. *Let $\mathscr{J}_L^{\mathbf{F}}$ be a spectral $\mathscr{J}^{\mathbf{O}}$-algebra as in Definition 7.2.19. In a homotopical structure (see Definition 5.1.1) on the category of spectra $[\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$ in which each morphism in $\mathcal{S}$ (as in (7.4.9)) is a weak equivalence, each morphism in $\hat{\mathcal{S}}$ as in (7.4.7) is also a weak equivalence.*

Proof We will make use of the direct summand condition of Definition 7.2.19(ii). For given objects V and W, choose an object W' such that $S^W \wedge S^{W'} \cong L^{\wedge n}$ for some $n \geqslant 0$. Then consider the diagram

$$\begin{tikzcd}
L^{\wedge n} \wedge S^W \wedge S^{-V\oplus W} \wedge L^{-n} \arrow[rr, "\cong"] \arrow[d, "L^{\wedge n} \wedge \xi_{V,W} \wedge L^{-n}"'] & & S^W \wedge L^{\wedge n} \wedge S^{-V\oplus W} \wedge L^{-n} \arrow[d, "\simeq"', "S^W \wedge \xi_{n,V+W}"] \\
L^{\wedge n} \wedge S^{-V} \wedge L^{-n} \arrow[r, "\xi_{V,n}", "\simeq"'] \arrow[d, "\cong"'] & S^{-V} & S^W \wedge S^{-V\oplus W} \arrow[l, "\xi_{V,W}"'] \\
S^W \wedge S^{W'} \wedge S^{-V\oplus W} \wedge S^{-W'} \arrow[urr, "S^W \wedge \xi_{V+W,W'}"'] & &
\end{tikzcd}$$

Note that if we merge the two pairs of isomorphic nodes and place S^{-V} at the bottom, we get a diagram having the same shape as that of (7.2.47), so the 2-of-6 property of Definition 5.1.1 applies. Since $\xi_{V,n}$ and $S^W \wedge \xi_{n,V+W}$ are weak equivalences, the other morphisms including $\xi_{V,W}$ are as well. □

The following is an analog of Proposition 7.3.20.

Theorem 7.4.12. The stabilizing maps of (7.4.9) are stable homotopy equivalences.

Proof The nth component of $\xi_{V,m}$ is the map $\xi_{V,m,n}$ of (7.2.22). We have

$$\begin{aligned}
\pi_\alpha(L^{\wedge m} \wedge S^{-V} \wedge L^{-m}) &= \operatorname*{colim}_n \pi_{\alpha,n}(L^{\wedge m} \wedge \mathscr{J}_L^{\mathbf{F}}(V+m,n)) \\
\text{and} \qquad \pi_\alpha(S^{-V}) &= \operatorname*{colim}_n \pi_{\alpha,n}\mathscr{J}_L^{\mathbf{F}}(V,n) \\
\pi_\alpha \xi_{V,m} &= \operatorname*{colim}_n \pi_{\alpha,n}\xi_{V,m,n}
\end{aligned}$$

The map $\pi_{\alpha,n}\xi_{V,m,n}$ is an isomorphism for large n by Definition 7.2.19(iii), so $\pi_\alpha\xi_{V,m}$ is an isomorphism. □

We remind the reader that by Corollary 6.2.7, this result implies that **all** stable equivalences are stable homotopy equivalences in these cases.

The following is a special case of Corollary 5.6.27. A very similar statement is Proposition 7.4.41 below.

Proposition 7.4.13. Some projectively cofibrant smashable spectra. *The spectra $L^{\wedge n} \wedge S^{-V} \wedge L^{-n}$ and S^{-V} of Definition 7.4.8 are projectively cofibrant. When V is an object in the ideal $\mathscr{L}_L^{\mathbf{F}}$, they are cofibrant in the positive model structure of Definition 7.4.1.*

Proposition 7.4.14. Smashing a cofibrant object in $\mathcal{M}$ with a stable equivalence. *Suppose $f : X \to Y$ is a stable equivalence and a cofibration between cofibrant objects in the category $\mathcal{S}p = [\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$. (Recall that stable and projective cofibrance are the same.) Then for any cofibrant object A in $\mathcal{M}$, the map $A \wedge f$ is a stable equivalence.*

Corollary 7.4.15. Smashing a cofibrant object in $\mathcal{M}$ with $\widetilde{\xi}_{V,n}$. *For any cofibrant object A in $\mathcal{M}$, the map $A \wedge \widetilde{\xi}_{V,n}$ (for $\widetilde{\xi}_{V,n}$ as in Remark 7.3.5) is a stable equivalence.*

Proof of Proposition 7.4.14. By Corollary 5.6.18(ii), f being a stable weak equivalence is equivalent to the map $f^* : \mathcal{S}p(Y,Z) \to \mathcal{S}p(X,Z)$ being a weak equivalence in $\mathcal{M}$ for every stably fibrant spectrum Z. We need to show that this implies that for any cofibrant object A in $\mathcal{M}$, the map

$$(A \wedge f)^* : \mathcal{S}p(A \wedge Y, Z) \to \mathcal{S}p(A \wedge X, Z)$$

is also a weak equivalence in $\mathcal{M}$. The fact that $\mathcal{S}p$ is a Quillen $\mathcal{M}$-module as in Definition 5.6.3 means that there are natural isomorphisms as in (3.1.48), namely

$$\begin{tikzcd}
\mathcal{S}p(X, Z^A) & \mathcal{S}p(A \wedge X, Z) \arrow[l, "\phi_\ell"', "\cong"] \arrow[r, "\phi_r", "\cong"'] & \mathcal{M}(A, \mathcal{S}p(X, Z)) \\
\mathcal{S}p(Y, Z^A) \arrow[u, "f^*"] & \mathcal{S}p(A \wedge Y, Z) \arrow[l, "\phi_\ell"', "\cong"] \arrow[r, "\phi_r", "\cong"'] \arrow[u, "(A \wedge f)^*"'] & \mathcal{M}(A, \mathcal{S}p(Y, Z)). \arrow[u, "\mathcal{M}(A, f^*)"']
\end{tikzcd} \tag{7.4.16}$$

By Corollary 5.6.19 the functor $\mathcal{M}(-,-)$ is homotopical when the first variable is cofibrant and the second is fibrant. We know that A is cofibrant while $\mathcal{S}p(X, Z)$ and $\mathcal{S}p(Y, Z)$ are fibrant. This makes the map $\mathcal{M}(A, f^*)$ a weak equivalence, so $(A \wedge f)^*$ is one also. Since this holds for each stably fibrant Z, the map $A \wedge f$ is a stable equivalence as desired.

Alternatively, we can show that the map f^* on the left in (7.4.16) is a weak equivalence in $\mathcal{M}$ as follows. We will show in Theorem 7.4.35 below that a spectrum is stably fibrant iff it is a Ω_L-spectrum. Corollary 5.6.19 also says that the functor $\mathcal{S}p(-,-)$ is homotopical (in the stable model structure) when the first variable is cofibrant and the second is stably fibrant. We can apply this to the map f^* since X and Y are cofibrant by assumption and Z^A is a Ω_L-spectrum by Proposition 7.2.51 and hence stably fibrant. □

For future reference we look at the Uth component of the factorization of $\xi_{V,n}$ of Remark 7.3.5, for an object U in $\mathscr{J}_L^{\mathbf{F}}$. We get

$$\begin{tikzcd}
L^{\wedge n} \wedge \mathscr{J}_L^{\mathbf{F}}(V \oplus i_{\mathbf{\Sigma}}^{\mathbf{F}}(\mathbf{n}), U) \arrow[rr, "(\xi_{V,n})_U"] \arrow[dr, "(\widetilde{\xi}_{V,n})_U"'] & & \mathscr{J}_L^{\mathbf{F}}(V, U) \\
& \left(\widetilde{S}_n^{-V}\right)_U \arrow[ur, "(\widehat{\xi}_{V,n})_U"'] &
\end{tikzcd} \tag{7.4.17}$$

where the map $(\xi_{V,n})_U$ is the composite of (7.2.22), which is a map between cofibrant objects in $\mathcal{M}$. Since $\widehat{\xi}_{V,n}$ is a projective weak equivalence, $(\widehat{\xi}_{V,n})_U$ is a weak equivalence in $\mathcal{M}$. Since $\widetilde{\xi}_{V,n}$ is a cofibration, each component of it is by Proposition 5.4.4(i). This means that each component of $\widetilde{S}_n^{-V}$ is cofibrant in $\mathcal{M}$.

Remark 7.4.18. Alternative stabilizing maps for presymmetric spectra. *We could use a similar set of maps in the presymmetric case, namely*

$$\left\{\widetilde{\xi}_n : K^{\wedge n} \wedge K^{-n} \to \widetilde{K}^{-0} : n > 0\right\},$$

in place of the maps of (7.3.3). We leave the details to the interested reader.

7.4B The functors Θ and Θ^∞ for smashable and symmetric spectra

Defining the functors Θ and Θ^∞ for smashable and symmetric spectra is more delicate than in the case of Hovey spectra described in Proposition 7.3.50. First we need the following.

Recall that the category of smashable and symmetric spectra $[\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}] = \mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$ is closed symmetric monoidal by Theorem 7.2.60. We denote its internal Hom functor by $F(-,-)$.

When $V = i_{\boldsymbol{\Sigma}}^{\mathbf{F}}\mathbf{n}$, we may write n instead of V, and $W+n$ instead $W \oplus V$. There is a compact cofibrant object L (S^{ρ_G} in the case of orthogonal equivariant G-spectra) such that $\mathscr{J}^{\mathbf{F}}(0,n) = L^{\wedge n}$ for each $n \geqslant 0$.

Definition 7.4.19. The desuspension and delooping endofunctors Σ^{-V} *and* Ω^{-V} *in* $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$*, where* $\mathbf{F}$ *is any of the four values of Definition 7.2.33 (excluding the presymmetric case), are defined by*

$$\Sigma^{-V} X = S^{-V} \wedge X \qquad \textit{and} \qquad \Omega^{-V} X = F(S^{-V}, X),$$

where S^{-V} *is the Yoneda spectrum of Definition 7.2.52.*

We define natural transformations

$$\sigma^V : \Sigma^V \Sigma^{-V} \Rightarrow 1_{\mathcal{S}p} \qquad \textit{and} \qquad \theta^V : 1_{\mathcal{S}p} \Rightarrow \Theta^V := \Omega^V \Omega^{-V},$$

as in Definition 5.7.3(i), by

$$\sigma_X^V = \xi_{0,V} \wedge X : S^V \wedge S^{-V} \wedge X \to S^{-0} \wedge X \cong X$$

and

$$\theta_X^V = \xi_{0,V}^* : X \cong F(S^{-0}, X) \to F(S^V \wedge S^{-V}, X) \cong \Omega^V F(S^{-V}, X) \cong \Theta^V X,$$

where

$$\xi_{0,V} : S^V \wedge S^{-V} \to S^{-0}$$

is the map of (7.2.22).

We cannot define formal desuspension for presymmetric spectra this way, because that category does not have a smash product.

Proposition 7.4.20. For each object V **in** $\mathscr{J}$**, the functor** $\Omega^{-V}(-)$ *is given by*

$$(\Omega^{-V} X)_W \cong X_{V+W}.$$

Proof Using the Yoneda adjunction of Remark 2.2.35, we have

$$\begin{aligned} F(S^{-V}, X)_W &\cong F(S^{-W}, F(S^{-V}, X))_0 \\ &\cong F(S^{-W} \wedge S^{-V}, X)_0 && \text{by Theorem 7.2.60} \\ &\cong F(S^{-V \oplus W}, X)_0 && \text{by Proposition 3.3.14} \\ &\cong X_{V+W}. \end{aligned}$$

□

Corollary 7.4.21. Structure and costructure maps for $\Omega^{-V}X$. *To define its structure map, let $J^{\mathbf{F}}_{U,W}$ denote $\mathscr{J}^{\mathbf{F}}_L(U, U\oplus W)$. Then the structure map for $\Omega^{-V}X$ is the composite*

$$\begin{tikzcd}
J^{\mathbf{F}}_{U,W}\wedge(\Omega^{-V}X)_U \arrow[rr, "\epsilon^{\Omega^{-V}X}_{U,W}"] \arrow[d, equal] & & (\Omega^{-V}X)_{U+W} \arrow[d, equal] \\
J^{\mathbf{F}}_{U,W}\wedge X_{U+V} \arrow[r, "\alpha\wedge X_{V+W}"] & J^{\mathbf{F}}_{U+V,W}\wedge X_{V+U} \arrow[r, "\epsilon^{X}_{U+V,W}"] & X_{U+V+W},
\end{tikzcd}$$

where

$$\alpha=\alpha_{V,U,U+W}: J^{\mathbf{F}}_{U,W}\to J^{\mathbf{F}}_{U+V,W}$$

is the addition morphism of Definition 2.6.6.

Adjointly, its costructure map is the composite

$$\begin{tikzcd}
(\Omega^{-V}X)_U \arrow[rr, "\eta^{\Omega^{-V}X}_{U,W}"] \arrow[d, equal] & & \mathcal{M}(J^{\mathbf{F}}_{U,W},(\Omega^{-V}X)_{U+W}) \arrow[d, equal] \\
X_{U+V} \arrow[r, "\eta^{X}_{U+V,W}"] & \mathcal{M}(J^{\mathbf{F}}_{U+V,W},X_{U+W+V}) \arrow[r, "\alpha^*"] & \mathcal{M}(J^{\mathbf{F}}_{U,W},X_{U+W+V}).
\end{tikzcd}$$

Proposition 7.4.22. Formal delooping commutes with cotensors. *For any smashable or symmetric spectrum X and any object M in $\mathcal{M}$,*

$$(\Omega^{-V}X)^M\cong\Omega^{-V}(X^M).$$

The interested reader can verify that both Ω^{-V} and Σ^{-V} commute with tensors, but Σ^{-V} does not commute with cotensors over $\mathcal{M}$.

Proof We will verify the isomomorphism componentwise.

$$\begin{aligned}\left((\Omega^{-V}X)^M\right)_W &\cong \left((\Omega^{-V}X)_W\right)^M\cong (X_{V+W})^M\\ &\cong \left(X^M\right)_{V+W}\cong\left(\Omega^{-V}X^M\right)_W.\end{aligned}$$ □

The following is an exercise for the reader.

Proposition 7.4.23. Properties of smashable and symmetricdesuspension and delooping. *The functors Σ^{-V} and Ω^{-V} of Definition 7.4.19 are left and right Quillen functors respectively with $\Sigma^{-V}\dashv\Omega^{-V}$. The left Quillen functors Σ^V, Σ^W, Σ^{-V} and Σ^{-W} all commute with each other up to natural isomorphism. The same is true of the right Quillen functors Ω^V, Ω^W, Ω^{-V} and Ω^{-W}.*

Here is our generalization of the functor Θ of Definition 5.7.3.

Definition 7.4.24. The functor Θ for smashable and symmetricspectra. *We will denote this functor by $\Theta_{\mathbf{F}}$. We will often omit the subscript. It is defined by*

$$\Theta X = (\Omega_L^{-1} X)^L = F(L \wedge L^{-1}, X),$$

so $(\Theta X)_V = (\Omega_L X)_{V+1}$. Thus its structure map

$$\epsilon^{\Theta X}_{V,W} : J^{\mathbf{F}}_{V,W} \wedge \Omega_L X_{V+1} \to \Omega_L X_{V+W+1}$$

is adjoint to a map

$$\begin{aligned} \Omega_L X_{V+1} \to \mathcal{M}(J^{\mathbf{F}}_{V,W}, \Omega_L X_{V+W+1}) &\cong \mathcal{M}(J^{\mathbf{F}}_{V,W} \wedge L, X_{V+W+1}) \\ &\cong \Omega_L \mathcal{M}(J^{\mathbf{F}}_{V,W}, X_{V+W+1}), \end{aligned}$$

namely $\Omega_L \eta^{\Omega_L^{-1} X}_{V+1,W}$.

Definition 7.4.25. The coaugmentation for Θ *on a smashable or symmetric spectrum X is the map*

$$\theta_X = (\xi_{0,1})^* : X = F(S^{-0}, X) \to F(L \wedge L^{-1}, X),$$

where $\xi_{0,1}$ is as in (7.4.6).

The following is proved by Hovey in [Hov01b, Lemma 4.5] for Hovey spectra, for which Θ and θ_X are defined in Definition 5.7.3.

With this in hand we can define Θ^∞ and prove Lemma 7.3.22 for smashable spectra.

Definition 7.4.26. Θ^∞ for smashable spectra. *For a spectral $\mathscr{J}^{\mathbf{O}}$-algebra $\mathscr{J}^{\mathbf{F}}_L$ let $\Theta^\infty X$ be the homotopy colimit (meaning the telescope as in Example 5.8.5 (iv)) in the category $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$ of*

$$X \xrightarrow{\theta_X} \Theta X \xrightarrow{\theta_{\Theta X}} \Theta^2 X \xrightarrow{\theta_{\Theta^2 X}} \Theta^3 X \xrightarrow{\theta_{\Theta^3 X}} \cdots . \tag{7.4.27}$$

where θ_X is the coaugmentation map of Definition 7.4.25.

Let $\theta^\infty_X : X \to \Theta^\infty X$ be the obvious natural map. It is the X-component of a natural transformation $\theta^\infty : 1_{\mathcal{S}p} \Rightarrow \Theta^\infty$, which is also a coaugmentation θ^∞ for Θ^∞. We will denote the composite map $X \to \Theta^k X$ by θ^k_X.

The following has essentially the same proof as Lemma 7.3.22.

Lemma 7.4.28. Properties of Θ^∞ for smashable and symmetric spectra. *With notation as in Definition 7.4.24 and Definition 7.4.26,*

(i) The map $\theta_{\Theta^\infty X} : \Theta^\infty X \to \Theta(\Theta^\infty X)$ is a weak equivalence. In particular $\Theta^\infty X$ is a Ω-spectrum as in Definition 7.2.45.

(ii) If Z is a Ω_L-spectrum, then the map $\theta_Z^\infty : Z \to \Theta^\infty Z$ is a strict equivalence, so Z is Θ^∞-local. In particular for any spectrum X, the map

$$\theta_{\Theta^\infty X}^\infty : \Theta^\infty X \to \Theta^\infty(\Theta^\infty X)$$

is a projective weak equivalence and Θ^∞ is homotopy idempotent as in Definition 6.2.15.

(iii) A morphism $f : X \to Y$ in $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$ is a stable equivalence iff the induced map $f^* : \mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)(Y, Z) \to \mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)(X, Z)$ is a weak equivalence for all Z for which the map $\theta_Z^\infty : Z \to \Theta^\infty Z$ (or equivalently the map $\theta_Z : Z \to \Theta Z$) is an isomorphism.

Proof (i) $\Theta^\infty X$ is a Ω-spectrum because it satisfies the condition of Proposition 7.2.46.

The rest of the argument is similar to that of Lemma 7.3.22. □

The following has the same proof as Theorem 7.3.23.

Theorem 7.4.29. Stable equivalences and $\Theta_{\mathbf{F}}^\infty$ for smashable spectra. *Let $\mathcal{M}$ be a stabilizable model category as in Definition 7.2.1 with a compact cofibrant object $L \cong S^1 \wedge \overline{L}$, and let $\mathscr{J}_L^{\mathbf{F}}$ be a spectral $\mathscr{J}^{\mathbf{O}}$-algebra as in Definition 7.2.19.*

(i) If $f : X \to Y$ is a map in $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, K)$ such that $\Theta_{\mathbf{F}}^\infty f$ is a projective weak equivalence, then f is a stable equivalence as in Definition 7.4.8.

(ii) The map $\theta_X^\infty : X \to \Theta_{\mathbf{F}}^\infty X$ is a stable equivalence for all spectra X.

(iii) For all X in $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, K)$ the map θ_X^∞ is a stable equivalence into a Ω-spectrum as in Definition 7.2.45, and therefore a fibrant approximation for the stable model structure.

(iv) If a map $f : X \to Y$ is a stable equivalence, then $\Theta_{\mathbf{F}}^\infty f$ is a projective weak equivalence.

(v) Every stable homotopy equivalence of smashable spectra is a stable equivalence.

Remark 7.4.30. $\Theta_{\mathbf{F}}^\infty$ is a homotopy idempotent functor *by Lemma 7.4.28 (ii), so the notions of Definition 6.2.15 apply to it. A map of smashable spectra is a $\Theta_{\mathbf{F}}^\infty$-equivalence iff it is a stable equivalence by Theorem 7.4.29(ii) and (iv).*

Remark 7.4.31. Stable equivalances of symmetric spectra need not be stable homotopy equivalences*, as we explained in §7.0E. Hence Corollary 7.3.25 holds only for smashable ones. Indeed our reason for defining the latter is to exclude the problems of symmetric spectra. The proof carries over to the smashable case with no difficulty and is left to the reader.*

7.4C Stabilizing maps for smashable spectra

Recall the restricted costructure map $\overline{\eta}^X_{V,W}$ for smashable spectra of Definition 7.2.42.

Lemma 7.4.32. An alternate description of the restricted costructure map. *The map $\overline{\eta}^X_{V,W}$ is the composite*

$$\begin{tikzcd}
X_V \arrow[r, "\cong"] & \mathcal{S}p(S^{-V}, X) \arrow[d, "(\xi_{V,W})^*"'] & & \\
& \mathcal{S}p(S^W \wedge S^{-W\oplus V}, X) \arrow[r, "\cong"] & \mathcal{S}p(S^{-W\oplus V}, \Omega^W X) \arrow[r, "\cong"] & \Omega^W X_{W+V},
\end{tikzcd}$$

where $\xi_{V,W}$ is the map of (7.2.63), and the three isomorphisms are associated with the three adjunctions $S^{-V} \wedge (-) \dashv \mathrm{Ev}_V$, $S^W \wedge (-) \dashv \Omega^W$ and $S^{-W\oplus V} \wedge (-) \dashv \mathrm{Ev}_{V+W}$.

The first and third adjunctions above are cases of the enriched Yoneda adjunction of Proposition 3.1.71 and Proposition 5.6.28.

Proof The statement is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
S^W \wedge X_V \arrow[r, "\cong"] \arrow[ddd, "\omega^{\mathbf{F}}_{V,\mathbf{o},W} \wedge X_V"'] & S^W \wedge \mathcal{S}p(S^{-V}, X) \arrow[d, "(\xi_{V,W})^*"] \\
& S^W \wedge \mathcal{S}p(S^W \wedge S^{-W\oplus V}, X) \arrow[d, "\cong"] \\
& S^W \wedge \mathcal{M}(S^W, X_{W+V}) \arrow[d, "\mathrm{Ev}"] \\
J^{\mathbf{F}}_{V,W} \wedge X_V \arrow[r, "\epsilon^X_{V,W}"] & X_{W+V},
\end{tikzcd} \tag{7.4.33}$$

where Ev is the evaluation map of Example 2.1.16(v), since $\overline{\eta}^X_{V,W}$ is the right adjoint of the counterclockwise composition. By writing the morphism objects in $\mathcal{S}p$ as ends as in Definition 3.2.18, we write the vertical isomorphism as the composite

$$
\begin{array}{c}
S^W \wedge \int^{U\in\mathscr{J}_L^{\mathbf{F}}} \mathcal{M}(S^W \wedge \mathscr{J}_L^{\mathbf{F}}(W\oplus V, U), X_U) \\
\Big\downarrow \cong \\
S^W \wedge \int^{U\in\mathscr{J}_L^{\mathbf{F}}} \mathcal{M}(S^W, \mathcal{M}(\mathscr{J}_L^{\mathbf{F}}(W\oplus V, U), X_U)) \\
\Big\downarrow \cong \\
S^W \wedge \mathcal{M}\left(S^W, \int^{U\in\mathscr{J}_L^{\mathbf{F}}} \mathcal{M}(\mathscr{J}_L^{\mathbf{F}}(W\oplus V, U), X_U)\right) \\
\Big\downarrow \cong \\
S^W \wedge \mathcal{M}(S^W, X_{W+V}).
\end{array}
\tag{7.4.34}
$$

Then

- the first isomorphism in (7.4.34) follows from the fact that $\mathcal{M}$ is a closed symmetric monoidal category, so for any objects A, B and C

$$
\mathcal{M}(A \wedge B, C) \cong \mathcal{M}(A, \mathcal{M}(B, C)),
$$

- the second one follows from the fact that the functor $\mathcal{M}(A, -)$ commutes with ends by Proposition 3.2.16(i) and
- the third one is the enriched Yoneda reduction of Proposition 3.2.25.

From (7.2.22) we see that the Uth component of $\xi_{V,W}$ is involves the map

$$
\mathscr{J}_L^{\mathbf{F}}(W\oplus V, U) \wedge \omega^{\mathbf{F}}_{V,0,W}.
$$

The result follows. □

Theorem 7.4.35. Stably fibrant smashable spectra are Ω-spectra. *Let $\mathcal{M}$ and K be as in Definition 7.4.8 and let $\mathcal{S}$ be as in (7.4.9). Then a smashable spectrum is stably fibrant (equivalently $\mathcal{S}$-local by Proposition 6.2.12) iff it is a Ω-spectrum as in Definition 7.2.45. The map $\widetilde{\xi}_{V,n}$ of (7.4.17) is a stable equivalence for each V and n.*

Proof A spectrum X is stably fibrant iff the morphism $(\xi_{V,n})^*$ below is a weak equivalence for each V and n.

$$
\begin{tikzcd}
\mathcal{S}p(S^{-V}, X) \arrow[rr, "\cong"] \arrow[d, "(\xi_{V,n})^*"'] & & X_V \arrow[d] \\
\mathcal{S}p(L^{\wedge n} \wedge S^{-V} \wedge L^{-n}, X) \arrow[r, "\cong"] & \mathcal{M}(L^{\wedge n}, X_{V+n}) & = \Omega_L^n X_{V+n}
\end{tikzcd}
$$

The vertical map on the right is $\overline{\eta}^X_{V,n}$ by Lemma 7.4.32. This is equivalent to X being a Ω_L-spectrum by Proposition 7.2.46.

It follows from Proposition 7.4.11 that the map $\xi_{V,W}$ of (7.2.63) is a stable equivalence. In the factorization of (7.4.17), $\hat{\xi}_{V,n}$ is a projective equivalence and therefore a stable one. This makes the cofibration $\tilde{\xi}_{V,n}$ a stable equivalence and hence a stably trivial cofibration as claimed. □

We remind the reader that there is more than one model structure on the functor category $[\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$. In addition to the projective model structure, we have the one induced up (using a left Kan extension and the Crans-Kan Transfer Theorem 5.2.27) from the projective structure on $[\mathscr{K}, \mathcal{M}]$ for any full subcategory $\mathscr{K}$ of $\mathscr{J}_L^{\mathbf{F}}$, as explained in Remark 5.4.23. **We will assume for the rest of this section that $\mathscr{K}$ is a fixed positive ideal $\mathscr{L}_L^{\mathbf{F}}$** as in Definition 7.2.19.

Definition 7.4.36. Four model structures for smashable spectra. *In addition to the* **projective model structure** *on the functor category*

$$\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L) = [\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}],$$

there are three others. When the full subcategory $\mathscr{K}$ of $\mathscr{J}_L^{\mathbf{F}}$ is a positive ideal $\mathscr{L}_L^{\mathbf{F}}$ as above, we will refer to the model structure on $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$ induced up from the projective one on $[\mathscr{L}_L^{\mathbf{F}}, \mathcal{M}]$ as the **positive model structure***. Its localization with respect to the set $\mathcal{S}$ of (7.4.9) is the* **positive stable model structure***. We will refer to the localization of the projective model structure with respect to $\mathcal{S}$ as simply the* **stable model structure***.*

Proposition 7.4.37. Stable equivalences and positive ideals. *Let $\mathscr{L}_L^{\mathbf{F}}$ be a positive ideal in a spectral $\mathscr{J}^{\mathbf{O}}$-algebra $\mathscr{J}_L^{\mathbf{F}}$ as in Definition 7.2.19. Suppose $f : X \to Y$ is a morphism in $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L) = [\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$ such that f_V is a weak equivalence in $\mathcal{M}$ for each object V in the ideal. Then f is a stable equivalence as in Definition 7.4.8.*

Proof We will use Theorem 7.4.29(iv), which says $f : X \to Y$ is a stable equivalence if and only if $\Theta_{\mathbf{F}}^{\infty} f$ is a projective equivalence. It suffices to show that if f_V is a weak equivalence for each object V in the ideal, then $(\Theta^{\infty} f)_V$ is one for all V. Given the definition of Θ^{∞} (Definition 7.4.26), it suffices to show that $(\Theta f)_V$ is a weak equivalence. Since the ideal is positive, it contains $V \oplus i^{\mathbf{F}\boldsymbol{\Sigma}}(\mathbf{1})$. It follows that

$$(\Theta f)_V = \Omega_L f_{1+V}.$$

The map f_{1+V} is a weak equivalence of fibrant objects, and the right Quillen functor Ω_L converts such a morphism into another weak equivalence. □

This means that even though the positive model structure on $[\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$ has more weak equivalences than the projective one, the collection of stable equivalences is larger still and is **the same for both**. Localizing either model structure with respect to the set $\mathcal{S}$ stabilizing maps of (7.4.9) gives the same homotopical structure, but differing collections of cofibrations and fibrations. The positive and stable positive model structures have fewer cofibrations (more fibrations) than the projective and stable ones.

7.4D Cofibrant generation for the positive stable model structure

We want to describe the positive stable model structure (as in Definition 7.4.36) in terms of cofibrant generation. Consider the set of maps

$$\mathcal{S}^+ = \left\{ \widetilde{\xi}_{V,n} \colon V \in ob\mathscr{L}_L^{\mathbf{F}},\ n > 0 \right\}. \tag{7.4.38}$$

with $\widetilde{\xi}_{V,n}$ as in (7.4.17). We remind the reader that the requirement that $n > 0$ is **not** related to the positive ideal $\mathscr{L}_L^{\mathbf{F}}$. It is instead a nontriviality condition, since the map $\widetilde{\xi}_{V,0}$ for any V is an identity map.

Remark 7.4.39. The purpose of the set $\mathcal{S}^+$. *The set $\mathcal{S}^+$ above requires V to be in the ideal because we need such a requirement in the generating set $\mathcal{K}_L^{\mathbf{F},+}$ of positive stably trivial cofibrations below. We are* **not** *using it to define stabilization, which is Bousfield localization with respect to the set $\mathcal{S}$ of (7.4.9). Then it is clear from Proposition 7.4.11 that each map in $\mathcal{S}^+$ is an $\mathcal{S}$-equivalence, so each map in the set $\mathcal{K}_L^{\mathbf{F},+}$ below is a stably trivial cofibration.*

Let $\mathcal{I}$ and $\mathcal{J}$ be cofibrant generating sets of $\mathcal{M}$. Let

$$\left.\begin{aligned}
\mathcal{I}_L^{\mathbf{F}} &= \left\{ \mathcal{I} \wedge S^{-V} \colon V \in ob\,\mathscr{J}_L^{\mathbf{F}} \right\}, \\
\mathcal{J}_L^{\mathbf{F}} &= \left\{ \mathcal{J} \wedge S^{-V} \colon V \in ob\,\mathscr{J}_L^{\mathbf{F}} \right\} \\
\mathcal{K}_L^{\mathbf{F}} &= \mathcal{J}_L^{\mathbf{F}} \cup (\mathcal{I} \square \mathcal{S}), \\
\mathcal{I}_L^{\mathbf{F},+} &= \left\{ \mathcal{I} \wedge S^{-V} \colon V \in ob\mathscr{L}_L^{\mathbf{F}} \right\}, \\
\mathcal{J}_L^{\mathbf{F},+} &= \left\{ \mathcal{J} \wedge S^{-V} \colon V \in ob\mathscr{L}_L^{\mathbf{F}} \right\} \\
\text{and}\quad \mathcal{K}_L^{\mathbf{F},+} &= \mathcal{J}_L^{\mathbf{F},+} \cup \left(\mathcal{I} \square \mathcal{S}^+ \right).
\end{aligned}\right\} \tag{7.4.40}$$

The following is similar to Proposition 7.4.13.

Proposition 7.4.41. Some positive cofibrant smashable spectra. *When V is an object in the ideal $\mathscr{L}_L^{\mathbf{F}}$, the spectra $K^{\wedge n} \wedge S^{-V} \wedge K^{-n}$ and S^{-V} of Definition 7.4.8 are cofibrant in the positive model structure.*

Proof Recall that Theorem 5.6.38 gives cofibrant generating sets for an induced model structure such as the positive one of Definition 7.4.36. In this case they are $\mathcal{I}_L^{\mathbf{F},+}$ and $\mathcal{J}_L^{\mathbf{F},+}$. It follows that for any cofibrant object A in $\mathcal{M}$ and any object W in $\mathscr{L}_L^{\mathbf{F}}$, the spectrum $A \wedge S^{-W}$ is positively cofibrant. □

Here is the analog of Proposition 7.3.27. It requires a proof since the model structure we are studying is not the Bousfield localization of the projective model structure, but of the positive one.

Proposition 7.4.42. Toward trivial positive stable fibrations in $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$. *Any map $p : X \to Y$ in $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$ having the right lifting property with respect to $\mathcal{I}_L^{\mathbf{F},+}$ as in (7.4.40) is a positive weak equivalence and hence a trivial positive fibration.*

Proof Let $f : A \to B$ be a map in $\mathcal{I}$, the set of generating cofibrations for $\mathcal{M}$. The right lifting property means that for each object V in the ideal and each $n \geqslant 0$, we have a lifting

$$\begin{tikzcd} A \wedge S^{-V} \arrow[r] \arrow[d, "i \wedge S^{-V}"'] & X \arrow[d, "p"] \\ B \wedge S^{-V} \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

where S^{-V} is the smashable Yoneda spectrum of Definition 7.2.52. By Proposition 7.2.55 this is adjoint to a lifting

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X_V \arrow[d, "p_V"] \\ B \arrow[r] \arrow[ur, dashed] & Y_V \end{tikzcd}$$

in $\mathcal{M}$. Hence the map p_V is a trivial fibration, and in particular a weak equivalence, in $\mathcal{M}$. This means that p is a positive weak equivalence. □

Here is a partial analog of Theorem 7.3.29.

Theorem 7.4.43. Trivial positive stable fibrations for smashable spectra. *A morphism in $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$, $p : X \to Y$ has the right lifting property with respect to the set $\mathcal{K}_L^{\mathbf{F},+}$ of (7.4.40) iff for each object V in the ideal $\mathscr{L}_L^{\mathbf{F}}$ and each integer $n > 0$, the map p_V is a fibration in $\mathcal{M}$ and the following diagram is homotopy Cartesian:*

$$\begin{tikzcd} X_V \arrow[r, "p_V"] \arrow[d, "\overline{\eta}^X_{V,n}"'] & Y_V \arrow[d, "\overline{\eta}^Y_{V,n}"] \\ \Omega_L^n X_{n+V} \arrow[r, "\Omega_L^n p_{n+V}"] & \Omega_L^n Y_{n+V}, \end{tikzcd} \tag{7.4.44}$$

where the vertical maps are as in Definition 7.2.42, and functor $\Omega_L^n = \mathcal{M}(L^{\wedge n}, -)$ as in Definition 7.2.29.

Proof As before we will abbreviate $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$ by $\mathcal{S}p$. Any map p with the right lifting property with respect to $\mathcal{J}_L^{\mathbf{F},+}$ is a projective fibration, so p_V is a fibration in $\mathcal{M}$ for each V.

To analyze the right lifting property with respect to the pushout corner maps in $\mathcal{K}_L^{\mathbf{F},+}$, we use Proposition 3.1.53 with the categories $\mathcal{C}$ and $\mathcal{E}$ replaced by $\mathcal{M}$ and $\mathcal{S}p$, and the maps g, i and f replaced by p, $\widetilde{\xi}_{V,n}$ (see Definition 7.3.6) and f. It says that p has the right lifting property with respect to $f \,\square\, \widetilde{\xi}_{V,n}$ (for a morphism $f : A \to B$ in $\mathcal{I}$, with $\widetilde{\xi}_{V,n}$ as in (7.4.17)) in $\mathcal{S}p$ iff f has the left lifting property with respect to the lifting test map $\mathcal{S}p_{\diamondsuit}(\widetilde{\xi}_{V,n}, p)$ (Definition 2.3.14) in $\mathcal{M}$. This is the pullback corner map for the following diagram in $\mathcal{M}$.

$$\begin{tikzcd}
\mathcal{S}p(\widetilde{S}_n^{-0} \wedge S^{-V}, X) \arrow[r, "p_*"] \arrow[d, "(\widetilde{\xi}_{V,n})^*"'] & \mathcal{S}p(\widetilde{S}_n^{-0} \wedge S^{-V}, Y) \arrow[d, "(\widetilde{\xi}_{V,n})^*"] \\
\mathcal{S}p(L^{\wedge n} \wedge S^{-W} \wedge L^{-n}, X) \arrow[r, "p_*"] & \mathcal{S}p(L^{\wedge n} \wedge S^{-V} \wedge L^{-n}, Y).
\end{tikzcd} \tag{7.4.45}$$

If each such f has the left lifting property with respect to the pullback corner map of (7.4.45), the latter is a trivial fibration and hence a weak equivalence, so the diagram is homotopy Cartesian.

We claim that the vertical maps in

$$\begin{tikzcd}
\mathcal{S}p(S^{-V}, X) \arrow[r, "p_*"] \arrow[d, "(\widehat{\xi}_{V,n})^*"'] & \mathcal{S}p(S^{-V}, Y) \arrow[d, "(\widehat{\xi}_{V,n})^*"] \\
\mathcal{S}p(\widetilde{S}_n^{-0} \wedge S^{-V}, X) \arrow[r, "p_*"] & \mathcal{S}p(\widetilde{S}_n^{-0} \wedge S^{-V}, Y),
\end{tikzcd}$$

with $\widehat{\xi}_{V,n}$ and $\widetilde{S}_n^{-0}$ as in (7.4.17), are weak equivalences. We will derive this from Lemma 5.8.51. Our assumption $\mathcal{M}$ is telescopically closed as in Definition 5.8.28 implies that all spectra, including X and Y, are projectively fibrant. We can deduce from (7.4.40) that S^{-V} and $S^{-V} \wedge L^{-n}$ are both projectively cofibrant. Since $L^{\wedge n}$ is by definition a compact cofibrant object in $\mathcal{M}$, $L^{\wedge n} \wedge S^{-W \oplus V}$ is also projectively cofibrant. Since $\widetilde{\xi}_{V,n}$ is a cofibration, $\widetilde{S}_n^{-0} \wedge S^{-V}$ is again projectively cofibrant, and the claim is verified.

Recall the object $L^{\wedge n}$ and the functor Ω^W of Definition 7.2.29. Using the adjunctions $L^{-W} \wedge (-) \dashv \mathrm{Ev}_n$, and $L^{\wedge n} \wedge (-) \dashv \Omega_L^n$, we can embed (7.4.45) in a larger diagram comparable to (7.3.31),

$$
\begin{tikzcd}
X_V \arrow[r, "p_V"] \arrow[d, "\cong"'] & Y_V \arrow[d, "\cong"] \\
\mathcal{S}p(S^{-V}, X) \arrow[r, "p_*"] \arrow[d, "(\hat{\xi}_{V,n})^*"', "\simeq"] \arrow[dd, bend right=60, "(\xi_{V,n})^*"'] & \mathcal{S}p(S^{-V}, Y) \arrow[d, "\simeq"', "(\hat{\xi}_{V,n})^*"] \arrow[dd, bend left=60, "(\xi_{V,n})^*"] \\
\mathcal{S}p(\widetilde{S}_n^{-0} \wedge S^{-V}, X) \arrow[r, "p_*"] \arrow[d, "(\tilde{\xi}_{V,n})^*"'] & \mathcal{S}p(\widetilde{S}_n^{-0} \wedge S^{-V}, Y) \arrow[d, "(\tilde{\xi}_{V,n})^*"] \\
\mathcal{S}p(L^{\wedge n} \wedge S^{-V} \wedge L^{-n}, X) \arrow[r, "p_*"] \arrow[d, "\cong"'] & \mathcal{S}p(L^{\wedge n} \wedge S^{-V} \wedge L^{-n}, Y) \arrow[d, "\cong"] \\
\mathcal{M}(L^{\wedge n}, X_{n+V}) \arrow[r, "(p_{n+V})_*"] \arrow[d, equal] & \mathcal{M}(L^{\wedge n}, Y_{n+V}) \arrow[d, equal] \\
\Omega_L^n X_{n+V} \arrow[r, "\Omega^W p_{n+V}"] & \Omega_L^n Y_{n+V}.
\end{tikzcd}
$$

Lemma 7.4.32 implies that the outer diagram above is that of (7.4.44), which is equivalent to (7.4.45) and is therefore homotopy Cartesian. Every step in this argument can be reversed, so the result follows. □

Corollary 7.4.46. Positive stable fibrant smashable spectra. *A smashable spectrum X is fibrant in the positive stable model structure if and only if for objects V in the ideal $\mathscr{L}_L^{\mathbf{F}}$, the map*

$$\overline{\eta}_{V,1}^X : X_V \to \Omega_L X_{1+V}$$

is a weak equivalence in $\mathcal{M}$.

Proof By Definition 4.1.19, X is fibrant iff the map $X \to *$ is a fibration, meaning it has the right lifting property with respect to $\mathcal{J}_L^{\mathbf{F}}$ as in (7.4.40). We apply Theorem 7.4.43 to the case $Y = *$ and $n = 1$. The map above is the one on the left in (7.4.44), which is a weak equivalence since the one on the right is. □

Remark 7.4.47. Not all positive stable fibrant spectra are Ω-spectra *because the fibrancy condition of Corollary 7.4.46 is weaker than that of Theorem 7.4.35. The latter requires $\overline{\eta}_{V,1}^X$ to be a weak equivalence for* **all** *V, while the former requires it only when V is in the ideal $\mathscr{L}_L^{\mathbf{F}}$. Recall that the positive stable model structure has the same weak equivalences as the stable model structure, but fewer cofibrations and therefore more fibrations and more fibrant objects. The positive stable fibrancy of X does not depend on the values of X_V for objects V outside the ideal.*

Since the positive stable model structure has more fibrant objects that the stable one, it has fewer cofibrant objects. The following necessary condtion for positive stable cofibrancy excludes many spectra.

Proposition 7.4.48. The 0th component of a positive stably cofibrant spectrum. *If W is a positive stably cofibrant spectrum, then the map*

$$(\eta_W^\infty)_0 : W_0 \to (\Theta^\infty W)_0 = \underset{n}{\mathit{hocolim}}\, \Omega_L^n W_n$$

is null homotopic, where the map η_W^∞ is as in Definition 7.4.26.

Proof Consider the spectrum $Y = \Theta^\infty W$. We will construct a spectrum X with $X_0 = *$ and a positive stably trivial fibration $p : X \to Y$. Then if W is cofibrant, then map $* \to W$ is a cofibration, so there must be a lifting the diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
W \arrow[r, "\eta_W^\infty"'] \arrow[ur, dashed] & \Theta^\infty W.
\end{tikzcd}$$

The 0th component of this digram gives the desired null homotopy.

We define the spectrum X by

$$X_V = \begin{cases} * & \text{for } V = 0 \\ (\Theta^\infty W)_V & \text{otherwise.} \end{cases}$$

Then the evident map $X \to Y$, which is the identity in positive degrees, is a positive stably trivial fibration by Theorem 7.4.43. □

Many familiar spectra fail to satisfy the condition of Proposition 7.4.48 and thus are not positive stably cofibrant.

Corollary 7.4.49. The sphere spectrum S^{-0} is not positive stably cofibrant.

The following will be needed to verify Dwyer-Hirschhorn-Kan's fourth condition in the proof of Theorem 7.4.52.

Proposition 7.4.50. More about positively stable trivial fibrations. *If a map $p : X \to Y$ in $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$ is a stable equivalence and has the right lifting property with respect to $\mathcal{K}_L^{\mathbf{F},+}$ as in (7.4.40), then it has the right lifting property with respect to $\mathcal{I}_L^{\mathbf{F}}$.*

Proof By Theorem 7.4.43, the right lifting property with respect to $\mathcal{K}_L^{\mathbf{F},+}$ implies that the map $p_V : X_V \to Y_V$ for each V in $\mathscr{L}_L^{\mathbf{F}}$ is a fibration and that (7.4.44) is a homotopy Cartesian diagram.

It follows that the diagram

$$\begin{tikzcd} X_V \arrow[r, "p_V"] \arrow[d] & Y_V \arrow[d] \\ \operatorname*{hocolim}_n \Omega_L^n(X_{V+n}) \arrow[r] \arrow[d, equal] & \operatorname*{hocolim}_n \Omega_L^n(Y_{V+n}) \arrow[d, equal] \\ (\Theta^\infty X)_V \arrow[r, "(\Theta^\infty p)_V"] & (\Theta^\infty Y)_V \end{tikzcd} \tag{7.4.51}$$

is also a homotopy Cartesian diagram. The assumption that p is a stable equivalence of spectra implies that the lower horizontal map is a weak equivalence in $\mathcal{M}$ by Theorem 7.4.29(iv), so the same is true of p_V. We also know by Theorem 7.4.43 that this map is a fibration in $\mathcal{M}$. This means that p has the right lifting property with respect to $\mathcal{I}_L^{\mathbf{F},+}$ as desired. □

The following is a generalization of Theorem 7.3.36. Its application to orthogonal G-spectra will be stated below as Theorem 9.2.11. Again, we note the similarity between the generating sets of trivial cofibrations with the $\mathcal{S}$-horns of Definition 6.3.8.

Theorem 7.4.52. The stable and positive stable model structures on $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$, the corner map theorem for smashable spectra. *The sets $\mathcal{I}_L^{\mathbf{F},+}$ and $\mathcal{K}_L^{\mathbf{F},+}$ ($\mathcal{I}_L^{\mathbf{F}}$ and $\mathcal{K}_L^{\mathbf{F}}$) as in (7.4.40) define a cofibrantly generated model structure on $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$. It is the Bousfield localization of the positive (projective) model structure of Definition 7.4.36 with respect to the morphism set $\mathcal{S}^+$ of (7.4.38), or equivalently with respect to the set $\mathcal{S}$ of (7.4.9).*

Proof We will treat only the positive case, the argument for the projective case being the same. We will prove this by showing that $\mathcal{I}_L^{\mathbf{F},+}$ and $\mathcal{K}_L^{\mathbf{F},+}$ satisfy the four conditions of the Dwyer-Hirschhorn-Kan Recognition Theorem 5.2.24. As in the proof of Theorem 7.3.36, this will mean that we have a model structure with the right weak equivalences and the right cofibrations. Since any model structure is uniquely determined by such data, we have the one we are looking for.

The numbers in the following list refer to Dwyer-Hirschhorn-Kan's conditions.

(i) We need to show that the domains of $\mathcal{I}_L^{\mathbf{F},+}$ are small with respect to it, and similarly for $\mathcal{K}_L^{\mathbf{F},+}$. The key point here is that the domains of $\mathcal{I}_L^{\mathbf{F},+}$, $\mathcal{J}_L^{\mathbf{F},+}$ and $\mathcal{S}^+$ are all cofibrant and the maps in them are cofibrations.

Any spectrum in which the underlying objects of $\mathcal{M}$ are cofibrant is small relative to $\mathcal{I}_L^{\mathbf{F},+}$. The domains of $\mathcal{I}_L^{\mathbf{F},+}$ and $\mathcal{J}_L^{\mathbf{F},+}$ fit this description, so they are small relative to $\mathcal{I}_T$. Moreover each of the maps in $\mathcal{K}_L^{\mathbf{F},+}$ is a cofibration and therefore in the saturated class (Definition 4.8.13) generated by $\mathcal{I}_L^{\mathbf{F},+}$

by Proposition 5.2.2. This means that any object small relative to $\mathcal{I}_L^{\mathbf{F},+}$ is also small relative to $\mathcal{K}_L^{\mathbf{F},+}$ by Proposition 4.8.19, so Dwyer-Hirschhorn-Kan's first condition is satisfied.

(ii) We need to show that each map in $\mathcal{K}_L^{\mathbf{F},+}$ is an $\mathcal{I}_L^{\mathbf{F},+}$-cofibration and a weak equivalence. This is true of the maps in $\mathcal{J}_L^{\mathbf{F},+}$.

We need to show the same for the corner maps $i \,\square\, \widetilde{\xi}_{V,n}$ for $i : A \to B$ a map in $\mathcal{I}$ and $\widetilde{\xi}_{V,n}$ as in (7.4.17).

For each object U of $\mathscr{J}_L^{\mathbf{F}}$, the Uth component of this map is $i \,\square\, (\widetilde{\xi}_{V,n})_U$, where $(\widetilde{\xi}_{V,n})_U$ is as in (7.4.17). The latter is a cofibration in $\mathcal{M}$, so the corner map is as well by Lemma 5.5.1. It follows that the corner map $i \,\square\, \widetilde{\xi}_{V,n}$ is a strict $\mathcal{I}$-cofibration.

The corner map $i \,\square\, \widetilde{\xi}_{V,n}$ is defined by the diagram

$$\begin{tikzcd}
A \wedge L^{\wedge n} \wedge S^{-V} \wedge L^{-n} \arrow[r, "A \wedge \widetilde{\xi}_{V,n}"] \arrow[d, "f = i \wedge L^{\wedge n} \wedge S^{-V} \wedge L^{-n}"] & A \wedge \widetilde{S}_n^{-V} \arrow[d, "\beta"] \arrow[ddr, bend left, "i \wedge \widetilde{S}_n^{-V}"] & \\
B \wedge L^{\wedge n} \wedge S^{-V} \wedge L^{-n} \arrow[r, "\alpha"] \arrow[drr, "B \wedge \widetilde{\xi}_{V,n}"'] & P(f, A \wedge \widetilde{\xi}_{V,n}) \arrow[dr, "i \,\square\, \widetilde{\xi}_{V,n}"] & \\
 & & B \wedge \widetilde{S}_n^{-V}
\end{tikzcd}$$

The maps $A \wedge \widetilde{\xi}_{V,n}$ and $B \wedge \widetilde{\xi}_{V,n}$ are stable equivalences by Corollary 7.4.15. The former implies that α is an equivalence, which means that $i \,\square\, \widetilde{\xi}_{V,n}$ is one as desired.

(iii) We need to show that each map $f : X \to Y$ having the right lifting property with respect to $\mathcal{I}_L^{\mathbf{F},+}$ also has it for $\mathcal{J}_L^{\mathbf{F},+}$ and is a weak equivalence. Such a map f is a weak equivalence by Proposition 7.4.42. Maps f having the right lifting property with respect to $\mathcal{J}$ are characterized in Theorem 7.4.43 by two properties:

 (a) The map f_V is a fibration in $\mathcal{M}$ for each V in the ideal. We saw in Proposition 7.4.42 that for such an f, f_V is a trivial fibration in $\mathcal{M}$.
 (b) The diagram of (7.4.44) is a homotopy pullback diagram. This follows from the fact that both horizontal maps are weak equivalences.

(iv) We need to show either the converse of (iii) above or that an $\mathcal{I}_L^{\mathbf{F},+}$-cofibration that is a weak equivalence is also a $\mathcal{J}_L^{\mathbf{F},+}$-cofibration. The desired converse is Proposition 7.4.50. □

Theorem 7.4.53. $\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, L)$ **is a symmetric monoidal model category** *as in Definition 5.5.9 under the projective, positive projective, stable and positive stable model structures.*

Proof The projective and positive projective model structures are instances of the two rightmost model structures of (5.6.41), as noted in Remark 5.6.42. Thus they are monoidal by Corollary 5.6.40.

For the two stable model structures, we need to show that stabilization is a monoidal Bousfield localization as in Definition 6.2.18. This follows from Theorem 6.2.19, whose hypotheses the reader can easily verify. □

7.4E Exact sequences for smashable spectra.

In this subsection we will show that for each spectral $\mathscr{J}^{\mathbf{O}}$-algebra $\mathscr{J}_L^{\mathbf{F}}$ as in Definition 7.2.19, the corresponding category of spectra, which we will abbreviate here by $\mathcal{S}p$, is exactly stable as in Definition 5.7.3. This will enable us to apply Theorem 5.7.6 and Theorem 5.7.11 to get the expected long exact sequences of homotopy groups.

The following is the smashable analog and strengthening of the stable equivalences of Lemma 7.3.43.

Lemma 7.4.54. The suspension and loop isomorphisms for smashable spectra. *For a cofibrant spectrum A, the map $\epsilon_A^V : \Sigma^V \Sigma^{-V} A \to A$ is a stable equivalence. Its Wth component is the restricted structure map*

$$\overline{\epsilon}_{W,V}^A : S^V \wedge A_W \to A_{V+W}$$

of Definition 7.2.38.

For a fibrant spectrum B, the map $\eta_B^V : B \to \Omega^V \Omega^{-V} B$ is a stable equivalence. Its Wth component is the restricted costructure map

$$\overline{\eta}_{W,V}^B : B_W \to \Omega^W B_{V+W}$$

of Definition 7.2.42.

Proof The natural transformations

$$\epsilon^V : \Sigma^V \Sigma^{-V} \Rightarrow 1_{\mathcal{S}p} \qquad \text{and} \qquad \eta^V : 1_{\mathcal{S}p} \Rightarrow \Omega^V \Omega^{-V}$$

are both induced by the stable equivalence $\xi_{0,V}$ of (7.4.5), namely $\epsilon_A^V = \xi_{0,V} \wedge A$ and $\eta_B^V = F(\xi_{0,V}, B)$. These are stable equivalences for A and B as indicated. □

The generalization of the isomorphism of Lemma 7.3.43(i) requires some care. We might write

$$\begin{tikzcd} \pi_{V+W}\Sigma^W A := \operatorname*{colim}_n \pi_{n+V+W}\Sigma^W A_n \arrow[r, dashed, "\cong ?"] & \operatorname*{colim}_n \pi_{n+V}\Sigma^W A_{n-W} \arrow[d] \\ & \operatorname*{colim}_n \pi_{n+V} A_n =: \pi_V A, \end{tikzcd} \tag{7.4.55}$$

but now the "reindexing" leads us to consider two different collections of spaces, namely

$$\left\{\Sigma^W A_n : n \geqslant 0\right\} \qquad \text{and} \qquad \left\{\Sigma^W A_{n-W} : n \gg 0\right\}.$$

Thus it is not even clear that the homomorphism exists, let alone that it is a isomorphism.

We can use the direct summand condition of Definition 7.2.19(ii) to fix this as follows. It says that the indexing category has an object W' such that $S^W \wedge S^{W'} \cong L^{\wedge m}$ for some $m > 0$. **Suppose for simplicity that $m = 1$, so $W' = 1 - W$.** Then for any spectrum X we have a diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "\overline{\eta}^X_{0,1}"] \arrow[dr, "\overline{\eta}^X_{0,1-W}"'] & & \Omega_L X_1 \arrow[rr, "\overline{\eta}^X_{1,1}"] \arrow[dr, "\overline{\eta}^X_{1,1-W}"] & & \cdots \\
& \Omega^{W'} X_{1-W} \arrow[ur, "\overline{\eta}^X_{1-W,W}"] \arrow[rr, "\overline{\eta}^X_{1-W,1}"] & & \Omega^{W'}\Omega_L X_{2-W} \arrow[ur, "\overline{\eta}^X_{2-W,W}"] \arrow[r, "\overline{\eta}^X_{2-W,1}", "\overline{\eta}^X_{2-W,1}"'] & \cdots
\end{tikzcd}$$

where the maps are those of Definition 7.2.42. Then the homotopy colimits of the rows are each weakly equivalent to that of the "zig zag" sequence of maps.

Now let $X = \Sigma^W A$. Then taking π_{V+W} of everything in the diagram above gives the first two rows of

$$\begin{tikzcd}
\pi_{V+W}\Sigma^W A_0 \arrow[rr] \arrow[dr] & & \pi_{1+V+W}\Sigma^W A_1 \arrow[rr] \arrow[dr] & & \cdots \\
& \pi_{V+1}\Sigma^W A_{1-W} \arrow[ur] \arrow[rr] \arrow[d, "\pi_{V+1}\overline{\epsilon}^A_{1-W,W}"'] & & \pi_{V+2}\Sigma^W A_{2-W} \arrow[ur] \arrow[r] \arrow[d, "\pi_{V+2}\overline{\epsilon}^A_{2-W,W}"'] & \cdots \\
& \pi_{V+1}A_1 \arrow[rr] & & \pi_{V+2}A_2 \arrow[r] & \cdots ,
\end{tikzcd}$$

where $\overline{\epsilon}^A_{1-W,W}$ is the restricted structure map of Definition 7.2.38. Now the colimits of the top two rows are the domain and codomain of the purported homomorphism of (7.4.55), so it exists and it is an isomorphism. The colimit of the third row is $\pi_V A$, and the vertical arrows above induce the vertical arrow of (7.4.55).

We leave it to the reader to generalize this argument to larger values of m.

Thus we have proved

Corollary 7.4.56. The suspension isomorphism for homotopy groups of smashable spectra. *For a smashable cofibrant spectrum A, and objects V and W in its indexing category, there is a natural isomorphism $\pi_{V+W}\Sigma^W A \to \pi_V A$.*

Theorem 7.4.57. Exact stability for smashable spectra. *The category $\mathcal{S}p = [\mathscr{J}_L^{\mathbf{F}}, \mathcal{M}]$, where $\mathscr{J}_L^{\mathbf{F}}$ is as in Definition 7.2.19 and $\mathcal{M}$ and L are as in Definition 7.2.4, is exactly stable as in Definition 5.7.3.*

Proof The conditions of Definition 5.7.3 are implied by Lemma 7.4.54. □

Thus we get the exact sequences of Theorem 5.7.6 and Theorem 5.7.11.

Remark 7.4.58. Two stable stable equivalences of smashable spectra. *This is the analog of Remark 7.1.29 for smashable spectra. The suspepnsion and desuspension functors are $\Sigma^V = S^V \wedge (-)$ and $\Sigma^{-V} = S^{-V} \wedge (-)$. Hence the difference between the functors $\Sigma^W \Sigma^{-V \oplus W}$ and $^{-V}$ is related to that between the spectra $S^W \wedge S^{-V \oplus W}$ and S^{-V}. They are the domain and codomain of the stabilizing map $\xi_{V,W}$ of (7.2.63), which is a stable equivalence.*

Similarly the difference between the functors $\Sigma^{V \oplus W} \Sigma^{-V}$ and Σ^W is related to that between the spectra $S^{V \oplus W} \wedge S^{-W}$ and S^V. They are the the domain and codomain of the map $S^V \wedge \xi_{0,W}$, which is also a stable equivalence.

7.4F The canonical homotopy presentation

We will need the following in our study of geometric fixed points in §9.11 and in our construction of $MU_{\mathbf{R}}$ in §12.2. It is not to be confused with the tautological presentation of Proposition 3.2.33.

As before we will denote by $\mathcal{S}p$ the category of smashable spectra corresponding to spectral $\mathscr{J}^{\mathbf{O}}$-algebra $\mathscr{J}_L^{\mathbf{F}}$ as in Definition 7.2.19, and we will abbreviate the indexing category $\mathscr{J}_L^{\mathbf{F}}$ by $\mathscr{J}$.

Consider the transition diagram

$$\begin{tikzcd} & L^{-(n+1)} \wedge \mathscr{J}(n, n+1) \wedge X_n \arrow[dl, "j(n,n+1)\wedge X_n"'] \arrow[dr, "L^{-(n+1)} \wedge \epsilon^X_{n,\rho}"] & \\ L^{-n} \wedge X_n & & L^{-(n+1)} \wedge X_{n+1} \end{tikzcd} \tag{7.4.59}$$

where $j(n, n+1)$ is as in (7.2.53) and $\epsilon^X_{n,\rho}$ is as in (7.2.36).

We have an embedding

$$L = \mathscr{J}(0, 1) \to \mathscr{J}(n, n+1),$$

and so from (7.4.59) a diagram

$$\begin{tikzcd} & L^{-(n+1)} \wedge L \wedge X_n \arrow[dl] \arrow[dr] & \\ L^{-n} \wedge X_n & & L^{-(n+1)} \wedge X_{n+1} \end{tikzcd} \tag{7.4.60}$$

Putting these together as n varies results in a system

$$\begin{tikzcd}[column sep=small] & B_0 \arrow[dl, "\sim"'] \arrow[dr] & & B_1 \arrow[dl, "\sim"'] \arrow[dr] & & B_2 \arrow[dl, "\sim"'] \arrow[dr] & & B_3 \arrow[dl, "\sim"'] \arrow[dr] & \\ A_0 & & A_1 & & A_2 & & A_3 & & A_4 \quad \dots . \end{tikzcd} \tag{7.4.61}$$

The system (7.4.61) maps to X and a simple check of equivariant stable homotopy groups shows that the map from its homotopy colimit to X is a weak equivalence. Now for each n let C_n be the homotopy colimit of the portion

$$\begin{tikzcd}[column sep=small] & B_0 \arrow[dl, "\sim"'] \arrow[dr] & & \cdots & & B_{n-1} \arrow[dl, "\sim"'] \arrow[dr] & \\ A_0 & & A_1 & \cdots & A_{n-1} & & A_n \end{tikzcd}$$

of (7.4.61). Then C_n is naturally weakly equivalent to $A_n = L^{-n} \wedge X_n$, and the C_n fit into a sequence

$$C_0 \to C_1 \to C_2 \to \dots \tag{7.4.62}$$

whose homotopy colimit coincides with that of (7.4.61). This is the **canonical homotopy presentation of** X. One can functorially replace the sequence (7.4.62) with a weakly equivalent sequence of cofibrations between cofibrant-fibrant objects as in Definition 4.1.19. The colimit of this sequence is naturally weakly equivalent to X. It will be cofibrant automatically, and fibrant since the model category $\mathcal{S}p^G$ is compactly generated.

Definition 7.4.63. The canonical homotopy presentation *of a smashable spectrum X is the stably equivalent telescope*

$$\underset{n}{hocolim}\, (L^{-n} \wedge X_n)_{cf},$$

or when more precision is needed, as a diagram

$$X \xleftarrow{\simeq} \underset{n}{hocolim}\, (L^{-n} \wedge X_n)_c \xrightarrow{\simeq} \underset{n}{hocolim}\, (L^{-n} \wedge X_n)_{cf},$$

where $(L^{-n} \wedge X_n)_c$ is a cofibrant replacement of $L^{-n} \wedge X_n$ and the map on the right is fibrant replacement. In the notation of (4.1.23),

$$(L^{-n} \wedge X_n)_c = Q(L^{-n} \wedge X_n) \qquad \text{and} \qquad (L^{-n} \wedge X_n)_{cf} = RQ(L^{-n} \wedge X_n).$$

8

Equivariant homotopy theory

> We shall sometimes restrict to finite groups to avoid technicalities, but most of what we say applies in technically modified form to general compact Lie groups. The reader unused to equivariant topology may find it helpful to concentrate on the case when G is a group of order 2. Even this simple case well illustrates most of the basic ideas.
>
> *John Greenlees and Peter May, [GM95, page 2]*

In this chapter we will introduce some tools from equivariant homotopy theory, that is the homotopy theory of spaces equipped with an action by a finite group G, that we will need later to study equivariant **stable** homotopy theory starting in Chapter 9. Our treatment is eclectic rather than comprehensive, the choice of topics being dictated solely by the needs of the rest of the book. This includes our decision to deal only with finite groups rather than compact Lie groups. The only groups figuring in the proof of the main theorem are cyclic 2-groups, specifically C_8 and its subgroups, but the cost of generalizing to arbitrary finite groups is minimal.

Remark 8.0.1. Notation for cyclic p-groups. *There are two common notations for the cyclic group of order p^n for a prime p, C_{p^n} and $\mathbf{Z}/p^n$. We will use the former* **when the group is acting on something***, such as a set, a topological space or a vector space. We will use the latter when the group occurs as the value of some functor such as a homotopy or homology group.*

8.0A G-sets, coefficient systems and Mackey functors

The first two sections concern algebraic infrastructure. It has been said (see [GM95, page 9] for the statement about points) that the equivariant analogs of points and abelian groups are G-orbits and Mackey functors.

In ordinary homotopy theory, the space of continuous maps from a point to a space X is of course X itself. In equivariant homotopy theory, the space of equivariant maps from G/H to a G-space X is X^H, the subspace fixed by H. For $K \subseteq H \subseteq G$, precomposition with the map $G/K \to G/H$ gives the restriction map $i_K^H : X^H \to X^K$. Any point of X that is fixed by $H \subseteq G$ is also fixed by K. Hence we have a contravariant functor from $\mathscr{O}_G \subseteq \mathcal{S}et^G$ (see Definition 8.6.22), the subcategory of that of G-sets consisting of single orbits. An $\mathcal{A}b$-valued functor on $\mathscr{O}_G^{op}$ is called a **coefficient system**; see Definition 8.6.24.

Finite G-sets are studied in §8.1. The set of isomorphism classes forms a semi-ring with disjoint union as addition and Cartesian product as multiplication. The corresponding Grothendieck group is the **Burnside ring** $A(G)$; see Definition 8.1.3.

A Mackey functor is a coefficient system F with additional structure. For $K \subseteq H \subseteq G$ one has a **restriction map**

$$\mathrm{Res}_K^H : F(G/H) \to F(G/K).$$

In a Mackey functor one also has a **transfer map**

$$\mathrm{Tr}_K^H : F(G/K) \to F(G/H)$$

going the other way. The algebraic details are spelled out in §8.2B. Here we will discuss informally but at some length why this additional structure is relevant to equivariant stable homotopy theory.

It has to do with equivariant Spanier-Whitehead duality, an early account of which was given by Adams in [Ada84, §8]. This can be described as categorical duality in the sense of §2.6E in the category of orthogonal G-spectra, the subject of Chapter 9 below. However it was originally constructed geometrically, as we shall now describe.

8.0B Ordinary Spanier-Whitehead duality

The original sources for this are [SW55] and [Spa59], and nice accounts of it were given by Adams in [Ada74b, III.5], and by Albrecht Dold (1928–2011) and Puppe in [DP80, §3]. For simplicity we will work in the original category of spectra. This may appear to be a cheat since that category is not closed symmetric monoidal and therefore lacks categorical duality. Nevertheless this duality was originally developed there. For a pointed space X, let $\Sigma^{-k}X$ be the spectrum defined by

$$(\Sigma^{-k}X)_m = \begin{cases} * & \text{for } m < k \\ \Sigma^{m-k}X & \text{for } m \geqslant k. \end{cases} \tag{8.0.2}$$

This is comparable to the generalized suspension spectrum of Definition 7.1.30. In particular $\Sigma^{-0}X$ is the suspension spectrum of X, sometimes written as $\Sigma^\infty X$.

The alert reader may notice that we also used the symbol Σ^{-k} for k-fold desuspension in Definition 5.7.3. This coincidence is intentional. In the original case, we can define $\Sigma^{-k}E$ for a spectrum E by

$$(\Sigma^{-k}E)_m = \begin{cases} E_{m-k} & \text{for } m \geqslant k \\ * & \text{otherwise} \end{cases}$$

For $E = \Sigma^\infty X$, meaning that $E_m = \Sigma^m X$, this coincides with (8.0.2).

In a category $\mathcal{S}p$ of smashable spectra as in Definition 7.2.33, we define

$$\Sigma^{-k}E := S^{-k} \wedge E, \tag{8.0.3}$$

where S^{-k} is the Yoneda spectrum of Definition 7.2.52.

Any finite CW complex X can be embedded in S^{n+1} for sufficiently large n. We denote the complement of X in S^{n+1} by D_nX, the **n-dual of** X. The homotopy type of D_nX depends not just on X and n, but on the choice of embedding. For example if $X = S^1$ and $n = 2$, then the embedding could be any knot, so there are infinitely many possibilities for the homotopy type of D_2S^1. Fortunately they are all stably equivalent.

The complement $D_{n+1}X$ of the composite embedding

$$X \to S^{n+1} \to S^{n+2}$$

(where the second map is the standard linear embedding) is ΣD_nX, and the complement of the suspended embedding

$$\Sigma X \to \Sigma S^{n+1} \cong S^{n+2},$$

that is $D_{n+1}\Sigma X$, is homotopy equivalent to D_nX.

It follows that the spectrum $\Sigma^{-k}D_nX$ satisfies

$$(\Sigma^{-k}D_nX)_m = \begin{cases} * & \text{for } m < k \\ \Sigma^{m-k}D_nX = D_{n+m-k}X & \text{for } m \geqslant k \end{cases} \tag{8.0.4}$$

and its stable homotopy type is determined by $n - k$. It is a finite spectrum as in Remark 7.1.31.

It is evident from the definition that for $\ell \geqslant 0$,

$$D_{n+\ell}D_nX = \Sigma^\ell X.$$

Replacing X by D_nX in (8.0.4), we recover (8.0.2).

Remark 8.0.5. Stable duality. *This suggests that for $n \geqslant k$,* **the spectra $\Sigma^{-k}D_nX$ and $\Sigma^{k-n}X$ are strongly dualizable as in Definition 2.6.54 and, up to stable equivalence, categorically dual to each other.** *This means there should be maps*

$$\widetilde{\epsilon}: \Sigma^{-k}D_nX \wedge \Sigma^{k-n}X \to S^{-0} \text{ and } \widetilde{\eta}: S^{-0} \to \Sigma^{-k}D_nX \wedge \Sigma^{k-n}X. \tag{8.0.6}$$

We will see that on the space level there are maps between $X \wedge D_nX$ and S^n in both directions. The composite $\widetilde{\epsilon}\widetilde{\eta}: S^{-0} \to S^{-0}$ is mutiplication by the Euler charcateristic of the finite complex X. On the spectrum level we need to *replace the domain* S^{-0} *of* $\widetilde{\eta}$ *by* $S^m \wedge S^{-m}$ *(which is stably equivalent to it) for sufficiently large* m*. Thus we get a diagram*

$$\begin{tikzcd} S^m \wedge S^{-m} \arrow[d, "\widetilde{\eta}"] \\ \Sigma^{-k} D_n X \wedge \Sigma^{k-n} X \arrow[d, "\widetilde{\epsilon}"] \\ S^{-0}, \end{tikzcd} \tag{8.0.7}$$

in which the composite map is in general **not** *a stable equivalence.*

The Spanier map. Suppose (X, x_0) and (Y, y_0) are disjoint pointed finite CW complexes of dimension less than n embedded in S^{n+1}. Each could be homotopy equivalent to the complement of the other, but we need not assume that for now. We define the **Spanier map**

$$u : X \wedge Y \to S^n \tag{8.0.8}$$

as follows. Choose a point in S^{n+1} outside of both X and Y, and remove it. This means that X and Y are now disjoint CW complexes in $\mathbf{R}^{n+1}$. Define

$$\tilde{u} : X \times Y \to S^n \quad \text{by} \quad \tilde{u}(x, y) = \frac{x - y}{|x - y|},$$

where $x \in X$ and $y \in Y$ are distinct vectors in $\mathbf{R}^{n+1}$, making $\tilde{u}(x, y)$ a unit vector and hence a point in S^n. Our assumption about the dimensions of X and Y means that the restrictions of $\tilde{u}$ to $x_0 \times Y$ and $X \times y_0$ are both null homotopic, so $\tilde{u}$ is homotopic to a map that factors through $X \wedge Y$, giving us the desired map u.

Now suppose in addition that X and Y are each homotopy equivalent to the complement of the other, so

$$X \simeq D_n Y \qquad \text{and} \qquad Y \simeq D_n X.$$

The Alexander duality theorem [Ale15] says that $H^i D_n X$ is naturally isomorphic to $H_{n-i} X$. The Spanier map of (8.0.8) leads to homomorphisms

$$H_i X \otimes H_{n-i} D_n \to H_n S^n \cong \mathbf{Z} \qquad \text{for } 0 \leqslant i \leqslant n,$$

and these lead to Alexander duality. It also gives us maps

$$X \wedge D_n X \to S^n \qquad \text{and} \qquad D_n Y \wedge Y \to S^n,$$

which are analogous to the map $\widetilde{\epsilon}$ of (8.0.6).

Atiyah duality.

Now suppose X is a smooth closed manifold M that is smoothly embedded in S^n (rather than S^{n+1}). Hence it has an closed tubular neighborhood T that is the total space of its normal unit disk bundle ν. If we collapse its boundary (the normal unit sphere bundle) we get the Thom space M^ν. We can also think of this as the quotient of S^n obtained by collapsing everything outside the interior of T to a point. This leads to the **Pontryagin-Thom map**

$$p_{M,S^n} : S^n \to M^\nu. \tag{8.0.9}$$

Next consider the diagonal map $\Delta : M \to M \times M$, and suppose that the target is equipped with the bundle ν on its first factor. This means that the Thom space for the target is $M^\nu \rtimes M$. Thus we get a map

$$\begin{tikzcd} & M^\nu \arrow[dr, "T\Delta"] & \\ S^n \arrow[ur, "p_{M,S^n}"] \arrow[rr, "\tilde{\eta}_M"'] & & M^\nu \rtimes M, \end{tikzcd} \tag{8.0.10}$$

the **Thom diagonal**. It is related to the similarly named map of Definition 2.6.54 and (8.0.6), meaning that the suspension spectrum of M^ν is the dual of $\Sigma^{-n}M_+$.

We can also construct a map $\tilde{\epsilon}$ going the other way. Recall that the normal bundle of the diagonal embedding $\Delta : M \to M \times M$ is isomorphic to the tangent bundle of M. Let $s : M \to \nu$ denote the zero section of the normal bundle ν of M in S^n. Then the normal bundle of the composite

$$M \xrightarrow{\Delta} M \times M \xrightarrow{s \times M} \nu \times M$$

is the direct sum of ν with the tangent bundle, namely the trivial n-plane bundle over M. Thus we get a composite

$$\begin{tikzcd} & \Sigma^n M_+ \arrow[dr] & \\ M^\nu \rtimes M \arrow[ur, "p_{M,\nu\times M}"] \arrow[rr, "\tilde{\epsilon}_M"'] & & S^n, \end{tikzcd} \tag{8.0.11}$$

where $p_{M,\nu\times M}$ is the Pontryagin-Thom map for the embedding $(s \times M)\Delta$, and the unnamed map is obtained by projecting M to a point.

It turns out that our finite CW complex X could be a manifold with boundary Y. It still has a normal bundle ν, but in the above constructions one need to replace M_+ (meaning M with a disjoint base point) by the pointed quotient X/Y. Then we get maps

$$\begin{tikzcd} X^\nu \wedge X/Y \arrow[r, shift left, "\tilde{\epsilon}_X"] & S^n. \arrow[l, shift left, "\tilde{\eta}_X"] \end{tikzcd}$$

The following was originally proved by Atiyah in [Ati61].

Theorem 8.0.12. Atiyah duality. *With notation as above suspension spectrum of X^ν is dual to $\Sigma^{-n}X/Y$.*

8.0C Equivariant Spanier-Whitehead duality

We now turn to the G-equivariant case for a finite group G. A detailed treatment is given by Lewis and May in [LMSM86, Chapter III]. Let X be a finite G-CW complex as in Definition 8.4.13 below. If the G-action is nontrivial, we cannot embed it equivariantly in any S^{n+1}, but we can embed it in some representation sphere S^{V+1} as in Definition 8.3.26. We denote its complement there by $D_V X$, the V**-dual of** X. For any representation W, the complement of the composite embedding $X \to S^{V+1} \to S^{V+W+1}$ is

$$D_{V+W}X \cong \Sigma^W D_V X.$$

With this understanding, it is possible to do Spanier-Whitehead duality equivariantly. The diagram of (8.0.7) reads

$$\begin{tikzcd} S^{m\rho} \wedge S^{-m\rho} \arrow[d, "\tilde{\eta}"] \\ \Sigma^{-V} D_W X \wedge \Sigma^{V-W} X \arrow[d, "\tilde{\epsilon}"] \\ S^{-0} \end{tikzcd}$$

for $m \gg 0$, where ρ denotes the regular representation of G.

We now specialize to the case where the G-CW complex is G/H_+, a G-orbit with disjoint base point. It is underlain by a discrete space with $1 + |G/H|$ points. Let $V_{G/H}$ be the representation given by the action of G on the real vector space having the set G/H as its basis. This is the **permutation representation** associated with the G-set G/H. This vector space has in invariant one dimensional summand generated by the sum of its basis elements. Its orthogonal complement $\overline{V}_{G/H}$ is the **reduced permutation representation**.

Then we have pointed equivariant embedding $G/H_+ \to S^{V_{G/G}} \cong S^{1+\overline{V}_{G/H}}$ that sends each element of G/H to the corresponding basis element, and the base point of G/H_+ to the point at infinity. The complement

$$D_{\overline{V}_{G/H}}(G/H_+)$$

of its image is equivariantly equivalent to a wedge of copies of $S^{|G/H|-1}$ indexed by G/H. These wedge summands are permuted by the group as expected, so we have

$$D_{V_{G/H}}(G/H_+) \simeq G \underset{H}{\ltimes} S^{V_{G/H}}.$$

The action of the subgroup H is trivial on both objects. From this we can deducethe following. We leave the details to the reader.

Proposition 8.0.13. The suspension spectrum $\Sigma^{-0}G/H_+$ is equivariantly self dual.

Remark 8.0.14. Equivariant suspension. *In the category of ordinary spectra, one has the suspension and loop functors* Σ *and* Ω*, which are adjoint to each other. The induced functors in the homotopy category are inverse to each other. The category of spectra is sometimes described as that of spaces with the suspension functor formally inverted.*

In the discussion above we made essential use of the **twisted suspension functor** S^V *(see Definition 8.9.3) in our discussion of equivariant duality. We will see in Proposition 8.9.4 that it is adjoint to the* **twisted loop functor** Ω^V*.* **The stable equivariant category** $\mathcal{S}p^G$ **needs to be defined in such a way that for each** V **these two functors are homotopy inverses to each other in the way that** Σ **and** Ω **are in the ordinary stable category.**

This self duality means that for $K \subseteq H \subseteq G$, the usual map $G/K \to G/H$ leads via stable duality to a map of spectra

$$\Sigma^{-0}G/H_+ \to \Sigma^{-0}G/K_+. \tag{8.0.15}$$

Recall that on the space level there is a map $G/K \to G/H$, which is a morphism in the orbit category $\mathscr{O}_G$. This leads us to consider coefficient systems, that is $\mathcal{A}b$-valued functors on $\mathscr{O}_G^{op}$. Equivalently one can consider $\mathcal{A}b$-valued functors on the categoy $\mathscr{F}_G^{op}$, where $\mathscr{F}_G$ denotes the category of finite G-sets, that convert disjoint unions (coproducts) to direct sums, which are coproducts in $\mathcal{A}b$.

The existence of the maps of (8.0.15) in the stable world means that we need to replace the category $\mathscr{F}_G$ by another category having the same objects but **more morphisms**. The one we want is the Lindner category $\mathscr{B}_G^+$ of Definition 8.2.4 below. A **Mackey functor** then is a contravariant coproduct preserving $\mathcal{A}b$-valued functor on it. The variance here is a moot point because $\mathscr{B}_G^+$ is self dual.

For a G-spectrum X, the cotensor $X^{G/H}$ by definition is the fixed point spectrum X^H; see Definition 9.1.9 below. This means that for $K \subseteq H \subseteq G$, the map of (8.0.15) induces the **transfer map**

$$X^K \to X^H.$$

It is **not** defined in general for G-spaces. For a pointed G-space X, the groups $\pi_*^H X := \pi_*(X^H)$ are components of a coefficient system, while for a G-spectrum X they are components of a Mackey functor.

We will see below that the homology of G-space is not merely a coefficient system but a Mackey functor. The reason for this is that homology is a stable invariant since it commutes with suspension, while homotopy groups do not.

8.0D The categories $\mathcal{T}op^G$ and $\mathcal{T}^G$

In §8.3 we discuss various objects associated with a G-space. For a finite group G, let $\mathcal{T}op^G$ be the category of topological G-spaces and equivariant maps as in Definition 3.1.59. Similarly let $\mathcal{T}^G$ be the category of pointed topological G-spaces (the base point is always fixed by G) and equivariant maps. In the literature (e.g. in [MM02] and [BDS16]) it is common to denote the category of G-objects and equivariant maps in $\mathcal{C}$ by $G\mathcal{C}$ rather than $\mathcal{C}^G$. To repeat, **we will assume that all topological spaces in sight are compactly generated weak Hausdorff.** See Definition 2.1.48 and the paragraph preceding it.

For a G-space X and a subgroup $H \subseteq G$, the fixed point set and orbit space of H are denoted respecrtively by X^H and X_H or X/H. They have homotopy analogs denoted by X^{hH} and X_{hH}; see Definition 8.3.8. For such spaces X and Y, $\mathcal{T}^G(X,Y)$ will denote the space of equivariant maps. Let $\mathcal{T}_G$ (denoted by $\underline{\mathcal{T}}_G$ in [HHR16]) be the category of pointed G-spaces and all continuous maps (equivariant or not) between them. This means that $\mathcal{T}_G(X,Y)$ is pointed G-space, where the group action is defined by conjugation and the base point is the constant map. The continuous maps fixed by G are the equivariant ones, that is

$$\mathcal{T}^G(X,Y) = \mathcal{T}_G(X,Y)^G.$$

For a subgroup $H \subseteq G$ one has a forgetful functor i_H^G (denoted by i_H^* in [HHR16]) from either category ($\mathcal{T}^G$ or $\mathcal{T}_G$) of pointed G-spaces to the corresponding category of pointed H-spaces. It has left and right adjoints as in Definition 2.2.25 sending a pointed H-space Y to the pointed G-spaces $G \ltimes_H Y$ and $\mathcal{T}^H(G,Y)$. Thus for pointed H-spaces X and Z and a pointed G-space Y we have

$$\mathcal{T}^G(G \ltimes_H X, Y) \cong \mathcal{T}^H(X, i_H^G Y) \tag{8.0.16}$$

and

$$\mathcal{T}^H(i_H^G Y, Z) \cong \mathcal{T}^G(Y, \mathcal{T}^H(G_+, Z)). \tag{8.0.17}$$

These are the **Wirthmüller isomorphisms** of [Wir74]. For $X = *$, (8.0.16) reads

$$\mathcal{T}^G(G/H_+, Y) \cong \mathcal{T}^H(*, i_H^G Y) = (i_H^G Y)^H,$$

which we often abbreviate by Y^H.

Remark 8.0.18. The pointed G-space $G \underset{H}{\ltimes} Y$ *is the H-orbit space of $G \ltimes Y$ where H acts on G by right multiplication and on the smash product by the diagonal action. This means that $(\gamma\eta \wedge y)$ and $(\gamma \wedge \eta y)$ (for $\gamma \in G$, $\eta \in H$ and $y \in Y$) are in the same H-orbit of $G \ltimes Y$. The underlying space of $G \underset{H}{\ltimes} Y$ is* *the finite wedge of $|G/H|$ copies of Y The G-action is defined in terms of left multiplication. Thus we have*

$$\gamma\eta\gamma^{-1}(\gamma \wedge y) = (\gamma\eta\gamma^{-1}\gamma \wedge y) = (\gamma\eta \wedge y) = (\gamma \wedge \eta y).$$

This means that the summand corresponding to γ is invariant under the action of the subgroup $\gamma H\gamma^{-1}$ rather than H.

We record the following here for future reference.

Proposition 8.0.19. A change of group isomorphism. *Let $\mathcal{B}_{G/H}G$ be the category of Example 2.9.1 for a subgroup $H \subseteq G$. Then the functor category $(\mathcal{T}^G)^{\mathcal{B}_{G/H}G}$ is isomorphic to $\mathcal{T}^H$.*

Proof The object set of $\mathcal{B}_{G/H}G$ is the G-set G/H, and the morphisms in it are isomorphisms between its objects. Hence a $\mathcal{T}^G$-valued functor on it is determined by its value on one object, the coset H. The functor determines an action of H on this object. Hence such a functor is the same thing as a pointed space with and H-action, i.e., an object in $\mathcal{T}^H$. □

G-CW complexes (Definition 8.4.13) are the subject of §8.4. While an ordinary CW complex X has a collection of n-cells indexed by a set K_n, in a G-CW complexes they are indexed by a G-set K_n, and the attaching maps are required to be equivariant. This means that the action of G permutes cells rather than acting on them individually. Thus an ordinary CW complex with a cellular G-action need not be a G-CW complex; see Example 8.4.15. Definition 8.4.13 also means that for any subgroup $H \subseteq G$, the fixed point set X^H is an ordinary CW complex in which the n-cells are indexed by the set K_n^H.

The homology of G-CW complexes is the subject of §8.5. For such a space X, the set K_n of n-cells is a G-set, so the resulting cellular chain complex $C_*(X)$ is a complex of modules over the group ring $\mathbf{Z}[G]$. Given such a module M, we can define a Mackey functor $\underline{M}$ by $\underline{M}(G/H) := M^H$. We call it the **fixed point Mackey functor of M**. It is functorial in M, so we get a chain complex of Mackey functors $\underline{C}_*(X)$, and we denote its homology by $\underline{H}_*X$. For $H \subseteq G$, the graded group $\underline{H}_*X(G/H)$ is **not** the same as $H_*(X^H)$; see Remark 8.5.2 below.

Model structures for G-spaces. Theorem 8.4.18, proved by Glen Bredon (1932–2000) in 1967, says that an equivariant map $f : X \to Y$ between G-CW complexes in an equivariant homotopy equivalence (meaning a homotopy equivalence for which the homotopies are equivariant) iff for each $H \subseteq G$ the induced map $f^H : X^H \to Y^H$ is an ordinary homotopy equivalence. **Fixed points tell all**. This condition is equivalent to requiring f to induce isomorphisms

$$\pi^H X := \pi_* X^H \to \pi_* Y^H =: \pi_*^H Y \qquad \text{for all } H \subseteq G.$$

We define a G-equivariant map $f : X \to Y$ in $\mathcal{T}^G$ to be a **Bredon equivalence** (Definition 8.6.1) if it satisfies this condition. The map f is a **Bredon fibration** if each f^H is a fibration for each subgroup H. Thus we get the **Bredon model structure** on $\mathcal{T}op^G$ and $\mathcal{T}^G$.

We can relax these definitions by saying that f is a weak equivalence or a fibration if f^H is one for each subgroup $H \subseteq G$ in a family $\mathscr{F}$ of subgroups (as in Definition 8.6.10) that is closed under inclusion and conjugation. When $\mathscr{F}$ contains only the trivial subgroup, then the model structure is the usual one on $\mathcal{T}op$ or $\mathcal{T}$. In other words, we are ignoring the group action. In general if $\mathscr{F}$ does not contain all subgroups of G, then the resulting model structure has more weak equivalences and fibrations, and hence fewer cofibrations than the Bredon model structure. For each $\mathscr{F}$ the model structure is cofibrantly generated with generating sets indicated in Theorem 8.6.13.

Some universal spaces and Elmendorf's theorem. The next two sections describe some technical results that will be needed later. The most interesting of these is Theorem 8.8.4, which asserts the existence of an equivariant Eilenberg-Mac Lane space associated with an arbitrary Mackey functor.

Orthogonal representations of G and related structures are the subject of §8.9, the final section of this chapter. The notions introduced here will be used extensively in Chapter 9.

Given a finite dimensional orthogonal representation V of a finite group G we define in Definition 8.3.26 its unit sphere $S(V)$ and its one point compactification S^V. The set of equivariant homotopy classes of maps from S^V to a pointed G-space X is denoted by $\pi_V^G X$. It usually but does not always have a natural abelian group structure. One can also look at the spaces $\Omega^V X$, the twisted loop space of X, and $\Sigma^V X = S^V \wedge X$, the twisted suspension of X; see Definition 8.9.3. The expected suspension loop adjunction $\Sigma^V \dashv \Omega^V$ is established in Proposition 8.9.4.

In Definition 8.9.10 we generalize the notion of a representation of G to that of a finite G-set T. Roughly speaking, when T is the disjoint union of orbits G/H_α for various subgroups H_α, a representation of T amounts to a representation of each H_α. Then in §8.9C we study two categories, enriched over $\mathcal{T}op^G$ and $\mathcal{T}^G$ respectively, whose objects are representations of finite G-sets T. We call them the **Stiefel category** $\mathscr{I}_G$ (Definition 8.9.19) and the **Mandell-May category** $\mathscr{J}_G$ (Definition 8.9.24). The latter is the indexing category for orthogonal G-spectra, the subject of Chapter 9.

In both categories there are morphisms only between representations of the same G-set T. Given two such representations V and W, a morphism from V to W in the Stiefel category $\mathscr{I}_G$ is an orthogonal embedding, suitably defined. Such embeddings exist only when the dimension of the representation W_α associated with the orbit G/H_α is no less than that of V_α **for each** α. In that case the morphism space is a product of Stiefel manifolds, with one factor for each orbit of T. The embeddings are not required to be equivariant, so G acts on the morphism space by conjugation.

The Mandell-May pointed morphism space $\mathscr{J}_G(V,W)$ is the Thom space of a certain vector bundle over $\mathscr{I}_G(V,W)$. Each embedding $t: V \to W$ determines an orthogonal complement $t(V)^\perp \subseteq W$, and it is the fiber of the vector bundle at t. in particular $\mathscr{J}_G(V,W)$ is a point when $\mathscr{I}_G(V,W)$ is empty for dimensional reasons. When V_α and W_α have the same dimension for each α, $\mathscr{I}_G(V,W)$ is a product of orthogonal groups. The vector bundle over it has dimension 0, so the Thom space is the same product with a disjoint base point.

In Theorem 8.9.34 we show that the Mandell-May category $\mathscr{J}_G$ is a spectral $\mathscr{J}^{\mathbf{O}}$-algebra as in Definition 7.2.19, in the case where G is a cyclic p-group. This is likely to be true for arbitrary finite G, but we leave that question for the future. The result means that the model structures studied in §7.4 exist for orthogonal G-spectra.

8.1 Finite G-sets and the Burnside ring of a finite group

Definition 8.1.1. *For a finite group G, let $\mathcal{F}_G$* **denote the category of finite G-sets** *and equivariant maps.*

Example 8.1.2. The power set $\mathscr{P}(G)$ of a finite group *G has an action of G by left multiplication. This action preserves cardinality, so $\mathscr{P}(G)$ splits as a G-set accordingly. Each subgroup $H \subseteq G$ is also a subset, so $H \in \mathscr{P}(G)$. Its orbit there consists of the left cosets of H and is therefore isomorphic to G/H as a G-set. It follows that every orbit G/H is contained in $\mathscr{P}(G)$. This is analogous to the fact that the regular representation ρ_G of G (see Definition 8.3.27 below) contains each irreducible orthogonal representation of G as a summand.*

Similarly the power set $\mathscr{P}(G/H)$ of the G-set G/H is also a G-set under left multiplication that contains a copy of G/K for each subgroup $K \subseteq H$.

Definition 8.1.3. The Burnside ring *$A(G)$ of a group G is the Grothendieck group of the abelian monoid (under disjoint union) of isomorphism classes of finite G-sets, with multiplication induced by Cartesian product. We will denote the isomorphism class of a finite G-set T by $[T]$.*

The Burnside ring (or the abelian monoid of actual finite G-sets inside it) was first considered by William Burnside (1852—1927) in [Bur11, §180, page

236], although he did not use the term "ring" for it, nor did he have a symbol for it. It is discussed as a ring by Dress in [Dre69], where it is denoted by $\Omega(G)$. Both show that $A(G)$ is additively isomorphic to the free abelian group generated by the set of isomorphism classes of orbits G/H, i.e., the set of conjugacy classes of subgroups $H \subseteq G$. Its multiplicative structure can be determined in the following way; see [Dre69, Lemma 1] for the proof.

Theorem 8.1.4. Detecting the Burnside ring with fixed points. *Given a finite G-set T and a subgroup $H \subseteq G$, let the* **Burnside mark of H on T** *be defined by*

$$\langle H, T\rangle := |T^H|,$$

the cardinality of the fixed point set of T under the action of H. Then two G-sets T_1 and T_2 are isomorphic iff $\langle H, T_1\rangle = \langle H, T_2\rangle$ for all H. Furthermore,

$$\langle H, T_1 \amalg T_2\rangle = \langle H, T_1\rangle + \langle H, T_2\rangle$$

and

$$\langle H, T_1 \times T_2\rangle = \langle H, T_1\rangle\langle H, T_2\rangle,$$

so these data determine an injective ring homomorphism φ from $A(G)$ to the ring $C(G)$ (sometimes called the **ghost ring***) of* $\mathbf{Z}$*-valued functions on the set of conjugacy classes of subgroups of G.*

The term "ghost" in a context similar to this (when the group is the p-adic integers or a finite quotient thereof) appears to be due to Witt. It was used by Lang in connection with Witt vectors in [Lan65].

It is known [tD79, Proposition 1.2.3] that a basis of $C(G)$ is

$$\{\varphi[G/H]/|W_H|\},$$

where φ is the ring homomorphism of Theorem 8.1.4, H ranges over the conjugacy classes of subgroups, and W_H is the Weyl group of H, meaning N_H/H where N_H is the normalizer of H in G. The group W_H acts freely on the G-set G/H and therefore on its fixed point set $(G/H)^K$ for any subgroups $K \subseteq G$. It follows that $\varphi[G/H]$ is divisible by $|W_H|$. More algebraic properties of $A(G)$ are discussed in [tD79, §1].

The term **mark** above was used by Burnside in [Bur11, §180]. The notation is due to [Dre69]. The following was observed by Burnside. Suppose that G has s conjugacy classes of subgroups and that

$$\{G_1, G_2, \ldots, G_s\}$$

is a collection of subgroups representing each conjugacy class. Suppose further that they are chosen so that

$$|G_1| \leqslant |G_2| \leqslant \cdots \leqslant |G_s|$$

(there could be more than one way to do this), so $G_1 = e$ and $G_s = G$. Let $m_i^j = \langle G_i, G/G_j \rangle = |(G/G_j)^{G_i}|$.

Proposition 8.1.5. Burnside's table of marks. *The integers m_i^j defined above satisfy the following.*

(i) $m_1^1 = |G|$ *and* $m_i^i > 0$ *for* $i > 0$.
(ii) $m_i^s = 1$ *for* $1 \leqslant i \leqslant s$.
(iii) $m_i^j = 0$ *for* $i > j$.

Proof For the first statement, $m_i^i = |(G/G_i)^{G_i}|$ by definition and G_i fixes the coset of the identity element. In the case $i = 1$, we have the trivial group acting on G itself. For the second statement, $m_i^s = |(G/G)^{G_i}|$. For the third statement, the condition $i > j$ means that G_i is not conjugate to any subgroup of G_j and therefore acts without fixed points on the set G/G_j. □

It follows that the integers m_i^j form a lower triangular matrix with nonzero determinant. Since an arbitrary finite G-set X has the form

$$X \cong \coprod_{1 \leqslant j \leqslant s} \coprod_{a_j} G/G_j,$$

the integers $a_j \geqslant 0$ are determined by the marks

$$\langle G_i, X \rangle = \sum_{1 \leqslant j \leqslant s} a_j m_i^j.$$

Thus Burnside proved that the ring homomorphism φ of Theorem 8.1.4 is injective. He stated this as [Bur11, §181, Theorem I].

Example 8.1.6. The Burnside ring for the symmetric group S_3. *We apply Theorem 8.1.4 to the case $G = S_3$, the symmetric group on three letters. The following table shows the values of the Burnside marks. Each row corresponds to a subgroup H, and each column corresponds to a G-set T.*

	G/e	G/C_2	G/C_3	G/G	$(G/C_2)^2$	$G/C_2 \times G/C_3$	$(G/C_3)^2$
e	*6*	*3*	*2*	*1*	*9*	*6*	*4*
C_2	*0*	*1*	*0*	*1*	*1*	*0*	*0*
C_3	*0*	*0*	*2*	*1*	*0*	*0*	*4*
S_3	*0*	*0*	*0*	*1*	*0*	*0*	*0*

The left half of the above is the transpose of Burnside's table of marks for S_3. Hence Burnside's lower triangular matrix is

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that the row vector corresponding to G/H is divisible by the order of the Weyl group $|W_H|$.

From the right half of the table above we learn that in $A(G)$

$$\begin{aligned} [G/C_2]^2 &= [G/C_2] + [G/e], \\ [G/C_2] \times [G/C_3] &= [G/e] \\ \text{and} \qquad [G/C_3]^2 &= 2[G/C_3]. \end{aligned}$$

Burnside did a similar calculation with the alternating group $G = A_4$ (which has subgroups of orders 1, 2, 3, 4 and 12, that of order 4 being isomorphic to C_2^2) in [Bur11, page 241]. The resulting matrix is

$$\begin{bmatrix} 12 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \\ 3 & 3 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

from which he deduced that in $A(G)$

$$\begin{aligned} [G/G_2]^2 &= 2[G/G_2] + 2[G/G_1], & [G/G_3]^2 &= [G/G_3] + [G/G_1], \\ [G/G_4]^2 &= 3[G/G_4], & [G/G_2][G/G_3] &= 2[G/G_1], \\ [G/G_2][G/G_4] &= 3[G/G_2], & \text{and } [G/G_3][G/G_4] &= [G/G_1]. \end{aligned}$$

Here G_i denotes the subgroup (unique up to conjugacy) of order i.

More generally for $n \geqslant 4$, we claim

$$\begin{aligned} [S_n/S_{n-1}]^2 &= [S_n/S_{n-1}] + [S_n/S_{n-2}] \\ \text{and} \qquad [A_n/A_{n-1}]^2 &= [A_n/A_{n-1}] + [A_n/A_{n-2}]. \end{aligned}$$

Note that S_n/S_{n-1} is a set with n elements being permuted by S_n and there is a diagonal embedding $S_n/S_{n-1} \to (S_n/S_{n-1})^2$. The complement of its image is the S_n-set

$$\{(i,j)\colon 1 \leqslant i,j \leqslant n, i \neq j\},$$

which is S_n/S_{n-2}. The same goes for A_n.

Example 8.1.7. The Burnside ring for the quaternion group Q_8. *We will denote the elements of this group by ± 1, $\pm i$, $\pm j$ and $\pm k$, and one has*

$$i^2 = j^2 = k^2 = ijk = -1.$$

It has a subgroup of order 2, $C_2 = \{\pm 1\}$. The elements i, j and k generate cyclic subgroups of order 4 which we denote by H_i, H_j, H_k. They are **not** *conjugate to each other. The following table indicates Burnside's marks on the six orbits.*

	G/e	G/C_2	G/H_i	G/H_j	G/H_k	G/G
e	8	4	2	2	2	1
C_2	0	4	2	2	2	1
H_i	0	0	2	0	0	1
H_j	0	0	0	2	0	1
H_k	0	0	0	0	2	1
G	0	0	0	0	0	1

In the commutative ring $A(G)$, let $x = [G/e]$, $y = [G/C_2]$, $z_i = [G/H_i]$, $z_j = [G/H_j]$ and $z_k = [G/H_k]$. The unit is $[G/G]$. From the above we can deduce the following multiplication table for $A(G)$.

	x	y	z_i	z_j	z_k
x	$8x$	$4x$	$2x$	$2x$	$2x$
y		$4y$	$2y$	$2y$	$2y$
z_i			$2z_i$	y	y
z_j				$2z_j$	y
z_k					$2z_k$

Definition 8.1.8. Some subgroups of a general finite group G. *For subgroups H_1 and H_2 and elements γ_1 and γ_2 of a group G, let*

$$H_i^{\gamma_i} := \gamma_i H_i \gamma_i^{-1} \quad \text{and} \quad L^{\gamma_1,\gamma_2} := H_1^{\gamma_1} \cap H_2^{\gamma_2}.$$

Proposition 8.1.9. Isotropy subgroups in $G/H_1 \times G/H_2$. *With notation as in Definition 8.1.8, the isotropy subgroup (as in Definition 2.1.29(iv)) of $G/H_1 \times G/H_2$ at $(\gamma_1 H_1, \gamma_2 H_2)$ is L^{γ_1,γ_2}.*

Theorem 8.1.10. The product of two orbits. *Given two subgroups H_1 and H_2 of G,*

$$[G/H_1][G/H_2] = \sum_{\substack{\gamma_1 H_1 \in G/H_1 \\ \gamma_2 H_2 \in G/H_2}} \frac{[G/L^{\gamma_1,\gamma_2}]}{|G/L^{\gamma_1,\gamma_2}|} = \sum_{H_2\gamma H_1 \in H_2\backslash G/H_1} [G/L^{e,\gamma}].$$

in $A(G)$, with L^{γ_1,γ_2} as in Definition 8.1.8.

Note that the first sum in Theorem 8.1.10 appears to lie in $A(G) \otimes \mathbf{Q}$. We will see in the proof that it actually lies in $A(G)$ itself.

Proof The first sum in the statement is over the points of $G/H_1 \times G/H_2$, and each term can be regarded formally as a fraction of a G-set. If two points $(\gamma_1 H_1, \gamma_2 H_2)$ and $(\gamma_1' H_1, \gamma_2' H_2)$ are in the same G-orbit, then their isotropy subgroups L^{γ_1,γ_2} and $L^{\gamma_1',\gamma_2'}$ are conjugate in G, so the corresponding terms in the sum are the same in $A(G) \otimes \mathbf{Q}$. If we sum over all $|G/L^{\gamma_1,\gamma_2}|$ points in that orbit, we get $G/L^{\gamma_1,\gamma_2}$. Summing over all orbits gives us the claimed value of $[G/H_1][G/H_2]$.

The second sum is over the points of the set of double cosets $H_2\backslash G/H_1$. Conjugating the subgroup L^{γ_1,γ_2} by γ_1^{-1} gives

$$\gamma_1^{-1}L^{\gamma_1,\gamma_2}\gamma_1 = \gamma_1^{-1}(H_1^{\gamma_1} \cap H_2^{\gamma_2})\gamma_1 = H_1 \cap H_2^{\gamma_1^{-1}\gamma_2} = L^{e,\gamma_1^{-1}\gamma_2},$$

so $[G/L^{\gamma_1,\gamma_2}] = [L^{e,\gamma_1^{-1}\gamma_2}]$.

To show the two sums are equal, we will use the fact that

$$|H_2\gamma H_1| = \frac{|H_1||H_2|}{|L^{e,\gamma}|}$$

to rewrite the second sum as one over the elements of G. We have

$$\begin{aligned}\sum_{H_2\gamma H_1 \in H_2\backslash G/H_1} [G/L^{e,\gamma}] &= \sum_{\gamma\in G} \frac{[G/L^{e,\gamma}]}{|H_2\gamma H_1|} = \sum_{\gamma\in G} \frac{[G/L^{e,\gamma}]|G|}{|H_2||H_1||G/L^{e,\gamma}|} \\ &= \sum_{\gamma_1,\gamma_2\in G} \frac{[G/L^{\gamma_1,\gamma_2}]}{|H_2||H_1||G/L^{\gamma_1,\gamma_2}|} \\ &= \sum_{\substack{\gamma_1 H_1\in G/H_1 \\ \gamma_2 H_2 \in G/H_2}} \frac{[G/L^{\gamma_1,\gamma_2}]}{|G/L^{\gamma_1,\gamma_2}|}.\end{aligned}$$

□

Corollary 8.1.11. The product of cosets normal subgroups. *For normal subgroups H_1 and H_2 of a finite group G with $L = H_1 \cap H_2$,*

$$G/H_1 \times G/H_2 = \coprod_{\frac{|G||L|}{|H_1||H_2|}} G/L$$

as G-sets. In particular this is the case for all H_1 and H_2 when G is abelian.

Proof For normal subgroups H_i, each subgroup L^{γ_1,γ_2} is L, so the result is a special case of Theorem 8.1.10. □

We also need to describe the G-set $(G/K)^{G/H}$, the set of nonequivariant maps $G/H \to G/K$. It contains an orbit of constant functions isomorphic to G/K.

Its fixed point set under the action of G is the set of equivariant maps,

$$\left((G/K)^{G/H}\right)^G = (G/K)^H,$$

which is empty unless H is congruent to a subgroup of K.

8.2 Mackey functors

Mackey functors play the role in equivariant stable homotopy theory that abelian groups play in ordinary stable homotopy theory **as coefficients** for various functors. A spectrum has homotopy groups and ordinary homology with coefficients in abelian groups. The analogs for G-spectra are homotopy

Mackey functors and ordinary homology with Mackey functor coefficients. In this section we will define Mackey functors in a purely algebraic way. We will explain their use in equivariant stable homotopy theory in §9.4B below.

8.2A Motivating the definition of a Mackey functor

A Mackey functor $\underline{M}$ (we will almost always use underlines) is a functor $\mathcal{F}_G \to \mathcal{A}b$ (finite G-sets to abelian groups) which is additive in the sense of converting disjoint unions to direct sums, and has certain additional properties. Since every finite G-set decomposes uniquely as a disjoint union of orbits of the form G/H, the additivity of $\underline{M}$ implies that it is determined by its values on such orbits. Before giving the formal definition, which is originally due to Dress [Dre73], we give a motivating example, possibly the original one.

Let $RO(G)$ be the orthogonal representation ring of G. This is the Grothedieck group of the abelian monoid (under direct sum) of isomorphism classes of finite dimensional orthogonal representations V of G. It has a multiplication induced by tensor product. This ring is very well understood. Serre's book [Ser67] is an excellent introduction, but **the reader unfamiliar with this topic would profit greatly from working out the structure of $RO(G)$ for some small groups G with minimal assistance**.

It is known that the number of isomorphism classes of irreducible real representations of G has the following description. Take the set of conjugacy classes of elements in G. It has an involution sending the class of an element γ to that of γ^{-1}. Then the number of orbits under this involution is the number of real irreducible representations. Similarly, the number of complex irreducible representations is the number of conjugacy classes of elements. In both cases there is however **no natural bijection between irreducible representations and conjugacy classes or orbits thereof**.

Example 8.2.1. Real orthogonal representations of some small groups.

(i) *Let $G = S_3$, the symmetric group on three letters. It has three conjugacy classes of elements, namely those of elements of orders 1, 2 and 3. The involution acts trivially on this set. Each element of order 1 or 2 is equal to its own inverse. The two elements or order 3 are inverse to each other but also conjugate to each other.*

There are three irreducible real representations: the trivial and sign representations, each having degree 1, and the 2-dimensional representation obtained by letting the group act on the vertices of an equilateral triangle. The three three irreducible complex representations are obtained by tensoring each of the reral ones with the complex numbers.

(ii) *Let $G = C_4$ with generator γ. Since G is abelian, each conjugacy class* *consists of a single element. There are four irreducible complex representations, each having degree 1. The eigenvalue of γ can be any fourth root of unity.*

The involution acts nontrivially on the ste of conjugacy classes, sending γ to $\gamma^{-1} = \gamma^3$, which is in a different conjugacy class. Thus the number of irreducible real representations is three rather than four. They are the trivial and sign representations, each having degree 1, and a 2-dimensional representation in which the plane gets rotated by $\pi/2$. Complexifying the latter gives the direct sum of the two representations with imaginary eigenvalues.

(iii) Let $G = C_8$ with generator γ. The conjugacy classes fall into five orbits under the involution, namely

$$\{e\}, \quad \{\gamma, \gamma^{-1}\}, \quad \{\gamma^2, \gamma^{-2}\}, \quad \{\gamma^3, \gamma^{-3}\} \quad \textit{and} \quad \{\gamma^4\}.$$

The five irreducible real representations are the trivial and sign representations, each having degree 1, and three 2-dimensional representations in which γ rotates the plane through angles of $\pi/2$, $\pi/4$ and $3\pi/4$. The last two have **2-locally equivalent representation spheres***, which makes them effectively isomorphic to each other for the purposes of any 2-local calculation, such as the ones needed to study the Kervaire invariant.*

For each subgroup $H \subseteq G$, we have a forgetful or **restriction map**

$$\mathrm{Res}_H^G : RO(G) \to RO(H)$$

obtained by restricting the action of G to an H-action. There is also a map $\mathrm{Tr}_H^G : RO(H) \to RO(G)$ called the **transfer map** or **induction** defined as follows. An orthogonal representation W of H is the same thing as a module over the real group ring $\mathbf{R}[H]$. This means that

$$\mathrm{Tr}_H^G W := \mathbf{R}[G] \otimes_{\mathbf{R}[H]} W =: \mathrm{Ind}_H^G W \tag{8.2.2}$$

(as in Example 2.5.8(iv)) is an $\mathbf{R}[G]$-module and hence a representation of G called the **induced representation of** W. As a representation of H is it the direct sum of $|G/H|$ copies of V, and elements in G outside of H permute the summands, each of which is invariant under H. Its dimension as a vector space is $|G/H|$ times that of W since $\mathbf{R}[G]$ is a $\mathbf{R}[H]$-module of rank $|G/H|$, and in $RO(H)$ we have

$$\mathrm{Res}_H^G \mathrm{Tr}_H^G(W) = |G/H|W.$$

Alternatively, let $\mathcal{C}$ and $\mathcal{D}$ be the one object categories associated with H and G with inclusion functor $K : \mathcal{C} \to \mathcal{D}$ as in Example 2.5.8(iv). Then $\mathrm{Tr}_H^G(W)$ is a Kan extension of the functor W to the category of real vector spaces.

We can define a Mackey functor $\underline{RO} : \mathcal{F}_G \to \mathcal{A}b$ (or $\underline{RO}(G)$) by setting $\underline{RO}(G/H) := RO(H)$ (regarded as an abelian group) and using additivity to define it on an arbitrary finite G-set. Alternatively, for a finite G-set T, $\underline{RO}(T)$

is the Grothendieck group of the semiring (under pointwise direct sum and tensor product) of functors to the category of finite dimensional orthogonal real vector spaces from $\mathcal{B}_T G$, the groupoid whose objects are the elements of T with morphisms defined by the action of G.

Two comments are in order:

(i) For subgroups $K \subset H \subseteq G$ we have maps

$$\mathrm{Res}^H_K : \underline{RO}(G/H) \to \underline{RO}(G/K) \quad \text{and} \quad \mathrm{Tr}^H_K : \underline{RO}(G/K) \to \underline{RO}(G/H)$$

(**restriction** and **transfer**, sometimes called **induction)** with suitable properties. This will be a feature of Mackey functors in general. In the category $\mathcal{F}_G$ one has a morphism in just one direction, from G/K to G/H. This suggests that a Mackey functor $\underline{M}$ consists of **two functors** from $\mathcal{F}_G$ to $\mathcal{A}b$, one covariant and one contravariant, having the same behavior on objects.

(ii) The functor $\underline{RO}$ actually takes values in commutative rings. A ring valued Mackey functor satisfying certain conditions is called a **Green functor**; see [TW95]. The restriction map Res^H_K is a ring homomorphism and the transfer satisfies the **Frobenius relation**

$$\mathrm{Tr}^H_K(\mathrm{Res}^H_K(a)b) = a(\mathrm{Tr}^H_K(b)) \qquad \text{for } a \in RO(H) \text{ and } b \in RO(K).$$

8.2B Two equivalent definitions of Mackey functors

Definition 8.2.3. A Mackey functor $\underline{M}$ *for a finite group* G *(or* G**-Mackey functor***) is a pair of functors*

$$M_* : \mathcal{F}_G \to \mathcal{A}b \qquad \textit{and} \qquad M^* : (\mathcal{F}_G)^{op} \to \mathcal{A}b$$

that agree on objects, convert finite disjoint unions to direct sums, and such that for every pullback diagram in $\mathcal{F}_G$*,*

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\beta"'] & S \arrow[d, "\gamma"] \\ T \arrow[r, "\delta"'] & U, \end{tikzcd}$$

we have $M^*(\gamma)M_*(\delta) = M_*(\alpha)M^*(\beta)$*. For a* G*-set* T *we define*

$$\underline{M}(T) := M^*(T) = M_*(T).$$

For subgroups $K \subseteq H \subseteq G$ *with projection* $p : G/K \to G/H$*, the* **restriction map** *is* $\mathrm{Res}^H_K = M^*(p)$ *and the* **transfer map** *is* $\mathrm{Tr}^H_K = M_*(p)$*. We denote the category of Mackey functors and natural transformations by* $\mathfrak{M}_G$*.*

We will sometimes refer to the maps Res_K^H and Tr_K^H as the **fixed point restriction and transfer maps** to distinguish them from the group action restriction and transfer maps of Definition 9.4.19 below.

The category of Mackey functors $\mathfrak{M}_G$ is abelian with kernels and cokernels defined objectwise.

There is an equivalent and more elegant definition due to Lindner [Lin76] of a Mackey functor as a **single $\mathcal{A}b$-valued functor** on a different category having the same objects as $\mathcal{F}_G$, which we now define.

Definition 8.2.4. The Lindner category $\mathscr{B}_G^+$ *for a finite group G has finite G-sets as objects. For G-sets X and Y, morphisms $X \to Y$ are equivalence classes of pushout diagrams (also known as* **spans***) of the form*

$$X \leftarrow U \to Y$$

in $\mathcal{F}_G$, where two such diagrams are equivalent if they are related by an isomorphism on the middle objects. The morphism set $\mathscr{B}_G(X, Y)$ is the abelian monoid under disjoint union of middle objects with the zero morphism being the class of the diagram with U empty.

Given a second such diagram $Y \leftarrow V \to Z$ representing a morphism $Y \to Z$, the composite morphism $X \to Z$ is represented by $X \leftarrow W \to Z$ coming from the diagram

$$\begin{tikzcd}[column sep=small]
 & & W \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge"] & & \\
 & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\
X & & Y & & Z,
\end{tikzcd}$$

where the square is a pullback diagram, meaning that $W = U \underset{Y}{\times} V$.

The **Burnside category** $\mathscr{B}_G$ *has the same objects as $\mathscr{B}_G^+$, and the morphism set $\mathscr{B}_G(X, Y)$ is the Grothendieck group of the abelian monoid $\mathscr{B}_G^+(X, Y)$ with composition induced by that in $\mathscr{B}_G^+$.*

The idea of passing from $\mathscr{B}_G^+$ to $\mathscr{B}_G$ is due to Lewis [Lew80]. Note that $\mathscr{B}_G$ is self dual ($\mathscr{B}_G \cong \mathscr{B}_G^{op}$) since interchanging source and target does not alter the morphism set. For a discussion of a closely related object, the effective Burnside ∞-category, in which morphisms are spans rather than equivalence classes thereof, see [Bar17].

Definition 8.2.3 is easily shown to be equivalent to the following.

Definition 8.2.5. A Mackey functor $\underline{M}$ *for a finite group G is an additive functor $\mathscr{B}_G^+ \to \mathcal{A}b$, meaning a functor of categories enriched over abelian monoids which converts disjoint unions to direct sums. Equivalently it is an additive functor $\mathscr{B}_G \to \mathcal{A}b$, meaning a functor of categories enriched over abelian groups with the same property.*

To see the equivalence between Definition 8.2.5 and Definition 8.2.3, let $\alpha : X \to Y$ be a map of G-sets. Then the morphisms $M_*(\alpha)$ and $M^*(\alpha)$ in Definition 8.2.3 correspond to the values in Definition 8.2.5 of $\underline{M}$ on the morphisms represented by the diagrams $X = X \to Y$ and $Y \leftarrow X = X$ respectively. An arbitrary morphism $X \leftarrow U \to Y$ in $\mathscr{B}_G$ can be written as the composite of two morphisms of this type with the diagram

$$\begin{tikzcd}
 & & U \arrow[dl, equal] \arrow[dr, equal] & & \\
 & U \arrow[dl] \arrow[dr, equal] & & U \arrow[dl, equal] \arrow[dr] & \\
X & & U & & Y.
\end{tikzcd}$$

8.2C Examples of Mackey functors

Definition 8.2.6. *The* **constant Mackey functor** $\underline{\mathbf{Z}}$ *is the functor represented on the category of finite G-sets by the abelian group* $\mathbf{Z}$ *with trivial G-action. Equivalently it is the fixed point Mackey functor (Definition 8.2.8) for the* $\mathbf{Z}[G]$*-module* $\mathbf{Z}$ *with trivial G-action. The value of* $\underline{\mathbf{Z}}$ *on a finite G-set B is the group of functions*

$$\underline{\mathbf{Z}}(B) = \hom^G(B, \mathbf{Z}) = \hom(B/G, \mathbf{Z}).$$

The restriction maps are given by precomposition, and the transfer maps by summing over the fibers. For $K \subset H \subset G$, the transfer map associated by $\underline{\mathbf{Z}}$ *to*

$$G/K \to G/H$$

is the map $\mathbf{Z} \to \mathbf{Z}$ *given by multiplication by the index of K in H.*

Definition 8.2.7. *The* **Burnside Mackey functor** $\underline{A}$ *(or $\underline{A(G)}$) for a group G is given by letting $\underline{A}(S)$ be the Grothendieck group of the abelian monoid (under disjoint union) of isomorphism classes of finite G-sets over S, meaning G-sets equipped with a map to S. A map $\alpha : S \to T$ of G-sets induces a map $\alpha_* : \underline{A}(S) \to \underline{A}(T)$ by composition and $\alpha^* : \underline{A}(T) \to \underline{A}(S)$ by pullback. Equivalently, $\underline{A}(S)$ is the abelian group $\mathscr{B}_G(G/G, S)$.*

There is an **augmentation map** $\epsilon : \underline{A} \to \underline{\mathbf{Z}}$ *which sends the isomorphism class of a virtual G-set over G/H to the element in* $\underline{\mathbf{Z}}(G/H) = \mathbf{Z}$ *corresponding to its cardinality. We denote its kernel by $\underline{I}$, the* **augmentation ideal Mackey functor***.*

The **free Mackey functor** $\underline{A}_S$ *on a finite G-set T is given by*

$$\underline{A}_S(T) := \underline{A}(S \times T).$$

A similar definition of $\underline{M}_S$ for a general Mackey functor $\underline{M}$ will be given below in Definition 8.2.9.

More information on $\underline{A}$ can be found in [Gre92]. There it is shown that for each finite G-set S, $\underline{A}_S$ is a projective object in the abelian category $\mathfrak{M}_G$, and that the latter has enough projective s. The group $\underline{A}(X)$ is a ring under fiber product over X. The ring $\underline{A}(G/H)$ is isomorphic to the Burnside ring $A(H)$ of Definition 8.1.3, where the H-set corresponding to $X \to G/H$ is the preimage of the coset of the identity element.

Definition 8.2.8. *Let M be a module over the group ring $\mathbf{Z}[G]$. The associated* **fixed point Mackey functor** $\underline{M}$ *is given by*

$$\underline{M}(G/H) = M^H := \mathrm{Hom}_{\mathbf{Z}[H]}(\mathbf{Z}, M),$$

the abelian subgroup of M fixed by H. For $K \subseteq H \subseteq G$, the restriction map Res_K^H is the restriction map of fixed point sets. The transfer map on $x \in M^K$ is

$$\mathrm{Tr}_K^H(x) = \sum_{\gamma K \in H/K} \gamma K(x),$$

where $\gamma K(x)$ is well defined since x is fixed by K. We denote by FP the resulting functor to $\mathfrak{M}_G$ from the category $Mod_{\mathbf{Z}[G]}$ of $\mathbf{Z}[G]$-modules given by $M \mapsto \underline{M}$.

Dually, the associated **fixed quotient Mackey functor** $\widehat{\underline{M}}$ *is given by*

$$\widehat{\underline{M}}(G/H) = M_H := \mathbf{Z} \otimes_{\mathbf{Z}[H]} M,$$

the quotient of M by the action of H. For $K \subseteq H \subseteq G$, the transfer map Tr_K^H is the surjection $M_K \to M_H$. The restriction map on an orbit Hx is

$$\mathrm{Res}_K^H(Hx) = \sum_{\gamma K \in H/K} K(\gamma x),$$

We denote by FQ the resulting functor $Mod_{\mathbf{Z}[G]} \to \mathfrak{M}_G$ given by $M \mapsto \widehat{\underline{M}}$.

In contrast to the Burnside Mackey functor, the restriction maps of $\underline{M}$ above are all one to one, but transfer maps need not be onto.

The functors FP and FQ above are known (see [TW90, 6.1]) to be the right and left adjoints respectively of the functor $\mathfrak{M}_G \to Mod_{\mathbf{Z}[G]}$ given by $\underline{M} \mapsto \underline{M}(G/e)$. They are **not** exact. In particular for $M = \mathbf{Z}[G]$ itself, $\underline{M}$ is not projective because its restriction maps are not onto, but they are known to be onto for all projective Mackey functors.

Definition 8.2.9. Mackey functor induction, restriction and precomposition. *For groups $H \subseteq G$ and an H-Mackey functor $\underline{M}$, the* **induced G-Mackey functor** $\uparrow_H^G \underline{M}$ *is given by*

$$(\uparrow_H^G \underline{M})(T) = \underline{M}(i_H^G T)$$

for each finite G-set T, where i_H^G denotes the forgetful functor from G-sets to H-sets.

For a G-Mackey functor $\underline{N}$, the **restricted H-Mackey functor** $\downarrow_H^G \underline{N}$ *is given by*

$$(\downarrow_H^G \underline{N})(S) = \underline{N}(G \underset{H}{\times} S)$$

for each finite H-set S.

For a G-Mackey functor $\underline{M}$ and a finite G-set S, the **precomposite Mackey functor** $\underline{M}_S$ *is given by*

$$\underline{M}_S(T) = \underline{M}(S \times T)$$

for each finite G-set T. In particular, $\underline{M}_{G/H} = \uparrow_H^G \downarrow_H^G \underline{M}$.

This notation for induction and restriction is due to Thévenaz-Webb [TW95]. They put the decorated arrow on the right, denoting $G \underset{H}{\times} S$ by $S \uparrow_H^G$ and $i_H^G T$ by $T\downarrow_H^G$. These two functors relating $\mathfrak{M}_G$ and $\mathfrak{M}_H$ are both left and right adjoints of each other, and they are both exact. The precomposite functor $\underline{M}_S$ is so named because it is the composition

$$\mathscr{B}_G \xrightarrow{S\times -} \mathscr{B}_G \xrightarrow{\underline{M}} \mathcal{A}b$$

The special where $\underline{M}$ is the Burnside Mackey functor $\underline{A}$ is the free Mackey functor $\underline{A}_S$ of Definition 8.2.7.

Example 8.2.10. The precomposite Mackey functor $\underline{M}_{G/e}$. *From Definition 8.2.9 we have for an arbitrary Mackey functor $\underline{M}$ and $S = G/e$,*

$$\begin{aligned}\underline{M}_{G/e}(G/H) = \underline{M}(G/e \times G/H) &= \bigoplus_{G/M} \underline{M}(G/e) \\ &= \mathbf{Z}[G/H] \otimes \underline{M}(G/e) \\ &= \underline{M}(G/e) \otimes \underline{\mathbf{Z}[G]}(G/H),\end{aligned}$$

so $\underline{M}_{G/e} = \underline{M}(G/e) \otimes \underline{\mathbf{Z}[G]}$.

Definition 8.2.11. *Suppose that S is a finite G-set, and write $\mathbf{Z}\{S\}$ for the free abelian group generated by S. The* **permutation Mackey functor** $\underline{\mathbf{Z}}\{S\}$ *is given by*

$$\underline{\mathbf{Z}}\{S\}(B) = \hom^G(B, \mathbf{Z}\{S\}),$$

with restriction maps are given by precomposition and transfer maps by summing over the fibers. Equivalently it is the fixed point Mackey functor (Definition 8.2.8) for $\mathbf{Z}\{S\}$ with $\mathbf{Z}[G]$-module structure induced by the action of G on S.

The permutation Mackey functor $\underline{\mathbf{Z}}\{S\}$ is naturally isomorphic to the Mackey functor $\underline{\pi}_0 H\underline{\mathbf{Z}} \otimes S_+$, where the Eilenberg-Mac Lane spectrum $H\underline{\mathbf{Z}}$ will be defined below in Theorem 9.1.47. It is also related to the Eilenberg-Mac Lane space $K(\underline{\mathbf{Z}}, n)$ of Theorem 8.8.4 by $\mathbf{Z}\{S\} = \underline{\pi}_n K(\underline{\mathbf{Z}}, n) \otimes S_+$. To see the former note that restricting to underlying non-equivariant spectra gives a map

$$\underline{\pi}_0 H\underline{\mathbf{Z}} \otimes S_+(B) = [B_+, H\underline{\mathbf{Z}} \otimes S_+]^G \to [B_+, H\mathbf{Z} \otimes S_+],$$

whose image lies in the G-invariant part. Since

$$[B_+, H\underline{\mathbf{Z}} \otimes S_+] = \hom(B, \mathbf{Z}\{S\})$$

this gives a natural transformation

$$\underline{\pi}_0 H\underline{\mathbf{Z}} \wedge S_+ \to \underline{\mathbf{Z}}\{S\}.$$

Since both sides take filtered colimits in S to filtered colimits, to check that it is an isomorphism, it suffices to do so when S is finite. In that case we can use the self duality of finite G-sets to compute

$$[B_+, H\underline{\mathbf{Z}} \rtimes S]^G \approx [(B \times S)_+, H\underline{\mathbf{Z}}]^G,$$

and then observe that by definition of the constant Mackey functor $\underline{\mathbf{Z}}$, the forgetful map

$$[(B \times S)_+, H\underline{\mathbf{Z}}]^G \to [(B \times S)_+, H\mathbf{Z}]$$

is an isomorphism with the G-invariant part of the target. The claim then follows from the compatibility of equivariant Spanier-Whitehead duality with the restriction functor to non-equivariant spectra.

The properties of permutation Mackey functors listed in the Lemma below follow immediately from the definition. They will be used in §11.3 to establish some of our basic tools for investigating the slice tower. To formulate part (ii), note that every G-set B receives a functorial map from a free G-set, namely $G \times B$, and the group of equivariant automorphisms of $G \times B$ over B is canonically isomorphic to G. For instance, one can give $G \times B$ the product of the left action on G and the trivial action on B, and take the map $G \times B \to B$ to be the original action mapping. With this choice the automorphisms $G \times B$ over B are of the form $(g, b) \mapsto (g\gamma, \gamma^{-1}b)$ with $\gamma \in G$.

Lemma 8.2.12. *Let $\underline{M}$ be a permutation Mackey functor and B finite G-set.*

(i) If $B' \to B$ is a surjective map of finite G-sets, then

$$\underline{M}(B) \to \underline{M}(B') \rightrightarrows \underline{M}(B' \underset{B}{\times} B')$$

is an equalizer.

(ii) Restriction along the action map $G \times B \to B$ gives an isomorphism

$$\underline{M}(B) \to \underline{M}(G \times B)^G.$$

(iii) The restriction mapping $\underline{M}(G/H) \to \underline{M}(G)$ gives an isomorphism

$$\underline{M}(G/H) \to \underline{M}(G)^H$$

of $\underline{M}(G/H)$ with the H-invariant part of $\underline{M}(G)$.

(iv) A map $\underline{M} \to \underline{M}'$ *of permutation Mackey functors is an isomorphism if and only if* $\underline{M}(G/e) \to \underline{M}'(G/e)$ *is an isomorphism.*

8.2D Lewis diagrams

When G is a finite cyclic 2-group, the main case of interest in this book, it has a linearly ordered sequence of subgroups. **When a subgroup of such a G appears as an index, we will often replace it by its order.** We can specify Mackey functors $\underline{M}$ for the group C_2, $\underline{M}'$ for C_4 and $\underline{M}''$ for C_8 by means of Lewis diagrams (first introduced in [Lew88]),

$$
\begin{tikzcd}
\underline{M}(C_2/C_2) \arrow[d, bend right, "\mathrm{Res}_1^2"'] \\
\underline{M}(C_2/e) \arrow[u, bend right, "\mathrm{Tr}_1^2"']
\end{tikzcd}
,\qquad
\begin{tikzcd}
\underline{M}'(C_4/C_4) \arrow[d, bend right, "\mathrm{Res}_2^4"'] \\
\underline{M}'(C_4/C_2) \arrow[u, bend right, "\mathrm{Tr}_2^4"'] \arrow[d, bend right, "\mathrm{Res}_1^2"'] \\
\underline{M}'(C_4/e) \arrow[u, bend right, "\mathrm{Tr}_1^2"']
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\underline{M}''(C_8/C_8) \arrow[d, bend right, "\mathrm{Res}_4^8"'] \\
\underline{M}''(C_8/C_4) \arrow[u, bend right, "\mathrm{Tr}_4^8"'] \arrow[d, bend right, "\mathrm{Res}_2^4"'] \\
\underline{M}''(C_8/C_2) \arrow[u, bend right, "\mathrm{Tr}_2^4"'] \arrow[d, bend right, "\mathrm{Res}_1^2"'] \\
\underline{M}''(C_8/e). \arrow[u, bend right, "\mathrm{Tr}_1^2"']
\end{tikzcd}
\tag{8.2.13}
$$

We omit Lewis' looped arrow indicating the Weyl group action on $\underline{M}(G/H)$ for proper subgroups H. This notation is prohibitively cumbersome in spectral sequence charts, so we will abbreviate specific examples by more concise symbols. Table 8.1 indicates some for the case $G = C_2$. **Admittedly some of them are arbitrary and take some getting used to, but we have to start somewhere.** Lewis denotes the fixed point Mackey functor for a $\mathbf{Z}G$-module M by $R(M)$. He abbreviates $R(\mathbf{Z})$and $R(\mathbf{Z}_-)$ by R and R_-. He also defines (with similar abbreviations) the orbit group or fixed quotient Mackey functor (see Definition 8.2.8) $L(M)$ by

$$L(M)(G/H) = M_H.$$

In this case each transfer map is the surjection of the orbit space for a smaller subgroup onto that of a larger one. The functors L and R are the left and right adjoints of the forgetful functor $\underline{M} \mapsto \underline{M}(G/e)$ from Mackey functors to $\mathbf{Z}[G]$-modules. In particular his R is the functor FQ of Definition 8.2.8.

8.2E The box product of Mackey functors

We now describe a closed symmetric monoidal structure on the category $\mathfrak{M}_G$ of G-Mackey functors. We will make use of the Day convolution of Definition 3.3.2. The Burnside category $\mathscr{B}_G$ (Definition 8.2.4) is symmetric monoidal under cartesian product, with the unit object being G/G. We will make use of the tensor product structure on the category $\mathcal{A}b$ of abelian groups, for which the unit object is $\mathbf{Z}$.

Table 8.1 *Some C_2-Mackey functors*

Symbol	$\square$	$\overline{\square}$	$\bullet$	◩	$\dot{\square}$	$\widehat{\square}$
Lewis diagram	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2}\\ \mathbf{Z}\end{array}$	$\begin{array}{c}0\\ \downarrow\ \uparrow\\ \mathbf{Z}_-\end{array}$	$\begin{array}{c}\mathbf{Z}/2\\ \downarrow\ \uparrow\\ 0\end{array}$	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle 2}\downarrow\ \uparrow{\scriptstyle 1}\\ \mathbf{Z}\end{array}$	$\begin{array}{c}\mathbf{Z}/2\\ {\scriptstyle 0}\downarrow\ \uparrow{\scriptstyle 1}\\ \mathbf{Z}_-\end{array}$	$\begin{array}{c}\mathbf{Z}\\ {\scriptstyle \Delta}\downarrow\ \uparrow{\scriptstyle \nabla}\\ \mathbf{Z}[C_2]\end{array}$
Lewis symbol	R	R_-	$\langle \mathbf{Z}/2\rangle$	L	L_-	$R(\mathbf{Z}^2)$

Symbol	$\underline{A}$	$\underline{I}$
Lewis diagram	$\begin{array}{c}A(C_2)\\ {\scriptstyle \epsilon}\downarrow\ \uparrow{\scriptstyle [C_2/e]}\\ \mathbf{Z}\end{array}$	$\begin{array}{c}\mathbf{Z}\\ \downarrow\ \uparrow\\ 0\end{array}$
Description	Burnside Mackey functor of Definition 8.2.7	Augmentation ideal, the kernel of $\epsilon : \underline{A} \to \square$

Definition 8.2.14. *For a finite group G* **the box product** $\underline{M} \square \underline{N}$ *of two G-Mackey functors $\underline{M}$ and $\underline{N}$ is the left Kan extension (see §2.5)*

$$\begin{tikzcd} \mathscr{B}_G \times \mathscr{B}_G \arrow[r, "\underline{M}\times\underline{N}"] \arrow[drr, "\times"'] & \mathcal{A}b \times \mathcal{A}b \arrow[rr, "\otimes"] & & \mathcal{A}b \\ & & \mathscr{B}_G \arrow[ur, dashed, "\underline{M}\square\underline{N}"'] & \end{tikzcd}$$

The coend formula (2.5.11) for a left Kan extension implies that for a finite G-set X,

$$(\underline{M} \square \underline{N})(X) = \int^{\mathscr{B}_G \times \mathscr{B}_G} \mathscr{B}_G(Y \times Z, X) \otimes \underline{M}(Y) \otimes \underline{N}(Z).$$

the Day Convolution Theorem 3.3.5 implies that the box product defines a closed symmetric monoidal structure on $\mathfrak{M}_G$ in which the unit is the Burnside Mackey functor $\underline{A}$ of Definition 8.2.7, which is also the Yoneda functor $よ^{G/G}$ of Yoneda Lemma 2.2.10.

Finding an explicit description of the box product for a given group G is not easy. For the case $G = C_p$, see [Maz15, Definition 2.3].

8.3 Some formal properties of G-spaces

Recall from Definition 3.1.59 that $\mathcal{T}op_G$ and $\mathcal{T}op^G$ denote the categories of G-spaces with all continuous maps and with equivariant maps respectively. Their pointed analogs are denoted by $\mathcal{T}_G$ and $\mathcal{T}^G$. In a pointed G-space the base point is fixed by G. Each category is symmetric monoidal as in Definition 2.6.1, but only $\mathcal{T}op^G$ and $\mathcal{T}^G$ are bicomplete as in Definition 2.3.25. Since maps in $\mathcal{T}op_G$ and $\mathcal{T}_G$ are not required to be equivariant, there are no natural group actions on limits or colimits.

Remark 8.3.1. The space underlying a G-space. *There are forgetful functors*

$$\mathcal{T}op^G \to \mathcal{T}op_G \to \mathcal{T}op \qquad \text{and} \qquad \mathcal{T}^G \to \mathcal{T}_G \to \mathcal{T}.$$

We will refer to the image of each as the **underlying space**. *We will sometimes say that a G-space is* **underlain** *by its image in $\mathcal{T}op$ or $\mathcal{T}$.*

Definition 8.3.2. *A* **free pointed G-space** *is pointed G-space in which the action a free away from the base point.*

Recall that $\mathcal{T}op$ and $\mathcal{T}$ are both closed symmetric monoidal, meaning they have internal Hom functors and are thus enriched over themselves. The same is true in the equivariant case, but we need to be more careful. As explained in Definition 3.1.59, morphism objects in $\mathcal{T}op_G$ have group actions and equivariant composition morphisms, but those in $\mathcal{T}op^G$ do not. This means that $\mathcal{T}op_G$ is enriched over $\mathcal{T}op^G$, and therefore over itself since $\mathcal{T}op^G$ is a subcategory of $\mathcal{T}op_G$. Hence it is closed symmetric monoidal. As we saw in Proposition 3.1.64 $\mathcal{T}op^G$ has an internal Hom functor, $\mathcal{T}op_G(-,-)$ that differs from its categorical Hom functor $\mathcal{T}opG(-,-)$, so it is also closed symmetric monoidal. Similar remarks apply to the pointed analogs $\mathcal{T}_G$ and $\mathcal{T}^G$.

$\mathcal{T}op^G$ and $\mathcal{T}^G$ are also the categories of functors to $\mathcal{T}op$ and $\mathcal{T}$ from the one object category $\mathcal{B}G$ of Definition 2.1.31. Such functor categories were discussed in Example 2.9.8, and we will discuss the corresponding norm induction functor in Definition 8.3.23. They were also discussed in the context of model categories in §5.4. We will use that perspective below in §8.6.

8.3A Orbit spaces, homotopy orbit spaces, fixed point sets and homotopy fixed point sets

We introduced these in Example 5.8.5(i) in connection with homotopy limits and colimits. Their importance in what follows warrants redefining them formally. First we need the following.

Definition 8.3.3. The classifying space BG of a group G and the contractible free G-space EG. *The former is the geometric realization of the nerve of the one object category $\mathcal{B}G$ as in Definition 3.4.12, and the latter is the same for the category $\mathcal{B}G\downarrow *$ of Definition 2.1.51, where $*$ denotes the single object of $\mathcal{B}G$.*

Definition 8.3.3 requires some unpacking. An n-simplex in the nerve $N(\mathcal{B}G)$ is a diagram in $\mathcal{B}G$ of the form

$$* \xrightarrow{\gamma_1} * \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_n} * \qquad \text{with } \gamma_i \in G. \tag{8.3.4}$$

Hence the set of such simplices is the n-fold Cartesian product G^n. In particular is has a single vertex and an edge for each element in G. Hence the geometric realization

$$BG = |N(\mathcal{B}G)|$$

is a suitable quotient of the space

$$\coprod_{n \geqslant 0} G^n \times \Delta^n.$$

The k-skeleton of BG is sometimes denoted by B_kG. It is the corresponding quotient of the space

$$\coprod_{0 \leqslant n \leqslant k} G^n \times \Delta^n.$$

When G is a topological group, the nerve is a simplicial space rather than a simplicial set, and the topologies of BG and B_kG are modifiled accordingly.

Similarly an n-simplex in the nerve $N(\mathcal{B}G{\downarrow}*)$ is a diagram in $\mathcal{B}G$ of the form

$$\begin{tikzcd} * \arrow[r, "\gamma_1"] & * \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_n"] & * \arrow[d, "\gamma"] \\ & & & * \end{tikzcd} \qquad \text{with } \gamma_i, \gamma \in G. \tag{8.3.5}$$

The set of n-simplices is G^{n+1}. In particular there is a vertex for each element of G.

There is a free action of G given by composition with the vertical arrow. The orbit space is BG, and the map $EG \to BG$ sends the diagram of (8.3.5) to that of (8.3.4).

The n-skeleton E_nG of EG is the $(n+1)$-fold join

$$G * G * \cdots * G,$$

with diagonal G-action, which is known to be $(n-1)$-connected, making EG itself contractible. This space is a quotient of the space

$$\left\{ (\gamma_0, \ldots \gamma_n; t_0, \ldots t_n) \in G^{n+1} \times I^{n+1} \colon 0 \leqslant t_i \leqslant 1, \sum_{0 \leqslant i \leqslant n} t_i = 1 \right\} \tag{8.3.6}$$

Two such points are identified if they have the same coordinates in all but the ith position, and both have $t_i = 0$. In other words, γ_i can be ignored when $t_i = 0$.

Example 8.3.7. Some classifying spaces.

(i) **The case $G = C_2$.** *The $(n+1)$-fold join is homeomorphic to S^n with the antipodal group action, and B_nC_2 is the n-dimensional real projective space $\mathbf{R}P^n$.*

(ii) **The case $G = S^1$.** *Regard the circle group S^1 as the mutiplicative group of complex numbers with modulus 1. The $(n+1)$-fold join is homeomorphic to S^{2n+1}, the space of unit vectors in the complex vector space $\mathbf{C}^{n+1}$. The group action is by scalar multiplication. It follows that B_nS^1 is the n-dimensional complex projective space $\mathbf{C}P^n$.*

(iii) **The case $G = C_p$ for an odd prime p.** *The $(n+1)$-fold join is a free G-CW complex (to be defined below in Definition 8.4.13) underlain by a space homotopy equivalent to a wedge of $(p-1)^{n+1}$ copies of S^n and having a complicated orbit space. By embedding C_p in S^1 as explained in Remark 3.4.17, we see that S^{2n+1} is a free G-space whose orbit space is a lens space.*

(iv) **The case $G = O(k)$, the kth orthogonal group.** *We recall the description of $BO(k)$ given by Milnor-Stasheff in [MS74, §5]. They denote the Grassmannian manifold of real k-planes in $\mathbf{R}^{n+k}$ by $G_k(\mathbf{R}^{n+k})$, and the Stiefel manifold $O(k, n+k)$ (in the notation we use below in Definition 8.9.15) of orthonormal k-frames in $\mathbf{R}^{n+k}$ by $V_k^O(\mathbf{R}^{n+k})$. A point in the latter can be specified by a $k \times (n+k)$ real matrix with orthonormal row vectors. The group $O(k)$ acts freely on it by left multiplication. The orbit space is $G_k(\mathbf{R}^{n+k})$, with the orbit of a matrix identified with its row space. The connectivity of $V_k^O(\mathbf{R}^{n+k})$ increases with n, so the colimit over all n is contractible. Hence the classifying space $BO(k)$ can described as the space of real k-planes in an infinite dimensional Euclidean space.*

This is **not** *the space of Definition 8.3.3. The space*

$$E_nO(k) = O(k)^{*(n+1)}$$

can be equivariantly embedded in the Stiefel manifold $V_k^O(\mathbf{R}^{k(n+1)})$ by sending a point as in (8.3.6) to the matrix in $V_k^O(\mathbf{R}^{(n+1)k})$ in which the $(i+1)$th set of k columns is the matrix$\sqrt{t_i}\,\gamma_i$. We leave the details to the reader.

Definition 8.3.8. Four spaces associated with a G-space. *Let X be a G-space, that is a $\mathcal{T}op$-valued functor from the one object category $\mathcal{B}G$ of Definition 2.1.31. Then*

(i) *Its* **orbit space** *X_G or X/G is the colimit of the functor. Equivalently it is the quotient of the space X obtained by collapsing each G-orbit to a single point, that is identifying x with γx for each $x \in X$ and $\gamma \in G$.*

(ii) *Its* **homotopy orbit space** *X_{hG}, also known as the* **Borel construction**, *is the homotopy colimit of the functor. Equivalently it is the space*

$$X \times_G EG = (X \times EG)_G,$$

the orbit space of the product $X \times EG$ equipped with the diagonal group *action. For a pointed G-space X, which is by definition a pointed space with a G-action fixing the base point, the* **pointed homotopy orbit space** X_{hG*} *is*

$$EG \underset{G}{\ltimes} X$$

which is the orbit space under the diagonal action of the pointed G-space

$$(EG \times X)/(EG \times \{x_0\})$$

by (2.6.16), where $x_0 \in X$ is the base point.

(iii) The **fixed point space** X^G *is the limit of the functor. Equivalently it is the subspace*

$$\{x \in X : \gamma x = x \text{ for each } \gamma \in G\},$$

which is the same as $\mathcal{T}op^G(*, X)$, the space of equivariant maps of a point into X.

(iv) The **homotopy fixed point space** X^{hG} *is the homotopy limit of the functor. Equivalently it is $\mathcal{T}op^G(EG, X)$, the space of equivariant maps of a contractible free G-space into X. Varying the choice of EG as in Remark 3.4.17 does not alter the homotopy type of X^{hG}.*

The following two results are exercises for the reader. The former was discussed in Example 2.3.35(iii).

Proposition 8.3.9. Properties of the orbit and fixed point spaces. *Let $\Delta : \mathcal{T} \to \mathcal{T}^G$ be the functor sending a pointed space to the same space with trivial G-action. Its left and right adjoints send a pointed G-space Y to its orbit space Y_G andits fixed point space Y^G respectively.*

Proposition 8.3.10. Properties of the homotopy orbit and homotopy fixed point spaces.

(i) The homotopy types of X_{hG} and X^{hG} are independent of the choice of contractible free G-space EG.

(ii) The map $X \to *$ induces a fibration $p : X_{hG} \to BG$ in which the preimage of each point is homeomorphic to X. Thus X_{hG} is a fiber bundle over BG with fiber X.

We learned the following from Jesper Grodal.

Proposition 8.3.11. The homotopy fixed point space X^{hG} is the space of sections of the bundle of Proposition 8.3.10(ii).

Proof Consider the map $p_2 : X \times EG \to EG$. A section s of it is the same thing as a map $EG \to X$. Consider the diagram

$$\begin{tikzcd} X \arrow[d] & X \times EG \arrow[l, "p_1"'] \arrow[d] \arrow[r, bend left, "p_2"] & EG \arrow[d] \arrow[l, bend left, "s"] \\ X_G & X_{hG} \arrow[l, "\epsilon"] \arrow[r, bend left, "p"] & BG. \arrow[l, bend left, dashed, "\overline{s}"] \end{tikzcd}$$

The spaces in the top row have G-actions with orbit spaces shown in the bottom row. The section s of p_2 is determined by its composite with p_1. It induces a section $\overline{s}$ of p iff $p_1 s$ is equivariant, making $p_1 s$ a point in X^{hG}. □

The following should be compared with Definition 7.2.30.

Definition 8.3.12. Equivariant homotopy groups and homotopy classes. *For a pointed G-space Y, $\pi_*^H(Y) := \pi_*(Y^H)$. Equivalently $\pi_k^H(Y)$ is the group of homotopy classes of H-equivariant maps $S^k \to Y$, where H acts trivially on S^k.*

More generally for pointed G-spaces X and Y,

$$[X, Y]^G = \pi_0 \mathcal{T}^G(X, Y),$$

which is the set of equivariant homotopy classes of pointed G-maps from X to Y. When $X = S^V$ for a representation V, we denote this set by $\pi_V^G X$.

Recall that $\mathcal{T}^G(X, Y)$ is not a G-space.

Proposition 8.3.13. Equivariant maps from a space with trivial G-action. *Let W, X and Y be pointed G-spaces with G acting trivially on W. Then*

$$\mathcal{T}^G(W \wedge X, Y) \cong \mathcal{T}(W, \mathcal{T}^G(X, Y)).$$

In particular when $W = S^k$ with trivial G-action, we have

$$\mathcal{T}^G(\Sigma^k X, Y) \cong \Omega^k \mathcal{T}^G(X, Y). \tag{8.3.14}$$

Proof Since $\mathcal{T}_G$ is closed symmetric monoidal, we have

$$\mathcal{T}_G(W \wedge X, Y) \cong \mathcal{T}_G(W, \mathcal{T}_G(X, Y)).$$

Taking the fixed points of both sides gives

$$\begin{aligned} \mathcal{T}^G(W \wedge X, Y) &\cong \mathcal{T}^G(W, \mathcal{T}_G(X, Y)) \\ &\cong \mathcal{T}(W, \mathcal{T}_G(X, Y)^G) = \mathcal{T}(W, \mathcal{T}^G(X, Y)), \end{aligned}$$

where the second isomorphism holds by the triviality of the action of G on W. An equivariant map out of W must land in the fixed point to the target, which in this case is $\mathcal{T}^G(X, Y)$. □

Remark 8.3.15. The Sullivan conjecture. *It is known that for a finite p-group G and a finite nilpotent G-CW complex X, the map $X^G \to X^{hG}$ (induced by the map $EG \to *$) is an equivalence after p-adic completion. This is stated as [May96, Theorem VIII.1.2], where it is attributed to Miller, Carlsson and Lannes. The term "nilpotent" here has to do with the action of $\pi_1 X$ on its higher homotopy groups. The condition is satisfied when X is simply connected.*

See Proposition 8.6.20 below for another description of these groups.

8.3B Change of group

As in Definition 2.2.25, for each subgroup $H \subseteq G$ we have a forgetful functor i_H^G (denoted by i_H^* in [HHR16, §2.2.3]) from $\mathcal{T}op^G$ ($\mathcal{T}^G$) to $\mathcal{T}op^H$ ($\mathcal{T}^H$).

Remark 8.3.16. The forgetful functor i_H^G is neither faithful nor full. *It is not faithful because one could have two different G-actions on a space which agree as H-actions. It is not full because there could be maps between two G-spaces which are H-equivariant but not G-equivariant.*

Remark 8.3.17. The forgetful functor and enrichment. *Enriched functors as in Definition 3.1.13 are functors between categories enriched over the symmetric monoidal category $\mathcal{V}$. Thus if we regard $\mathcal{T}^G$ and $\mathcal{T}^H$ as categories enriched over themselves, then the ordinary functor i_H^G is not enriched. The same goes for any functor from a $\mathcal{T}^G$-category to a $\mathcal{T}^H$-category.*

On the other hand, i_H^G is strictly monoidal as in Definition 2.6.20, and hence lax monoidal as in Definition 2.6.19. Thus Proposition 3.1.21 and Proposition 3.1.22 give us a way to convert a $\mathcal{T}^G$-category into a $\mathcal{T}^H$-category.

Definition 8.3.18. The induction functor for G-spaces. *For a subgroup $H \subseteq G$ and a (pointed) H-space Y, the* **induced** *(pointed) G-space is*

$$G \underset{H}{\times} Y \qquad \left(G \underset{H}{\ltimes} Y \right).$$

The following is immediate and is the topological analog of part of Definition 2.2.25.

Proposition 8.3.19. Induction is the left adjoint of restriction. *The functor*

$$G \underset{H}{\times} (-) : \mathcal{T}op^H \to \mathcal{T}op^G \qquad \left(G \underset{H}{\ltimes} (-) : \mathcal{T}^H \to \mathcal{T}^G \right)$$

is the left adjoint of i_H^G. This is the **first change of group adjunction**, *or simply the change of group adjunction.*

We will see in Proposition 8.3.24 that i_H^G also has a right adjoint known as **coinduction**.

For a pointed G-space X and a pointed H-space Y, the analogs of the maps in (2.2.26) are maps

$$\mu_H^G : G \underset{H}{\ltimes} i_H^G X \to X \qquad \text{and} \qquad \psi_H^G : Y \to i_H^G (G \underset{H}{\ltimes} Y). \tag{8.3.20}$$

given by

$$\mu_H^G(\gamma \wedge x) = \gamma(x) \qquad \text{and} \qquad \psi_H^G(y) = (e \wedge y)$$

for $y \in Y$, $\gamma \in G$ and $x \in X$. When X is induced up from a pointed H-space W, we have an extended action map

$$\hat{\mu}_H^G : G \underset{H}{\ltimes} i_H^G (G \underset{H}{\ltimes} W) = (G \underset{H}{\times} G) \underset{H}{\ltimes} W \to G \underset{H}{\ltimes} W \tag{8.3.21}$$

generalizing (2.2.27). When Y is the restriction of a pointed G-space Z, we have a lifted coaction map of pointed G-spaces

$$\widetilde{\psi}_H^G : Z \to G \underset{H}{\ltimes} i_H^G Z \tag{8.3.22}$$

generalizing (2.2.28), with $i_H^G(\widetilde{\psi}_H^G) = \psi_H^G$.

Definition 8.3.23. The indexed product and norm functors

$$(-)^{G/H} : \mathcal{T}op^H \to \mathcal{T}op^G \qquad \textit{and} \qquad N_H^G : \mathcal{T}^H \to \mathcal{T}^G$$

are special cases of the functor $p_*^{\otimes}$ of (2.9.9). For a (pointed) H-space X (Y), these are given by

$$X \mapsto \mathcal{T}op^H(G, X) \qquad \left(Y \mapsto \mathcal{T}^H(G_+, Y)\right),$$

in which G (G_+) is a (pointed) H-space under right multiplication.

The G-space $X^{G/H}$ and the pointed G-space $N_H^G Y$ are underlain by the Cartesian and smash products

$$X^{|G/H|} \qquad \text{and} \qquad Y^{\wedge |G/H|}$$

respectively. The action of G permutes the factors of each with the subgroup H leaving them invariant and acting on each one in the prescribed way.

We saw in Proposition 8.3.19 that i_H^G has a left adjoint called induction. Now we will deal with its right adjoint, **coinduction**, which was described in the category of sets in Definition 2.2.29.

Proposition 8.3.24. The forgetful functor is a left adjoint: the second change of group adjunction. *The restriction functors i_H^G of Definition 8.3.23 are right adjoints of i_H^G in their respective categories.*

Proof To identify the right adjoint F of i_H^G, let X and Y be a G-space and an H-space respectively. Then we have

$$\mathcal{T}op^H(i_H^G X, Y) \cong \mathcal{T}op^G(X, FY) \tag{8.3.25}$$

If we set $X = G$, then $i_H^G X$ is the disjoint union of $|G/H|$ copies of H and (8.3.25) reads

$$Y^{|G/H|} \cong FY$$

as topological spaces. This means the G-space FY is underlain by the $|G/H|$-fold Cartesian power of Y. Since the action of G on itself permutes the $|G/H|$ copies of H within it, its action on FY permutes its factors. This functor is the indexed product of (2.9.9) for $\mathcal{V} = \mathcal{T}op$. This means that the action of G permutes the factors of the Cartesian product, each of which is invariant under and acted upon by the subgroup H.

Similar considerations in the pointed case lead to the norm functor, which is an indexed smash (rather than Cartesian) power. □

We will define a similar functor of spectra below in Definition 9.7.3.

The following G-spaces will be used repeatedly in this book.

Definition 8.3.26. Representation spheres. *Let V be a finite dimensional orthogonal representation of a finite group G, that is a module over $\mathbf{R}[G]$, the group ring of G over the real numbers. Then its* **representation sphere** S^V *the one point compactification of V. Its* **unit sphere** $S(V)$ *is the space of unit vectors in V, the equator of S^V. Its* **unit disk** $D(V)$ *is the space vectors of length $\leqslant 1$ in V, the northern hemisphere of S^V.*

Note that S^V is the mapping cone of the map $S(V) \to *$.

Given another such representation V', one has $S^{V \oplus V'} \cong S^V \wedge S^{V'}$, and $S(V \oplus V') \cong S(V) * S(V')$, the join of the two unit spheres.

Definition 8.3.27. *The* **regular representation** ρ_G **of a finite group** G *is $\mathbf{R}[G]$, the real vector space with the set G as a basis, with the group G acting by left multiplication. The* **reduced regular representation** $\overline{\rho}_G$ *is the subspace in which the sum of the coordinates is zero.*

Thus we have $\rho_G = 1 + \overline{\rho}_G$ in the orthogonal representation ring $RO(G)$. It is known that every nontivial irreducible orthogonal representation of G is a summand of $\overline{\rho}_G$.

An important example of the norm of Definition 8.3.23 is the following. Let $H \subseteq G$ be a subgroup with an orthogonal representation V. Then classically one has the induced representation $\mathrm{Ind}_H^G V$ of G of (8.2.2).

Proposition 8.3.28. The induced representation as an indexed product. *Let V be an orthogonal representation of a subgroup $H \subseteq G$. Then* $\mathrm{Ind}_H^G V$

is the indexed product $V^{G/H}$ of Definition 8.3.23. Its one point compactification $S^{\mathrm{Ind}_H^G V}$ is the norm $N_H^G S^V$.

Now suppose that in addition we have a subgroup $K \subseteq H$ and a pointed K-space Z. Then we have pointed maps

$$\begin{tikzcd}
G \underset{K}{\ltimes} i_K^G X \arrow[d, equal] \arrow[rrd, bend left, "\mu_K^G"] & & \\
G \underset{H}{\ltimes} (H \underset{K}{\ltimes} i_K^G X) \arrow[r, "G \underset{H}{\ltimes} \mu_L^H"'] & G \underset{H}{\ltimes} i_H^G X \arrow[r, "\mu_H^G"'] & X,
\end{tikzcd} \tag{8.3.29}$$

with μ_L^H and μ_H^G as in (8.3.20), and

$$\begin{tikzcd}
Z \arrow[r, "\psi_K^H"] \arrow[rrd, bend right, "\psi_K^G"'] & i_K^H(H \underset{K}{\ltimes} Z) \arrow[r, "i_K^H(\psi_H^G)"] & i_K^G(G \underset{H}{\ltimes} (H \underset{K}{\ltimes} Z)) \arrow[d, equal] \\
& & i_K^G(G \underset{K}{\ltimes} Z),
\end{tikzcd} \tag{8.3.30}$$

with ψ_H^G and ψ_K^G as in (8.3.20).

A similar argument to that of Theorem 8.1.10 gives the following generalization from products of finite G-sets induced up from subgroups to products of similarly constructed (pointed) G-spaces.

Proposition 8.3.31. The (smash) product of (pointed) G-spaces induced up from two subgroups. *Let H_1 and H_2 be subgroups of G, and let X_i be an H_i-space for both values of i. Using the notation of Definition 8.1.8, we will also regard X_i as an H_i^γ-space for each $\gamma \in G$. Then*

$$(G \underset{H_1}{\times} X_1) \times (G \underset{H_2}{\times} X_2) \cong \coprod_{\substack{\gamma_1 H_1 \in G/H_1 \\ \gamma_2 H_2 \in G/H_2}} \frac{G \underset{L^{\gamma_1,\gamma_2}}{\times} \left(i_{L^{\gamma_1,\gamma_2}}^{H_1^{\gamma_1}} X_1 \times i_{L^{\gamma_1,\gamma_2}}^{H_2^{\gamma_2}} X_2 \right)}{|G/L^{\gamma_1,\gamma_2}|},$$

where i_L^H for $l \subseteq H$ is the forgetful functor from $\mathcal{T}op^H$ to $\mathcal{T}op^L$.

For pointed H_i-space X_i,

$$(G \underset{H_1}{\ltimes} X_1) \wedge (G \underset{H_2}{\ltimes} X_2) \cong \bigvee_{\substack{\gamma_1 H_1 \in G/H_1 \\ \gamma_2 H_2 \in G/H_2}} \frac{G \underset{L^{\gamma_1,\gamma_2}}{\ltimes} \left(i_{L^{\gamma_1,\gamma_2}}^{H_1^{\gamma_1}} X_1 \wedge i_{L^{\gamma_1,\gamma_2}}^{H_2^{\gamma_2}} X_2 \right)}{|G/L^{\gamma_1,\gamma_2}|}.$$

When the subgroups H_i are both normal with $L = H_1 \cap H_2$,

$$(G \underset{H_1}{\times} X_1) \times (G \underset{H_2}{\times} X_2) \cong \coprod_{|G||L|/|H_1||H_2|} G \underset{L}{\times} \left(i_L^{H_1} X_1 \times i_L^{H_2} X_2 \right),$$

and in the pointed case,

$$(G \underset{H_1}{\ltimes} X_1) \wedge (G \underset{H_2}{\ltimes} X_2) \cong \bigvee_{|G||L|/|H_1||H_2|} G \underset{L}{\ltimes} \left(i_L^{H_1} X_1 \wedge i_L^{H_2} X_2 \right).$$

Proof The pointed and unpointed cases are similar. In the latter for general subgroups H_i, as in Theorem 8.1.10, the disjoint union on the right is taken over the points of $G/H_1 \times G/H_2$ with each term being formally a fraction of the indicated G-space. If we sum over the $|G/L^{\gamma_1,\gamma_2}|$ points in the orbit of $(\gamma_1 H_1, \gamma_2 H_2)$, the numerators are all the same and the denominator is 1. Hence the right hand side is actually a disjoint union of G-spaces rather than fractions thereof, one for each G-orbit of $G/H_1 \times G/H_2$. □

8.4 G-CW complexes

Recall the definition of a *CW* complex X. For each integer $n \geqslant 0$ there is a (possibly empty) discrete set K_n of n-cells and a collection of spaces (called **skeleta**)

$$K_0 = X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$$

with X being their union. For $n > 0$, X^n is obtained from X^{n-1} as a pushout (see §2.3A)

$$\begin{tikzcd} K_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d, "K_n \times i_n"'] & X^{n-1} \arrow[d] \\ K_n \times D^n \arrow[r] & X^n. \end{tikzcd} \tag{8.4.1}$$

where $i_n : S^{n-1} \to D^n$ is the inclusion of the boundary, and f_n is called the n**th attaching map.** The image of each of the disks D^n is called an n**-cell** in X. A map $f : X \to Y$ of CW complexes is **cellular** if it sends X^n to Y^n for each n.

Remark 8.4.2. The word "cellular." *The above use of the term "cellular" is different from that of Definition 6.3.1.*

Associated with this structure is the **cellular chain complex** $C_*(X)$ in which the nth chain group $C_n(X)$ is the free abelian group generated by the set K_n. To define its boundary operator, note that

$$X^n/X^{n-1} \cong \bigvee_{K_n} S^n,$$

so $H_n(X^n/X^{n-1}) = C_n(X)$. From the cofiber sequence

$$X^{n-1}/X^{n-2} \to X^n/X^{n-2} \to X^n/X^{n-1}$$

we get a short exact sequence of chain complexes of the form

$$0 \to C_{n-1}(X) \to C_*(X^n/X^{n-2}) \to C_n(X) \to 0$$

in which the end terms are chain complexes with a single nontrivial chain group, and X^n/X^{n-2} is a CW complex with cells only in dimensions $n-1$ and n. The resulting connecting homomorphism $C_n(X) \to C_{n-1}(X)$, which is induced by the map $X^n/X^{n-1} \to \Sigma X^{n-1}/X^{n-2}$, is the boundary operator in $C_*(X)$.

As explained in Example 4.8.20, a CW complex is an $\mathcal{I}$-cell complex in $\mathcal{T}$, for

$$\mathcal{I} = \{i_{n+} : n \geqslant 0\} \quad \text{where } i_{n+} \text{ is the map } S_+^{n-1} = \partial D_+^n \to D_+^n,$$

in which cells are attached in order according to their diemnsions. In the equivariant setting we need to replace $\mathcal{I}$ by

$$\mathcal{I}_G = \left\{ (G \underset{H}{\times} i_n)_+ : n \geqslant 0,\ H \subseteq G \right\}. \tag{8.4.3}$$

Definition 8.4.4. *A* G**-CW complex** *[Bre67] is a CW complex as above in which each set K_n and each space X_{n-1} has a G-action and each attaching map f_n is equivariant. The action of G on S^{n-1} and D^n in (8.4.10) is trivial. The diagram of (8.4.10) is a pushout in $\mathcal{T}op^G$, and X_n gets a G-action from those on the other three spaces.*

Equivalently it is an $\mathcal{I}_G$-cell complex as in Definition 4.8.18 in which cells are attached in dimensional order.

Thus a G-CW complex is an ordinary CW complex equipped with a cellular G-action of a particular form, one that can be described in terms of permuting cells in each dimension. Since each of the G-sets K_n is a disjoint union of sets of the form G/H for some subgroup H (defined up to conjugacy), we refer to the images of each $G/H \times D^n$ as an n**-dimensional** G**-cell**.

Definition 8.4.5. Types of G-cells. *We says that a G-cell of the form $G/H \times D^n$ in a G-CW complex is* **moving** *if $H \subseteq G$ is a proper subgroup,* **stationary** *if $H = G$,* **free** *if H is trivial, and* **bound** *if H is nontrivial.*

Example 8.4.6. A CW complex with cellular G-action that is not a G-CW complex. *Let V be a nontrivial finite dimensional representation of G and consider the space S^V, the one point compactification of V. The underlying space $S^{|V|}$ can be described as an ordinary CW complex with a single 0-cell and a single $|V|$-cell with constant attaching map. The 0-cell is fixed by the G-action, but the action on the $|V|$-cell, the image of the unit disk of V, is nontrivial. The self map of S^V induced by each element of the group is cellular since the 0-skeleton is fixed. However this action is not determined by the (necessarily trivial) action on the singleton sets K_0 and $K_{|V|}$, so this CW complex with G-action is not a G-CW complex.*

If X is an ordinary CW complex with cellular G-action as in the previous example, there is always a way to convert it to a G-CW complex by altering the cellular structure. If it is a simplicial complex with a simplicial G-action, barycentric subdivision will do the job, as is illustrated in Example 8.5.4 below.

The following is an exercise for the reader.

Proposition 8.4.7. Representation spheres for cyclic p-groups. *Let $G = C_{p^\ell}$ for a prime p and positive integer ℓ, and let $G^i \subseteq G$ denote the subgroup of index p^i. For brevity we will denote G/G^i (which is cyclic of order p^i) by G_i. The action of G on a G-space X induces an action of G_i on the fixed point set X^{G^i}.*

(i) Let V be a nontrivial representation of G, and let S^V denote its one point compactification. Then we have fixed point sets

$$S^{V^G} \subseteq S^{V^{G'}} \subseteq S^{V^{G''}} \subseteq \cdots \subseteq S^V.$$

(Note here that for $H \subseteq G$, $(S^V)^H = S^{(V^H)}$.) Since the action of G on S^{V^G} is trivial, we can form it by attaching a single $|V^G|$-cell to a point. In particular, if $V^G = 0$, $S^{V^G} = S^0$ is obtained by attaching a single 0-cell to a point. We can obtain $S^{V^{G^i}}$ from $S^{V^{G^{(i-1)}}}$ by attaching a single G-cell of the form $G_i \times D^m$ for each m with $|V^{G^{(i-1)}}| < m \leqslant |V^{G^i}|$.

(ii) Recall the regular and reduced regular representations of a finite group H of Definition 8.3.27. Then for each $n > 0$, we have

$$\left(S^{n\rho_G}\right)^{G^i} \cong S^{n\rho_{G_i}} \qquad \text{and} \qquad \left(S^{n\overline{\rho}_G}\right)^{G^i} \cong S^{n\overline{\rho}_{G_i}}.$$

Hence $K := S^{n\overline{\rho}_G}$ has a G-CW structure with a single G-cell in each dimension up to $n(p^\ell - 1)$ as described in (i). More explicitly, K has skeleta K^i and a diagram of cofiber sequences

$$\begin{tikzcd} S^0 = K^0 \arrow[r] & K^1 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & K^3 \arrow[r] \arrow[d] & \cdots \arrow[r] & K^{n(p^\ell-1)} = S^{n\overline{\rho}_G} \arrow[d] \\ & L_1 & L_2 & L_3 & & L_{n(p^\ell-1)}, \end{tikzcd} \tag{8.4.8}$$

where for $1 \leqslant i \leqslant \ell$,

$$L_j = G \underset{G^i}{\ltimes} S^j \qquad \text{for} \quad n(p^{i-1} - 1) < j \leqslant n(p^i - 1).$$

The action of G_i on S^j is trivial, and L_j is a wedge of p^i copies of S^j that are cyclically permuted by the action of G. It follows that $\mathcal{T}^G(L_j, X) \cong \Omega^j X^{G^i}$ for a pointed G-space X. The action of G on $\Omega^j X$ induces an action of G_i on this fixed point set, **but we are ignoring it** *and treating $\Omega^j X^{G^i}$ as an ordinary pointed space.*

We will use (8.4.17) below to prove Theorem 8.9.34 for $G = C_{p^\ell}$.

A different cell structure on $S^{\overline{\rho}_G}$ will be given below in Example 8.5.17.

The following is due to Bredon [Bre67]. It generalizes Burnside's statement (Theorem 8.1.4) about a finite G-set's being determined by its marks, i.e., by the cardinalities of its fixed point sets.

Theorem 8.4.9. Equivariant homotopy equivalences of G-CW complexes. *An equivariant map of G-CW complexes $f : X \to Y$ is an equivariant homotopy equivalence (meaning a homotopy equivalence for which the homotopies are equivariant) iff the induced maps $X^H \to Y^H$ of fixed point sets are ordinary homotopy equivalences for all subgroups $H \subseteq G$.*

Thus Theorem 8.4.18 says an equivariant map of G-CW complexes is an equivalence iff it induces an isomorphism in π_*^H for all subgroups $H \subseteq G$. Recall that in the Quillen model structure for the category of pointed topological spaces $\mathcal{T}$, described in §4.2A, a map is defined to be a weak equivalence if it induces and isomorphism in homotopy groups. We will see in Theorem 8.6.2 below that there is a model structure on $\mathcal{T}^G$, the category of pointed G-spaces, in which a weak equivalence is defined to be a map which induces an isomorphism in the equivariant homotopy groups π_*^H for Definition 8.3.12 for all subgroups $H \subseteq G$.

Recall the definition of a *CW* complex X. For each integer $n \geqslant 0$ there is a (possibly empty) discrete set K_n of n-cells and a collection of spaces (called **skeleta**)

$$K_0 = X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$$

with X being their union. For $n > 0$, X^n is obtained from X^{n-1} as a pushout (see §2.3A)

$$\begin{tikzcd} K_n \times S^{n-1} \arrow[r, "f_n"] \arrow[d, "K_n \times i_n"'] & X^{n-1} \arrow[d] \\ K_n \times D^n \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & X^n. \end{tikzcd} \tag{8.4.10}$$

where $i_n : S^{n-1} \to D^n$ is the inclusion of the boundary, and f_n is called the **nth attaching map.** The image of each of the disks D^n is called an **n-cell** in X. A map $f : X \to Y$ of CW complexes is **cellular** if it sends X^n to Y^n for each n.

Remark 8.4.11. The word "cellular." *The above use of the term "cellular" is different from that of Definition 6.3.1.*

Associated with this structure is the **cellular chain complex** $C_*(X)$ in which the nth chain group $C_n(X)$ is the free abelian group generated by the set K_n. To define its boundary operator, note that

$$X^n/X^{n-1} \cong \bigvee_{K_n} S^n,$$

so $H_n(X^n/X^{n-1}) = C_n(X)$. From the cofiber sequence

$$X^{n-1}/X^{n-2} \to X^n/X^{n-2} \to X^n/X^{n-1}$$

we get a short exact sequence of chain complexes of the form

$$0 \to C_{n-1}(X) \to C_*(X^n/X^{n-2}) \to C_n(X) \to 0$$

in which the end terms are chain complexes with a single nontrivial chain group, and X^n/X^{n-2} is a CW complex with cells only in dimensions $n-1$ and n. The resulting connecting homomorphism $C_n(X) \to C_{n-1}(X)$, which is induced by the map $X^n/X^{n-1} \to \Sigma X^{n-1}/X^{n-2}$, is the boundary operator in $C_*(X)$.

As explained in Example 4.8.20, a CW complex is an $\mathcal{I}$-cell complex in $\mathcal{T}$, for

$$\mathcal{I} = \{i_{n+} : n \geqslant 0\} \quad \text{where } i_{n+} \text{ is the map } S^{n-1}_+ = \partial D^n_+ \to D^n_+,$$

in which cells are attached in order according to their diemnsions. In the equivariant setting we need to replace $\mathcal{I}$ by

$$\mathcal{I}_G = \left\{ (G \underset{H}{\times} i_n)_+ : n \geqslant 0,\ H \subseteq G \right\}. \tag{8.4.12}$$

Definition 8.4.13. *A G-***CW complex** *[Bre67] is a CW complex as above in which each set K_n and each space X_{n-1} has a G-action and each attaching map f_n is equivariant. The action of G on S^{n-1} and D^n in (8.4.10) is trivial. The diagram of (8.4.10) is a pushout in $\mathcal{T}op^G$, and X_n gets a G-action from those on the other three spaces.*

Equivalently it is an $\mathcal{I}_G$-cell complex as in Definition 4.8.18 in which cells are attached in dimensional order.

Thus a G-CW complex is an ordinary CW complex equipped with a cellular G-action of a particular form, one that can be described in terms of permuting cells in each dimension. Since each of the G-sets K_n is a disjoint union of sets of the form G/H for some subgroup H (defined up to conjugacy), we refer to the images of each $G/H \times D^n$ as an **n-dimensional G-cell**.

Definition 8.4.14. Types of G-cells. *We says that a G-cell of the form $G/H \times D^n$ in a G-CW complex is* **moving** *if $H \subseteq G$ is a proper subgroup,* **stationary** *if $H = G$,* **free** *if H is trivial, and* **bound** *if H is nontrivial.*

Example 8.4.15. A CW complex with cellular G-action that is not a G-CW complex. *Let V be a nontrivial finite dimensional representation of G and consider the space S^V, the one point compactification of V. The* *underlying space $S^{|V|}$ can be described as an ordinary CW complex with a single 0-cell and a single $|V|$-cell with constant attaching map. The 0-cell is fixed by the G-action, but the action on the $|V|$-cell, the image of the unit disk of V, is nontrivial. The self map of S^V induced by each element of the group is cellular since the 0-skeleton is fixed. However this action is not determined by the (necessarily trivial) action on the singleton sets K_0 and $K_{|V|}$, so this CW complex with G-action is not a G-CW complex.*

If X is an ordinary CW complex with cellular G-action as in the previous example, there is always a way to convert it to a G-CW complex by altering the cellular structure. If it is a simplicial complex with a simplicial G-action, barycentric subdivision will do the job, as is illustrated in Example 8.5.4 below.

The following is an exercise for the reader.

Proposition 8.4.16. Representation spheres for cyclic p-groups. *Let $G = C_{p^\ell}$ for a prime p and positive integer ℓ, and let $G^i \subseteq G$ denote the subgroup of index p^i. For brevity we will denote G/G^i (which is cyclic of order p^i) by G_i. The action of G on a G-space X induces an action of G_i on the fixed point set X^{G^i}.*

(i) Let V be a nontrivial representation of G, and let S^V denote its one point compactification. Then we have fixed point sets

$$S^{V^G} \subseteq S^{V^{G'}} \subseteq S^{V^{G''}} \subseteq \cdots \subseteq S^V.$$

(Note here that for $H \subseteq G$, $(S^V)^H = S^{(V^H)}$.) Since the action of G on S^{V^G} is trivial, we can form it by attaching a single $|V^G|$-cell to a point. In particular, if $V^G = 0$, $S^{V^G} = S^0$ is obtained by attaching a single 0-cell to a point. We can obtain $S^{V^{G^i}}$ from $S^{V^{G^{(i-1)}}}$ by attaching a single G-cell of the form $G_i \times D^m$ for each m with $|V^{G^{(i-1)}}| < m \leqslant |V^{G^i}|$.

(ii) Recall the regular and reduced regular representations of a finite group H of Definition 8.3.27. Then for each $n > 0$, we have

$$(S^{n\rho_G})^{G^i} \cong S^{n\rho_{G_i}} \qquad \text{and} \qquad \left(S^{n\overline{\rho}_G}\right)^{G^i} \cong S^{n\overline{\rho}_{G_i}}.$$

Hence $K := S^{n\overline{\rho}_G}$ has a G-CW structure with a single G-cell in each dimension up to $n(p^\ell - 1)$ as described in (i). More explicitly, K has skeleta K^i and a diagram of cofiber sequences

$$\begin{tikzcd} S^0 = K^0 \arrow[r] & K^1 \arrow[r] \arrow[d] & K^2 \arrow[r] \arrow[d] & K^3 \arrow[r] \arrow[d] & \cdots \arrow[r] & K^{n(p^\ell-1)} = S^{n\overline{\rho}_G} \arrow[d] \\ & L_1 & L_2 & L_3 & & L_{n(p^\ell-1)}, \end{tikzcd} \tag{8.4.17}$$

where for $1 \leqslant i \leqslant \ell$,

$$L_j = G \underset{G^i}{\ltimes} S^j \qquad \text{for} \quad n(p^{i-1} - 1) < j \leqslant n(p^i - 1).$$

The action of G_i on S^j is trivial, and L_j is a wedge of p^i copies of S^j that are cyclically permuted by the action of G. It follows that $\mathcal{T}^G(L_j, X) \cong \Omega^j X^{G^i}$ for a pointed G-space X. The action of G on $\Omega^j X$ induces an action of G_i on this fixed point set, **but we are ignoring it** *and treating $\Omega^j X^{G^i}$ as an ordinary pointed space.*

We will use (8.4.17) below to prove Theorem 8.9.34 for $G = C_{p^\ell}$.

A different cell structure on $S^{\overline{\rho}_G}$ will be given below in Example 8.5.17.

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Theorem 8.4.18. Equivariant homotopy equivalences of G-CW complexes. *An equivariant map of G-CW complexes $f : X \to Y$ is an equivariant homotopy equivalence (meaning a homotopy equivalence for which the homotopies are equivariant) iff the induced maps $X^H \to Y^H$ of fixed point sets are ordinary homotopy equivalences for all subgroups $H \subseteq G$.*

Thus Theorem 8.4.18 says an equivariant map of G-CW complexes is an equivalence iff it induces an isomorphism in π_*^H for all subgroups $H \subseteq G$. Recall that in the Quillen model structure for the category of pointed topological spaces $\mathcal{T}$, described in §4.2A, a map is defined to be a weak equivalence if it induces and isomorphism in homotopy groups. We will see in Theorem 8.6.2 below that there is a model structure on $\mathcal{T}^G$, the category of pointed G-spaces, in which a weak equivalence is defined to be a map which induces an isomorphism in the equivariant homotopy groups π_*^H for Definition 8.3.12 for all subgroups $H \subseteq G$.

8.5 The homology of a G-CW complex

For each G-CW complex X we have a cellular chain complex C_*X defined as before, but now it is a chain complex of $Z[G]$-modules. This means its homology H_*X is also a $Z[G]$-module with the G-action induced by the one X.

Definition 8.5.1. Mackey functor homology and cohomology of G-CW complexes. *Let X be a G-CW complex (Definition 8.4.13) with cellular chain complex C_*X. Since the later is a chain complex of $Z[G]$-modules we can apply the fixed point functor FP of Definition 8.2.8 and get a chain complex of Mackey functors which we denote by $\underline{C}_*X$. We denote its homology by $\underline{H}_*X$.*

For cohomology, we consider the cochain complex $C^*(X) = \mathrm{Hom}(C_*X, \mathbf{Z})$. It is a cochain complex of $Z[G]$-modules, and again we can apply the fixed point functor FP of Definition 8.2.8 and get a cochain complex of Mackey functors which we denote by $\underline{C}^*X$. Equivalently, $\underline{C}^*X$ is $\mathbf{Z}$-linear dual of

$\widehat{C_* X}$, *the value of the fixed quotient functor* FQ *of Definition 8.2.8 on* $C_* X$. *We denote its homology by* $\underline{H}^* X$.

The nonexactness of FP implies that the graded Mackey functor $\underline{H}_* X$ is **not** the one obtained by application of FP to the graded $Z[G]$-module $H_* X$, nor is $\underline{H}_* X(G/H)$ the same as $H_*(X^H)$. However it is true that $\underline{H}_* X(G/e) = H_* X$, the underlying homology of X, as a $Z[G]$-module. The same goes for cohomology with $\underline{H}^* X(G/e) = H^* X$.

Remark 8.5.2. Mackey functor homology and the homology of the fixed point set. *To see why* $\underline{H}_* X(G/H)$ *differs from* $H_*(X^H)$, *consider the following. Let* K_n *denote the* G*-set of* n*-cells for* X. *Then* $C_n X$ *is the free abelian group on* K_n, *which is a* $\mathbf{Z}[G]$*-module. The fixed point set* X^H *is a CW complex for which the set of* n*-cells is* K_n^H. *It follows that* $C_n(X^H)$ *is the free abelian group on* K_n^H. *This is contained in but is generally* **not equal to** $(C_n X)^H$. *For example if* $K_n = G/e$, *the fixed point set* K_n^G *is empty, so* $C_n(X^G) = 0$. *On the other hand* $(C_n X)^G = (\mathbf{Z}[G])^G$ *is nontrivial since it contains the sum of all elements in* G.

Proposition 8.5.3. The cohomology of the orbit space. *For a* G*-CW complex* X, $\underline{H}^* X(G/G) = H^* X_G$, *the ordinary cohomology of the orbit space* X_G.

Proof It follows from the definitions that

$$\underline{H}^* X(G/G) = H^*(\mathrm{Hom}(C_*(X)_G, Z)),$$

and $C_*(X)_G$ is the cellular chain complex for X_G. □

The following will figure in Proposition 13.2.1 below, a step toward proving the Gap Theorem of §1.1C(iii).

Example 8.5.4. The group $\pi^G_{\rho_G-2} H\underline{\mathbf{Z}}$. *The regular representation* ρ_G *of a nontrivial finite group* G *can be written as* $1 + \overline{\rho}_G$, *where* $\overline{\rho}_G$ *is the reduced regular representation as in Definition 8.3.27. The group*

$$\pi^G_{\rho_G-2} H\underline{\mathbf{Z}} \approx H^1_G(S^{\overline{\rho}_G}; \underline{\mathbf{Z}}),$$

is isomorphic to

$$H^1(S^{\overline{\rho}_G}/G; \mathbf{Z}).$$

by Proposition 8.5.3. The G*-space* $S^{\overline{\rho}_G}$ *is the unreduced suspension of the unit sphere* $S(\overline{\rho}_G)$, *and so the orbit space is also a suspension. If* $|G| > 2$ *then* $S(\overline{\rho}_G)$ *is connected, hence so is the orbit space. If* $G = C_2$, *then* $S(\overline{\rho}_G) \approx G$ *and the orbit space is again connected. In all cases then, the unreduced suspension* $S^{\overline{\rho}_G}/G$ *is simply connected. Thus*

$$\pi^G_{\rho_G-2} H\underline{\mathbf{Z}} \approx H^2_G(S^{\rho_G}; \underline{\mathbf{Z}}) \approx H^1_G(S^{\overline{\rho}_G}; \underline{\mathbf{Z}}) = 0.$$

In fact, the same argument shows that for $m > 0$ the orbit space $S^{m\overline{\rho}_G}/G$ is simply connected, and hence

$$H^0_G(S^{m\overline{\rho}_G}; \underline{\mathbf{Z}}) = H^1_G(S^{m\overline{\rho}_G}; \underline{\mathbf{Z}}) = 0$$

or, equivalently

$$\pi^G_{m\overline{\rho}_G} H\underline{\mathbf{Z}} = \pi^G_{m\overline{\rho}_G - 1} H\underline{\mathbf{Z}} = 0.$$

Example 8.5.5. The case $S^{n\rho}$ for $G = C_2$. *Let σ denote the sign representation for G. Then the regular representation is $\rho = 1 + \sigma$. Let $X = S^{n\rho}$ for $n > 0$. It has a reduced cellular chain complex C with*

$$C_i^{n\rho} = \begin{cases} \mathbf{Z}[G]/(\gamma - 1) & \text{for } i = n \\ \mathbf{Z}[G] & \text{for } n < i \leqslant 2n \\ 0 & \text{otherwise.} \end{cases} \tag{8.5.6}$$

Let $c_i^{(n)}$ denote a generator of $C_i^{n\rho}$. The boundary operator d is given by

$$d(c_{i+1}^{(n)}) = \begin{cases} c_i^{(n)} & \text{for } i = n \\ \gamma_{i+1-n}(c_i^{(n)}) & \text{for } n < i \leqslant 2n \\ 0 & \text{otherwise} \end{cases} \tag{8.5.7}$$

where $\gamma_i = 1 - (-1)^i\gamma$. It is determined by the fact that the homology of the underlying chain complex must be that of the underlying space S^{2n}

Applying the fixed point Mackey functor of Definition 8.2.8 gives a chain complex of Mackey functors (for which we use the symbols of Table 8.1) of the form

$$\begin{tikzcd}
n & n+1 & n+2 & n+3 & & 2n \\
\square & \widehat{\square} \arrow[l, "\nabla"'] & \widehat{\square} \arrow[l, "\gamma_2"'] & \widehat{\square} \arrow[l, "\gamma_3"'] & \cdots \arrow[l] & \widehat{\square} \arrow[l, "\gamma_n"'] \\
\mathbf{Z} \arrow[d, bend right, "1"'] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[l] & \mathbf{Z} \arrow[l, "\epsilon_n"'] \arrow[d, bend right, "\Delta"'] \\
\mathbf{Z} \arrow[u, bend right, "2"'] & \mathbf{Z}[G] \arrow[l, "\nabla"'] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[G] \arrow[l, "\gamma_2"'] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[G] \arrow[l, "\gamma_3"'] \arrow[u, bend right, "\nabla"'] & \cdots \arrow[l] & \mathbf{Z}[G] \arrow[l, "\gamma_n"'] \arrow[u, bend right, "\nabla"']
\end{tikzcd} \tag{8.5.8}$$

where

$$\epsilon_n = \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Passing to homology we get

$$\begin{array}{cccccc} n & n+1 & n+2 & n+3 & & 2n \\ \bullet & 0 & \bullet & 0 & \cdots & \underline{H}_{2n} \end{array}$$

$$\begin{tikzcd} \mathbf{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbf{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & \cdots & \underline{H}_{2n}(G/G) \arrow[d, bend right, "(1+(-1)^n)/2"'] \\ 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & \underline{H}_{2n}(G/e) \arrow[u, bend right, "1+(-1)^n"'] \end{tikzcd}$$

where

$$\underline{H}_{2n}(G/G) = \begin{cases} \mathbf{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \quad \text{and} \quad \underline{H}_{2n}(G/e) = \begin{cases} \mathbf{Z} & \text{for } n \text{ even} \\ \mathbf{Z}_- & \text{for } n \text{ odd.} \end{cases}$$

For $n \geqslant 0$

$$\underline{H}_{n+i}(S^{n\rho}) = \begin{cases} \bullet & \text{for } 0 \leqslant i < n \text{ and } i \text{ even} \\ \square & \text{for } i = n \text{ and } n \text{ even} \\ \overline{\square} & \text{for } i = n \text{ and } n \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{8.5.9}$$

Here $\square$ and $\overline{\square}$ are fixed point Mackey functors but $\bullet$ is not.

Hence we see that $\underline{H}_* S^{n\rho}(G/G)$ is quite different from

$$H_*(S^{n\rho})^G = H_* S^n.$$

This Mackey functor homology commutes with ordinary suspension, so we can read off the value $\underline{H}_* S^{m+n\sigma}$ for any m. See Theorem 9.9.19 below for more discussion. We record the answer for $m = 0$ for future reference.

$$\underline{H}_i(S^{n\sigma}) = \begin{cases} \bullet & \text{for } 0 \leqslant i < n \text{ and } i \text{ even} \\ \square & \text{for } i = n \text{ and } n \text{ even} \\ \overline{\square} & \text{for } i = n \text{ and } n \text{ odd} \\ 0 & \text{otherwise.} \end{cases} \tag{8.5.10}$$

In Example 9.9.21 below we will generalize this to larger cyclic 2-groups.

For a more general discussion of S^V, where V is a representation of a finite cyclic 2-group, see §9.9.

Example 8.5.11. The classical Hopf map as a map of C_2-spaces. *The Hopf map $\eta : S^3 \to S^2$ is defined by sending each unit vector in $\mathbf{C}^2$ to the corresponding complex line through the origin. Both source and target come equipped with a C_2-action defined in terms of complex conjugation. With this in mind we can rewrite the map as*

$$\eta : S(2\rho) = S^{1+2\sigma} \to S^\rho = S^{1+\sigma}.$$

The induced map on fixed points is the double covering $S^1 \to \mathbf{R}P^1 = S^1$.

All of its iterates are essential, so all iterates of η are equivariantly essential. Thus, even though the composite

$$S^6 \xrightarrow{\Sigma^3\eta} S^5 \xrightarrow{\Sigma^2\eta} S^4 \xrightarrow{\Sigma\eta} S^3 \xrightarrow{\eta} S^2$$

is known to be null, the equivariant composite

$$S^{1+5\sigma} \xrightarrow{\Sigma^{3\sigma}\eta} S^{1+4\sigma} \xrightarrow{\Sigma^{2\sigma}\eta} S^{1+3\sigma} \xrightarrow{\Sigma^{\sigma}\eta} S^{1+2\sigma} \xrightarrow{\eta} S^{1+\sigma}$$

is equivariantly essential because the induced map of fixed points is the degree 16 map on S^1.

Example 8.5.12. The complex projective plane as a C_2-space. *Now consider the cofiber sequence*

$$S^{1+2\sigma} \xrightarrow{\eta} S^{1+\sigma} \xrightarrow{i} \mathbf{C}P^2 \xrightarrow{j} S^{2+2\sigma},$$

where the complex projective plane $\mathbf{C}P^2$ has a C_2-action via complex conjugation, and the Hopf map η of Example 8.5.11 has degree 2 on the bottom cell. It leads to a short exact sequence of reduced cellular chain complexes

$$0 \to C_*(S^{1+\sigma}) \to C_*(\mathbf{C}P^2) \to C_*(S^{2+2\sigma}) \to 0.$$

From this we find that the cellular chain complex for $\mathbf{C}P^2$ is

$$\begin{tikzcd}
1 & 2 & 3 & 4 \\
\mathbf{Z} & \mathbf{Z}[G] \oplus \mathbf{Z} \arrow[l, "{[-\nabla \quad 2]}"'] & \mathbf{Z}[G] \arrow[l, "{\begin{bmatrix} 1+\gamma \\ \nabla \end{bmatrix}}"'] & \mathbf{Z}[G] \arrow[l, "1-\gamma"']
\end{tikzcd} \tag{8.5.13}$$

Applying the fixed point Mackey functor of Definition 8.2.8 gives a chain complex of Mackey functors of the form

$$\begin{tikzcd}
1 & 2 & 3 & 4 \\
\square & \widehat{\square} \oplus \square \arrow[l, "{[-\nabla \quad 2]}"'] & \widehat{\square} \arrow[l, "{\begin{bmatrix} 1+\gamma \\ \nabla \end{bmatrix}}"'] & \widehat{\square} \arrow[l, "1-\gamma"'] \\
\mathbf{Z} \arrow[d, bend right, "1"'] & \mathbf{Z} \oplus \mathbf{Z} \arrow[l, "{[-2 \quad 2]}"'] \arrow[d, bend right, "\Delta\oplus 1"'] & \mathbf{Z} \arrow[l, "{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] \\
\mathbf{Z} \arrow[u, bend right, "2"'] & \mathbf{Z}[G] \oplus \mathbf{Z} \arrow[l, "{[-\nabla \quad 2]}"'] \arrow[u, bend right, "\nabla\oplus 2"'] & \mathbf{Z}[G] \arrow[l, "{\begin{bmatrix} 1+\gamma \\ \nabla \end{bmatrix}}"'] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[G]. \arrow[l, "1-\gamma"'] \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

Passing to homology we get

$$\begin{tikzcd}
1 & 2 & 3 & 4 \\
\bullet & \overline{\Box} & 0 & \Box \\
\mathbf{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbf{Z} \arrow[d, bend right, "1"'] \\
0 \arrow[u, bend right] & \mathbf{Z}_- \arrow[u, bend right] & 0 \arrow[u, bend right] & \mathbf{Z}. \arrow[u, bend right, "2"']
\end{tikzcd}$$

We can define homology and cohomology with coefficients in a Mackey functor $\underline{M}$ as follows.

Definition 8.5.14. Bredon homology and cohomology. *Let X be a G-CW complex with*

$$X^n/X^{n-1} \cong S^n \wedge K_{n+}$$

where K_n is a possibly infinite G-set, which we write as

$$K_n = \coprod_\alpha G/H_\alpha.$$

Suppose X has finite type, meaning that each K_n is finite. For a Mackey functor $\underline{M}$ we define the **Bredon chain complex** *$C_*(X;\underline{M})$ and the* **Bredon cochain complex** *$C^*(X;\underline{M})$ by*

$$\begin{aligned} C_n(X;\underline{M}) &= \underline{M}_{K_n} \\ \text{and} \qquad C^n(X;\underline{M}) &= \underline{M}_{K_n}, \end{aligned}$$

where $\underline{M}_{K_n}$ is the precomposite Mackey functor of Definition 8.2.9. The map

$$X^n/X^{n-1} \to \Sigma X^{n-1}/X^{n-2}$$

defines boundary and coboundary maps

$$\begin{aligned} C_n(X;\underline{M}) &\to C_{n-1}(X;\underline{M}) \\ \text{and} \qquad C^{n-1}(X;\underline{M}) &\to C^n(X;\underline{M}). \end{aligned}$$

The equivariant homology and cohomology groups of X with coefficients in $\underline{M}$ are the homology and cohomology groups of these complexes of Mackey functors.

Without the finiteness assumption we define

$$\begin{aligned} C_n(X;\underline{M}) &= \bigoplus_\alpha \underline{M}_{G/H_\alpha} \\ \text{and} \qquad C^n(X;\underline{M}) &= \prod_\alpha \underline{M}_{G/H_\alpha} \end{aligned}$$

with similar boundary and coboundary maps.

Remark 8.5.15. The nonfinite case. *For infinite K_n, the functors $C_n(X;\underline{M})$ and $C^n(X;\underline{M})$ defined above are Mackey functors, even though the expression $\underline{M}_{K_n}$ as in Definition 8.2.9 does not make sense since $K_n \times T$ for finite T is not a finite G-set.*

Remark 8.5.16. Defining Bredon homology and cohomology in terms of Eilenberg-Mac Lane spaces. *For each Mackey functor $\underline{M}$ there is an Eilenberg-Mac Lane space $K(\underline{M}, n)$ that will be given below in Theorem 8.8.4; the Eilenberg-Mac Lane spectrum $H\underline{M}$ will be the subject of Theorem 9.1.47 below. The definitions of $C_n(X;\underline{M})$ and $C^n(X;\underline{M})$ above are equivalent to*

$$\begin{aligned} C_n(X;\underline{M}) &= \underline{\pi}_n K(\underline{M}, n) \wedge K_{n+} \\ \text{and} \qquad C^n(X;\underline{M}) &= \underline{\pi}_n K(\underline{M}, n)^{K_n}. \end{aligned}$$

When $\underline{M} = \underline{\mathbf{Z}}$, the Bredon homology and cohomology groups are those of Definition 8.5.1.

Example 8.5.17. The reduced regular representation sphere. *Let $\rho = \rho_G$ be the (real) regular representation of a finite group G, so $\rho_G - 1$ is the reduced regular representation $\overline{\rho}_G$.*

The groups

$$H^*(S^{\overline{\rho}_G};\underline{M})$$

play an important role in equivariant stable homotopy theory. To describe them we need an equivariant cell decomposition of $S^{\overline{\rho}_G}$. Since $S^{\overline{\rho}_G}$ is the mapping cone of the map

$$S(\overline{\rho}_G) \to *$$

from the unit sphere in $(\overline{\rho}_G)$, it suffices to construct an equivariant cell decomposition of $S(\overline{\rho}_G)$. Let $g = |G|$ and think of $\mathbf{R}^g$ as the vector space whose basis is the set G. The boundary of the standard $(g-1)$-simplex in this space (see Definition 3.4.2) is equivariantly homeomorphic to $S(\overline{\rho}_G)$.

The simplicial decomposition of this simplex is not an equivariant cell decomposition, because the group action rotates certain simplices rather than permuting them. A rotation of a simplex induces a permutation of the simplices in its barycentric subdivision (Definition 3.4.21). This means the barycentric subdivision of the simplex is an equivariant cell decomposition.

It follows that $S(\overline{\rho}_G)$ is homeomorphic to the geometric realization of the nerve of the poset of non-empty proper subsets of G. This leads to the complex

$$\underline{M}(G/G) \to \underline{M}(S_0) \to \underline{M}(S_1) \to \cdots \to \underline{M}(S_{g-1}) \tag{8.5.18}$$

in which S_k is the G-set of flags $F_0 \subset \cdots \subset F_k \subset G$ of proper inclusions of proper subsets of G, with G acting by translation. The coboundary map is the alternating sum of the restriction maps derived by omitting one of the sets in a flag.

A different cell structure for the group $G = C_{p^\ell}$ was given in (8.4.17).

Corollary 8.5.19. H^0 of the reduced regular representation sphere. *For any Mackey functor $\underline{M}$, the group*

$$\pi^G_{\overline{\rho}_G} H\underline{M} = H^0_G(S^{\overline{\rho}_G}; \underline{M})$$

is given by

$$\bigcap_{H \subsetneq G} \ker\big(\underline{M}(G/G) \to \underline{M}(G/H)\big).$$

Proof Using the complex (8.5.18) it suffices to show that the orbit types occurring in S_0 are precisely the transitive G-sets of the form G/H with H a proper subgroup of G. The set S_0 is the set of non-empty proper subsets $S \subset G$. Any proper subgroup H of G occurs as the stabilizer of itself, regarded as a subset of G. Since the subsets are proper, the group G does not occur as a stabilizer. □

8.6 Model structures

The following is discussed in [MM02, III.1 and IV.6] and [BDS16, Chapter 1]. We will first describe two different cofibrantly generated model structures on $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps for a finite group G. Later we will see that there is one for each family $\mathscr{F}$ of subgroups of G closed under inclusion and conjugation.

Definition 8.6.1. The underlying and Bredon model structures on $\mathcal{T}^G$. *(In [BDS16] these are called the naive and genuine model structures.) An equivariant map $f : X \to Y$ of pointed G-spaces is an* **underlying fibration (underlying weak equivalence)** *if the same is true of f as a morphism in $\mathcal{T}$. It is a* **Bredon fibration (Bredon weak equivalence)** *if $f^H : X^H \to Y^H$ is a Serre fibration (weak equivalence) for each subgroup $H \subseteq G$. It is an* **underlying or Bredon cofibration** *if it has the appropriate left lifting property.*

The projective model structure of Theorem 5.6.26 is the underlying one above.

Bredon's Theorem 8.4.18 then says that a map $f : X \to Y$ of G-CW complexes is an equivariant homotopy equivalence iff it is a Bredon weak equivalence.

The following will be proved later in this section.

Theorem 8.6.2. Two model structures on $\mathcal{T}^G$. *The two sets of fibrations and weak equivalences of Definition 8.6.1 each define a compactly generated* *model category structure on $\mathcal{T}^G$. In the underlying case the sets of generating (trivial) cofibrations are*

$$\begin{aligned} & \mathcal{I}_G^e = \{(G \times i_n)_+ : n \geqslant 0\} \\ \text{and} \qquad & \mathcal{J}_G^e = \{(G \times j_n)_+ : n \geqslant 0\}, \end{aligned}$$

for i_n and j_n as in (5.2.10) and (5.2.11), while in the Bredon case they are

$$\begin{aligned} & \mathcal{I}_G^{\mathcal{A}\ell\ell} = \left\{ (G \underset{H}{\times} i_n)_+ : n \geqslant 0,\, H \subseteq G \right\} \\ \text{and} \qquad & \mathcal{J}_G^{\mathcal{A}\ell\ell} = \left\{ (G \underset{H}{\times} j_n)_+ : n \geqslant 0,\, H \subseteq G \right\}, \end{aligned}$$

where G (and hence H) acts trivially on S^{n-1}, D^n, I^n and I^n.
Similar statements hold for $\mathcal{T}op^G$.

The Bredon model structure has fewer weak equivalences and fibrations than the underlying one since the requirements to be such maps are more demanding. It follows that it has more .

Example 8.6.3. A Bredon cofibration that is not un underlying cofibration. *Let $G = C_2$, let σ denote the sign representation, and let $i : S(\sigma) \to D(\sigma)$ (see Definition 8.3.26) be the evident inclusion. It is easily seen to be a Bredon cofibration in $\mathcal{T}op^G$. We will see that* **it is not an underlying cofibration in $\mathcal{T}op^G$ even though the underying map is a cofibration in $\mathcal{T}op$.** *Let EG be a contractible free G-space, and let $p : EG \to *$ be the unique map to a point. It is an underlying trivial fibration since EG is contractible. However the map of fixed points is $\varnothing \to *$, so p is not a Bredon weak equivalence.*

We can construct an equivariant map $\alpha : S(\sigma) \to EG$ by choosing a point in the target as the image of one of the two points in the domain, and using the group action to determine the image of the other point. Thus we get a lifting diagram

$$\begin{tikzcd} S(\sigma) \arrow[r, "\alpha"] \arrow[d, "i"'] & EG \arrow[d, "p"] \\ D(\sigma) \arrow[r, "\beta"'] \arrow[ur, dashed] & * \end{tikzcd}$$

An equivariant lift does not exist because $D(\sigma)$ has a fixed point, but EG does not.

There is a similar example in $\mathcal{T}^G$ obtained from the above by adding a disjoint base point to every space in sight.

The reason for the notation of Theorem 8.6.2 will become apparent in Theorem 8.6.13 below. The Bredon model structure above is established in [MM02, Theorem III.1.8], and we will outline the proof later in this section. The underlying one can be established with the Crans-Kan Transfer Theorem 5.2.27 as follows. Let $F : \mathcal{T} \to \mathcal{T}^G$ be the functor given by $X \mapsto (X \times G)_+$. It is the left adjoint of the forgetful functor $U : \mathcal{T}^G \to \mathcal{T}$. Then F sends the classical model structure on $\mathcal{T}$ to the underlying one on $\mathcal{T}^G$.

Remark 8.6.4. Equivariant model structures require equivariant maps. *There is no reasonable model structure on $\mathcal{T}_G$, the category of pointed G-spaces and* **nonequivariant** *maps. One reason for this is that if $f : X \to Y$ is such a map between pointed G-spaces, then its fiber and cofiber will not have G-actions. Even worse, the same goes for limits and colimits, so $\mathcal{T}_G$ is neither complete nor cocomplete. It does support a surjective (on objects and morphisms) forgetful functor to the bicomplete category $\mathcal{T}$.*

Lemma 8.6.5. Underlying equivalences of free G-spaces. *An equivariant map $f : X \to Y$ of free G-spaces that is an underlying equivalence is a Bredon equivalence.*

Proof An equivariant map f is a Bredon equivalence if it induces an ordinary weak equivalence on the fixed point sets for each subgroup of G. Since X and Y are free G-spaces, the fixed point set for each nontrivial subgroup is empty, so Bredon's conditions are met. □

Proposition 8.6.6. The pointed homotopy orbit space of a pointed free G-space. *Suppose X is a free pointed G-space as in Definition 8.3.2. Then the map $X_{hG} \to X_G$ (see Definition 8.3.8(ii)) is a weak equivalence.*

Proof Consider the equivariant map $p_2 : EG \ltimes X \to X$. For any nontrivial subgroup $H \subseteq G$, the map of H fixed points is the identity on the base point, and for the trivial subgroup it is a the underlying map p_2, which is a weak equivalence since the space underlying EG is contractible. This means that P_2 is an equivariant equivalence, so the map of orbit spaces is a weak equivalence. □

Definition 8.6.7. *An* **hG-equivalence** *is an equivariant map of G-spaces underlain by an ordinary weak equivalence.*

Theorem 8.6.8. An hG-equivalence induces a weak equivalence on homotopy fixed point spaces. *An equivariant map $f : X \to Y$ of G-spaces that is an underlying weak equivalence induces a weak equivalence $f^{hG} : X^{hG} \to Y^{hG}$.*

Proof Consider the diagram

$$\begin{tikzcd} X \times EG \arrow[r, "f \times EG"] \arrow[d, "p_1"'] & Y \times EG \arrow[d, "p_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The actions of G on the spaces in the top row are free since EG is free. Hence the underlying weak equivalence $f \times EG$ is a Bredon weak equivalence by Lemma 8.6.5. This means the map $f \times_G EG$ is a weak equivalence in the diagram

$$\begin{tikzcd}
X \times_G EG \arrow[rr, "f \times_G EG"] \arrow[dr, "p_X"'] & & Y \times_G EG \arrow[dl, "p_Y"] \\
& BG. &
\end{tikzcd}$$

It follows that the induced map from the space of sections of p_X to that of p_Y, meaning (by Proposition 8.3.11) from X^{hG} to Y^{hG}, is also a weak equivalence. $\square$

Example 8.6.9. The map $EG \to *$ *is not a Bredon equivalence because for each nontrivial subgroup* $H \subseteq G$ *the map of fixed points sends the empty set to a point. However it satisfies the hypothesis of Theorem 8.6.8 since both spaces are contractible. Hence* EG^{hG} *is contractible.*

Without Theorem 8.6.8 we can see that it is nonempty since it is by Definition 8.3.8(iv) the space $Map^G(EG, EG)$*, which contains the identity map.*

8.6A Families of subgroups of G

The two model structures of Theorem 8.6.2 can be generalized in the following way.

Definition 8.6.10. *A* **family** $\mathscr{F}$ *of subgroups of a finite group* G *is a collection that is closed under conjugation and inclusion. If a subgroup* H *is in* $\mathscr{F}$*, so are all of its subgroups and all of its conjugates.*

Please note the difference between the symbols $\mathscr{F}$ (`\mathscr{F}`), which we use for a family of subgroups, and $\mathcal{F}$ (`\mathcal{F}`) as in $\mathcal{F}_G$, our symbol for the category of finite G-sets as in Definition 8.1.1. The symbol $\mathscr{F}$ is also used in Chapter 4 to denote the class of fibrations in a model category. Hopefully these two uses of it will not lead to any confusion.

Example 8.6.11. Some families $\mathscr{F}$ of subgroups of a finite group G.

(i) *The trivial case* e*, consisting of just the trivial subgroup* e *of* G*.*
(ii) $\mathscr{F} = \mathscr{A\ell\ell}$*, the family of all subgroups of* G*.*
(iii) *The family of subgroups not containing any conjugate of a given subgroup* K*.*
(iv) *The family* $\mathcal{P}$ *of proper subgroups, the case above for* $K = G$*.*
(v) *The set of all abelian subgroups of* G*.*
(vi) *The set of all p-subgroups of* G *for a fixed prime* p*.*
(vii) *The family of subgroups having trivial intersection with any conjugate of a given subgroup* K*. For* $K = G$*, this is (i) above.*

Associated with each family $\mathscr{F}$ is a model structure on $\mathcal{T}^G$ defined as follows.

Definition 8.6.12. *An equivariant map $f : X \to Y$ of pointed G-spaces is an* **$\mathscr{F}$-fibration** **($\mathscr{F}$-weak equivalence)** *if same is true of $f^H : X^H \to Y^H$ for each subgroup $H \in \mathscr{F}$. It is an* **$\mathscr{F}$-cofibration** *if it has the appropriate left lifting property.*

Theorem 8.6.13. The $\mathscr{F}$-model structure on $\mathcal{T}^G$. *For each family of subgroups $\mathscr{F}$, the classess of fibrations and weak equivalences of Definition 8.6.12 each define a cofibrantly generated model structure on $\mathcal{T}^G$.*

The cofibrant generating sets are

$$\mathcal{I}_G^{\mathscr{F}} = \left\{(G \underset{H}{\times} i_n)_+ : n \geqslant 0,\ H \in \mathscr{F}\right\}$$
$$\textit{and} \qquad \mathcal{J}_G^{\mathscr{F}} = \left\{(G \underset{H}{\times} j_n)_+ : n \geqslant 0,\ H \in \mathscr{F}\right\}.$$

The two cases in Theorem 8.6.2 correspond to the two extreme values of $\mathscr{F}$. When $\mathscr{F}$ does not contain all subgroups of G, then the resulting model structure has more weak equivalences and fibrations, and hence fewer cofibrations than the Bredon structure. This method of modifying the latter is essentially the second of the three listed in Table 6.1, that is a form of induction as in Theorem 5.4.21. See the discussion following Proposition 8.6.29 below.

Proposition 8.6.14. All pointed G-spaces are $\mathscr{F}$-fibrant. *For any finite group G and any family of subgroups $\mathscr{F}$, all objects of $\mathcal{T}^G$ are fibrant in the $\mathscr{F}$-model structure.*

Proof A pointed G-space X is $\mathscr{F}$-fibrant if the map $X \to *$ is an $\mathscr{F}$-fibration, namely if $X^H \to *$ is a fibration in $\mathcal{T}$ for each H in $\mathscr{F}$. This is the case since every object in $\mathcal{T}$ is fibrant. □

Definition 8.6.15. *Let $\mathscr{F}$ be a family of subgroups of a finite group G. A G-CW complex $E\mathscr{F}$ (without base point) is a* **universal $\mathscr{F}$-space** *if its fixed point set $E\mathscr{F}^H$ is contractible for $H \in \mathscr{F}$ and empty otherwise.*

Such a space can be constructed by taking infinite joins of orbits of the form G/H for $H \in \mathscr{F}$. It is also characterized by the property that for each orbit G/K, the space of G-equivariant maps $G/K \to E\mathscr{F}$ is contractible if $K \in \mathscr{F}$ and empty otherwise. Details can be found in [Lüc05, §1].

Example 8.6.16. Some universal $\mathscr{F}$-spaces.

(i) A contractible free G-space EG is universal for the trivial family.
(ii) For $\mathscr{F} = \mathscr{A}\ell\ell$, the one point space is universal.
(iii) For a finite cyclic p-group G, $E(G/G')$, where G' is the subgroup of index p, is universal for $\mathcal{P}$, the family of proper subgroups. See Example 9.11.6 below for another description.

(iv) *Let G be a finite group of order g, and let $\overline{\rho} = \overline{\rho}_G$ be its reduced regular representation as in Example 8.5.17. Its unit sphere $S(\overline{\rho})$ has an empty G-fixed point set G-space (but is not a free G-space), as does*

$$S(k\overline{\rho}) = S(\overline{\rho}) * S(\overline{\rho}) * \cdots S(\overline{\rho}),$$

its k-fold join, which is underlain by S^{kg-1}.

In paricular, for $G = C_4$, the unit sphere $S(\overline{\rho})$ is underlain by S^2. A generator γ reflects through the equator and rotates about the vertical axis by $\pi/2$. This means that γ^2 rotates by π and fixes the poles.

In general, let $H \subset G$ be a proper subgroup of order h. We denote its regular and reduced regular representations by ρ' and $\overline{\rho}'$. The restriction of ρ to H is $(g/h)\rho'$, so that of $\overline{\rho} = \rho - 1$ is $(g/h)\overline{\rho}' + g/h - 1$. It follows that

$$i_H^G S(k\overline{\rho}) = S(\ell\overline{\rho}' + \ell - 1),$$

where $\ell = kg/h$, so

$$\left(i_H^G S(k\overline{\rho})\right)^H = S^{\ell-2}.$$

We denote the infinite join of $S(\overline{\rho})$ by $E\mathcal{P}$. Our justification for this notation is the fact that

$$E\mathcal{P}^H \simeq \begin{cases} \varnothing & \text{for } H = G \\ * & \text{otherwise} \end{cases}$$

It serves as a universal space for the family $\mathcal{P}$ of proper subgroups of G.

The space $E\mathcal{P}$ is needed in §9.11A below where geometric fixed points are defined. Some other examples will be discussed in Definition 8.7.1.

Remark 8.6.17. Homotopy fixed points and homotopy orbits. *For G-space X, the fixed point space X^G can be identified with the space of equivariant maps to X from $*$. We could instead consider the space of equivariant maps to X from EG. This is the* **homotopy fixed point set** *of X of Definition 8.3.8(iv), which we denote by X^{hG}.*

The equivariant map $EG \to *$ induces a map $X^G \to X^{hG}$ which is not an equivalence in general. This is not surprising because the map $EG \to *$ is not an equivariant equivalence, since the induced map of fixed points for any nontrivial subgroup is the map from the empty set to a point.

Dually, the orbit space X_G is the coequalizer of the diagram

$$X \times G \times * \rightrightarrows X \times *$$

where the two maps represent the actions of G on X and on a point. If we replace $*$ by EG, the resulting coequalizer is the **homotopy orbit space** *X_{hG}, which supports a map to X_G.*

In both cases one could replace EG by $E\mathscr{F}$ for any family of subgroups $\mathscr{F}$, and obtain spaces $X^{h\mathscr{F}}$ and $X_{h\mathscr{F}}$.

Lemma 8.6.18. Equifibrancy of the family model structures. *For a subgroup $H \subseteq G$, let $\mathscr{F}_H$ and $\mathscr{F}_G$ be families of subgroups of H and G with $\mathscr{F}_H \subseteq \mathscr{F}_G$. Then the corresponding model structures on $\mathcal{T}^H$ and $\mathcal{T}^G$ are related by a Quillen pair (Definition 4.5.1)*

$$G \underset{H}{\ltimes} (-) : \mathcal{T}^H \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{T}^G : i_H^G.$$

Proof The right adjoint clearly preserves fibrations and trivial fibrations. This is sufficient by Proposition 4.5.12(iii). □

Remark 8.6.19. Equifibrancy. *We will refer to the adjunction of Lemma 8.6.18 and others like it as* **change of group adjunctions**. *The word* **equifibrant** *has appeared in the category theory literature before, but as far as we know this is its first use in equivariant homotopy theory. We will use it to describe a model category in which the change of group adjunctions are Quillen pairs. In [HHR16] we used the term "complete" to describe a model structure with a similar property on the category of G-spectra, to be introduced below in Chapter 9. We prefer not to use that term here due to its prior use in Definition 2.3.25.*

Proposition 8.6.20. Equivariant homotopy groups and maps between G-spaces. *For any pointed G-space X,*

$$\pi_k^H X \cong [G \underset{H}{\ltimes} S^k, X]^G,$$

where $\pi_k^H X$ is as in Definition 8.3.12 and the expression on the right denotes the set of G-equivariant homotopy classes of pointed maps $G \underset{H}{\ltimes} S^k \to X$.

Proof The change of group adjunction of Lemma 8.6.18 gives an isomorphism

$$\mathcal{T}^H(S^k, i_H^G X) \cong \mathcal{T}^G(G \underset{H}{\ltimes} S^k, X).$$

Passing to path component sets gives the desired isomorphism. □

Proposition 8.6.21. Properties of the restriction functor i_H^G. *Let $\mathcal{T}^G$ and $\mathcal{T}^H$ each have the Bredon model structure. Then*

(i) the functor i_H^G preserves all weak equivalences,
(ii) it preserves all fibrations, and
(iii) it preserves all cofibrations.

Proof (i) and (ii) A map $f : X \to Y$ in $\mathcal{T}^G$ is a Bredon weak equivalence or fibration if the induced map $f^K : X^K \to Y^K$ is one in $\mathcal{T}$ for each subgroup $K \subseteq G$. In particular this holds for each subgroup $K \subseteq H$, making $i_H^G f$ a Bredon weak equivalence or fibration in $\mathcal{T}^H$.

(iii) We know that i_H^G is a left adjoint by Proposition 8.3.24. This means it suffices to show that it the sends generating cofibrations in $\mathcal{T}^G$ (listed in

Theorem 8.6.2) to cofibrations in $\mathcal{T}^H$. Thus we need to show that for each subgroup $K \subseteq G$ and each $n \geqslant 0$, the map

$$i_H^G(G \underset{K}{\ltimes} i_{n+}) \cong i_H^G(G/K \ltimes i_{n+}) \cong (i_H^G G/K) \ltimes i_{n+}$$

is a cofibration in $\mathcal{T}^H$. We know that $i_H^G G/K$ is a finite H-set and hence a disjoint union of H-orbits. This means the map is a finite wedge of generating cofibrations in $\mathcal{T}^H$ as required. □

8.6B The orbit category

The proof of Theorem 8.6.2 can be found in [MM02, III.1], and that of its generalization Theorem 8.6.13 is in [MM02, IV.6] and [BDS16, §1.3]. The proof we give here is due to Marc Stephan [Ste16] and Guillou-May-Rubin [GMR10]. Like the others, it makes use of the following category.

Definition 8.6.22. The orbit category $\mathcal{O}_G$ *for a finite group* G *is the based topological category whose objects are orbits of the form* G/H*, denoted by* $[G/H]$ *to avoid confusion with the* G*-spaces of the same name, and equivariant maps given by multiplication by suitable elements* $\gamma \in G$*. It is a full subcategory of the category* $\mathcal{F}_G$ *of finite* G*-sets of Definition 8.1.1. Thus the morphism sets are*

$$\mathcal{O}_G([G/H],[G/K]) = (G/K)_+^H. \tag{8.6.23}$$

In particular $\mathcal{O}_G([G/H],[G/K])$ *has more than one point only when* K *contains a conjugate of* H*.*

For each family of subgroups $\mathscr{F}$ *as in Definition 8.6.10, there is a subcategory* $\mathcal{O}_{\mathscr{F}}$ *of* $\mathcal{O}_G$ *whose objects are the* $[G/H]$ *with* $H \in \mathscr{F}$*.* **Thus** $\mathcal{O}_G = \mathcal{O}_{\mathscr{A}\ell\ell}$*.*

Definition 8.6.24. *A* **coefficient system for** G *is a functor* $F : \mathcal{O}_G^{op} \to \mathcal{A}b$*. For* $K \subseteq H \subseteq G$ *the homomorphism* $F[G/H] \to F[G/K]$ *induced by the projection map* $G/K \to G/H$ *is called the* **restriction map** Res_K^H*.*

As noted in the introduction to this chapter, $F[G/H]$ has a natural $\mathbf{Z}[W_H]$-module structure, where

$$W_H = N_H/H$$

is the Weyl group of H. In particular $F[G/e]$ is a $\mathbf{Z}[G]$-module.

We can extend such a functor from $\mathcal{O}_G^{op}$ to $(\mathcal{S}et^G)^{op}$ by requiring it to convert products in $(\mathcal{S}et^G)^{op}$ (meaning coproducts in $\mathcal{S}et^G$, that is disjoint unions of G-sets) to products in $\mathcal{A}b$.

Example 8.6.25. *For an abelian group* A*, the* **ordinary** A**-valued coefficient system** *is the functor sending a* G*-set* X *to the product* A^X*, the group of* A*-valued functions on* X*.*

Definition 8.6.26. *A* **(pointed)** $\mathscr{O}_G$**-space** *is a functor* $\mathscr{O}_G^{op} \to \mathcal{T}op$ *(*$\mathscr{O}_G^{op} \to \mathcal{T}$*) and we denote the category of such functors by* $\mathscr{O}_G\mathcal{T}op$ *(*$\mathscr{O}_G\mathcal{T}$*). We denote the value of such a functor* F *on* $[G/H]$ *by* $F_{[G/H]}$*.*

For a family $\mathscr{F}$ *of subgroups of* G*, a* **(pointed)** $\mathscr{O}_{\mathscr{F}}$**-space** *is a functor* $\mathscr{O}_{\mathscr{F}}^{op} \to \mathcal{T}op$ *(*$\mathscr{O}_{\mathscr{F}}^{op} \to \mathcal{T}$*) and we denote the category of such functors by* $\mathscr{O}_{\mathscr{F}}\mathcal{T}op$ *(*$\mathscr{O}_{\mathscr{F}}\mathcal{T}$*).*

An $\mathscr{O}_{\mathscr{F}}$**-CW complex** *is a CW complex as in (8.4.10) in which the diagram is in the category* $\mathscr{O}_{\mathscr{F}}\mathcal{T}op$*, where* S^{n-1} *and* D^n *are understood to be constant functors on* $\mathscr{O}_{\mathscr{F}}^{op}$*. Thus each* K_n *is an* $\mathscr{O}_{\mathscr{F}}$*-set, each* X_n *is an* $\mathscr{O}_{\mathscr{F}}$*-space, and each* f_n *is an* $\mathscr{O}_{\mathscr{F}}$*-map.*

A **pointed** $\mathscr{O}_{\mathscr{F}}$**-CW complex** *is similarly defined.*

In particular when $H = K = e$ we have $\mathscr{O}_G([G/e],[G/e]) = G_+$, so an $\mathscr{O}_G$-space X determines a pointed G-space $X_{G/e}$. This implies that the fixed point functor Φ below is faithful.

In [MM02, page 39], $\mathscr{O}_G$ (denoted there by $G\mathscr{O}$) is described as an **unbased** topological category with morphism spaces $(G/K)^H$. However in [MMSS01, §1] (where the theory of model structures on diagram categories is developed), it is explained that when a topological indexing category is unbased, it is implicitly converted to a based one by adding a disjoint base point to each of its morphism spaces, as we have done in Definition 8.6.22.

Definition 8.6.27. *The* **fixed point functor** $\Phi : \mathcal{T}^G \to \mathscr{O}_G\mathcal{T}$ *sends a pointed* G*-space* X *to the pointed* $\mathscr{O}_G$*-space* $X^{(-)}$ *given by* $[G/H] \mapsto X^H$*. We will abusively denote the induced functor on homotopy categories by* Φ *as well.*

Remark 8.6.28. The symbol Φ**.** *We will also use the symbol* Φ *below in Definition 9.11.7 in connection with geometric fixed points. This second usage is compatible with the one here in the following sense. For each subgroup* $H \subseteq G$ *and each* G*-spectrum* X *(see Definition 9.0.2) one has a spectrum* $\Phi^H X$*. When* $X = \Sigma^\infty A$*, the suspension spectrum (see Remark 7.1.25) of a pointed* G*-space* A*, then* $\Phi^H X$ *is weakly equivalent to the suspension spectrum* $\Sigma^{-0} A^H$*.*

Proposition 8.6.29. The left adjoint of Φ *is the functor*

$$\Gamma : \mathscr{O}_G\mathcal{T} \to \mathcal{T}^G$$

given by $X \mapsto X_{[G/e]}$*.*

Proof We will make use of Theorem 2.2.22 and define the unit $\eta : 1_{\mathscr{O}_G\mathcal{T}} \Rightarrow \Phi\Gamma$ and counit $\epsilon : \Gamma\Phi \Rightarrow 1_{\mathcal{T}^G}$. Note first that $\Gamma\Phi = 1_{\mathcal{T}^G}$ by definition, since $X_{[G/e]}$ comes equipped with a G-action by functoriality. Hence we have our counit ϵ.

For the unit, note that in any $\mathscr{O}_G$-space X and for any subgroups $K \subseteq H \subseteq G$, the map $X_{[G/H]} \to X_{[G/K]}$ must factor through the fixed point set $X^H_{[G/K]}$.

Since for each $H \subseteq G$, we have

$$(\Phi\Gamma X)_{[G/H]} = (\Phi X_{[G/e]})_{[G/H]} = X^H_{[G/e]},$$

the map factorization of the map $X_{[G/H]} \to X_{[G/e]}$ through $X^H_{[G/e]}$ defines η.

The reader can easily verify that ϵ and η have the properties required by Theorem 2.2.22. $\square$

Now we can apply the Crans-Kan Transfer Theorem 5.2.27 to the adjoint pair

$$\Gamma : \mathscr{O}_G\mathcal{T} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{T}^G : \Phi \tag{8.6.30}$$

to lift the cofibrantly generated model structure on the functor category $\mathscr{O}_G\mathcal{T}$ to one on $\mathcal{T}^G$. The former is given by the projective model structure of Theorem 5.6.26. Its generating sets of cofibrations and trivial cofibrations are

$$\begin{aligned} \mathcal{I} &= \left\{ よ^{[G/H]} \wedge i_{n+} : n \geqslant 0,\ H \subseteq G \right\} \\ \text{and} \qquad \mathcal{J} &= \left\{ よ^{[G/H]} \wedge j_{n+} : n \geqslant 0,\ H \subseteq G \right\}, \end{aligned} \tag{8.6.31}$$

where $よ^{[G/H]}$ is the Yoneda functor of Yoneda Lemma 2.2.10. It follows from the definitions that

$$よ^{[G/H]}([G/e]) = \mathscr{O}_G^{op}([G/H], [G/e]) = \mathscr{O}_G([G/e], [G/H]) = G/H_+.$$

This means the generating sets for $\mathcal{T}^G$ are

$$\begin{aligned} \Gamma\mathcal{I} &= \{G/H \ltimes i_{n+} : n \geqslant 0,\ H \subseteq G\} \\ \text{and} \qquad \Gamma\mathcal{J} &= \{G/H \ltimes j_{n+} : n \geqslant 0,\ H \subseteq G\}. \end{aligned} \tag{8.6.32}$$

The other model structures of Theorem 8.6.13 can be similarly obtained by replacing $\mathscr{O}_G$ by an appropriate subcategory $\mathscr{O}_{\mathscr{F}}$ for each family $\mathscr{F}$. See Remark 5.4.23.

Remark 8.6.33. Extending to the category of finite G-sets. *The orbit category $\mathscr{O}_G$ is a full subcategory of $\mathcal{F}_G$, the category of finite G-sets. A functor from $\mathscr{O}_G^{op}$ to $\mathcal{T}op$ ($\mathcal{T}$) can be extended to $\mathcal{F}_G^{op}$ by requiring it to convert disjoint unions to Cartesian (smash) products. We denote the category of such functors by $\mathcal{F}_G\mathcal{T}op$ ($\mathcal{F}_G\mathcal{T}$).*

We can get an adjunction similar to that of (8.6.30) with $\mathscr{O}_G\mathcal{T}$ replaced by $\mathcal{F}_G\mathcal{T}$ as follows. We define Γ as before, as evaluation at the G-set $[G/e]$. For the functor Φ, note that for a pointed G-space X, the fixed point set X^H is the same thing as $\mathcal{T}^G(G/H_+, X)$, the space of equivariant maps to X from G/H. We could replace G/H by a finite G-set T, and $\mathcal{T}^G(T_+, X)$ would be the appropriate smash product of fixed point sets. Thus we could define a functor

$$\Phi : \mathcal{T}^G \to \mathcal{F}_G\mathcal{T} \quad \textit{by} \quad (\Phi X)_T = \mathcal{T}^G(T_+, X).$$

The resulting cofibrant generating sets on $\mathcal{T}^G$ are then

$$\begin{aligned} \Gamma\mathcal{I} &= \{T \ltimes i_{n+} : n \geqslant 0,\, T \in \mathcal{F}_G\} \\ \text{and} \qquad \Gamma\mathcal{J} &= \{T \ltimes j_{n+} : n \geqslant 0,\, T \in \mathcal{F}_G\}\,. \end{aligned}$$

Since each finite G-set T is a union of orbits, these will lead to the same model structure on $\mathcal{T}^G$ as the generating sets of (8.6.32).

8.7 Some universal spaces

In this section we will discuss some universal $\mathscr{F}$-spaces (Definition 8.6.15) that we will need in our study of symmetric powers in §10.5–§10.9 below.

Definition 8.7.1. *Let Λ be a finite group acted on by another finite group G, and let $\widetilde{G}$ denote the semidirect product $\Lambda \rtimes G$. A G-**equivariant universal** Λ-**space** $E_G\Lambda$ is a universal space $E\mathscr{F}$ (Definition 8.6.15) for the family $\mathscr{F}$ of subgroups of $\widetilde{G}$ having trivial intersection with the normal subgroup Λ. In particular for trivial G, it is a contractible free Λ-space $E\Lambda$.*

As noted in the paragraph following Definition 8.6.15, this space may be characterized up to $\widetilde{G}$-equivariant homotopy equivalence either in terms of its fixed point sets or in terms of maps to it from orbits. We will use the latter, namely

$$\mathcal{T}op^{\widetilde{G}}(\widetilde{G}/H, E_G\Lambda) \simeq \begin{cases} * & \text{for } H \in \mathscr{F} \\ \varnothing & \text{otherwise.} \end{cases} \tag{8.7.2}$$

Now let S be a finite G-set and consider the group Λ^S of Λ-valued functions on S. It has a G-action defined by

$$\gamma(\phi)(s) = \gamma^{-1}\phi(\gamma(s)) \in \Lambda \qquad \text{for } \gamma \in G,\ \phi \in \Lambda^S \text{ and } s \in S.$$

Thus we can form the semidirect product $\Lambda^S \rtimes G$, which we denote by $\widetilde{G}^{(S)}$. In §10.6 we will need the following.

Lemma 8.7.3. *Let S be a finite G-set. If $E_G\Lambda$ is a G-equivariant universal Λ-space (Definition 8.7.1) then, under the product action, $(E_G\Lambda)^S$ is a G-equivariant universal Λ^S-space $E_G(\Lambda^S)$.*

Proof Recall that $\widetilde{G}$ and $\widetilde{G}^{(S)}$ denote the semidirect products $\Lambda \rtimes G$ and $\Lambda^S \rtimes G$. The evaluation map $\mathrm{Ev} : S \times \Lambda^S \to \Lambda$ induces a similar map

$$\mathrm{Ev} : S \times \widetilde{G}^{(S)} \to \widetilde{G}.$$

The functor $\mathcal{T}op^{\widetilde{G}} \to \mathcal{T}op^{\widetilde{G}^{(S)}}$ given by $X \mapsto X^S$ has a left adjoint F. To describe it, let M be the set $\widetilde{G} \times S$. It has a left action of $\widetilde{G}$ via left multiplication on the first coordinate and via the action of G on the second. There is a commuting right action of $\widetilde{G}^{(S)}$ defined by

$$(g, s)\gamma = (g \cdot eval(s, \gamma), s)$$

for $g \in \widetilde{G}$, $s \in S$, $\gamma \in \widetilde{G}^{(S)}$ and $eval$ as above. This action has one orbit for each element of S, and the isotropy group (as in Definition 2.1.29(iv)) for the sth orbit is $\Lambda^{S-\{s\}}$.

The functor $X \mapsto X^S$ can be identified with

$$X \mapsto \hom_{\widetilde{G}}(M, X)$$

and so its left adjoint F is given by

$$Y \mapsto M \underset{\widetilde{G}^{(S)}}{\times} Y \qquad \text{for a } \widetilde{G}^{(S)}\text{-space } Y.$$

Breaking M into right $\widetilde{G}^{(S)}$-orbits as described above gives the decomposition

$$M \underset{\widetilde{G}^{(S)}}{\times} Y = \coprod_{s \in S} Y/\Lambda^{S-\{s\}}.$$

In this latter expression, the action of $\sigma \in \Lambda$ on $y \in Y/\Lambda^{S-\{s\}}$ can be computed as the orbit class of ϕ^y, where $\phi \in \Lambda^S$ is any element with sending s to σ. For example, the entire Λ-action can be computed by restricting to the diagonal subgroup of Λ^S.

Observe that a $\widetilde{G}^{(S)}$-space Y is Λ^S-free if and only if

$$M \underset{\widetilde{G}^{(S)}}{\times} Y$$

is Λ-free. Clearly if Y is Λ^S-free then for each $s \in S$, $Y/\Lambda^{S-\{s\}}$ is Λ-free. On the other hand if $\phi \in \Lambda^S$ is a non-identity element fixing $y \in Y$, then there is a $s \in S$, with $\phi(s)$ not the identity element. For this s we have $\phi(s) \cdot \Lambda^{S-\{s\}} = \Lambda^{S-\{s\}}s$.

Now to the proof. Let K be a finite $\widetilde{G}^{(S)}$-set. We need to show that the space of $\widetilde{G}^{(S)}$-maps

$$\mathcal{T}op^{\widetilde{G}^{(S)}}(K, E_G\Lambda^S)$$

is empty or contractible depending on whether or not K has a point fixed by a nontrivial element of Λ^S. By adjunction, this space can be identified with the space of $\widetilde{G}$-maps from

$$\mathcal{T}op^{\widetilde{G}}(M \underset{\widetilde{G}^{(S)}}{\times} K, E_G\Lambda),$$

so the result follows from the observation above. □

8.8 Elmendorf's theorem

In addition to the adjunction of Proposition 8.6.29 there is an equivalence of the two categories due to Elmendorf [Elm83]. As before let $\mathcal{T}op^G$ denote the category of G-spaces and equivariant maps, i.e., functors from the one object category $\mathcal{B}G$ of Definition 2.1.31 to that of topological spaces $\mathcal{T}op$.

Bredon's theorem [Bre67] states that a map $f : X \to Y$ of G-CW complexes is an equivariant equivalence iff the maps $f^H : X^H \to Y^H$ are ordinary equivalences for all subgroups $H \subseteq G$. We take this condition as a definition of weak equivalence in $\mathcal{T}op^G$ and use it to define the homotopy category $\mathrm{Ho}\mathcal{T}op^G$. The theorem is that this homotopy category is equivalent to the homotopy category of $\mathscr{O}_G$-spaces given in Definition 8.6.26.

The category $\mathcal{F}_G$ as in Definition 8.1.1 contains $\mathscr{O}_G$ as a full subcategory. Since every finite G-set is a finite disjoint union of orbits, an $\mathscr{O}_G$-space extends uniquely to a functor $(\mathcal{F}_G)^{op} \to \mathcal{T}op$ taking disjoint unions to Cartesian products. A pointed $\mathscr{O}_G$-space extends uniquely to a functor $(\mathcal{F}_G)^{op} \to \mathcal{T}$ taking disjoint unions to smash products.

Next recall the fixed point functor $\Phi : \mathcal{T}^G \to \mathscr{O}_G\mathcal{T}$ of Definition 8.6.27, where $\mathscr{O}_G\mathcal{T}$ is the category of (pointed) $\mathscr{O}_G$-spaces of Definition 8.6.26.

Elmendorf's theorem states that this functor is an equivalence of categories. His theorem has both pointed and unpointed versions; we will state only the former.

Theorem 8.8.1. Elmendorf's equivalence. *There is a functor*

$$\Psi : \mathscr{O}_G\mathcal{T} \to \mathcal{T}^G$$

(called the **coalescence functor** *C in [Elm83]) with a natural transformation $\epsilon : \Phi\Psi \Rightarrow 1_{\mathscr{O}_G\mathcal{T}}$ (which he denotes by η) such that for each $\mathscr{O}_G$-space T and each subgroup H, the map $\epsilon_T : \Psi(T)^H \to T_{[G/H]}$ is a weak equivalence. If T is a pointed $\mathscr{O}_G$-CW complex, so is ΨT.*

Corollary 8.8.2. *For a pointed G-space X, there is a natural weak equivalence $\Psi\Phi(X) \to X$ obtained by restricting ϵ to G/e.*

Elmendorf attributes the following result to Jim McClure.

Theorem 8.8.3. McClure's adjunction. *For a pointed G-CW complex X and pointed $\mathscr{O}_G$-CW complex T, there is a natural bijection*

$$\mathrm{Ho}\,\mathcal{T}^G(X, \Psi T) \cong \mathrm{Ho}\,\mathscr{O}_G\mathcal{T}(\Phi X, T).$$

The functor Ψ is described in terms of the two sided bar construction of Definition 5.8.11. For an $\mathscr{O}_G$-space T,

$$\Psi(T) = B(T, \mathscr{O}_G, S)$$

where S is the covariant functor $\mathscr{O}_G \to \mathcal{T}$, $[G/H] \mapsto G/H_+$ and the action of G is on the third variable. It follows that

$$\begin{aligned}\Psi(T)^H &= B(T, \mathscr{O}_G, S)^H = B(T, \mathscr{O}_G, S^H) \\ &= B(T, \mathscr{O}_G, \mathscr{O}_G([G/H], -)) \qquad \text{by (8.6.23).}\end{aligned}$$

A formal property of the two-sided bar construction in general is a map

$$\epsilon : B(T, J, J(j, -)) \to T(j).$$

for an object j of the small category J, which is known to be an equivalence. In the case at hand it is

$$\epsilon : \Psi(T)^H = B(T, \mathscr{O}_G, \mathscr{O}_G([G/H], -)) \to T([G/H]).$$

This is the key step in showing that Φ is an equivalence of categories.

An important application is the following.

Theorem 8.8.4. Eilenberg-Mac Lane spaces for Mackey functors. *For each Mackey functor $\underline{M}$ as in Definition 8.2.3 and each integer $n \geqslant 0$ there is a G-space $K(\underline{M}, n)$ with*

$$\pi_k^H(K(\underline{M}(G/H), n)) = \begin{cases} \underline{M}(G/H) & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof The restriction of the functor $M^* : (\mathcal{F}_G)^{op} \to \mathcal{A}b$ to $\mathscr{O}_G$ is the discrete $\mathscr{O}_G$-space (more precisely a discrete $\mathscr{O}_G$-abelian group) given by $[G/H] \mapsto \underline{M}(G/H)$. Composing with the nth iterated classifying space functor B^n (see Proposition 3.4.15(ii) and Remark 3.4.22) gives us an $\mathscr{O}_G$-space given by $[G/H] \mapsto K(\underline{M}(G/H), n)$, the nth Eilenberg-Mac Lane space for the abelian group $\underline{M}(G/H)$. We denote this $\mathscr{O}_G$-space by $K'(\underline{M}, n)$. We can regard it as a pointed $\mathscr{O}_G$-space for any choice of base point in $K(\underline{M}, n)_{[G/G]}$. Now let $K(\underline{M}, n) = \Psi K'(\underline{M}, n)$.

We can use McClure's adjunction (Theorem 8.8.3) to compute its equivariant homotopy groups as in Definition 8.3.12. We have

$$\begin{aligned}\pi_k^H K(\underline{M}, n) &= \pi_k^H i_H^G \Psi K'(\underline{M}, n) = \pi_k^H \Psi i_H^G K'(\underline{M}, n) \\ &= \operatorname{Ho} \mathcal{T}^H(S^k, \Psi i_H^G K'(\underline{M}, n)) \\ &\cong \operatorname{Ho} \mathscr{O}_H \mathcal{T}(\Phi S^k, i_H^G K'(\underline{M}, n)) \\ &\cong \operatorname{Ho} \mathcal{T}(S^k, K'(\underline{M}, n)_{[G/H]}) \\ &= \pi_k(K(\underline{M}(G/H), n)) \\ &= \begin{cases} \underline{M}(G/H) & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \qquad \square$$

The Eilenberg-Mac Lane spectrum for $\underline{M}$ will be the subject of Theorem 9.1.47 below.

8.9 Orthogonal representations of G and related structures

8.9A Basic definitions

A finite dimensional orthogonal representation V of a finite group G can be regarded as a functor

$$V : \mathcal{B}G \to \mathcal{V}ect_{\mathbf{R}}^{O},$$

where $\mathcal{B}G$ is the one object category associated with the group G as in Definition 2.1.31, and $\mathcal{V}ect_{\mathbf{R}}^{O}$ denotes the category of finite dimensional orthogonal real vector spaces. Hence the set of orthogonal representations of G is the functor set $\mathcal{F}un(\mathcal{B}G, \mathcal{V}ect_{\mathbf{R}}^{O})$ as in Definition 2.1.12. We denote the dimension of V as a real vector space, also called the **degree of** V, by $|V|$. We will sometimes use the symbol $|V|$ to denote a real vector space with trivial G-action having the same dimension as V.

For a subgroup $H \subseteq G$ we can consider the composite functor

$$\begin{tikzcd} \mathcal{B}H \arrow[r, "i"] \arrow[dr, "\mathrm{Res}_H^G V"'] & \mathcal{B}G \arrow[d, "V"] \\ & \mathcal{V}ect_{\mathbf{R}}^{O}, \end{tikzcd}$$

the **restriction of** V **to** H. Thus precomposition with i induces the **restriction functor**

$$\mathrm{Res}_H^G : \mathcal{F}un(\mathcal{B}G, \mathcal{V}ect_{\mathbf{R}}^{O}) \to \mathcal{F}un(\mathcal{B}H, \mathcal{V}ect_{\mathbf{R}}^{O}). \tag{8.9.1}$$

Recall the representation sphere S^V of Definition 8.3.26. The following is a special case of Definition 8.3.12.

Definition 8.9.2. Homotopy groups indexed by representations of G**.** *For a finite group G, let X be a pointed G-space and let V be a finite dimensional orthogonal representation of G. Then*

$$\pi_V^G X = \pi_0 \mathcal{T}^G(S^V, X) = [S^V, X]^G,$$

is the set of homotopy classes of pointed G-maps from S^V to X. For a representation W of a subgroup $H \subseteq G$, $\pi_W^H X$ is defined similarly.

Note here that V must be an actual, as opposed to a virtual, representation of G. In Definition 9.1.1 below we will define $RO(G)$-graded homotopy groups in the stable category.

Definition 8.9.3. Twisted suspensions and twisted loop spaces. *For an orthogonal representation V of G and a pointed G-space X,*

(i) $\Sigma^V X$, *the* **twisted suspension**, *is* $S^V \wedge X$, *and*
(ii) $\Omega^V X$, *the* **twisted loop space**, *is the* G*-space* $\mathcal{T}_G(S^V, X)$ *of pointed maps* $S^V \to X$.

Proposition 8.9.4. The equivariant suspension loop adjunction. *For representations* V *and* W *of a finite groups* G *and a pointed* G*-space* X, *we have a natural isomorphism*

$$\pi^G_{V+W} X \cong \pi^G_V \Omega^W X,$$

where $\Omega^W X$ *is the pointed* G*-space* $\mathcal{T}_G(S^W, X)$.

Proof Since $\mathcal{T}_G$ is closed symmetric monoidal, we have

$$\mathcal{T}_G(S^V \wedge S^W, X) \cong \mathcal{T}_G(S^V \wedge \mathcal{T}_G(S^W, X)).$$

Taking the fixed points of both sides gives

$$\mathcal{T}^G(S^{V+W}, X) \cong \mathcal{T}^G(S^V \wedge \Omega^W X).$$

Taking π_0 of both sides gives the desired isomorphism. □

Remark 8.9.5. The group structure in $\pi_V X$. *When* $V^G \neq 0$, *we can write* $V = 1 \oplus \overline{V}$, *so* $S^V \cong S^1 \wedge S^{\overline{V}}$. *Then the usual pinch map* $\between_{S^0} : S^1 \to S^1 \vee S^1$ *(Definition 4.6.22) leads to a pinch map for* S^V. *It can be used to define a natural group structure on* $\pi_V X$ *in the usual way. It is abelian when* $|V^G| \geqslant 2$.

Example 8.9.6. A case where $\pi_V X$ is not a group. *Let* $G = C_2$ *and let* V *be the sign representation* σ. *Then* $V^G = 0$, *so we do not have a group structure on* $\pi_\sigma X$. *The space* S^σ *is a circle on which the nontrivial element* $\gamma \in G$ *acts by a reflection having two fixed points. Pinching them to a single point gives an equivariant map*

$$f : S^\sigma \to G \ltimes S^1,$$

in which the target is the wedge of two circles interchanged by γ. *A pointed equivariant map* $G \ltimes S^1 \to X$ *is equivalent to a pointed map* $S^1 \to X$. *This means that* f *induces a map*

$$\pi^u_1 X \to \pi^G_\sigma X,$$

where where $\pi^u_* X$ *denotes the* **underlying homotopy** *of the* G*-space* X.

The map f *above fits into an equivariant cofiber sequence*

$$S^0 \xrightarrow{i} S^\sigma \xrightarrow{f} G \ltimes S^1 \longrightarrow S^1 \longrightarrow S^{1+\sigma},$$

which leads to an exact sequence similar to that of Proposition 4.7.11(ii),

$$\pi_0 X^G \xleftarrow{i^*} \pi^G_\sigma X \xleftarrow{f^*} \pi^u_1 X \longleftarrow \pi^G_1 X \longleftarrow \pi^G_{1+\sigma} X.$$

Here the three objects on the right are groups, but the two on the left are *merely pointed sets. The image of f^* is the preimage of the base point under i^*.*

Applying the forgetful functor gives $i_H^G S^V = S^{\mathrm{Res}_H^G V}$. **We will sometimes omit the notation i_H^G and Res_H^G when the group under consideration is clear from the context.**

For a finite dimensional orthogonal representation W of H, we have the induced representation of G, namely

$$\mathrm{Ind}_H^G W := \mathbf{R}[G] \otimes_{\mathbf{R}[H]} W.$$

This induction can be regarded as a functor

$$\mathrm{Ind}_H^G : \mathcal{F}un(\mathcal{B}H, \mathcal{V}ect_{\mathbf{R}}^O) \to \mathcal{F}un(\mathcal{B}G, \mathcal{V}ect_{\mathbf{R}}^O). \tag{8.9.7}$$

Then

$$S^{\mathrm{Ind}_H^G W} = \mathcal{T}^H(G_+, S^W),$$

where $\mathcal{T}^H$ is the category pointed H-spaces and equivariant maps as in Definition 3.1.59. Here we are regarding G_+ as a pointed H-space via right multiplication, and $\mathcal{T}^H(G_+, S^W)$ as a pointed G-space by left multiplication on the source. The underlying space here is the smash power $(S^{|W|})^{\wedge|G/H|}$.

Example 8.9.8. The regular representation $\rho = \rho_G$ of a finite group *G is the real group ring $\mathbf{R}[G]$ (a vector space of dimension $|G|$) where G acts by left multiplication. This vector space has a basis corresponding to the set of all elements $\gamma_i \in G$,*

$$\{[\gamma_i] : \gamma_i \in G\}.$$

These elements are permuted by the action of G. The element

$$\delta = \sum_i [\gamma_i]$$

is fixed by this action and generates a one dimensional summand, the **diagonal subspace**, *on which G acts trivially. It orthogonal complement is the subspace*

$$\left\{ \sum_i x_i[\gamma_i] \colon x_i \in \mathbf{R}, \sum_i x_i = 0 \right\}$$

It is invariant under G and we call it the **reduced regular representation** *$\overline{\rho} = \overline{\rho}_G$.*

It follows that $S^\rho \cong S^1 \wedge S^{\overline{\rho}}$.

For other examples of representations, see Example 8.2.1.

Definition 8.9.9. A partial ordering on the set of orthogonal representations of G. *Let V_1 and V_2 be two nonzero representations of G. We say that $V_1 < V_2$ if for every irreducible representation U,*

$$\dim \hom^G(U, V_1) < \dim \hom^G(U, V_2) - 1.$$

Without subtracting 1 on the right, this condition would insure that V_1 embeds equivariantly in V_2. The extra dimension assures that the space $O(V_1, V_2)^G$ of equivariant orthogonal embeddings is connected, so all such embeddings are homotopic. It also implies that $O(V_1, V_2)$ is simply connected.

8.9B Representations of finite G-sets

For a finite G-set T, recall the split groupoid $\mathcal{B}_T G$ of Definition 2.1.31. When T has one element, this is the one object category $\mathcal{B}G$ associated with the group G, and a finite dimensional orthogonal representation V of G is a functor from it to the category of finite dimensional orthogonal real vector spaces, which we denote by $\mathcal{V}ect_{\mathbf{R}}^{O}$.

Definition 8.9.10. A finite dimensional orthogonal representation V **of a finite** G**-set** T *is a functor from the split groupoid* $\mathcal{B}_T G$ *of Definition 2.1.31 to the category of finite dimensional orthogonal real vector spaces* $\mathcal{V}ect_{\mathbf{R}}^{O}$*, in which morphisms are orthogonal embeddings. For each* $t \in T$ *we denote its image under* V *by* V_t*. It can also be thought of as a* G*-equivariant vector bundle over* T*. (This language is used in [HHR16, §B.5], which is similar to the first four sections of Chapter 10 below.)*

(i) Its **representation sphere** *is*

$$S^V = \bigwedge_{t \in T} S^{V_t}$$

for S^{V_t} *as in Definition 8.3.26,*

(ii) Given two pairs (T', V') *and* (T'', V'')*, each consisting of a finite* G*-set and a representation thereof, we define the* **direct sum**

$$(T', V') \oplus (T'', V'') := (T' \times T'', V' \oplus V''),$$

where the functor $V' \oplus V''$ *on the* G*-set* $T' \times T''$ *is defined by*

$$(V' \oplus V'')_{(t_1, t_2)} = V'_{t_1} \oplus V''_{t_2} \qquad \text{for } (t_1, t_2) \in T' \times T''.$$

In particular $(G/G, 0) \oplus (T, V) = (T, V) \oplus (G/G, 0) = (T, V)$*.*

(iii) The **disjoint union** *of* (T', V') *and* (T'', V'') *is*

$$(T', V') \amalg (T'', V'') := (T' \amalg T'', V' \amalg V'')$$

where the functor $V' \amalg V''$ *is defined by*

$$(V' \amalg V'')_t = \begin{cases} V'_t & \text{for } t \in T' \\ V''_t & \text{for } t \in T''. \end{cases}$$

(iv) The **degree** $|(T, V)|$ **of** (T, V) *is*

$$|(T, V)| \cong \sum_{t \in T/G} |V_t|.$$

Here the sum is over all G-orbits in T. It is well defined because, if t and t' are in the same orbit, the vector spaces V_t and $V_{t'}$ are necessarily isomorphic.

(v) *A representation (T, V) is* **positive** *if the following condition holds. The finite G-set T is a finite union of orbits G/H_α, where each isotropy group H_α is defined up to conjugacy. For each such orbit we have a representation V_α of H_α. We require that that the invariant subspace $V_\alpha^{H_\alpha}$ be nontrivial for each α.*

(vi) *For a subgroup $H \subseteq G$, $i_H^G T$ denotes the finite H-set obtained by applying the forgetful functor to the finite G-set T. Thus $\mathcal{B}_{i_H^G T}H$, which we will denote abusively by $\mathcal{B}_T H$, is a wide subcategory (Definition 2.1.4) of $\mathcal{B}_T G$, and the* **restriction of** V **to** H*, $\mathrm{Res}_H^G V$ is the composite functor*

$$\mathcal{B}_T H \xrightarrow{j_H^G} \mathcal{B}_T G \xrightarrow{V} \mathcal{V}ect_{\mathbf{R}}^{O}$$

for j_H^G as in Definition 2.1.31. Note that $\mathrm{Res}_H^G V$ is positive if V is.

(vii) *Given a representation V of a finite G-set T and a surjective map $p : T \to T'$ of G-sets, the* **representation of** T' **induced by** *p is the left Kan extension $p_! V$. (See Example 2.5.8(iv).)*

Proposition 8.9.11. Induced representations of G/G. *With notation as in Definition 8.9.10(vii), when $T = G/H$, making V and representation of H, and $T' = G/G$, then $p_! V = \mathrm{Ind}_H^G V$, the representation of G induced up from V. When T is a union of such orbits, and $T' = G/G$, then $p_! V$ is the direct sum of the corresponding induced representations.*

Example 8.9.12. Some representations of finite G-sets.

(i) *Let $T = G/H$ for a subgroup $H \subseteq G$. Then a finite dimensional orthogonal representation V of G/H assigns a finite dimensional real vector space $V_{\gamma H}$ to each element γH of T. Functoriality requires them all to be isomorphic to V_{eH}, which comes equipped with an orthogonal action of H. Thus a representation of the G-set G/H is equivalent to a representation of the group H in the usual sense.* **We will usually make no notational distinction between the two.** *In particular a representation of G/e is simply finite dimensional orthogonal real vector space. When T is a finite union of orbits G/H_α, a representation of it consists of a representation of H_α for each α.*

(ii) *When $T' = T'' = G/G$, then the direct sum of two representations as defined in Definition 8.9.10(ii) corresponds to the usual direct sum of two representations of G. When instead $T' = G/H$, then the direct sum as above coincides with that of V' and the restriction of V'' to H.*

(iii) *When $T' = G/H'$ and $T'' = G/H''$, then $T' \times T''$ can be described as a union of orbits using the methods of §8.1. The degree of the resulting representation on the isotropy group of each of them is the sum of the degrees of V' and V''.*

(iv) Consider the specific case $G = S_3$ and $H' = H'' = C_2$ with V' and V'' both being the sign representation σ. We saw in Example 8.1.6 that

$$S_3/C_2 \times S_3/C_2 \cong S_3/e \amalg S_3/C_2$$

The corresponding representations of e and C_2 each have degree two, with the one on C_2 being 2σ.

Suppose H is a subgroup of G, T is a finite H-set and V is an orthogonal representation of T. Then $G \times_H T$ is a G-set. Its elements are pairs (γ, t) for $\gamma \in G$ and $t \in T$ subject to the relation $(\gamma\eta, t) \sim (\gamma, \eta t)$ for $\eta \in H$. The categories $\mathcal{B}_T H$ and $\mathcal{B}_{G\times_H T} G$ are equivalent by Proposition 2.1.38. It follows (Corollary 2.1.40) that the same is true for the categories of functors from them to $\mathcal{V}ect^{O}_{\mathbf{R}}$, that is the categories of orthogonal representations of the H-set T and of the G-set $G \times_H T$, are also equivalent.

More explicitly we can extend the functor V from T to $G \times_H T$ by defining

$$V_{(\gamma,t)} = V_t.$$

This makes $V_{(\gamma\eta,t)} = V_{(\gamma,\eta t)} = V_{\eta t}$, which is canonically isomorphic to V_t, so the functor is well defined on $G \times_H T$. If T contains an orbit of the form H/K, then $G \times_H T$ contains one of the form $G \times_H H/K = G/K$. The restriction of the original V to this copy of H/K is equivalent to an orthogonal representation of K, as is the restriction of the extended V to the orbit G/K. We denote this extended representation by

$$G \times_H (T, V). \tag{8.9.13}$$

In paricular we have an isomorphism

$$(G/H, V) \cong G \times_H (H/H, V). \tag{8.9.14}$$

Definition 8.9.15. Orthogonal embeddings. *Given representations V and W of a finite G-set T as in Definition 8.9.10, an* **orthogonal embedding**

$$f : (T, V) \to (T, W)$$

consists of an orthogonal embedding $f_t : V_t \to W_t$ for each t. These need not respect the action of G.

We denote the space of all such embeddings by $O_T(V, W)$. When $T = G/G$, we denote it simply by $O(V, W)$ or $O(|V|, |W|)$ as in (7.2.7). Such orthogonal embeddings can be composed in an obvious way.

The proof of the following is an exercise for the reader.

Proposition 8.9.16. Properties of the space $O_T(V, W)$.

(i) The space $O_T(V, W)$ is the product of Stiefel manifolds

$$O_T(V, W) \cong \prod_{t\in T} O(V_t, W_t).$$

(ii) It has an action of G defined as follows. Given an embedding f and an element $\gamma \in G$, consider the diagram

$$\begin{tikzcd} V_t \arrow[r, "f_t"] \arrow[d, "\gamma"'] & W_t \\ V_{\gamma t} \arrow[r, "f_{\gamma t}"] & W_{\gamma t}. \arrow[u, "\gamma^{-1}"'] \end{tikzcd}$$

The embedding $\gamma(f)$ is given by $\gamma(f)_t = \gamma^{-1} f_{\gamma t}\gamma$.

(iii) For $T = G/G$, let $V^\perp$ and $W^\perp$ denote the orthogonal complements of V^G and W^G in V and W. Then

$$O(V, W)^G \cong O(V^G, W^G) \times O(V^\perp, W^\perp)^G.$$

(iv) For

$$T \cong \coprod_\alpha G/H_\alpha,$$

the fixed point set is

$$O_T(V, W)^G \cong \prod O(V_t, W_t)^{H_\alpha} \cong \prod \left(O(V_t^{H_\alpha}, W_t^{H_\alpha}) \times O(V_t^\perp, W_t^\perp)^{H_\alpha} \right),$$

where the product is over all G-orbits of T, with one t taken from each.

(v) For $T = G/G$, the orthogonal group $O(V)$ acts freely on $O(V, W)$.

There is a finer version of Proposition 8.9.16(iii) that involves splitting V and W further into summands corresponding to (and consisting of a multiples of) each irreducible representation of G. We do not need it here, so we leave the details to the reader.

Example 8.9.17. The case $G = C_4$. *The group has three irreducible representations, the trivial representation 1, the sign representation σ, and the 2-dimensional representation λ in which a generator γ of the group gives a rotation of order 4. Let $V = V_0 \oplus V_1 \oplus V_2 = v_0 + v_1\sigma + v_2\lambda$ where $V_0 = V^G$, γ acts on V_1 by multiplication by -1, and V_2 is a sum of v_2 copies of λ. Similarly, let $W = W_0 \oplus W_1 \oplus W_2 = w_0 + w_1\sigma + w_2\lambda$.*

Then an equivariant orthogonal embedding of V into W must preserve this decomposition into three subspaces. Any orthogonal embedding $V_0 \subseteq W_0$ and $V_1 \subseteq W_1$ is equivariant, but more care is needed for $V_2 \subseteq W_2$. We can define complex structures on V_2 and W_2 in which multiplication by i is the action of γ. Thus structure must be preserved by an equivariant embedding, so we have

$$O(V_2, W_2)^{C_4} \cong U(V_2, W_2),$$

where the space on the right is a complex Stiefel manifold.

It follows that

$$O(V, W)^{C_4} \cong O(V_0, W_0) \times O(V_1, W_1) \times U(V_2, W_2).$$

In particular, this space is nonempty only if $v_i \leqslant w_i$ for each i. When each v_i is positive, the sum of the connectivities of the three factors is one more that of the underlying space of $O(V, W)$.

There is a similar decomposition and connectivity statement for any finite cyclic group G.

Remark 8.9.18. Orthogonal embeddings as natural transformations. *Morphisms in the category $\mathcal{V}ect^O_{\mathbf{R}}$ are orthogonal embeddings by definition. It follows that morphisms in the functor category $(\mathcal{V}ect^O_{\mathbf{R}})^{\mathcal{B}_T G}$, that is natural transformations between functors on $\mathcal{B}_T G$, are* **equivariant** *orthogonal embeddings. The morphisms we are consider are those in the larger functor category $(\mathcal{V}ect^O_{\mathbf{R}}))^{\mathcal{B}_T}$, where $\mathcal{B}_T$ is the discrete groupoid (in which the only morphisms are identities) for the set T.*

8.9C The Stiefel and Mandell-May categories

The categories $\mathscr{I}_G$ and $\mathscr{J}_G$ (to be defined below) we will study in this subsection (and use extensively in the next two chapters) have representations of finite G-sets as in Definition 8.9.10 as objects. Similar categories were studied in [MM02], but only for representations of the G-set G/G, that is ordinary representations of G. The only morphisms we consider here are between representations of the same G-set T, so in effect we get a pair of categories $\mathscr{I}_G^T$ and $\mathscr{J}_G^T$ for each T. The ones studied by Mandell and May are $\mathscr{I}_G^{G/G}$ and $\mathscr{J}_G^{G/G}$. We will need this level of generality for the constructions of Chapter 10 below. For now the reader may find it convenient to assume that $T = G/G$.

The reader is advised to take careful note of the difference between the symbols $\mathscr{I}$ and $\mathscr{J}$ (`\mathscr{I}` and `\mathscr{J}`) used here for the Stiefel and Mandell-May categories!

Definition 8.9.19. The Stiefel category $\mathscr{I}_G$ *is the topological G-category (as in Definition 3.1.66) whose objects are representations V of finite G-set T. Given two representations V and W of the same T, the morphism spaces $\mathscr{I}_G(V, V)$ is the embedding space $O_T(V, W)$ of Definition 8.9.15. No morphisms are defined between representations of distinct G-sets.*

The **positive Stiefel category** $\mathscr{I}_G^+ \subset \mathscr{I}_G$ *is the full subcategory of* **positive** *representations V as in Definition 8.9.10(v). We denote the inclusion functor by*

$$i_G : \mathscr{I}_G^+ \to \mathscr{I}_G. \tag{8.9.20}$$

When the dimension of V exceeds that of W, the morphism space $\mathscr{I}_G(V, W)$ is empty. When the two dimensions are equal, it is the orthogonal group $O(V)$ with approprate G-action. When $V \cong 0$, the morphism space is a point.

Proposition 8.9.21. Equivariance of composition morphisms in the Stiefel category. *Given representations U, V and W for a finite G-sets T, the composition morphism*

$$i_{U,V,W} : \mathscr{I}_G(V,W) \times \mathscr{I}_G(U,V) \to \mathscr{I}_G(U,W)$$

is G-equivariant.

Proof We can embed the Stiefel category $\mathscr{I}_G$ as a subcategory of $\mathcal{T}op_G$ by

$$V \mapsto \bigoplus_{t \in T} V_t.$$

The result then follows from Proposition 3.1.63. □

The following is similar to Definition 7.2.4(iii).

Definition 8.9.22. Mandell-May morphism spaces. *Given an orthogonal embedding $f : (T,V) \to (T,W)$ as in Definition 8.9.15, for each $t \in T$ we get an orthogonal compliment $f_t(V_t)^\perp \subseteq W_t$. We define*

$$f(V)^\perp = \bigoplus_{t \in T} (V_t)^\perp \subseteq \bigoplus_{t \in T} W_t.$$

Thus we can define a space

$$\begin{aligned} E(V,W) &= \left\{ (f,w) \in \mathscr{I}_G(V,W) \times \bigoplus_{t \in T} W_t : w \in f(V)^\perp \subseteq W \right\} \\ &= \prod_{t \in T} \left\{ (f,w) \in O(V_t, W_t) \times W_t : w_t \in f_t(V_t)^\perp \subseteq W_t \right\} \end{aligned}$$

The **Mandell-May vector bundle over** $\mathscr{I}_G(V,W)$ *is the evident map*

$$E(V,W) \to \mathscr{I}_G(V,W),$$

and the **Mandell-May morphism space** $\mathscr{J}_G(V,W)$ *is its Thom space.*

Here is an example of the embedding of Remark 7.2.25.

Definition 8.9.23. An embedding of $\mathscr{J}_G(V,W)$ into $\mathcal{T}_G(S^V, S^W)$. *Let V and W be representations of a finite G-set T. Each nonbase point of $\mathscr{J}_G(V,W)$ is a pair (f,a), where $f : (T,V) \to (T,W)$ is an orthogonal embedding (as in Definition 8.9.15) and $a \in f(V)^\perp \subseteq W$ is a vector in the orthogonal complement of $f(V) \subseteq W$. From this we get a map $g : S^V \to S^W$ (see Definition 8.9.10(i)), the one point compactification of the product (over $t \in T$) of maps sending $v_t \in V_t$ to $f_t(v_t) + a_t \in W_t$. We denote the resulting map by*

$$e_G(V,W) : \mathscr{J}_G(V,W) \to \mathcal{T}_G(S^V, S^W).$$

Definition 8.9.24. The Mandell-May category $\mathscr{J}_G$ for a finite group G *is the pointed topological G-category (as in Definition 3.1.66) whose objects are finite dimensional orthogonal representations (actual rather than virtual)*

V of G and whose morphism objects are the Thom spaces $\mathscr{J}_G(V, W)$ above. Given representations U, V and W, there is a composition morphism

$$j_{U,V,W} : \mathscr{J}_G(V, W) \wedge \mathscr{J}_G(U, V) \to \mathscr{J}_G(U, W)$$

induced by composition of affine isometric embeddings.

We denote by $\mathscr{J}^G$ the **topological Mandell-May category**, *which has the same objects as $\mathscr{J}_G$, in which the morphism spaces are the fixed sets*

$$\mathscr{J}^G(V, W) := \mathscr{J}_G(V, W)^G, \tag{8.9.25}$$

which will be described in Proposition 8.9.30 below. The composition pairing in $\mathscr{J}_G$ restricts to one in this category by prop-MM-eqvr below.

The **positive Mandell-May category** $\mathscr{J}_G^+ \subset \mathscr{J}_G$ *is the full subcategory in which all objects V are positive as in Definition 8.9.19. We denote the inclusion functor by*

$$j_G : \mathscr{J}_G^+ \to \mathscr{J}_G. \tag{8.9.26}$$

Here is the Mandell-May analog of Proposition 8.9.21.

Proposition 8.9.27. Equivariance of composition morphisms in the Mandell-May category. *Given representations U, V. Given representations U, V and W for a finite G-set T, the composition morphism*

$$j_{U,V,W} : \mathscr{J}_G(V, W) \wedge \mathscr{J}_G(U, V) \to \mathscr{J}_G(U, W)$$

is G-equivariant.

Proof We can use Definition 8.9.23 to construct a faithful functor

$$e_G : \mathscr{J}_G \to \mathcal{T}_G$$

analogous to the functor of (7.2.17). As in the proof of Proposition 8.9.21, result then follows from Proposition 3.1.63. $\square$

When the dimension of V exceeds that of W, the morphism space $\mathscr{J}_G(V, W)$ is a point. When the two dimensions are equal, it is the orthogonal group $O(V)$ with disjoint base point. When $V \cong 0$, the morphism space is S^0.

Example 8.9.28. Some Mandell-May spaces $\mathscr{J}(m, n)$ for the trivial group.

(i) For $m > n$ the embedding space is empty, so the Thom space consists only of the point at infinity and $\mathscr{J}(m, n) = *$.
(ii) For $m = n$ the embedding space is the orthogonal group $O(n)$, and the vector bundle has dimension 0, so $\mathscr{J}(n, n) = O(n)_+$.
(iii) For $m = 0$ the embedding space is a point and $\mathscr{J}(0, n) \cong S^n$.

(iv) *For $m = 1$ and $n > 0$, the embedding space is S^{n-1} and the vector bundle is its tangent bundle. Thus $\mathscr{J}(1,n)$ is a CW complex of the form $S^{n-1} \cup e^{2n-2}$. Since the Whitney sum of the tangent bundle with the trivial line bundle is the trivial $\mathbf{R}^n$-bundle, $\Sigma\mathscr{J}(1,n)$ is weakly equivalent to $S^n \vee S^{2n-1}$.*

Like Proposition 8.9.16, the following is an exercise for the reader.

Proposition 8.9.29. The free action of the orthogonal group. *The free action of $O(m)$ on $\mathscr{I}(m,n)$ (Proposition 8.9.16(v)) induces an action of it on $\mathscr{J}(m,n)$ that is free away from the base point, namely the point at infinity.*

Note that both Proposition 8.9.16(v) and Proposition 8.9.29 are true even when $n < m$. In the former case, the Stiefel manifold $O(m,n)$ is empty, so any action on it is free by definition. In the latter case we have the Thom space of a vector bundle over the empty set, which is a point by definition.

We will now look at the fixed point set of $\mathscr{J}_G(V,W)$. Recall (Proposition 3.1.62) that the fixed point set of $\mathcal{T}_G(X,Y)$ is $\mathcal{T}^G(X,Y)$, the space of pointed equivariant maps from X to Y. On the other hand, the fixed point set of $\mathscr{J}_G(V,W)$ is **not** the Thom space associated with $\mathscr{I}_G(V,W)^G$, the space of equivariant orthogonal embeddings of Definition 8.9.15.

While it is true that each point in $\mathscr{J}_G(V,W)^G$ is associated with such an orthogonal equivariant embedding $f : V \to W$, one has to consider the action of G on the orthogonal complement of $f(V)$ in W, $f(V)^\perp$. Recall that $V^\perp$ denotes the orthogonal complement of V^G in V. Since f is equivariant it sends V^G to W^G and $V^\perp$ to $W^\perp$, so $f(V)^\perp$ splits accordingly. The action of G fixes each point in the complement of $f(V^G)$ in W^G, but it **fixes only the origin** in the complement of $f(V^\perp)$ in $W^\perp$.

Hence we have the following analog of Proposition 8.9.16(iii).

Proposition 8.9.30. The fixed point set of the Mandell-May morphism space $\mathscr{J}_G(V,W)$. *For $H \subseteq G$, let N_H be the normalizer of H in G. For representations V and W of G, let $V_H^\perp$ and $W_H^\perp$ be the orthogonal complements of V^H in V and W^H in W. We will drop the subscript when $H = G$. Then*

$$\begin{aligned}\mathscr{J}_G(V,W)^H &\cong \mathscr{J}_{N_H/H}(V^H,W^H) \rtimes \mathscr{I}_G(V_H^\perp,W_H^\perp)^H \\ &\cong \mathscr{J}_{N_H/H}(V^H,W^H) \rtimes O(V_H^\perp,W_H^\perp)^H\end{aligned}$$

as pointed N_H/H-spaces. It has the same connectivity as $\mathscr{J}_{N_H/H}(V^H,W^H)$.

Again, as in Proposition 8.9.16(iii), the equivariant embedding space

$$\mathscr{I}_G(V^\perp,W^\perp)^G$$

has a finer splitting which we do not need and which we leave to the interested reader.

Example 8.9.31. The case $G = C_4$ again. *We use the notation of Example 8.9.17. We find that*

$$\mathscr{J}_G(V,W)^G \cong \mathscr{J}(V_0,W_0) \rtimes (O(V_1,W_1) \times U(V_2,W_2)).$$

It is $(w_0 - v_0 - 1)$-connected.

Proposition 8.9.32. The categories $\mathscr{I}_G$ and $\mathscr{J}_G$ are symmetric monoidal *under direct sum $\oplus$ with the trivial representation 0 as the unit.*

Proof It is easy to verify that the direct sum has the unitors and the associator required by Definition 2.6.1. For symmetry one has the required map $\tau_{V,W}$, namely the evident isomorphism from $V \oplus W$ to $W \oplus V$. □

When G is the trivial group we will denote the category $\mathscr{J}_G$ simply by $\mathscr{J}$. For any G there is an inclusion $\mathscr{J} \subset \mathscr{J}_G$ identifying $\mathscr{J}$ with the full subcategory of objects with trivial G-action. There is also a forgetful functor $\mathscr{J}_G \to \mathscr{J}$ which refines in the evident manner to a functor from $\mathscr{J}_G$ to the G-category of objects in $\mathscr{J}$ equipped with a G-action. One can easily check that this is an equivalence.

The following will be used to prove Proposition 9.7.8 below.

Proposition 8.9.33. *The forgetful functor described above gives an equivalence of $\mathscr{J}_G$ with the topological G-category of objects in $\mathscr{J}$ equipped with a G-action. Passage to fixed points gives an equivalence of $\mathscr{J}^G$ with the topological category of objects in $\mathscr{J}$ equipped with a G-action.* □

Theorem 8.9.34. *When G is a cyclic p-group,* **the Mandell-May category $\mathscr{J}_G$ is a spectral $\mathscr{J}^{\mathbf{O}}$-algebra** *as in Definition 7.2.19, with $\mathcal{M} = \mathcal{T}^G$ and $L = S^{\rho_G}$. Here ρ_G is the regular real representation of G as in Definition 8.3.27. The functor of (7.2.20), which we denote here by*

$$i_{\mathbf{O}}^G : \mathscr{J}^{\mathbf{O}} \to \mathscr{J}_G \tag{8.9.35}$$

sends $\mathbf{n}$ to $n\rho_G$. The map of morphism objects are induced by the homomorphisms $O(n) \to O(n\rho_g)$ of Example 7.2.18(ii).

The subcategory $\mathscr{J}_G^+$ is a positive ideal as in Definition 7.2.19.

This theorem is likely to hold for any finite group G. The p-cyclic case is adequate for the applications in this book, since the only groups we need in Part THREE are cyclic 2-groups.

Our proof relies on an explicit description of $S^{n\overline{\rho}_G}$ (for $G = C_{p^\ell}$) as a G-CW complex given in (8.4.17). A different description of $S^{\overline{\rho}_G}$ for arbitrary finite G is given in Example 8.5.17. One may be able to generalize it to one for $S^{n\overline{\rho}_G}$ and use it in a proof similar to ours.

Proof The first two conditions of Definition 7.2.19 and the positivity of the ideal are easy. The hard part is to show that $\mathscr{J}_G$ satisfies the stable homotopy condition of (iii). We know that a complete set of homotopy invariants for $\mathcal{T}^G$ is

$$\left\{\pi_k^H : H \subseteq G, k \geqslant 0\right\} = \left\{\pi_0 \mathcal{T}^G(G \underset{H}{\ltimes} S^k, -) : H \subseteq G, k \geqslant 0\right\}.$$

Let

$$\pi_{k,n}^H := \pi_0 \mathcal{T}^G((G \underset{H}{\ltimes} S^k) \wedge S^{n\rho_G}, -) \cong \pi_0 \mathcal{T}^G(G \underset{H}{\ltimes} S^{k+|G/H|n\rho_H}, -).$$

Then the condition is that for each representation V, the map $\pi_{k,n}^H \xi_{V,m,n}$ of (7.2.21) is an isomorphism for large n.

We will argue by induction on the order of G. We start the induction with the trivial group, which is the subject of Proposition 7.2.24. For each proper subgroup H, the statement about $\pi_{k,n}^H \xi_{V,m,n}$ is part of the stable homotopy condition for $\mathscr{J}_H$, which holds by the inductive hypothesis.

The inductive step then is the statement that $\pi_{k,n}^G \xi_{V,m,n}$ is an isomorphism for large n. For this, note that

$$\pi_{k,n}^G(-) = \pi_0 \mathcal{T}^G(G \underset{G}{\ltimes} S^{k+n\rho_G}, -) = \pi_0 \mathcal{T}^G(S^{k+n\rho_G}, -) = \pi_{k+n\rho_G}^G(-),$$

where the last group is as in Definition 8.9.2. For a pointed G-space X, we have

$$\pi_{k+n\rho_G}^G X \cong \pi_{k+n(1+\overline{\rho}_G)}^G X \cong \pi_{k+n}^G \Omega^{n\overline{\rho}_G} X \cong \pi_{k+n}(\Omega^{n\overline{\rho}_G} X)^G,$$

where $\overline{\rho}_G$ is the reduced regular representation as in Definition 8.3.27.

We now specialize to the case $G = C_{p^\ell}$ for a prime p and a positive integer ℓ. As before, we denote the subgroup of index p^i by G^i, with $G_i := G/G^i$. Recall the description of $S^{n\overline{\rho}_G}$ as a G-CW complex of (8.4.17). Applying the functor $\mathcal{T}^G(-, X)$ (for any pointed G-space X) to that diagram gives a diagram of fiber sequences in $\mathcal{T}$,

$$\begin{tikzcd}[column sep=small] X^G = (X^{K_0})^G & (X^{K_1})^G \arrow[l] & (X^{K_2})^G \arrow[l] & \cdots \arrow[l] & (X^{K^{n(p^\ell-1)}})^G = (\Omega^{n\overline{\rho}_G} X)^G \arrow[l] \\ & (X^{L_1})^G \arrow[u] & (X^{L_2})^G \arrow[u] & & (X^{L_{n(p^\ell-1)}})^G, \arrow[u] \end{tikzcd} \tag{8.9.36}$$

where for $1 \leqslant i \leqslant \ell$,

$$(X^{L_j})^G = \Omega^j \left(X^{G^i}\right) \quad \text{for} \quad n(p^{i-1}-1) < j \leqslant n(p^i - 1). \tag{8.9.37}$$

Note here that the fixed point set X^{G^i} has an action of the group G_i (which is cyclic of order p^i) induced by that of G on X. **We are ignoring that action here**, as explained in Proposition 8.4.16(ii).

The stable homotopy condition requires that for each $k \geqslant 0$, the map $\pi_{k+n}(\Omega^{n\overline{\rho}_G} \xi_{V,m,n})^G$ is an isomorphism for large n. Using (8.9.36), we see that this follows if $\pi_{k+n}(\xi_{V,m,n})^G$ and $\pi_{k+n+j}(\xi_{V,m,n})^{G^i}$, with i and j as in (8.9.37), are isomorphisms for large n.

The map $\xi_{V,m,n}$ has the form

$$\begin{tikzcd}
\mathscr{J}_G(V \oplus \mathbf{m}, \mathbf{n}) \wedge S^{m\rho_G} \arrow[d, equal] & & \\
\mathscr{J}_G(V \oplus \mathbf{m}, \mathbf{n}) \wedge \mathscr{J}_G(0, \mathbf{m}) \arrow[rr, "\xi_{V,m,n}"] \arrow[dr, "\mathscr{J}_G(V\oplus\mathbf{m},\mathbf{n})\wedge\omega^{\mathbf{F}}_{V,0,m}"'] & & \mathscr{J}_G(V, \mathbf{n}) \\
& \mathscr{J}_G(V \oplus \mathbf{m}, \mathbf{n}) \wedge \mathscr{J}_G(V, V \oplus \mathbf{m}), \arrow[ur, "j_{V,V+m,n}"'] &
\end{tikzcd} \tag{8.9.38}$$

Using Proposition 8.9.30, we see that the map of G-fixed points is

$$\begin{tikzcd}
\begin{array}{c}\big(\mathscr{J}(V^G + m, n) \rtimes \mathscr{I}_G(V^\perp + m\overline{\rho}_G, n\overline{\rho}_G)^G\big) \\ \wedge \big(S^m \rtimes \mathscr{I}_G(0, m\overline{\rho}_G)^G\big)\end{array} \arrow[d, "\cong"] \\
\begin{array}{c}\big(\mathscr{J}(V^G + m, n) \wedge S^m\big) \\ \rtimes \big(\mathscr{I}_G(V^\perp + m\overline{\rho}_G, n\overline{\rho}_G)^G \times \mathscr{I}_G(0, m\overline{\rho}_G)^G\big)\end{array} \arrow[d, "\mathscr{J}(V^G+m,m)\wedge\omega^{\mathbf{O}}_{V^G,0,m}\rtimes -"] \arrow[dd, bend right=60, "\xi_{V^G,m,n}\rtimes\left(i_{V^\perp,V^\perp+m\overline{\rho}_G,n\overline{\rho}_G}\right)^G"'] \\
\begin{array}{c}\big(\mathscr{J}(V^G + m, n) \wedge \mathscr{J}(V^G, V^G + m)\big) \\ \rtimes \big(\mathscr{I}_G(V^\perp + m\overline{\rho}_G, n\overline{\rho}_G)^G \times \mathscr{I}_G(V^\perp, V^\perp + m\overline{\rho}_G)^G\big)\end{array} \arrow[d, "j_{V^G,V^G+m,n}\rtimes\left(i_{V^\perp,V^\perp+m\overline{\rho}_G,n\overline{\rho}_G}\right)^G"] \\
\mathscr{J}(V^G, n) \rtimes \mathscr{I}_G(V^\perp, n\overline{\rho}_G)^G
\end{tikzcd} \tag{8.9.39}$$

We saw above that the stable homotopy condition on $\mathscr{J}_G$ is equivalent to the requirement that for each $k, m \geqslant 0$ and each representation V, the map $(\Omega^{n\overline{\rho}_G}\xi_{V,m,n})^G$ is $(n+k)$-connected (meaning that it induces an isomorphism is π_{n+k}) for large n.

The diagram of (8.9.36) shows that this is equivalent to requiring that $(\xi_{V,m,n})^G$ is $(n+k)$-connected (which we have just shown), and that for $0 \leqslant i < \ell$, $(\xi_{V,m,n})^{G^i}$ is $(n+k+j)$-connected for j as in (8.9.37).

The diagram (8.9.39) shows that the connectivity of $(\xi_{V,m,n})^G$ is that of $\xi_{V^G,m,n}$, which is $(2n - 2|V^G| - m - 1)$-connected by Proposition 7.2.24. This connectivity grows without bound as n increases. Hence for large enough n, $(\xi_{V,m,n})^G$ is $(n+k)$-connected, as required.

Now we need to look at the connectivity of the map $(\xi_{V,m,n})^{G^i}$ for $1 \leqslant i \leqslant \ell$. For brevity, let $G_i = G_i$, which isomorphic to C_{p^i}. It is the group which acts in V^{G^i}. Then the map $\xi_{V^G,m,n}$ in (8.9.39) gets replaced by the left map in

$$\begin{tikzcd}
\mathscr{J}_{G_i}(V^{G^i}+m\rho_{G_i},n\rho_{G_i})\wedge S^{m\rho_{G_i}} \arrow[r, mapsto, "i_0^{G_i}"] \arrow[d, "\xi_{V^{G^i},m,n}"'] & \mathscr{J}(|V^{G^i}|+p^im,p^in)\wedge S^{p^im} \arrow[d, "\xi_{|V^{G^i}|,p^im,p^in}"] \\
\mathscr{J}_{G_i}(V^{G^i},n\rho_{G_i}) \arrow[r, mapsto, "i_0^{G_i}"] & \mathscr{J}(|V^{G^i}|,p^in)
\end{tikzcd}$$

and its connectivity is that of $(\xi_{V,m,n})^{G^i}$. We are applying the forgetful functor $i_0^{G_i}$ because we are ignoring the action of G_i as explained in Proposition 8.4.16(ii). By Proposition 7.2.24, the connectivity of the map on the right is

$$2p^in-2|V^{G^i}|-p^im-1=\big(n(p^i-1)+n\big)+n(p^i-1)-2|V^{G^i}|-p^im-1,$$

which exceeds the required $k+j+n$ (for $j\leqslant n(p^i-1)$) for large n. $\square$

Remark 8.9.40. Exhausting sequences $V_0\subseteq\cdots V_n\subseteq V_{n+1}\subseteq\cdots$ *of representations of* G *are defined in [HHR16, Definition 2.16] to be collections such that every finite dimensional orthogonal representation of* G *is isomorphic to a summand of some* V_n. *Such sequences are then used to define various homotopy colimits such as the fibrant replacement functor of [HHR16, Proposition B.24]. A similar homotopy colimit is used here in Definition 7.4.24. It is understood, if not explicitly stated in [HHR16], that that such homotopy colimits are independent (up to weak equivalence of fibrant objects) of the choice of exhausting sequence, and that one such choice consists of multiples of the regular representation* ρ_G *of* G.

We do **not** *make such a definition here. Such a choice is effectively built into the definition of a spectral* $\mathscr{J}^{\mathbf{O}}$*-algebra in Definition 7.2.19 with its direct summand condition (ii) and choice of the object* L*, which is* S^{ρ_G} *for our category* $\mathscr{J}_G$.

9

Orthogonal G-spectra

We are now ready to introduce our main objects of study, orthogonal G-spectra for a finite group G. They are smashable spectra in the sense of Definition 7.2.33. This means they are $\mathcal{M}$-valued functors on an indexing category $\mathscr{J}_L^{\mathbf{F}}$.

The underlying model category $\mathcal{M}$ is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps, with the Bredon model structure of Theorem 8.6.2. Weak equivalences are maps $X \to Y$ that induce ordinary weak equivalences of fixed point sets $X^H \to Y^H$ for each subgroup $H \subseteq G$. It is cofibrantly generated with cofibrant generating sets

$$\begin{aligned} \mathcal{I}_G &= \left\{ \left(G \underset{H}{\times} i_n \right)_+ : n \geqslant 0,\, H \subseteq G \right\} \\ \text{and} \qquad \mathcal{J}_G &= \left\{ \left(G \underset{H}{\times} j_n \right)_+ : n \geqslant 0,\, H \subseteq G \right\}, \end{aligned} \tag{9.0.1}$$

with $i_n : S^{n-1} \to D^n$ and $j_n : I^n \to I^{n+1}$ as in (5.2.10) and (5.2.11).

Our indexing category is the Mandell-May category $\mathscr{J}_G$ of Definition 8.9.24. The object K is S^{ρ_G}, where ρ_G denotes the real regular representation of G, which we often abbreviate by ρ. This means that the n-fold smash product $K^{\wedge n}$ is $S^{n\rho}$. It has an action of the orthogonal group $O(n)$ described in Example 7.2.18(ii). We saw in Theorem 8.9.34 that $\mathscr{J}_G$ is an algebra over the category $\mathscr{J}^{\mathbf{O}}$ of Definition 7.2.4(iii), at least in the case where G is a cyclic p-group. This means the results of §7.4 apply here..

The following should be compared with [MM02, II.4.3].

Definition 9.0.2. *An* **orthogonal G-spectrum** *E for a finite group G is a $\mathcal{T}_G$-valued functor on $\mathscr{J}_G$, the Mandell-May category of Definition 8.9.24. We will denote its value on V by E_V (the* **Vth space of** *E), e.g. by E_n when V is an n-dimensional vector space with trivial G-action. $\mathcal{S}p_G$ ($\underline{\mathcal{S}}_G$ in [HHR16] and $\mathscr{I}_G\mathscr{S}$ in [MM02]) denotes the category of orthogonal G-spectra and nonequivariant maps. The corresponding category with equivariant maps is* *denoted by $\mathcal{S}p^G$ ($\mathcal{S}^G$ in [HHR16]). The latter is the functor category $[\mathscr{J}_G, \mathcal{T}^G]$ as in Definition 3.2.18.*

A **positive orthogonal G-spectrum** *is a corresponding functor on the positive Mandell-May category $\mathscr{J}_G^+$ of Definition 8.9.24.*

When the group G is trivial we omit it from the notation.

Remark 9.0.3. The spectrum underlying an orthogonal G-spectrum. *As in Remark 8.3.1 we can speak of the underying orthogonal spectrum of orthogonal G-spectrum. It is given by precomposition with the inclusion functor $i = i_{\mathbf{O}}^G : \mathscr{J} \to \mathscr{J}_G$ of (8.9.35), which sends $\mathscr{J}$ to the full subcategory of trivial representations in $\mathscr{J}_G$. We will sometimes say that an orthogonal G-spectrum X is* **underlain** *by i^*X.*

Note that both $\mathscr{J}_G$ and $\mathcal{T}_G$ are enriched over $\mathcal{T}^G$, meaning that their morphism objects are pointed G-spaces. This means that the functor category

$$\mathcal{S}p_G = [\mathscr{J}_G, \mathcal{T}_G]$$

is also enriched over $\mathcal{T}^G$. Since $\mathcal{T}^G$ is enriched over $\mathcal{T}$, it is also enriched over $\mathcal{T}^G$, which contains $\mathcal{T}$ as a full subcategory. The same is true of $\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G]$.

The category $\mathcal{S}p^G$ is bicomplete, with limits and colimits defined objectwise. On the other hand, the category $\mathcal{S}p_G$ is neither complete nor cocomplete since the same is true of $\mathcal{T}_G$.

We will study model structures on $\mathcal{S}p^G$. (There are none on $\mathcal{S}p_G$ since it is not complete or cocomplete.) The first thing one might try is to start with the projective model structure of Definition 5.4.2 and then stabilize it by applying left Bousfield localization with respect to a set of stabilizing maps as in §7.4C.

This turns out to be unsatisfactory for several reasons. It fails to have some properties required for some constructions that are needed below to prove the main theorem. These properties include the following.

Model structure conditions 9.0.4. *A model structure on the category $\mathcal{S}p^G$ of orthogonal G-spectra should satisfy the following.*

(i) **Equifibrancy.** *(See Remark 8.6.19.) We need the model structure to be compatible with change of group in the sense that for each subgroup $H \subseteq G$, the change of group adjunction*

$$\begin{tikzcd} \mathcal{S}p^H \arrow[r, shift left, "G \underset{H}{\ltimes} (-)"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G \arrow[l, shift left, "i_H^G"] \end{tikzcd}$$

(see (9.1.18) below) is a Quillen pair. The Bredon model structure on $\mathcal{T}^G$ of Theorem 8.6.2 has such an adjunction by Lemma 8.6.18. It was constructed with the help of the auxiliary indexing category $\mathscr{O}_G^{op}$ and functor category $[\mathscr{O}_G^{op}, \mathcal{T}]$ of Definition 8.6.26. We need a different approach here.

The problem is that projective cofibrations in $\mathcal{S}p^H$ do not map to projective cofibrations in $\mathcal{S}p^G$. We will use the construction of Theorem 5.2.34 to enlarge the class of cofibrations in the latter.

(ii) **Indexing compatibility.** *Indexed wedges and smash products, such as the norm N_H^G of Definition 9.7.3, should be homotopical on cofibrant objects. Equifibrancy will be shown to imply this for indexed wedges, but indexed smash products require a separate argument. It is the subject of the first four sections of Chapter 10, culminating in Theorem 10.4.7.*

(iii) **Positivity.** *The symmetric power functor should be homotopical on cofibrant objects. This is the subject the next five sections of Chapter 10, §10.5–§10.9. We need it to get a model structure on* $\mathbf{Comm}^G$, *the category of commutative algebras in $\mathcal{S}p^G$, which will be produced in §10.7. It will give us a Quillen adjunction*

$$\begin{tikzcd} \mathcal{S}p^J \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathbf{Comm}^G \arrow[l, shift left, "U"] \end{tikzcd}$$

where the right adjoint U is the forgetful functor and its left adjoint Sym *is the free commutative algebra functor of Lemma 2.6.66,*

$$X \mapsto \mathrm{Sym}\, X := \bigvee_{i \geqslant 0} X^{\wedge i}/\Sigma_i = S^{-0} \vee X \vee \mathrm{Sym}^2 X \vee \cdots .$$

We also have a change of group adjunction

$$\begin{tikzcd} \mathbf{Comm}^H \arrow[r, shift left, "N_H^G"] \arrow[r, phantom, "\perp"] & \mathbf{Comm}^G; \arrow[l, shift left, "i_H^G"] \end{tikzcd}$$

see Corollary 10.7.5.

(iv) **Geometric fixed point compatibility.** *The geometric fixed point functor of Definition 9.11.7 below should preserve cofibrant objects.*

We will return to these issues in §9.2.

9.1 Categorical properties of orthogonal G-spectra

Definition 9.0.2 requires some unpacking. For each orthogonal G-spectrum E and each pair of representations V and W, we have the following maps.

$$\begin{aligned}
\epsilon^E_{V,W} &: \mathscr{J}_G(V, V+W) \wedge E_V \to E_{V+W} && \text{as in (7.2.36)},\\
\overline{\epsilon}^E_{V,W} &: S^W \wedge E_V \to E_{V+W} && \text{as in (7.2.39)},\\
\widetilde{\epsilon}^E_{V,W} &: \mathscr{J}_G(V, V+W) \underset{\mathscr{J}_G(V,V)}{\wedge} E_V \to E_{V+W} && \text{as in (7.2.40)},\\
\eta^E_{V,W} &: E_V \to \mathcal{T}_G(\mathscr{J}_G(V, V \oplus W), E_{V+W}) && \text{as in (7.2.43)},\\
\text{and } \overline{\eta}^E_{V,W} &: E_V \to \Omega^W E_{V+W} = \mathcal{T}_G(S^W, E_{V+W}) && \text{as in (7.2.44)}.
\end{aligned}$$

9.1A Equivariant homotopy groups

With these maps in hand we can define homotopy groups of G-spectra as in Definition 7.3.14 using Definition 8.3.12 in place of Definition 7.2.30.

Definition 9.1.1. The equivariant homotopy groups of an orthogonal G-spectrum. *Let X be a G-spectrum as in Definition 9.0.2 and let V be an object in the Mandell-May category $\mathscr{J}_G$. Then its V***th homotopy group** *(also known as the V***th stable homotopy group***) is*

$$\pi_V^G X = \operatorname*{colim}_n \pi_V^G \Omega^{n\rho} X_{n\rho} \cong \operatorname*{colim}_n \pi_{V+n\rho}^G X_{n\rho}, \tag{9.1.2}$$

where the colimit is the sequential one associated with the following diagram in $\mathcal{T}^G$.

$$X_0 \xrightarrow{\overline{\eta}^X_{0,\rho}} \Omega^\rho X_\rho \xrightarrow{\Omega^\rho \overline{\eta}^X_{\rho,2\rho}} \cdots . \tag{9.1.3}$$

Here ρ is short for ρ_G, the regular representation of G, the homotopy groups of objects in $\mathcal{T}^G$ are as in Definition 8.3.12, and the maps $\overline{\eta}^X_{k\rho,\rho}$ are the restricted costructure maps of (7.2.44).

We can extend this definition from actual representations V (objects of $\mathscr{J}_G$) to virtual ones, meaning elements in the representation ring $RO(G)$. For each virtual representation V, $V + n\rho$ is an actual representation of G for sufficiently large n. In the second colimit of (9.1.2) we can define $\pi_{V+n\rho}X_{n\rho}$ to be trivial when $V + n\rho$ is in $RO(G)$ but not an object of $\mathscr{J}$.

For a virtual representation W of a subgroup $H \subseteq G$, we define $\pi_W^H X$ to be $\pi_W^H(i_H^G X)$, which is defined in a similar manner to (9.1.2).

Note that the homotopy colimit of (9.1.3) is none other than $\Theta^\infty X$ as in Definition 7.4.26. We know by Theorem 7.4.29 that a map of G-spectra $f : X \to Y$ is a stable equivalence iff $\Theta^\infty f$ is a projective equivalence. This implies the following.

Proposition 9.1.4. Stable equivalences and equivariant homotopy groups. *A map of G-spectra $f : X \to Y$ is a stable equivalence if $\pi_*^H f$ (as Definition 9.1.1) is an isomorphism for all subgroups $H \subseteq G$.*

Corollary 9.1.5. The restriction $i_H^G f$ of a stable G-equivalence f is a stable H-equivalence for all subgroups $H \subseteq G$.

The following is a special case of Corollary 7.4.56.

Proposition 9.1.6. The suspension isomorphism for G-spectra. *For any cofibrant G-spectrum A, and representation W of G, there is a natural isomorphism*

$$\pi_{V+W}^G \Sigma^W A \to \pi_V^G A$$

for all $V \in RO(G)$.

The following connection between the spectra of Definition 9.0.2 and the orthogonal and symmetric spectra of Definition 7.2.4 is straightforward.

Proposition 9.1.7. Orthogonal G-spectra as orthogonal and symmetric spectra with G-action. *For an orthogonal G-spectrum X, precomposition with the functor $i_{\mathbf{O}}^G$ of (8.9.35) gives an orthogonal spectrum with G-action $(i_{\mathbf{O}}^G)^*X$, i.e., a $\mathcal{T}$-functor $\mathscr{J}_K^{\mathbf{O}} \to \mathcal{T}_G$. Similarly, precomposition with $i_{\mathbf{\Sigma}}^G$ a symmetric spectrum with G-action.*

Sometimes we will call orthogonal G-spectra **genuine G-spectra** to distinguish them from the naive G-spectra of Definition 9.3.2 below.

In the following, the symbols $\epsilon_{V,W-V}$ and $\tilde{\epsilon}_{V,W-V}$ are our new notations (Remark 7.2.37) for the structure map of (3.2.27) and the reduced structure map of (3.2.30). The notation makes sense even though $W - V$ may not be an object in the category $\mathscr{J}_G$.

Lemma 9.1.8. *[MM02, Lemma V.1.1]* **The independence of the underlying space of E_V of the G-action on V.** *Suppose V and W have the same dimension. The structure map $\epsilon^E_{V,W-V}$ factors through a G-homeomorphism*

$$\tilde{\epsilon}^E_{V,W-V} : \mathscr{J}_G(V,W) \underset{\mathscr{J}_G(V,V)}{\wedge} E_V \to E_W.$$

whose domain is homeomorphic (but possibly not G-homeomorphic) to E_V. In particular (the case $W = |V|$) E_V is nonequivariantly homeomorphic to the G-space

$$\mathscr{J}_G(|V|,V) \underset{O(|V|)_+}{\wedge} E_{|V|} \cong O(|V|,V) \underset{O(|V|)}{\ltimes} E_{|V|},$$

the orbit space of $O(|V|,V) \ltimes E_{|V|}$ under the diagonal action of $O(|V|)$.

Proof Since $|V| = |W|$, its source is

$$O(V,W) \underset{O(V)}{\times} E_V.$$

To show that it is a homeomorphism, let $f : V \to W$ be a not necessarily equivariant orthogonal isomorphism, that is an element of $O(V,W)$ and hence of $\mathscr{J}_G(V,V)$ inducing an invertible map

$$E_f : E_V \to E_W.$$

Then the map that sends $y \in E_W$ to the equivalence class of $(f, E_{f^{-1}}(y))$ gives the inverse homeomorphism. Mapping x to the equivalence class of (f,x) gives the homeomorphism

$$E_V \xrightarrow{\cong} O(V,W) \underset{O(V)}{\times} E_V. \qquad \square$$

This means that a G-spectrum E is determined by its values on vector spaces V with trivial G-action. **We will come back to this in §9.3.**

9.1B Fixed point and orbit spectra

Definition 9.1.9. Fixed point and orbit spectra. *Let X be an orthogonal G-spectrum as in Definition 9.0.2 and let $H \subseteq G$ be a subgroup. Then the H*-**fixed point spectrum** *X^H of X is the orthogonal spectrum (as in Definition 7.2.33 for $\mathcal{M} = \mathcal{T}$ and $K = S^1$) given by*

$$(X^H)_n = (X_n)^H,$$

the H-fixed point set of the pointed G-space X_n. We will sometimes call this the **naive fixed point spectrum**.

The H-**orbit point spectrum** *X_H* **or** *X/H of X is the orthogonal spectrum (as in Definition 7.2.33 for $\mathcal{M} = \mathcal{T}$ and $K = S^1$) given by*

$$(X_H)_n = (X_n)_H,$$

the H-orbit space of the pointed G-space X_n.

The H-**homotopy fixed point spectrum** *X^{hH} is the orthogonal spectrum given by*

$$(X^{hH})_n = (X_n)^{hH},$$

the H-homotopy fixed point set (see Example 5.8.5(i) and Definition 8.3.8(iv)) of the pointed G-space X_n.

The H-**homotopy orbit spectrum** *X_{hH} is the orthogonal spectrum given by*

$$(X_{hH})_n = (X_n)_{hH},$$

the homotopy orbit space of X_n.

Recall Example 2.2.30(iii), which says that in the category of G-sets, the fixed point functor $(-)^G$ from the category of G-sets $\mathcal{S}et^G$ to $\mathcal{S}et$ is right adjoint to the diagonal functor Δ which assigns to an ordinary set X the same set with trivial G-action. There is a similar adjunction relating $\mathcal{T}^G$ and $\mathcal{T}$.

The situation with spectra is a little more complicated. We have the Mandell-May categories $\mathscr{J}$ (for the trivial group) and $\mathscr{J}_G$ as in Definition 8.9.24. The former is $\mathscr{J}^{\mathbf{O}}_{S^1}$ in the notation of Definition 7.2.4. In the language of Definition 7.2.19, $\mathscr{J}_G$ is a $\mathscr{J}^{\mathbf{O}}_{S^1}$-algebra as explained in Example 7.2.31(i). The functor $i^{\mathbf{F}}_{\mathbf{O}}$ used to define that structure on $\mathscr{J}_G$ sends $\mathbf{n}$ to $n\rho_G$.

Now we need to consider a **different functor** $i : \mathscr{J} \to \mathscr{J}_G$ that instead sends $\mathbf{n}$ to $\mathbf{R}^n$, the n-dimensional representation with trivial G-action. Its left Kan extension gives a functor

$$i_! : \mathcal{S}p \to \mathcal{S}p^G, \tag{9.1.10}$$

where the category on the left is that of orthogonal spectra, $\mathcal{S}p^{\mathbf{O}}(\mathcal{T}, S^1)$ in the notation of Definition 7.2.33. For an orthogonal spectrum X, Lemma 9.1.8 implies that

$$(i_! X)_V \cong O(|V|, V) \times_{O(|V|)} X_{|V|},$$

the space $X_{|V|}$ twisted by the action of G on V.

With this in mind, the spectral analog of the fixed point adjunction is the following.

Proposition 9.1.11. The fixed point adjunction for spectra. *With notation as above, there is an adjunction (enriched over $\mathcal{T}$)*

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "i_!"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G, \arrow[l, shift left, "(-)^G"] \end{tikzcd}$$

where $i_!$ is the left Kan extension of (9.1.10) and the fixed point functor $(-)^G$ is as in Definition 9.1.9.

More generally for each subgroup $H \subseteq G$, there is a composite adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "i_!"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^H \arrow[l, shift left, "(-)^H"] \arrow[r, shift left, "G \ltimes_H (-)"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G. \arrow[l, shift left, "i_H^G"] \end{tikzcd}$$

We will discuss fixed point spectra further in §9.10 below.

9.1C Morphism objects

The morphism G-space $\mathcal{S}p_G(E, F)$ can be described categorically as an enriched end (Definition 3.2.12 and Definition 3.2.18),

$$\mathcal{S}p_G(E, F) = \int^{\mathscr{J}_G} \mathcal{T}_G(E_V, F_V). \tag{9.1.12}$$

More explicitly, it is a certain subspace of the product

$$\prod_V \mathcal{T}_G(E_V, F_V),$$

since a map of spectra $E \to F$ induces maps $E_V \to F_V$ for each V. That subspace is the equalizer of

$$\mathcal{S}p_G(E, F) \dashrightarrow \prod_V \mathcal{T}_G(E_V, F_V) \underset{\nu}{\overset{\mu}{\rightrightarrows}} \prod_{V,W} \mathcal{T}_G(\mathscr{J}_G(V, W), \mathcal{T}_G(E_V, F_W)).$$

To define the maps μ and ν, note that in $\mathcal{T}_G$ the categorical and internal Hom spaces are that same, so we have an adjunction isomorphism

$$\mathcal{T}_G(A \wedge B, C) = \mathcal{T}_G(A, \mathcal{T}_G(B, C)).$$

The image under this isomorphism of the structure map of (7.2.36) for E,

$$\epsilon^E_{V,W-V} \in \mathcal{T}_G(\mathscr{J}_G(V, W-V) \wedge E_V, E_W),$$

is the costructure map of (7.2.43),

$$\eta^E_{V,W-V} \in \mathcal{T}_G(\mathscr{J}_G(V, W-V), \mathcal{T}_G(E_V, E_W)).$$

Then for $f = \{f_V\} \in \prod_V \mathcal{T}_G(E_V, F_V)$, we have

$\mu(f)_{V,W-V} = \epsilon^F_{V,W-V}(\mathscr{J}_G(V, W-V) \wedge f_V)$ and $\nu(f)_{V,W-V} = f_{(W)*}\eta^E_{V,W-V}$.

Similarly we have

$$\mathcal{S}p^G(E,F) = \int^{\mathscr{J}_G} \mathcal{T}^G(E_V, F_V). \tag{9.1.13}$$

Proposition 9.1.14. Equivariant mapping spaces as fixed point sets. *The space $\mathcal{S}p^G(E,F)$ of (9.1.13) is isomorphic to the fixed point set $(\mathcal{S}p_G(E,F))^G$.*

Proof The fixed point functor $(-)^G$ is a limit, so it commutes with ends, and

$$\begin{aligned}\mathcal{S}p^G(E,F) &\cong \int^{\mathscr{J}_G} \mathcal{T}^G(E_V, F_V) \cong \int^{\mathscr{J}_G} \mathcal{T}_G(E_V, F_V)^G \\ &\cong \left(\int^{\mathscr{J}_G} \mathcal{T}_G(E_V, F_V)\right)^G \cong \mathcal{S}p_G(E,F)^G.\end{aligned}$$

□

9.1D Change of group

Let $H \subseteq G$ be a subgroup. The category $\mathscr{J}_G$ is enriched over $\mathcal{T}^G$, and therefore over $\mathcal{T}$. The latter enrichment is obtained by forgetting the G-action on the morphism objects of the former. For a subgroup $H \subseteq G$, there is a restriction functor

$$\mathrm{Res}^G_H : \mathscr{J}_G \to \mathscr{J}_H \tag{9.1.15}$$

given on objects by $V \mapsto \mathrm{Res}^G_H V$ as in (8.9.1). Since

$$i^G_H(\mathscr{J}_G(V,W)) \cong \mathscr{J}_H(\mathrm{Res}^G_H V, \mathrm{Res}^G_H W),$$

the functor of (9.1.15), which is between categories enriched over $\mathcal{T}$, sends morphism objects in the domain category to isomorphic (as underlying spaces) morphism objects in the codomain category.

Given an orthogonal G-spectrum X, consider the diagram

$$\begin{tikzcd} \mathscr{J}_G \arrow[r, "X"] \arrow[dr, "\mathrm{Res}^G_H"'] & \mathcal{T}^G \arrow[r, "i^G_H"] & \mathcal{T}^H \\ & \mathscr{J}_H, \arrow[ur, dashed] & \end{tikzcd}$$

where the unidentified map is the left Kan extension $Lan_{\mathrm{Res}^G_H}(i^G_H \cdot X)$, where $i^G_H \cdot X$ denotes the composite functor in the top row. In order to steamline notation, we will denote it from now on by $i^G_H X$ or sometimes $(i^G_H X)$, giving us a restriction or forgetful functor

$$\mathcal{S}p^G \xrightarrow{\ i^G_H\ } \mathcal{S}p^H. \tag{9.1.16}$$

Alternatively, for each G-spectrum X we have a diagram

$$\begin{tikzcd}
V \arrow[rr, mapsto] & & X_V \arrow[r, equal] \arrow[dd, mapsto] & O(|V|,V) \underset{O(|V|)}{\times} X_{|V|} \\
\mathscr{J}_G \arrow[r, "X"] \arrow[d, "\mathrm{Res}^G_H"'] & \mathcal{T}^G \arrow[d, "i^G_H"] & & \\
\mathscr{J}_H \arrow[r, "(i^G_H X)"] & \mathcal{T}^H & i^G_H(X_V) = O(|V|, \mathrm{Res}^G_H V) \underset{O(|V|)}{\times} i^G_H(X_{|V|}) & \\
W \arrow[r, mapsto] & (i^G_H X)_W. & &
\end{tikzcd}$$

Proposition 9.1.17. The restriction of a G-spectrum. *Let X be a G-spectrum and $H \subseteq G$ a subgroup. Then the H-spectrum $i^G_H X$ in (9.1.16) is given by*

$$(i^G_H X)_W \cong O(|W|, W) \underset{O(|W|)}{\times} i^G_H(X_{|W|}).$$

for each representation W of H.

Proof We will use Lemma 9.1.8 with G and E replaced by H and $i^G_H X$. It says that $(i^G_H X)_W$ is H-homeomorphic to

$$O(|W|, W) \underset{O(|W|)}{\times} (i^G_H X)_{|W|},$$

so it suffices to show that

$$(i^G_H X)_{|W|} \cong i^G_H(X_{|W|})$$

both as $O(|W|)$-spaces and as H-spaces.

Using Proposition 3.2.35 we see that

$$(i^G_H X)_{|W|} \cong \int_{\mathscr{J}_G} \mathscr{J}_H(\mathrm{Res}^G_H V, |W|) \wedge i^G_H(X_V),$$

which is isomorphic to $i^G_H(X_{|W|})$ by the enriched Yoneda coreduction, Proposition 3.2.25. $\square$

The forgetful functor $i^G_H : \mathcal{S}p_G \to \mathcal{S}p_H$ (and $i^G_H : \mathcal{S}p^G \to \mathcal{S}p^H$) has a left adjoint (induction) sending an H-spectrum E to the G-spectrum $G \underset{H}{\ltimes} E$, defined objectwise by

$$(G \underset{H}{\ltimes} E)_V = G \underset{H}{\ltimes} (E_{\mathrm{Res}^G_H V}).$$

This may be written as a wedge indexed by the G-set G/H,

$$G \underset{H}{\ltimes} E = \bigvee_{i \in G/H} E_i \qquad \text{where } E_i = (H_i) \underset{H}{\times} E$$

with $H_i \subseteq G$ the coset indexed by i. Thus we have a change of group adjunction similar to that of Lemma 8.6.18 (see Remark 8.6.19),

$$\begin{tikzcd} \mathcal{S}p^H \arrow[r, shift left, "G \underset{H}{\ltimes}(-)"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G \arrow[l, shift left, "i_H^G"] \end{tikzcd} \tag{9.1.18}$$

that is enriched over $\mathcal{T}$.

9.1E The tautological presentation, smash products and function spectra

The category $\mathcal{S}p_G$ ($\mathcal{S}p^G$) is bitensored (as in Definition 3.1.31) over $\mathcal{T}_G$ ($\mathcal{T}^G$).

Since $\mathcal{T}^G$ is bicomplete we can define limits and colimits in $\mathcal{S}p^G$ objectwise,

$$(\operatorname{colim} E^\alpha)_V = \operatorname{colim}(E^\alpha_V) \quad \text{and} \qquad (\lim_E {}^\alpha)_V = \lim (E^\alpha_V).$$

Any spectrum E has a **tautological presentation** as in Proposition 3.2.33. We abbreviate it by

$$E = \lim_V S^{-V} \wedge E_V = \int_{\mathscr{J}_G} S^{-V} \wedge E_V. \tag{9.1.19}$$

Remark 9.1.20. Sifted colimit preserving functors on spectra. *The coequalizer defined by the above coend is reflexive by Proposition 3.2.33. It follows that a functor on the category of G-spectra that preserves sifted colimits is determined by its behavior on spectra of the form $S^{-V} \wedge K$ for pointed G-spaces K.*

Such spectra are sometimes called (e.g. in [BDS16] and [Sch14] where they are denoted by $\mathcal{F}_V K$ and $F_V K$) **free G-spectra**. When K also has an $O(V)$-action, we can define the **semifree G-spectrum** $S^{-V} \wedge_{O(V)} K$ (denoted by $\mathcal{G}_V K$ in [BDS16] and $G_V K$ in [Sch14]) by

$$(S^{-V} \wedge_{O(V)} K)_W = \mathscr{J}_G(V, W) \wedge_{O(V)} K.$$

Note that

$$S^{-V} \wedge K = S^{-V} \wedge_{O(V)} (O(V) \ltimes K),$$

so every free G-spectrum is also semifree.

In a G-spectrum E, each space E_V comes equipped with an $O(V)$-action, and the map $S^{-V} \wedge E_V \to E$ of (9.1.19) factors through $S^{-V} \wedge_{O(V)} E_V$. Hence every G-spectrum is a colimit of semifree ones.

The following is a special case of Theorem 7.2.60.

Definition 9.1.21 (Jeff Smith)**. The smash product of two spectra.** *Using the map $\oplus$ of Proposition 8.9.32 we define the smash product of two spectra E and F using the Day Convolution Theorem 3.3.5, i.e., the reflexive coequalizer*

$$E \wedge F = \int^{\mathscr{J}_G \times \mathscr{J}_G} S^{-V'\oplus V''} \wedge E_{V'} \wedge F_{V''}.$$

Equivalently, the smash product of spectra E and F as a functor $\mathscr{J}_G \to \mathcal{T}_G$ is the left Kan extension (Definition 2.5.3) in the diagram

$$\begin{tikzcd}
\mathscr{J}_G \times \mathscr{J}_G \arrow[r, "E\times F"] \arrow[dr, "\oplus"'] & \mathcal{T}_G \times \mathcal{T}_G \arrow[r, "\wedge"] & \mathcal{T}_G \\
& \mathscr{J}_G. \arrow[ur, dashed, "E\wedge F"'] &
\end{tikzcd}$$

In other words,

$$E \wedge F = Lan_{\oplus}(\wedge(E \times F)).$$

The Wth space in $E \wedge F$ is

$$\begin{aligned}
(E \wedge F)_W &= \int^{\mathscr{J}_G \times \mathscr{J}_G} (S^{-V'\oplus V''})_W \wedge E_{V'} \wedge F_{V''} \\
&= \int^{\mathscr{J}_G \times \mathscr{J}_G} \mathscr{J}_G(V' \oplus V'', W) \wedge E_{V'} \wedge F_{V''}.
\end{aligned}$$

This is isomorphic to the finite enriched colimit in which we only consider those V' and V'' for which $|V'| + |V''| \leqslant |W|$.

In particular, by Proposition 3.3.14

$$S^{-U'} \wedge S^{-U''} = S^{-U'\oplus U''}. \tag{9.1.22}$$

Using this and the formal properties of coends, we can write

$$\begin{aligned}
E \wedge F &= \int^{\mathscr{J}_G \times \mathscr{J}_G} S^{-V'\oplus V''} \wedge E_{V'} \wedge F_{V''} \\
&= \int^{\mathscr{J}_G \times \mathscr{J}_G} S^{-V''} \wedge S^{-V'} \wedge E_{V'} \wedge F_{V''} \\
&= \int^{\mathscr{J}_G} S^{-V''} \wedge \left(\int^{\mathscr{J}_G} S^{-V'} \wedge E_{V'} \right) \wedge F_{V''}
\end{aligned}$$

$$= \int_{\mathscr{J}_G} S^{-U} \wedge E \wedge F_U.$$

Now let $E = S^{-V}$, so we have

$$S^{-V} \wedge F = \int_{\mathscr{J}_G} S^{-U} \wedge S^{-V} \wedge F_U = \int_{\mathscr{J}_G} S^{-V \oplus U} \wedge F_U$$
$$(S^{-V} \wedge F)_W = \int_{\mathscr{J}_G} (S^{-V \oplus U})_W \wedge F_U = \int_{\mathscr{J}_G} \mathscr{J}_G(V \oplus U, W) \wedge F_U.$$

Here the representations V and W are fixed, so the only U's that matter are those whose dimension does not exceed $|W| - |V|$. This implies the following.

Proposition 9.1.23. Smashing with a Yoneda spectrum. *Let X be a G-spectrum. If $|W| < |V|$, then $(S^{-V} \wedge X)_W = *$. If $|W| \geqslant |V|$, then there is a natural isomorphism of G-spaces*

$$(S^{-V} \wedge X)_W \approx \mathscr{J}_G(V \oplus U, W) \underset{O(U)}{\ltimes} X_U$$

where U is any orthogonal G-representation with

$$|U| + |V| = |W|.$$

Proposition 3.3.12 implies the following, which justifies using the same symbol for the smash products in $\mathcal{T}^G$ and $\mathcal{S}p^G$, as explained in Remark 3.3.13.

Proposition 9.1.24. The smash product of a spectrum with a suspension spectrum. *For a pointed G space K, the spectrum $E \wedge K$ as defined in Proposition 7.2.49 is the same as $E \wedge (S^{-0} \wedge K)$ as defined in Definition 9.1.21.*

The Day Convolution Theorem 3.3.5 implies

Theorem 9.1.25. Smash products and function spectra in $\mathcal{S}p_G$. *The smash product defined above makes $\mathcal{S}p_G$ into a closed symmetric monoidal category with unit S^{-0}. The right adjoint to the smash product with Y as a functor from $\mathcal{S}p_G$ to itself is the internal Hom functor which we will denote by $F_G(Y, -)$ with*

$$\mathcal{S}p_G(X \wedge Y, Z) \cong \mathcal{S}p_G(X, F_G(Y, Z)) \tag{9.1.26}$$

for spectra X, Y and Z.

This means the smash product is strictly associative and commutative, **thereby solving decades of technical problems in stable homotopy theory!** As we have seen, the proof follows easily from the theory of symmetric monoidal categories, **once one has the right perspective**. In Theorem 9.8.4 below we will see that $\mathcal{S}p^G$ (in which morphisms are required to be equivariant) is a closed symmetric monoidal model category.

Remark 9.1.27. The failure of fixed points to commute with smash products of spectra. *The fixed point functor is a type of limit, so it can be defined objectwise on spectra, meaning that for a G-spectrum X,*

$$(X^G)_V \cong (X_V)^G.$$

With this in mind, we can compare $(X^G \wedge Y^G)$ with $(X \wedge Y)^G$. For the former we have

$$(X^G \wedge Y^G)_W \cong \int_{V',V'' \in \mathscr{J}_G} \mathscr{J}_G(V' \oplus V'', W) \wedge X^G_{V'} \wedge Y^G_{V''} \tag{9.1.28}$$

by (3.3.3).

For the latter, recall that the fixed point functor **does** *commute with smash products on the space level. Since it is a finite limit, it commutes with reflexive coequalizers such as that of (3.3.3); see Proposition 3.3.11. Using (3.3.3) again, we have*

$$\begin{aligned}(X \wedge Y)^G_W &\cong \left(\int_{V',V'' \in \mathscr{J}_G} \mathscr{J}_G(V' \oplus V'', W) \wedge X_{V'} \wedge Y_{V''}\right)^G \\ &\cong \int_{V',V'' \in \mathscr{J}_G} \mathscr{J}_G(V' \oplus V'', W)^G \wedge X^G_{V'} \wedge Y^G_{V''}.\end{aligned}$$

This differs from the expression of (9.1.28) since G acts nontrivially on the spaces $\mathscr{J}_G(V' \oplus V'', W)$. We get a map

$$(X \wedge Y)^G \to X^G \wedge Y^G$$

which is not an isomorphism or even a stable equivalence in general.

The following is a special case of Proposition 7.2.61.

Proposition 9.1.29. The relation between function spectra and morphism spaces.

(i) *For G-spectra X and Y, let $F_G(X,Y)$ be the function spectrum defined in Theorem 9.1.25. Then for each representation V,*

$$F_G(X,Y)_V = \mathcal{S}p_G(S^{-V} \wedge X, Y).$$

In particular $F_G(X,Y)_0 = \mathcal{S}p_G(X,Y)$.

(ii) *For $X = S^{-V}$ we have $F_G(S^{-V}, Y)_W = Y_{V \oplus W}$. In particular*

$$F_G(S^{-0}, X) = X.$$

(iii) *For a pointed G-space K, we have*

$$F_G(S^{-V} \wedge K, Y)_W = \mathcal{T}_G(K, Y_{V \oplus W}).$$

In particular we define $\Omega^V Y$ for a spectrum Y to be $F_G(S^{-0} \wedge S^V, Y)$.

Corollary 9.1.30. Unstable homotopy theory embeds in stable homotopy theory. *The functor* $\Sigma^\infty : \mathcal{T}_G \to \mathcal{S}p_G$ *is an embedding as a full subcategory.*

Proof Using Proposition 9.1.29 (iii) we have for pointed G-spaces K and L,

$$\mathcal{S}p_G(\Sigma^\infty K, \Sigma^\infty L) \cong F(\Sigma^\infty K, \Sigma^\infty L)_0 \cong \mathcal{T}_G(K, (\Sigma^\infty L)_0) \cong \mathcal{T}_G(K, L). \qquad \square$$

Let T be a finite G-set and $K = T_+$. Then

$$F_G(\Sigma^\infty T_+, Y)_V = \mathcal{T}_G(T_+, Y_V) = \prod_{t\in T} Y_V$$

so

$$F_G(\Sigma^\infty T_+, Y) = \prod_{t\in T} Y \tag{9.1.31}$$

This is a **finite indexed product** in $\mathcal{S}p_G$, meaning that G acts on the finite indexing set as well as the factors Y; see §2.9.

Now consider the spectrum

$$T \ltimes Y = \bigvee_{t\in T} Y \tag{9.1.32}$$

defined by

$$(T \ltimes Y)_V = T \ltimes Y_V = \bigvee_{t\in T} Y_V.$$

This is a **finite indexed coproduct** as in §2.9.

9.1F G-CW spectra

Recall Example 4.8.20 and Definition 8.4.13. In both cases a CW complex is defined to be an $\mathcal{I}$-cell complex as in Definition 4.8.18 for a collection of cofibrations $\mathcal{I}$, in which cells are attached in dimensional order. In the category of orthogonal G-spectra, the relevant collection of morphisms is

$$\widetilde{\mathcal{I}}^G = \bigcup_{H\subseteq G} \left\{ G \underset{H}{\ltimes} \left(i_{n+} \wedge S^{-V}\right) : n \geqslant 0,\ V \in ob\,\mathscr{J}_H \right\} \tag{9.1.33}$$

where as usual, $i_{n+} : S^{n-1}_+ \to D^n_+$ is the inclusion of the boundary with dispoint base point.

Definition 9.1.34. A connective G-CW spectrum *is an* $\widetilde{\mathcal{I}}^G$*-cell complex as in Definition 4.8.18 in which the cell dimension* $n - |V|$ *is bounded below and the cells are attached in dimensional order. It is* **finite** *if it has finitely many cells.*

Example 9.1.35. The generalized suspension spectrum of a (finite) G-CW complex K *as in Definition 7.2.52,* $S^{-V} \wedge K$*, is a (finite) connective G-CW spectrum.*

9.1G Thom spectra

Classically a Thom spectrum T is one for which the nth space T_n is the Thom space of an $(n+k)$-plane bundle ξ_{n+k} (for a fixed integer k independent of n) over a space B_n. It has structure maps $B_n \to B_{n+1}$ pulling ξ_{n+k+1} back to $1 \oplus \xi_{n+k}$ and therefore Thomifying to a map $\Sigma T_n \to T_{n+1}$. In some cases one wants to generalize the notion of a vector bundle, replacing it with a spherical fibration, but we will leave that aside for now. For us each $(n+k)$-plane bundle ξ_{n+k} is induced by a map $B_n \to BO(n+k)$ pulling back the universal bundle γ_{n+k} to ξ_{n+k}. The Thom spectra associated with classical cobordism theories, such as MO, MU, MSO and MSU, are cases where $k = 0$. Ordinary kth suspension for $k > 0$ is achieved by adding a trivial k-plane bundle to each ξ_n. Desuspension, the case $k < 0$, is less concretely rooted in explicit geometry.

In our language, for each orthogonal representation V of a finite group G one has an orthogonal group $O(V)$ on which G acts by conjugation. It has a classifying space $BO(V)$; see Definition 3.4.12 and Proposition 3.4.15(iii) for the definition of the classifying space of a group. Since it is the orbit space of a contractible free $O(V)$-space $EO(V)$ on which G acts via a homomorphism to $O(V)$, the action of G on $BO(V)$ is trivial. The Thom space is

$$MO(V) = EO(V) \ltimes_{O(V)} S^V, \tag{9.1.36}$$

which has G acting linearly on each fiber. An orthogonal embedding $\tau \colon V \to W$ induces a monomorphism $O(V) \to O(W)$ in which the image acts trivially on the orthogonal complement $W - \tau(V)$. This in turn induces a map

$$BO(V) \to BO(W).$$

The space of all such τ is $O(V, W)$, so we get a map

$$\ell_{V,W} : O(V, W) \times BO(V) \to BO(W) \tag{9.1.37}$$

in which the universal bundle over $BO(W)$ pulls back to the product of the canonical bundle over $O(V, W)$ with the universal bundle over $BO(V)$.

Definition 9.1.38. The G-equivariant unoriented cobordism spectrum *MO has as its Vth space MO_V the Thom space $MO(V)$ of (9.1.36). The structure map*

$$\epsilon_{V,W} : \mathscr{J}_G(V, W) \wedge MO_V \to MO_W$$

is the Thomification of the map $\ell_{V,W}$ of (9.1.37).

Proposition 9.1.39. Commutativity of MO. *The spectrum MO is a commutative ring spectrum.*

Proof It follows from (3.4.16) that

$$BO(V) \times BO(W) \cong BO(V \oplus W).$$

By Thomifying the two factors on the left and the space on the right, we get a map

$$MO(V) \wedge MO(W) \to MO(V \oplus W).$$

This shows that the functor $V \mapsto MO(V)$ defining MO is lax symmetric monoidal, so the result follows from Proposition 7.2.62. □

Remark 9.1.40. The spectrum $MU_{\mathbf{R}}$. *We will construct the C_2-spectrum $MU_{\mathbf{R}}$ below in Chapter 12. It is the starting point in the calculations related to the Kervaire invariant. In order to make things work out correctly we need model structures on $\mathcal{S}p^{C_2}$ and on the subcategory of commutative ring spectra in which $MU_{\mathbf{R}}$ is cofibrant. This, along with the reduction theorem, is one of the major technical challenges in the solution to the Kervaire invariant problem.*

Given a collection of maps $f_V : X_V \to BO(V)$ making the diagrams

$$\begin{tikzcd} O(V,W) \times BO(V) \arrow[r] & BO(W) \\ O(V,W) \times X_V \arrow[u, "O(V,W)\times f_V"] \arrow[r] & X_W \arrow[u, "f_W"'] \end{tikzcd} \tag{9.1.41}$$

commute, we get a Thom spectrum T where T_V is the Thom space for the bundle induced by f_V. The Thomification of the bottom row of (9.1.41) is the structure map for T,

$$\epsilon_{V,W-V} : \mathscr{J}_G(V,W) \wedge T_V \to T_W.$$

A more categorical way to formulate this is the following. Define a **Stiefel space** to be a $\mathcal{T}op^G$ enriched functor $\mathscr{I}_G \to \mathcal{T}op_G$, from the Stiefel category $\mathscr{I}_G$ of Definition 8.9.19. In the construction above, $f : X \to BO$ is a natural transformation between such such functors, a map of Stiefel spaces. In other words, X **is a Stiefel space over** BO. Thomification as above associates an orthogonal G-spectrum to each such f.

For each representation K define a Stiefel space $BO^{\oplus K}$ by

$$(BO^{\oplus K})_V = BO(V \oplus K), \tag{9.1.42}$$

where the structure map is the composite

$$\begin{tikzcd} O(V,W) \times BO(V \oplus K) \arrow[d, "\alpha_{K,V,W} \times BO(V\oplus K)"] & \\ O(V \oplus K, W \oplus K) \times BO(V \oplus K) \arrow[r] & BO(W \oplus K) \end{tikzcd}$$

for $\alpha_{K,V,W}$ as in Definition 2.6.6. Then a Stiefel space over $BO^{\oplus K}$ also leads to a Thom spectrum. The Thomification of $BO^{\oplus K}$ itself is the spectrum $MO^{\oplus K}$ for which $(MO^{\oplus K})_V = MO_{V \oplus K}$.

Example 9.1.43. Thom spectra associated to Stiefel spaces over $BO^{\oplus K}$.

(i) For each representation K there is map of Stiefel spaces $BO \to BO^{\oplus K}$ induced by taking the Whitney sum of the universal V-bundle over $BO(V)$ with the trivial vector bundle over $BO(V)$ with fiber K. The resulting Thom spectrum is $\Sigma^K MO = MO \wedge S^K$.

(ii) For each representation K we have the trivial V-bundle over

$$\mathscr{I}_G(K, V) = O(K, V).$$

It is induced by the constant map of Stiefel spaces

$$\text{⅄}^K = O(K, -\,) \to BO.$$

The resulting Thom spectrum is $S^{-K} \wedge S^K$.

As a coend,

$$MO = \int_{\mathscr{J}_G} S^{-V} \wedge MO_V = \int_{\mathscr{J}_G} S^{-V} \wedge MO(V),$$

so for each representation K,

$$\begin{aligned} \Sigma^{-K} MO &= S^{-K} \wedge \int_{\mathscr{J}_G} S^{-V} \wedge MO_V \\ &= \int_{\mathscr{J}_G} S^{-K} \wedge S^{-V} \wedge MO_V \\ &= \int_{\mathscr{J}_G} S^{-K \oplus V} \wedge MO_V \qquad \text{by(9.1.22)}. \end{aligned}$$

Hence

$$(\Sigma^{-K} MO)_V = \begin{cases} MO_{V'} & \text{when } V = V' \oplus K \text{ for some } V' \\ * & \text{otherwise.} \end{cases}$$

We learned about the following example from Stefan Schwede.

Example 9.1.44. The Thom spectrum of an inverse bundle over BG. *Let V be a faithful representation of G, that is one for which there is no nontrivial element of G fixing all of V. Then for each $n \geqslant 0$ we have a G-action on the spaces $\mathscr{I}(V, n)$ and $\mathscr{J}(V, n)$ induced by the one on V. Consider the ordinary spectrum $T(V)$ defined by*

$$T(V)_n = \mathscr{J}(V, n)/G,$$

the orbit space of the G-action on $\mathscr{J}(V, n)$. For $n \geqslant |V|$, $\mathscr{J}(V, n)$ is the Thom *space of an $\mathbf{R}^{n-|V|}$ bundle over over $\mathscr{I}(V,n)$. For $n < |V|$, $\mathscr{J}(V,n)$ is a point. It follows that for $n \geqslant |V|$, $T(V)_n = \mathscr{J}(V,n)/G$ is the Thom space of an $\mathbf{R}^{n-|V|}$ bundle $\xi_{n-|V|}$ over over $\mathscr{I}(V,n)_G$. By construction, the Whitney sum $\xi_{n-|V|} \oplus V$ is a trivial n-plane bundle.*

The connectivity of $\mathscr{I}(V,n)$ increases with n and the G-action on it is free, so the colimit of the orbit spaces $\mathscr{I}(V,n)/G$ has the homotopy type of the classifying space BG. **It follows that $T(V)$ is the Thom spectrum of the Whitney sum inverse of V over BG.**

The spectrum $T(V)$ is contravariant as a functor of the representation V. Thus a diagram of faithful representations

$$V_0 \to V_1 \to V_2 \to \cdots$$

leads to a spectrum $\lim T(V_k)$. For $G = C_2$, $\lim_k T(k\sigma)$ (where σ is the sign representation) is the spectrum commonly known as $\mathbf{R}P^{\infty}_{-\infty}$. It was first introduced by Adams in [Ada74a]. In [Lin79] and [LDMA80] it was shown to have the homotopy type of the 2-adic completion of S^{-1}, which implies the Segal Conjecture for the group C_2 as explained in [Ada80].

9.1H G-equivariant Eilenberg-Mac Lane spectra

In this section we will construct the Eilenberg-Mac Lane spectrum $H\underline{M}$ for a Mackey functor $\underline{M}$ as in Definition 8.2.3.

We start by constructing a naive G-spectrum $H'\underline{M}$ whose nth space is the Eilenberg-Mac Lane space $K(\underline{M},n)$ of Theorem 8.8.4. To do this we need a structure map

$$\mathscr{J}(n,n+k) \wedge K(\underline{M},n) \to K(\underline{M},n+k). \tag{9.1.45}$$

for each $n,k \geqslant 0$. We construct it by applying Elmendorf's functor Ψ (Theorem 8.8.1) to the map of pointed $\mathscr{O}_G$-spaces (Definition 8.6.26)

$$\Phi\mathscr{J}(n,n+k) \wedge K'(\underline{M},n) \to K'(\underline{M},n+k)$$

defined as follows. The space $\mathscr{J}(n,n+k)$ (Definition 8.9.24) has trivial G-action, so the $\mathscr{O}_G$-space $\Phi\mathscr{J}(n,n+k)$ (where Φ is the functor of Definition 8.6.27) is the constant $\mathscr{J}(n,n+k)$-valued functor on $\mathscr{O}_G^{op}$. Thus on the orbit $[G/H]$, the map above is

$$\mathscr{J}(n,n+k) \wedge K(\underline{M}(G/H),n) \to K(\underline{M}(G/H),n+k),$$

the structure map for the classical Eilenberg-Mac Lane spectrum $H\underline{M}(G/H)$. Thus applying Ψ gives us the desired map (9.1.45) and we have our naive G-spectrum $H'\underline{M}$. It is an Ω-spectrum since $K(\underline{M},n)$ is equivalence to $\Omega^k K(\underline{M},n+$

k). Applying the Kan extension of (9.3.8) gives us a genuine G-spectrum, which we also denote by $H'\underline{M}$, with

$$H'\underline{M}_V = O(|V|, V) \ltimes_{O(|V|)} K(\underline{M}, |V|). \tag{9.1.46}$$

Theorem 9.1.47. The Eilenberg-Mac Lane spectrum for a Mackey functor $\underline{M}$. *For a Mackey functor $\underline{M}$ (Definition 8.2.3), let $H\underline{M}$ be the G-spectrum with*

$$H\underline{M}_V = \operatorname*{hocolim}_n \Omega^{n\rho_G} H'\underline{M}_{V\oplus\gamma_G}$$

*for $H'\underline{M}$ as in (9.1.46), with the evident structure maps. Then $H\underline{M}$ is a fibrant G-spectrum (**??**) with*

$$\pi_k^H H\underline{M} = \begin{cases} \underline{M}(G/H) & \text{for } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

for all integers k.

Remark 9.1.48. The $RO(G)$-graded homotopy groups of $H\underline{M}$. *Theorem 9.1.47 only specifies the* **integer graded** *homotopy groups of $H\underline{M}$. The noninteger graded groups could be nontrivial, and their determination is a nontrivial problem. For instance, in Example 8.5.5 we found the groups $\underline{H}_i S^{n\rho}$ for $G = C_2$. They can be reinterpreted as*

$$\underline{H}_i S^{n\rho} = \underline{\pi}_i S^{n\rho} \wedge H\underline{\mathbf{Z}} = \underline{\pi}_{i-n\rho} H\underline{\mathbf{Z}} = \underline{\pi}_{i-n-n\sigma} H\underline{\mathbf{Z}},$$

where the Mackey functors associated with homotopy groups are as in (9.4.9) below, and some of them are nontrivial for $n \neq 0$.

9.2 Model structures for orthogonal G-spectra

Since our category of spectra $\mathcal{S}p^G$ is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$, the results of §7.4 (with $L = S^{\rho_G}$) apply to it. It has a cofibrantly generated projective model structure and a set of stabilizing maps. We can use the latter to apply Bousfield localization to the former.

Our indexing category $\mathscr{J}_G$ also has a positive ideal (see Definition 7.2.19) $\mathscr{J}_G^+$ identified in Definition 8.9.24. Using it we can define the **positive stable model structure** as in Definition 7.4.36. It was first defined and studied by Mandell-May in [MM02, §III.5] following the ideas of [MMSS01, §14].

However this positive stable model structure is **not adequate** for our purposes because it is not equifibrant. Equifibrancy is needed for the reasons indicated in Model structure conditions 9.0.4. Roughly speaking, the positive and positive stable model structures are not equifibrant because they do not have enough cofibrations. (Recall that Bousfield localization does not alter the class of cofibrations.)

More precisely, the set of generating cofibrations in this case is

$$\left\{ S^{-V} \wedge \mathcal{I}_G \colon V^G \neq 0 \right\}, \tag{9.2.1}$$

where V ranges over the finite dimensional orthogonal representations of G having a nontrivial invariant vector and $\mathcal{I}_G$ is as in (9.0.1). This follows from Theorem 5.6.26. A simplification of this will be given in Remark 9.2.6 below.

Now consider the image of the corresponding set for a subgroup $H \subseteq G$ under the induction functor $G \underset{H}{\ltimes} (-)$,

$$\left\{ G \underset{H}{\ltimes} S^{-W} \wedge \mathcal{I}_H \colon W^H \neq 0 \right\}. \tag{9.2.2}$$

Here W ranges over the positive representations of H. These cofibrations are not all generated by the set of cofibrations in (9.2.1) because not every representation W of H is the restriction of a representation of G.

Thus equifibrancy requires that we replace (9.2.1) with

$$\left\{ G \underset{H}{\ltimes} S^{-W} \wedge \mathcal{I}_H \colon H \subseteq G, W^H \neq 0 \right\}, \tag{9.2.3}$$

where H ranges over the subgroups of G and W ranges over the positive representations of H. We did this in [HHR16, (B.62)] and defined the desired model structure by specifying its cofibrant generating sets and weak equivalences.

We learned the following from Mike Mandell.

Example 9.2.4. Why we need equifibrancy. *Let $G = C_4$, $H = C_2$, and let σ and σ_2 denote their sign representations. Then $S^{-(1+\sigma)}$ and $S^{-(1+\sigma_2)}$ are cofibrant in the positive projective model structures on $\mathcal{S}p^G$ and $\mathcal{S}p^H$ respectively. We will show that*

$$X = G \underset{H}{\ltimes} S^{-(1+\sigma_2)} \tag{9.2.5}$$

is **not** *positive projectively cofibrant in $\mathcal{S}p^G$. This means that the change of group adjunction of (9.1.18) is not a Quillen adjunction.*

The following table shows values of $(S^{-V})_n$ as pointed G-spaces for $n \leqslant 2$ and for various positive representations V of G.

V	$(S^{-V})_0$	$(S^{-V})_1$	$(S^{-V})_2$
1	$*$	$O(1)_+$	$\mathscr{J}(1,2) \simeq S^1 \vee S^2$
2	$*$	$*$	$O(2)_+ \cong (S^1 \amalg S^1)_+$
$1+\sigma$	$*$	$*$	$O(1+\sigma, 2)_+ \cong (S^\sigma \amalg S^\sigma)_+$
$\lvert V \rvert > 2$	$*$	$*$	$*$

In each case the action of H is trivial, and that of G is nontrivial only for $(S^{-(1+\sigma)})_2$. It follows that for positive V, the space

$$\left(S^{-V} \rtimes G/K\right)_2 = (S^{-V})_2 \rtimes G/K$$

contains a free orbit only for $V = 1+\sigma$ and $K = e$, in which case G acts freely away from the base point. This means that for **any** *positive projectively cofibrant orthogonal G-spectrum X, if H acts nontrivially on X_2, then G acts freely on that space away from the base point.*

On the other hand, for X as in (9.2.5), we have

$$X_2 = \left(G \underset{H}{\times} (S^{\sigma_2} \amalg S^{\sigma_2})\right)_+,$$

$$\text{so} \qquad X_2^H = \left(G \underset{H}{\times} (S^0 \amalg S^0)\right)_+$$

Hence X_2 has free orbits but G does not act freely away from the base point. This means that X is not positive projectively cofibrant as a G-spectrum even though $S^{-(1+\sigma_2)}$ is positive projectively cofibrant as an H-spectrum.

Remark 9.2.6. Simplifying the generating sets. *Recall that the set $\mathcal{I}_G$ appearing in (9.2.1) is*

$$\mathcal{I}_G = \left\{(G \underset{H}{\times} i_n)_+ \colon n \geqslant 0,\, H \subseteq G\right\}.$$

For a subgroup $H \subseteq G$, a pointed H-space Y and a representation V of G, we have

$$(G \underset{H}{\ltimes} Y) \wedge S^{-V} \cong G \underset{H}{\ltimes} (Y \wedge S^{-\mathrm{Res}_H^G V})$$

using the homeomorphism $\widetilde{u}_H^G(\Sigma^\infty Y, S^{-V})$ of Definition 9.4.10 below. It follows that the set of (9.2.1) is the same as

$$\bigcup_{H \subseteq G} G \underset{H}{\ltimes} (i_{n+} \wedge S^{-\mathrm{Res}_H^G V} \colon n \geqslant 0)$$

where V ranges over all the positive representations of G. The only representations of H appearing here are ones that are restrictions of representations of G. Meanwhile the set of (9.2.2) involves **all** *positive representations of H.*

Now we will give an alternate approach this definition. Consider the following diagram of categories and adjoint functors.

$$\begin{tikzcd} \prod\limits_{H \subseteq G} \mathcal{S}p^H \arrow[r, shift left, "\prod G \underset{H}{\ltimes} -"] \arrow[r, phantom, "\perp"] & \prod\limits_{H \subseteq G} \mathcal{S}p^G \arrow[l, shift left, "\prod i_H^G"] \arrow[r, shift left, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G \arrow[l, shift left, "\Delta"] \end{tikzcd} \tag{9.2.7}$$

This is a composite adjunction similar to the enlarging adjunction of Theorem 5.2.34, which was formulated with precisely this application in mind. **We**

will use it to enlarge the set generating cofibrations of $\mathcal{S}p^G$ from that of (9.2.1) to that of (9.2.3).

The adjunction on the right in (9.2.7) is the coproduct diagonal adjunction of Example 4.5.6(i). The one on the left is the product over all subgroups $H \subseteq G$ of the change of group adjunctions of (9.1.18). We will verify below that for each H the forgetful functor i_H^G satisfies the hypotheses of the right adjoint of (5.2.35) in Theorem 5.2.34. This means that the composite right adjoint satisfies those of the Crans-Kan Transfer Theorem 5.2.27.

Note also that

$$\prod_{H \subseteq G} \mathcal{S}p^H \cong \mathcal{S}p^G \times \prod_{H \subset G} \mathcal{S}p^H. \tag{9.2.8}$$

This means the product on the left of (9.2.7) is that of the category on the right (corresponding to $\mathcal{M}$ in Theorem 5.2.34) with the product of $\mathcal{S}p^H$ over all proper subgroups $H \subset G$, the $\mathcal{M}'$ in this case. The product in the middle of (9.2.7) is that of copies of the category on the right indexed by the subgroups of G.

For each group G there are stable and positive stable model structures on $\mathcal{S}p^G$ with cofibrant generating sets described in Theorem 7.4.52. In order to make them equifibrant we need to enlarge their classes of cofibrations. **We could proceed by induction on the order of** G, there being no need for additional cofibrations when G is trivial. Assume inductively that this has been done for each proper subgroup of G, so it has been done for the second factor on the right in (9.2.8). Then the enlarged model structure given by Theorem 5.2.34 imports all the extra cofibrations for the proper subgroups into $\mathcal{S}p^G$ itself.

Having said that, it is in fact **not necessary** to enlarge the model structures on the categories of spectra for proper subgroups in order to enlarge the one on $\mathcal{S}p^G$. Every cofibration in the set of (9.2.3) is induced up from ones in (9.2.1) for some subgroup. Thus we will get the same enlarged generating set for $\mathcal{S}p^G$ whether or not we enlarge the generating sets for its proper subgroups ahead of time.

We will now state Theorem 7.4.52 for the case at hand. In terms of its notation, the present case for a finite group G is $\mathcal{M} = \mathcal{T}^G$ and $L = S^{\rho_G}$ (where ρ_G is the real regular representation) of G. The cofibrant generating sets of $\mathcal{T}^G$ are $\mathcal{I}_G$ and $\mathcal{J}_G$ as in (9.0.1).

The indexing category and its positive ideal are

$$\mathscr{I}_L^{\mathbf{F}} = \mathscr{J}_G \qquad \text{and} \qquad \mathscr{L}_L^{\mathbf{F}} = \mathscr{J}_G^{+}$$

the Mandell-May category and its positive subcategory as in Definition 8.9.24.

The generating sets of (7.4.40) are now

$$\left.\begin{aligned} \mathcal{I}^G &= \left\{\mathcal{I}_G \wedge S^{-V} : V \in ob\,\mathscr{J}_G\right\}, \\ \mathcal{J}^G &= \left\{\mathcal{J}_G \wedge S^{-V} : V \in ob\,\mathscr{J}_G\right\}, \\ \mathcal{K}^G &= \mathcal{J}^G \cup (\mathcal{I}_G \,\square\, \mathcal{S}_G), \\ \mathcal{I}^{G,+} &= \left\{\mathcal{I}_G \wedge S^{-V} : V \in ob\,\mathscr{J}_G^{+}\right\}, \\ \mathcal{J}^{G,+} &= \left\{\mathcal{J}_G \wedge S^{-V} : V \in ob\,\mathscr{J}_G^{+}\right\} \\ \text{and}\quad \mathcal{K}^{G,+} &= \mathcal{J}^{G,+} \cup \left(\mathcal{I}_G \,\square\, \mathcal{S}_G^{+}\right), \end{aligned}\right\} \tag{9.2.9}$$

where

$$\begin{aligned} \mathcal{S}_G &= \left\{\tilde{\xi}_{V,n} : V \in ob\,\mathscr{J}_G,\ n > 0\right\} \\ \text{and}\qquad \mathcal{S}_G^{+} &= \left\{\tilde{\xi}_{V,n} : V \in ob\,\mathscr{J}_G^{+},\ n > 0\right\}. \end{aligned} \tag{9.2.10}$$

These can be rewritten as in Remark 9.2.6. Here $\tilde{\xi}_{V,n}$ is the inclusion into the reduced mapping cylinder of (3.5.4),

$$\begin{tikzcd} S^{\wedge n\rho} \wedge S^{-V} \wedge S^{-n\rho} \arrow[rr, "\xi_{V,n}"] \arrow[dr, "\tilde{\xi}_{V,n}"'] & & S^{-V} \\ & M'_{\xi_{V,n}}, \arrow[ur, "\hat{\xi}_{V,n}"'] & \end{tikzcd}$$

where $\xi_{V,n}$ (as in (7.2.22)) is the map whose Uth component is the composite

$$\begin{tikzcd} \mathscr{J}_G(0,n\rho) \wedge \mathscr{J}_G(V\oplus n\rho, U) \arrow[dr, "\omega^G_{V,0,n\rho} \wedge \mathscr{J}_G(V\oplus n\rho, U)"'] & & \mathscr{J}_G(V,U) \\ & \mathscr{J}_G(V, V\oplus n\rho) \wedge \mathscr{J}_G(V\oplus n\rho, U). \arrow[ur, "j_{V,V+n\rho,U}\,t"'] & \end{tikzcd}$$

Thus $\tilde{\xi}_{V,n}$ is a projective cofibration and $\hat{\xi}_{V,n}$ is a projective weak equivalence. We exclude the case $n = 0$ only because $\xi_{V,0}$ is the identity map on S^{-V}.

The following is a special case of Theorem 7.4.52 and hence does not require a proof.

Theorem 9.2.11. The stable and positive stable model structures on $\mathcal{S}p^G$, the corner map theorem for orthogonal G-spectra. *For a finite group G, the sets $\mathcal{I}^{G,+}$ and $\mathcal{K}^{G,+}$ ($\mathcal{I}^G$ and $\mathcal{K}^G$) as in (9.2.9) define a cofibrantly generated model structure on $\mathcal{S}p^G$, the positive stable (stable) model structure as in Definition 7.4.36. It is the Bousfield localization of the positive (projective) model structure of Definition 7.4.36, which is cofibrantly generated by $\mathcal{I}^{G,+}$ and $\mathcal{J}^{G,+}$ ($\mathcal{I}^G$ and $\mathcal{J}^G$), with respect to the morphism set $\mathcal{S}_G^{+}$ ($\mathcal{S}_G$) of (9.2.10).*

For each of the four model structures of Definition 7.4.36 on $\mathcal{S}p^G$, we can define similar ones on $\mathcal{S}p^H$ for every subgroup $H \subseteq G$. For each H the object K in $\mathcal{T}^H$ is understood to be S^{ρ_H}, the representation sphere for the regular representation of H. Thus we get four different model structures on the product on the left of (9.2.7). In each case we want to use Theorem 5.2.34 to enlarge the corresponding model structure in $\mathcal{S}p^G$.

We have a set of adjunctions as in (5.2.35), one for each proper subgroup $H \subset G$, namely the change of group adjunction of (9.1.18),

$$\begin{tikzcd} \mathcal{S}p^H \arrow[r, shift left, "G \underset{H}{\ltimes} (-)"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G \arrow[l, shift left, "i_H^G"] \end{tikzcd}$$

The hypotheses are

(i) The images under the left adjoint of the two generating sets of $\mathcal{S}p^H$ ($\mathcal{I}'$ and $\mathcal{J}'$, one of the four pairs described in Theorem 9.2.11) permit the small object argument in $\mathcal{S}p^G$. This is easy.

(ii) The image under the right adjoint of a relative $G \underset{H}{\ltimes} \mathcal{J}'$-complex is a weak equivalence in $\mathcal{S}p^H$. If $j' : A \to B$ is a map in $\mathcal{J}'$, then

$$i_H^G\left(G \underset{H}{\ltimes} j'\right) = \bigvee_{|G/H|} j',$$

the coproduct of $|G/H|$ copies of j'. Furthermore the right adjoint i_H^G is also a left adjoint, so it preserves transfinite compositions, pushouts and retracts as required.

(iii) The image under the right adjoint i_H^G of a weak equivalence in $\mathcal{S}p^G$ is a weak equivalence in $\mathcal{S}p^H$. This holds by Corollary 9.1.5 in stable case and by Proposition 8.6.21 in the projective case.

In order to describe their generating sets, let

$$\left.\begin{aligned} \widetilde{\mathcal{I}}^G &= \bigcup_{H \subseteq G} G \underset{H}{\ltimes} \mathcal{I}^H, & \widetilde{\mathcal{I}}^{G,+} &= \bigcup_{H \subseteq G} G \underset{H}{\ltimes} \mathcal{I}^{H,+}, \\ \widetilde{\mathcal{J}}^G &= \bigcup_{H \subseteq G} G \underset{H}{\ltimes} \mathcal{J}^H, & \widetilde{\mathcal{J}}^{G,+} &= \bigcup_{H \subseteq G} G \underset{H}{\ltimes} \mathcal{J}^{H,+}, \\ \widetilde{\mathcal{K}}^G &= \bigcup_{H \subseteq G} G \underset{H}{\ltimes} \mathcal{K}^H, \quad \text{and} & \widetilde{\mathcal{K}}^{G,+} &= \bigcup_{H \subseteq G} G \underset{H}{\ltimes} \mathcal{K}^{H,+}, \end{aligned}\right\} \tag{9.2.12}$$

where $\mathcal{I}^H$, $\mathcal{I}^{H,+}$, $\mathcal{J}^H$, $\mathcal{J}^{H,+}$, $\mathcal{K}^H$ and $\mathcal{K}^{H,+}$ are as in (9.2.9).

Theorem 9.2.13. The eight model structures on $\mathcal{S}p^G$. *The cofibrant generating sets for the eight model structures of (7.1) are as shown in the following table, using the notation of (9.2.9) and (9.2.12). Here the term "prestable" means before stabilization. See Figure 7.1 and (5.4.34).*

Model structure	*Generating cofibrations*	*Generating trivial cofibrations*	
		Prestable	*Stable*
Projective	$\mathcal{I}^G$	$\mathcal{J}^G$	$\mathcal{K}^G$
Positive	$\mathcal{I}^{G,+}$	$\mathcal{J}^{G,+}$	$\mathcal{K}^{G,+}$
Equifibrant	$\widetilde{\mathcal{I}}^G$	$\widetilde{\mathcal{J}}^G$	$\widetilde{\mathcal{K}}^G$
Positive equifibrant	$\widetilde{\mathcal{I}}^{G,+}$	$\widetilde{\mathcal{J}}^{G,+}$	$\widetilde{\mathcal{K}}^{G,+}$

For the reasons why we are considering such model structures we refer the reader to the discussion at the start of Chapter 7, specifically to Remark 7.0.7, and to the Model structure conditions 9.0.4.

Remark 9.2.14. Properties of the eight model structures of Theorem 9.2.13.

(i) Each morphism set with "+" in its superscript is **smaller** *than the corresponding set without it.*

(ii) The set of generating trivial cofibrations in the stable case is bigger than that for the prestable case while set of generating cofibrations is the same.

(iii) The morphism set in the equifibrant case is larger than the corresponding one in the nonequifibrant case.

(iv) The identity morphism from a stabilized or enlarged model structure to the prestable or unenlarged one is a right adjoint, while the one from a positivized structure is a left adjoint; see Table 6.1. Positivization is the odd man out.

(v) Enlargement does not alter the class of weak equivalences. Positivization makes it a little bigger. For a map $f : X \to Y$ to be a weak equivalence, f_V must be one only for positive V. Stabilization makes the class **a lot bigger***. A sufficient (but not necessary) condition for f to be a stable equivalence is that f_V is weak equivalence for sufficiently large V.*

We will see in Theorem 9.8.4 below that $\mathcal{S}p^G$ with the positive stable equifibrant model structure is a symmetric monoidal model category or Quillen ring as in Definition 5.5.9.

Our model structure of choice is the positive stable equifibrant one, which has fewer cofibrant objects than the stable equifibrant one. It turns out that some nice properties enjoyed by cofibrant objects in the former are also enjoyed by the more plentiful cofibrant objects in the latter, so we give such spectra a name.

Definition 9.2.15. Bredon cofibrant G-spectra. *An equivariant orthogonal spectrum is* **Bredon cofibrant** *if it is in the smallest subcategory of $\mathcal{S}p^G$ containing the spectra of the form*

$$G \underset{H}{\ltimes} S^k \wedge S^{-V}$$

with V a representation of s subgroup $H \subseteq G$ and $k \geqslant 0$ and which is closed under the formation of arbitrary coproducts, the formation of mapping cones, retracts, and the formation of filtered colimits along h-cofibrations.

Equivalently it is one that is cofibrant in the stable equifibrant model structure, or equivalently in the projective equifibrant model structure, without a positivity condition. See (7.1) and Theorem 9.2.13.

In [HHR16, Definition B.57] we called Bredon cofibrant G-spectra "cellular." We prefer not to use that term here in order to avoid confusion with its use in Definition 6.3.1 and in §8.4.

Note that the representation V above is **not** required to have a nonzero invariant vector as in Theorem 9.2.11. Here there is no positivity condition as in Remark 7.4.3.

Example 9.2.16. Connective G-CW spectra *as in Definition 9.1.34 are Bredon cofibrant, but the converse is not true. A Bredon cofibrant spectrum need not be connective and its cells need not be attached in dimensional order.*

Remark 9.2.17. Bredon cofibrant and cofibrant spectra. *Of the three constructions used to to form the diagram of Figure 7.1, equifibrant enlargement enlarges the class of cofibrations, stabilization leaves it unchanged and positivization makes it smaller, as indicated in Table 6.1. Thus the two model structures in Figure 7.1 having the largest class of cofibrations are the equifibrant and stable equifibrant model structures. They are also the ones with the biggest class of cofibrant objects, the subject of Definition 9.2.15.*

Bredon cofibrant G-spectra are **not** *all cofibrant with respect to the* **positive stable equifibrant model structure** *of Theorem 9.2.11, which, for reasons having to do with commutative ring spectra, is our model structure of choice. Nevertheless they have some pleasant properties such as flatness as we shall see in Proposition 9.6.5.*

One could make an analogous definition of Bredon cofibrant G-spaces as above but without mentioning the Yoneda spectrum S^{-V}. Such pointed G-spaces are precisely the ones that are cofibrant with respect to the Bredon model structure of Theorem 8.6.2, hence the name used in Definition 9.2.15.

Cofibrant approximation in the equifibrant model structure gives functorial weak equivalence $\tilde{X} \to X$ from a Bredon cofibrant $\tilde{X}$ to each orthogonal G-spectrum X.

9.3 Naive and genuine G-spectra.

An ordinary orthogonal spectrum is a $\mathcal{T}$-functor (see Definition 3.1.13) $\mathscr{J} \to \mathcal{T}$ for $\mathscr{J}$ as in Definition 8.9.24. An orthogonal G-spectrum is a $\mathcal{T}^G$-functor $\mathscr{J}_G \to \mathcal{T}_G$ as in Definition 9.0.2. The functor category $\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G]$ is that of orthogonal G-spectra and **equivariant maps**.

Since $\mathscr{J}$ is a full subcategory of $\mathscr{J}_G$, an orthogonal G-spectrum X induces a functor $\mathscr{J} \to \mathcal{T}_G$. We know that X is determined by its values on the subcategory $\mathscr{J}$ by Lemma 9.1.8. We will write the inclusion functor as

$$i : \mathscr{J} \to \mathscr{J}_G.$$

It induces a precomposition functor

$$i^* : \mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G] \to [\mathscr{J}, \mathcal{T}^G]. \tag{9.3.1}$$

We will see that is an equivalence of categories in Theorem 9.3.10. This result is originally due to [MM02, Theorem V.1.5]. The latter category is isomorphic to $[\mathcal{B}G, \mathcal{S}p]$ (where $\mathcal{B}G$ is the one object category associated with the group G as in Example 2.9.1), that is the category of **ordinary orthogonal spectra equipped with G-actions**.

In §10.1 we will consider the category $[\mathcal{B}_T G, \mathcal{S}p]$ for a finite G-set T. Its objects are diagrams of spectra indexed by the groupoid $\mathcal{B}_T G$. When $T = G/H$, this category is equivalent to $\mathcal{S}p^H$. The category $[\mathcal{B}_T G, \mathcal{S}p]$ is the product of such categories over the orbits of T, so it could be the product of categories involving more than one subgroup of G. Using it we can define indexed (by a finite G-set T) wedges and indexed smash products of orthogonal spectra.

Since this is what one might first guess what G-equivariant spectra should be, such objects are commonly called **naive G-spectra**, the term we use in Definition 9.3.2, to distinguish them from the **genuine G-spectra** of Definition 9.0.2.

However these terms are misleading. The category $[\mathscr{J}, \mathcal{T}^G]$ has all the information we need. Indeed Schwede in [Sch14, Definition 2.1] **defines** orthogonal G-spectra this way, adding

> Readers familiar with other accounts of equivariant stable homotopy theory may wonder immediately why no orthogonal representations of the group G show up in the definition of equivariant spectra. The reason is that they are secretly already present: the actions of the orthogonal groups encode enough information so that we can evaluate an orthogonal G-spectrum on a G-representation.

What one must be careful about is the **homotopical structure** (see Definition 5.1.1) one puts on this category. There is what we call the **naive homotopical structure** on $[\mathscr{J}, \mathcal{T}^G]$ specified in (9.3.3) which is **not** homotopically equivalent to the stable homotopical structure on $\mathcal{S}p^G$. This homotopical distinction between the two categories is illustrated in Example 9.3.11. Then there is another one we call the **genuine homotopical structure**, specified in (9.3.13).

While the set of (9.3.13) does not contain that of (9.3.3), the set of weak equivalences generated by the former does contain all of those generated by the latter. Hence the genuine homotopical structure has more weak equivalences than the naive one.

Both structures are associated with Bousfield localizations of the projective model structure on $[\mathscr{J}, \mathcal{T}^G]$ in which a map $f : X \to Y$ of spectra is a weak equivalence iff $f_n : X_n \to Y_n$ is a weak equivalence in $\mathcal{T}^G$ (meaning that for each $H \subseteq G$, f_n^H is a weak equivalence in $\mathcal{T}$) for each $n \geqslant 0$. Then, as in §7.4A we enlarge the set of weak equivalences by requiring it to include a specified set of additional maps. These are indicated in (9.3.3) in the naive case and by (9.3.13) in the genuine case. The latter is the image under the functor i^* of (9.3.1) of a the set of maps used to define the stable model structure on $\mathcal{S}p^G$. Thus the genuine homotopical structure on $[\mathscr{J}, \mathcal{T}^G]$ is pulled back from the stable one on $\mathcal{S}p^G$ as in Proposition 5.1.6 and is therefore equivalent to it.

9.3A Homotopical structures

Definition 9.3.2. *A* **naive G-spectrum** *is a $\mathcal{T}^G$-functor (see Definition 3.1.13) $\mathscr{J} \to \mathcal{T}_G$, or equivalently an ordinary orthogonal spectrum equipped with an action of G. (Here we are regarding $\mathscr{J}$ as a category enriched over $\mathcal{T}^G$. This enrichment is derived from its usual one over $\mathcal{T}$ by regarding its morphism objects as G-spaces with trivial G-action.) The category of naive G-spectra (and equivariant maps) will be denoted by $\mathcal{S}p_G^{naive}$ ($\mathcal{S}p^G_{naive}$). It is the functor category $[\mathscr{J}, \mathcal{T}_G]$ ($[\mathscr{J}, \mathcal{T}^G]$) as in Definition 3.2.18.*

Since $\mathscr{J} = \mathscr{J}_{S^1}^{\mathbf{O}}$ is a $\mathscr{J}_{S^1}^{\mathbf{\Sigma}}$-algebra as in Definition 7.2.19, naive G-spectra are smashable spectra and the machinery of §7.4 applies to the category $\mathcal{S}p^G_{naive}$. The category $\mathscr{J}$ has a positive ideal $\mathscr{L}$ as in Definition 7.2.19, the subcategory of positive dimensional vector spaces. $\mathcal{S}p^G_{naive}$ has a set of stabilizing maps as in Definition 7.4.8, namely

$$\mathcal{S}_{naive} = \left\{ \xi_{m,n} \colon S^n \wedge S^{-m} \wedge S^{-n} \to S^{-m} \colon m \geqslant 0, n > 0 \right\}. \tag{9.3.3}$$

This set defines a stable homotopical structure on the category of naive G-spectra, which we will refer to as **the naive homotopical structure**. The genuine alternative is given below in (9.3.13).

The fibrant replacement functor Θ^∞ of Definition 7.4.26 is such that

$$(\Theta^\infty_{naive} X)_k = \operatorname*{hocolim}_n \Omega^n X_{n+k}, \tag{9.3.4}$$

which is a pointed G-space. We will call it the **naive fibrant replacement**. As in Theorem 7.4.29, a map $f : X \to Y$ is a stable equivalence iff $\Theta^\infty f$ is a projective weak equivalence, and this condition involves the Bredon model structure of Definition 8.6.1 on $\mathcal{T}^G$. This leads to naive stable homotopy groups, namely

$$\pi^G_{V,naive}X = \operatorname*{colim}_n \pi^G_V \Omega^n X_n \cong \operatorname*{colim}_n \pi^G_{V+n} X_n, \tag{9.3.5}$$

which should be compared to (9.1.2).

$\mathcal{S}p^G_{naive}$ has the four model structures of Definition 7.4.36, and they have cofibrant generating sets similar to those indicated in Theorem 9.2.13. We will not dwell on this because the naive homotopical structure associated with (9.3.3) **is not the one we want to use**. The reasons for this will be indicated in Example 9.3.11 below. An alternative homotopical structure on $\mathcal{S}p^G_{naive}$ that **is** equivalent to the stable one on $\mathcal{S}p^G$ will be given below in Proposition 9.3.16.

Recall that a **genuine G-spectrum** is a $\mathcal{T}^G$-functor $\mathscr{J}_G \to \mathcal{T}_G$ as in Definition 9.0.2.

Remark 9.3.6. The relation between $\mathscr{J}$ and $\mathscr{J}_G$. *There is a functor $i : \mathscr{J} \to \mathscr{J}_G$ endowing a finite dimensional orthogonal real vector space V with trivial G-action, and a forgetful functor $u : \mathscr{J}_G \to \mathscr{J}$ sending an orthogonal representation W to $|W|$. Then $\mathscr{J}(V, |W|)$ and $\mathscr{J}_G(i(V), W)$ are isomorphic as topological spaces but not as G-spaces since the former has trivial G-action while the latter may not. Hence $\mathscr{J}$ and $\mathscr{J}_G$ are equivalent as $\mathcal{T}$-categories but not as $\mathcal{T}_G$-categories.*

As noted above in Lemma 9.1.8, a functor on $\mathscr{J}_G$ is determined by its value on $\mathscr{J}$, meaning its precomposition with the inclusion functor $i : \mathscr{J} \to \mathscr{J}_G$. For a spectrum E, for each representation V we have an equivariant homeomorphism

$$E_V \approx O(|V|, V) \underset{O(|V|)}{\times} E_{|V|}, \tag{9.3.7}$$

where $|V|$ here denotes the vector space V with trivial G-action. We will show that the categories of naive and genuine G-spectra are equivalent. **However the homotopy theories of the two categories are different.** The category $\mathcal{S}p^G$ has more stable weak equivalences than $\mathcal{S}p^G_{naive}$. We will give illustrate this below in Example 9.3.11. This means that that $\mathcal{S}p^G$ and $\mathcal{S}p^G_{naive}$, with the homotopical structure of (9.3.3) on the latter, are **not** equivalent as homotopical categories as in Definition 5.1.1.

More explicitly, for a naive G-spectrum E, consider the diagram

$$\begin{tikzcd} \mathscr{J} \arrow[rr, "E"] \arrow[dr, "i"'] & & \mathcal{T}^G \\ & \mathscr{J}_G \arrow[ur, dashed, "i_! E"'] & \end{tikzcd} \tag{9.3.8}$$

where $i_!E$ is the left Kan extension of E along i. Using the formula of Proposition 3.2.35, we have

$$\begin{aligned}(i_!E)_V &= \int_{\mathscr{J}} \mathscr{J}_G(\mathbf{R}^n, V) \wedge E_n \cong \mathscr{J}_G(|V|, V) \wedge_{O(|V|)} E_{|V|} \\ &\cong O(|V|, V) \ltimes_{O(|V|)} E_{|V|}.\end{aligned}$$

It follows that for a genuine G-spectrum X,

$$\begin{aligned}(i_!i^*X)_V &= \mathscr{J}_G(|V|, V) \wedge_{O(|V|)} (i^*X)_{|V|} = \mathscr{J}_G(|V|, V) \wedge_{O(|V|)} X_{|V|} \\ &\cong X_V \qquad \text{by Lemma 9.1.8},\end{aligned}$$

so

$$i_!i^*X \cong X. \tag{9.3.9}$$

On the other hand, the functor $i^*i_!$ is the identity functor on $\mathcal{S}p^G_{naive}$ by definition. Hence we have proved

Theorem 9.3.10. Categorical equivalence of naive and genuine G-spectra. *Let $i : \mathscr{J} \to \mathscr{J}_G$ be the inclusion functor. Then the adjoint functors*

$$\begin{tikzcd} \mathcal{S}p^G_{naive} \arrow[r, shift left, "i_!"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G \arrow[l, shift left, "i^*"] \end{tikzcd}$$

given by restriction and left Kan extension along i are inverse equivalences of enriched symmetric monoidal categories. The same functors give equivalences relating $\mathcal{S}p_G^{naive}$ and $\mathcal{S}p_G$.

Alternate proof that $\mathcal{S}p^G$ and $\mathcal{S}p^G_{naive}$ are equivalent The following argument makes no use of Kan extensions. From Definition 9.0.2 we have

$$\begin{aligned}\mathcal{S}p^G &= [\mathscr{J}_G, \mathcal{T}^G] = [\mathscr{J}_G, [\mathcal{B}G, \mathcal{T}]] \\ &\cong [\mathscr{J}_G \wedge \mathcal{B}G, \mathcal{T}] &&\text{by Proposition 3.2.23} \\ &\cong [\overline{\mathscr{J}}_G \wedge \mathcal{B}G, \mathcal{T}] &&\text{by Proposition 3.1.50,}\end{aligned}$$

where $\overline{\mathscr{J}}_G$ denotes the $\mathcal{T}_G$-category $\mathscr{J}_G$ with trivial G-action on its morphism objects. Using Proposition 3.2.23 again we have

$$[\overline{\mathscr{J}}_G \wedge \mathcal{B}G, \mathcal{T}] \cong [\overline{\mathscr{J}}_G, [\mathcal{B}G, \mathcal{T}]] = [\overline{\mathscr{J}}_G, \mathcal{T}^G].$$

Next we claim that $\overline{\mathscr{J}}_G$ is equivalent to $\mathscr{J}$. The objects in the former are nominally finite dimensional orthogonal representations of G, but the G-action plays no role in the morphism objects. We have functors $u : \overline{\mathscr{J}}_G \to \mathscr{J}$ (the forgetful functor) and $i : \mathscr{J} \to \overline{\mathscr{J}}_G$, the evident inclusion functor. The composite ui is the identity functor on $\mathscr{J}$. There is a natural equivalence $\theta : 1_{\overline{\mathscr{J}}_G} \Rightarrow iu$ where $\theta_V : V \to |V|$ is the isomorphism underlain by the identity map on each vector space V.

This equivalence means that the functor categories $[\overline{\mathscr{J}}_G, \mathcal{T}^G]$ and $[\mathscr{J}, \mathcal{T}^G]$ are equivalent by Proposition 3.2.24.

Thus we have

$$\begin{aligned}
[\overline{\mathscr{J}}_G, \mathcal{T}^G] &\simeq [\mathscr{J}, \mathcal{T}^G] = [\mathscr{J}, [\mathcal{B}G, \mathcal{T}]] \\
&\cong [\mathscr{J} \wedge \mathcal{B}G, \mathcal{T}] \qquad \text{by Proposition 3.2.23 again} \\
&= [\mathcal{B}G \wedge \mathscr{J}, \mathcal{T}] \\
&\cong [\mathcal{B}G, [\mathscr{J}, \mathcal{T}]] = [\mathcal{B}G, \mathcal{S}p] = \mathcal{S}p^G_{naive}. \qquad \square
\end{aligned}$$

The following example shows that while $\mathcal{S}p^G$ and $\mathcal{S}p^G_{naive}$ for a nontrivial group G are equivalent as categories, **they are not equivalent as homotopical categories**, assuming we use the naive homotopical structure on $\mathcal{S}p^G_{naive}$ associated with (9.3.3).

Example 9.3.11. Why we need genuine G-spectra. *Let $G = C_2$ and let σ be its sign representation. The regular representation ρ is $\sigma + 1$. We will show that the map of (7.2.68)*

$$s_\rho : S^{-\rho} \wedge S^\rho \to S^{-0},$$

which is one of the stabilizing maps of Definition 7.4.8, is a stable equivalence in $\mathcal{S}p^G$ but **not** *in $\mathcal{S}p^G_{naive}$. Here we are using the naive homotopical structure of (9.3.3) on $\mathcal{S}p^{C_2}_{naive}$ and the usual stable one on $\mathcal{S}p^{C_2}$. This means that the forgetful functor i^* of Theorem 9.3.10 is* **not homotopical.**

Each representation V of C_2 has the form $m\sigma \oplus n$ for integers $m, n \geqslant 0$. We have

$$\mathscr{J}_G(a\sigma \oplus b, c\sigma \oplus d)^G \cong O(a,c) \ltimes \mathscr{J}(b,d) \tag{9.3.12}$$

by Proposition 8.9.30. In particular it is a point if $a > c$ or $b > d$.

Working in $\mathcal{S}p^G_{naive}$, we have

$$(S^{-\rho})_n = \mathscr{J}_G(\sigma + 1, n),$$

so $(S^{-\rho} \wedge S^\rho)^G_n = *$ for all n, and $\pi^G_*(S^{-\rho}) = 0$. Hence the fixed point spectrum of $S^{-\rho}$ is contractible, so the same is true of $(S^{-\rho} \wedge S^\rho)$.

On the other hand, $(S^{-0})_n = S^n$ with trivial G-action, so $\pi^G_*(S^{-0})$ is nontrivial. **This means that $S^{-\rho} \wedge S^\rho$ and S^{-0} are homotopically distinct as naive G-spectra** *because they have homotopically distinct fixed point sets, namely $*$ and S^{-0} respectively.*

In $\mathcal{S}p^G$, we have

$$(S^{-\rho} \wedge S^\rho)_{m\sigma\oplus n} = \mathscr{J}_G(\sigma + 1, m\sigma \oplus n) \wedge S^{\sigma+1},$$

so for $m > 0$

$$\begin{aligned}
(S^{-\rho} \wedge S^\rho)^G_{m\sigma\oplus n} &= \left(\mathscr{J}_G(\sigma + 1, m\sigma \oplus n) \wedge S^{\sigma+1}\right)^G \\
&\cong O(1,m) \ltimes \mathscr{J}(1,n) \wedge S^1
\end{aligned}$$

$$\begin{aligned} &\qquad \textit{by Proposition 8.9.30} \\ &\simeq (S^0 \vee S^{m-1}) \wedge (S^n \vee S^{2n-1}) \\ &\qquad \textit{by Example 8.9.28(iv)} \\ &\cong S^n \wedge (S^0 \vee S^{m-1}) \wedge (S^0 \vee S^{n-1}) \\ \textit{and} \qquad (S^{-0})^G_{m\sigma\oplus n} &= (S^{m\sigma\oplus n})^G = S^n, \end{aligned}$$

and the map s_ρ induces an isomorphism in π_*^G. In particular we have

$$\begin{aligned} (S^{-\rho} \wedge S^\rho)^G_{n\rho} &\cong O(1,n) \ltimes \mathscr{J}(1,n) \wedge S^1 \\ &\cong S^n \wedge (S^0 \vee S^{n-1})^{\wedge 2} \\ \textit{and} \qquad (S^{-0})^G_{n\rho} &= (S^{n\rho})^G = S^n. \end{aligned}$$

The map underlying s_ρ is s_2, which we have already seen to be a stable equivalence of ordinary spectra. **It follows that s_ρ also induces an isomorphism in π_* and is therefore a stable equivalence of genuine G-spectra.**

In order to avoid the difficulty of Example 9.3.11, we will replace the naive homotopical structure on $\mathcal{S}p^G_{naive}$ by one pulled back along $i_!$ as in Proposition 5.1.5 from the stable structure on $\mathcal{S}p^G$. This means we also change the definition of stable homotopy groups from that of (9.3.5) to that of (9.1.2), with the spaces $X_{n\rho}$ defined by (9.3.15) below.

We will refer to it as the **genuine homotopical structure**. As Example 9.3.11 illustrates, this structure has more weak equivalences than the naive one.

This means replacing the set of stabilizing maps of (9.3.3) with $i^*\mathcal{S}$ for $\mathcal{S}$ as in (7.4.9). More explicitly we get

$$\mathcal{S}^G_{genuine} = \left\{\xi_{V,n} \colon S^{n\rho} \wedge S^{-V} \wedge S^{-n\rho} \to S^{-V} : n > 0\right\}. \tag{9.3.13}$$

Here we are abusing notation because the spectrum we are calling S^{-V} is not a Yoneda spectrum with respect to the indexing category $\mathscr{J}$. In this setting the Yoneda functor $よ^V$ is defined only if V has trivial G-action. Nonetheless we can define a naive G-spectrum S^{-V} for arbitrary V by

$$\left(S^{-V}\right)_n = \mathscr{J}_G(V, n),$$

where the object on the right is the pointed G-space of Definition 8.9.24. This spectrum is the image under the forgetful functor i^* of the spectrum of the same name in $\mathcal{S}p^G$.

The corresponding fibrant replacement functor $\Theta^\infty_{genuine}$, the replacement for that of (9.3.4), is given by

$$(\Theta^\infty_{genuine} X)_k = \operatorname*{hocolim}_n \Omega^{n\rho_G} X_{n\rho_G + k}, \tag{9.3.14}$$

where the definition of the space $X_{n\rho_G+k}$ is suggested by the homeomorphism of (9.3.7), namely

$$X_V = O(|V|, V) \underset{O(|V|)}{\times} V_{|V|}, \tag{9.3.15}$$

for each representation V, in particular for $V = n\rho_G + k$.

Proposition 9.3.16. The homotopical equivalence of naive and genuine G-spectra. *The functors i^* and $i_!$ of Theorem 9.3.10,*

$$\begin{tikzcd} \mathcal{S}p^G_{naive} \arrow[r, shift left, "i_!"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G \arrow[l, shift left, "i^*"] \end{tikzcd},$$

are homotopical with respect to the genuine homotopical structure on $\mathcal{S}p^G_{naive}$ defined by (9.3.13) and the stable homotopical structure on $\mathcal{S}p^G$.

Proof It suffices to show that the two functors send stabilizing maps in one category to weak equivalences in the other. For i^* this is obvious since the stabilizing maps of $\mathcal{S}p^{naive}_G$ are defined to be the images under i^* of those in $\mathcal{S}p^G$. For the converse, (9.3.9) implies that $i_!$ sends the set of genuine stabilizing maps of $\mathcal{S}p^{naive}_G$, $\mathcal{S}^G_{genuine}$ as in (9.3.13), to those of $\mathcal{S}p^G$. □

9.3B Model structures

The hypotheses of Corollary 5.2.23 are met by the categorical equivalence of Theorem 9.3.10 and Proposition 9.3.16, so we have the following.

Corollary 9.3.17. Eight left induced model structures. *Each of the eight model structures on $\mathcal{S}p^G$ of Theorem 9.2.13 is Quillen equivalent to one on $\mathcal{S}p^G_{naive}$ that is left induced as in Definition 5.2.19.*

Recall that $\mathcal{S}p^{\mathcal{B}H} = \mathcal{S}p^H_{naive}$ is equivalent to $\mathcal{S}p^{\mathcal{B}_{G/H}G}$ by Corollary 2.1.40. The following will be helpful in §10.1.

Theorem 9.3.18. Eight model structures on $\mathcal{S}p^{\mathcal{B}_{G/H}G}$. *Consider the diagram*

$$\begin{tikzcd} \mathcal{S}p^{\mathcal{B}_{G/H}G} \arrow[r, shift right, "j^*"'] \arrow[r, phantom, "\perp"] & \mathcal{S}p^{\mathcal{B}H} = \mathcal{S}p^H_{naive} \arrow[l, shift right, "k^*"'] \arrow[r, shift left, "i_!"] \arrow[r, phantom, "\perp"] & \mathcal{S}p^H \arrow[l, shift left, "i^*"] \end{tikzcd}.$$

The eight cofibrantly generated model structures on $\mathcal{S}p^H$ transfer to ones on $\mathcal{S}p^{\mathcal{B}H}$ and on $\mathcal{S}p^{\mathcal{B}_{G/H}G}$ and both adjunctions are Quillen equivalences.

Proof Both composite functors on the middle category, j^*k^* and $i^*i_!$, are the identity functor. We have seen in Corollary 9.3.17 that each of the eight cofibrantly generated model structures on $\mathcal{S}p^H$ transfer to ones on $\mathcal{S}p^{\mathcal{B}H}$ through the adjunction on the right. The adjunction on the left satisfies the hypotheses of Corollary 5.2.31, so the model structures on $\mathcal{S}p^{\mathcal{B}H}$ transfer to ones on $\mathcal{S}p^{\mathcal{B}_{G/H}G}$. □

Remark 9.3.19. The equifibrant model structure via coverings. *In this setting there is alternative interpretation to enlarging the model structure via Theorem 5.2.34. For each subgroup $K \subseteq H$ we have a surjective map of G-sets $r : G/K \to G/H$, which induces a finite covering $p : \mathcal{B}_{G/K}G \to \mathcal{B}_{G/H}G$ as in Example 2.9.1. This in turn induces an indexed wedge*

$$p_*^{\vee} : \mathcal{S}p^{\mathcal{B}_{G/K}G} \to \mathcal{S}p^{\mathcal{B}_{G/H}G} \tag{9.3.20}$$

as in (5.5.34). Here the superscript on p refers to the wedge operation, the monoidal structure with respect to which the indexed product is defined.

For a representation V of K, consider the map

$$H \underset{K}{\ltimes} S^{-V} \wedge (S_+^{n-1} \to D_+^n) \qquad \text{in } \mathcal{S}p^{\mathcal{B}_{G/H}G},$$

where it is the pullback of a generating cofibration in the equifibrant model structure for $\mathcal{S}p^H$. It is the image under the functor $p_*^{\vee}$ as in (9.3.20) of the map

$$S^{-V} \wedge (S_+^{n-1} \to D_+^n) \qquad \text{in } \mathcal{S}p^{\mathcal{B}_{G/K}G},$$

which is a generating cofibration in the model structure pulled back from the projective one in $\mathcal{S}p^K$.

The generating trivial cofibrations are the maps of the form

$$p_*^{\vee}(S^{-V} \wedge (I_+^{n-1} \to I_+^n))$$

and those constructed as the corner map formed by smashing

$$p_*^{\vee}\left(S^{-V\oplus W} \wedge S^W \to \tilde{S}_W^{-V}\right) \tag{9.3.21}$$

with the maps $S_+^{n-1} \to D_+^n$. As in (7.4.17), the map (9.3.21) is extracted from the factorization

$$S^{-V\oplus W} \wedge S^W \to \widetilde{S}_W^{-V} \to S^{-V} \tag{9.3.22}$$

by applying the small object construction in the category of equivariant G/K-diagrams using the generating cofibrations. The map $\widetilde{S}_W^{-V} \to S^{-V}$ is a stable weak equivalence.

We will now generalize and consider the diagram category $\mathcal{S}p^{\mathcal{B}_T G}$ for a finite G-set T. Each such T is a union of orbits G/G_t, so we have

$$\mathcal{B}_T G \cong \coprod_t \mathcal{B}_{G/G_t}G \qquad \text{and} \qquad \mathcal{S}p^{\mathcal{B}_T G} \cong \prod_t \mathcal{S}p^{\mathcal{B}_{G/G_t}G}. \tag{9.3.23}$$

This isomorphism depends on the choice of an element t in each orbit of T. It leads to the following, whose proof is similar to that of Theorem 9.3.18.

Corollary 9.3.24. Eight model structures on $\mathcal{S}p^{\mathcal{B}_T G}$. *Let T be a finite G-set as in (9.3.23), and consider the diagram*

$$\begin{tikzcd} \mathcal{S}p^{\mathcal{B}_T G} \arrow[r, shift right, "j^*"'] \arrow[r, phantom, "\perp"] & \prod_t \mathcal{S}p^{\mathcal{B}G_t} \arrow[l, shift right, "k^*"'] = \mathcal{S}p^{G_t}_{naive} \arrow[r, shift left, "i_!"] \arrow[r, phantom, "\perp"] & \prod_t \mathcal{S}p^{G_t}. \arrow[l, shift left, "i^*"] \end{tikzcd}$$

The eight cofibrantly generated model structures on the right (which are products over the orbits of T of the ones given in Theorem 9.2.13) transfer to ones on the two other categories and both adjunctions are Quillen equivalences.

To be more explicit, a map of T-diagrams $X \to Y$ is a weak equivalence iff for each orbit $G/G_t \subseteq T$, the map $X_t \to Y_t$ is a weak equivalence in $\mathcal{S}p^{\mathcal{B}G_t}$. Note here that t is an element of T rather than of $\mathscr{J}_{G_t}$, and X_t is a G_t-spectrum rather than a pointed space.

The generating cofibrations are maps in which the tth component has the form

$$G_{t+} \underset{H_t}{\wedge} S^{-V_t} \wedge (S_+^{n_t-1} \to D_+^{n_t}) \tag{9.3.25}$$

in which V_t is a representation of $H_t \subseteq G_t$. They can be expressed without reference to points and stabilizers as an indexed wedge

$$p_*^{\vee} \left(S^{-V} \wedge \left(S_+^{n_*-1} \to D_+^{n_*} \right) \right) \tag{9.3.26}$$

as in (5.5.34), in which $p : T' \to T$ a finite surjective map of G-sets in which the preimage of the orbit G/G_t is G/H_t, and V is a representation of T' as in Definition 8.9.10. Here n_* is a function assigning a nonnegative integer n_t to each orbit. The dimensions of the sphere and disk may vary with the orbit in T'.

The generating trivial cofibrations can be described in similar terms, generalizing the description of the single orbit case given in Remark 9.3.19.

9.4 Homotopical properties of G-spectra

9.4A Exact sequences

The category $\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G]$ of G-spectra is exactly stable as in Definition 5.7.3, so we have the Puppe exact sequences of Theorem 5.7.6 and the Adams exact sequence of Theorem 5.7.11, which we repeat here for convenience.

Corollary 9.4.1. Exact sequences for G-spectra.

(i) *Given a stable fiber sequence in $\mathcal{S}p^G$*

$$F \xrightarrow{i} X \xrightarrow{p} Y \qquad \text{with a right action } F \wedge \Omega Y \xrightarrow{m} F$$

as in (4.7.7) and a cofibrant spectrum A, we have the Puppe long exact sequence

$$\cdots \xrightarrow{(\Omega^q\partial)_*} \pi(A, \Omega^q F) \xrightarrow{(\Omega^q i)_*} \pi(A, \Omega^q X) \xrightarrow{(\Omega^q p)_*} \pi(A, \Omega^q Y) \xrightarrow{(\Omega^{q-1}\partial)_*} \cdots$$

for all integers q.

(ii) Dually, given a cofiber sequence

$$X \xrightarrow{u} Y \xrightarrow{v} C \qquad \text{with a right coaction } C \xrightarrow{m'} C \vee \Sigma X$$

as in (4.7.8) and a stably fibrant spectrum Z, we have the Puppe long exact sequence

$$\cdots \xrightarrow{(\Sigma^q\delta)^*} \pi(\Sigma^q C, Z) \xrightarrow{(\Sigma^q v)^*} \pi(\Sigma^q Y, Z) \xrightarrow{(\Sigma^q u)^*} \pi(\Sigma^q X, Z) \xrightarrow{(\Sigma^{q-1}\delta)^*} \cdots$$

for all integers q.

(iii) For a cofibrant spectrum W, we have the Adams long exact sequence

$$\cdots \xrightarrow{(\Sigma^{q-1}\delta)_*} \pi(W, \Sigma^q X) \xrightarrow{(\Sigma^q u)_*} \pi(W, \Sigma^q Y) \xrightarrow{(\Sigma^q v)_*} \pi(W, \Sigma^q C) \xrightarrow{(\Sigma^q\delta)_*} \cdots$$

for all integers q.

In particular, when $A = G \underset{H}{\ltimes} S^V \wedge S^{-0}$ for a representation V of a subgroup $H \subseteq G$, Corollary 9.4.1(i) gives us the following.

Proposition 9.4.2. Fiber sequences in $\mathcal{S}p^G$. *Let $f : X \to Y$ be a map of G-spectra, and let F be its fiber, as in Definition 4.7.6. Then for each subgroup $H \subseteq G$ and representation V of H, there is a long exact sequence*

$$\cdots \xrightarrow{\partial_*} \pi_V^H(F) \xrightarrow{i_*} \pi_V^H(X) \xrightarrow{p_*} \pi_V^H(Y) \xrightarrow{\partial_*} \pi_{V-1}^H(F) \xrightarrow{i_*} \cdots$$

The above and the following results are originally proved in [MM02, III.3.5] and with more generality in [MMSS01, 7.4 (iv)]. For us they are special cases of Corollary 9.4.1.

Proposition 9.4.3. Cofiber sequences in $\mathcal{S}p^G$. *Let $f : X \to Y$ be an equivariant map of G-spectra with mapping cone C_f as in Definition 4.7.6.*

(i) For any G-spectrum Z and subgroup $H \subseteq G$ there is a natural long exact sequence

$$\cdots \leftarrow [X, Z]^H \leftarrow [Y, Z]^H \leftarrow [C_f, Z]^H \leftarrow [\Sigma X, Z]^H \leftarrow \cdots$$

where the map $[C_f, Z]^H \leftarrow [\Sigma X, Z]^H$ is induced by the map $C_f \to S^1 \wedge X$ and the suspension isomorphism of Proposition 9.1.6 for $W = 1$.

(ii) *For any G-spectrum W and subgroup $H \subseteq G$ there is a natural long exact sequence*

$$\cdots \to [W,X]^H \to [W,Y]^H \to [W,C_f]^H \to [\Sigma^{-1}W,X]^H \to \cdots$$

where the map $[W,C_f]^H \to [\Sigma^{-1}W,X]^H$ is induced by the map $C_f \to S^1 \wedge X$ as above.

In particular (the case where W is the sphere spectrum S^{-0}) there is a natural long exact sequence

$$\cdots \to \pi_k^H X \to \pi_k^H Y \to \pi_k^H C_f \to \pi_{k-1}^H X \to \cdots . \qquad (9.4.4)$$

(iii) *If f is an h-cofibration (Definition 5.6.7), then map $C_f \to Y/X$ is a stable equivalence and we can replace C_f by Y/X in the long exact sequences above.*

Proof The first two statements follow from Corollary 9.4.1, while the third follows from Corollary 5.6.9. □

Proposition 9.4.3 implies that the formation of mapping cones is homotopical as is the formation of quotients of h-cofibrations. It also gives parts (i) and (iii) of the Proposition below. Part (ii) follows from the fact that the formation of unstable homotopy groups commutes with products and the fact that filtered colimits commute with finite products.

Proposition 9.4.5. Products and coproducts in $\mathcal{S}p^G$.

(i) *For any set of spectra $\{X_\alpha\}$ the map*

$$\bigoplus \pi_*^G X_\alpha \to \pi_*^G \bigvee X_\alpha$$

is an isomorphism, hence the formation of wedges is homotopical.

(ii) *For any any finite set of spectra $\{X_\alpha\}$ the map*

$$\pi_*^G \prod X_\alpha \to \prod \pi_*^G X_\alpha$$

is an isomorphism, hence the formation of finite products is homotopical.

(iii) *For any finite set of spectra $\{X_\alpha\}$ the map*

$$\bigvee X_\alpha \to \prod X_\alpha$$

is a weak equivalence.

We will see an indexed analog of the above and the following in Proposition 9.6.4 below.

Corollary 9.4.6. *The category $\operatorname{Ho}\mathcal{S}p^G$ is additive, and admits finite products and arbitrary coproducts. The coproducts are given by wedges and the finite products by finite wedges.*

Proof We start with the case of coproducts. Let J be a set. The adjoint functors

$$\bigvee : (\mathcal{S}p^G)^J \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}p^G : \Delta.$$

are homotopical by Proposition 9.4.5. They therefore induce adjoint functors

$$\bigvee : (\mathrm{Ho}\,\mathcal{S}p^G)^J \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{Ho}\,\mathcal{S}p^G : \Delta.$$

on the homotopy categories. This shows that arbitrary coproducts exist in $\mathrm{Ho}\,\mathcal{S}p^G$ and that they may be computed as wedges. A similar argument shows that finite products exist, are computed as products in $\mathcal{S}p^G$, and that the map from a finite coproduct to a finite product is an isomorphism. Given two morphisms

$$f, g : X \to Y,$$

consider the following diagram in $\mathcal{S}p^G$:

$$\begin{tikzcd} & X \vee X \arrow[r, "f \vee g"] \arrow[d] & Y \vee Y \arrow[r, "\varphi"] & Y \\ X \arrow[r, "\delta"] & X \times X, & & \end{tikzcd}$$

where δ is the diagonal map, φ is the fold map and the vertical map is a weak equivalence. In the corresponding diagram in $\mathrm{Ho}\,\mathcal{S}p^G$ (Definition 4.3.16) the vertical map is an isomorphism and hence has an inverse. The resulting composite is the morphism $f + g$. This endows the morphism sets in $\mathrm{Ho}\,\mathcal{S}p^G$ with the structure of commutative monoids.

To see that these monoids are abelian groups, it suffices to define a morphism of degree -1 on the sphere spectrum S^{-0} in $\mathrm{Ho}\,\mathcal{S}p_G$. Recall (Corollary 7.2.67(i)) that the sphere spectrum itself has no such morphism. However there is one on the space S^1 and hence on the spectrum $S^{-1} \wedge S^1$, and there is a map of (7.2.68)

$$s_1 : S^{-1} \wedge S^1 \to S^{-0}$$

which is a stable equivalence (without an inverse in $\mathcal{S}p_G$ by Corollary 7.2.67(ii)!) by definition. Thus we have a diagram

$$\begin{tikzcd} S^{-1} \wedge S^1 \arrow[r, "{[-1]}"] \arrow[d, "e_1"'] & S^{-1} \wedge S^1 \arrow[d, "e_1"] \\ S^{-0} \arrow[r, dashed] & S^{-0} \end{tikzcd}$$

in which the map on the left has an inverse in $\mathrm{Ho}\,\mathcal{S}p^G$, so we have the desired morphism there. $\square$

Proposition 9.4.7. A connective G-CW spectrum (Definition 9.1.34) is flat *as in Definition 5.1.20, meaning that the endofunctor $X \wedge (-)$ preserves colimits and weak equivalences.*

This will be generalized to Bredon cofibrant spectra as in Definition 9.2.15 below in Proposition 9.6.5.

Proof Smashing with S^{-V} shifts homotopy groups and therefore preserves stable equivalences. Since $\mathcal{S}p^G$ is closed symmetric monoidal, it is a left adjoint and therefore preserves colimits by Proposition 2.3.36.

We now proceed by skeletal induction. Attaching a cell to X leads to a cofiber sequence in the usual way. Thus we can use the long exact sequence of (9.4.4) to show that attaching a cell preserves flatness. □

9.4B Homotopy groups of G-spectra and Mackey functors

We first discuss some structure on the equivariant homotopy groups of a G-spectrum X. They can be defined in terms of finite G-sets T. For a finite G-set T let

$$\underline{\pi}_0^G X(T) = [\Sigma^\infty T_+, X]^G = \pi_0 \mathcal{S}p^G(\Sigma^\infty T_+, X), \tag{9.4.8}$$

be the set of homotopy classes of equivariant maps from $\Sigma^\infty T_+$ to the spectrum X. We will often omit G from the notation when it is clear from the context. This set has a natural abelian group structure, so we have an $\mathcal{A}b$-valued functor on $\mathscr{B}_G$ (see Definition 8.2.4). We will see that it is a Mackey functor; see Definition 8.2.3 and Definition 8.2.5.

For an orthogonal representation V of G, we define

$$\underline{\pi}_V X(T) = [S^V \wedge \Sigma^\infty T_+, X]^G. \tag{9.4.9}$$

As an $RO(G)$-graded contravariant abelian group valued functor of T, this converts disjoint unions to direct sums. This means it is determined by its values on the sets G/H for subgroups $H \subseteq G$.

When G is abelian, H is normal and $\underline{\pi}_V X(G/H)$ is a $Z[G/H]$-module. More generally it is a module over the Weyl group $W_H = N_H/H$, where N_H denotes the normalizer of H.

The following definition should be compared with [Ada84, (2.3)]. It is related in spirit to Proposition 3.1.50.

Definition 9.4.10. An equivariant homeomorphism. *Let X be a G-space and Y an H-space for a subgroup $H \subseteq G$. We define the equivariant homeomorphism*

$$\widetilde{u}_H^G(Y, X) : G \underset{H}{\times} (Y \times i_H^G X) \to (G \underset{H}{\times} Y) \times X$$

by $(g, y, x) \mapsto (g, y, g(x))$ for $g \in G$, $y \in Y$ and $x \in X$. We will use the same notation for a similarly defined homeomorphism

$$\widetilde{u}_H^G(Y, X) : G \underset{H}{\ltimes} (Y \wedge i_H^G X) \to (G \underset{H}{\ltimes} Y) \wedge X$$

for a G-spectrum X and H-spectrum Y. We will abbreviate

$$\widetilde{u}_H^G(S^{-0}, X) : G \underset{H}{\ltimes} i_H^G X \to G/H \ltimes X$$

by $\widetilde{u}_H^G(X)$.

For representations V and V' of G both restricting to W on H, but having distinct restrictions to all larger subgroups, we define

$$\widetilde{u}_{V-V'} = \widetilde{u}_H^G(S^V)\widetilde{u}_H^G(S^{V'})^{-1},$$

so the following diagram of equivariant homeomorphisms commutes:

$$\begin{tikzcd}
 & G/H \wedge S^V \\
G \underset{H}{\ltimes} S^W \arrow[ur, "\widetilde{u}_H^G(S^V)"] \arrow[dr, "\widetilde{u}_H^G(S^{V'})"'] & \\
 & G/H \wedge S^{V'}. \arrow[uu, "\widetilde{u}_{V-V'}"']
\end{tikzcd} \tag{9.4.11}$$

When $V' = |V|$ (meaning that $H = G_V$ acts trivially on W), then we abbreviate $\widetilde{u}_{V-V'}$ by $\widetilde{u}_V$.

The following will be proved below as Proposition 9.6.1.

Proposition 9.4.12. Equivalence of finite products and coproducts of G-spectra. *Let T and Y be a finite G-set and a G-spectrum. Then the standard map*

$$T \ltimes Y = \bigvee_{t \in T} Y \to \prod_{t \in T} Y = F_G(\Sigma^\infty T_+, Y)$$

(for $T \ltimes Y$ as in Definition 2.1.49 and the indexed coproduct and product as defined as in (9.1.32) and (9.1.31)) is a weak equivalence.

Given subgroups $K \subset H \subseteq G$, a fold map between the H-spectra $\Sigma^\infty H/H_+$ and $\Sigma^\infty H/K_+$ induced by the map $p : H/K \to H/H$ of H-sets. There is also a pinch map $\Sigma^\infty H/H_+ \to \Sigma^\infty H/K_+$ of H-spectra which we now describe. Note that the target $F_G(\Sigma^\infty T_+, Y)$ in Proposition 9.4.12 is a contravariant in T, so p induces a map

$$F_G(\Sigma^\infty H/K_+, S^{-0}) \leftarrow F_G(\Sigma^\infty H/H_+, S^{-0}),$$

which by Proposition 9.4.12 is weakly equivalent to a map

$$\Sigma^\infty H/K_+ \leftarrow \Sigma^\infty H/H_+.$$

Alternatively, choose a representation V of H with trivial restriction to K on which each element of H not in K acts nontrivially and choose a point $x \in S^V$ with isotropy group (see Definition 2.1.29(iv)) K. Then the Pontryagin-Thom construction along its orbit H/K leads to an H-equivariant map $S^V \to H \underset{K}{\times} S^{|V|}$. Composing with the inverse of the homeomorphism $\tilde{u}^H_K(*, S^V)$ we get

$$S^V \longrightarrow H \underset{K}{\times} S^{|V|} \xrightarrow{\tilde{u}^H_K(*,S^V)^{-1}} H/K \times S^V.$$

Applying Σ^∞ gives

$$S^{-0} \wedge S^V = \Sigma^\infty (H/H)_+ \wedge S^V \to (H/K) \ltimes S^V.$$

This leads to a diagram

$$\begin{tikzcd} & \Sigma^\infty H/H_+ \arrow[r, shift left, "\text{pinch}"] \arrow[d, Rightarrow, "G \underset{H}{\ltimes} (\cdot)"] & \Sigma^\infty H/K_+ \arrow[l, shift left, "\text{fold}"] & \\ \Sigma^\infty G/H_+ \arrow[r, equal] & G \underset{H}{\ltimes} \Sigma^\infty H/H_+ \arrow[r, shift left, "\text{pinch}"] & G \underset{H}{\ltimes} \Sigma^\infty H/K_+ \arrow[l, shift left, "\text{fold}"] \arrow[r, equal] & G/K_+. \end{tikzcd} \tag{9.4.13}$$

Definition 9.4.14. The Mackey functor structure maps in $\underline{\pi}^G_V X$. *The* **fixed point transfer and restriction maps**

$$\underline{\pi}_V X(G/H) \underset{\mathrm{Res}^H_K}{\overset{\mathrm{Tr}^H_K}{\rightleftarrows}} \underline{\pi}_V X(G/K)$$

are the ones induced by the composite maps in the bottom row of (9.4.13) and specified in Definition 8.2.3.

These satisfy the formal properties needed to make $\underline{\pi}_V X$ into a Mackey functor as in §8.2B. They are usually referred to simply as the transfer and restriction maps. We use the words "fixed point" to distinguish them from another similar pair of maps specified below in Definition 9.4.19.

When X is a ring spectrum, we have the **fixed point Frobenius relation**

$$\mathrm{Tr}^H_K(\mathrm{Res}^H_K(a)b) = a(\mathrm{Tr}^H_K(b)) \tag{9.4.15}$$

for $a \in \underline{\pi}_\star X(G/H)$ and $b \in \underline{\pi}_\star X(G/K)$. In particular this means that

$$a(\mathrm{Tr}^H_K(b)) = 0 \qquad \text{when} \mathrm{Res}^H_K(a) = 0. \tag{9.4.16}$$

For a representation V of G, the group

$$\underline{\pi}^G_V X(G/H) = \pi^H_V X = [S^V, X]^H$$

is isomorphic to

$$[S^{-0}, S^{-V} \wedge X]^H = \pi_0(S^{-V} \wedge X)^H.$$

However fixed points do not respect smash products of spectra (see Remark 9.1.27), so we cannot equate this group with

$$\pi_0(S^{-V^H} \wedge X^H) = [S^{V^H}, X^H] = \pi_{|V^H|} X^H = \underline{\pi}^G_{|V^H|} X(G/H).$$

Conversely a G-equivariant map $S^V \to X$ represents an element in

$$[S^V, X]^G = \pi^G_V X = \underline{\pi}^G_V X(G/G).$$

We also need notation for X as an H-spectrum for subgroups $H \subseteq G$. For this purpose we will enlarge the orthogonal representation ring of G, $RO(G)$, to the representation ring Mackey functor $\underline{RO}(G)$ of §8.2A.

Definition 9.4.17. **$\underline{RO}(G)$-graded homotopy groups.** *For each G-spectrum X and each pair (H, V) consisting of a subgroup $H \subseteq G$ and a virtual orthogonal representation V of H, let the G-Mackey functor $\underline{\pi}_{H,V}(X)$ be defined by*

$$\begin{aligned}\underline{\pi}_{H,V}(X)(T) &:= \left[(G \underset{H}{\ltimes} S^V) \wedge T_+, X\right]^G \\ &\cong \left[S^V \wedge i^G_H T_+, i^G_H X\right]^H = \underline{\pi}^H_V(i^G_H X)(i^G_H T),\end{aligned}$$

for each finite G-set T. Equivalently, $\underline{\pi}_{H,V}(X) = \uparrow^G_H \underline{\pi}^H_V(i^G_H X)$ (see Definition 8.2.9) as Mackey functors. We will often denote $\underline{\pi}_{G,V}$ by $\underline{\pi}^G_V$ or $\underline{\pi}_V$.

If V is a representation of H restricting to W on K, we can smash the diagram (9.4.13) with S^V and get

$$\begin{array}{c} S^V \underset{\text{fold}}{\overset{\text{pinch}}{\rightleftarrows}} H/K \ltimes S^V \\ \Big\Downarrow {\scriptstyle G \underset{H}{\ltimes} (\cdot)} \\ G \underset{H}{\ltimes} S^V \underset{\text{fold}}{\overset{\text{pinch}}{\rightleftarrows}} G \underset{H}{\ltimes} (H/K \ltimes S^V) \xrightarrow{\approx} G \underset{H}{\ltimes} (H \underset{K}{\ltimes} S^W) = G \underset{K}{\ltimes} S^W, \end{array} \tag{9.4.18}$$

where the homeomorphism is induced by that of Definition 9.4.10.

Definition 9.4.19. The group action restriction and transfer maps. *For subgroups $K \subseteq H \subseteq G$, let V be a representation of H restricting to W on K. The group action transfer and restriction maps*

$$\uparrow^G_H \underline{\pi}^H_V(i^G_H X) = \underline{\pi}_{H,V} X \underset{\underline{r}^H_K}{\overset{\underline{t}^{H,V}_K}{\rightleftarrows}} \underline{\pi}_{K,W} X = \uparrow^G_K \underline{\pi}^K_W(i^*_K X)$$

(see Definition 8.2.9) are the ones induced by the composite maps in the bottom row of (9.4.18). The symbols t and r here are underlined because they are maps **between** *Mackey functors rather than maps within Mackey functors as in Definition 8.2.3.*

We include V as an index for the group action transfer $\underline{t}_K^{H,V}$ because its target is not determined by its source.

Thus we have abelian groups $\underline{\pi}_{H',V}(X)(G/H'')$ for all subgroups $H', H'' \subseteq G$ and representations V of H'. Most of them are redundant in view of Theorem 9.4.21 below. In what follows, we will use the notation $H_\cap := H' \cap H''$ and $H_\cup := H' \cup H''$, the smallest subgroup containing both H' and H''.

Lemma 9.4.20. An equivariant module structure. *For a G-spectrum X and H'-spectrum Y,*

$$[G \underset{H'}{\ltimes} Y, X]^{H''} = \mathbf{Z}[G/H_\cup] \otimes [H_\cup \underset{H'}{\ltimes} Y, X]^{H''}$$

as $\mathbf{Z}[G/H'']$-modules.

Proof As abelian groups,

$$\begin{aligned} [G \underset{H'}{\ltimes} Y, X]^{H''} &\cong [i_{H''}^G(G \underset{H'}{\ltimes} Y), X]^{H''} \\ &\cong \left[\bigvee_{|G/H_\cup|} H_\cup \underset{H'}{\ltimes} Y, X\right]^{H''} \\ &\cong \bigoplus_{|G/H_\cup|} [H_\cup \underset{H'}{\ltimes} Y, X]^{H''} \end{aligned}$$

and G/H'' permutes the wedge summands of $\bigvee_{|G/H_\cup|} H_\cup \underset{H'}{\ltimes} Y$ as it permutes the elements of $G/H_\cup$. $\square$

Theorem 9.4.21. The module structure for $\underline{RO(G)}$-graded homotopy groups. *For subgroups $H', H'' \subseteq G$ with $H_\cup = \overline{H' \cup H''}$ and $H_\cap = H' \cap H''$, and a representation V of H' restricting to W on $H_\cap$,*

$$\begin{aligned} \underline{\pi}_{H',V} X(G/H'') &\cong \mathbf{Z}[G/H_\cup] \otimes \underline{\pi}_{H_\cap,W} X(G/G) \\ &\cong \mathbf{Z}[G/H_\cup] \otimes \underline{\pi}_W^{H_\cap} i_{H_\cap}^* X(H_\cap/H_\cap) \end{aligned}$$

as $\mathbf{Z}[G/H'']$-modules.

Suppose that H'' is a proper subgroup of H' and $\gamma \in H'$ is a generator. Then as an element in $\mathbf{Z}[G/H'']$, γ induces multiplication by -1 in $\underline{\pi}_{H',V} X(G/H'')$ iff V is nonorientable.

Proof We start with the definition and use the homeomorphism of Definition 9.4.10 and the module structure of Lemma 9.4.20.

$$\begin{aligned} \underline{\pi}_{H',V} X(G/H'') &\cong [(G \underset{H'}{\ltimes} S^V) \wedge G/H''_+, X]^G \\ &\cong [G \underset{H''}{\ltimes} (G \underset{H'}{\ltimes} S^V), X]^G \\ &\cong [G \underset{H'}{\ltimes} S^V, X]^{H''} \end{aligned}$$

$$
\begin{aligned}
& \cong \mathbf{Z}[G/H_\cup] \otimes [H_\cup \underset{H'}{\ltimes} S^V, X]^{H''} \\
\text{and} \qquad [H_\cup \underset{H'}{\ltimes} S^V, X]^{H''} & \cong [S^W, X]^{H_\cap} \cong [G \underset{H_\cap}{\ltimes} S^W, X]^G \\
& \cong \underline{\pi}_W^{H_\cap}(i^*_{H_\cap} X)(H_\cap/H_\cap) \cong \underline{\pi}_{H_\cap, W} X(G/G).
\end{aligned}
$$

For the statement about nonoriented V, we have

$$\underline{\pi}_{H',V} X(G/H'') \cong \mathbf{Z}[G/H'] \otimes \underline{\pi}_W^{H''} i^*_{H''} X(H''/H'') \cong \mathbf{Z}[G/H'] \otimes [S^W, X]^{H''}.$$

Then γ induces a map of degree ± 1 on the sphere depending on the orientability of V. $\square$

Theorem 9.4.21 means that we need only consider the groups

$$\underline{\pi}_{H,V} X(G/G) = \underline{\pi}_V^H i^*_H X(H/H).$$

When $H \subset G$ and V is a representation of G restricting to W on H, we have

$$\underline{\pi}_V X(G/H) \cong \underline{\pi}_{H,W} X(G/G). \tag{9.4.22}$$

This isomorphism makes the following diagram commute for $K \subseteq H$.

$$
\begin{tikzcd}
\underline{\pi}_V X(G/H) \arrow[r, "\cong"] \arrow[d, shift right, "\mathrm{Res}^H_K"'] & \underline{\pi}_{H,W} X(G/G) \arrow[d, shift right, "\underline{r}^H_K"'] \\
\underline{\pi}_V X(G/K) \arrow[r, "\cong"] \arrow[u, shift right, "\mathrm{Tr}^H_K"'] & \underline{\pi}_{K, i^*_K W} X(G/G) \arrow[u, shift right, "\underline{t}^{H,W}_K"']
\end{tikzcd}
$$

We will use these two groups of (9.4.22) interchangeably as convenient. Note that the group on the left is indexed by $RO(G)$ while the one the right is indexed by $RO(H)$. This means that if V and V' are representations of G each restricting to W on H, then $\underline{\pi}_V X(G/H)$ and $\underline{\pi}_{V'} X(G/H)$ are canonically isomorphic. The first of these is

$$[G/H \ltimes S^V, X]^G \cong [G \underset{H}{\ltimes} S^W, X]^G \cong [S^W, i^G_H X]^H$$

where the first isomorphism is induced by the homeomorphism $\widetilde{u}^G_H(X)$ of Definition 9.4.10 and the second is the fact that $G \underset{H}{\ltimes} (\cdot)$ is the left adjoint of the forgetful functor i^G_H.

For a ring spectrum X, such as the one we are studying in this paper, an indecomposable element in $\underline{\pi}_\star X(G/H)$ may map to a product in $\underline{\pi}_{H,\star} X(G/G)$ of elements in groups indexed by representations of H that are not restrictions of representations of G. This factoring can make some computations easier.

9.5 A homotopical approximation to the category of G-spectra

In this section we will introduce an auxiliary homotopical category $\pi^{\mathrm{st}}\mathcal{S}p^G$ (Definition 9.5.4) which approximates $\mathcal{S}p^G$ in that it has the same objects. On the other hand it is easier to compute with because it is enriched over abelian groups instead of topological spaces. It enables us to strengthen (in Proposition 9.5.18) our earlier statement (Proposition 9.4.3) about cofiber sequences and homotopy groups. It also enables us to prove (in Proposition 9.5.6) that for the map s_V of (7.2.68), the map $s_V \wedge X$ is a stable equivalence for any X.

In the next section we will use it to prove that finite products one coproducts coincide in $\mathcal{S}p^G$; see Proposition 9.6.1 and Corollary 9.6.3.

The following category is a variant of one introduced by Adams in [Ada84, §4]. The letters $\mathcal{SW}$ stand for Spanier-Whitehead. For Adams the colimit was the filtered one over the poset of all representations of G, while for us it is the sequential one over multiples of the regular representation. They can be shown to be the same using Theorem 2.3.82.

Definition 9.5.1. The Adams category $\mathcal{SW}^G$ *has finite pointed G-CW complexes (see Definition 8.4.13) as objects with*

$$\mathcal{SW}^G(X,Y) := \operatorname*{colim}_n\, [S^{n\rho}\wedge X, S^{n\rho}\wedge Y]^G,$$

where $\rho = \rho_G$ denotes the regular representation of G and $[-,-]^G$ denotes homotopy classes of maps in $\mathcal{T}^G$. The colimit is defined by smashing both source and target of a map

$$S^{n\rho}\wedge X \to S^{n\rho}\wedge Y$$

with S^ρ to get a map

$$S^{(n+1)\rho}\wedge X \to S^{(n+1)\rho}\wedge Y.$$

The composite of morphisms represented by

$$f : S^{m\rho}\wedge X \to S^{m\rho}\wedge Y \qquad \text{and} \qquad g : S^{n\rho}\wedge Y \to S^{n\rho}\wedge Z$$

is represented by $(S^{m\rho}\wedge g)(S^{n\rho}\wedge f)$.

We will construct a category $\pi^{\mathrm{st}}\mathcal{S}p^G$ (see Definition 9.5.4) that is tensored over $\mathcal{SW}^G$ (see Corollary 9.5.15 below) in which the objects are orthogonal G-spectra. For G-spectra X and Y, we define

$$(\pi^{\mathrm{st}}\mathcal{S}p^G)(X,Y) := \operatorname*{colim}_n \pi_0(\mathcal{S}p^G(X\wedge S^{n\rho}\wedge S^{-n\rho}, Y)). \tag{9.5.2}$$

Note here that $S^{-n\rho}$, X and Y are spectra while $S^{n\rho}$ is a space.

To define this sequential colimit we use the map of (7.2.69) for $V = \rho$, namely

$$S^\rho \wedge \xi_{n\rho,\rho} : S^{(n+1)\rho} \wedge S^{-(n+1)\rho} \to S^{n\rho} \wedge S^{-n\rho}, \tag{9.5.3}$$

which induces

$$\mathcal{S}p^G(X \wedge S^{(n+1)\rho} \wedge S^{-(n+1)\rho}, Y) \xleftarrow{(X \wedge S^\rho \wedge \xi_{n\rho,\rho})^*} \mathcal{S}p^G(X \wedge S^{n\rho} \wedge S^{-n\rho}, Y)$$

We want this to be the group of morphisms in a homotopical category having the same objects and same homotopy category as $\mathcal{S}p^G$, so we need to define composition. Given $f \in \pi^{\mathrm{st}}\mathcal{S}p^G(X,Y)$ and $g \in \pi^{\mathrm{st}}\mathcal{S}p^G(Y,Z)$ represented by

$$f_{m\rho} : X \wedge S^{m\rho} \wedge S^{-m\rho} \to Y \qquad \text{and} \qquad g_{n\rho} : Y \wedge S^{n\rho} \wedge S^{-n\rho} \to Z$$

the composition $g \cdot f$ is defined to be the equivalence class of the map

$$(g \cdot f)_{(m+n)\rho} : X \wedge S^{(m+n)\rho} \wedge S^{-(m+n)\rho} \to Z$$

constructed from the isomorphism

$$S^{(m+n)\rho} \wedge S^{-(m+n)\rho} \cong S^{n\rho} \wedge S^{-n\rho} \wedge S^{m\rho} \wedge S^{-m\rho}$$

and the composite is

$$X \wedge S^{n\rho} \wedge S^{-n\rho} \wedge S^{m\rho} \wedge S^{-m\rho} \xrightarrow{S^{n\rho} \wedge S^{-n\rho} \wedge f_{m\rho}} Y \wedge S^{n\rho} \wedge S^{-n\rho} \xrightarrow{g_{n\rho}} Z.$$

Associativity of this composition follows from that of the smash product.

Definition 9.5.4. The category $\pi^{\mathrm{st}}\mathcal{S}p^G$ *has the same objects as* $\mathcal{S}p^G$*. It is enriched over abelian groups with morphism groups* $\pi^{\mathrm{st}}\mathcal{S}p^G(X,Y)$ *as in (9.5.2) and composition as defined above.*

Proposition 9.5.5. *For all* $k \in \mathbf{Z}$*, there is a natural isomorphism*

$$\pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^k \wedge S^{-0}, Y) \cong \pi_k^H Y.$$

This means that a map in $\mathcal{S}p^G$ that is an isomorphism in $\pi^{\mathrm{st}}\mathcal{S}p^G$ is also a stable equivalence.

Proof Suppose $k \geqslant 0$. Then

$$\begin{aligned} \pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^k, Y) &= \operatorname*{colim}_n \pi_0 \mathcal{S}p^G(G/H \ltimes S^k \wedge S^{n\rho} \wedge S^{-n\rho}, Y) \\ &= \operatorname*{colim}_n \pi_0 \mathcal{S}p^H(S^k \wedge S^{n\rho} \wedge S^{-n\rho}, Y) \\ &= \operatorname*{colim}_n \pi_0 \mathcal{T}^H(S^{n\rho} \wedge S^k, Y_{n\rho}) \\ &= \operatorname*{colim}_n \pi^H_{k+n\rho} Y_{n\rho} = \pi_k^H Y. \end{aligned}$$

Here we are using the same notation for a G-space or G-spectrum X and its image under the forgetful functor i_H^G. Similarly,

$$\pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^{-k}, Y) = \operatorname*{colim}_n \pi_0 \mathcal{S}p^G(G/H \ltimes S^{-k} \wedge S^{n\rho} \wedge S^{-n\rho}, Y)$$

$$
\begin{aligned}
&= \operatorname*{colim}_n \pi_0 \mathcal{S}p^H(S^{-k} \wedge S^{n\rho} \wedge S^{-n\rho}, Y) \\
&= \operatorname*{colim}_n \pi_0 \mathcal{T}^H(S^{n\rho}, Y_{n\rho+k}) \\
&= \operatorname*{colim}_n \pi^H_{n\rho} Y_{n\rho+k} = \operatorname*{colim}_{n>k} \pi^H_{n\rho-k} Y_{n\rho} = \pi^H_{-k} Y. \qquad \square
\end{aligned}
$$

The following is the promised statement about smashing with the map s_V.

Proposition 9.5.6. *Suppose that V is a representation of G. For every X, the map*

$$
X \wedge s_V : X \wedge S^V \wedge S^{-V} \to X, \tag{9.5.7}
$$

where s_V is as in (7.2.68), is an isomorphism in $\pi^{\mathrm{st}}\mathcal{S}p^G$ and hence a stable equivalence.

Proof We will show that for all Y, the map

$$
\pi^{\mathrm{st}}\mathcal{S}p^G(X, Y) \to \pi^{\mathrm{st}}\mathcal{S}p^G(S^V \wedge S^{-V} \wedge X, Y)
$$

is an isomorphism. By definition,

$$
\pi^{\mathrm{st}}\mathcal{S}p^G(X, Y) = \operatorname*{colim}_n \pi_0(\mathcal{S}p^G(X \wedge S^{n\rho} \wedge S^{-n\rho}, Y)) \tag{9.5.8}
$$

while

$$
\begin{aligned}
&\pi^{\mathrm{st}}\mathcal{S}p^G(X \wedge S^V \wedge S^{-V}, Y) \\
&\qquad = \operatorname*{colim}_n \pi_0\, \mathcal{S}p^G(S^V \wedge S^{-V} \wedge X \wedge S^{-n\rho} \wedge S^{n\rho}, Y).
\end{aligned} \tag{9.5.9}
$$

Now we use the fact that V is a summand of some multiple of ρ. **This is the equivariant instance of the direct summand condition of Definition 7.2.19(ii).**

Suppose there is a representation W with $V \oplus W = k\rho$ for some $k > 0$. We can replace the colimit of (9.5.8) by

$$
\operatorname*{colim}_n \pi_0(\mathcal{S}p^G(X \wedge S^{nk\rho} \wedge S^{-nk\rho}, Y)),
$$

and similarly for the colimit of (9.5.9). This means replacing the maps

$$
S^\rho \wedge \xi_{n\rho,\rho} \qquad \text{and} \qquad S^\rho \wedge \xi_{V+n\rho,\rho}
$$

$n \geqslant 0$ of (9.5.3) by the maps

$$
S^{k\rho} \wedge \xi_{nk\rho,k\rho} \qquad \text{and} \qquad S^{k\rho} \wedge \xi_{V+nk\rho,k\rho} \tag{9.5.10}
$$

for $n \geqslant 0$. Then because $k\rho \cong V \oplus W$, these maps factor as indicated in the following diagram.

$$\begin{tikzcd}
S^{V\oplus(n+1)k\rho} \wedge S^{-V\oplus(n+1)k\rho} \arrow[d, "S^{k\rho}\wedge\xi_{V+nk\rho,k\rho}"'] \arrow[rr, "S^{V}\wedge\xi_{(n+1)k\rho,V}"] & & S^{(n+1)k\rho} \wedge S^{-(n+1)k\rho} \arrow[dll, "S^{W}\wedge\xi_{V+nk\rho,W}"'] \arrow[d, "S^{k\rho}\wedge\xi_{nk\rho,k\rho}"] \\
S^{V\oplus nk\rho} \wedge S^{-V\oplus nk\rho} \arrow[rr, "S^{V}\wedge\xi_{nk\rho,V}"'] & & S^{k\rho} \wedge S^{-nk\rho}.
\end{tikzcd}$$

It follows that the colimits of (9.5.8) and (9.5.9) are the same. □

Remark 9.5.11. *The stable equivalence (9.5.7) is often written in the form*

$$S^{-V\oplus W} \wedge S^{W} \wedge X \to S^{-V} \wedge X.$$

This is gotten from (9.5.7) by writing $S^{-V\oplus W}$ as $S^{-V} \wedge S^{-W}$ and writing the map as

$$S^{-W} \wedge S^{W} \wedge \big(S^{-V} \wedge X\big) \to \big(S^{-V} \wedge X\big).$$

Lemma 9.5.12. *For a map $X \to Y$ in $\pi^{\mathrm{st}}\mathcal{S}p^{G}$, the following are equivalent*

(i) The map $X \to Y$ is a stable equivalence.

(ii) For all $H \subset G$ and all $k \in \mathbf{Z}$ the map

$$\pi^{\mathrm{st}}(G/H \ltimes S^{k}, X) \to \pi^{\mathrm{st}}(G/H \ltimes S^{k}, Y)$$

is an isomorphism.

(iii) For **some** *representation V of G, all $H \subset G$ and all $k \in \mathbf{Z}$ the map*

$$\pi^{\mathrm{st}}(G/H \ltimes S^{k} \wedge S^{V}, X) \to \pi^{\mathrm{st}}(G/H \ltimes S^{k} \wedge S^{V}, Y)$$

is an isomorphism.

(iv) For **all** *representations V of G, all $H \subset G$ and all $k \in \mathbf{Z}$ the map*

$$\pi^{\mathrm{st}}(G/H \ltimes S^{k} \wedge S^{V}, X) \to \pi^{\mathrm{st}}(G/H \ltimes S^{k} \wedge S^{V}, Y)$$

is an isomorphism.

Proof The equivalence of the first two statements is Proposition 9.5.23 below, and they imply (iv) by Proposition 9.5.25. The fourth statement obviously implies the third. That the third statement implies the first two is proved by induction on $|G|$, the assertion being trivial when G is trivial. We may therefore assume that (iii) holds, and that (ii) holds for all proper $H \subset G$. Let $V_0 \subset V$ be the subspace of invariant vectors. Using the long exact sequence of Proposition 9.4.3(ii), and working by downward induction through an equivariant cell decomposition of S^{V}, one sees that for all $k \in \mathbf{Z}$ and all $H \subset G$, our assumptions imply that the map

$$\pi^{\mathrm{st}}(G/H \ltimes S^{k} \wedge S^{V_0}, X) \to \pi^{\mathrm{st}}(G/H \ltimes S^{k} \wedge S^{V_0}, Y)$$

is an isomorphism. But in $\pi^{\mathrm{st}}\mathcal{S}p^G$ there is an isomorphism $S^k \wedge S^{V_0} \approx S^{k+\ell}$ with $\ell = \dim V_0$, so this implies (ii). $\square$

Proposition 9.5.13. *Let V be a representation of G. The following conditions on a map $X \to Y \in \pi^{\mathrm{st}}\mathcal{S}p^G$ are equivalent*

(i) The map $X \to Y$ is a weak equivalence
(ii) The map $S^V \wedge X \to S^V \wedge Y$ is a stable equivalence
(iii) The map $S^{-V} \wedge X \to S^{-V} \wedge Y$ is a stable equivalence.

Proof Since smashing with S^V is the inverse equivalence of smashing with S^{-V} it suffices to establish the equivalence of the first two assertions. Now for any X, smashing with S^V gives an isomorphism

$$\pi^{\mathrm{st}}(G/H \ltimes S^k, S^{-V} \wedge X) \cong \pi^{\mathrm{st}}(G/H \ltimes S^k \wedge S^V, X),$$

so the equivalence of the first two assertions is a consequence of Lemma 9.5.12. $\square$

Corollary 9.5.14. Suspension and desuspension in $\pi^{\mathrm{st}}\mathcal{S}p^G$. *For any representation V of G, smashing with S^V and S^{-V} are inverse equivalences in $\pi^{\mathrm{st}}\mathcal{S}p^G$.*

Corollary 9.5.15. $\pi^{\mathrm{st}}\mathcal{S}p^G$ is tensored over $\mathcal{SW}^G$.

Proof Let X be a spectrum and $f : K \to L$ a morphism in $\mathcal{SW}^G$ represented by a map $K \wedge S^V \to L \wedge S^V$. Smashing this map of spaces with X gives a map of spectra $X \wedge K \wedge S^V \to X \wedge L \wedge S^V$ and hence an element in $\pi^{\mathrm{st}}\mathcal{S}p^G(X \wedge K \wedge S^V, X \wedge L \wedge S^V)$. This is isomorphic to $\pi^{\mathrm{st}}\mathcal{S}p^G(X \wedge K, X \wedge L)$ by Corollary 9.5.14, so we have the desired structure on $\pi^{\mathrm{st}}\mathcal{S}p^G$. $\square$

This fact leads to a form of Spanier-Whitehead duality (see § 8.0C) in $\pi^{\mathrm{st}}\mathcal{S}p^G$. Suppose that K is a finite G-CW complex, and that L is a "V-dual" in the sense that there is a representation V of G and maps in SW^G

$$K \wedge L \xrightarrow{\epsilon} S^V \xrightarrow{\eta} L \wedge K \tag{9.5.16}$$

with the property that the composites

$$S^V \wedge L \xrightarrow{\eta \wedge L} L \wedge K \wedge L \xrightarrow{L \wedge \epsilon} L \wedge S^V$$
$$\text{and} \qquad K \wedge S^V \xrightarrow{K \wedge \epsilon} K \wedge L \wedge S^V \xrightarrow{\epsilon \wedge K} S^V \wedge K$$

are symmetry isomorphisms. Then for $X, Y \in \pi^{\mathrm{st}}\mathcal{S}p^G$ the composite

$$\begin{tikzcd}
\pi^{\mathrm{st}}\mathcal{S}p^G(X, Y \wedge K) \arrow[d, "\omega_{L,X,Y \wedge K}"] & \\
\pi^{\mathrm{st}}\mathcal{S}p^G(X \wedge L, Y \wedge K \wedge L) \arrow[d, "(Y \wedge \epsilon)_*"] & \\
\pi^{\mathrm{st}}\mathcal{S}p^G(X \wedge L, Y \wedge S^V) \arrow[r, "\cong"] & \pi^{\mathrm{st}}\mathcal{S}p^G(S^{-V} \wedge X \wedge L, Y)
\end{tikzcd} \tag{9.5.17}$$

(for $\omega_{L,X,Y\wedge K}$ as in Definition 2.6.6) is an isomorphism, by the standard duality manipulation.

The following should be compared with Proposition 9.4.3 above.

Proposition 9.5.18. Long exact sequences in $\pi^{\mathrm{st}}\mathcal{S}p^G$. *Given a morphism $f : X \to Y$ in $\mathcal{S}p^G$, and any G-spectra W and Z there are long exact sequences*

$$\cdots \longrightarrow \pi^{\mathrm{st}}\mathcal{S}p^G(S^k \wedge C_f, Z) \longrightarrow \pi^{\mathrm{st}}\mathcal{S}p^G(S^k \wedge Y, Z) \to \pi^{\mathrm{st}}\mathcal{S}p^G(S^k \wedge X, Z)$$
$$\longrightarrow \pi^{\mathrm{st}}\mathcal{S}p^G(S^{k-1} \wedge C_f, Z) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \pi^{\mathrm{st}}\mathcal{S}p^G(W, S^k \wedge X) \longrightarrow \pi^{\mathrm{st}}\mathcal{S}p^G(W, S^k \wedge Y) \to \pi^{\mathrm{st}}\mathcal{S}p^G(W, S^k \wedge C_f)$$
$$\longrightarrow \pi^{\mathrm{st}}\mathcal{S}p^G(W, S^{k+1} \wedge X) \longrightarrow \cdots$$

where C_f denotes the mapping cone $(Y \cup CX)$.

Proof For the first one, consider the fiber sequence $Z^{C_f} \to Z^Y \to Z^X$. Then the long exact sequence of Proposition 9.4.2 reads

$$\cdots \to \pi_k^G Z^{C_f} \to \pi_k^G Y \to \pi_k^G X \to \pi_{k-1}^G Z^{C_f} \to \cdots .$$

The isomorphism of Proposition 9.5.5 converts this the long exact sequence we want.

For the second sequence, the long exact sequence of Proposition 9.4.3 leads to

$$\cdots \longrightarrow \pi_{-k}\mathcal{S}p^G(W, X) \longrightarrow \pi_{-k}\mathcal{S}p^G(W, Y) \to \pi_{-k}\mathcal{S}p^G(W, Y)$$
$$\longrightarrow \pi_{-k-1}\mathcal{S}p^G(W, X) \longrightarrow \cdots$$

Again Proposition 9.5.5 enables us to rewrite this as the desired long exact sequence. □

Definition 9.5.19. $\pi^{\mathrm{st}}\mathcal{S}p^G$ as a homotopical category. *A map in $\pi^{\mathrm{st}}\mathcal{S}p^G$ is a* **weak equivalence** *if it induces isomorphisms in π_*^H for all subgroups $H \subseteq G$ via the isomorphism of Proposition 9.5.5.*

Since this condition also defines stable equivalences in $\mathcal{S}p^G$, we have the following.

Proposition 9.5.20. *The homotopy categories of $\mathcal{S}p^G$ and $\pi^{\mathrm{st}}\mathcal{S}p^G$ are isomorphic.*

This and Corollary 5.1.10 gives the following.

Lemma 9.5.21. *If $X \in \mathcal{S}p^G$ has the property that $\pi^{\mathrm{st}}\mathcal{S}p^G(X, -)$ is a homotopy functor, then for all Y, the maps*

$$\pi^{\mathrm{st}}\mathcal{S}p^G(X, Y) \to \mathrm{Ho}\,\pi^{\mathrm{st}}\mathcal{S}p^G(X, Y) \xleftarrow{\sim} \mathrm{Ho}\,\mathcal{S}p^G(X, Y) \qquad (9.5.22)$$

are isomorphisms, and $\operatorname{Ho}\mathcal{S}p^G(X,Y)$ *may be computed as* $\pi^{\mathrm{st}}\mathcal{S}p^G(X,Y)$.

Proposition 9.5.23. *For* $k \in \mathbf{Z}$ *the maps of Proposition 9.5.5 and (9.5.22) give isomorphisms*

$$\pi_k^H X \cong \pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^k, X) \cong \operatorname{Ho}\mathcal{S}p^G(G/H \ltimes S^k, X).$$

Proof The first isomorphism is given by Proposition 9.5.5, and it implies that $\pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^k, X)$ is a homotopy functor of X. Lemma 9.5.21 then gives the second isomorphism. □

Corollary 9.5.24. *A map* $X \to Y$ *in* $\mathcal{S}p^G$ *is a stable equivalence if and only if it becomes an isomorphism in* $\operatorname{Ho}\mathcal{S}p^G$. □

Proposition 9.5.25. *When* X *is of the form* $X = S^{-0} \wedge S^\ell \wedge K$ *with* K *a finite* G*-CW complex, and* $\ell \in \mathbf{Z}$*, the functor* $\pi^{\mathrm{st}}\mathcal{S}p^G(X, -)$ *is a homotopy functor, and so for all spectra* Y, $\operatorname{Ho}\mathcal{S}p^G(X,Y)$ *may be computed as* $\pi^{\mathrm{st}}\mathcal{S}p^G(X,Y)$.

Proof Working through the skeletal filtration of K and using the first exact sequence of Proposition 9.5.18 reduces the claim to the case in which $K = G/H \ltimes S^n$. But that case is Proposition 9.5.5. □

Note that

$$\pi^{\mathrm{st}}\mathcal{S}p^G(S^{-0} \wedge K, S^{-0} \wedge L) = \operatorname*{colim}_n \pi_0\,\mathcal{T}^G(S^{n\rho} \wedge K, S^{n\rho} \wedge L).$$

When L is a finite G-CW complex, this is the definition of $\mathcal{SW}^G(K,L)$. Thus Proposition 9.5.25 contains it as a special case.

Proposition 9.5.26. *The functor* Σ^∞ *(smashing with the sphere spectrum* S^{-0}*) induces a fully faithful embedding* $\mathcal{SW}^G \to \operatorname{Ho}\mathcal{S}p^G$. □

Proposition 9.5.27. Smashing with generalized suspension spectra. *Let* $X = S^{-V} \wedge K$ *for a representation* V *and* G*-CW complex* K*. Then smashing with* X *is homotopical.*

Proof Since every G-CW complex K is a filtered colimit of finite complexes, we can use Proposition 9.4.7 to reduce to the case where K is finite. In addition, it suffices to show that smashing with $S^{-W} \wedge K$ is homotopical as a functor from $\pi^{\mathrm{st}}\mathcal{S}p^G$ to itself. Suppose that $Y \to Y'$ is a stable equivalence. Let $L \in SW^G$ be a V-dual of K. By the isomorphism of Proposition 9.5.5 it suffices to show that for all $H \subset G$ and all $k \in \mathbf{Z}$, the map

$$\pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^k, Y \wedge X) \to \pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^k, Y' \wedge X)$$

is an isomorphism. Using the first part of the duality isomorphism (9.5.17), we can identify this map with

$$\pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^k \wedge S^W \wedge L, S^V \wedge Y) \to \pi^{\mathrm{st}}\mathcal{S}p^G(G/H \ltimes S^k \wedge S^W \wedge L, S^V \wedge Y'),$$

and finally by Proposition 9.5.25, with

$$\mathrm{Ho}\,\mathcal{S}p^G(G/H \ltimes S^k \wedge S^W \wedge L, S^V \wedge Y) \to \mathrm{Ho}\,\mathcal{S}p^G(G/H \ltimes S^k \wedge S^W \wedge L, S^V \wedge Y').$$

But this latter map is an isomorphism since $S^V \wedge Y \to S^V \wedge Y'$ is a stable equivalence by Proposition 9.5.13. □

9.6 Homotopical properties of indexed wedges and indexed smash products

The object of this section is to show that the formations of indexed wedges and, in favorable cases, indexed smash products are homotopical. This means that both constructions preserve stable equivalences.

9.6A Indexed wedges

We remind the reader how such wedges are defined. Given a finite G-set T and a G-spectrum X, we define the indexed wedge and product by

$$\left(\bigvee_{t\in T} X\right)_V = T \ltimes X_V, \quad \text{i.e.,} \bigvee_{t\in T} X = T \ltimes X$$

$$\text{and} \qquad \left(\prod_{t\in T} X\right)_V = \mathcal{T}_G(T_+, X_V), \quad \text{i.e.,} \prod_{t\in T} X = X^{T_+}$$

for X^{T_+} as in Proposition 7.2.49.

Proposition 9.6.1. *Let T be a finite G-set. For any $X \in \mathcal{S}p^G$, the canonical map*

$$\bigvee_{t\in T} X \to \prod_{t\in T} X$$

is an isomorphism in $\pi^{\mathrm{st}}\mathcal{S}p^G$, hence a stable equivalence.

Proof The finite G-sets are self-dual in $\mathcal{SW}^G$ by Proposition 8.0.13. Since

$$\bigvee_{t\in T} X \cong T \ltimes X,$$

the result follows from the duality isomorphism

$$\pi^{\mathrm{st}}\mathcal{S}p^G(Z, T \ltimes X) \cong \pi^{\mathrm{st}}\mathcal{S}p^G(T \ltimes Z, X) \cong \pi^{\mathrm{st}}\mathcal{S}p^G(Z, \prod_{t\in T} X)$$

once one checks that the composite map is the same as the one coming from the canonical map from the (constant) finite indexed wedge to the finite indexed product. We leave this to the reader. □

The next result concerns equivariant T-diagrams. Recall the category $\mathcal{B}_T G$ of Example 2.9.1 associated with a finite G-set T. A T-diagram of spectra is a functor $F : \mathcal{B}_T G \to \mathcal{S}p$. Since a spectrum E is itself a functor $E : \mathscr{J} \to \mathcal{T}$ and T-diagram is equivalent to a functor $\mathcal{B}_T G \times \mathscr{J} \to \mathcal{T}$, which we will also denote by F. We denote the category of such diagrams by $\mathcal{S}p^{\mathcal{B}_T G}$. It has model structures indicated in Corollary 9.3.24.

For each object V of $\mathscr{J}$ we have a functor $F_V : \mathcal{B}_T G \to \mathcal{T}$. Since T is a disjoint union of orbits of the form G/H_α for various subgroups H_α, F_V amounts to a collection pointed G-spaces of the form

$$\{(G/H_\alpha) \ltimes X_{\alpha,V}\}$$

where each $X_{\alpha,V}$ is a pointed H_α-space. Thus we have

$$\begin{array}{l} \mathcal{S}p^G \\ \quad \downarrow{\scriptstyle i^*} \\ \mathcal{S}p^G_{naive} \xrightarrow{\cong} [\mathscr{J}, \mathcal{T}^G] \xrightarrow{U_*} [\mathscr{J}, \mathcal{T}^{\mathcal{B}_T G}] \cong [\mathscr{J}, \mathcal{T}]^{\mathcal{B}_T G} \cong \mathcal{S}p^{\mathcal{B}_T G}, \end{array} \tag{9.6.2}$$

where i^* is induced by the inclusion map i of (9.3.8), and U_* is induced by the pullback functor $U : \mathcal{T}^G \to \mathcal{T}^{\mathcal{B}_T G}$ of (2.9.2).

Corollary 9.6.3. *Let T be a finite G-set and X an equivariant T-diagram in the category $\mathcal{S}p$ of orthogonal spectra. The map*

$$\bigvee_{t \in T} X_t \to \prod_{t \in T} X_t$$

is an isomorphism in $\pi^{\mathrm{st}}\mathcal{S}p^G$, hence a stable equivalence.

Proof Consider the functor $U_* i^* : \mathcal{S}p^G \to \mathcal{S}p^{\mathcal{B}_T G}$ of (9.6.2). The indexed wedge and product are its left and right adjoints. The natural transformation from the former to the latter is easily seen to satisfy the condition of Lemma 2.2.38. This reduces us to checking the case in which the T-diagram is constant at a G-spectrum X. But that case is covered by Proposition 9.6.1. □

Corollary 9.6.3 implies the second part of the following indexed analogue of Proposition 9.4.5.

Proposition 9.6.4. Indexed products and coproducts.

(i) The formation of finite indexed products is homotopical.

(ii) Suppose that T is a finite G-set, and $X : \mathcal{B}_T G \to \mathcal{S}p$ is a functor, namely a collection of spectra X_t indexed by T with suitable maps between them. The map

$$\bigvee_{t \in T} X_t \to \prod_{t \in T} X_t$$

is a stable equivalence in $\mathcal{S}p^G$. Hence the formation of finite indexed wedges is homotopical.

(iii) The formation of all indexed wedges is homotopical.

9.6B Indexed smash products

The smash product of spectra is not known to preserve weak equivalences in general, but it does so in favorable cases.

Proposition 9.6.5. *If a spectrum X in $\mathcal{S}p^G$ is Bredon cofibrant as in Definition 9.2.15, then it is flat as in Definition 5.1.20.*

Proof By Proposition 9.5.27 and the fact that the formation of indexed wedges is homotopical (Proposition 9.6.4) the result is true when

$$X = G \underset{H}{\ltimes} \wedge S^{-V} \wedge S^k.$$

The functor $(-) \wedge X$ is built from

$$(-) \wedge G \underset{H}{\ltimes} \wedge S^{-V} \wedge S^k$$

by forming wedges, mapping cones, and filtered colimits along h-cofibrations, all of which are homotopical by Proposition 9.6.4. $\square$

Since every spectrum is weakly equivalent to a Bredon cofibrant one, and such spectra are flat as in Definition 5.1.20 by Proposition 9.6.5, Proposition 5.1.26 implies

Proposition 9.6.6. *Suppose that $X \to Y$ is a weak equivalence of flat spectra. Then for any Z, the map $X \wedge Z \to Y \wedge Z$ is a weak equivalence.* $\square$

Let $\mathcal{S}p^G_{\mathrm{fl}} \subset \mathcal{S}p^G$ be the full subcategory of flat objects, considered as a homotopical category using the stable weak equivalences. Since every object of $\mathcal{S}p^G$ is weakly equivalent to an object of $\mathcal{S}p^G_{\mathrm{fl}}$, the functor

$$\operatorname{Ho}\mathcal{S}p^G_{\mathrm{fl}} \to \operatorname{Ho}\mathcal{S}p^G \tag{9.6.7}$$

is an equivalence of categories. The above results show

Proposition 9.6.8. *The smash product functor*

$$\mathcal{S}p^G_{\mathrm{fl}} \times \mathcal{S}p^G \to \mathcal{S}p^G$$

is homotopical. $\square$

The equivalence (9.6.7) and Proposition 9.6.5 are enough to show that the smash product descends to give $\operatorname{Ho}\mathcal{S}p^G$ a symmetric monoidal structure, and that the map $SW^G \to \operatorname{Ho}\mathcal{S}p^G$ is symmetric monoidal. For a more refined statement, see §9.8.

9.7 The norm functor

9.7A Equivariant commutative and associative algebras

Using the notions described in §2.6G one can transport many algebraic structures to $\mathcal{S}p^G$ using the symmetric monoidal smash product.

Definition 9.7.1. *A* **G-equivariant commutative (associative) algebra** *is a commutative (associative) algebra with unit in $\mathcal{S}p^G$. We will abbreviate the categories* $\mathbf{Comm}\,\mathcal{S}p^G$ *and* $\mathbf{Assoc}\,\mathcal{S}p^G$ *(as in Definition 2.6.58) of such spectra and equivariant maps by* $\mathbf{Comm}^G$ *and* $\mathbf{Alg}^G$ *respectively. The corresponding categories with all continuous maps will be denoted by* $\mathbf{Comm}_G$ *and* $\mathbf{Alg}_G$*. We will sometimes refer to such objects as* **rings** *(with suitable adjectives) or* **ring spectra***.*

Since $\mathcal{S}p^G$ is a closed symmetric monoidal category under $\wedge$, Lemma 2.6.66 implies that both $\mathbf{Comm}^G$ and $\mathbf{Alg}^G$ are complete and cocomplete, and that the forgetful functors

$$\begin{aligned} \mathbf{Comm}^G &\to \mathcal{S}p^G \\ \mathbf{Alg}^G &\to \mathcal{S}p^G \end{aligned}$$

create enriched limits, sifted colimits, and have left adjoints

$$\begin{aligned} \mathrm{Sym} : \mathcal{S}p^G &\to \mathbf{Comm}^G \\ T : \mathcal{S}p^G &\to \mathbf{Alg}^G. \end{aligned}$$

Similarly, there are categories of left and right modules over an associative algebra A. We will use the symbol $\mathcal{M}_A$ for the category of **left A-modules.** As described in §2.6G, when A is commutative, the category $\mathcal{M}_A$ inherits a symmetric monoidal product $M \underset{A}{\wedge} N$ defined by the reflexive coequalizer diagram

$$M \wedge A \wedge N \rightrightarrows M \wedge N \dashrightarrow M \underset{A}{\wedge} N. \tag{9.7.2}$$

9.7B Defining the norm functor

For each subgroup $H \subseteq G$ we have a forgetful functor $i_H^G : \mathcal{S}p^G \to \mathcal{S}p^H$ of Proposition 9.1.17 and the change of group adjunction of (9.1.18).

Using the notation and constructions of Example 2.9.8 and Example 2.9.12, note that the functor categories $\mathcal{S}p^{\mathcal{B}G} = [\mathcal{B}G, \mathcal{S}p]$ and $\mathcal{S}p^{\mathcal{B}H} = [\mathcal{B}H, \mathcal{S}p]$ are $\mathcal{S}p^G_{naive}$ and $\mathcal{S}p^H_{naive}$. The latter is equivalent to $\mathcal{S}p^{\mathcal{B}_{G/H}G}$. Let $p : \mathcal{B}_{G/H}G \to \mathcal{B}G$ be the functor induced by the G-map $G/H \to G/G$. Recall that inclusion of the identity coset in G/H gives a functor $j : \mathcal{B}H \to \mathcal{B}_{G/H}H$ which is an equivalence of categories.

Definition 9.7.3. *The* **norm functor** $N_H^G : \mathcal{S}p^H \to \mathcal{S}p^G$ *is the composite*

$$\begin{tikzcd} \mathcal{S}p^H \arrow[r, "i^*"] \arrow[drr, "N_H^G"'] & \mathcal{S}p^H_{naive} = [\mathcal{B}H, \mathcal{S}p] \arrow[rr, "j_!"] & & {[\mathcal{B}_{G/H}G, \mathcal{S}p]} \arrow[d, "p_*^{\wedge}"] \\ & & \mathcal{S}p^G & \mathcal{S}p^G_{naive} = [\mathcal{B}G, \mathcal{S}p]. \arrow[l, "i_!"'] \end{tikzcd}$$

When the groups G and H are cyclic with $|G| = g$ and $|H| = h$, we will sometimes write N_h^g for N_H^G.

In proving the Kervaire invariant theorem we will use this for $H = C_2$, $G = C_8$ and apply it to the C_2-spectrum $MU_{\mathbf{R}}$.

The following is a consequence of Proposition 2.9.7.

Proposition 9.7.4. Properties of the norm. *The functor N_H^G of Definition 9.7.3 is symmetric monoidal, and it commutes with sifted colimits as in Definition 2.3.73.*

We will study the norm further in Chapter 10.

Remark 9.7.5. A relation between the norms for spectra and for spaces. *We have defined the norm on the topological categories of equivariant spectra. Since it is symmetric monoidal it naturally extends to a functor of enriched categories*

$$N_H^G : \mathcal{S}p_H \to \mathcal{S}p_G$$

compatible with the norm on spaces (as in Definition 8.3.23) in the sense that it gives for every $X, Y \in \mathcal{S}p_H$ a G-equivariant map

$$N_H^G(\mathcal{S}p_H(X,Y)) \to \mathcal{S}p_G(N_H^G X, N_H^G Y).$$

By Proposition 2.9.55, on equivariant commutative algebras the norm is the left adjoint of the restriction functor.

Corollary 9.7.6. *The following diagram, in which U denotes the forgetful functor, commutes up to a natural isomorphism given by the symmetry of the smash product:*

$$\begin{tikzcd} \mathbf{Comm}^H \arrow[r] \arrow[d, "U"'] & \mathbf{Comm}^G \arrow[d, "U"] \\ \mathcal{S}p^H \arrow[r, "N_H^G"] & \mathcal{S}p^G. \end{tikzcd}$$

The top arrow is the left adjoint to the restriction functor.

Remark 9.7.7. *Because of Corollary 9.7.6 we will refer to the left adjoint to the restriction functor*

$$i_H^G : \mathbf{Comm}^G \to \mathbf{Comm}^H$$

as the **commutative algebra norm**, *and denote it*

$$N_H^G : \mathbf{Comm}^H \to \mathbf{Comm}^G.$$

The Yoneda embedding (Definition 3.1.68) is the functor

$$\begin{tikzcd} \mathscr{J}^{op} \arrow[r, "よ"] & \mathcal{S}p \\ V \arrow[r, mapsto] & S^{-V} \end{tikzcd}$$

By the definition of $\wedge$, this is a symmetric monoidal functor, and we are in the situation described in Proposition 2.9.10. Thus if $p : I \to J$ is a covering category, there is a natural isomorphism between the two ways of going around

$$\begin{tikzcd} \left(\mathscr{J}^{op}\right)^I \arrow[r, "よ_*"] \arrow[d, "p_*^{\oplus}"'] & \mathcal{S}p^I \arrow[d, "p_*^{\wedge}"] \\ \left(\mathscr{J}^{op}\right)^J \arrow[r, "よ_*"] & \mathcal{S}p^J . \end{tikzcd}$$

Take $I = \mathcal{B}_{G/H}G$ and $J = \mathcal{B}G$. Then the functor category $\left(\mathscr{J}^{op}\right)^I$ is equivalent to the category $\left(\mathscr{J}^H\right)^{op}$ by Proposition 8.9.33, and $\mathcal{S}p^I$ is equivalent to $\mathcal{S}p^H$ by Theorem 9.3.10. By naturality, the functor

$$(\mathscr{J}^H)^{op} \to \mathcal{S}p^H$$

corresponding to

$$\left(\mathscr{J}^{op}\right)^I \to \mathcal{S}p^I$$

is just the Yoneda embedding よ, and so sends an orthogonal H-representation V to S^{-V}. Similarly $\left(\mathscr{J}^{op}\right)^J$ is equivalent to $(\mathscr{J}^G)^{op}$, $\mathcal{S}p^J$ is equivalent to the category of orthogonal G-spectra, and the functor between them sends an orthogonal G-representation W to S^{-W}. One easily checks (as in Example 2.9.11) that the functor $p_*^{\oplus}$ corresponds to additive induction. We therefore have a commutative diagram

$$\begin{tikzcd} \left(\mathscr{J}^H\right)^{op} \arrow[r, "よ"] \arrow[d, "\mathrm{Ind}_H^G"'] & \mathcal{S}p^H \arrow[d, "N_H^G"] \\ \left(\mathscr{J}^G\right)^{op} \arrow[r, "よ"] & \mathcal{S}p^G \end{tikzcd}$$

This proves

Proposition 9.7.8. The norm of a Yoneda spectrum. *There is a natural isomorphism*

$$N_H^G\, S^{-V} \cong S^{-\mathrm{Ind}_H^G V}$$

of functors $\left(\mathscr{J}^H\right)^{op} \to \mathcal{S}p^G$. □

9.7C Other uses of the norm

There are several important constructions derived from the norm functor which also go by the name of "the norm."

Suppose that R is a G-equivariant commutative ring spectrum, and X is an H-spectrum for a subgroup $H \subset G$. Write

$$R^0_H(X) := [X, i^G_H R]^H.$$

There is a norm map

$$N^G_H : R^0_H(X) \to R^0_G(N^G_H X) \tag{9.7.9}$$

defined by sending an H-equivariant map $X \to R$ to the composite

$$N^G_H X \to N^G_H(i^G_H R) \to R,$$

in which the second map is the counit of the restriction-norm adjunction of Corollary 9.7.6. This is the **norm map on equivariant spectrum cohomology**, and is the form in which the norm is described in [GM97]. For an explicit comparison with [GM97], see [Boh14]. We will use the map of (9.7.9) below in Definition 13.3.12.

When V is a representation of H and $X = S^V$ the above gives a map

$$N = N^G_H : \pi^H_V R \to \pi^G_{\mathrm{Ind} V} R$$

in which $\mathrm{Ind}V$ is the induced representation which we call the **norm map on the $RO(G)$-graded homotopy groups of commutative rings**

Now suppose that X is a pointed G-space. There is a norm map

$$N^G_H : R^0_H(X) \to R^0_G(X)$$

sending

$$x \in R^0_H(X) = [S^0 \wedge X, i^G_H R]^H$$

to the composite

$$S^0 \wedge X \to S^0 \wedge N(X) \cong N(S^0 \wedge X) \to N(i^G_H R) \to R,$$

in which the equivariant map of pointed G-spaces

$$X \to N^G_H(X)$$

is the "diagonal"

$$X \to \prod_{j \in G/H} X_j \to \bigwedge_{j \in G/H} X_j$$

whose jth component is the inverse to the isomorphism

$$X_j = (H_j) \underset{H}{\rtimes} X \to X$$

given by the action map. That this is actually equivariant is probably most easily seen by making the identification

$$X_j \cong \hom_H(H_j^{-1}, X)$$

in which H_j^{-1} denotes the *left* H-coset consisting of the inverses of the elements of H_j, and then writing

$$\prod_{j \in G/H} X_j \cong \hom_H(G, X).$$

Under this identification, the "diagonal" map is the map

$$X \to \hom_H(G, X)$$

adjoint to the action map

$$G \underset{H}{\times} X \to X,$$

which is clearly equivariant.

One can combine these construction to define the **norm on $RO(G)$-graded cohomology** of a G-space X

$$N_H^G : R_H^V(X) \to R_G^{\mathrm{Ind}V}(X)$$

sending

$$S^0 \wedge X \xrightarrow{a} S^V \wedge i_H^G R$$

to the composite

$$S^0 \wedge X \to S^0 \wedge NX \xrightarrow{Na} S^{\mathrm{Ind}V} \wedge Ni_H^G R \to S^{\mathrm{Ind}V} \wedge R.$$

9.8 Change of group and smash product

The restriction functor

$$i_H^G : \mathcal{S}p^G \to \mathcal{S}p^H$$

preserves weak equivalences, fibrations and cofibrations in the positive stable equifibrant model structure. This implies

Proposition 9.8.1. The change of group adjunction for G-spectra. *Let $H \subset G$ be a subgroup. The restriction functor and its left adjoint form a Quillen pair*

$$G \underset{H}{\ltimes} (-) : \mathcal{S}p^H \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}p^G : i_H^G,$$

as do the restriction functor and its right adjoint

$$i_H^G : \mathcal{S}p^G \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}p^H : \prod_{j \in G/H} (-)_j.$$

Corollary 9.8.2. *An indexed wedge of cofibrations is a cofibration.*

Corollary 9.8.2 is one of our reasons for introducing the positive stable equifibrant model structure. The positive stable model structure of [MM02, §III.5] does not have this property.

Associated to any map $\phi : G' \to G$ of finite groups is a functor

$$\phi^* : \mathcal{S}p^G \to \mathcal{S}p^{G'}.$$

This functor has both a left and right adjoint. The functor ϕ^* sends the generating cofibrations to indexed wedges of generating cofibrations, hence cofibrations by Corollary 9.8.2. Since it is a left adjoint it therefore sends cofibrations to cofibrations. It also sends the generating trivial cofibrations to weak equivalences. To see this note that the generators of the form $X \wedge (I_+^{n-1} \to I_+^n)$ are homotopy equivalences hence go to homotopy equivalences. To check that the corner maps in (9.2.9) go to weak equivalences, it suffices to show that the maps $G \underset{H}{\ltimes} \widetilde{\xi}_{V,W}$ (see (7.2.63), (7.4.17) and Remark 7.3.5) go to weak equivalences. Since $G \underset{H}{\ltimes} \widehat{\xi}_{V,W}$ is a homotopy equivalence, this is equivalent to showing that maps of the form

$$G \underset{H}{\ltimes} \wedge \xi_{V,W} : G \underset{H}{\ltimes} \wedge \left(S^{-V\oplus W} \wedge S^W\right) \to G \underset{H}{\ltimes} \wedge S^{-V}$$

go to weak equivalences. But these maps go to an indexed wedge of maps of the form

$$\xi_{V',W'} : \left(S^{-V'\oplus W'} \wedge S^{W'}\right) \to S^{-V'}$$

which are weak equivalences. Thus ϕ^* also sends trivial cofibrations to trivial cofibrations. This gives

Proposition 9.8.3. *If $\phi : G' \to G$ is any homomorphism of finite groups, then the pullback functor*

$$\phi^* : \mathcal{S}p^G \to \mathcal{S}p^{G'}$$

is a left Quillen functor (Definition 4.5.1). In particular the restriction functor (the case where $\phi : H \to G$ is the inclusion) is a left Quillen functor.

Theorem 9.8.4. $\mathcal{S}p^G$ as a closed symmetric monoidal model category. *Equipped with the smash product, the positive stable equifibrant model category structure on $(\mathcal{S}p^G, \wedge, S^{-0})$ makes it a closed Quillen ring (Definition 5.5.9) satisfying the monoid axiom of Definition 5.5.22.*

Proof We already know it is a closed symmetric monoidal category by Theorem 9.1.25. We need to show that the smash product satisfies the pushout product and unit axioms of Definition 5.5.9 in addition to the monoid axiom.

The pushout product axiom asserts that given cofibrations $f_i : A_i \to B_i$ for $i = 1$ and 2, then $f_1 \square f_2$ is a cofibration which is trivial if either f_1 or f_2

is trivial. By Proposition 5.5.14 it suffices to check the cofibration condition when f_1 and f_2 are in $\mathcal{I}$ and so of the form

$$G \underset{H_i}{\wedge} S^{-V_i} \wedge \left(S^{k_i-1} \to D^{k_i}\right) \qquad \text{for } i = 1, 2.$$

But in that case the corner map is the smash product of

$$\left(G \underset{H_1}{\wedge} S^{-V_1}\right) \wedge \left(G \underset{H_2}{\wedge} S^{-V_2}\right)$$

(which can be described more explicitly with the help of Theorem 8.1.10) with the pushout product of $S^{k_1-1} \to D^{k_1}$ and $S^{k_2-1} \to D^{k_2}$, namely the inclusion $S^{k_1+k_2-1} \to D^{k_1+k_2}$ as in Example 2.6.17. This is an indexed wedge of cofibrations hence a cofibration by Proposition 5.5.39.

In order to show the second part of the pushout product axiom, we claim that if $g : X \to Y$ is a trivial cofibration in $\mathcal{S}p^G$, and Z is arbitrary then $g \wedge Z$ is a flat weak equivalence. Since g is a cofibration it is an h-cofibration, so it suffices to show that $(Y/X) \wedge Z$ is weakly contractible if Y/X is. But Y/X is cofibrant, hence flat, so the claim follows from Proposition 9.6.6. Now let $f : A \to B$ be a cofibration and consider the diagram (similar to that of Definition 2.6.12)

$$\begin{tikzcd}
A \wedge X \arrow[rr, "f \wedge X"] \arrow[d, "A \wedge g"', "\simeq"] & & B \wedge X \arrow[d, "\simeq"'] \arrow[ddr, bend left, "B \wedge g"] & \\
A \wedge Y \arrow[rr] \arrow[drrr, "f \wedge Y"'] & & P(f \wedge X, A \wedge g) \arrow[dr, "f \square g"] & \\
& & & B \wedge Y
\end{tikzcd}$$

Then $B \wedge g$ is an equivalence since g is a trivial cofibration, so $f \square g$ is one by the two out of three property.

The unit axiom follows from Proposition 9.6.6 since cofibrant objects are Bredon cofibrant (Remark 9.2.17), hence flat (Proposition 9.6.5).

The monoid axiom asserts that maps obtained from those in $\mathcal{J}$ through certain constructions are weak equivalences; see Lemma 5.5.23. The maps in $\mathcal{J}$ are all precofibrations as in Definition 5.1.15, so this follows from Proposition 5.1.21. $\square$

9.9 The $RO(G)$-graded homotopy of $H\underline{\mathbf{Z}}$

We describe part of the $RO(G)$-graded Mackey functor $\underline{\pi}_\star(H\underline{\mathbf{Z}})$, where $H\underline{\mathbf{Z}}$ is the integer Eilenberg-Mac Lane spectrum $H\underline{\mathbf{Z}}$ in the G-equivariant category (see Theorem 9.1.47), for a finite cyclic 2-group G. For each actual (as opposed to virtual) G-representation V we have an equivariant reduced cellular chain complex C_*^V for the space S^V. It is a complex of $\mathbf{Z}[G]$-modules with $H_*(C^V) = H_*(S^{|V|})$. It has Mackey functor homology as in Definition 8.5.1.

Given a finite G-CW spectrum X, meaning a suspension spectrum (see Remark 7.1.25) of a finite G-CW complex as in Definition 8.4.13, we get a reduced cellular chain complex of $\mathbf{Z}[G]$-modules C_*X, leading to a chain complex of fixed point Mackey functors $\underline{C}_*X$, as in Definition 8.5.1. Its homology is a graded Mackey functor $\underline{H}_*X$ with

$$\underline{H}_*X(G/H) = \underline{\pi}_*(X \wedge H\underline{\mathbf{Z}})(G/H) = \pi_*(X \wedge H\underline{\mathbf{Z}})^H.$$

In particular $\underline{H}_*X(G/\{e\}) = H_*X$, the underlying homology of X. In general $\underline{H}_*X(G/H)$ is not the same as $H_*(X^H)$ because fixed points do not commute with smash products of spectra; see Remark 9.1.27.

For a finite cyclic 2-group $G = C_{2^k}$, the irreducible representations are the 2-dimensional ones $\lambda(m)$ corresponding to rotation through an angle of $2\pi m/2^k$ for $0 < m < 2^{k-1}$, the sign representation σ and the trivial one of degree one, which we denote by 1. The 2-local equivariant homotopy type of $S^{\lambda(m)}$ depends only on the 2-adic valuation of m, so we will only consider $\lambda(2^j)$ for $0 \leqslant j \leqslant k-2$ and denote it by λ_j. The planar rotation λ_{k-1} though angle π is the same representation as 2σ. **We will denote $\lambda(1) = \lambda_0$ simply by λ.**

Proposition 9.9.1. The regular representation ρ_G for $G = C_{2^k}$ *is, in the notation defined above,*

$$\rho_G = 1 + \sigma + \sum_{0<m<2^{k-1}} \lambda(m)$$

We will describe the chain complex C^V for

$$V = a + b\sigma + \sum_{2\leqslant j\leqslant k} c_j\lambda_{k-j}. \tag{9.9.2}$$

for nonnegative integers a, b and c_j. This generalizes the discussion of Example 8.5.5, which deals with the case $k = 1$ and $a = b = n$. The isotropy group of V (the largest subgroup fixing all of V) is

$$G_V = \begin{cases} C_{2^k} = G & \text{for } b = c_2 = \cdots = c_k = 0 \\ C_{2^{k-1}} =: G' & \text{for } b > 0 \text{ and } c_2 = \cdots = c_k = 0 \\ C_{2^{k-\ell}} & \text{for } c_\ell > 0 \text{ and } c_{1+\ell} = \cdots = c_k = 0 \end{cases}$$

The proof of the following is an exercise for the reader.

Proposition 9.9.3. The representation sphere S^V as a G-CW complex. *Let $G = C_{2^k}$ and let V be as in (9.9.2). Then the sphere S^V has a*

G-CW structure with reduced cellular chain complex C^V of the form

$$C_n^V = \begin{cases} \mathbf{Z} & \text{for } n = d_0 \\ \mathbf{Z}[G/G'] & \text{for } d_0 < n \leqslant d_1 \\ \mathbf{Z}[G/C_{2^{k-j}}] & \text{for } d_{j-1} < n \leqslant d_j \text{ and } 2 \leqslant j \leqslant \ell \\ 0 & \text{otherwise.} \end{cases} \tag{9.9.4}$$

where

$$d_j = \begin{cases} a & \text{for } j = 0 \\ a+b & \text{for } j = 1 \\ a+b+2c_2+\cdots+2c_j & \text{for } 2 \leqslant j \leqslant \ell, \end{cases}$$

so $d_\ell = |V|$.

The boundary map $\partial_n : C_n^V \to C_{n-1}^V$ is determined by the fact that $H_*(C^V) = H_*(S^{|V|})$. More explicitly, let γ be a generator of G, and let

$$e_j = \sum_{0 \leqslant t < 2^j} \gamma^t \qquad \text{for } 1 \leqslant j \leqslant k.$$

Then we have

$$\partial_n = \begin{cases} \nabla & \text{for } n = 1 + d_0 \\ (1-\gamma)x_n & \text{for } n - d_0 \text{ even and } 2 + d_0 \leqslant n \leqslant d_n \\ x_n & \text{for } n - d_0 \text{ odd and } 2 + d_0 \leqslant n \leqslant d_n \\ 0 & \text{otherwise,} \end{cases}$$

where ∇ is the fold map sending $\gamma \mapsto 1$, and x_n denotes multiplication by an element in $\mathbf{Z}[G]$ to be named below. We will use the same symbol below for the quotient map $\mathbf{Z}[G/H] \to \mathbf{Z}[G/K]$ for $H \subseteq K \subseteq G$. The elements $x_n \in \mathbf{Z}[G]$ for $2 + d_0 \leqslant n \leqslant |V|$ are determined recursively by $x_{2+d_0} = 1$ and

$$x_n x_{n-1} = e_j \qquad \text{for } 2 + d_{j-1} < n \leqslant 2 + d_j.$$

Then $H_{|V|}C^V = \mathbf{Z}$ generated by either $x_{1+|V|}$ or its product with $1-\gamma$, depending on the parity of b.

Corollary 9.9.5. The low dimensional homology of S^V. *Let $G = C_{2^k}$ and let V be as in (9.9.2). Then the map*

$$S^{a+b\sigma} \to S^V$$

induces an isomorphism in homology below dimension $a+b$.

The homology of $S^{n\sigma}$ will be computed below in Example 9.9.21.

The chain complex of Proposition 9.9.3 is

$$C^V = \Sigma^{|V_0|} C^{V/V_0}$$

where $V_0 = V^G$. This means we can assume without loss of generality that $V_0 = 0$.

An element

$$x \in H_n\underline{C}^V(G/H) = \underline{H}_n S^V(G/H)$$

corresponds to an element $x \in \underline{\pi}_{n-V} H\mathbf{Z}(G/H)$.

We will denote the dual complex $\mathrm{Hom}_{\mathbf{Z}}(C^V, \mathbf{Z})$ by C^{-V}. Its chains lie in dimensions $-n$ for $0 \leqslant n \leqslant |V|$. An element $x \in \underline{H}_{-n}(S^{-V})(G/H)$ corresponds to an element $x \in \underline{\pi}_{V-n} H\mathbf{Z}(G/H)$.

The method we have just described determines only a portion of the $RO(G)$-graded Mackey functor $\underline{\pi}_{(G,\star)} H\underline{\mathbf{Z}}$, namely the groups in which the index differs by an integer from an actual representation V or its negative. For example it does not give us $\underline{\pi}_{\sigma-\lambda} H\underline{\mathbf{Z}}$ for $|G| \geqslant 4$.

We leave the proof of the following as an exercise for the reader.

Proposition 9.9.6. The top (bottom) homology groups for S^V (S^{-V}). *Let G be a finite cyclic 2-group and V a nontrivial representation of G of degree d with $V^G = 0$ and isotropy group G_V (see Definition 2.1.29(iv)). Then*

$$C_d^V = C_{-d}^{-V} = \mathbf{Z}[G/G_V]$$

and

(i) *If V is oriented then $\underline{H}_d S^V = \underline{\mathbf{Z}}$, the constant $\mathbf{Z}$-valued Mackey functor in which each restriction map is an isomorphism and each transfer Tr_H^K is multiplication by $|K/H|$.*

(ii) *$\underline{H}_{-d} S^{-V} = \underline{\mathbf{Z}}(G, G_V)$, the constant $\mathbf{Z}$-valued Mackey functor in which*

$$\mathrm{Res}_H^K = \begin{cases} 1 & \text{for } K \subseteq G_V \\ |K/H| & \text{for } G_V \subseteq H \end{cases}$$

and

$$\mathrm{Tr}_H^K = \begin{cases} |K/H| & \text{for } K \subseteq G_V \\ 1 & \text{for } G_V \subseteq H. \end{cases}$$

(The above completely describes the cases where $|K/H| = 2$, and they determine all other restrictions and transfers.) The functor $\underline{\mathbf{Z}}(G, e)$ is also known as the dual $\underline{\mathbf{Z}}^*$. These isomorphisms are induced by the maps

$$\begin{array}{ccccc} \underline{H}_d S^V & & & & \underline{H}_{-d} S^{-V} \\ \| & & & & \| \\ \underline{\mathbf{Z}} & \xrightarrow{\Delta} & \underline{\mathbf{Z}[G/G_V]} & \xrightarrow{\nabla} & \underline{\mathbf{Z}}(G, G_V). \end{array}$$

(iii) *If V is not oriented then $\underline{H}_d S^V = \underline{\mathbf{Z}}_-$, where*

$$\underline{\mathbf{Z}}_-(G/H) = \begin{cases} 0 & \text{for } H = G \\ \mathbf{Z}_- := \mathbf{Z}[G]/(1+\gamma) & \text{otherwise} \end{cases}$$

where each restriction map Res_H^K is an isomorphism and each transfer Tr_H^K is multiplication by $|K/H|$ for each proper subgroup K.

(iv) We also have $\underline{H}_{-d}S^{-V} = \underline{\mathbf{Z}}(G, G_V)_-$, *where*

$$\underline{\mathbf{Z}}(G, G_V)_-(G/H) = \begin{cases} 0 & \text{for } H = G \text{ and } V = \sigma \\ \mathbf{Z}/2 & \text{for } H = G \text{ and } V \neq \sigma \\ \mathbf{Z}_- & \text{otherwise} \end{cases}$$

with the same restrictions and transfers as $\underline{\mathbf{Z}}(G, G_V)$. *These isomorphisms are induced by the maps*

$$\begin{array}{ccccc} \underline{H}_d S^V & & & & \underline{H}_{-d}S^{-V} \\ \| & & & & \| \\ \underline{\mathbf{Z}}_- & \xrightarrow{\Delta_-} & \underline{\mathbf{Z}[G/G_V]} & \xrightarrow{\nabla_-} & \underline{\mathbf{Z}}(G, G_V)_- . \end{array}$$

The Mackey functor $\underline{\mathbf{Z}}(G, G_V)$ is one of those defined (with different notation) in [HHR17a, Definition 2.1].

Definition 9.9.7. Three elements in $\underline{\pi}_\star^G(H\mathbf{Z})$. *Let V be an actual (as opposed to virtual) representation of the finite cyclic 2-group G with $V^G = 0$ and isotropy group G_V.*

(i) The equivariant inclusion $S^0 \to S^V$ defines an element in $\underline{\pi}_{-V}S^0(G/G)$ via the isomorphisms

$$\underline{\pi}_{-V}S^0(G/G) \cong \underline{\pi}_0 S^V(G/G) \cong \pi_0 S^{V^G} \cong \pi_0 S^0 \cong \mathbf{Z},$$

and we will use the symbol a_V to denote its image in $\underline{\pi}_{-V}H\underline{\mathbf{Z}}(G/G)$.

(ii) The underlying equivalence $S^V \to S^{|V|}$ defines an element in

$$\underline{\pi}_V S^{|V|}(G/G_V) \cong \underline{\pi}_{V-|V|}S^0(G/G_V)$$

and we will use the symbol e_V to denote its Hurewicz image in

$$\underline{\pi}_{V-|V|}H\underline{\mathbf{Z}}(G/G_V).$$

(iii) If W is an oriented representation of G (we do not require that $W^G = 0$), there is a map

$$\Delta : \mathbf{Z} \to C^W_{|W|} = \mathbf{Z}[G/G_W]$$

as in Proposition 9.9.6 giving an element

$$u_W \in \underline{H}_{|W|}S^W(G/G) \cong \underline{\pi}_{|W|-W}H\mathbf{Z}(G/G).$$

For nonoriented W, Proposition 9.9.6 gives a map

$$\Delta_- : \mathbf{Z}_- \to C^W_{|W|}$$

and an element

$$u_W \in \underline{H}_{|W|}S^W(G/G') \cong \underline{\pi}_{|W|-W}H\mathbf{Z}(G/G').$$

The element u_W above is related to the element $\widetilde{u}_V$ of (9.4.11) as follows.

Lemma 9.9.8. The restriction of u_W to a unit and permanent cycle. *Let W be a nontrivial representation of G with $H = G_W$. Then the homeomorphism*

$$\Sigma^{-W}\widetilde{u}_W : G/H \ltimes S^{|W|-W} \to G/H_+$$

of (9.4.11) induces an isomorphism $\underline{\pi}_0 H\underline{\mathbf{Z}}(G/H) \to \underline{\pi}_{|W|-W} H\underline{\mathbf{Z}}(G/H)$ sending the unit to $\mathrm{Res}^K_H(u_W)$ for u_W as defined in (iii) above and $K = G$ or G' depending on the orientability of W.

The product

$$\mathrm{Res}^K_H(u_W)e_W \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/H) = \mathbf{Z}$$

is a generator, so e_W and $\mathrm{Res}^K_H(u_W)$ are units in the ring $\underline{\pi}_\star H\underline{\mathbf{Z}}(G/H)$, and $\mathrm{Res}^K_H(u_W)$ is in the Hurewicz image of $\underline{\pi}_\star S^0(G/H)$.

Proof The diagram

$$G/K \ltimes S^{|W|-W} \xleftarrow{\text{fold}} G/H \ltimes S^{|W|-W} \xrightarrow{\widetilde{u}_W} G/H_+$$

induces (via the functor $[\cdot, H\underline{\mathbf{Z}}]^G$)

$$\begin{tikzcd}
\underline{\pi}_{|W|-W} H\underline{\mathbf{Z}}(G/K) \arrow[r, "\mathrm{Res}^K_H"] \arrow[d, equal] & \underline{\pi}_{|W|-W} H\underline{\mathbf{Z}}(G/H) \arrow[d, equal] & \underline{\pi}_0 H\underline{\mathbf{Z}}(G/H) \arrow[l, "\cong"'] \arrow[d, equal] \\
\underline{H}_{|W|} S^W(G/K) & \underline{H}_{|W|} S^W(G/H) & \mathbf{Z}
\end{tikzcd}$$

The restriction map is an isomorphism by Proposition 9.9.6 and the group on the left is generated by u_W.

The product is the composite of H-maps

$$S^W \xrightarrow{\quad e_W \quad} S^{|W|} \xrightarrow{\mathrm{Res}^K_H(u_W)} \Sigma^W H\underline{\mathbf{Z}},$$

which is the standard inclusion. □

Remark 9.9.9. The elements a_V and e_V are permanent cyles but u_V may not be. *Note that a_V and e_V are induced by maps to equivariant spheres while u_W is not. This means that in any spectral sequence based on a filtration where the subquotients are equivariant $H\mathbf{Z}$-modules, elements defined in terms of a_V and e_V will be permanent cycles, while multiples and powers of u_W can support nontrivial differentials. Lemma 9.9.8 says a certain restriction of u_W is a permanent cycle.*

Each nonoriented V has the form $W + \sigma$ where σ is the sign representation and W is oriented. It follows that

$$u_V = u_\sigma \mathrm{Res}^G_{G'}(u_W) \in \underline{\pi}_{|V|-V} H\mathbf{Z}(G/G').$$

Note also that $a_0 = e_0 = u_0 = 1$. The trivial representations contribute nothing to $\pi_\star(H\mathbf{Z})$. We can limit our attention to representations V with $V^G = 0$. Among such representations of cyclic 2-groups, the oriented ones are precisely the ones of even degree.

Lemma 9.9.10. Properties of a_V, e_V and u_W. *The elements*

$$a_V \in \underline{\pi}_{-V} H\mathbf{Z}(G/G),\ e_V \in \underline{\pi}_{V-|V|} H\mathbf{Z}(G/G_V) \text{ and } u_W \in \underline{\pi}_{|W|-W} H\mathbf{Z}(G/G)$$

for W oriented of Definition 9.9.7 satisfy the following.

(i) $a_{V+W} = a_V a_W$ *and* $u_{V+W} = u_V u_W$.
(ii) $|G/G_V| a_V = 0$ *where G_V is the isotropy group of V.*
(iii) For oriented V, $\mathrm{Tr}_{G_V}^G(e_V)$ *and* $\mathrm{Tr}_{G_V}^{G'}(e_{V+\sigma})$ *have infinite order, while* $\mathrm{Tr}_{G_V}^G(e_{V+\sigma})$ *has order 2 if $|V| > 0$ and* $\mathrm{Tr}_{G_V}^G(e_\sigma) = \mathrm{Tr}_{G'}^G(e_\sigma) = 0$.
(iv) For oriented V and $G_V \subseteq G' \subseteq G$

$$\mathrm{Tr}_{G_V}^G(e_V)u_V = |G/G_V| \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/G) = \mathbf{Z}$$
$$\text{and} \qquad \mathrm{Tr}_{G_V}^{G'}(e_{V+\sigma})u_{V+\sigma} = |G'/G_V| \in \underline{\pi}_0 H\underline{\mathbf{Z}}(G/G') = \mathbf{Z} \qquad \text{for } |V| > 0.$$

(v) $a_{V+W}\mathrm{Tr}_{G_V}^G(e_{V+W}) = 0$ *if* $|V| > 0$.
(vi) For V and W oriented, $u_W \mathrm{Tr}_{G_V}^G(e_{V+W}) = |G_V/G_{V+W}|\mathrm{Tr}_{G_V}^G(e_V)$.
(vii) **The gold (or au) relation.** *For V and W oriented representations of degree 2 with $G_V \subseteq G_W$,* $a_W u_V = |G_W/G_V| a_V u_W$.

For nonoriented W similar statements hold in $\underline{\pi}_\star H\mathbf{Z}(G/G')$. $2W$ is oriented and u_{2W} is defined in $\underline{\pi}_{2|W|-2W} H\underline{\mathbf{Z}}(G/G)$ with $\mathrm{Res}_{G'}^G(u_{2W}) = u_W^2$.

Proof (i) This follows from the existence of the pairing

$$C^V \otimes C^W \to C^{V+W}.$$

It induces an isomorphism in H_0 and (when both V and W are oriented) in $H_{|V+W|}$.

(ii) This holds because $H_0(V)$ is killed by $|G/G_V|$.
(iii) This follows from Proposition 9.9.6.
(iv) Using the Frobenius relation we have

$$\begin{aligned}
\mathrm{Tr}_{G_V}^G(e_V)u_V &= \mathrm{Tr}_{G_V}^G(e_V \mathrm{Res}_{G_V}^G(u_V)) \\
&= \mathrm{Tr}_{G_V}^G(1) \qquad \text{by Lemma 9.9.8} \\
&= |G/G_V| \\
\mathrm{Tr}_{G_V}^{G'}(e_{V+\sigma})u_{V+\sigma} &= \mathrm{Tr}_{G_V}^{G'}(e_{V+\sigma}\mathrm{Res}_{G_V}^{G'}(u_{V+\sigma})) \\
&= \mathrm{Tr}_{G_V}^{G'}(1) = |G'/G_V|.
\end{aligned}$$

(v) We have

$$a_{V+W}\mathrm{Tr}_{G_V}^G(e_{V+U}) : S^{-|V|-|U|} \to S^{W-U}.$$

It is null because the bottom cell of S^{W-U} is in dimension $-|U|$.

(vi) Since V is oriented, then we are computing in a torsion free group so we can tensor with the rationals. It follows from (iv) that

$$\begin{aligned} \mathrm{Tr}_{G_{V+W}}^{G}(e_{V+W}) &= \frac{|G/G_{V+W}|}{u_V u_W} \\ \text{and} \qquad \mathrm{Tr}_{G_V}^{G}(e_V) &= \frac{|G/G_V|}{u_V} \\ \text{so} \qquad u_W \mathrm{Tr}_{G_{V+W}}^{G}(e_{V+W}) &= \frac{|G/G_{V+W}|}{u_V} = |G_V/G_{V+W}|\mathrm{Tr}_{G_V}^{G}(e_V). \end{aligned}$$

(vii) For $G = C_{2^n}$, each oriented representation of degree 2 is 2-locally equivalent to a λ_j for $0 \leqslant j < n$. The isotropy group is $G_{\lambda_j} = C_{2^j}$. Hence the assumption that $G_V \subset G_W$ is can be replaced with $V = \lambda_j$ and $W = \lambda_k$ with $0 \leqslant j < k < n$. the statment we wish to prove is

$$a_{\lambda_k} u_{\lambda_j} = 2^{k-j} a_{\lambda_j} u_{\lambda_k}.$$

One has a map $S^{\lambda_j} \to S^{\lambda_k}$ which is the suspension of the 2^{k-j}th power map on the equatorial circle. Hence its underlying degree is 2^{k-j}. We will denote it by $a_{\lambda_k}/a_{\lambda_j}$ since there is a diagram

$$\begin{tikzcd} & S^{\lambda_j} \arrow[d, "a_{\lambda_k}/a_{\lambda_j}"] \\ S^0 \arrow[ur, "a_{\lambda_j}"] \arrow[r, "a_{\lambda_k}"'] & S^{\lambda_k}. \end{tikzcd}$$

We claim there is a similar diagram

$$\begin{tikzcd} & S^{\lambda_k} \wedge H\underline{\mathbf{Z}} \arrow[d, "u_{\lambda_j}/u_{\lambda_k}"] \\ S^2 \arrow[ur, "u_{\lambda_k}"] \arrow[r, "u_{\lambda_j}"'] & S^{\lambda_j} \wedge H\underline{\mathbf{Z}} \end{tikzcd} \tag{9.9.11}$$

in which the underlying degree of the vertical map is one.

Smashing $a_{\lambda_k}/a_{\lambda_j}$ with $H\underline{\mathbf{Z}}$ and composing with $u_{\lambda_j}/u_{\lambda_k}$ gives a factorization of the degree 2^{k-j} map on $S^{\lambda_j} \wedge H\underline{\mathbf{Z}}$. Thus we have

$$\begin{aligned} \frac{u_{\lambda_j}}{u_{\lambda_k}} \frac{a_{\lambda_k}}{a_{\lambda_j}} &= 2^{k-j} \\ u_{\lambda_j} a_{\lambda_k} &= 2^{k-j} u_{\lambda_k} a_{\lambda_j} \end{aligned}$$

as desired.

The vertical map in (9.9.11) would follow from a map

$$S^{\lambda_k - \lambda_j} \to H\underline{\mathbf{Z}}$$

with underlying degree one. Let $G = C_{2^n}$ and $G \supset H = C_{2^j}$. Then $S^{-\lambda_j}$ has a cellular structure of the form

$$G/H \ltimes S^{-2} \cup G/H \ltimes e^{-1} \cup e^0.$$

We need to smash this with S^{λ_k}. Since λ_k restricts trivially to H,

$$G/H \ltimes S^{\lambda_k} = G/H \ltimes S^2.$$

This means

$$S^{\lambda_k-\lambda_j} = S^{\lambda_k} \wedge S^{-\lambda_j} = G/H \ltimes S^0 \cup G/H \ltimes e^1 \cup e^0 \wedge S^{\lambda_k}.$$

Thus its cellular chain complex has the form

$$\begin{tikzcd}
2 & \mathbf{Z}[G/K] \arrow[d, "1-\gamma"] \arrow[dr, "\Delta"] & \\
1 & \mathbf{Z}[G/K] \arrow[d, "\nabla"] \arrow[dr, "-\Delta"] & \mathbf{Z}[G/H] \arrow[d, "1-\gamma"] \\
0 & \mathbf{Z} & \mathbf{Z}[G/H]
\end{tikzcd}$$

where $K = G/C_{p^k}$ and the left column is the chain complex for S^{λ_k}.

There is a corresponding chain complex of fixed point Mackey functors. Its value on the G-set G/L for an arbitrary subgroup L is

$$\begin{tikzcd}
2 & \mathbf{Z}[G/\max(K,L)] \arrow[d, "1-\gamma"] \arrow[dr, "\Delta"] & \\
1 & \mathbf{Z}[G/\max(K,L)] \arrow[d, "\nabla"] \arrow[dr, "-\Delta"] & \mathbf{Z}[G/\max(H,L)] \arrow[d, "1-\gamma"] \\
0 & \mathbf{Z} & \mathbf{Z}[G/\max(H,L)]
\end{tikzcd}$$

For each L the map Δ is injective and maps the kernel of the first $1-\gamma$ isomorphically to the kernel of the second one. This means we can replace the above by a diagram of the form

$$\begin{tikzcd}
1 & \operatorname{coker}(1-\gamma) \arrow[d, "\nabla"] \arrow[dr, "-\Delta"] & \\
0 & \mathbf{Z} & \operatorname{coker}(1-\gamma)
\end{tikzcd}$$

where each cokernel is isomorphic to $\mathbf{Z}$ and each map is injective.

This means that $\underline{H}_* S^{\lambda_k-\lambda_j}$ is concentrated in degree 0 where it is the pushout of the diagram above, meaning a Mackey functor whose value on each subgroup is $\mathbf{Z}$. Any such Mackey functor admits a map to $\underline{\mathbf{Z}}$ with underlying degree one. This proves the claim of (9.9.11). $\square$

The elements a_V and u_V behave well with respect to the norm. The following result is a simple consequence of the fact (Proposition 8.3.28) that $NS^V = S^{\mathrm{Ind}V}$.

Lemma 9.9.12. The norms of a_V and u_V. *Suppose that V is a d-dimensional representation of a subgroup $H \subset G$, let $W = \mathrm{Ind}_H^G V$ be the induced representation over G, and let $L = \mathrm{Ind}_H^G d$, the representation of G induced up from the d-dimensional trivial representation of H. Then*

$$\begin{aligned} a_W &= N_H^G a_V \\ \textit{and} \qquad u_W &= u_L \cdot N_H^G u_V. \end{aligned}$$

When the subgroup H above is normal, the representation L is d copies of the composition of $G \to G/H$ with the regular representation $\rho_{G/H}$.

Corollary 9.9.13. The case where H has index 2 in G and $d = 2k$. *Here $L = 2k(1+\sigma)$, where σ is the sign representation of G (which factors through G/H), so we have*

$$u_W = u_{2\sigma}^k N_H^G u_V.$$

Remark 9.9.14. Notation for multiplication. *As is standard in algebra, we will adopt the convention that the operation of mutiplication by an element of a ring on a module is denoted by the element of the ring. We will also use it in closely related contexts. For example, for a G-spectrum X we will refer to the to the maps*

$$\begin{aligned} a_V \wedge 1_X &: S^{-V} \wedge X \to X \\ u_V \wedge 1_X &: S^{d-V} \wedge X \to H\underline{\mathbf{Z}} \wedge X \end{aligned}$$

as multiplication *by a_V and u_V respectively, and, when no confusion is likely, denote them simply by a_V and u_V. Note that X might be a virtual representation sphere. This means that we will not usually distinguish in notation between these maps and their suspensions. Similarly, if R is any equivariant algebra, and $x \in \pi_V^G S^0$ then the product of x with $1 \in \pi_0^G R$ will be denoted $x \in \pi_V^G R$. In accordance with this, at various places in this paper the symbol a_V might refer to a map $S^{-V} \to S^0$, or its suspension $S^0 \to S^V$ or the Hurewicz image $S^0 \to H\underline{\mathbf{Z}} \wedge S^V$ or equivalently an element of $\pi_0^G H\underline{\mathbf{Z}} \wedge S^V$.*

The $\mathbf{Z}$-valued Mackey functor $\underline{H}_0 S^{\lambda_k - \lambda_j}$ is discussed in more detail in [HHR17a], where it is denoted by $\underline{\mathbf{Z}}(k,j)$.

Example 9.9.15. Inverting a_V. *Let $S^{\infty V}$ be the colimit of the spaces S^{nV} under the standard inclusions. Each of these inclusions is "multiplication by a_V." Smashing with a G-spectrum X we find that $S^{\infty V} \wedge X$ is the colimit of the sequence*

$$X \xrightarrow{a_V} S^V \wedge X \xrightarrow{a_V} S^{V \oplus V} \wedge X \cdots \xrightarrow{a_V} S^{nV} \wedge X \xrightarrow{a_V} \cdots.$$

Using the suspension isomorphism to replace $\pi_\star^G S^{nV} \wedge X$ with $\pi_{\star - nV}^G X$ the sequence of the $RO(G)$-graded groups becomes

$$\pi_\star^G X \xrightarrow{a_V} \pi_{\star - V}^G X \cdots \xrightarrow{a_V} \pi_{\star - nV}^G X \cdots$$

from which one gets an isomorphism

$$\pi_\star^G S^{\infty V} \wedge X \cong a_V^{-1} \pi_\star^G X.$$

Under this isomorphism the effect in $RO(G)$-graded homotopy groups induced by the inclusion

$$S^{nV} \wedge X \to S^{\infty V} \wedge X$$

sends $x \in \pi_\star^G X \cong \pi_{\star+V}^G S^{nV} \wedge X$ to $a_V^{-n} x \in a_V^{-1} \pi_\star^G X$.

Example 9.9.16. Inverting a_σ. *Passing to the colimit as $d \to \infty$ and using the last part of Example 9.9.15 we find that $a_{(d-k)\sigma} \cdot u_{k\sigma}$ is sent to*

$$a_{d\sigma}^{-1} \cdot a_{(d-k)\sigma} \cdot u_{k\sigma} = a_{k\sigma}^{-1} u_{k\sigma} \in \pi_k S^{\infty\sigma}.$$

Writing $b = a_{2\sigma}^{-1} u_{2\sigma}$ we find that the homogeneous component

$$\pi_{2n}^{C_2} H\underline{\mathbf{Z}} \wedge S^{\infty\sigma} \subset \pi_\star^{C_2} H\underline{\mathbf{Z}} \wedge S^{\infty\sigma} = a_{2\sigma}^{-1} \pi^{C_2} H\underline{\mathbf{Z}}$$

is cyclic of order 2, generated by b^n. See Remark 9.9.20 below.

In §13.1 below we will need a description of of $\underline{\pi}_\star S^{n\rho} \wedge H\mathbf{Z}$, the $RO(G)$-graded homotopy of $S^{n\rho} \wedge H\mathbf{Z}$ for $G = C_2$, where ρ denotes the regular representation. The representation ring $RO(C_2)$ is the free abelian group generated by 1, meaning the trivial one dimensional representation, and σ, the sign representation. The regular representation ρ is $\sigma + 1$. It follows that

$$\begin{aligned} \underline{\pi}_{a+b\sigma} S^{n\rho} \wedge H\mathbf{Z} &= \underline{\pi}_{a+b\sigma} S^{n+n\sigma} \wedge H\mathbf{Z} \\ &= \underline{\pi}_{a-n+(b-n)\sigma} H\mathbf{Z}. \end{aligned}$$

Hence we need to find $\underline{\pi}_{a+b\sigma} H\mathbf{Z}$ for all integers a and b.

In (8.5.9) we determined

$$\underline{H}_i S^{n\rho} = \underline{\pi}_i S^{n+n\sigma} \wedge H\mathbf{Z} = \underline{\pi}_{i-n-n\sigma} \wedge H\mathbf{Z}$$

for $n \geqslant 0$ and all integers i. Thus we know $\underline{\pi}_{a+b\sigma} H\mathbf{Z}$ for all integers a and b with $b \leqslant 0$.

It remains to compute

$$\underline{\pi}_{a+b\sigma} H\mathbf{Z} = \underline{\pi}_{a+b} S^{-b\rho} H\mathbf{Z} \qquad \text{for } b > 0.$$

Since $S^{-n\rho}$ is the Spanier-Whitehead dual (see (9.5.16)) of $S^{n\rho}$, we need to look at the $\mathbf{Z}$-linear dual of the chain complex of (8.5.8). It has the form

$$\begin{tikzcd}
-n & -n-1 & -n-2 & -n-3 & & -2n \\
\square \arrow[r, "\Delta"] & \widehat{\square} \arrow[r, "\gamma_2"] & \widehat{\square} \arrow[r, "\gamma_3"] & \widehat{\square} \arrow[r, "\gamma_4"] & \cdots \arrow[r, "\gamma_n"] & \widehat{\square} \\
\mathbf{Z} \arrow[r, "1"] \arrow[d, bend right, "1"'] & \mathbf{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[r, "2"] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[r, "\epsilon_n"] & \mathbf{Z} \arrow[d, bend right, "\Delta"'] \\
\mathbf{Z} \arrow[r, "\Delta"] \arrow[u, bend right, "2"'] & \mathbf{Z}[G] \arrow[r, "\gamma_2"] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[G] \arrow[r, "\gamma_4"] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[G] \arrow[r, "\gamma_4"] \arrow[u, bend right, "\nabla"'] & \cdots \arrow[r, "\gamma_n"] & \mathbf{Z}[G] \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

A key point here is that $\Delta : \square \to \widehat{\square}$ is the dual of $\nabla : \widehat{\square} \to \square$. The former induces an isomorphism when evaluated on G/G while the latter induces multiplication by 2.

Passing to homology we get

$$\begin{tikzcd}[row sep=small]
-n & -n-1 & -n-2 & -n-3 & & -2n \\
0 & 0 & 0 & \bullet & \cdots & \underline{H}_{-2n} \\
0 \arrow[d, bend right] & 0 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbf{Z}/2 \arrow[d, bend right] & \cdots & \underline{H}_{-2n}(G/G) \arrow[d, bend right, "\Delta"'] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & \mathbf{Z}[G]/(\gamma_n) \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

We need to pay careful attention to the bottom homology group $\underline{H}_{-2n}$, which depends on the parity of n. For $n > 1$ there is an exact sequence of Mackey functors

$$\widehat{\square} \xrightarrow{\gamma_n} \widehat{\square} \longrightarrow \underline{H}_{-2n} \longrightarrow \underline{0}. \tag{9.9.17}$$

For even n it reads

$$\begin{tikzcd}
Z \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & Z \arrow[r, "1"] \arrow[d, bend right, "\Delta"'] & Z \arrow[r] \arrow[d, bend right, "2"'] & 0 \arrow[d, bend right] \\
\mathbf{Z}[G] \arrow[r, "1-\gamma"] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[G] \arrow[r, "\nabla"] \arrow[u, bend right, "\nabla"'] & Z \arrow[r] \arrow[u, bend right, "1"'] & 0. \arrow[u, bend right]
\end{tikzcd}$$

while for odd n we have

$$\begin{tikzcd}
Z \arrow[r, "2"] \arrow[d, bend right, "\Delta"'] & Z \arrow[r, "1"] \arrow[d, bend right, "\Delta"'] & Z/2 \arrow[r] \arrow[d, bend right, "0"'] & 0 \arrow[d, bend right] \\
\mathbf{Z}[G] \arrow[r, "1+\gamma"] \arrow[u, bend right, "\nabla"'] & \mathbf{Z}[G] \arrow[r, "\nabla_-"] \arrow[u, bend right, "\nabla"'] & Z_- \arrow[r] \arrow[u, bend right, "1"'] & 0. \arrow[u, bend right]
\end{tikzcd}$$

These homology groups are respectively the Mackey functors $\blacksquare$ and $\dot{\square}$ of Table 8.1.

For $n = 1$, instead of the exact sequence of (9.9.17) we have

$$\square \xrightarrow{\delta} \widehat{\square} \longrightarrow \underline{H}_{-2n} \longrightarrow \underline{0},$$

which reads

$$\begin{tikzcd}
Z \arrow[r, "1"] \arrow[d, bend right, "1"'] & Z \arrow[r, "1"] \arrow[d, bend right, "\Delta"'] & 0 \arrow[r] \arrow[d, bend right] & 0 \arrow[d, bend right] \\
\mathbf{Z} \arrow[r, "1+\gamma"] \arrow[u, bend right, "2"'] & \mathbf{Z}[G] \arrow[r, "\nabla_-"] \arrow[u, bend right, "\nabla"'] & Z_- \arrow[r] \arrow[u, bend right] & 0. \arrow[u, bend right]
\end{tikzcd}$$

Thus for $n > 0$ we have

$$\underline{H}_i(S^{-n\rho}) = \begin{cases} \bullet & \text{for } -n-3 \geqslant i > -2n \text{ and } i+n \text{ odd} \\ \boxbslash & \text{for } i = -2n \text{ and } n \text{ even} \\ \dot{\square} & \text{for } i = -2n \text{ and } n \text{ odd and } n > 1 \\ \overline{\square} & \text{for } i = -2n \text{ and } n \text{ odd and } n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{9.9.18}$$

The vanishing of $\underline{H}_{-1-n}S^{-n\rho}$, which contrasts with the nonvanishing of $\underline{H}_n S^{n\rho}$ in (8.5.9), is part of the phenomenon we call **the gap**.

By combining (9.9.18) with (8.5.9) we can describe $\underline{\pi}_{a+b\sigma}\Sigma^{n\rho}H\mathbf{Z}$ in all cases. The following is illustrated in Figure 9.1 below.

Theorem 9.9.19. The $RO(C_2)$-graded homotopy of $H\mathbf{Z}$. *For $G = C_2$, σ the sign representation and $\rho = \sigma + 1$ the regular representation, the Mackey functor*

$$\underline{\pi}_{a+b\sigma}\Sigma^{n\rho}H\mathbf{Z} = \underline{\pi}_{a+b\sigma-n}\Sigma^{n\sigma}H\mathbf{Z}$$

is as follows, with Mackey functor notation as in Table 8.1.

- *For $a + b = 2n$ it is*

$$\begin{aligned} \underline{\pi}_{2(n-b)}\Sigma^{(n-b)\rho}H\mathbf{Z} &= \underline{H}_{2(n-b)}S^{(n-b)\rho} = \underline{H}_{(n-b)}S^{(n-b)\sigma} \\ &= \begin{cases} \square & \text{for } n-b \geqslant 0 \text{ and } n-b \text{ even} \\ \overline{\square} & \text{for } n-b \geqslant -1 \text{ and } n-b \text{ odd} \\ \boxbslash & \text{for } n-b \leqslant 2 \text{ and } n-b \text{ even} \\ \dot{\square} & \text{for } n-b \leqslant -3 \text{ and } n-b \text{ odd.} \end{cases} \end{aligned}$$

- *It is $\bullet$ for $n > b$ (which is equivalent to $n < 2n - b$), $a + n$ even and $n \leqslant a < 2n - b$.*
- *It is also $\bullet$ for $n < b$ (which is equivalent to $n > 2n - b$), $a + n$ odd and $n - 3 \geqslant a > 2n - b$.*
- *It is trivial in all other cases.*

Moreover, in the notation of Definition 9.9.7, we have the following.

- *For $n \geqslant b$ and $n - b$ even, $\underline{H}_{2(n-b)}S^{(n-b)\rho}(G/G)$ is generated by $u_{2\sigma}^{(n-b)/2}$.*
- *For $n \geqslant b$ and $n - b$ odd, $\underline{H}_{2(n-b)}S^{(n-b)\rho}(G/e)$ is generated by $u_{\sigma}^{(n-b)}$.*
- *For $n > b$ and $0 \leqslant i < (n-b)/2$ the group $\underline{H}_{n-n+2i}S^{(n-b)\rho}(G/G)$ (which has order 2) is generated by $u_{2\sigma}^{i}a_{\sigma}^{b-n-2i}$.*
- *For $n < b$, $\underline{H}_{2(n-b)}S^{(n-b)\rho}(G/e)$ is generated by $e_{(b-n)\sigma}$. Its transfer in $\underline{H}_{2(n-b)}S^{(n-b)\rho}(G/G)$ is trivial for $b - n = 1$, has infinite order for $b - n$ even, and has order two for $b - n$ odd and $b - n \geqslant 3$.*
- *For $b - n$ odd and $b - n \geqslant 3$, the element of order 2*

$$x_{n-b} := \mathrm{Tr}_e^G e_{(b-n)\sigma} \in \underline{H}_{2(n-b)}S^{(n-b)\rho}(G/G)$$

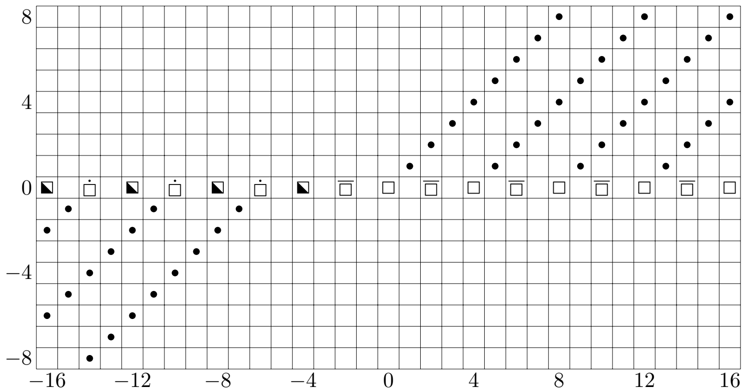

Figure 9.1 **The $RO(C_2)$-graded homotopy of $H\underline{\mathbf{Z}}$ as in Theorem 9.9.19.** The Mackey functor $\underline{H}_i S^{n\rho} = \underline{\pi}_i \Sigma^{n\rho} H\underline{\mathbf{Z}}$ is shown at $(i, 2n-i)$. The symbols used are defined in Table 8.1. Multiplication by a_σ, u_σ and $u_{2\sigma}$ are in directions $(1,1)$, $(2,0)$ and $(4,0)$ respectively.

is infinitely divisible by $a_\sigma \in \underline{H}_1 S^\rho$. For such b and n, for each $i \geqslant 0$, $a_\sigma^{-i} x_{n-b}$ generates the group $\underline{H}_{2(n-b)-i} S^{(n-b-i)\rho}(G/G)$, which also has order 2. Equivalently the generator of this group is the product of a_σ with that of $\underline{H}_{2(n-b)-i-1} S^{(n-b-i-1)\rho}(G/G)$.

The following is closely related to Example 9.9.16.

Remark 9.9.20. Inverting a_σ in $\underline{\pi}_\star H\underline{\mathbf{Z}}(G/G)$. *In Theorem 9.11.20 below we will see that formally inverting a_σ in $\underline{\pi}_\star H\underline{\mathbf{Z}}(G/G)$ is of interest because its Z-graded part is the homotopy of the geometric fixed point spectrum $\Phi^G H\underline{\mathbf{Z}}$ of Definition 9.11.7 below. Since a_σ has order 2, inerting it necessarily kills 2 and thus converts the object to 2-torsion. We see from Figure 9.1 that it kills everything to the left of the origin, leaving $\mathbf{Z}/2[a_\sigma^{\pm 1}, u_{2\sigma}]$. The $\mathbf{Z}$-graded portion of this is $\mathbf{Z}/2[b]$ where $b = u_{2\sigma} a_\sigma^{-2}$ is in dimension 2.*

We will use the following in the proof of Lemma 13.3.2 below. It is closely related to the computation of Example 8.5.5.

Example 9.9.21. The homology of the space $S^{n\sigma}$ for a general cyclic 2-group. *Let G be a finite cyclic 2-group with with index 2 subgroup G'. Let σ denote the composition of the map $G \to G/G'$ with the sign representation of $G/G' \cong C_2$. The case where $G = C_2$ was done in Example 8.5.5 with the answer given explicitly in (8.5.10). The relevant chain complex of Mackey functors is that of (8.5.8) desuspended n times.*

For a general finite cyclic 2-group G with generator γ, the cellular chain *complex is a variant of that of (8.5.6), namely*

$$C_i^{n\sigma} = \begin{cases} \mathbf{Z}[G]/(\gamma - 1) & \text{for } i = 0 \\ \mathbf{Z}[G/G'] & \text{for } 0 < i \leqslant n \\ 0 & \text{otherwise.} \end{cases} \tag{9.9.22}$$

Let $c_i^{(n)}$ denote a generator of $C_i^{n\sigma}$. The boundary operator d is given by

$$d(c_{i+1}^{(n)}) = \begin{cases} c_i^{(n)} & \text{for } i = 0 \\ \gamma_{i+1}(c_i^{(n)}) & \text{for } 0 < i \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{9.9.23}$$

where $\gamma_i = 1 - (-1)^i\gamma$. It is determined by the fact that the homology of the underlying chain complex must be that of the underlying space S^{2n}.

As before, applying the fixed point Mackey functor of Definition 8.2.8 gives a chain complex of Mackey functors similar to the nth desuspension of that of (8.5.8). We need to adjust the shape of the Lewis diagrams according to the cyclic group under consideration. In the following we will abbreviate $\mathbf{Z}[G/G']$ by $\mathbf{Z}G/G'$. For $G = C_4$ we get

$$\begin{tikzcd}
0 & 1 & 2 & 3 & & n \\
\mathbf{Z} \arrow[d, bend right, "1"'] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "0"'] \arrow[d, bend right, "\Delta"'] & \mathbf{Z} \arrow[l, "2"'] \arrow[d, bend right, "\Delta"'] & \cdots \arrow[l] & \mathbf{Z} \arrow[l, "\epsilon_n"'] \arrow[d, bend right, "\Delta"'] \\
\mathbf{Z} \arrow[u, bend right, "2"'] \arrow[d, bend right, "1"'] & \mathbf{Z}G/G' \arrow[l, "\nabla"'] \arrow[u, bend right, "\nabla"'] \arrow[d, bend right, "1"'] & \mathbf{Z}G/G' \arrow[l, "\gamma_2"'] \arrow[u, bend right, "\nabla"'] \arrow[d, bend right, "1"'] & \mathbf{Z}G/G' \arrow[l, "\gamma_3"'] \arrow[u, bend right, "\nabla"'] \arrow[d, bend right, "1"'] & \cdots \arrow[l] & \mathbf{Z}G/G' \arrow[l, "\gamma_n"'] \arrow[u, bend right, "\nabla"'] \arrow[d, bend right, "1"'] \\
\mathbf{Z} \arrow[u, bend right, "2"'] & \mathbf{Z}G/G' \arrow[l, "\nabla"'] \arrow[u, bend right, "2"'] & \mathbf{Z}G/G' \arrow[l, "\gamma_2"'] \arrow[u, bend right, "2"'] & \mathbf{Z}G/G' \arrow[l, "\gamma_3"'] \arrow[u, bend right, "2"'] & \cdots \arrow[l] & \mathbf{Z}G/G', \arrow[l, "\gamma_n"'] \arrow[u, bend right, "2"']
\end{tikzcd}$$

where, as before, $\gamma_i = 1 - (-1)^i\gamma$. Let $k_n = 1 + (-1)^n$, which is either 0 or 2 depending on th eparity of n. Passing to homology we get

$$\begin{tikzcd}
0 & 1 & 2 & 3 & & n \\
\mathbf{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbf{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & \cdots & \underline{H}_n(G/G) \arrow[d, bend right, "k_n/2"'] \\
0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & \cdots & \underline{H}_n(G/G') \arrow[u, bend right, "k_n"'] \arrow[d, bend right, "1"'] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & \underline{H}_n(G/e), \arrow[u, bend right, "2"']
\end{tikzcd}$$

where

$$\underline{H}_n(G/G) = \begin{cases} \mathbf{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

and

$$\underline{H}_n(G/G') = \underline{H}_n(G/e) = \begin{cases} \mathbf{Z} & \text{for } n \text{ even} \\ \mathbf{Z}_- & \text{for } n \text{ odd.} \end{cases}$$

For $G = C_{2^k}$ for $k \geqslant 3$ a similar computation gives a similar answer, which is also valid for $k = 1$ and 2, namely

$$\underline{H}_i(S^{n\sigma})(C_{2^k}/C_{2^\ell}) = \begin{cases} \mathbf{Z}/2 & \text{for } 0 \leqslant i < n,\ i \text{ even and } \ell = k \\ \mathbf{Z} & \text{for } i = n,\ n \text{ is even and } 0 \leqslant \ell \leqslant k \\ \mathbf{Z}_- & \text{for } i = n,\ n \text{ is odd and } 0 \leqslant \ell < k. \\ 0 & \text{otherwise.} \end{cases}$$

Each restriction map is an isomorphism when both its domain and codomain are nontrivial, and the degree of each corresponding transfer map is twice as much. This can be written in the same form as (8.5.10), namely

$$\underline{H}_i(S^{n\sigma}) = \begin{cases} \bullet & \text{for } 0 \leqslant i < n \text{ and } i \text{ even} \\ \square & \text{for } i = n \text{ and } n \text{ even} \\ \overline{\square} & \text{for } i = n \text{ and } n \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

after redefining the first three symbols of Table 8.1 as follows.

$$\square = \underline{M}(\mathbf{Z}), \quad \overline{\square} = \underline{M}(\mathbf{Z}_-), \quad \text{and} \quad \bullet(C_{2^k}/C_{2^\ell}) = \begin{cases} \mathbf{Z}/2 & \text{for } \ell = k \\ 0 & \text{otherwise,} \end{cases}$$

where $\underline{M}$ is the fixed point Mackey functor of Definition 8.2.8.

The group $\underline{H}_{2j}S^{n\sigma}(G/G)$ is generated by $a_\sigma^{n-2j}u_{2\sigma}^j$ for $0 \leqslant j \leqslant \lfloor n/2 \rfloor$.

9.10 Fixed point spectra

The fixed point spectrum E^H for a subgroup $H \subseteq G$ and a G-spectrum E was given in Definition 9.1.9. Now we can discuss its homotopical properties. We begin with a cautionary example.

Example 9.10.1. The fixed point spectrum is not a homotopy invariant. *Let $G = C_2$ and let σ denote the sign representation. Consider the map*

$$s_\sigma : S^{-\sigma} \wedge S^\sigma \to S^{-0}$$

of (7.2.68). The group action in the target is trivial, so $(S^{-0})^G = S^{-0}$. On the other hand we have

$$\begin{aligned} ((S^{-\sigma} \wedge S^s)_n)^G &\cong (\mathscr{J}_G(\sigma, n) \wedge S^\sigma)^G \cong \mathscr{J}_G(\sigma, n)^G \wedge (S^\sigma)^G \\ &\cong * \wedge S^0 \qquad \text{by (9.3.12),} \end{aligned}$$

so

$$(S^{-\sigma} \wedge S^{\sigma})^G = *.$$

Hence the stable equivalence s_σ does **not** *induce an equivalence of fixed point sets.*

It also fails to induce an equivalence of homotopy fixed point sets. See Remark 8.3.15.

Definition 9.1.9 describes what Schwede calls the **naive fixed point spectrum** in [Sch14, §7.1]. The way out of its nonhomotopical nature is to replace the naive fixed spectrum of X by that of its fibrant replacement as in Definition 7.4.26 and (9.3.14). Schwede [Sch14, Definition 7.1] has a less drastic solution.

Definition 9.10.2. The Schwede and fibrant fixed point spectra $F^G X$ and $R^G X$. *Let X be a genuine or naive G-spectrum.*

(i) Let FX be the G-spectrum defined by

$$(FX)_V = \Omega^{V \otimes \overline{\rho}} X_{V \otimes \rho},$$

where $\rho = \rho_G$, and $\overline{\rho}$ denotes the reduced regular representation of G as in Example 8.5.17. Since

$$V \otimes \rho = |V|\rho,$$

we have

$$(FX)_V = \Omega^{|V|\rho - V} X_{|V|\rho}. \tag{9.10.3}$$

Then the **Schwede fixed point spectrum** *$F^G X$ is given by*

$$(F^G X)_k = ((FX)_k)^G,$$

so $F^G X$ is the naive fixed point spectrum (as in Definition 9.1.9) of FX.

(ii) Let $RX = \Theta^{\infty}_{genuine} X$ as in (9.3.14), so that

$$(RX)_k = \operatorname*{hocolim}_n \Omega^{n\rho} X_{n\rho+k}.$$

Then **fibrant fixed point spectrum** *$R^G X$ is given by*

$$(R^G X)_k = ((RX)_k)^G,$$

so $R^G X$ is the naive fixed point spectrum (as in Definition 9.1.9) of RX.

Note that for any naive G-spectrum, that is any G-object in the category of orthogonal spectra, **the naive fixed point spectrum is the categorical limit of the original functor**. In particular the spectrum $R^G X$ is fibrant (hence the name) because RX is and any limit of fibrant objects is fibrant by Proposition 4.1.13.

Remark 9.10.4. Properties of Schwede's functor F. *Like the functor Θ of Definition 7.4.24, F is coaugmented, meaning that there is a natrual transformation to it from the identity functor. Therefore it can be iterated and we can define F^∞ as a homotopy sequential colimit or telescope. Unlike Θ, F does not alter the 0-space of a spectrum, so $(F^\infty X)_0$ is equivalent to X_0.*

One can show by induction on d that for $d \geqslant 0$,

$$
\begin{aligned}
& (F^{d+1}X)_V \cong \Omega^{V\otimes\overline{\rho}}\Omega^{|V|(g^d-1)\rho}X_{|V|g^d\rho} \\
\text{so for } |V|>0, \quad & (F^\infty X)_V := (\operatorname*{hocolim}_d F^{d+1}X)_V \\
& \cong \Omega^{V\otimes\overline{\rho}} \operatorname*{hocolim}_d \Omega^{|V|(g^d-1)\rho}X_{|V|g^d\rho} \\
& \cong \Omega^{V\otimes\overline{\rho}}(RX)_{|V|\rho} \cong (FRX)_V, \\
\text{and} \quad & (RFX)_V \cong \operatorname*{hocolim}_m \Omega^{m\rho}(FX)_{V+m\rho} \\
& \cong \operatorname*{hocolim}_m \Omega^{m\rho}\Omega^{(V+m\rho)\otimes\overline{\rho}}X_{(|V|+mg)\rho} \\
& \cong \Omega^{V\otimes\overline{\rho}} \operatorname*{hocolim}_m \Omega^{mg\rho}X_{(|V|+mg)\rho} \cong (FRX)_V.
\end{aligned}
$$

A spectrum Y which is weakly equivalent to FY must satisfy

$$Y_V \simeq \Omega^{V\otimes\overline{\rho}}Y_{V\otimes\rho} = \Omega^{V\otimes\overline{\rho}}Y_{|V|\rho}.$$

These two spaces are $\mathcal{S}p_G(S^{-V},Y)$ and $\mathcal{S}p_G(S^{-|V|\rho}\wedge S^{V\otimes\overline{\rho}},Y)$ respectively. This means Y is local (as in Definition 6.2.1) with respect to the set

$$\mathcal{S}_F = \left\{\xi_{V,V\otimes\overline{\rho}} : S^{V\otimes\overline{\rho}}\wedge S^{-|V|\rho} \to S^{-V} : V \in \mathscr{J}_G\right\},$$

where $\xi_{V,V\otimes\overline{\rho}}$ is the map of (7.2.63). Note here that the only map in this set with codomain S^{-0} is the identity map.

We could use the set $\mathcal{S}_F$ to define a Bousfield localization of the projective model structure which differs only slightly from stabilization. Its fibrant objects are spectra Y as above. For each positive dimensional V, Y_V is required to be an infinite loop space, but there is no condition on Y_0.

Proposition 9.10.5. Equivariant homotopy groups as ordinary homotopy groups of fixed point spectra. *For a genuine or naive G-spectrum X, the groups $\pi_*^G X$, $\pi_* F^G X$ and $\pi_* R^G X$ are naturally isomorphic. In particular, $F^G X$ and $R^G X$ equivariantly homotopical on X.*

Proof For the isomorphism between $\pi_*^G X$ and $\pi_* F^G X$ we have

$$
\begin{aligned}
\pi_k^G X &= \operatorname*{colim}_m \pi_{k+m\rho}^G X_{m\rho} && \text{by (9.1.2)} \\
&= \operatorname*{colim}_m \pi_0 \mathcal{T}^G(S^{k+m\rho}, X_{m\rho}) \\
&= \operatorname*{colim}_m \pi_0 \mathcal{T}^G(S^{k+m}, \Omega^{m\overline{\rho}}X_{m\rho}) && \text{by Proposition 8.9.4} \\
&= \operatorname*{colim}_m \pi_{k+m}(\Omega^{m\overline{\rho}}X_{m\rho})^G && \text{by Proposition 8.3.13}
\end{aligned}
$$

$$\begin{aligned} &= \operatorname*{colim}_m \pi_{k+m}(F^G X)_m \\ &= \pi_k F^G X. \end{aligned}$$

Similarly,

$$\begin{aligned} \pi_k^G X &= \operatorname*{colim}_m \pi_{k+m\rho}^G X_{m\rho} = \operatorname*{colim}_m \pi_0 \mathcal{T}^G(S^{k+m\rho}, X_{m\rho}) \\ &= \operatorname*{colim}_m \pi_0 \mathcal{T}^G(S^k, \Omega^{m\rho} X_{m\rho}) = \operatorname*{colim}_m \pi_k(\Omega^{m\rho} X_{m\rho})^G \\ &= \pi_k(\operatorname*{hocolim}_m \Omega^{m\rho} X_{m\rho})^G = \pi_k((RX_0)^G) = \pi_k R^G X. \qquad \square \end{aligned}$$

Recall (Example 2.2.30 (iii)) that the fixed point functor on G-sets has a left adjoint Δ that assigns to each set X the same set with trivial G-action. The same can be done with pointed topological spaces. For spectra the fixed point functor of Definition 9.1.9 has a left adjoint

$$\Delta : \mathcal{S}p \to \mathcal{S}p^G \tag{9.10.6}$$

which sends $S^{-V} \wedge X_V \in \mathcal{S}p$ to $S^{-V} \wedge X_V \in \mathcal{S}p^G$, where in the latter expression G acts trivially on both V and X_V. It can be computed for general X in terms of the tautological presentation

$$\bigvee_{V,W} \mathscr{J}(V,W) \ltimes S^{-W} \wedge X_V \rightrightarrows \bigvee_V S^{-V} \wedge X_V \to X$$

for the trivial group (see (9.1.19)), once one observes that

$$\mathscr{J}(V,W) = \mathscr{J}_G(V,W)$$

when V and W have trivial G-action.

Under the equivalence between $\mathcal{S}p^G$ and the category of objects in $\mathcal{S}p$ equipped with a G-action, the fixed point spectrum functor is formed by passing to objectwise fixed points, and its left adjoint is given by regarding a non-equivariant spectrum as a G-object with trivial G-action.

Proposition 9.10.7. The diagonal fixed poitn adjunction for spectra. *The fixed point functor $(-)^G$ Definition 9.1.9 and its left adjoint Δ (9.10.6) form a Quillen pair (Definition 4.5.1).*

$$\Delta : \mathcal{S}p \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}p^G : (-)^G$$

Proof The functor Δ preserves both cofibrations and trivial cofibrations, so the result follows from Proposition 4.5.12. $\square$

Neither the fixed point functor nor its left adjoint is homotopical and so both need to be derived. The right derived functor of the former is $R^G(\cdot)$ by Proposition 9.10.5.

The (derived) fixed point functor on spectra does not always have the properties one might be led to expect by analogy with spaces. For example even though the composition of the fixed point functor with its left adjoint is the identity, the composition of the derived functors is not. The derived fixed point functor does not generally commute with smash products, or with the formation of suspension spectra.

9.11 Geometric fixed points

The ordinary fixed point functor, even when it is adjusted as in Definition 9.10.2 so as to be homotopical, has some inconvenient features.

- It does not commute with smash products, that is $(X \wedge Y)^G$ is not the same as $X^G \wedge Y^G$.
- It does not behave on suspension spectra as one would like. For a pointed G-space K, $(\Sigma^\infty K)^G$ need not be the same as $\Sigma^\infty(K^G)$.

Example 9.11.1. The suspension spectrum of a pointed space K with trivial G-action. *Then the naive fixed point (as in Definition 9.1.9) is of $\Sigma^\infty K$ is $\Sigma^\infty K$ as expected.*

However, the 0th space of the fibrant replacement of $\Sigma^\infty K$ is

$$\underset{m}{hocolim}\, \Omega^{m\rho}\Sigma^{m\rho} K.$$

The action of G here is nontrivial and the fixed point set is not the space itself. This means that $R^G\Sigma^\infty K$ is not the spectrum $R\Sigma^\infty K$. The fixed point set of the fibrant replacement of $\Sigma^\infty K$ is the subject of the **tom Dieck splitting theorem** *of [tD75, Satz 2], also given in [LMSM86, §V.11].*

Similarly the kth space of the Schwede spectrum $F\Sigma^\infty K$ of Definition 9.10.2(i) is

$$\Omega^{k\overline{\rho}}\Sigma^{k\rho} K,$$

which also has a nontrivial G-action. It follows that $(F^G\Sigma^\infty K)_k$ is not the space above, and $F^G\Sigma^\infty K$ is not equivalent to $\Sigma^\infty K$.

Since $(S^{-0})^G$ is not S^{-0} and S^{-0} is the unit for the smash product of spectra, it follows that $X^G \cong (X \wedge S^{-0})^G$ is not the same as $X^G \wedge (S^{-0})^G$.

The purpose of this section is to describe an alternative functor Φ^G, the **geometric fixed point functor** of Definition 9.11.7. **We will see in Theorem 9.11.8 that it suffers from neither of the defects above.**

9.11A Isotropy separation and geometric fixed points

A standard approach to getting at the equivariant homotopy type of a G-spectrum X is to put X in a cofiber sequence between two other spectra, one an aggregate of information about the spectra $i_H^G X$ for all proper subgroups

$H \subset G$, and the other a localization of X at a "purely G" part. This is the **isotropy separation sequence** of X.

More formally, let $\mathcal{P}$ denote the family of proper subgroups of G, and $E\mathcal{P}$ the universal $\mathcal{P}$-space of Definition 8.6.15. It is characterized up to equivariant weak equivalence by the property that the space of fixed points $E\mathcal{P}^G$ is empty, while for any proper $H \subset G$, $E\mathcal{P}^H$ is weakly contractible. Such a space was explicitly described in Example 8.6.16(iv) for an arbitrary finite group G, and in Example 8.6.16(iii) a simpler description when G is a cyclic p-group. Any such G-CW complex $E\mathcal{P}$ admits an equivariant cell decomposition into moving cells as in Definition 8.4.14.

Definition 9.11.2. *For a finite group G,* **isotropy separation space** $\widetilde{E}\mathcal{P}$ *is the mapping cone of $E\mathcal{P} \to *$, or equivalently that of $E\mathcal{P}_+ \to S^0$, with the cone point taken as base point. The G-CW complexes $E\mathcal{P}_+$ and $\widetilde{E}\mathcal{P}$ are characterized up to equivariant homotopy equivalence by the properties*

$$\left(E\mathcal{P}_+\right)^H \simeq \begin{cases} * & H = G \\ S^0 & H \neq G \end{cases} \qquad \text{and} \qquad \left(\widetilde{E}\mathcal{P}\right)^H \simeq \begin{cases} S^0 & H = G \\ * & H \neq G. \end{cases} \tag{9.11.3}$$

The **isotropy separation sequence** *is constructed by smashing a G spectrum X with the defining cofibration sequence for $\widetilde{E}\mathcal{P}$,*

$$E\mathcal{P} \ltimes X \to X \to \widetilde{E}\mathcal{P} \wedge X. \tag{9.11.4}$$

The term on the left can be described in terms of the action of proper subgroups $H \subset G$ on X.

Proposition 9.11.5. Smash products involving $E\mathcal{P}_+$ and $\widetilde{E}\mathcal{P}$. *For the G-spaces $E\mathcal{P}_+$ and $\widetilde{E}\mathcal{P}$ defined above,*

$$\begin{aligned} E\mathcal{P} \ltimes E\mathcal{P}_+ &\simeq E\mathcal{P}_+, \\ \widetilde{E}\mathcal{P} \wedge \widetilde{E}\mathcal{P} &\simeq \widetilde{E}\mathcal{P} \\ \text{and} \qquad E\mathcal{P} \ltimes \widetilde{E}\mathcal{P} &\simeq *. \end{aligned}$$

Proof Since fixed points commute with smash products of pointed G-spaces, we see from (9.11.3) that in each case the two sides of the asserted equivalence have the same fixed point sets. The first equivalence is induced by the smash product of $E\mathcal{P}_+$ with the map $E\mathcal{P}_+ \to S^0$, the second by that of $\widetilde{E}\mathcal{P}$ with the map $S^0 \to \widetilde{E}\mathcal{P}$, and the third by the unique map $E\mathcal{P} \ltimes \widetilde{E}\mathcal{P} \to *$. □

Example 9.11.6. $E\mathcal{P}$ for a finite cyclic p-group. *When $G = C_{2^n}$, the space $E\mathcal{P}$ is the space EC_2 with G acting through the epimorphism $G \to C_2$. Taking S^∞ with the antipodal action as a model of EC_2, this leads to an identification*

$$\widetilde{E}\mathcal{P} \cong \operatorname*{colim}_{n\to\infty} S^{n\sigma},$$

in which $S^{n\sigma}$ denotes the one point compactification of the direct sum of n copies of the real sign representation of G.

For p an odd prime we use the degree 2 representation λ of the order p quotient C_p that sends a generator of the latter to a rotation of order p. Then we have

$$\widetilde{E}\mathcal{P} \cong \operatorname*{colim}_{n\to\infty} S^{n\lambda}.$$

See Example 8.6.16(iii) and (iv) for two other descriptions of $E\mathcal{P}$. The latter implies that

$$\widetilde{E}\mathcal{P} \cong \operatorname*{colim}_{n\to\infty} S^{n\overline{\rho}},$$

where $\overline{\rho}$ is the reduced regular representation for a finite group G.

The homotopy type of the term on the right in (9.11.4) is determined by its right derived fixed point spectrum, namely the following.

Definition 9.11.7. *For a G-spectrum X, the* **geometric fixed point spectrum** *is*

$$\Phi^G(X) = \left((\widetilde{E}\mathcal{P} \wedge X)_f\right)^G,$$

in which the subscript f indicates a functorial fibrant replacement, such as Θ^∞ as in Definition 5.7.3. For a subgroup $H \subseteq G$, we define $\Phi^H X$ to be $\Phi^H(i_H^G X)$. In particular when H is the trivial group e, $\Phi^e X = (i_e^G X)_f$. We will call the connectivity of $\Phi^H X$ for various H the **geometric connectivity of X**.

The functor Φ^G has many remarkable properties.

Theorem 9.11.8. Properties of Φ^G.

(i) The functor Φ^G sends weak equivalences to weak equivalences, i.e., it is homotopical.
(ii) The functor Φ^G commutes with filtered homotopy colimits.
(iii) For a G-space A and a representation V of G there is a weak equivalence $\Phi^G(S^{-V} \wedge A) \cong S^{-V^G} \wedge A^G$ where $V^G \subset V$ is the subspace of G-invariant vectors. In particular (the case $V = 0$), $\Phi^G(\Sigma^\infty A) \cong \Sigma^\infty A^G$.
(iv) For G-spectra X and Y the spectra

$$\Phi^G(X \wedge Y) \quad \text{and} \quad \Phi^G(X) \wedge \Phi^G(Y).$$

are related by a natural chain of weak equivalences.

Before giving the proof we recall the canonical homotopy presentation of §7.4F. It was described there for smashable spectra in general. For orthogonal

G-spectra in particular, the diagrams of (7.4.59) and (7.4.60) read

$$\begin{tikzcd}[column sep=small] & S^{-(n+1)\rho} \wedge \mathscr{J}_G(n\rho,(n+1)\rho) \wedge X_{n\rho} \arrow[dl, "j(n\rho,(n+1)\rho)\wedge X_{n\rho}"'] \arrow[dr, "S^{-(n+1)\rho}\wedge \epsilon^X_{n\rho,\rho}"] & \\ S^{-n\rho} \wedge X_{n\rho} & & S^{-(n+1)\rho} \wedge X_{(n+1)\rho} \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small] & S^{-(n+1)\rho} \wedge S^{\rho} \wedge X_{n\rho} \arrow[dl] \arrow[dr] & \\ S^{-n\rho} \wedge X_{n\rho} & & S^{-(n+1)\rho} \wedge X_{(n+1)\rho}, \end{tikzcd}$$

where ρ denotes the regular representation of G. Thus Definition 7.4.63 gives

$$X \xleftarrow{\ \simeq\ } \operatorname*{hocolim}_n \, (S^{-n\rho} \wedge X_{n\rho})_c \xrightarrow{\ \simeq\ } \operatorname*{hocolim}_n \, (S^{-n\rho} \wedge X_{n\rho})_{\mathrm{cf}},$$

and properties (i)–(iii) of Theorem 9.11.8 imply that

$$\Phi^G X \cong \operatorname*{hocolim}_n \left(S^{-n\rho}\right)^G \wedge X^G_{n\rho} \cong \operatorname*{hocolim}_n S^{-n} \wedge X^G_{n\rho}. \tag{9.11.9}$$

Sketch of proof of Theorem 9.11.8 The first assertion follows from the fact that smashing with $\widetilde{E}\mathcal{P}$ is homotopical (Proposition 9.6.5), so need not be derived, and that the fixed point functor is homotopical on the full subcategory of fibrant objects (Proposition 9.10.5). The second is straightforward. Part (iii) will be proved below as Corollary 9.11.19. By (ii), the canonical homotopy presentation of §7.4F reduces (iv) to the case $X = S^{-m\rho} \wedge A$, $Y = S^{-n\rho} \wedge B$, for G-CW complexes A and B. One easily checks the assertion in this case using (iii). □

Proposition 9.11.10. The simplifying effect of smashing with $\widetilde{E}\mathcal{P}$. *An equivariant a map $f : X \to Y$ of cofibrant (or Bredon cofibrant as in Definition 9.2.15) G-spectra induces a weak equivalence*

$$\widetilde{E}\mathcal{P} \wedge X \to \widetilde{E}\mathcal{P} \wedge Y$$

iff the map of geometric fixed point spectra $\Phi^G X \to \Phi^G Y$ is a weak equivalence.

Proof The map $\widetilde{E}\mathcal{P} \wedge f$ is an equivariant equivalence iff it induces an ordinary weak equivalence on each fixed point set. Since for every proper $H \subset G$,

$$\pi_*^H \widetilde{E}\mathcal{P} \wedge X = \pi_*^H \widetilde{E}\mathcal{P} \wedge Y = 0,$$

this is equivalent to showing that the map $\pi_*^G \widetilde{E}\mathcal{P} \wedge X \to \pi_*^G \widetilde{E}\mathcal{P} \wedge X$ is an isomorphism. Now $\pi_* \Phi^G X = \pi_*^G(\widetilde{E}\mathcal{P} \wedge X)$ by Definition 9.11.7 and the fact that fibrant replacement is a stable equivalence. □

Proposition 9.11.11. Geometric fixed points detect equivariant contractibility and equivalences.

(i) Suppose that X is a G-spectrum with the property that for all $H \subset G$, the geometric fixed point spectrum $\Phi^H X$ is contractible. Then X is contractible as a G-spectrum.
(ii) If $f : Y \to Z$ is an equivariant map such $\Phi^H f$ is a weak equivalence for each H, then it is an equivariant equivalence.
(iii) The spectrum $E\mathcal{P} \ltimes X$ is equivariantly contractible iff $\Phi^H X$ is contractible for all proper subgroups $H \subset G$.
(iv) The spectrum $\tilde{E}\mathcal{P} \wedge X$ is equivariantly contractible iff $\Phi^G X$ is contractible.

Proof (i) By induction on $|G|$ we may assume that for proper $H \subset G$, the spectrum $i_H^G X$ is contractible. Since both $G \underset{H}{\ltimes} \wedge(-)$ and the formation of mapping cones are homotopical, it follows that $T \wedge X$ is contractible for any G-CW complex built entirely from cells of the form $G \underset{H}{\ltimes} D^n$ with $H \subset G$ proper. This applies in particular to $T = E\mathcal{P}_+$. The isotropy separation sequence then shows that

$$X \to \tilde{E}\mathcal{P} \wedge X$$

is a weak equivalence. But Proposition 9.11.10 and our assumption that $\Phi^G X$ is contractible imply that $\tilde{E}\mathcal{P} \wedge X$ is contractible.

(ii) For the map f, we can apply the previous argument to its cofiber to conclude that it is an equivariant equivalence.

(iii) Let $W = E\mathcal{P} \wedge X$ and consider its isotropy separation sequence,

$$\begin{tikzcd}
E\mathcal{P} \wedge W \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d, equal] & \tilde{E}\mathcal{P} \wedge W \arrow[d, equal] \\
E\mathcal{P} \wedge E\mathcal{P} \wedge X \arrow[d, "\simeq"'] & E\mathcal{P} \wedge X & \tilde{E}\mathcal{P} \wedge E\mathcal{P} \wedge X \arrow[d, "\simeq"'] \\
E\mathcal{P} \wedge X & & *
\end{tikzcd}$$

where the two equivalences follow from the facts that $E\mathcal{P} \wedge E\mathcal{P} \simeq E\mathcal{P}$ and $\tilde{E}\mathcal{P} \wedge E\mathcal{P}$ is contractible.

The argument for (iv) is similar to that of (iii). $\square$

Proposition 9.11.12. Isotropy separation and induced G-cells. *If a pointed G-space T is obtained from a G-space T_0 by attaching G-cells induced from proper subgroups, then the restriction map*

$$[T, \tilde{E}\mathcal{P} \wedge X]_*^G \to [T_0, \tilde{E}\mathcal{P} \wedge X]_*^G$$

is an isomorphism. This holds in particular when $T_0 \subset T$ is the subcomplex of G-fixed points.

Proof Suppose W is a G-CW complex, built entirely from G-cells of the form

$G/H \times D^n$ with H a proper subgroup of G. Then since $\pi_*^H \widetilde{E}\mathcal{P} \wedge X = 0$ for every proper $H \subset G$,

$$[W, \widetilde{E}\mathcal{P} \wedge X]_*^G = 0.$$

The quotient T/T_0 is such a W, so the result follows. □

Since the formation of mapping cones is homotopical, for a map $A \to X$, the map

$$\Phi^G(X) \cup C\Phi^G(A) \xrightarrow{\sim} \Phi^G(X \cup CA) \tag{9.11.13}$$

is a weak equivalence. Among other things this provides a long exact sequence of homotopy groups $\pi_*\Phi^G(X)$ associated to a cofiber sequence in the X variable.

The characterizing property of $\widetilde{E}\mathcal{P}$ implies that for any G**-space** Z and any G-CW complex A, the restriction map

$$[A, \widetilde{E}\mathcal{P} \wedge Z]^G \to [A^G, \widetilde{E}\mathcal{P} \wedge Z]^G$$

is an isomorphism. Since G-acts trivially on A^G, the right hand side is isomorphic to

$$[A^G, \left(\widetilde{E}\mathcal{P} \wedge Z\right)^G] = [A^G, Z^G].$$

Combining these gives the isomorphism

$$[A, \widetilde{E}\mathcal{P} \wedge Z]^G \cong [A^G, Z^G]. \tag{9.11.14}$$

This isomorphism is the foundation for our investigation into Φ^G.

For spectra which are Bredon cofibrant in the sense of Definition 9.2.15, the geometric fixed point functor is an inverse to the functor Δ of (9.10.6).

Proposition 9.11.15. Geometric fixed points for Bredon cofibrant spectra with trivial G-action. *For a Bredon cofibrant spectrum $X \in \mathcal{S}p$ as in Definition 9.2.15, the map*

$$X \to \Phi^G(\Delta X) \tag{9.11.16}$$

adjoint (under the adjunction of Proposition 9.10.7) to

$$\Delta X \to \widetilde{E}\mathcal{P} \wedge \Delta X \to (\widetilde{E}\mathcal{P} \wedge \Delta X)_f.$$

is a weak equivalence.

Proof The long exact sequence of homotopy groups coming from (9.11.13) reduces the claim to the case in which X has the form $S^{-V} \wedge A$ with V a vector space and A a CW complex. This case can be checked by a direct computation. For a G-representation W we have

$$\Delta X_W = \mathscr{J}_G(V, W) \wedge A,$$

and

$$(\Delta X_W)^G = \mathscr{J}_G(V,W)^G \wedge A = \mathscr{J}_G(V,W^G) \wedge A = X_{W^G}. \tag{9.11.17}$$

We can then compute

$$\begin{aligned}
\pi_k \Phi^G(\Delta X) &\cong \operatorname{Ho}\mathcal{S}p(S^k, (\widetilde{E}\mathcal{P} \wedge X)^G_f) \\
&\cong \operatorname{Ho}\mathcal{S}p(S^k, (\widetilde{E}\mathcal{P} \wedge X)^G) \\
&\cong \operatorname{Ho}\mathcal{S}p^G(S^k, \widetilde{E}\mathcal{P} \wedge X) \\
&\cong \operatorname*{colim}_{W > -k} \pi^G_{k+W} \widetilde{E}\mathcal{P} \wedge X_W \\
&\cong \operatorname*{colim}_{W > -k} \pi_{k+W^G} (X_W)^G \\
&\cong \operatorname*{colim}_{W > -k} \pi_{k+W^G} X_{W^G}
\end{aligned}$$

with the penultimate isomorphism coming from (9.11.14), and the last isomorphism from (9.11.17). Under the composite isomorphism, the map on stable homotopy groups induced by (9.11.16) is

$$\operatorname*{colim}_{V > -k} \pi_{k+V} X_V \to \operatorname*{colim}_{W > -k} \pi^G_{k+W^G} X_{W^G},$$

in which V is ranging through the poset of finite dimensional orthogonal vector spaces and W through the poset of G-representations. This is clearly an isomorphism. □

Since $\widetilde{E}\mathcal{P}$ is H-equivariantly contractible when H is a proper subgroup of G, the smash product $\widetilde{E}\mathcal{P} \wedge X$ is contractible if X is a Bredon cofibrant spectrum built entirely from G-cells induced from a proper subgroup of G. More generally

Lemma 9.11.18. Attaching moving cells does not alter geometric fixed points. *Let A and Y be G-spectra. If X is constructed from A by attaching G-cells as in Definition 8.4.14, then the inclusion $A \to X$ induces a weak equivalence*

$$\widetilde{E}\mathcal{P} \wedge A \wedge Y \xrightarrow{\sim} \widetilde{E}\mathcal{P} \wedge X \wedge Y$$

hence a weak equivalence

$$\Phi^G(A \wedge Y) \xrightarrow{\sim} \Phi^G(X \wedge Y).$$

Corollary 9.11.19. Geometric fixed points for generalized suspension spectra. *Let V be a G-representation and A a G-CW complex. Then*

$$\Phi^G(S^{-V} \wedge A) \sim S^{-V^G} \wedge A^G.$$

In particular (the case $V = 0$) $\Phi^G \Sigma^\infty A = \Sigma^\infty A^G$.

Proof We will show that the maps

$$S^{-V^G} \wedge A^G \to S^{-V^G} \wedge A \leftarrow S^{-V} \wedge A,$$

constructed from the inclusions $A^G \subset A$ and $V^G \subset V$, induce weak equivalences

$$S^{-V^G} \wedge A^G \sim \Phi^G\big(S^{-V^G} \wedge A^G\big) \xrightarrow{\sim} \Phi^G\big(S^{-V^G} \wedge A\big) \xleftarrow{\sim} \Phi^G\big(S^{-V} \wedge A\big),$$

giving a zigzag of weak equivalences

$$\Phi^G\big(S^{-V} \wedge A\big) \leftrightarrow \sim S^{-V^G} \wedge A^G.$$

We work our way from the left. The weak equivalence

$$S^{-V^G} \wedge A^G \cong \Phi^G(A^G \wedge S^{-V^G})$$

is Proposition 9.11.15. The next map is a weak equivalence by Lemma 9.11.18 since A is constructed from A^G by adding induced G-cells. The last map can be constructed by applying Φ^G to the composition

$$S^{-V} \wedge A \to S^{-V} \wedge S^{V-V^G} \wedge A \to S^{-V^G} \wedge A.$$

The right arrow is a weak equivalence. Since S^{V-V^G} is a G-CW complex with fixed point space S^0, it is constructed from S^0 by adding induced G-cells. The left map therefore induces an equivalence of geometric fixed points by Lemma 9.11.18. □

We now explicitly describe the geometric fixed point spectrum of $H\underline{\mathbf{Z}}$ when $G = C_{2^n}$. The computation plays an important role in the proof of the Reduction Theorem in §12.4E.

Theorem 9.11.20. Geometric fixed points of $H\underline{\mathbf{Z}}$ for a finite cyclic 2-group. *Let $G = C_{2^n}$. For any G-spectrum X, the $RO(G)$-graded homotopy groups of $\widetilde{E}\mathcal{P} \wedge X$ are given by*

$$\pi_\star^G(\widetilde{E}\mathcal{P} \wedge X) = a_\sigma^{-1}\pi_\star^G(X),$$

and $\pi_*\Phi^G X$ is the $\mathbf{Z}$-graded part of the indicated $RO(G)$-graded group.

In particular the homotopy groups of the commutative algebra $\Phi^G H\underline{\mathbf{Z}}$ are given by

$$\pi_*(\Phi^G H\underline{\mathbf{Z}}) = \mathbf{Z}/2[b],$$

where $b = u_{2\sigma}a_\sigma^{-2} \in \pi_2(\Phi^G H\underline{\mathbf{Z}}) = \pi_2^G(\widetilde{E}\mathcal{P} \wedge H\underline{\mathbf{Z}}) \subset a_\sigma^{-1}\pi_\star^G H\underline{\mathbf{Z}}$, for $u_{2\sigma}$ and a_σ as in Definition 9.9.7.

Proof As mentioned in Example 9.11.6, the space $\widetilde{E}\mathcal{P}$ can be identified with

$$\lim_{n\to\infty} S^{n\sigma}.$$

The first assertion therefore follows from Example 9.9.15. The second assertion follows from Example 9.9.16 and the fact that the map $a_\sigma^{-1}\pi_\star^G X \to \pi_\star^G \widetilde{E}\mathcal{P} \wedge X$ is a ring homomorphism when X is an equivariant algebra.

For the computation of $\pi_*(\Phi^G H\underline{\mathbf{Z}})$ see Remark 9.9.20. □

9.11B Homotopy fixed points revisited

Recall the homotopy fixed point spectrum of Definition 9.1.9. We saw in Example 9.10.1 that the homotopy fixed point functor is not homotopical on spectra.

Here are the stable analogs of Definition 8.6.7 and Theorem 8.6.8.

Definition 9.11.21. *An* **stable hG-equivalence** *is an equivariant map of G-spectra underlain by an ordinary stable equivalence.*

Theorem 9.11.22. A stable hG-equivalence induces a stable equivalence on homotopy fixed point spectra. *An equivariant map $f : X \to Y$ of G-spectra that is an underlying stable equivalence of orthogonal spectra induces a stable equivalence $f^{hG} : X^{hG} \to Y^{hG}$.*

Proof By Theorem 7.4.29, f is an underlying stable equivalence iff $\Theta_{\mathbf{O}}^\infty f$ is a projective weak equivalence. Thus it suffices to show that if f is an underlying projective weak equivalence, so is f^{hG}, which follows directly from Theorem 8.6.8. □

Recall (Proposition 7.2.49) that for a pointed G-space K and an orthogonal G-spectrum X we have a spectrum X^K defined by

$$(X^K)_V = (X_V)^K = \mathcal{T}_G(K, X_V),$$

and an adjunction isomorphism as in (7.2.50),

$$\mathcal{S}p^G(Z \wedge K, X) \cong \mathcal{S}p^G(Z, X^K).$$

Definition 9.11.23. *A G-spectrum X is* **cofree** *if the map*

$$X \to X^{EG_+} \tag{9.11.24}$$

adjoint to the projection map $EG \ltimes X \to X$ is a weak equivalence.

If X is cofree then the map

$$\pi_*^G X \to \pi_*^G X^{EG_+} = \pi_* X^{hG}$$

is an isomorphism. The Homotopy Fixed Point Theorem 13.3.28 below asserts that any module over $D^{-1}MU^{((G))}$ is cofree.

The map of (9.11.24) is an equivalence of underlying spectra, and hence becomes an equivalence after smashing with any G-CW complex built entirely out of free G-cells. In particular, the map

$$EG \ltimes X \xrightarrow{\sim} EG \ltimes (X^{EG_+}) \tag{9.11.25}$$

is an equivariant equivalence. One exploits this, as in [Car84], by making use of the pointed G-space $\tilde{E}G$ defined by the cofibration sequence

$$EG_+ \to S^0 \to \tilde{E}G. \tag{9.11.26}$$

Lemma 9.11.27. Cofreeness conditions. *For a G-spectrum X, the following are equivalent:*

(i) For all non-trivial $H \subset G$, the spectrum $\Phi^H X$ (as in Definition 9.11.7) is contractible.
(ii) The map $EG \ltimes X \to X$ is a weak equivalence.
(iii) The G-spectrum $\tilde{E}G \wedge X$ is contractible.

Proof The equivalence of the second and third conditions is immediate from the cofibration sequence defining $\tilde{E}G$. Since EG_+ is built from free G-cells, condition 2 implies condition 1. For $H \subset G$ non-trivial, we have

$$\Phi^H(\tilde{E}G \wedge X) \approx \Phi^H(\tilde{E}G) \wedge \Phi^H(X) \approx S^0 \wedge \Phi^H(X).$$

Since the non-equivariant spectrum underlying $\tilde{E}G$ is contractible, condition 1 therefore implies that $\Phi^H(\tilde{E}G \wedge X)$ is contractible for **all** $H \subset G$. But this means that $\tilde{E}G \wedge X$ is contractible (Proposition 9.11.11). □

Corollary 9.11.28. Modules over a cofree ring are cofree. *If R is an equivariant ring spectrum (as in Definition 9.7.1) satisfying the equivalent conditions of Lemma 9.11.27 then any module over R is cofree.*

The condition of Corollary 9.11.28 requires R to be an equivariant ring spectrum in the weakest possible sense, namely that R possesses a unital multiplication (not necessarily associative) in $\mathrm{Ho}\,\mathcal{S}p^G$. Similarly, the "module" condition is also one taking place in the homotopy category.

Proof Let M be an R-module. Consider the diagram

$$\begin{tikzcd} EG \ltimes M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \tilde{E}G \wedge M \arrow[d] \\ EG \ltimes M^{EG_+} \arrow[r] & M^{EG_+} \arrow[r] & \tilde{E}G \wedge M^{EG_+} \end{tikzcd} \tag{9.11.29}$$

obtained by smashing $M \to M^{EG_+}$ with the sequence (9.11.26). The fact that R satisfies the condition 1 of Lemma 9.11.27 implies that any R-module M' does since $\Phi^H(M)$ is a retract of $\Phi^H(R \wedge M) \approx \Phi^H(R) \wedge \Phi^H(M)$. Thus both M and M^{EG_+} satisfy the conditions of Lemma 9.11.27, and the terms on the right in (9.11.29) are contractible. The left vertical arrow is the weak equivalence of (9.11.25). It follows that the middle vertical arrow is a weak equivalence. □

9.11C Monoidal geometric fixed points

The geometric fixed point functor was studied in §9.11A. For some purposes it is useful to have a version of it which is lax symmetric monoidal. For example, such a functor automatically takes (commutative) algebras to (commutative) algebras.

In this section we describe the variation constructed by Mandell-May in [MM02, §V.4]. We refer to the Mandell-May construction as the **monoidal geometric fixed point functor** and denote it Φ_M^G, in order not to confuse it with the usual geometric fixed point functor. It is so named because it is lax monoidal as in Definition 2.6.19; see (9.11.46) below. Its construction is simpler in that it does not require the use of the isotropy separation sequence and fibrant replacement.

The following is a compendium of results from [MM02, §V.4]. The construction and proofs are described in §9.11D below.

Proposition 9.11.30. Basic properties. *The monoidal geometric fixed point functor has the following properties:*

(i) It preserves trivial cofibrations.
(ii) It is lax symmetric monoidal.
(iii) If X and Y are cofibrant, the map

$$\Phi_M^G(X) \wedge \Phi_M^G(Y) \to \Phi_M^G(X \wedge Y)$$

is an isomorphism.
(iv) It commutes with cobase change along a closed inclusion.
(v) It commutes with directed colimits.

Property(iii) implies that Φ_M^G is weakly symmetric monoidal in the sense of the definition below.

Definition 9.11.31 ([SS03a]). *A functor $F : \mathcal{C} \to \mathcal{D}$ between (symmetric) monoidal model categories is* **weakly (symmetric) monoidal** *if it is lax (symmetric) monoidal, and the map*

$$F(X) \wedge F(Y) \to F(X \wedge Y)$$

is a weak equivalence when X and Y are cofibrant.

The next result is [MM02, Proposition V.4.17] and is discussed in more detail as Proposition 9.11.49, where the middle object is defined.

Proposition 9.11.32. The left derived functor of Φ_M^G is Φ^G. *More specifically, there are natural transformations*

$$\Phi^G(X) \to \tilde{\Phi}_M^G(X) \xleftarrow{\sim} \Phi_M^G(X)$$

in which the rightmost arrow is always a weak equivalence and the leftmost arrow is a weak equivalence when X is cofibrant. □

Because Φ^G is lax monoidal, it determines functors

$$\begin{aligned} &\Phi^G_M : \mathbf{Alg}^G \to \mathbf{Alg} \\ \text{and} \qquad &\Phi^G_M : \mathbf{Comm}^G \to \mathbf{Comm}, \end{aligned}$$

and for each associative algebra R a functor

$$\Phi^G_M : \mathcal{M}_R \to \mathcal{M}_{\Phi^G_M R}.$$

In addition, if R is an associative algebra, M a right R-module and N a left R-module there is a natural map

$$\Phi^G_M(M \underset{R}{\wedge} N) \to \Phi^G_M M \underset{\Phi^G_M R}{\wedge} \Phi^G_M N. \tag{9.11.33}$$

We will see in Proposition 10.8.7 that it is a an isomorphism if M and N are cofibrant. Blumberg and Mandell [BM15, Appendix A] have shown that one need only require one of M or N to be cofibrant in order to guarantee that this map is an isomorphism.

While these properties if Φ^G_M are very convenient, they must be used with caution. The value $\Phi^G_M(X)$ is only guaranteed to have the "correct" homotopy type on cofibrant objects. The spectrum underlying a commutative algebra is rarely known to be cofibrant, making the monoidal geometric fixed point functor difficult to use in that context. The situation is a little better with associative algebras. The weak equivalence (9.11.33) leads to an expression for the geometric fixed point spectrum of a quotient module which we will use in §12.4E. In order to do so, we need criteria guaranteeing that the monoidal geometric fixed point functor realizes the correct homotopy type. Such criteria are described in §9.11F.

9.11D Definition and categorical properties

To motivate the definition, for an orthogonal representation V of G let $V^G \subset V$ be the space of invariant vectors, and $V^\perp$ the orthogonal complement of V^G. Recall (Proposition 8.9.30) that

$$\mathscr{J}_G(V,W)^G \cong \mathscr{J}(V^G,W^G) \rtimes O(V^\perp,W^\perp)^G, \tag{9.11.34}$$

so that there is a canonical map

$$\mathscr{J}_G(V,W)^G \to \mathscr{J}(V^G,W^G),$$

given in terms of (9.11.34) by smashing the identity map with the map

$$O(V^\perp,W^\perp)^G \to *.$$

We wish to define a functor Φ^G_M with the property that

$$\Phi^G_M(S^{-V} \wedge A) = S^{-V^G} \wedge A^G \tag{9.11.35}$$

and which commutes with colimits as far as is possible. A value needs to be assigned to the effect of Φ^G_M on the map

$$S^{-W} \wedge \mathscr{J}_G(V, W) \to S^{-V}.$$

The only obvious choice is to take

$$\Phi^G_M(S^{-W} \wedge \mathscr{J}_G(V, W)) \to \Phi^G_M(S^{-V})$$

to be the composite

$$S^{-W^G} \wedge \mathscr{J}_G(V, W)^G \to S^{-W^G} \wedge \mathscr{J}(V^G, W^G) \to S^{-V^G}. \tag{9.11.36}$$

If Φ^G_M actually **were** to commute with colimits, it would be determined by the specifications given by (9.11.35) and (9.11.36). Indeed, using the tautological presentation to write a general equivariant orthogonal spectrum X as a reflexive coequalizer

$$\bigvee_{V,W} S^{-W} \wedge \mathscr{J}_G(V, W) \wedge X_V \rightrightarrows \bigvee_{V} S^{-V} \wedge X_V \to X,$$

the value of $\Phi^G_M(X)$ would be given by the reflexive coequalizer diagram

$$\bigvee_{V,W} S^{-W^G} \wedge \mathscr{J}_G(V, W)^G \ltimes X^G_V \rightrightarrows \bigvee_{V} S^{-V^G} \wedge X^G_V \to \Phi^G_M X. \tag{9.11.37}$$

We take this as the definition of $\Phi^G_M(X)$.

Definition 9.11.38. *The* **monoidal geometric fixed point functor**

$$\Phi^G_M : \mathcal{S}p^G \to \mathcal{S}p$$

is the functor defined by the coequalizer diagram (9.11.37).

Remark 9.11.39. *In the case $X = S^{-V} \wedge A$, the tautological presentation is a split coequalizer, and one recovers both (9.11.35) and (9.11.36).*

A fundamental property of the usual geometric fixed point functor Φ^G is that for proper $H \subset G$, the spectrum $\Phi^G(G \underset{H}{\ltimes} X)$ is contractible. The monoidal geometric fixed point functor has this property on the nose.

Proposition 9.11.40. The functor Φ^G_M on an indexed wedge. *Suppose that T is a G-set and X an equivariant T-diagram. If T has no G-fixed points then the map*

$$\Phi^G_M\Big(\bigvee_{t \in T} X_t\Big) \to *$$

is an isomorphism. In particular, if $H \subset G$ is a proper subgroup and X an orthogonal H-spectrum, then the map

$$\Phi^G_M(G \underset{H}{\ltimes} X) \to *$$

is an isomorphism.

Proof Since indexed wedges are computed componentwise, the assumption that T has no fixed points implies that for all representations W of G,

$$\big(\bigvee_{t\in T} X_t\big)^G_W = \big(\bigvee_{t\in T}(X_t)_W\big)^G = *.$$

The claim then follows from the definition of Φ^G_M. □

Working through an equivariant cell decomposition gives the following analog of Lemma 9.11.18.

Corollary 9.11.41. Attaching moving cells does not alter monoidal geometric fixed points. *Let A and Y be G-spectra. If X is constructed from A by attaching moving G-cells as in Definition 8.4.14, then the map*

$$\Phi^G_M(A \wedge Y) \to \Phi^G_M(X \wedge Y)$$

is an isomorphism.

There is a natural map

$$X^G \to \Phi^G_M X \tag{9.11.42}$$

from the fixed point spectrum of X to the monoidal geometric fixed point spectrum. To construct it note that the fixed point spectrum of X is computed termwise, and so is given by the coequalizer diagram

$$\bigvee_{V,W\in\mathscr{J}} S^{-W} \wedge \mathscr{J}(V,W) \ltimes X^G_V \rightrightarrows \bigvee_{V\in\mathscr{J}} S^{-V} \wedge X^G_V \to X^G. \tag{9.11.43}$$

The map (9.11.42) is given by the evident inclusion of (9.11.43) into (9.11.37).

The functor Φ^G_M cannot commute with all colimits. However, since colimits of orthogonal G-spectra are computed objectwise, the definition implies that Φ^G_M commutes with whatever enriched colimits are preserved by the fixed point functor on G-spaces. This means that there is a functorial isomorphism

$$\Phi^G_M(X \wedge A) \cong \Phi^G_M(X) \wedge A^G \tag{9.11.44}$$

for each pointed G-space A, and that Φ^G_M commutes with the formation of wedges, directed colimits and cobase change along a closed inclusion. Because h-cofibrations and hence cofibrations are objectwise closed inclusions (Lemma 3.5.18), the functor Φ^G_M has good homotopy theoretic properties.

9.11E Homotopy properties of Φ^G_M

Several variations on the following appear in in [MM02, §V.4].

Proposition 9.11.45. The functor Φ^G_M preserves cofibrations. *The functor Φ^G_M sends cofibrations to cofibrations and acyclic cofibrations to trivial cofibrations. It therefore sends weak equivalences between cofibrant objects to weak equivalences.*

Proof That Φ_M^G sends cofibrations to cofibrations follows from the fact that it preserves cobase change along closed inclusions and sends generating cofibrations to generating cofibrations. A similar argument applies to the trivial cofibrations, once one checks that Φ_M^G sends both maps in the factorization (7.4.17)

$$S^W \wedge S^{-V\oplus W} \to \widetilde{S}_W^{-V} \to S^{-V}$$

to weak equivalences. But the second map is a homotopy equivalence and the composite map is sent to a weak equivalence by (9.11.35). The last assertion is a consequence of Ken Brown's Lemma 5.1.7. □

Proposition 9.11.45 implies that the monoidal geometric fixed point functor has a left derived functor which can be computed on any cofibrant approximation. A similar argument with a slightly different model structure could be used to show that the left derived functor can be computed on a cellular approximation. We will show in §9.11G that the left derived functor $\mathbf{L}\Phi_M^G$ is the geometric fixed point functor Φ^G.

9.11F Monoidal geometric fixed points and smash products

The properties (9.11.35) and (9.11.36) give an identification

$$\Phi_M^G(S^{-V} \wedge A \wedge S^{-W} \wedge B) \cong \Phi_M^G(S^{-V} \wedge A) \wedge \Phi_M^G(S^{-W} \wedge B)$$

making the diagram

$$\begin{tikzcd}
\Phi_M^G(S^{-V_1} \wedge \mathscr{J}_G(W_1,V_1)) \wedge \Phi_M^G(S^{-V_2} \wedge \mathscr{J}_G(W_2,V_2)) \arrow[r] \arrow[d] & \Phi_M^G(S^{-W_1}) \wedge \Phi_M^G(S^{-W_2}) \arrow[d] \\
\Phi_M^G(S^{-V_1} \wedge \mathscr{J}_G(W_1,V_1) \wedge S^{-V_2} \wedge \mathscr{J}_G(W_2,V_2)) \arrow[r] & \Phi_M^G(S^{-W_1} \wedge S^{-W_2})
\end{tikzcd}$$

commute. Applying Φ_M^G termwise to the smash product of the tautological presentations of X and Y, and using the above identifications, gives a natural transformation

$$\Phi_M^G(X) \wedge \Phi_M^G(Y) \to \Phi_M^G(X \wedge Y), \tag{9.11.46}$$

making Φ_M^G lax monoidal (Definition 2.6.19. From the formula (9.11.35) this map is an isomorphism if $X = S^{-V} \wedge A$ and $Y = S^{-W} \wedge B$. This leads to

Proposition 9.11.47 ([MM02], Proposition V.4.7)**. The functor Φ_M^G is lax monoidal.** *The map (9.11.46) is an isomorphism if X and Y are Bredon cofibrant.*

Proof The class of spectra X and Y for which (9.11.46) is an isomorphism is stable under smashing with a G-space, the formation of wedges, directed colimits, and cobase change along an objectwise closed inclusion. Since (9.11.46) is an isomorphism when

$$X = G \underset{H}{\ltimes} S^{-V} \wedge A \qquad \text{and} \qquad Y = G \underset{H}{\ltimes} S^{-W} \wedge B$$

this implies it is an isomorphism when X and Y are Bredon cofibrant. Since isomorphisms are weak equivalences, the result follows. □

Remark 9.11.48. *Blumberg and Mandell [BM15, Appendix A] have shown that Proposition 9.11.47 remains true under the assumption that only one of X or Y is Bredon cofibrant. This implies that Proposition 10.8.7 below remains true if only one of N or N' is cofibrant.*

9.11G Relation with the geometric fixed point functor

We now turn to identifying the left derived functor $\mathbf{L}\Phi^G_M$ with the geometric fixed point functor Φ^G. The inclusion $S^0 \to \tilde{E}\mathcal{P}$ and the fibrant replacement functor give maps

$$X \to \tilde{E}\mathcal{P} \wedge X \to (\tilde{E}\mathcal{P} \wedge X)_f.$$

Proposition 9.11.49 ([MM02], Proposition V.4.17)**. The functors Φ^G and Φ^G_M agree on cofibrant spectra.** *If X is cofibrant, then the maps*

$$\Phi^G X = (\tilde{E}\mathcal{P} \wedge X_f)^G \to \Phi^G_M((\tilde{E}\mathcal{P} \wedge X)_f) \leftarrow \Phi^G_M(X)$$

are weak equivalences. The middle object is denoted by $\tilde{\Phi}^G_M(X)$ in Proposition 9.11.32.

Sketch of proof: For the arrow on the left, note that both functors are homotopical and, up to weak equivalence, preserve filtered colimits along h-cofibrations. Using the canonical homotopy presentation of Definition 7.4.63, it suffices to check that the arrow on the left is a weak equivalence when $X = S^{-V} \wedge A$, with A a G-CW complex. This follows from Corollary 9.11.19, the identity (9.11.35), and a little diagram chasing to check compatibility.

The right arrow is the composition of

$$\Phi^G_M(X) \to \Phi^G_M(\tilde{E}\mathcal{P} \wedge X)$$

which is an isomorphism by (9.11.44), and

$$\Phi^G_M(\tilde{E}\mathcal{P} \wedge X) \to \Phi^G_M((\tilde{E}\mathcal{P} \wedge X)_f),$$

which is a trivial cofibration by Proposition 9.11.45. □

9.11H Geometric fixed points and the norm

The geometric fixed point construction interacts well with the norm. Suppose $H \subset G$ is a subgroup, and that X is an H-spectrum. The following is due to Andrew Blumberg and Mike Mandell.

Proposition 9.11.50. *Suppose $H \subset G$. There is a natural transformation*

$$\Phi^H_M(\,-\,) \to \Phi^G_M \circ N^G_H(\,-\,)$$

which is an isomorphism, hence a weak equivalence on Bredon cofibrant objects.

Because of Proposition 9.11.32 and the fact that the norm preserves cofibrant objects (Theorem 10.2.4 below), the above result gives a natural zigzag of weak equivalences relating $\Phi^H(X)$ and $\Phi^G(N^G_H X)$ when X is cofibrant. In fact there is a natural zigzag of maps

$$\Phi^H X \leftrightarrow \Phi^G(N^G_H X)$$

which is a weak equivalence not only for cofibrant X, but for suspension spectra of cofibrant G-spaces and for the spectra underlying cofibrant commutative rings. The actual statement is somewhat technical, and is one of the main results of this section. The condition is described in the statement of Proposition 9.11.55. See also Remark 9.11.57 and Remark 9.11.58.

Corollary 9.11.51. *For the spectra satisfying the condition of Proposition 9.11.55 below, the composite functor*

$$\Phi^G \circ N^G_H : \mathcal{S}p^H \to \mathcal{S}p$$

preserves, up to weak equivalence, wedges, directed colimits along closed inclusions and cofiber sequences.

Proof The properties obviously hold for Φ^H. $\square$

There is another useful result describing the interaction of the geometric fixed point functor with the norm map in $RO(G)$-graded cohomology described in §9.7C. Suppose that R is a G-equivariant commutative algebra, X is a G-space, and $V \in RO(H)$ a virtual real representation of a subgroup $H \subset G$. In this situation one can compose the norm

$$N : R^V_H(X) \to R^{\mathrm{Ind}\,V}_G(X)$$

with the geometric fixed point map

$$\Phi^G : R^{\mathrm{Ind}\,V}_G(X) \to (\Phi^G R)^{V^H}(X^G),$$

where $V^H \subset V$ is the subspace of H-fixed vectors, and X^G is the space of G-fixed points in X.

Proposition 9.11.52. *The composite*

$$\Phi^G \circ N : R^V_H(X) \to (\Phi^G R)^{V^H}(X^G)$$

is a ring homomorphism.

Proof Multiplicativity is a consequence of the fact that both the norm and the geometric fixed point functors are weakly monoidal. Additivity follows from the fact that the composition $\Phi^G \circ N$ preserves wedges (Corollary 9.11.51). □

Proof of Proposition 9.11.50. To construct the natural transformation, first note that there is a natural isomorphism

$$A^H \cong (N_H^G A)^G$$

for H-equivariant **spaces** A. Next note that for an orthogonal representation V of H, Proposition 9.7.8 and the property (9.11.35) give isomorphisms

$$\Phi^G_M N^G_H S^{-V} \cong \Phi^G_M S^{-\mathrm{Ind}^G_H V} \cong S^{-V^H} \cong \Phi^H S^{-V}.$$

The monoidal properties of Φ^G_M and the norm then combine to give an isomorphism

$$\Phi^H(S^{-V} \wedge A) \cong \Phi^G N^G_H(S^{-V} \wedge A) \tag{9.11.53}$$

which one easily checks to be compatible with the maps

$$S^{-V} \wedge \mathscr{J}_H(W,V) \to S^{-W}.$$

To construct the transformation, write a general H-spectrum X in terms of its tautological presentation

$$\bigvee_{V,W} S^{-W} \wedge \mathscr{J}_H(V,W) \wedge X_V \rightrightarrows \bigvee_{V} S^{-V} \wedge X_V \to X,$$

and apply (9.11.53) termwise to produce a diagram

$$\bigvee_{V,W} S^{-W^H} \wedge \mathscr{J}_H(V,W)^H \wedge X_V^H \rightrightarrows \bigvee_{V} S^{-V^H} \wedge X_V^H \to \Phi^G N^G_H X.$$

The coequalizer of the two arrows is, by definition, $\Phi^H_M(X)$. This gives the natural transformation.

The isomorphism assertion for Bredon cofibrant X reduces to the special case (9.11.53), once one shows that $\Phi^G_M \circ N^G_H(-)$ commutes with the formation of wedges, cobase change along cofibrations between cofibrant objects, and filtered colimits along closed inclusions. The last property is clear since both of the functors being composed commutes with filtered colimits along closed inclusions. For the other two assertions it will be easier to work in terms of equivariant J-diagrams for $J = G/H$.

Suppose that T is an indexing set, and X_t, $t \in T$ a set of equivariant J-diagrams. We wish to show that the natural map

$$\bigvee_{t \in T} \Phi^G_M X_t^{\wedge J} \to \Phi^G_M \big(\bigvee_{t \in T} X_t \big)^{\wedge J}$$

is an isomorphism. For this use the distributive law to rewrite the argument of the right hand side as

$$\bigvee_{\gamma \in \Gamma} X^{\wedge \gamma}$$

where γ is the G-set of functions $J \to T$ and

$$X^{\wedge \gamma} = \bigwedge_{j \in J} X_{\gamma(j)}.$$

The map asserted to be an isomorphism on monoidal geometric fixed points is the inclusion of the summand indexed by the constant functions. But since G acts trivially on T, the other summands form an indexed wedge over a G-set with no fixed points. The claim then follows from Proposition 9.11.40.

The cobase change property is similar. Suppose we are given a pushout square of equivariant J-diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

in which $A \to B$ is a cofibration and A is cofibrant. We consider the filtration of $Y^{\wedge J}$ given in §2.9C whose stages fit into a pushout square

$$\begin{tikzcd} \displaystyle\bigvee_{\substack{J = J_0 \amalg J_1 \\ |J_1| = m}} X^{\wedge J_0} \wedge \partial_A B^{\wedge J_1} \arrow[r] \arrow[d] & \displaystyle\bigvee_{\substack{J = J_0 \amalg J_1 \\ |J_1| = m}} X^{\wedge J_0} \wedge B^{\wedge J_1} \arrow[d] \\ \mathrm{fil}_{m-1} Y^{\wedge J} \arrow[r] & \mathrm{fil}_m Y^{\wedge J} . \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

By Proposition 10.3.9, the upper arrow is an h-cofibration, so the resulting diagram of monoidal geometric fixed points is a pushout. But since J is a transitive G-set, unless $m = |J|$ the group G has no fixed points on the G-set indexing the wedges. Applying Proposition 9.11.40 then shows that for $m < |J|$ the map

$$\Phi^G_M X^{\wedge J} \to \Phi^G_M \mathrm{fil}_m Y^{\wedge J}$$

is an isomorphism, and that the pushout square when $m = |J|$ becomes

$$\begin{tikzcd} \Phi^G_M \partial_A B^{\wedge J} \arrow[r] \arrow[d] & \Phi^G_M B^{\wedge J} \arrow[d] \\ \Phi^G_M X^{\wedge J} \arrow[r] & \Phi^G_M Y^{\wedge J} . \end{tikzcd}$$

However the term $\partial_A B^{\wedge J}$ is the term $\mathrm{fil}_{|J|-1} B^{\wedge J}$ in the case in which $X = A$ and $Y = B$, and so $\Phi^G_M A^{\wedge J} \to \Phi^G_M \partial_A B^{\wedge J}$ is an isomorphism. This completes the proof. $\square$

Thinking in terms of left derived functors, one can get a slightly better result. As long as X has the property that the map $(\mathbf{L}N_H^G)X \to N_H^G X$ is a weak equivalence, there will be a weak equivalence between $\Phi^H X$ and $\Phi^G N_H^G X$. Since it plays an important role in our work, we spell it out. Start with $X \in \mathcal{S}p^H$ and let $X_c \to X$ be a cofibrant approximation. Now consider the diagram

$$\begin{tikzcd} \Phi^H X_c \arrow[r, leftrightarrow, "\sim", "\text{zig zag}"'] \arrow[d, "\sim"'] & \Phi^H_M X_c \arrow[r, "\sim"] & \Phi^G_M N_H^G X_c \arrow[r, leftrightarrow, "\sim", "\text{zig zag}"'] & \Phi^G N_H^G X_c \arrow[d] \\ \Phi^H X & & & \Phi^G N_H^G X \end{tikzcd} \tag{9.11.54}$$

The left vertical arrows are weak equivalences since the geometric fixed point functor preserves weak equivalences. The weak equivalences in the top row are given by Proposition 9.11.49, Theorem 10.2.4, and Proposition 9.11.50. Since Φ^G is homotopical we have

Proposition 9.11.55. *Suppose that $X \in \mathcal{S}p^H$ has the property that for some (hence any) cofibrant approximation $X_c \to X$ the map*

$$N_H^G X_c \to N_H^G X$$

is a weak equivalence. Then the functorial relationship between $\Phi^H X$ and $\Phi^G N_H^G X$ given by (9.11.54) is a weak equivalence. $\square$

Remark 9.11.56. *Proposition 9.11.55 can be proved without reference to Φ^G_M by using the canonical homotopy presentation of (7.4.62).*

Remark 9.11.57. *Proposition 9.11.55 applies in particular when X is* **equifibrantly flat** *in the sense of §10.9B below. By Theorem 10.9.9 this means that if $R \in \mathcal{S}p^H$ is a cofibrant commutative ring, then $\Phi^H R$ and $\Phi^G N_H^G R$ are related by a functorial zigzag of weak equivalences. The case of interest to us is $H = C_2$, $G = C_{2^n}$ and $R = MU_{\mathbf{R}}$. Then we have $N_H^G R = MU^{((G))}$, and we get an equivalence*

$$\Phi^G MU^{((G))} \cong \Phi^{C_2} MU_{\mathbf{R}} \cong MO.$$

Remark 9.11.58. *Proposition 9.11.55 also applies to the suspension spectra of cofibrant H-spaces. Indeed, if X is a cofibrant H-space then*

$$S^{-1} \wedge S^1 \wedge X \to X$$

is a cofibrant approximation. Applying N_H^G leads to the map

$$S^{-V} \wedge S^V \wedge N_H^G(X) \to N_H^G(X)$$

with $V = \mathrm{Ind}_H^G 1$, which is a weak equivalence (in fact a cofibrant approximation). This case is used to show that $\Phi^G \circ N_H^G$ is a ring homomorphism on the $RO(G)$-graded cohomology of G-spaces (Proposition 9.11.52).

10

Multiplicative properties of G-spectra

This is the most technically difficult chapter in the book. The first time reader may want to skip it until she encounters a point in remaining three chapters where its results are needed.

In §11.4 we will use some of them to study multiplicative properties of the slice spectral sequence. In particular we will use the fact that indexed smash products (Theorem 10.4.7) and indexed symmetric powers (Theorem 10.5.10) of cofibrant spectra are cofibrant. The latter objects will be specified in Definition 10.5.6.

The results of this chapter will be used again in §12.1 in our construction of the real cobordism spectrum $MU_{\mathbf{R}}$. In §12.2 they will be used to study the norms of $MU_{\mathbf{R}}$ to larger groups. The method of twisted monoid rings of §10.10 will be used in §12.2 and §12.4 to analyze the slice filtration of $MU_{\mathbf{R}}$ and its norms.

Indexed smash products. In the first four sections, we will study indexed smash products such as the norm of Definition 9.7.3. The purpose here is to establish Theorem 10.4.7, which asserts that such smash products have a total left derived functor (Definition 4.4.7) which may be computed on cofibrant objects. In other words, while the indexed smash product functor on certain diagrams of spectra is not homotopical in general, it becomes so if we replace the input diagram X by a cofibrant approximation.

This works for the stable equifibrant model structure, with or without positivity. This means that such smash products are also homotopical on Bredon cofibrant objects (Definition 9.2.15). These are spectra that are cofibrant with respect to the stable equifibrant model structure, in which there is no positivity condition.

Many of the technical results in these sections are also required for our analysis of symmetric powers and of commutative algebras later in the chapter.

Let $\mathcal{S}p$ denote the category of orthogonal spectra as in Definition 7.2.4. It is convenient to work in the category $\mathcal{S}p^J$ of functors to it from certain small categories J. Recall that when J is the one object category $\mathcal{B}G$ associated with a group G, then $\mathcal{S}p^J = \mathcal{S}p^G_{naive}$, the category of naive G-spectra, which is known by Theorem 9.3.10 to be equivalent to $\mathcal{S}p^G$, the category of genuine G-spectra. More generally we will consider small categories of the form $J = \mathcal{B}_T G$ for a finite G-set T, as in Definition 2.1.31. This is spelled out in §10.1. Thus $\mathcal{S}p^{\mathcal{B}_T G}$ is the category of T-shaped diagrams of spectra equipped with certain G-actions.

The resulting indexed smash products of cofibrations are studied in §10.2, where the main result is Theorem 10.2.4. We need to show that the indexed smash product of a trivial cofibration of cofibrant T-diagrams is a weak equivalence and therefore a trivial cofibration itself. This is established in Proposition 10.4.6, which leads directly to Theorem 10.4.7. It says that the left derived functor for the indexed smash product is the indexed smash product of the cofibrant replacements of the spectra in question.

How does this compare with the results of §9.6? Proposition 9.6.4 says that an indexed wedge of weak equivalences is a weak equivalence, and smashing with a flat spectrum is homotopical by definition. Theorem 10.2.4 says that an indexed smash product of cofibrations is an h-cofibration. Proposition 10.4.6 says that an indexed smash product of weak equivalences between cofibrant objects is again a weak equivalence between cofibrant objects.

Symmetric powers. In the next five sections, §10.5–§10.9, we study the homotopical properties of symmetric smash powers, or just "symmetric powers" for short. The nth symmetric power of a spectrum X is described in Definition 10.5.1, which is a special case of Definition 2.6.63. This will figure in the left adjoint of the forgetful functor from commutative algebras in the category of spectra to spectra as in Lemma 2.6.66. In order to apply the Crans-Kan Transfer Theorem 5.2.27 we need to know that the symmetric power functor is homotopical on cofibrant spectra. Example 10.5.2 illustrates why we need the sphere spectrum S^{-0} **not** to be cofibrant for this to happen.

In order to proceed we need to generalize the nth symmetric power of Definition 10.5.1 in two ways:

(i) We replace the n-fold smash product $X^{\wedge n}$ by a smash product $X^{\wedge T}$ indexed by a finite G-set T. This means that the group Σ_T of (not necessarily equivariant) isomorphisms of the set T has an action of G by conjugation.

(ii) For the sake of generality we replace the group Σ_T by a G-stable subgroup Λ, which could be Σ_T itself.

This leads to the notions of indexed symmetric powers and indexed symmetric corner maps given in Definition 10.5.6 and Definition 10.5.7. The former is acted on and the latter is equivariant with respect to that action by the group

$$\widetilde{G} := \Lambda \rtimes G.$$

The main result of §10.5 is Theorem 10.5.10, which says that good things happen when we have a cofibration $X \to Y$ between cofibrant $\widetilde{G}$-equivariant T-diagrams. Its proof occupies most of the section. We show that being cofibrant means that Λ acts freely away from the base point of $X^{\wedge T}$. Indeed the positivity condition is designed to make this happen. For more details see Remark 10.5.20. We also show that for a cofibrant $\widetilde{G}$-equivariant T-diagram X there is a cofibrant approximation to its indexed symmetric power $\mathrm{Sym}^T_\Lambda X$ involving the G-equivariant universal Λ-space $E_G\Lambda$ of Definition 8.7.1.

In §10.6 we study **iterated** indexed symmetric powers. Suppose that we have a second finite G-set S. The action of Λ on T leads to an action of Λ^S (the group of Λ-valued functions on S) on $T \times S$ that leaves the S-coordinate unchanged. Combining this with the actions of G on S, T and Λ leads to an action of the group

$$\widetilde{G}^{(S)} := \Lambda^S \rtimes G.$$

The analog of Theorem 10.5.10 in this case is Proposition 10.6.6, which concerns indexed smash products of indexed symmetric powers. For indexed smash symmetric powers of indexed symmetric powers, see Remark 10.6.7.

In §10.7 we will define a cofibrantly generated model structure on $\mathbf{Comm}^G$, the category of commutative algebras in $\mathcal{S}p^G$ as in Definition 9.7.1. The proof is easy now that we have the relevant machinery in place. The main tools in addition to the Crans-Kan Transfer Theorem 5.2.27 are the target exponent filtration of Definition 2.9.34 and Theorem 10.5.10.

The subject of §10.8 is the category $\mathcal{M}_R$ of left modules over an equivariant associative algebra R. We define a cofibrantly generated model structure on it in Proposition 10.8.1. In Corollary 10.8.2 we show that a map $f : R \to R'$ of equivariant associative algebras leads to a Quillen pair of functors between their model categories. In Proposition 10.8.3 we show that the functor $M \underset{R}{\wedge} (-)$ is flat if M is a cofibrant right R-module. In Corollary 10.8.4 we show that applying it to a map of left R-modules $N \to N'$ whose underlying map of spectra is an h-cofibration leads to the expected cofiber sequence.

In §10.9 we address a **serious technical issue**. The spectrum underlying a commutative ring R is almost never cofibrant, even when R is a cofibrant object in $\mathbf{Comm}^G$. This means that **there is no guarantee that the norm of a commutative ring has the correct homotopy type**. The fact that it does, Corollary 10.9.10, is one of the main results of this chapter. We also prove some results about indexed smash products of commutative rings that will be needed in §11.4.

Twisted monoid rings. Finally in §10.10 we discuss twisted monoid rings. These are associative algebras weakly equivalent to wedges of spheres, which can be manufactured by hand and mapped to commutative algbras. They are used in the proof Lemma 12.4.23, which is a key step in the proof the

Reduction Theorem 12.4.8, which in turn is pivotal in the proof of the Gap Theorem of §1.1C(iii).

10.1 Equivariant T-diagrams

Given a non-empty finite G-set T, consider the category $\mathcal{S}p^{\mathcal{B}_T G}$ of functors

$$\mathcal{B}_T G \to \mathcal{S}p = [\mathscr{J}_{S^1}^{\mathbf{O}}, \mathcal{T}]$$

for $\mathscr{J}_{S^1}^{\mathbf{O}}$ as in Definition 7.2.4; see Example 2.9.1 and Example 2.9.8. Recall that $\mathcal{S}p^{\mathcal{B}_{G/H}G}$ is equivalent to $\mathcal{S}p^{\mathcal{B}H} = \mathcal{S}p^H_{naive}$ by Corollary 2.1.40, which is equivalent to $\mathcal{S}p^H$ by Theorem 9.3.10. By Corollary 9.3.24, a choice of a point t in each G-orbit of T gives a Quillen equivalence

$$\mathcal{S}p^{\mathcal{B}_T G} \cong \prod_t \mathcal{S}p^{G_t}, \tag{10.1.1}$$

where G_t is the stabilizer of t; and the model structure on the right is the product of any of the eight model structures on each factor given by Theorem 9.2.13. We will use the positive stable equifibrant one on each factor, and refer to the corresponding one on $\mathcal{S}p^{\mathcal{B}_T G}$ as **the model category of equivariant T-diagrams of spectra**.

If $p : \widetilde{T} \to T$ is a map of finite G-sets, then the precomposition functor

$$p^* : \mathcal{S}p^{\mathcal{B}_T G} \to \mathcal{S}p^{\mathcal{B}_{\widetilde{T}} G}$$

has both a left and right adjoint, given by the two Kan extensions. All three functors are homotopical, and both the restriction functor and its left adjoint send cofibrations to cofibrations. This means that p^* is both a left and right Quillen functor.

Example 10.1.2. Precomposition and change of group. *Let $K \subseteq H \subseteq G$ be subgroups, $\widetilde{T} = G/K$, $T = G/H$ and let $p : G/K \to G/H$ be the usual map. Suppose also that we have chosen a point $t' \in G/K$ and its image is $t = p(t') \in G/H$. Then our precomposition functor p^* and its left adjoint $p_! = p_*^\vee$ (the indexed wedge of (9.3.20)) are related to the change of group adjunction of (9.1.18) as in the diagram*

$$\begin{tikzcd}
 & \mathcal{S}p^{\mathcal{B}_{G/K}G} \arrow[r, shift left, "p_*^\vee"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "j^*"] \arrow[d, phantom, "\vdash"] & \mathcal{S}p^{\mathcal{B}_{G/H}G} \arrow[l, shift left, "p^*"] \arrow[d, shift left, "j^*"] \arrow[d, phantom, "\vdash"] & \\
\mathcal{S}p^K_{naive} \arrow[r, equal] \arrow[d, shift left, "i_!"] \arrow[d, phantom, "\vdash"] & \mathcal{S}p^{\mathcal{B}K} \arrow[u, shift left, "k^*"] & \mathcal{S}p^{\mathcal{B}H} \arrow[u, shift left, "k^*"] \arrow[r, equal] & \mathcal{S}p^H_{naive} \arrow[d, shift left, "i_!"] \arrow[d, phantom, "\vdash"] \\
\mathcal{S}p^K \arrow[u, shift left, "i^*"] \arrow[rrr, shift left, "H \underset{K}{\ltimes} -"] \arrow[rrr, phantom, "\perp"] & & & \mathcal{S}p^H, \arrow[u, shift left, "i^*"] \arrow[lll, shift left, "i_K^H"]
\end{tikzcd}$$

where the upper vertical maps are induced by the functors $j : \mathcal{B}K \to \mathcal{B}_{G/K}G$, $k : \mathcal{B}_{G/K}G \to \mathcal{B}K$ and similar ones for H as in Proposition 2.2.31, and the lower vertical maps are each the left Kan extension of Theorem 9.3.10. The equalities in the diagram are there by Definition 9.3.2.

10.2 Indexed smash products and cofibrations

Let $p : \widetilde{T} \to T$ be an equivariant map of finite G-sets as above. The indexed smash product gives a functor

$$p_*^\wedge = (-)^{\wedge \widetilde{T}/T} : \mathcal{S}p^{\mathcal{B}_{\widetilde{T}}G} \to \mathcal{S}p^{\mathcal{B}_T G} \tag{10.2.1}$$

as in (5.5.34). When $\widetilde{T} \to T$ is the map $G/H \to *$, this is the norm. The various homotopical properties of indexed and symmetric smash products we require are most naturally expressed as properties of $(-)^{\wedge \widetilde{T}/T}$. Working fiberwise (Definition 2.9.4), establishing these reduces to the case $T = *$. To keep the discussion uncluttered we focus on that case in this chapter, leaving the extension to the case of more general T to the reader.

Let $p : \widetilde{T} \to *$ be the unique equivariant map and write the indexed smash product as $(-)^{\wedge \widetilde{T}}$. Note that if V is a representation of $\widetilde{T}$ then

$$(S^{-V})^{\wedge \widetilde{T}} = S^{-p_! V},$$

where $p_! V$ as in Definition 8.9.10(vii). When $T = G/H$, $p_! V = \operatorname{Ind}_H^G V$.

Example 10.2.2. The norm of a Yoneda spectrum. *Let $\widetilde{T} = G/H$. Then a representation of $\widetilde{T}$ is a representation of the subgroup H. Then $p_! V$ is the induced representation $\mathbf{R}[G] \otimes_{\mathbf{R}[H]} V$ and we have $N_H^G S^{-V} \cong S^{-p_! V}$. See Proposition 9.7.8.*

Lemma 10.2.3. The indexed corner map of a diagram of generating cofibrations is a cofibration. *Suppose that $i : A \to B$ is a generating cofibration in $\mathcal{S}p^{\mathcal{B}_T G}$ as in (9.3.26). The indexed corner map $\partial_A B^{\wedge T} \to B^{\wedge T}$ is an indexed wedge of the form*

$$\bigvee_{\Gamma} S^{-V'} \wedge \left(S(W')_+ \to D(W')_+\right)$$

in which Γ is a finite G-set (to be identified in the proof), V' and W' are representations of Γ, and V' is positive as in Definition 8.9.10(v). In particular, $\partial_A B^{\wedge T} \to B^{\wedge T}$ is a cofibration.

More generally for any cofibration $i : A \to B$ in $\mathcal{S}p^{\mathcal{B}_T G}$, the indexed corner map $\partial_A B^{\wedge T} \to B^{\wedge T}$ is again a cofibration.

Proof This is a straightforward consequence of the indexed distributive law

(Proposition 2.9.20) applied to (9.3.26), and the compatibility of the formation of $\partial_A B^{\wedge T}$ with indexed wedges, as described in (2.9.54).

In more detail, we will apply the indexed distributive law of Proposition 2.9.20 to the case where the symmetric monoidal category $\mathcal{V}$ is the category $\mathcal{S}p$ of orthogonal spectra, and the indexing categories J, K and L are instances of those in Example 2.9.22, namely

$$\begin{tikzcd} J \arrow[r, "p"] \arrow[d, equal] & K \arrow[r, "q"] \arrow[d, equal] & L \arrow[d, equal] \\ \mathcal{B}_{\widetilde{T}}G & \mathcal{B}_T G & \mathcal{B}G \end{tikzcd}$$

Hence the diagram of (2.9.51) is

$$\begin{tikzcd} \mathcal{S}p_1^{\mathcal{B}_{\widetilde{T}}G} \arrow[rr, "\mathrm{Ev}^*"] \arrow[d, "p_*^\vee"'] & & \mathcal{S}p_1^{\mathcal{B}_{T\times\Gamma}G} \arrow[d, "\varpi_*^\square"] \\ \mathcal{S}p_1^{\mathcal{B}_T G} \arrow[dr, "q_*^\square"'] & & \mathcal{S}p_1^{\mathcal{B}_\Gamma G} \arrow[dl, "r_*^\vee"] \\ & \mathcal{S}p_1^{\mathcal{B}G}, & \end{tikzcd}$$

where $\mathcal{S}p_1^-$ denotes the arrow category of $\mathcal{S}p^-$. Here $\mathcal{B}_\Gamma G$ is the G-set of sections $s : \mathcal{B}_T G \to \mathcal{B}_{\widetilde{T}}G$ with $ps = 1_T$. Its fiber product with $\mathcal{B}_T G$ over $\mathcal{B}G$ is the same as the Cartesian product because $\mathcal{B}G$ has only one object. The evaluation and projection functors

$$\mathcal{B}_{\widetilde{T}}G \xleftarrow{\mathrm{Ev}} \mathcal{B}_{T\times\Gamma}G \xrightarrow{\varpi} \mathcal{B}_\Gamma G$$

are defined in the obvious way, and the functor $r : \mathcal{B}_\Gamma G \to \mathcal{B}G$ is unique.

Our generating cofibration $i : A \to B$ is a morphism in $\mathcal{S}p^{\mathcal{B}_T G}$ that is the image under $p_*^\vee$ of morphism $i' : A' \to B'$ in $\mathcal{S}p^{\mathcal{B}_{\widetilde{T}}G}$ (also a generating cofibration) in which each component has the form $S^{-V_{t'}} \wedge i_{n_t+}$. The image of i' under $q_*^\square p_*^\vee$ is an indexed pushout product of an indexed wedge of these components.

The distributive law equates this pushout product of wedges with a wedge (the functor $r_*^\vee$) of large number, given by the functor Ev^*, of pushout products, the functor $\varpi_*^\square$. The image under $\varpi_*^\square \mathrm{Ev}^*$ of each component of i' is a map of the form specified in the Lemma. □

Theorem 10.2.4. The indexed smash product of cofibrations. *Suppose that T is a non-empty finite G-set. If $f : X \to Y$ is a cofibration of equivariant T-diagrams, then the indexed smash product*

$$f^{\wedge T} : X^{\wedge T} \to Y^{\wedge T}$$

is an h-cofibration. If X is cofibrant, then $f^{\wedge T}$ is a cofibration between cofibrant objects in $\mathcal{S}p^G$.

Proof The assertion that $f^{\wedge T}$ is an h-cofibration is contained in Proposition 3.5.26. For the cofibrant object assertion we work by induction on $|T|$, and may therefore assume the result to be known for any non-empty $T_0 \subset T$ and any $H \subset G$ stabilizing T_0 as a subset. In particular, we may assume that if X is cofibrant, then $X^{\wedge T_0}$ is a cofibrant H-spectrum for any non-empty proper subset T_0 of T and any $H \subset G$ stabilizing T_0 as a subset.

We will establish the theorem in the case where f arises from a pushout square of T-diagrams

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[d] \\ X \arrow[r, "f"] \arrow[ur, phantom, "\lrcorner"] & Y \end{tikzcd}$$

in which i is a generating cofibration. We will show in this case that $f^{\wedge T}$ is an h-cofibration, and is a cofibration between cofibrant objects if X is cofibrant. Since the formation of indexed smash products commutes with directed colimits and retracts, the proposition then follows from the small object argument.

We give $Y^{\wedge T}$ the target exponent filtration of Definition 2.9.34. The successive terms are related by the pushout square

$$\begin{tikzcd} \displaystyle\bigvee_{\substack{T=T_0\amalg T_1\\ |T_1|=n}} X^{\wedge T_0} \wedge \partial_A B^{\wedge T_1} \arrow[r] \arrow[d] & \displaystyle\bigvee_{\substack{T=T_0\amalg T_1\\ |T_1|=n}} X^{\wedge T_0} \wedge B^{\wedge T_1} \arrow[d] \\ \mathrm{fil}_{n-1} Y^{\wedge T} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathrm{fil}_n Y^{\wedge T}. \end{tikzcd} \tag{10.2.5}$$

By Lemma 10.2.3, each of the maps

$$\partial_A B^{\wedge T_1} \to B^{\wedge T_1}$$

is a cofibration. If X is cofibrant, then $X^{\wedge T_0}$ is either the sphere spectrum S^{-0} (if $T_0 = \varnothing$) or cofibrant by induction, hence

$$X^{\wedge T_0} \wedge \partial_A B^{\wedge T_1} \to X^{\wedge T_0} \wedge B^{\wedge T_1}$$

is a cofibration by the pushout product axiom. Since indexed wedges preserve cofibrations, the top row of (10.2.5) is then a cofibration and hence so is the bottom row. □

To show that the indexed smash product has a left derived functor (see Definition 5.1.11 and Theorem 5.1.13) we need to augment Theorem 10.2.4 and show that what when $X \to Y$ is a trivial cofibration, then $X^{\wedge T} \to Y^{\wedge T}$ is a weak equivalence. This can be proved with the above argument once we know that the indexed corner maps $\partial_A B^{\wedge T} \to B^{\wedge T}$ associated to the generating trivial cofibrations are weak equivalences. But the generating trivial cofibrations contain the maps of the form (9.3.21) so dealing with them requires understanding something about indexed corner maps of fairly general cofibrations.

In §5.5D we saw that the arrow category for a closed Quillen ring(such as $\mathcal{S}p^G$) has the pushout product of maps, i.e., the formation of corner maps, as its binary operation. This gives us a convenient way to study indexed corner maps.

10.3 The arrow category and indexed corner maps

Let $\mathcal{S}p_1^G$ denote the category whose objects are maps $X_1 \to X_2$ in $\mathcal{S}p^G$, with morphisms being the evident commutative diagrams. As mentioned in Definition 2.6.55, $\mathcal{S}p_1^G$ can be made into a closed symmetric monoidal category using the pushout product operation $\square$, for which the unit is $* \to S^{-0}$.

Starting with any one of the eight model structures on $\mathcal{S}p^G$ of Theorem 9.2.13, we can give $\mathcal{S}p_1^G$ the projective model structure (Definition 5.4.2) in which a map

$$(X_1 \to X_2) \to (Y_1 \to Y_2) \tag{10.3.1}$$

is a weak equivalence or fibration if each of $X_i \to Y_i$ is, and is a cofibration if both $X_1 \to Y_1$ and the corner map

$$X_2 \underset{X_1}{\cup} Y_1 \to Y_2 \tag{10.3.2}$$

are cofibrations. An object $X_1 \to X_2$ is cofibrant if X_1 is cofibrant and $X_1 \to X_2$ is a cofibration; see Proposition 5.5.29.

Each of these model structures on $\mathcal{S}p_1^G$ is compactly generated. The generating (trivial) cofibrations in $\mathcal{S}p_1^G$ are of two types. Type I are the maps

$$(K \to K) \to (L \to L) \tag{10.3.3}$$

and type II are the maps

$$(* \to K) \to (* \to L) \tag{10.3.4}$$

where $K \to L$ is running through the set of generating (trivial) cofibrations indicated in Theorem 9.2.13. The following is a special case of Proposition 5.5.32.

Proposition 10.3.5. The arrow category is symmetric monoidal. *Equipped with each of the model structures just described, $\mathcal{S}p_1^G$ is a Quillen ring satisfying the monoid axiom.*

Proposition 10.3.5 addresses the homotopy properties of ordinary smash products in $\mathcal{S}p_1^G$. For the indexed smash products we work in the arrow category $\mathcal{S}p_1^{\mathcal{B}_T G}$ of maps of equivariant T-diagrams, in the projective model structure. Our aim is to establish Proposition 10.3.8, which gives control over the indexed corner maps (Definition 2.9.29) in $\mathcal{S}p^G$ (Proposition 10.3.9). It is the analogue in $\mathcal{S}p_1^{\mathcal{B}_T G}$ of Theorem 10.2.4. In preparation, we need to identify the generating (trivial) cofibrations. Those in $\mathcal{S}p_1^G$ were identitied above.

Remark 10.3.6. *A map as in (10.3.1) is an h-cofibration if both $X_1 \to Y_1$ and the corner map (10.3.2) are. Since cofibrations in $\mathcal{S}p^G$ are h-cofibrations the same is true of cofibrations in $\mathcal{S}p_1^G$.*

Lemma 10.3.7. The behavior of indexed smash products in the arrow category. *If $i : A \to B$ is a generating cofibration in the category $\mathcal{S}p_1^{\mathcal{B}_T G}$, then the indexed corner map*

$$i_T : \partial_A B^{\wedge T} \to B^{\wedge T}$$

as in Definition 2.9.29 is a cofibration between cofibrant objects in $\mathcal{S}p_1^G$.

Proof First note that for generating cofibrations of type I (as in (10.3.3)), the corner map is

$$(\partial_K L^{\wedge T} \to \partial_K L^{\wedge T}) \to (L^{\wedge T} \to L^{\wedge T})$$

and in type II (as in (10.3.4)), it is

$$(* \to \partial_K L^{\wedge T}) \to (* \to L^{\wedge T}).$$

The assertion therefore reduces to Lemma 10.2.3. □

Proposition 10.3.8. The behavior of an indexed smash product of cofibrations in the diagram category indexed by a G-set. *Suppose that T is a finite G-set. If $i : X \to Y$ is a cofibration in $\mathcal{S}p_1^{\mathcal{B}_T G}$ and X is cofibrant, then the indexed smash product*

$$i^{\wedge T} : X^{\wedge T} \to Y^{\wedge T}$$

is a cofibration between cofibrant objects in $\mathcal{S}p_1^G$.

Proof The proof proceeds exactly as that of Theorem 10.2.4. The target exponent filtration of Definition 2.9.34 and induction on $|T|$ reduce the problem to showing that the indexed corner map (in $\mathcal{S}p_1^{\mathcal{B}_T G}$)

$$i_T : \partial_A B^{\wedge T} \to B^{\wedge T}$$

is a cofibration between cofibrant objects, when $A \to B$ is a cofibrant generator. This is the content of Lemma 10.3.7. □

Specializing, we now have

Proposition 10.3.9. The behavior of an indexed corner map of cofibrations in the diagram category indexed by a G-set. *If $i : X \to Y$ is a cofibration in $\mathcal{B}_T G$ and X is cofibrant, then the indexed corner map $i_T : \partial_X Y^{\wedge T} \to Y^{\wedge T}$ is a cofibration between cofibrant objects.*

Proof If $i : X \to Y$ is a cofibration of cofibrant T-diagrams, then $(X \to Y)$ is cofibrant T-diagram in $\mathcal{S}p_1^G$, and so

$$(X \to Y)^{\square T} = (i_T : \partial_X Y^{\wedge T} \to Y^{\wedge T})$$

is cofibrant by Proposition 10.3.8. □

10.4 Indexed smash products and trivial cofibrations

With the indexed corner maps of cofibrations under control we can now turn to the trivial cofibrations.

Lemma 10.4.1. The behavior of an indexed corner map of trivial cofibrations in the diagram category indexed by a G-set. *If $i : A \to B$ is a generating trivial cofibration in $\mathcal{S}p^{\mathcal{B}_T G}$, then the indexed corner map*

$$i_T : \partial_A B^{\wedge T} \to B^{\wedge T}$$

as in Definition 2.9.29 is a trivial cofibration of cofibrant objects in $\mathcal{S}p^G$.

Proof We know from Proposition 10.3.9 that the indexed corner maps are cofibrations between cofibrant objects, so what remains is the assertion that they are weak equivalences. This can be reduced further. Suppose that $i : A \to B$ is a trivial cofibration in $\mathcal{S}p^{\mathcal{B}_T G}$ and we wish to show that the indexed corner map $i_T : \partial_A B^{\wedge T} \to B^{\wedge T}$ is a weak equivalence. Give $B^{\wedge T}$ the target exponent filtration of Definition 2.9.34, in which the successive terms are related by the pushout square

$$\begin{tikzcd}
\displaystyle\bigvee_{\substack{T=T_0\amalg T_1\\ |T_1|=n}} A^{\wedge T_0} \wedge \partial_A B^{\wedge T_1} \arrow[r] \arrow[d] & \displaystyle\bigvee_{\substack{T=T_0\amalg T_1\\ |T_1|=n}} A^{\wedge T_0} \wedge B^{\wedge T_1} \arrow[d] \\
\mathrm{fil}_{n-1} B^{\wedge T} \arrow[r] & \mathrm{fil}_n B^{\wedge T}.
\end{tikzcd}$$

By Proposition 10.3.9 and the pushout product axiom, the upper arrow is a cofibration, which, by induction on $|T|$, we may assume to be trivial when $|T| < n$. Since the cofibrations are flat, this means that the bottom arrow is a trivial cofibration when $|T| < n$. It follows that in this case, the indexed corner map is a weak equivalence if and only if the **absolute** map $i^{\wedge T} : X^{\wedge T} \to B^{\wedge T}$ is.

We now turn to the generating trivial cofibrations. The generators of the form $A \wedge \left(I_+^{n-1} \to I_+^n\right)$ are homotopy equivalences, hence so are the absolute maps. The other generators are of the form

$$\left(S_+^{n-1} \to D_+^n\right) \square \left(p_*^\vee S^{-V \oplus W} \wedge S^W \to p_*^\vee \tilde{S}^{V,W}\right), \tag{10.4.2}$$

(see (9.3.21)) where $p : \widetilde{T} \to T$ is a map of finite G-sets and V and W are equivariant vector bundles over $\widetilde{T}$. The fact that the norm is symmetric monoidal by Proposition 9.7.4, together with the monoid axiom for $\mathcal{S}p_1^G$, reduces us to considering only the right hand factor in (10.4.2). The distributive law further reduces us to the case $\widetilde{T} = T$. Finally, since the map $\tilde{S}^{V,W} \to S^{-V}$ is a homotopy equivalence, we may replace $\tilde{S}^{V,W}$ with S^{-V}. Evaluating both sides using Proposition 9.7.8 we see that the assertion amounts to checking that

$$S^{-V' \oplus W'} \wedge S^{W'} \to S^{-V'}$$

is a weak equivalence, where V' and W' are the G-spaces of global sections. But this is a special case of Proposition 9.5.6. □

As with Lemma 10.3.7, the separate cases of type I and type II generators reduce the result below to Lemma 10.4.1.

Lemma 10.4.3. The indexed corner map of a generating trivial cofibration. *If $i : A \to Y$ is a generating trivial cofibration in the category of equivariant T-diagrams in $\mathcal{S}p_1^G$, then the indexed corner map*

$$i_T : \partial_A B^{\wedge T} \to B^{\wedge T}$$

is a trivial cofibration of cofibrant objects in $\mathcal{S}p_1^G$.

Proposition 10.4.4. The indexed smash product of trivial cofibrations between cofibrant objects. *Suppose that T is a finite G-set. The functor*

$$(-)^{\wedge T} : \mathcal{S}p_1^{\mathcal{B}_T G} \to \mathcal{S}p_1^G$$

sends trivial cofibrations between cofibrant objects to trivial cofibrations between cofibrant objects, and hence weak equivalences between cofibrant objects to weak equivalences between cofibrant objects.

Proof The proof proceeds exactly as in the case of Theorem 10.2.4. That the second assertion follows from the first is Ken Brown's Lemma 5.1.7. □

Specializing Proposition 10.4.4, we have

Proposition 10.4.5. The indexed corner map and indexed smash product of a trivial cofibration from a cofibrant object. *If $i : X \to Y$ is a trivial cofibration in $\mathcal{S}p^{\mathcal{B}_T G}$ and X is cofibrant, then both the indexed corner map $i_T : \partial_X Y^{\wedge T} \to Y^{\wedge T}$ and the absolute map $i^{\wedge T} : X^{\wedge T} \to Y^{\wedge T}$ are trivial cofibrations between cofibrant objects.*

With all this in hand we can now show that indexed smash products have left derived functors. From Theorem 10.2.4, Proposition 10.4.5, and Ken Brown's Lemma 5.1.7 we have

Proposition 10.4.6. The indexed smash product of weak equivalence between cofibrant objects. *The indexed smash product*

$$(-)^{\wedge T} : \mathcal{S}p^{\mathcal{B}_T G} \to \mathcal{S}p^G$$

takes weak equivalences between cofibrant objects to weak equivalences between cofibrant objects.

This gives the main result of this section.

Theorem 10.4.7. The left derived functor of the indexed smash product. *The indexed smash product has a left derived functor*

$$(-)^{\overset{\mathbf{L}}{\wedge} T} : \mathcal{S}p^{\mathcal{B}_T G} \to \mathrm{Ho}\,\mathcal{S}p^G$$

which may be computed as

$$X^{\overset{\mathbf{L}}{\wedge} T} = (X_c)^{\wedge T}$$

where $X_c \to X$ is a cofibrant approximation.

10.5 Indexed symmetric powers

The following is a special case of the nth symmetric product of Definition 2.6.63.

Definition 10.5.1. *Let Σ_n denote the symmetric group on n letters.* **The nth symmetric (smash) power** *of a G-spectrum X is the Σ_n-orbit spectrum*

$$\mathrm{Sym}^n(X) = X^{\wedge n}/\Sigma_n.$$

The homotopy properties of this functor are fundamental to understanding the homotopy theory of equivariant commutative algebras. Before proceeding further we offer the following.

Example 10.5.2. A badly behaved map of symmetric powers. *The map*

$$e_1 : S^{-1} \wedge S^1 \to S^{-0}$$

of (7.2.68) is a weak equivalence. However, the induced map

$$\mathrm{Sym}^n(S^{-1} \wedge S^1) \to \mathrm{Sym}^n S^{-0} \tag{10.5.3}$$

for $n > 1$ is not, **even if G is trivial**. *In other words, this bad behavior has to do with symmetric powers rather than G-equivariance.*

The right side of (10.5.3) is S^{-0} since it is the unit for the smash product, while the left side works out by Lemma 10.5.18(ii) below to be weakly equivalent to the suspension spectrum of the classifying space for G-equivariant principal Σ_n-bundles with disjoint base point.

Fortunately the sphere spectrum S^{-0} is **not** *cofibrant in the positive stable equifibrant model structure, so e_1 is not a weak equivalence of cofibrant spectra.*

This example illustrates the need for the positivity condition in our model structure of choice and why S^{-0} cannot be cofibrant. Another reason is given in Remark 5.5.28. We need a model structure in which the symmetric power functor is homotopical on cofibrant objects.

For a finite G-set T, let Σ_n^T be the $|T|$-fold Cartesian product of Σ_n equipped with a G-action induced by the one on T. It is a G-stable subgroup of $\Sigma_{\mathbf{n}\times T}$ (where $\mathbf{n} = \{1, \dots, n\}$), the group of nonequivariant isomorphisms of the G-set $\mathbf{n} \times T$, on which G acts by conjugation. It can also be thought of as the group of Σ_n-valued functions on T, with pointwise multiplication. It has an action of G induced by the one on T.

For indexed smash products of commutative algebras, the distributive law leads one to consider indexed smash products of symmetric powers

$$(\mathrm{Sym}^n X)^{\wedge T}.$$

These can be written as

$$(\mathrm{Sym}^n X)^{\wedge T} = (X^{\wedge n}/\Sigma_n)^{\wedge T} \cong X^{\wedge(\mathbf{n}\times T)}/\Sigma_n^T \tag{10.5.4}$$

The last expression in (10.5.4) is an **indexed symmetric power**. The definition (Definition 10.5.6) and homotopy properties of indexed symmetric powers are the subject of this section and the next.

Before turning to the definition, we consider a simpler situation. Recall that by Theorem 9.3.10 and Proposition 9.3.16, the category of orthogonal G-spectra $\mathcal{S}p^G$ is homotopically equivalent to the category $\mathcal{S}p^{\mathcal{B}G}$ of orthogonal spectra with G-actions.

The following is a variant of Definition 9.1.9. Recall that for groups G and $\tilde{G}$, $\mathcal{S}p^{\mathcal{B}G}$ and $\mathcal{S}p^{\mathcal{B}\tilde{G}}$ are the categories of naive G-spectra and naive $\tilde{G}$-spectra respectively.

Proposition 10.5.5. An orbit spectrum that is a left Kan extension. *Suppose that $i : \widetilde{G} \to G$ is a surjective group homomorphism with kernel N. Then the functor $i^* : \mathcal{S}p^{\mathcal{B}G} \to \mathcal{S}p^{\mathcal{B}\widetilde{G}}$ has both a left and a right adjoint. The left adjoint $i_! : \mathcal{S}p^{\widetilde{G}} \to \mathcal{S}p^G$ sends a naive $\tilde{G}$-spectrum X to the* **orbit spectrum** *X/N equipped with its residual G-action, as explained in Example 2.5.8(v).*

The expression on the right of (10.5.4) is a special case of this in which $\widetilde{G}$ is the semi-direct product $\Sigma_n^T \rtimes G$.

Proof The left and right adjoints of the precomposition functor i^* are the left and right Kan extensions $i_!$ and i_*. The adjunction $i_! \dashv i^*$ means that for any naive G-spectrum Y there is a natural isomorphism

$$\mathcal{S}p^{\mathcal{B}\tilde{G}}(X, i^*Y) \cong \mathcal{S}p^{\mathcal{B}G}(i_!X, Y),$$

so for each $n \geqslant 0$, there is an isomorphism

$$\mathcal{T}^{\tilde{G}}(X_n, (i^*Y)_n) \cong \mathcal{T}^G((i_!X)_n, Y_n).$$

The $\tilde{G}$-space $(i^*Y)_n$ is Y_n with $\tilde{G}$-action induced by the homomorphism i, so N acts trivially on it, so we have

$$\mathcal{T}^{\tilde{G}}(X_n, Y_n) \cong \mathcal{T}^{\tilde{G}}(X_n/N, Y_n) \cong \mathcal{T}^G(X_n/N, Y_n).$$

The last isomorphims follows from that fact that both X_n/N and Y_n have trivial N-action, so a $\tilde{G}$-equivariant map is the same thing as a G-equivariant map. Thus the adjunction isomorphism reads

$$\mathcal{T}^G(X_n/N, Y_n) \cong \mathcal{T}^G((i_!X)_n, Y_n),$$

which implies that

$$(i_!X)_n \cong X_n/N \cong (X/N)_n,$$

where the second isomorphism follows from the fact that colimits of spectra, such as orbit spectra, are defined objectwise. □

Definition 10.5.6. Indexed symmetric powers. *Let T be a finite G-set, and Σ_T the group of (not necessarily equivariant) automorphisms of T, with G acting on it by conjugation. Fix a G-stable subgroup $\Lambda \subset \Sigma_T$, and let $\widetilde{G} = \Lambda \rtimes G$. The actions of Λ and G on T define an action of $\widetilde{G}$ on it. For a $\widetilde{G}$-equivariant T-diagram X the* **indexed symmetric power** *is the orbit G-spectrum*

$$\mathrm{Sym}^T_\Lambda X = X^{\wedge T}/\Lambda.$$

When the indexing set T has a trivial G-action, Λ could be the full symmetry group of T. Then the equivariant T-diagram is the constant diagram with value $X \in \mathcal{S}p^G$, and this construction is the usual symmetric power $\mathrm{Sym}^{|T|}X$ discussed above. We will usually use the same notation for the $\widetilde{G}$-spectrum X and the constant equivariant T-diagram with value X.

Definition 10.5.7. *If $f : X \to Y$ is a map of $\widetilde{G}$-equivariant T-diagrams, the* **indexed symmetric corner map** *is the map of orbit G-spectra*

$$\mathrm{Sym}_\Lambda f_T : \partial_X \mathrm{Sym}^T_\Lambda Y \to \mathrm{Sym}^T_\Lambda Y$$

obtained by passing to Λ-orbits from the indexed corner map of Definition 2.9.29

$$esf_T : \partial_X Y^{\wedge T} \to Y^{\wedge T}.$$

It is the indexed symmetric power (as in Definition 10.5.6) of f in the arrow category $\mathcal{S}p_1^{\mathcal{B}_T\widetilde{G}}$.

Example 10.5.8. *In* **the case** $X = *$, *the domain of the map above is also the point valued spectrum. This can be derived from Example 2.6.14.*

Remark 10.5.9. *Since the orbit spectrum functor is a continuous left adjoint, it sends h-cofibrations to h-cofibrations. For example, suppose that $i : X \to Y$ is a cofibration of cofibrant $\widetilde{G}$-equivariant T-diagrams. By Theorem 10.2.4 and Proposition 10.3.9, both the indexed smash product*

$$i^{\wedge T} : X^{\wedge T} \to Y^{\wedge T}$$

and the corner map

$$i_T : \partial_X Y^{\wedge T} \to Y^{\wedge T}$$

are cofibrations, and hence h-cofibrations, of $\widetilde{G}$-spectra. This means that all four of the maps

$$\begin{aligned} \mathrm{Sym}_\Lambda^T i : \mathrm{Sym}_\Lambda^T X &\to \mathrm{Sym}_\Lambda^T Y, \\ \mathrm{Sym}_\Lambda i_T : \partial_X \mathrm{Sym}_\Lambda^T Y &\to \mathrm{Sym}_\Lambda^T Y, \\ (E_G\Lambda) \underset{\Lambda}{\ltimes} i : (E_G\Lambda) \underset{\Lambda}{\ltimes} X^{\wedge T} &\to (E_G\Lambda) \underset{\Lambda}{\ltimes} Y^{\wedge T} \\ \textit{and} \qquad (E_G\Lambda) \underset{\Lambda}{\ltimes} i_T : (E_G\Lambda) \underset{\Lambda}{\ltimes} \partial_X Y^{\wedge T} &\to (E_G\Lambda) \underset{\Lambda}{\ltimes} Y^{\wedge T} \end{aligned}$$

are h-cofibrations of G-spectra.

Note that $X^{\wedge T}$ with its $\widetilde{G}$-action is a special case of an indexed monoidal product, the subject of §2.9. This means that the distributive law (see §2.9B) applies to symmetric powers. Given a pushout square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\lrcorner"] & Z, \end{tikzcd}$$

there is a filtration of $\mathrm{Sym}_\Lambda^T Z$ whose successive terms are related by passing to Λ-orbits from the target exponent filtration of Definition 2.9.34.

As described in [MM02, §III.8], the homotopy theoretic analysis of indexed symmetric powers requires certain equivariant principal bundles. For the moment, let Λ be any finite group with a G-action. Let $\widetilde{G}$ denote the group $\Lambda \rtimes G$, and let $E_G\Lambda$ be a G-equivariant universal Λ-space of Definition 8.7.1.

The symmetric powers of a cofibrant spectrum are cofibrant, as we see from the following in the case where X is a point.

Theorem 10.5.10. An indexed symmetric power of a cofibration of cofibrant diagrams is a cofibration of cofibrant objects. *Suppose that*

$i : X \to Y$ is a cofibration between cofibrant $\widetilde{G}$-equivariant T-diagrams. In the commutative square of G-spectra

$$\begin{tikzcd}
(E_G\Lambda) \underset{\Lambda}{\ltimes} \partial_X Y^{\wedge T} \arrow[r, "(E_G\Lambda)\underset{\Lambda}{\ltimes} i_T"] \arrow[d, "\simeq"'] & (E_G\Lambda) \underset{\Lambda}{\ltimes} Y^{\wedge T} \arrow[d, "\simeq"] \\
\partial_X \mathrm{Sym}^T_\Lambda Y \arrow[r, "\mathrm{Sym}_\Lambda i_T"] & \mathrm{Sym}^T_\Lambda Y,
\end{tikzcd} \tag{10.5.11}$$

every object is flat, the upper row is a cofibration between cofibrant objects, the vertical maps are weak equivalences, and the bottom row is an h-cofibration. The horizontal maps are weak equivalences if i is one.

Remark 10.5.12. *By Proposition 9.6.6 the maps in (10.5.11) asserted to be weak equivalences remain so after smashing with any spectrum Z.*

Remark 10.5.13. *In studying the free commutative algebra functor, we have a cofibration $i : X \to Y$ of cofibrant G-spectra, regarded as $\widetilde{G}$-spectra through the map $\widetilde{G} \to G$, and then regarded as constant equivariant T-diagrams. This map of equivariant T-diagrams is a cofibration by Proposition 9.8.3, so Theorem 10.5.10 applies.*

Along the way to proving Theorem 10.5.10 we will also show

Proposition 10.5.14. *The functors $(E_G\Lambda) \underset{\Lambda}{\ltimes} (-)^{\wedge T}$ and $\mathrm{Sym}_\Lambda(-)_T$ take weak equivalences between cofibrant objects to weak equivalences.*

Remark 10.5.15. *Theorem 10.5.10 is part of the reason for the* **positive** *condition in the model structure we have chosen. The result is not true for general* **Bredon cofibrant** *objects of Definition 9.2.15, though it is true for Bredon cofibrant objects built from cells of the form $G \underset{H}{\ltimes} S^{-V} \wedge D^k_+$ with V non-zero. The condition that V is non-zero is used in the proof of Lemma 10.5.18. See Remark 10.5.20 below.*

The assertions about the top row in Theorem 10.5.10 are most easily analyzed in the arrow category $\mathcal{S}p_1^{\mathcal{B}_T G}$. Recall that Λ is a finite group with an action by another finite group G, $\widetilde{G}$ denotes the group $\Lambda \rtimes G$, and $E_G\Lambda$ denotes the G-equivariant universal Λ-space of Definition 8.7.1.

Lemma 10.5.16. The top row of (10.5.11). *The functor*

$$E_G\Lambda_+ \underset{\Lambda}{\wedge} (-)^{\wedge T} : \mathcal{S}p_1^{\mathcal{B}_T\widetilde{G}} \to \mathcal{S}p_1^{\mathcal{B}G}$$

takes trivial cofibrations between cofibrant objects to trivial cofibrations between cofibrant objects and hence weak equivalences between cofibrant objects to weak equivalences between cofibrant objects.

Proof The space $E_G\Lambda$ is built entirely of $\tilde{G}$-cells of the form

$$\tilde{G} \underset{\tilde{H}}{\ltimes} D_+^n$$

for subgroups $\tilde{H} \subseteq \tilde{G}$ having trivial intersection with Λ. This fact, along with the pushout product axiom (see Definition 5.5.9) means that it suffices to show that for a trivial cofibration $j : X \to Y$ between cofibrant objects in $\mathcal{S}p^{\mathcal{B}_T\tilde{G}}$, the map

$$\tilde{G}/\tilde{H} \underset{\Lambda}{\ltimes} j^{\wedge T}$$

is a trivial cofibration between cofibrant objects in $\mathcal{S}p^{\mathcal{B}G}$ for any subgroup $\tilde{H}$ as above. This is a wedge of maps, indexed by the orbit set $\tilde{G}/\Lambda\tilde{H}$ (where $\Lambda\tilde{H}$ denotes the subgroup generated by Λ and $\tilde{H}$), which is a G-set. This is the same as $j^{\wedge\widetilde{T}}$ where

$$\widetilde{T} = \tilde{G}/\tilde{H} \underset{\Lambda}{\times} T,$$

which is a trivial cofibration between cofibrant objects by Proposition 10.4.4. The assertion about weak equivalences between cofibrant objects follows from that about trivial cofibrations by Ken Brown's Lemma 5.1.7. □

The vertical maps in (10.5.11) require a more detailed analysis. Each of them is the map from the homotopy orbit spectrum to the actual orbit spectrum for an action of the group Λ. Recall that for a pointed Λ-space in which the group action is free away from the base point, this map is a weak equivalence by Proposition 8.6.6.

Definition 10.5.17. Λ-free $\widetilde{G}$-spectra. *Suppose that Λ is a group with an action of G, and that X is a $\widetilde{G}$-spectrum, with $\widetilde{G} = \Lambda \rtimes G$ as before. We will say that X is* **Λ-free as a $\widetilde{G}$-spectrum** *if for each orthogonal G-representation W (which is also a representation of $\widetilde{G}$), the Λ-action on the pointed $\tilde{G}$-space X_W is free away from the base point.*

Lemma 10.5.18. Properties of cofibrant $\widetilde{G}$-equivariant T-diagrams. *Let X be a cofibrant $\widetilde{G}$-equivariant T-diagram in $\mathcal{S}p$ for a finite G-set T, and let Z be any $\widetilde{G}$-spectrum. Then*

(i) $X^{\wedge T} \wedge Z$ is a Λ-free G-spectrum, where Λ acts on $X^{\wedge T}$ by permuting its factors through the inclusion of Λ as a subgroup of Σ_T, the group of (nonequivariant) isomorphisms of the finite G-set T, and

(ii) the map

$$E_G\Lambda \underset{\Lambda}{\ltimes} (X^{\wedge T} \wedge Z) \to (X^{\wedge T} \wedge Z)/\Lambda$$

is a weak equivalence in $\mathcal{S}p^{\mathcal{B}G}$.

Remark 10.5.19. *We will be mostly interested in the case in which the Λ-action on Z is trivial. In that case the equivalence of Lemma 10.5.18(ii) takes the form*

$$\left(E_G\Lambda \underset{\Lambda}{\ltimes} X^{\wedge T}\right) \wedge Z \xrightarrow{\simeq} \mathrm{Sym}_\Lambda^T(X) \wedge Z.$$

Remark 10.5.20. The role of the positivity condition. *Consider the assertion of Lemma 10.5.18(ii) when G is trivial, Z is the sphere spectrum S^{-0} and Λ is the full symmetric group Σ_i for $i = |T|$. The statement then is that the homotopy orbit spectrum $(E\Sigma_i) \underset{\Lambda}{\ltimes} X^{\wedge i}$ is stably equivalent to the actual orbit spectrum $X^{\wedge i}/\Sigma_i$. This is the case when the action of Σ_i is free away from the base point, but not in general. However for a pointed CW complex Y other than a point, the action of Σ_i on $Y^{\wedge i}$ for $i > 1$ is* **never** *free away from the base point, because there is always a diagonal subspace fixed by Σ_i.*

The positivity condition means that $S^{-1} \wedge Y$ is cofibrant but $S^{-0} \wedge Y$ is not, so we illustrate with $X = S^{-1} \wedge Y$. Then $X^{\wedge i} = S^{-i} \wedge Y^{\wedge i}$, so by definition $(X^{\wedge i})_n = \mathscr{J}(i, n) \wedge Y^{\wedge i}$. By Proposition 8.9.29, $\mathscr{J}(i, n)$ has a free (away from the base point) action of the orthogonal group $O(i)$, and hence of its subgroup Σ_i. It follows that the same is true of the spectrum $X^{\wedge i}$.

This is the very reason for the positivity condition. *We need the i-fold smash power of a cofibrant spectrum to have a free Σ_i-action.*

Proof of Lemma 10.5.18. For the Λ-freeness assertion, note first that the initial object in the category, the constant $*$-valued diagram, is Λ-free by definition. Thus it suffices to show that applying the functor $(-)^{\wedge T} \wedge Z$ to the process of attaching a cell in $\mathcal{S}p^{\mathcal{B}_T\widetilde{G}}$ preserves Λ-freeness. In other words it suffices to show that if $A \to B$ is a generating cofibration in $\mathcal{S}p^{\mathcal{B}_T\widetilde{G}}$ (see (9.3.25) and (9.3.26)),

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X_0 \arrow[r] & X_1 \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd},$$

is a pushout square, and $X_0^{\wedge T} \wedge Z$ is Λ-free, then $X_1^{\wedge T} \wedge Z$ is Λ-free.

We now take a closer look at the generating cofibration $A \to B$. The G-set T has orbits of the form G/G_t, where $G_t \subseteq G$ is the subgroup fixing $t \in T$. T is also a $\widetilde{G}$-set in which action of Λ permutes these G-orbits, so a $\widetilde{G}$-orbit is a finite union of isomorphic G-orbits. The orbit of t has the form $\widetilde{G}/\widetilde{G}_t$ where $\widetilde{G}_t = \Sigma_t \rtimes G_t$. A generating cofibration is a wedge, indexed by the set of $\widetilde{G}$-orbits of T, of maps

$$\widetilde{G}_{t+} \underset{\widetilde{H}_t}{\wedge} S^{-V_t} \wedge (S_+^{n_t - 1} \to D_+^{n_t})$$

with V_t a positive representation of $\widetilde{H}_t \subseteq \widetilde{G}_t$. The positivity condition on V_t

implies the source and target of this map are Λ-free by an argument similar to that of Remark 10.5.20.

It can also be described as in (9.3.26) as

$$p_*^\vee\left(S^{-V}\wedge\left(S_+^{n-1}\to D_+^n\right)\right)$$

with $p:\widetilde{T}\to T$ a finite surjective map of $\widetilde{G}$-sets, and V a positive representation of $\widetilde{G}$-set $\widetilde{T}$ as in Definition 8.9.10(v).

We use the target exponent filtration of Definition 2.9.34 to study $X_1^{\wedge T}$ and consider the pushout square below

$$\begin{tikzcd}
\bigvee\limits_{\substack{T=T_0\amalg T_1\\ |T_1|=m}} X_0^{\wedge T_0}\wedge\partial_A B^{\wedge T_1}\wedge Z \arrow[r] \arrow[d] & \bigvee\limits_{\substack{T=T_0\amalg T_1\\ |T_1|=m}} X_0^{\wedge T_0}\wedge B^{\wedge T_1}\wedge Z \arrow[d] \\
\mathrm{fil}_{m-1}X_1\wedge Z \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathrm{fil}_m X_1\wedge Z.
\end{tikzcd} \tag{10.5.21}$$

Since $A\to B$ is a cofibration, the map in the top row is an h-cofibration (Proposition 10.3.9) hence a closed inclusion. The object in the upper left is an indexed wedge of indexed smash products of Λ-free spectra, so it is also Λ-free. The object on the lower left is Λ-free by induction on m.

It therefore suffices to show that Λ acts freely away from the base point on the upper right term of (10.5.21); see Remark 2.1.47. Induction on $|T|$ reduces this to the case $m=|T|$. In this way the first assertion of the lemma reduces to checking the special case

$$X=p_*^\vee S^{-V}\wedge D_+^k,$$

with $p:\widetilde{T}\to T$ a surjective map of $\widetilde{G}$-sets and V a positive representation of $\widetilde{T}$ as in Definition 8.9.10(v). Since the factor $(D_+^k)^{\wedge T}$ can be absorbed into Z, we might as well suppose

$$X=p_*^\vee S^{-V}.$$

The indexed distributive law (Proposition 2.9.20) gives

$$X^{\wedge T}=\bigvee_{\gamma\in\Gamma}S^{-V_\gamma},$$

where Γ is the $\widetilde{G}$-set of sections $T\to\widetilde{T}$, and

$$V_\gamma=\bigoplus_{t\in T}V_{\gamma(t)}.$$

For an orthogonal $\widetilde{G}$-representation W we have, by Proposition 9.1.23,

$$(X^{\wedge T} \wedge Z)_W = \begin{cases} * & \text{if } |W| < |V_\gamma| \\ \bigvee_{\gamma\in\Gamma} O(V_\gamma \oplus U_\gamma, W) \underset{O(U_\gamma)}{\ltimes} Z_{U_\gamma} & \text{if } |W| \geqslant |V|_\gamma, \end{cases}$$

in which $U = \{U_\gamma\}$ is any representation of Γ with $|U_\gamma| = |W| - |V_\gamma|$. We are interested in representations W which are pulled back from the projection map $\widetilde{G} \to G$. In the first case there is nothing to prove. In the second case the complement of the base point is homeomorphic to

$$\coprod_{\gamma\in\Gamma} O(V_\gamma \oplus U_\gamma, W) \underset{O(U_\gamma)}{\times} (Z_{U_\gamma} - \{*\})$$

(see Remark 2.1.47). The Λ-freeness then follows from the fact that this space has an equivariant map to the disjoint union of Stiefel-manifolds

$$\coprod_{\gamma\in\Gamma} O(V_\gamma, W) = \coprod_{\gamma\in\Gamma} O(V_\gamma \oplus U_\gamma, W)/O(U_\gamma),$$

which is Λ-free since each $V_{\gamma(t)}$ is non-zero, and Λ acts faithfully on T (meaning there is no nontrivial $\lambda \in \Lambda$ that fixes all of T) but trivially on W.

With one additional observation, a similar argument reduces the assertion about weak equivalences to the same case

$$X = p_*^\vee S^{-V}. \tag{10.5.22}$$

To spell it out, abbreviate (10.5.21) as

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

and form

$$\begin{tikzcd} E_G\Lambda \underset{\Lambda}{\ltimes} K \arrow[r, "\flat"] \arrow[d] \arrow[dr, "\simeq"] & E_G\Lambda \underset{\Lambda}{\ltimes} L \arrow[dr, "\simeq"] & \\ E_G\Lambda \underset{\Lambda}{\ltimes} Y \arrow[dr, "\simeq"] & K/\Lambda \arrow[r, "\flat"'] \arrow[d] & L/\Lambda \\ & Y/\Lambda. & \end{tikzcd}$$

By Remark 10.5.9 the two horizontal maps are h-cofibrations, hence flat, as indicated by the musical symbol $\flat$. This means that if the diagonal maps are weak equivalences, then the map of pushouts

$$E_G\Lambda \underset{\Lambda}{\ltimes} Y' \to Y'/\Lambda$$

is also a weak equivalence by Proposition 5.1.24. With this in hand, one now reduces the second claim to the cases

$$X = p_*^\vee S^{-V} \wedge S_+^{k-1} \qquad \text{and} \qquad X = p_*^\vee S^{-V} \wedge D_+^k.$$

Absorbing the factors $(S_+^{k-1})^{\wedge T}$ and $(D_+^k)^{\wedge T}$ into Z completes the reduction to (10.5.22).

With this X, the map on Wth spaces induced by the map of Lemma 10.5.18(ii) is the identity map of the terminal object if $|W| < |V_\gamma|$ and otherwise the map of Λ-orbit spaces induced by

$$E_G\Lambda \ltimes \bigvee_{\gamma\in\Gamma} O(V_\gamma \oplus U_\gamma, W)_+ \underset{O(U_\gamma)}{\wedge} Z_{U_\gamma} \to \bigvee_{\gamma\in\Gamma} O(V_\gamma \oplus U_\gamma, W)_+ \underset{O(U_\gamma)}{\wedge} Z_{U_\gamma},$$

(where $X \ltimes Y$ is as in Definition 2.1.49) in which $U = \{U_\gamma\}$ is any $\widetilde{G}$-equivariant vector bundle over Γ with $|U_\gamma| = |W| - |V_\gamma|$. The proposition then follows from the fact that

$$E_G\Lambda \times \coprod_{\gamma\in\Gamma} O(V_\gamma \oplus U_\gamma, W) \to \coprod_{\gamma\in\Gamma} O(V_\gamma \oplus U_\gamma, W)$$

is an equivariant homotopy equivalence for the compact Lie group

$$\widehat{G} = \Big(\prod_{\gamma\in\Gamma} O(U_\gamma) \rtimes \Lambda\Big) \rtimes G.$$

To see this, note that by [Ill83, Ill78], both sides are $\widehat{G}$-CW complexes so it suffices to check that the map is a weak equivalence of H-fixed point spaces for all $H \subset \widehat{G}$. If the image of H in $\widetilde{G}$ is not a subgroup of Λ then $E_G\Lambda^H$ is contractible and the map of fixed points is a homotopy equivalence. If H is a subgroup of $\prod O(U_\gamma)$ then it acts trivially on $E_G\Lambda$, and once again $E_G\Lambda^H$ is contractible.

Finally, suppose that there is an element $h \in H$ whose image in $\widetilde{G}$ is a nontrivial element of Λ. Since W is pulled back from a G-representation, this element acts trivially on W. If $\gamma \in \Gamma$ is not fixed by h then no point of $O(V_\gamma \oplus U_\gamma, W)$ can be fixed by h. If $\gamma \in \Gamma$ **is** fixed by h, then h acts on V_γ. This action is non-trivial since Λ acts faithfully on T. This means that $O(V_\gamma \oplus U_\gamma, W)$ has no points fixed by h since h acts trivially on W. Both sides therefore have empty H-fixed points in this case. □

Proof of Theorem 10.5.10. The assertion that the upper arrow is a cofibration between cofibrant objects and a weak equivalence if $X \to Y$ is contained in Lemma 10.5.16. Indeed consider the map of arrows

$$(X \to Y) \to (Y \to Y).$$

If $X \to Y$ is a cofibration between cofibrant objects then both the domain and range of the above map of arrows are cofibrant. By Lemma 10.5.16 the map

$$\big(E_G\Lambda \underset{\Lambda}{\ltimes} \partial_X Y^{\wedge T} \to E_G\Lambda \underset{\Lambda}{\ltimes} Y^{\wedge T}\big) \to \big(E_G\Lambda \underset{\Lambda}{\ltimes} Y^{\wedge T} \to E_G\Lambda \underset{\Lambda}{\ltimes} Y^{\wedge T}\big)$$

is a map of cofibrant objects, which is a weak equivalence if $X \to Y$ is. This gives the assertion about the upper row. The fact that the bottom row is an h-cofibration is part of Remark 10.5.9.

For the remaining assertions it will be helpful to reference the expanded diagram

$$\begin{tikzcd}
E_G\Lambda \underset{\Lambda}{\ltimes} \partial_X Y^{\wedge T} \wedge Z \arrow[r] \arrow[d] & E_G\Lambda \underset{\Lambda}{\ltimes} Y^{\wedge T} \wedge Z \arrow[r] \arrow[d] & E_G\Lambda \underset{\Lambda}{\ltimes} (Y/X)^{\wedge T} \wedge Z \arrow[d] \\
\partial_X \mathrm{Sym}^T_\Lambda Y \wedge Z \arrow[r] & \mathrm{Sym}^T_\Lambda Y \wedge Z \arrow[r] & \mathrm{Sym}^T_\Lambda (Y/X) \wedge Z\,,
\end{tikzcd}$$

in which Z is any G-spectrum. By Lemma 10.5.18 the two right vertical maps are weak equivalences. Since the left horizontal maps are h-cofibrations, hence flat, this implies that the left vertical map is a weak equivalence. Taking $Z = S^{-0}$ gives the weak equivalence of the vertical arrows in the statement of Theorem 10.5.10. Letting Z vary through a weak equivalence and using the fact that cofibrant objects are flat gives the flatness assertion. By what we have already proved, when $X \to Y$ is a weak equivalence the vertical and top arrows in the left square are weak equivalences, hence so is the bottom left map. This completes the proof. □

Proof of Proposition 10.5.14. Suppose that $X \to Y$ is a weak equivalence of cofibrant objects, and consider the diagram

$$\begin{tikzcd}
E_G\Lambda \underset{\Lambda}{\ltimes} X^{\wedge T} \arrow[r] \arrow[d] & E_G\Lambda \underset{\Lambda}{\ltimes} Y^{\wedge T} \arrow[d] \\
\mathrm{Sym}^T_\Lambda X \arrow[r] & \mathrm{Sym}^T_\Lambda Y\,.
\end{tikzcd}$$

The vertical maps are weak equivalences by Lemma 10.5.18. The top horizontal map is a weak equivalence by Lemma 10.5.16 (applied to, say, the map $(* \to X) \to (* \to Y)$). The bottom map is therefore a weak equivalence. □

10.6 Iterated indexed symmetric powers

In our analysis of the norms of commutative rings in §10.9 we will encounter iterated indexed smash products and symmetric powers. These work out just to be other indexed smash or symmetric powers. The point of this section is to spell this out.

Suppose that S and T are G-sets and that X is an equivariant $(T \times S)$-diagram of orthogonal spectra. The factorization

$$T \times S \to S \to *$$

gives an isomorphism

$$(X^{\wedge T})^{\wedge S} \cong X^{\wedge (T \times S)}, \tag{10.6.1}$$

in which $X^{\wedge T}$ is shorthand for $p_*^{\wedge} X$ with $p : T \times S \to S$ the projection mapping. The spectrum on the left is an iterated indexed smash product, meaning an indexed smash product of indexed smash products, while the one on the right is simply an indexed smash product with a bigger indexing set.

Applying this to the arrow category, given a map $X \to Y$ of $(T \times S)$-diagrams, we get an isomorphism of the corner map

$$\partial_X Y^{\wedge (T \times S)} \to Y^{\wedge (T \times S)},$$

an indexed smash product of maps indexed by a bigger set, with the iterated corner map

$$\partial_W Z^{\wedge T} \to Z^{\wedge T}$$

in which $W \to Z$ is the map

$$\partial_X Y^{\wedge S} \to Y^{\wedge S}.$$

Hence in both $\mathcal{S}p$ and its arrow category $\mathcal{S}p_1$ iterated indexed smash products can be identified simply as indexed smash products.

There is also a version with symmetric powers. An indexed symmetric power of an indexed symmetric power of spectra or maps is itself an appropriately defined indexed symmetric power.

Suppose as before that $\Lambda \subset \Lambda_T$ is a G-stable subgroup. Then the action of Λ^S (the set of Λ-valued functions on S) on $T \times S$ defined by

$$\phi \cdot (t, s) = (\phi(s) \cdot t, s)$$

is G-stable, making $T \times S$ into a $\widetilde{G}^{(S)}$-set, where

$$\widetilde{G}^{(S)} = \Lambda^S \rtimes G.$$

The projection map $T \times S \to S$ is $\widetilde{G}^{(S)}$-equivariant, with $\widetilde{G}^{(S)}$ acting on S through G. When X is a $\widetilde{G}^{(S)}$-equivariant $(T \times S)$-diagram, the isomorphism (10.6.1) is $\widetilde{G}^{(S)}$-equivariant.

Passing to Λ^S-orbits from (10.6.1) gives an isomorphism of G-spectra

$$(\mathrm{Sym}^T_\Lambda X)^{\wedge S} \cong \mathrm{Sym}^{T \times S}_{\Lambda^S} X. \tag{10.6.2}$$

By working in the arrow category we get an isomorphism of the corner map

$$\partial_X \mathrm{Sym}^{T \times S}_{\Lambda^S} Y \to \mathrm{Sym}^{T \times S}_{\Lambda^S} Y$$

with the iterated indexed corner map

$$\partial_W Z^{\wedge T} \to Z^{\wedge T} \tag{10.6.3}$$

in which $W \to Z$ is the map

$$\partial_X \mathrm{Sym}^S_\Lambda Y \to \mathrm{Sym}^S_\Lambda Y.$$

Our analysis of the homotopy properties of symmetric powers depended on convenient cofibrant approximations, namely the vertical maps in (10.5.11). Their sources are defined in terms of the universal G-equivariant Λ-space $E_G\Lambda$ of Definition 8.7.1. For the iterated case, let $E_G\Lambda^S$ be $(E_G\Lambda)^{\times S}$ or equivalently the space of maps $S \to E_G\Lambda$. The above discussion leads to an isomorphism

$$\left(E_G\Lambda \underset{\Lambda}{\ltimes} X^{\wedge T}\right)^{\wedge S} \cong E_G\Lambda^S \underset{\Lambda^S}{\ltimes} X^{\wedge (T\times S)}, \tag{10.6.4}$$

and an identification of the corner map

$$\partial_{\tilde W} \widetilde{Z}^{\wedge T} \to \widetilde{Z}^{\wedge T},$$

in which $\tilde W \to \widetilde{Z}$ is the map

$$E_G\Lambda \underset{\Lambda}{\ltimes} \left(\partial_X Y^{\wedge S} \to Y^{\wedge S}\right),$$

with

$$\left(E_G\Lambda^S\right) \underset{\Lambda^S}{\ltimes} \left(\partial_X Y^{\wedge (T\times S)} \to Y^{\wedge (T\times S)}\right). \tag{10.6.5}$$

We know by Lemma 8.7.3 that $E_G\Lambda^S$ is a universal equivariant Λ^S-space $E_G(\Lambda^S)$. It follows that the isomorphisms above identify an iterated indexed symmetric power of a diagram or map of diagrams with an indexed symmetric power of same involving a larger group.

Lemma 8.7.3 and Theorem 10.5.10 imply the following.

Proposition 10.6.6. Nice properties of indexed smash products of indexed symmetric powers. *Suppose that $X \to Y$ is a cofibration of cofibrant $\widetilde{G}^{(S)}$-equivariant $(T \times S)$-diagrams. Then in the diagram*

$$\begin{tikzcd}
(E_G\Lambda)^S \underset{\Lambda^S}{\ltimes} \partial_X Y^{\wedge (S\times T)} \arrow[r] \arrow[d, "\sim"'] & (E_G\Lambda)^S \underset{\Lambda^S}{\ltimes} Y^{\wedge (T\times S)} \arrow[d, "\sim"] \\
\partial_X \mathrm{Sym}^{T\times S}_{\Lambda^S} Y \arrow[r] & \mathrm{Sym}^{T\times S}_{\Lambda^S} Y
\end{tikzcd}$$

every object is flat, the top row is a cofibration of cofibrant objects, the bottom row is an h-cofibration, and the vertical maps are weak equivalences and remain so after smashing with any spectrum Z.

Remark 10.6.7. Iterated symmetric powers. *In Proposition 10.6.6 we are looking at a smash power indexed by S of a symmetric power indexed by T, but we are* **not** *symmetrizing the former by passing to the orbit spectrum of the action by a nontrivial G-stable subgroup $\Lambda' \subseteq \Lambda_S$. We* **could** *do so by applying Theorem 10.5.10 a second time, with T and Λ replaced by S and Λ'.*

The same conclusion therefore holds for the corresponding diagram of iterated indexed symmetric powers

$$\begin{tikzcd}
\partial_{\tilde{W}}(\widetilde{Z}^{\wedge(T)}) \arrow[r] \arrow[d] & \widetilde{Z}^{\wedge(T)} \arrow[d] \\
\partial_W(Z^{\wedge(T)}) \arrow[r] & Z^{\wedge(T)}
\end{tikzcd}$$

in which

$$\begin{tikzcd}
\tilde{W} \arrow[r] \arrow[d] & \widetilde{Z} \arrow[d] \\
W \arrow[r] & Z
\end{tikzcd}$$

is the diagram

$$\begin{tikzcd}
E_G\Lambda \underset{\Lambda}{\ltimes} \partial_X Y^{\wedge S} \arrow[r] \arrow[d] & E_G\Lambda \underset{\Lambda}{\ltimes} Y^{\wedge S} \arrow[d] \\
\partial_X \mathrm{Sym}^S_\Lambda Y \arrow[r] & \mathrm{Sym}^S_\Lambda Y\,.
\end{tikzcd}$$

Working fiberwise leads to an analogous result about the indexed smash product along a map $q : S' \to S$ of finite G-sets. It plays an important role in our analysis of the homotopy properties of the norms of commutative rings. Aside from the map $S' \to S$ of finite G-sets, the situation is the same as what we have been discussing in this section. We have fixed a finite G-set T, a G-stable subgroup $\Lambda \subset \Sigma_T$, and a universal G-equivariant Λ-space $E_G\Lambda$ as in Definition 8.7.1.

Proposition 10.6.8. Nice properties of iterated indexed symmetric powers. *Let $X \to Y$ be a cofibration of cofibrant $\widetilde{G}^{(S')}$-equivariant $T \times S'$-diagrams and write*

$$\begin{tikzcd}
\tilde{W} \arrow[r] \arrow[d] & \widetilde{Z} \arrow[d] \\
W \arrow[r] & Z
\end{tikzcd}$$

for the diagram

$$\begin{tikzcd} E_G\Lambda \underset{\Lambda}{\ltimes} \partial_X Y^{\wedge S'} \arrow[r] \arrow[d] & E_G\Lambda \underset{\Lambda}{\ltimes} Y^{\wedge S'} \arrow[d] \\ \partial_X \mathrm{Sym}^{S'}_{\Lambda} Y \arrow[r] & \mathrm{Sym}^{S'}_{\Lambda} Y. \end{tikzcd}$$

In the G-equivariant S-diagram of corner maps

$$\begin{tikzcd} \partial_{\widetilde{W}}(\widetilde{Z}^{\wedge S'/S}) \arrow[r] \arrow[d] & \widetilde{Z}^{\wedge S'/S} \arrow[d] \\ \partial_W(Z^{\wedge S'/S}) \arrow[r] & Z^{\wedge S'/S} \end{tikzcd}$$

(see (10.2.1) for the meaning of $(-)^{\wedge S'/S}$) every object is flat, the vertical maps are weak equivalences after smashing with any object, the upper map is a cofibration of cofibrant objects and the lower map is an h-cofibration. The horizontal maps are weak equivalences if $X \to Y$ is.

Remark 10.6.9. *The actual hypothesis on $X \to Y$ required for the fiberwise argument is that for each $s \in S$, the map $X \to Y$ is a cofibration of $\Lambda^{S'_s} \rtimes G_s$-equivariant $T \times S'_s$-diagram, where $S'_s \subset S'$ is the inverse image of s, and G_s is the stabilizer of s. For the sake of a cleaner statement we have made the slightly stronger assumption that it is a cofibration of cofibrant $\widetilde{G}^{(S')}$-equivariant $T \times S'$-diagrams. That this implies the "fiberwise" hypothesis is a consequence of Proposition 9.8.3.*

Remark 10.6.10. *As in Remark 10.5.13, Proposition 10.6.8 applies to the situation in which $X \to Y$ is a cofibration of cofibrant G-equivariant T-diagrams, regarded as a $\widetilde{G}$-equivariant $T \times S$ diagram by pulling back along the projection mappings $\widetilde{G} \to G$ and $T \times S \to T$.*

10.7 Commutative algebras in the category of G-spectra

The purpose of this section is to define a cofibrantly generated model structure on $\mathbf{Comm}^G$, the category of commutative algebras in $\mathcal{S}p^G$. Recall that Theorem 5.5.25, proved by Schwede and Shipley as [SS00, Theorem 4.1], concerns the category of (not necessarily commutative) algebras in a Quillen ring.

The main tool is the Crans-Kan Transfer Theorem 5.2.27, which we apply to pair of adjoint functors

$$\mathrm{Sym} : \mathcal{S}p^G \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Comm}^G : U,$$

where U is the forgetful functor and its left adjoint Sym is the free commutative algebra functor

$$X \mapsto \operatorname{Sym} X := \bigvee_{i\geqslant 0} X^{\wedge i}/\Sigma_i = S^{-0} \vee X \vee \operatorname{Sym}^2 X \vee \cdots$$

of Lemma 2.6.66. The category of G-spectra $\mathcal{S}p^G$ is endowed with the positive stable equifibrant model category structure of Theorem 9.2.13 with generating sets $\widetilde{\mathcal{I}}^{G,+}$ and $\widetilde{\mathcal{K}}^{G,+}$ as in (9.2.12).

As we saw in Remark 4.5.2, the functor U need not preserve cofibrancy, so we make the following to avoid confusion.

Definition 10.7.1. Algebraic cofibrancy. *A commutative algebra in $\mathcal{S}p^G$, that is an object* $\mathbf{Comm}^G$, *is* **algebraically cofibrant** *if it is cofibrant in the model structure on that category to be described in Theorem 10.7.2, but not necessarily cofibrant as a G-spectrum, that is an object in $\mathcal{S}p^G$.*

Theorem 10.7.2. The model structure on $\mathbf{Comm}^G$. *The forgetful functor*

$$U : \mathbf{Comm}^G \to \mathcal{S}p^G$$

creates a topological model structure on $\mathbf{Comm}^G$ *in which the fibrations and weak equivalences in* $\mathbf{Comm}^G$ *are the maps that are fibrations and weak equivalences in $\mathcal{S}p^G$.*

Proof It is easy to check that both $\operatorname{Sym}\mathcal{I}$ and $\operatorname{Sym}\mathcal{J}$ permit the small object argument, which is the first of the two requirements of the Crans-Kan Transfer Theorem 5.2.27. The second one is that U takes relative $\operatorname{Sym}\mathcal{J}$-cell complexes (Definition 4.8.18) to weak equivalences. This means that if

$$\begin{tikzcd} \operatorname{Sym} A \arrow[r, "\operatorname{Sym} j"] \arrow[d] & \operatorname{Sym} B \arrow[d] \\ R \arrow[r] & R' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

(compare with the diagram of Definition 2.9.47) is a pushout diagram in $\mathbf{Comm}^G$ in which $j : A \to B$ is a generating trivial cofibration in $\mathcal{S}p^G$, then $R \to R'$ is a weak equivalence; see Remark 5.2.30. This is a special case Lemma 10.7.3 below. □

Lemma 10.7.3. A pushout diagram of equivariant commutative rings. *Suppose that $A \to B$ is a map of G-spectra, and*

$$\begin{tikzcd} \operatorname{Sym} A \arrow[r] \arrow[d] & \operatorname{Sym} B \arrow[d] \\ R \arrow[r] & R' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

is a pushout diagram of equivariant commutative rings. If $A \to B$ is a trivial cofibration of cofibrant objects, then $R \to R'$ is a weak equivalence.

Proof We use the pushout ring filtration of Definition 2.9.47. It suffices to show that the map $\mathrm{fil}^R_{k-1} R' \to \mathrm{fil}^R_k R'$ is a weak equivalence for each k. In the diagram (2.9.48), if $A \to B$ is a trivial cofibration between cofibrant objects, then

$$\partial_A \mathrm{Sym}^k B \to \mathrm{Sym}^k B$$

is a weak equivalence and an h-cofibration of flat spectra by Theorem 10.5.10. It follows that the bottom map is a weak equivalence. □

Corollary 10.7.4. The norm functor on commutative algebras

$$N_H^G : \mathbf{Comm}^H \to \mathbf{Comm}^G$$

is a left Quillen functor. *It preserves the classes of cofibrations and trivial cofibrations, hence weak equivalences between cofibrant objects.*

Proof This is immediate from Corollary 9.7.6. The assertion about weak equivalences is Ken Brown's Lemma 5.1.7. □

Corollary 10.7.5. For $H \subset G$, the adjoint functors

$$N_H^G : \mathbf{Comm}^H \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Comm}^G : i_H^G$$

form a Quillen pair.

Proof The restriction functor preserves the classes of fibrations and weak equivalences. □

Corollary 10.7.6. The norm of the restriction of a commutative algebra. *There is a natural isomorphism*

$$N_H^G(i_H^G R) \to R^{\wedge(G/H)}$$

under which the counit of the adjunction is identified with the map

$$R^{\wedge(G/H)} \to R$$

given by the unique G-map $G/H \to pt$.

Proof Since both $R^{\wedge(G/H)}$ and the left adjoint to restriction corepresent the same functor, this follows from Corollary 10.7.5. □

A useful consequence Corollary 10.7.6 is that the group of G-automorphisms of G/H, the Weyl group $N(H)/H$, acts naturally on $N_H^G(i_H^G R)$. The result below is used in the main computational assertion of Proposition 12.3.10.

Corollary 10.7.7. *For $\gamma \in N(H)/H$ the following diagram commutes:*

$$\begin{tikzcd} N_H^G(i_H^G R) \arrow[rr, "\gamma"] \arrow[dr] & & N_H^G(i_H^G R) \arrow[dl] \\ & R & \end{tikzcd}$$

Proof Immediate from Corollary 10.7.6. □

The following is a consequence of Theorem 10.9.9 below.

Proposition 10.7.8. *Suppose that R is an algebraically cofibrant commutative H-algebra, and that $\tilde{R} \to R$ is a cofibrant approximation of the underlying H-spectrum. If $\widetilde{Z} \to Z$ is a weak equivalence of G-spectra then*

$$N_H^G(\tilde{R}) \wedge \widetilde{Z} \to N_H^G(R) \wedge Z$$

is a weak equivalence.

We refer to the property exhibited in Proposition 10.7.8 by saying that cofibrant commutative rings are **equifibrantly flat**. See Definition 10.9.6 below.

10.8 R-modules in the category of spectra

The category $\mathcal{M}_R$ of left modules over an equivariant associative algebra R is defined in §9.7A. As pointed out there, when R is commutative, a left R-module can be regarded as a right R-module, and $\mathcal{M}_R$ becomes a symmetric monoidal category under the operation $M \underset{R}{\wedge} N$ of (9.7.2).

10.8A A model structure for R-modules

The following result is a consequence of Theorem 9.8.4 and [SS00, Theorem 4.1]. Except for the slight change of model structure, it is [MM02, Theorem III.7.6].

Proposition 10.8.1. A model structure for R-modules. *The forgetful functor*

$$\mathcal{M}_R \to \mathcal{S}p^G$$

creates a model structure on $\mathcal{M}_R$ in which the fibrations and weak equivalences are the maps which become fibrations and weak equivalences in $\mathcal{S}p^G$. When R is commutative, the operation (9.7.2) satisfies the pushout product and monoid axioms making $\mathcal{M}_R$ into a symmetric monoidal model category.

Though not explicitly stated, the following formal result was surely known to the authors of [SS00]; see the proof of [SS00, Theorem 4.3].

Corollary 10.8.2. A change of rings Quillen pair. *Let $f : R \to R'$ be a map of equivariant associative algebras. The functors*

$$R' \underset{R}{\wedge} (-) : \mathcal{M}_R \underset{U}{\overset{\perp}{\rightleftarrows}} \mathcal{M}_{R'} : U$$

given by restriction and extension of scalars form a Quillen pair. If R' is cofibrant as a left R-module, then the restriction functor is also a left Quillen functor.

Proof Proposition 10.8.1 implies that the restriction functor preserves fibrations and trivial fibrations. This gives the first assertion. The second follows from the fact that the restriction functor preserves colimits, and the consequence of Proposition 10.8.1 that the generating (trivial) cofibrations for $\mathcal{M}_{R'}$ are formed as the smash product of R' with the generating (trivial) cofibrations for $\mathcal{S}p^G$. □

The following result is [MM02, Proposition III.7.7]. Using the fact that h-cofibrations are flat, the proof reduces to checking the case

$$M = G \underset{H}{\ltimes} S^{-V} \wedge R,$$

which is Proposition 9.6.5.

Proposition 10.8.3. Tensoring over an associative ring R with a cofibrant module. *Suppose that R is an associative algebra, and M is a cofibrant right R-module. The functor $M \underset{R}{\wedge} (-)$ preserves weak equivalences.*

In other words, the functor $M \underset{R}{\wedge} (-)$ is flat if M is cofibrant, and so it need not be derived.

Corollary 10.8.4. Tensoring a cofibrant R-module with a short exact sequence of R-modules. *Suppose that R is an associative algebra, M a cofibrant right R-module. If $N \to N'$ a map of left R-modules whose underlying map of spectra is an h-cofibration, then the sequence*

$$M \underset{R}{\wedge} N \to M \underset{R}{\wedge} N' \to M \underset{R}{\wedge} (N'/N)$$

is weakly equivalent to a cofiber sequence.

Note that the assumption is **not** that $N \to N'$ is an h-cofibration in the category of left R-modules. In that case the result would not require any hypothesis on M.

Proof We must show that the map from the mapping cone of

$$M \underset{R}{\wedge} N \to M \underset{R}{\wedge} N' \tag{10.8.5}$$

to $M \underset{R}{\wedge} (N'/N)$ is a weak equivalence. The mapping cone of (10.8.5) is isomorphic to

$$M \underset{R}{\wedge} (N' \cup CN),$$

and the spectrum underlying the R-module mapping cone $N' \cup N$ is the mapping cone formed in spectra. Since $N \to N'$ is an h-cofibration, Corollary 5.6.9 says the map $N' \cup CN \to N'/N$ is a weak equivalence. The result now follows from Proposition 10.8.3. □

Corollary 10.8.4 can be used to show that many constructions derived from the formation of monomial ideals (as in Definition 2.9.57) have good homotopy theoretic properties. It is used in §10.10D and in §12.4. In those cases the map of spectra underlying $N \to N'$ is the inclusion of a wedge summand, and so obviously an h-cofibration.

10.8B The relative monoidal geometric fixed point functor

The functor Φ^G_M can be formulated relative to an equivariant commutative or associative algebra R. As described below, care must be taken in using the theory in this way.

Because it is lax monoidal, the functor Φ^G_M gives a functor

$$\Phi^G_M : \mathcal{M}_R \to \mathcal{M}_{\Phi^G_M R}$$

which is lax monoidal when R is commutative.

Proposition 10.8.6. Monoidal geometric fixed points of R-modules. *The functor*

$$\Phi^G_M : \mathcal{M}_R \to \mathcal{M}_{\Phi^G_M R}$$

commutes with cobase change along a cofibration and preserves the classes of cofibrations and trivial cofibrations.

Proof This follows easily from the fact that the maps of equivariant orthogonal spectra underlying the generating cofibrations for $\mathcal{M}_R$ are h-cofibrations. □

Proposition 10.8.7. Monoidal geometric fixed points of modules over a commutative ring. *When R is commutative, the functor*

$$\Phi^G_M : \mathcal{M}_R \to \mathcal{M}_{\Phi^G_M R}$$

is weakly monoidal, and the map

$$\Phi^G_M(M) \underset{\Phi^G_M(R)}{\wedge} \Phi^G_M(N) \to \Phi^G_M(M \underset{R}{\wedge} N) \tag{10.8.8}$$

is an isomorphism if M and N are cofibrant.

As noted in Remark 9.11.48, this holds if either M or N is cofibrant.

Proof The proof is the same as that of Proposition 9.11.47 once one knows that the class of modules M and N for which (10.8.8) is an isomorphism is stable under cobase change along a generating cofibration. This, in turn, is a consequence of the fact that both sides of (10.8.8) preserve h-cofibrations in each variable, since h-cofibrations are closed inclusions. The functor Φ^G_M does so since it commutes with the formation of mapping cylinders, and $M \underset{R}{\wedge} (-)$ does since $\mathcal{M}_R$ is a closed symmetric monoidal category. □

As promising as it looks, it is not so easy to make use of Proposition 10.8.7. The trouble is that unless X is cofibrant, $\Phi^G_M(X)$ may not have the weak homotopy type of $\Phi^G(X)$. So in order to use Proposition 10.8.7 one needs a condition guaranteeing that $M \underset{R}{\wedge} N$ is a cofibrant spectrum. The criterion of Proposition 10.8.9 below was suggested to us by Mike Mandell.

Proposition 10.8.9. Cofibrancy of smash products over R. *Suppose R is an associative algebra with the property that $S^{-1} \wedge R$ is cofibrant. If M is a cofibrant right R-module, and $S^{-1} \wedge N$ is a cofibrant left R-module, then*

$$M \underset{R}{\wedge} N$$

is cofibrant.

Proof First note that the condition on R guarantees that for every representation U with $|U^G| > 0$ and every cofibrant G-space T, the spectrum

$$S^{-U} \wedge R \wedge T \tag{10.8.10}$$

is cofibrant. Since the formation of $M \underset{R}{\wedge} N$ commutes with cobase change in both variables, the result reduces to the case $M = S^{-V} \wedge R \wedge X$ and $N = S^{-W} \wedge R \wedge Y$ with V having a non-zero fixed point space, and X and Y cofibrant G-spaces. But in that case

$$M \underset{R}{\wedge} N \cong S^{-V \oplus W} \wedge R \wedge X \wedge Y$$

which is of the form (10.8.10), and hence cofibrant. □

Corollary 10.8.11. Cofibrancy of R-modules. *Suppose R is an associative algebra with the property that $S^{-1} \wedge R$ is cofibrant. If M is a cofibrant right R-module, then the equivariant orthogonal spectrum underlying M is cofibrant.*

Proof Just take $N = R$ in Proposition 10.8.9. □

The following result plays an important role in determining $\Phi^G R(\infty)$ (§12.4E).

Proposition 10.8.12. Smashing over R with the sphere spectrum. *Suppose that R is an equivariant associative algebra whose underlying G-spectrum is Bredon cofibrant, and that $R \to S^{-0}$ is an equivariant associative* *algebra map. If M is a cofibrant right R-module, then $M \underset{R}{\wedge} S^{-0}$ is a cofibrant spectrum, and the map*

$$\Phi^G_M(M) \underset{\Phi^G_M R}{\wedge} S^{-0} \to \Phi^G_M(M \underset{R}{\wedge} S^{-0})$$

is an isomorphism.

Proof One easily reduces to the case $M = S^{-V} \wedge X \wedge R$, in which V is a representation with $V^G \neq 0$, and X is a cofibrant G-space. In this case $M \underset{R}{\wedge} S^{-0}$ is isomorphic to $S^{-V} \wedge X$ which is cofibrant. The assertion about monoidal geometric fixed points follows easily from Proposition 9.11.47. □

10.9 Indexed smash products of commutative rings

10.9A Description of the problem

Theorem 10.4.7 asserts that the indexed smash product functor

$$(\,-\,)^{\wedge T} : \mathcal{S}p^{\mathcal{B}_T G} \to \mathcal{S}p^G$$

for a finite G-set T has a left derived functor

$$(\,-\,)^{\overset{\mathbf{L}}{\wedge} T} : \operatorname{Ho}\mathcal{S}p^{\mathcal{B}_T G} \to \operatorname{Ho}\mathcal{S}p^G$$

which can be computed by applying the indexed smash product to a cofibrant approximation. We also know from Corollary 10.7.5 (and the fact that coproducts of weak equivalences are weak equivalences) that the restriction functor and its left adjoint form a Quillen pair

$$p_! : \mathbf{Comm}\,\mathcal{S}p^{\mathcal{B}_T G} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Comm}\,\mathcal{S}p^G : p^*,$$

where both maps are induced by $p : T \to *$. Furthermore, the following diagram, in which the vertical arrows are the forgetful functors of Corollary 9.7.6, commutes.

$$\begin{tikzcd} \mathbf{Comm}\,\mathcal{S}p^{\mathcal{B}_T G} \arrow[r, "p_!"] \arrow[d, "U"'] & \mathbf{Comm}\,\mathcal{S}p^G \arrow[d, "U"] \\ \mathcal{S}p^{\mathcal{B}_T G} \arrow[r, "(\,-\,)^{\wedge T}"'] & \mathcal{S}p^G \end{tikzcd} \tag{10.9.1}$$

However, what we really want is the commutativity of the diagram

$$\begin{tikzcd} \operatorname{Ho}\mathbf{Comm}\,\mathcal{S}p^{\mathcal{B}_T G} \arrow[r, "\mathbf{L}p_!"] \arrow[d, "\operatorname{Ho}U"'] & \operatorname{Ho}\mathbf{Comm}\,\mathcal{S}p^G \arrow[d, "\operatorname{Ho}U"] \\ \operatorname{Ho}\mathcal{S}p^{\mathcal{B}_T G} \arrow[r, "(\,-\,)^{\overset{\mathbf{L}}{\wedge} T}"'] & \operatorname{Ho}\mathcal{S}p^G \end{tikzcd} \tag{10.9.2}$$

in which the vertical maps are the forgetful functors (which are homotopical, so don't need to be derived), and the horizontal arrows are the left derived functors indicated. **The point of this section is to establish this as Corollary 10.9.10 below.**

In particular for $T = G/H$, using the equivalence of $\mathcal{B}_{G/H}G$ with $\mathcal{B}H$ of Proposition 2.1.38, (10.9.2) becomes

$$\begin{tikzcd} \mathrm{Ho}\,\mathbf{Comm}\,\mathcal{S}p^H \arrow[r, "\mathbf{L}N_H^G"] \arrow[d, "\mathrm{Ho}\,U"'] & \mathrm{Ho}\,\mathbf{Comm}\,\mathcal{S}p^G \arrow[d, "\mathrm{Ho}\,U"] \\ \mathrm{Ho}\,\mathcal{S}p^H \arrow[r, "\mathbf{L}N_H^G"] & \mathrm{Ho}\,\mathcal{S}p^G, \end{tikzcd} \tag{10.9.3}$$

the homotopy analog of the diagram of Corollary 9.7.6. **This means that the norm of a commutative ring spectrum has the expected homotopy type.**

To clarify the issue, suppose that $R \in \mathbf{Comm}\,\mathcal{S}p^{\mathcal{B}_T G}$ is a cofibrant T-diagram of commutative rings. Let $\tilde{R} \to R$ be a cofibrant approximation of the underlying T-diagram of spectra. What needs to be checked is that the map

$$(\tilde{R})^{\wedge T} \to (R)^{\wedge T} \tag{10.9.4}$$

is a weak equivalence. The proof involves an elaboration of the notion of flatness. To motivate it, we describe a bit of the argument.

The main point in the proof is to investigate the situation of a pushout diagram of equivariant T-diagrams of commutative rings

$$\begin{tikzcd} \mathrm{Sym}\,X \arrow[r] \arrow[d] & \mathrm{Sym}\,Z \arrow[d] \\ R_1 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & R_2 \end{tikzcd}$$

(as in Definition 2.9.47) in which the top row is constructed by applying the symmetric algebra functor Sym to a generating cofibration $X \to Z$, and in which one knows that the map (10.9.4) is a weak equivalence for $R = R_1$. We would like to conclude that (10.9.4) is also a weak equivalence for $R = R_2$.

To pass from R_1 to R_2 we use the pushout ring filtration of Definition 2.9.47, which is derived from the target exponent filtration of Definition 2.9.34 via the symmetric product filtration of Definition 2.9.45. Its stages are related by pushout squares

$$\begin{tikzcd} R_1 \wedge \partial_X \mathrm{Sym}^k Z \arrow[r] \arrow[d] & R_1 \wedge \mathrm{Sym}^k Z \arrow[d] \\ \mathrm{fil}_{k-1}^{R_1} R_2 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \mathrm{fil}_k^{R_1} R_2\,, \end{tikzcd} \tag{10.9.5}$$

as in (2.9.48), where

$$\partial_X \mathrm{Sym}^k Z = \left(\partial_X Z^{\wedge k}\right)/\Sigma_k.$$

To interpolate between

$$(\mathrm{fil}_{k-1}^{R_1} R_2)^{\wedge T} \qquad \text{and} \qquad (\mathrm{fil}_k^{R_1} R_2)^{\wedge T},$$

we will use the target exponent filtration of Definition 2.9.34 in which the diagram of (2.9.31) is the T-diagram of spectra in which each component is (10.9.5). The upper right hand corner of the diagram of Lemma 2.9.39 is a wedge of terms of the form

$$(\mathrm{fil}_{k-1} R_2)^{\wedge T_0} \wedge (R_1 \wedge \mathrm{Sym}^k Z)^{\wedge T_1},$$

indexed by the set-theoretic decompositions $T = T_0 \amalg T_1$.

We need to know two things about this expression. One is that the left derived functor of its formation (in all variables) is computed in terms of the expression itself, and the other is that formation of each of the pushout squares we encounter is homotopical. Motivated by this we are led to consider a technical condition slightly stronger than the requirement that (10.9.4) be a weak equivalence. That is the subject of the next subsection.

10.9B Equifibrantly flat diagrams

For a map of finite G-sets $q : K \to L$ and an equivariant K-diagram X, we will write

$$X^{\wedge(K/L)} := q_*^{\wedge} X$$

for the indexed smash product, as in § 10.6. This object is an equivariant L-diagram.

Definition 10.9.6. *An equivariant T-diagram X is* **equifibrantly flat** *if it has the following property: for every cofibrant approximation $\tilde{X} \to X$ of equivariant T-diagrams, every diagram of finite G-sets*

$$T \xleftarrow{p} K \xrightarrow{q} L \tag{10.9.7}$$

and every weak equivalence of equivariant L-diagrams $\widetilde{Z} \to Z$, the map

$$(p^* \tilde{X})^{\wedge(K/L)} \wedge \widetilde{Z} \to (p^* X)^{\wedge(K/L)} \wedge Z \tag{10.9.8}$$

is a weak equivalence.

Our goal is to establish the following result.

Theorem 10.9.9. Cofibrant commutative ring diagrams are equifibrantly flat. *If $R \in \mathcal{S}p^{\mathcal{B}_T G}$ is a cofibrant commutative ring, then the equivariant T-diagram of spectra underlying R is equifibrantly flat.*

The condition that R be equifibrantly flat certainly implies that (10.9.4) is a weak equivalence. Theorem 10.9.9 therefore implies

Corollary 10.9.10. Derived indexed smash products of commutative rings. *The diagram of (10.9.2),*

$$\begin{tikzcd}
\mathrm{Ho}\,\mathbf{Comm}\,\mathcal{S}p^{\mathcal{B}_T G} \arrow[r, "\mathbf{L}p_!"] \arrow[d, "\mathrm{Ho}\,U"'] & \mathrm{Ho}\,\mathbf{Comm}\,\mathcal{S}p^{G} \arrow[d, "\mathrm{Ho}\,U"] \\
\mathrm{Ho}\,\mathcal{S}p^{\mathcal{B}_T G} \arrow[r, "(\,-\,)^{\overset{\mathbf{L}}{\wedge} T}"'] & \mathrm{Ho}\,\mathcal{S}p^{G}
\end{tikzcd}$$

commutes up to natural isomorphism. Hence the diagram of (10.9.3) also commutes, and the norm of a commutative ring spectrum has the expected homotopy type.

Remark 10.9.11. *Since identity maps are weak equivalences, the condition of being equifibrantly flat implies that every arrow in the diagram*

$$\begin{tikzcd}
(p^*\tilde{X})^{\wedge(K/L)} \wedge \tilde{Z} \arrow[r] \arrow[d] & (p^*\tilde{X})^{\wedge(K/L)} \wedge Z \arrow[d] \\
(p^*X)^{\wedge(K/L)} \wedge \tilde{Z} \arrow[r] & (p^*X)^{\wedge(K/L)} \wedge Z
\end{tikzcd} \tag{10.9.12}$$

is a weak equivalence. In particular it implies that $p^*\tilde{X})^{\wedge(K/L)}$ are $(p^*X)^{\wedge(K/L)}$ is flat as in Definition 5.1.20.

Since $\tilde{X}^{\wedge(K/L)}$ is cofibrant (Theorem 10.2.4), and cofibrant objects are flat (Proposition 9.6.5), the top arrow in (10.9.12) is always a weak equivalence. It therefore suffices to check the equifibrantly flat condition when $\tilde{Z} \to Z$ is the identity map. This corresponds to the two vertical arrows in (10.9.12).

If (10.9.8) is a weak equivalence for one cofibrant approximation it is a weak equivalence for any cofibrant approximation. It therefore suffices to check the "equifibrantly flat" condition for a single cofibrant approximation $\tilde{X} \to X$.

Lemma 10.9.13. Wedges, smash products and filtered colimits preserve equifibrant flatness. *Arbitrary wedges of equifibrantly flat spectra are equifibrantly flat. Smash products of equifibrantly flat spectra are equifibrantly flat. Filtered colimits of equifibrantly flat equivariant T-diagrams along h-cofibrations are equifibrantly flat.*

Proof The first assertion follows easily from the distributive law, and the fact that the formation of indexed wedges is homotopical. The second follows from the fact that the formation of indexed smash products is symmetric monoidal. The third makes use of Proposition 3.5.26. The details are left to the reader. □

Example 10.9.14. *Here is one motivation for the definition of equifibrant flatness. Suppose we are given a pushout square of equivariant J-diagrams*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

and we are interested in the filtration of $Y^{\wedge(K/L)}$ described in §2.9C, whose stages are related by pushout squares

$$\begin{tikzcd} \bigvee_{(\ell,K_1)\in G_n} X^{\wedge K_0} \wedge \partial_A B^{\wedge K_1} \arrow[r] \arrow[d] & \bigvee_{(\ell,K_1)\in G_n} X^{\wedge K_0} \wedge B^{\wedge K_1} \arrow[d] \\ \mathrm{fil}_{n-1} Y^{\wedge K/L} \arrow[r] & \mathrm{fil}_n Y^{\wedge K/L} \end{tikzcd}, \tag{10.9.15}$$

where $G_n = G_n(K/L)$ is the G-set of pairs (ℓ, K_1) with $\ell \in L$ and $K_1 \subset q^{-1}(\ell)$ a subset of cardinality n, and the map $G_n \to L$ sends (ℓ, K_1) to ℓ. For $(\ell, K_1) \in G_n$ we have written K_0 to denote the complement of K_1 in $q^{-1}(\ell)$.

The condition that B be equifibrantly flat gives some control over the upper right term. To see this let $V_n = V_n(K/L)$ be the set of triples (ℓ, K_1, k) with $(\ell, K_1) \in G_n$ and $k \in K_1$. We define maps

$$\begin{tikzcd} J & V_n \arrow[l, "f"'] \arrow[r, "g"] & G_n \\ q(k) & (\ell, K_1, k) \arrow[l, mapsto] \arrow[r, mapsto] & (\ell, K_1) \end{tikzcd}$$

The spectra $X^{\wedge K_0}$ form an equivariant G_n-diagram, which we denote Z. The $B^{\wedge K_1}$ are the constituents of $(f^*B)^{\wedge(V_n/G_n)}$, and so the indexed wedge occurring in the pushout square is

$$\bigvee_{G_n} Z \wedge (f^*B)^{\wedge(V_n/G_n)}.$$

Since the formation of indexed wedges is homotopical, its homotopy properties come down to understanding the homotopy properties of the equivariant G_n-diagram $Z \wedge f^*B^{\wedge(V_n/G_n)}$, some of which are specified by the condition that B be equifibrantly flat.

By replacing the category of equivariant T-diagrams with its arrow category, we arrive at the notion of a **equifibrantly flat object** of $\mathcal{S}p_1^{\mathcal{B}_T G}$. The formal properties of being equifibrantly flat persist in this context, and in particular the analogues of Remark 10.9.11 and Lemma 10.9.13 hold.

To get a feel for the more particular aspects of equifibrantly flat arrows, suppose that $(W \to Y)$ is an object of $\mathcal{S}p_1^{\mathcal{B}_T G}$, and $(\tilde{W} \to \tilde{Y})$ is a cofibrant approximation to it. Consider a weak equivalence of the form

$$(\tilde{X} \to *) \to (X \to *).$$

In this case the equifibrantly flat condition becomes that

$$\begin{array}{c} (p^*(\tilde{Y}/\tilde{W})^{\wedge(K/L)} \wedge \tilde{X} \to *) \\ \downarrow \\ (p^*(Y/W)^{\wedge(K/L)} \wedge X \to *) \end{array}$$

is a weak equivalence. This is so if and only if Y/W is equifibrantly flat.

Next consider a weak equivalence of the form

$$(* \to \tilde{X}) \to (* \to X).$$

The equifibrantly flat condition in this case is that

$$\begin{array}{c} (\partial_{p^*\tilde{W}} p^*\tilde{Y}^{\wedge(K/L)} \to p^*\tilde{Y}^{\wedge(K/L)}) \wedge \tilde{X} \\ \downarrow \\ (\partial_{p^*W} p^*Y^{\wedge(K/L)} \to p^*Y^{\wedge(K/L)}) \wedge X \end{array}$$

is a weak equivalence. This holds if and only if Y is equifibrantly flat and $(W \to Y)$ satisfies the condition that

$$\partial_{p^*\tilde{W}} p^*\tilde{Y}^{\wedge(K/L)} \wedge \tilde{X} \to \partial_{p^*W} p^*Y^{\wedge(K/L)} \wedge X \tag{10.9.16}$$

is a weak equivalence. If we happen to know that the indexed corner maps

$$\partial_{p^*\tilde{W}} p^*\tilde{Y}^{\wedge(K/L)} \to \tilde{Y}^{\wedge(K/L)} \qquad \text{and} \qquad \partial_{p^*W} p^*Y^{\wedge(K/L)} \to Y^{\wedge(K/L)}$$

are h-cofibrations, then the leftmost horizontal maps in

$$\begin{tikzcd} \partial_{p^*\tilde{W}} p^*\tilde{Y}^{\wedge(K/L)} \wedge \tilde{X} \arrow[r] \arrow[d] & p^*\tilde{Y}^{\wedge(K/L)} \wedge \tilde{X} \arrow[r] \arrow[d] & p^*(\tilde{Y}/\tilde{W})^{\wedge(K/L)} \wedge \tilde{X} \arrow[d] \\ \partial_{p^*W} p^*Y^{\wedge(K/L)} \wedge X \arrow[r] & p^*Y^{\wedge(K/L)} \wedge X \arrow[r] & p^*(Y/W)^{\wedge(K/L)} \wedge X \end{tikzcd}$$

are h-cofibrations, hence flat. Thus the middle and left vertical arrows are weak equivalences if and only if the middle and right vertical arrows are, or in other words if and only if both Y and Y/W are equifibrantly flat. So in the presence of the condition above, a necessary condition that $(W \to Y)$ be a equifibrantly flat arrow is that Y and Y/W are equifibrantly flat. This turns out to be sufficient. We single out this condition for future reference.

Condition 10.9.17. *For every $T \xleftarrow{p} K \xrightarrow{q} L$ the corner map*

$$\partial_{p^*W}(p^*Y)^{\wedge(K/L)} \to (p^*Y)^{\wedge(K/L)}$$

is an h-cofibration.

Remark 10.9.18. *By Proposition 10.3.8 and the monoid axiom for $\mathcal{S}p_1^{\mathcal{B}_L G}$, a cofibrant object $(W \to Y)$ of $\mathcal{S}p_1^{\mathcal{B}_T G}$ is equifibrantly flat and satisfies Condition 10.9.17.*

Lemma 10.9.19. A condition that makes a morphism of diagrams equifibrantly flat. *If $W_1 \to W_2$ satisfies Condition 10.9.17, and both W_1 and W_2/W_1 are equifibrantly flat, then $W = (W_1 \to W_2)$ is equifibrantly flat.*

Proof Fix a diagram of finite G-sets

$$T \xleftarrow{p} K \xrightarrow{q} L$$

let $\tilde{W} = (\tilde{W}_1 \to \tilde{W}_2)$ be a cofibrant approximation to $W = (W_1 \to W_2)$, and

$$\begin{aligned} \tilde{X} &\to X \\ \tilde{X} &= (\tilde{X}_1 \to \tilde{X}_2) \\ X &= (X_1 \to X_2) \end{aligned}$$

a weak equivalence in $\mathcal{S}p_1^{\mathcal{B}_L G}$. By Remark 10.9.18, $\tilde{W}$ also satisfies the conditions of the lemma. Let

$$X' \to X \to X''$$

be the sequence

$$(* \to X_2) \to (X_1 \to X_2) \to (X_1 \to *).$$

and $\tilde{X}' \to \tilde{X} \to \tilde{X}'$ the analogous sequence for $\tilde{X}$. The maps $X' \to X$ and $\tilde{X}' \to \tilde{X}$ are not h-cofibrations, but they are so objectwise, and hence flat.

Consider the diagram

$$\begin{tikzcd} p^*\tilde{W}^{\wedge(K/L)} \wedge \tilde{X}' \arrow[r] \arrow[d] & p^*\tilde{W}^{\wedge(K/L)} \wedge \tilde{X} \arrow[r] \arrow[d] & p^*\tilde{W}^{\wedge(K/L)} \wedge \tilde{X}'' \arrow[d] \\ p^*W^{\wedge(K/L)} \wedge X' \arrow[r] & p^*W^{\wedge(K/L)} \wedge X \arrow[r] & p^*W^{\wedge(K/L)} \wedge X''. \end{tikzcd} \tag{10.9.20}$$

Our aim is to show that the middle vertical map is a weak equivalence.

The first step is to show that the left horizontal maps are flat. This reduces us to checking that the left and right vertical maps are weak equivalences. For this, let's examine the bottom left horizontal map in more detail. It is given by

$$\begin{tikzcd} (\partial_{p^*W_1} p^*W_2^{\wedge(K/L)} \wedge X_2 \to p^*W_2^{\wedge(K/L)} \wedge X_2) \arrow[d] \\ (C \to p^*W_2^{\wedge(K/L)} \wedge X_2) \end{tikzcd} \tag{10.9.21}$$

in which C is defined by the pushout diagram

$$\begin{tikzcd}
\partial_{p^*W_1} p^* W_2^{\wedge(K/L)} \wedge X_1 \arrow[r] \arrow[d] & p^* W_2^{\wedge(K/L)} \wedge X_1 \arrow[d] \\
\partial_{p^*W_1} p^* W_2^{\wedge(K/L)} \wedge X_2 \arrow[r] & C.
\end{tikzcd} \tag{10.9.22}$$

When $W_1 \to W_2$ satisfies Condition 10.9.17 the top map in (10.9.22) is an h-cofibration, hence so is the bottom map. This means that (10.9.21) is an objectwise h-cofibration, and so flat. Since $\tilde{W}_1 \to \tilde{W}_2$ also satisfies Condition 10.9.17 the upper left horizontal map in (10.9.20) is also flat. Thus we are reduced to checking that the maps

$$p^* \tilde{W}^{\wedge(K/L)} \wedge \tilde{X}' \to p^* W^{(\wedge K/L)} \wedge X'$$
$$p^* \tilde{W}^{\wedge(K/L)} \wedge \tilde{X}'' \to p^* W^{(\wedge K/L)} \wedge X''$$

are weak equivalences. As described above, this fact for the second map follows from the assumption that W_2/W_1 is equifibrantly flat. The assertion in the case of the first map is that the middle and left vertical arrows in

$$\begin{tikzcd}
\partial_{p^*\tilde{W}_1} p^* \tilde{W}_2^{\wedge(K/L)} \wedge \tilde{X}_2 \arrow[r] \arrow[d, "\therefore\sim"'] & p^* \tilde{W}_2^{\wedge(K/L)} \wedge \tilde{X}_2 \arrow[r] \arrow[d, "\sim"] & p^*(\tilde{W}_2/\tilde{W}_1)^{\wedge(K/L)} \wedge \tilde{X}_2 \arrow[d, "\sim"] \\
\partial_{p^*W_1} p^* W_2^{\wedge(K/L)} \wedge X_2 \arrow[r] & p^* W_2^{\wedge(K/L)} \wedge X_2 \arrow[r] & p^*(W_2/W_1)^{\wedge(K/L)} \wedge X_2
\end{tikzcd}$$

are weak equivalences. Since W_2 and W_2/W_1 are equifibrantly flat, the middle and right vertical maps are weak equivalences. Condition 10.9.17 shows that the left horizontal maps are h-cofibrations, hence flat. It follows that the left vertical map is a weak equivalence. □

We can now establish an important technical fact used in the proof of Theorem 10.9.9.

Lemma 10.9.23. A condition that makes a corner map of diagrams equifibrantly flat. *Suppose that $W \to Y$ is a cofibrant object of $\mathcal{S}p_1^{\mathcal{B}_T G}$, I is a G-set and $\Lambda \subset \Lambda_I$ a G-stable subgroup. Then*

$$\mathrm{Sym}_\Lambda^I (W \to Y) = (\partial_W \mathrm{Sym}_\Lambda^I Y \to \mathrm{Sym}_\Lambda^I Y)$$

is equifibrantly flat.

Proof Proposition 10.6.8 implies that in this situation the map $\mathrm{Sym}_\Lambda^I (W \to Y)$ satisfiesCondition 10.9.17, and that for **every** cofibrant Y, $\mathrm{Sym}_\Lambda^I Y$ is equifibrantly flat (so both $\mathrm{Sym}_\Lambda^I Y$ and $\mathrm{Sym}_\Lambda^I (Y/W)$ are equifibrantly flat). The result then follows from Lemma 10.9.19. □

Example 10.9.24. *Continuing with Example 10.9.14, the top map in (10.9.15) arises naturally in the arrow category as*

$$\bigvee_{G_n(K/L)} Z \wedge (p^*(W \to Y)^{\wedge(K/L)}),$$

where Z is the identity arrow of the diagram $X^{\wedge T_0}$. Since the formation of indexed wedges is homotopical, the information in the homotopy type of this expression is contained in the $G_n(K/L)$-diagram $Z \wedge (p^*(W \to Y)^{\wedge(K/L)})$. The condition that $(W \to Y)$ be equifibrantly flat thus specifies good homotopical properties of the top map in (10.9.15).

Lemma 10.9.25. A condition that makes a pushout of a diagram equifibrantly flat. *Consider a pushout square*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \tag{10.9.26}$$

in which $(W \to Y)$ is a equifibrantly flat object of $\mathcal{S}p_1^{\mathcal{B}_T G}$ satisfying Condition 10.9.17. If X is equifibrantly flat, then so is Z.

Proof Using the fact that cofibrations are flat, we can arrange things so that the cofibrant approximation $\widetilde{Z} \to Z$ fits into a pushout square

$$\begin{tikzcd} \tilde{W} \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\ \tilde{X} \arrow[r] & \widetilde{Z} \end{tikzcd} \tag{10.9.27}$$

of cofibrant approximations to (10.9.26), in which $\tilde{W} \to \tilde{Y}$ is a cofibration. We give $\widetilde{Z}^{\wedge(K/L)}$ and $Z^{\wedge(K/L)}$ the filtration described in §2.9C. We will prove by induction on n that for any weak equivalence $\widetilde{Z} \to Z$ of equivariant T-diagrams, the map

$$\mathrm{fil}_n \widetilde{Z}^{\wedge K} \wedge \widetilde{Z} \to \mathrm{fil}_n Z^{\wedge K} \wedge Z \tag{10.9.28}$$

is a weak equivalence. The case $n = 0$ is the assertion that X is equifibrantly flat, which is true by assumption. For the inductive step, consider the diagram

$$
\begin{tikzcd}[column sep=small]
& \bigvee\limits_{G_n(K/L)} \tilde{X}^{\wedge K_0} \wedge \partial_{\tilde{W}} \tilde{Y}^{\wedge K_1} \wedge \tilde{Z} \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathrm{fil}_{n-1} \tilde{Z}^{\wedge K} \wedge \tilde{Z} \arrow[dd] & & \bigvee\limits_{G_n(L)} \tilde{X}^{\wedge K_0} \wedge \tilde{Y}^{\wedge K_1} \wedge \tilde{Z} \arrow[dd] \\
& \bigvee\limits_{G_n(K/L)} sX^{\wedge K_0} \wedge \partial_W Y^{\wedge K_1} \wedge Z \arrow[dl] \arrow[dr] & \\
\mathrm{fil}_{n-1} Z^{\wedge K} \wedge Z & & \bigvee\limits_{G_n(K/L)} X^{\wedge K_0} \wedge Y^{\wedge K_1} \wedge Z \,.
\end{tikzcd}
$$

The map from the pushout of the top row to the pushout of the bottom row is (10.9.28). The rightmost horizontal maps are h-cofibrations by Condition 10.9.17. The left vertical map is a weak equivalence by induction, and the other two vertical maps are weak equivalences since $(W \to Y)$ is equifibrantly flat (Example 10.9.24). The map of pushouts is therefore a weak equivalence since h-cofibrations are flat. $\square$

10.9C The proof of Theorem 10.9.9

Since the class of equifibrantly flat T-diagrams is closed under the formation of filtered colimits along h-cofibrations by Lemma 10.9.13, it suffices to show that if $W \to Y$ is a generating cofibration in $\mathcal{S}p^{\mathcal{B}_T G}$,

$$
\begin{tikzcd}
\mathrm{Sym}\, W \arrow[r] \arrow[d] & \mathrm{Sym}\, Y \arrow[d] \\
X \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & Z
\end{tikzcd}
$$

is a pushout square of commutative T-algebras, and X is equifibrantly flat, then Z is equifibrantly flat. Working fiberwise, the pushout ring filtration of Definition 2.9.47 gives a filtration of Z by X-modules, whose stages are related by the pushout squares

$$
\begin{tikzcd}
X \wedge \partial_W \mathrm{Sym}^m Y \arrow[r] \arrow[d] & X \wedge \mathrm{Sym}^m Y \arrow[d] \\
\mathrm{fil}_{m-1} Z \arrow[r] \arrow[ur, phantom, "\lrcorner" very near end] & \mathrm{fil}_m Z \,.
\end{tikzcd}
\tag{10.9.29}
$$

We show by induction on m that each $\mathrm{fil}_m Z$ is equifibrantly flat. Since $\mathrm{fil}_0 Z = X$, the induction starts. The arrow $(\partial_W \mathrm{Sym}^m Y \to \mathrm{Sym}^m Y)$ is equifibrantly flat by Lemma 10.9.23. This means that the top row of (10.9.29) is a equifibrantly flat arrow, since smash products of equifibrantly flat objects are equifibrantly flat (Lemma 10.9.13). This places us in the situation of Lemma 10.9.25, which completes the inductive step.

10.10 Twisted monoid rings

We will describe a method of constructing various equivariant ring spectra. They will be used in §12.2 and §12.4.

10.10A Definitions

Definition 10.10.1. *Let V be a virtual representation of a subgroup $H \subseteq G$. A* **positive representative of** V *is a pair of representations (V', V'') of actual representations such that $V = V' - V''$ with $(V'')^H$ positive dimensional, i.e., V'' has a nonzero vector fixed by H. For such a choice we define S^V to be the spectrum $S^{-V''} \wedge S^{V'}$, where $S^{-V''}$ is the Yoneda spectrum as in Definition 7.2.52 and the space $S^{V'}$ is the one point compactification of V' as in §8.9.*

The existence of a nonzero H-invariant vector in V'' insures the cofibrancy of $S^{-V''} \wedge S^{V'}$ in the positive equifibrant model structure, with or without stabilization.

Definition 10.10.2. *Given a positive representative of V as in Definition 10.10.1, the* **free associative algebra or twisted monoid ring generated by** S^V *is*

$$S^{-0}[S^V] := \bigvee_{k \geqslant 0} \left(S^{-V''} \wedge S^{V'}\right)^{\wedge k} \cong \bigvee_{k \geqslant 0} S^{-kV''} \wedge S^{kV'}.$$

The underlying stable homotopy type of $S^{-0}[S^V]$ is that of a wedge of spheres, one in each nonnegative dimension divisible by $|V|$. In particular it has the stable homotopy type of a suspension spectrum. However the positivity condition of Definition 10.10.1 implies that it appears not to be isomorphic to a suspension spectrum (since $S^{-V''}$ is not one), hence our use of the word **twisted.**

Let $\overline{x} \in \pi_V^H S^{-0}[S^V]$ be the homotopy class of the generating inclusion. Then $\pi_\star^H S^{-0}[S^V]$ is a free module over $\pi_\star^H S^{-0}$ on the set $\{1, \overline{x}, \overline{x}^2, \ldots\}$. For this reason we will sometimes write

$$S^{-0}[S^V] = S^{-0}[\overline{x}]. \tag{10.10.3}$$

It is not commutative because the map $S^V \wedge S^V \to S^{2V}$ does not factor through the orbit spectrum $(S^V \wedge S^V)_{\Sigma_2}$.

Example 10.10.4. Noncommutativity. *Consider the case $S^V = S^{-1} \wedge S^1$, so we have a Σ_2 action on $S^V \wedge S^V = S^{-2} \wedge S^2$. As a coend we have*

$$S^{-2} \wedge S^2 = \int_{\mathscr{J}} S^{-n} \wedge \mathscr{J}(2,n) \wedge S^2 = \int_{\mathscr{J}} S^{-n} \wedge \mathscr{J}(2,n) \wedge \mathscr{J}(0,2).$$

The symmetric group Σ_2 acts nontrivially on both $\mathscr{J}(2,n)$ and $\mathscr{J}(0,2)$ via its permutation action on $\mathbf{R}^2$. Recall that the map $e_2 : S^{-2} \wedge S^2 \to S^{-0}$ of (7.2.68) is induced by the composition map $j_{0,2,n} : \mathscr{J}(2,n) \wedge \mathscr{J}(0,2) \to \mathscr{J}(0,n)$ of Definition 8.9.24. This j is Σ_2-equivariant with respect to the specified action on the source and the trivial action on the target. The orbit spectrum $(S^{-2} \wedge S^2)_{\Sigma_2}$ has as its nth space

$$\mathscr{J}(2,n)_{\Sigma_2} \wedge \mathscr{J}(0,2)_{\Sigma_2}.$$

Since $S^{-0}[S^V]$ is an H-spectrum, we can apply the norm functor N_H^G of Definition 9.7.3 to it and get

$$S^{-0}[G \cdot S^V] := N_H^G(S^{-0}[S^V]).$$

More explicitly we have

$$N_H^G\left(S^{-0}[S^V]\right) \simeq \bigwedge_{j \in G/H} \left(\bigvee_{n=0}^{\infty} S^{\wedge n V_j} \right),$$

where V_j is the virtual representation of gHg^{-1} whose positive representative (V_j', V_j'') is the precomposition of (V', V'') with the canonical isomorphism $gHg^{-1} \to H$. As in (10.10.3) we will write this as

$$S^{-0}[G \cdot S^V] = S^{-0}[G \cdot \overline{x}]. \tag{10.10.5}$$

We can smash together such ring spectra for a set of positive representatives (V_i', V_i'') of various virtual representations of various subgroups H_i of G. We will denote the resulting G-equivariant associative algebra by

$$R = S^{-0}[G \cdot \overline{x}_1, G \cdot \overline{x}_2, \dots] \qquad \text{with } \overline{x}_i \in \pi_{V_i}^{H_i}[S^{V_i}].$$

When the set we are considering is infinite, we will smash together the first m of them and then pass to the colimit as m increases. More explicitly, let

$$B_m = \coprod_{1 \leqslant i \leqslant m} G/H_i.$$

Then we have

$$R_m = \bigwedge_{b \in B_m} \left(\bigvee_{n=0}^{\infty} S^{\wedge n V_b} \right) \tag{10.10.6}$$

where V_b for $b \in G/H_i$ is the virtual representation of gH_ig^{-1} whose positive representative (V_b', V_b'') is the precomposition of (V_i', V_i'') with the canonical isomorphism $gH_ig^{-1} \to H_i$.

Since (10.10.6) describes R_m as a smash product of wedges, we can use the distributive law of Proposition 2.9.20 to rewrite it as a wedge of smash products. In the case at hand, the binary operations $\oplus$ and $\otimes$ are replaced by $\vee$ and $\wedge$. In the notation of Proposition 2.9.20, the category L is trivial, but J and K are not discrete as in Definition 2.1.7. Let $K_m = \mathcal{B}_{B_m}G$ and as in Example 2.9.1, and $J_m = K_m \times \mathcal{N}$ where $\mathcal{N}$ is the discrete category associated with the natural numbers $\mathbf{N}$, and the functor $p_m : J_m \to K_m$ is given by $(b, n) \mapsto b$. Then the category Γ_m of sections $K_m \to J_m$ is the set $\mathbf{N}^{B_m}$ of $\mathbf{N}$-valued functions on the G-set B_m.

Then Proposition 2.9.20 gives

$$R_m \cong \bigvee_{f \in \Gamma_m} S^{V_f} \quad \text{where } V_f = \sum_{b \in B_m} f(b) V_b.$$

Here V_f is a virtual representation of the stabilizer H_f of f with positive representative

$$\left(\bigoplus_{b \in B_m} f(b) V_b', \bigoplus_{b \in B_m} f(b) V_b'' \right).$$

The G-set Γ_m is an abelian monoid under addition of functions, and the ring structure on R_m is the indexed sum of the isomorphisms

$$S^{V_f} \wedge S^{V_g} \cong S^{V_f \oplus V_g} \cong S^{V_{f+g}}.$$

When our ring spectrum R involves infinitely many $\overline{x}_i$, we pass to the colimit of spectra T_m as above. This means the finite G-sets B_m get replaced by an infinite one,

$$B = \operatorname*{colim}_m B_m,$$

and the category and abelian monoid

$$\Gamma = \operatorname*{colim}_m \Gamma_m$$

is that of **finitely supported N-**valued functions on B. Thus we have

$$R = \operatorname*{colim}_m R_m \cong \bigvee_{f \in \Gamma} S^{V_f}. \tag{10.10.7}$$

with V_f defined as in the finite case.

10.10B Ideals

Definition 10.10.8. *A* **monoid ideal** *in an abelian monoid L is a subset I with $I + L \subseteq I$. Its n***th power** *is nI, the set of n-fold sums of elements in I.*

For a G-invariant monoid ideal $I \subseteq \Gamma$ for Γ as in (10.10.7), the corresponding **monomial ideal** *is the G-spectrum*

$$R_I = \bigvee_{f \in I} S^{V_f},$$

which is a sub-bimodule of R.

Monomial ideals in a monoidal product of free associative algebras in a closed symmetric monoidal category were discussed in §2.9G.

Example 10.10.9. Some monomial ideals.

(i) [(i)] Let I be the set of all nonzero elements of Γ, the **augmentation ideal***. We denote the corresponding spectrum by $(G\cdot\overline{x}_1, G\cdot\overline{x}_1,\dots)$.*

(ii) [(ii)] Let $\dim:\Gamma\to\mathbf{N}$ be given by

$$\dim f = |V_f| = \sum_{b\in B} f(b)|V_b|.$$

When $|V_b|>0$ for all $b\in B$, the set $I_d=\{f:\dim f\geqslant d\}$ is a monoid ideal, and the corresponding monoidal ideal M_d is the wedge of all "spheres" in R of dimension $\geqslant d$, and

$$M_d/M_{d-1} = \bigvee_{\dim f=d} S^{V_f}.$$

10.10C The method of twisted monoid rings

Definition 10.10.10. *Suppose that*

$$f_i : B_i \to R, \quad i = 1,\dots m$$

are algebra maps from associative algebras B_i to a commutative algebra R. The **smash product** *of the f_i is the algebra map*

$$\bigwedge_m B_i \xrightarrow{\ \wedge_m f_i\ } R^{\wedge m} \longrightarrow R$$

in which the right most map is the iterated multiplication.

If B is an H-equivariant associative algebra, and $f:B\to i_H^G R$ is an algebra map, we define the **norm of** *f to be the G-equivariant algebra map*

$$N_H^G B \to R$$

given by

$$N_H^G B \to N_H^G(i_H^G R) \to R,$$

in which the rightmost map is the counit of the adjunction described in Corollary 10.7.5.

These constructions make it easy to map a twisted monoid ring to a commutative algebra. Suppose that R is a fibrant G-equivariant commutative algebra, and we have a sequence

$$\overline{x}_i \in \pi_{V_i}^{H_i} R, \qquad i = 1, 2, \dots.$$

A choice of positive representative (V_i', V_i'') of V_i and a map

$$S^{V_i} \to R$$

representing $\overline{x}_i$ determines an associative algebra map

$$S^{-0}[\overline{x}_i] \to R.$$

Applying the norm gives a G-equivariant associative algebra map

$$S^{-0}[G \cdot \overline{x}_i] \to R.$$

By smashing these together we can make a sequence of equivariant algebra maps

$$S^{-0}[G \cdot \overline{x}_1, \dots, G \cdot \overline{x}_m] \to R.$$

Passing to the colimit gives an equivariant algebra map

$$S^{-0}[G \cdot \overline{x}_1, G \cdot \overline{x}_2, \dots] \to R \tag{10.10.11}$$

representing the sequence $\overline{x}_i$. We will refer to this process by saying that the map (10.10.11) is **constructed by the method of twisted monoid rings**.

The whole construction can also be made relative to a commutative algebra E. Let

$$E[G \cdot \overline{x}_1, G \cdot \overline{x}_2, \dots] := E \wedge S^{-0}[G \cdot \overline{x}_1, G \cdot \overline{x}_2, \dots] \tag{10.10.12}$$

Then we have an E-algebra map

$$E[G \cdot \overline{x}_1, G \cdot \overline{x}_2, \dots] \to R \tag{10.10.13}$$

when R is a commutative E-algebra.

10.10D Quotient modules

One important construction in ordinary stable homotopy theory is the formation of the quotient of a module M over a commutative algebra R by the ideal generated by a sequence $\{x_1, x_2, \dots\} \subset \pi_* R$. This is done by inductively forming the cofiber sequence of R-modules

$$\Sigma^{|x_n|} M/(x_1, \dots, x_{n-1}) \to M/(x_1, \dots, x_{n-1}) \to M/(x_1, \dots, x_n) \tag{10.10.14}$$

and passing to the homotopy colimit in the end. There is an evident equivalence

$$M/(x_1, \dots) \cong M \underset{R}{\wedge} R/(x_1, \dots)$$

when M is a cofibrant R-module. The situation is slightly trickier in equivariant stable homotopy theory, where the group G might act on the elements x_i, and prevent the inductive approach described above. The method of twisted monoid rings (§10.10C) can be used to get around this difficulty.

Suppose that R is a fibrant equivariant commutative algebra, and that

$$\overline{x}_i \in \pi_{V_i}^{H_i}(R) \qquad i = 1, 2, \ldots$$

is a sequence of equivariant homotopy classes. Using the method of twisted monoid rings, construct an associative R-algebra map

$$T = R[G \cdot \overline{x}_1, G \cdot \overline{x}_2, \ldots] \to R. \tag{10.10.15}$$

Using this map, we may regard an equivariant R-module M as a T-module. In addition to (10.10.15) we will make use of the augmentation $\epsilon : T \to R$ sending the $\overline{x}_i$ to zero.

Definition 10.10.16. *The* **quotient module** $M/(G\cdot\overline{x}_1, \ldots)$ *is the R-module*

$$M \overset{\mathbf{L}}{\underset{T}{\wedge}} R$$

in which T acts on M through the map (10.10.15) and on R through the augmentation.

The symbol $\overset{\mathbf{L}}{\wedge}$ denotes derived smash product. By Proposition 10.8.3 it can be computed by taking a cofibrant approximation in either variable.

Let us check that this construction reduces to the usual one when G is trivial and M is a cofibrant R-module. For ease of notation, write

$$\begin{aligned} T &= R[x_1, \ldots] \\ T_n &= R[x_1, \ldots, x_n]. \end{aligned}$$

Using the isomorphism

$$T \cong T_n \underset{R}{\wedge} R[x_{n+1}, \ldots]$$

one can construct an associative algebra map

$$T \to R[x_{n+1}, \ldots]$$

by smashing the augmentation

$$T_n \to R$$

sending each x_i to 0, with the identity map of $R[x_{n+1}, \ldots]$. By construction, the evident map of T-algebras

$$\operatorname*{colim}_n R[x_{n+1}, \ldots] \to R$$

is an isomorphism, and hence so is the map

$$\operatorname*{colim}_n M \underset{T}{\wedge} R[x_{n+1}, \ldots] \to M \underset{T}{\wedge} R.$$

In fact this isomorphism is also a derived equivalence. To see this, construct a sequence

$$\cdots \to N_{n+1} \to N_{n+2} \to \cdots$$

of cofibrations of cofibrant left T-module approximations to

$$\cdots \to R[x_{n+1},\dots] \to R[x_{n+2},\dots] \to \cdots .$$

We have

$$\pi_* \operatorname*{colim}_n N_n \cong \operatorname*{colim}_n \pi_* N_n \cong \operatorname*{colim}_n (\pi_* R)[x_n,\dots] \cong \pi_* R$$

from which one concludes that the map

$$\operatorname*{colim}_n N_n \to \operatorname*{colim}_n R[x_n,\dots]$$

is a cofibrant approximation. It follows that

$$M/(x_1,\dots) \cong \operatorname*{hocolim}_n M/(x_1,\dots,x_n).$$

To compare $M/(x_1,\dots,x_{n-1})$ with $M/(x_1,\dots,x_n)$ let $T_n \to R[x_n]$ be associative algebra map constructed from the isomorphism

$$T_n \cong T_{n-1} \underset{R}{\wedge} R[x_n].$$

by smashing the augmentation of T_{n-1} with the identity map of $R[x_n]$. We have

$$M/(x_1,\dots,x_{n-1}) \sim M \underset{T_{n-1}}{\wedge} R \cong M \underset{T_n}{\wedge} T_n \underset{T_{n-1}}{\wedge} R \cong M \underset{T_n}{\wedge} R[x_n].$$

By Proposition 10.8.3,

$$M \underset{T_n}{\wedge} R[x_n]$$

is a cofibrant $R[x_n]$-module. The cofiber sequence (10.10.14) is now constructed by applying the functor

$$M/(x_1,\dots,x_{n-1}) \underset{R[x_n]}{\wedge} (-).$$

to the pushout diagram of $R[x_n]$ bimodules

$$\begin{tikzcd}
(x_n) \arrow[r] \arrow[d] & R[x_n] \arrow[d] \\
* \arrow[r] & R
\end{tikzcd}$$

and appealing to Corollary 10.8.4.

A similar discussion applies to the equivariant situation, giving

$$M/(G\cdot \overline{x}_1,\dots) \cong \operatorname*{colim}_n M/(G\cdot \overline{x}_1,\dots,G\cdot \overline{x}_n),$$

a relation

$$M/(G\cdot \overline{x}_1,\dots,G\cdot \overline{x}_n) \cong M/(G\cdot \overline{x}_1,\dots,G\cdot \overline{x}_{n-1}) \underset{R[G\cdot \overline{x}_n]}{\wedge} R,$$

and a cofiber sequence

$$\begin{array}{c}(G\cdot\overline{x}_n)\cdot M/(G\cdot\overline{x}_1,\dots,G\cdot\overline{x}_{n-1})\\ \downarrow\\ M/(G\cdot\overline{x}_1,\dots,G\cdot\overline{x}_{n-1})\\ \downarrow\\ M/(G\cdot\overline{x}_1,\dots,G\cdot\overline{x}_n),\end{array}$$

derived by applying the functor

$$M/(G\cdot\overline{x}_1,\dots,G\cdot\overline{x}_{n-1})\underset{R[G\cdot\overline{x}_n]}{\wedge}(-)$$

to

$$(G\cdot\overline{x}_n)\to R[G\cdot\overline{x}_n]\to R.$$

One can also easily deduce the equivalences

$$R/(G\cdot\overline{x}_1,\dots,G\cdot\overline{x}_n)\cong R/(G\cdot\overline{x}_1)\underset{R}{\wedge}\cdots\underset{R}{\wedge}R/(G\cdot\overline{x}_1)$$

and

$$R/(G\cdot\overline{x}_1,\dots)\cong\operatorname*{colim}_n R/(G\cdot\overline{x}_1)\underset{R}{\wedge}\cdots\underset{R}{\wedge}R/(G\cdot\overline{x}_n).$$

These expressions play an important role in the proof Lemma 12.4.23, which is a key step in the proof the Reduction Theorem 12.4.8.

PART THREE

PROVING THE KERVAIRE INVARIANT THEOREM

11

The slice filtration and slice spectral sequence

The slice spectral sequence is our main computational tool. It is named after an analogous construction in motivic homotopy theory [Voe02, Voe04, Lev13, Hoy15].

The slice filtration, which we study in §11.1, is an equivariant analogue of the Postnikov tower, to which it reduces in the case of the trivial group. In this chapter we introduce the slice filtration and establish some of its basic properties. We work for the most part with a general finite group G, though our application to the Kervaire invariant problem involves only the case $G = C_{2^n}$. While the situation for general G exhibits many remarkable properties, the reader should regard as exploratory the apparatus of definitions at this level of generality.

There are two differences between the presentation here and that of [HHR16].

- Our slice spheres (called slice cells in [HHR16]) of Definition 11.1.3, involve multiples (both positive and negative) of regular representations of subgroups $H \subseteq G$. In [HHR16, Definition 4.1] slices cells are the objects identified in Definition 11.1.1 **and their single desuspensions**; see Remark 11.1.5. The current definition was not available when [HHR16] was written. It leads to better multiplicative properties than we had before.
- We use the recent work of Yarnall and the first author [HY18] to characterize slice connectivity (Definition 11.1.11) in terms of ordinary connectivity of geometric fixed points; see §11.1D.

In §11.2 we study the slice spectral sequence, which is the homotopy spectral sequence of the slice tower of Definition 11.1.42. We show that it converges and is concentrated in certain parts of the first and third quadrants when displayed with the Adams convention. This is illustrated in Figure 11.1. We also discuss an $RO(G)$-graded form of the spectral sequence.

In §11.3 we discuss cases when the slice filtration has a particularly convenient form: each slice or layer is the smash product of the Eilenberg-Mac Lane spectrum $H\mathbf{Z}$ with a wedge of slice spheres of the appropriate dimension. These are the spherical slices of Definition 11.3.14. We call a spectrum **pure**

(Definition 11.3.14) if all of its slices are spherical and bound, that is induced up from nontrivial subgroups. It turns out that all slices of the spectra we need to compute with have this property. Its convenience is apparent in Lemma 11.3.16 and Theorem 11.3.17. The latter says that a map between such spectra is a weak equivalence of G-spectra if the underlying map is an ordinary stable equivalence of spectra.

§11.4 is more technical and makes use of the machinery developed in §10.7. Here we have to be more careful and replace the slice spheres of Definition 11.1.1 with cofibrant approximations. We show that slice connectivity is preserved by indexed wedges (Proposition 11.4.2), indexed smash products (Proposition 11.4.4) and indexed symmetric powers (Proposition 11.4.10). Then we show in Theorem 11.4.13 that each stage of the slice tower of a commutative ring spectrum is again a commutative ring spectrum.

A word about notation and negative dimensions. In this chapter we will often speak of cells in both negative and positive dimensions. We remind the reader that for $n \geqslant 0$, S^n denotes the n-dimensional sphere, which is a pointed topological space, while S^{-n} denotes a Yoneda spectrum as in Definition 7.2.52. In particular, the symbols S^0 and S^{-0} have different meanings. The letter denotes the sphere spectrum, and the suspension spectrum of a pointed space X is $X \wedge S^{-0}$ or $\Sigma^\infty X$.

When $n < 0$, we will usually write $S^{-|n|}$ rather than S^n to emphasize that we mean a spectrum rather than a space. We will sometimes refer to S^{-n} (for $n > 0$) as the **Spanier-Whitehead dual** (see §8.0C) of $S^n \wedge S^{-0} = \Sigma^n S^{-0} = \Sigma^\infty S^n$. The map $\xi_{0,n} : S^n \wedge S^{-n} \to S^{-0}$ of (7.2.63) is a stable equivalence. **Similar remarks apply in the equivariant case.**

11.1 The filtration behind the spectral sequence

11.1A The Postnikov filtration of a spectrum

We begin by recalling the classical Postnikov filtration. Given a space or spectrum X, one can kill its homotopy groups above dimension n by attaching cells of dimension $> n+1$, and doing so does not alter the homotopy groups in dimensions $\leqslant n$. In this way one obtains a map $X \to P^n X$, where $P^n X$ is the nth Postnikov section of X satisfying

$$\pi_k P^n X = \begin{cases} \pi_k X & \text{for } k \leqslant n \\ 0 & \text{for } k > n. \end{cases}$$

Since $P^n X$ is obtained from X by attaching cells, the map $X \to P^n X$ is a cofibration. The fiber of this map is $P_{n+1} X$, the n-connected cover of X. We denote the fiber of the map $P^n X \to P^{n-1} X$ by $P_n^n X$. It is the Eilenberg-

Mac Lane space or spectrum satisfying

$$\pi_k P_n^n X = \begin{cases} \pi_n X & \text{for } k = n \\ 0 & \text{otherwise.} \end{cases}$$

The diagram

$$\begin{tikzcd} & P_n^n X \arrow[d] & P_{n-1}^{n-1} X \arrow[d] & \\ \cdots \arrow[r] & P^n X \arrow[r] & P^{n-1} X \arrow[r] & \cdots \end{tikzcd}$$

is the **Postnikov tower of X**. The limit and colimit of the bottom row are X and $*$ respectively.

The category τ_n of $(n-1)$-connected spaces is an example of a localizing subcategory as in Definition 6.3.12. Then, using the notation of Theorem 6.3.17, we have $P^n = P^{\tau_n}$ and $P_{n+1} = P_{\tau_n}$.

The category of spectra $\mathcal{S}p$ is known to be Hirschhorn as in Definition 6.3.2, so we can do a similar construction there with τ_n being the localizing subcategory generated by $\Sigma^\infty S^n$.

11.1B Slice spheres

We will define a nested sequence of localizing subcategories of $\mathcal{S}p^G$ analogous to the τ_ns above. They will be generated by certain finite G-CW complexes we call **slice spheres**, which are merely spheres when G is trivial. They will enable us to construct an equivariant analog of the Postnikov tower we call the **slice tower** in §11.1E. In §11.1D we will give an equivalent and easier to work with definition of these subcategories in terms of geometric connectivity.

Definition 11.1.1. Slice spheres. *For a subgroup $H \subset G$ let ρ_H denote its regular representation, and write*

$$\widehat{S}(m, H) = G \underset{H}{\ltimes} \begin{cases} S^{m\rho_H} \wedge S^{-0} & \text{for } m \geqslant 0 \\ S^{-|m|\rho_H} & \text{for } m < 0. \end{cases}$$

Note here that we are defining a G-spectrum $\widehat{S}(m, H)$. The symbol $S^{m\rho_H}$ for $m \geqslant 0$ denotes a pointed H-space (the one point compactification of the vector space $m\rho_H$), which we need to convert to a suspension spectrum by smashing with the sphere spectrum S^{-0}. For $m < 0$, the symbol $S^{-|m|\rho_H}$ already denotes an H-spectrum, namely the Yoneda spectrum of Definition 7.2.52. We refer to these spectra as **slice spheres**.

These spectra are Bredon cofibrant as in Definition 9.2.15, but for $m \geqslant 0$ they are not cofibrant in the positive stable equifibrant model structure of

Theorem 9.2.13. The spectra $S^{-1} \wedge S^1 \wedge \widehat{S}(m,H)$ are cofibrant replacements for them that we use in §11.4 below. For $m < 0$, we have

$$S^{m\rho_H} = S^{-|m|\rho_h} \cong S^{-|m|} \wedge S^{-|m|\overline{\rho}_H},$$

(where $\overline{\rho}_H$ denotes the reduced regular representation of H as in Example 8.9.8), which is cofibrant. Thus we define

$$\widehat{S}_c(m,H) = \begin{cases} S^{-1} \wedge S^1 \wedge \widehat{S}(m,H) & \text{for } m \geqslant 0 \\ \widehat{S}(m,H) & \text{otherwise.} \end{cases} \tag{11.1.2}$$

We refer to these spectra as **cofibrant slice spheres**.

Definition 11.1.3. *The set of* G**-slice spheres** *(or just* **slice spheres** *when the group G is clear from the context) is*

$$\{\widehat{S}(m,H) \mid m \in \mathbf{Z}, H \subset G\}.$$

The set of **cofibrant** G**-slice spheres** *is*

$$\{\widehat{S}_c(m,H) \mid m \in \mathbf{Z}, H \subset G\}.$$

Remark 11.1.4. Slice spheres and their cofibrant replacements. *Most definitions and statements in this chapter will be made in terms of slice spheres, but they could be stated in terms of cofibrant slice spheres. The distinction only become important in §11.4 where it is import to keep everything cofibrant to insure that certain constructions (such as symmetric products) are homotopical.*

Remark 11.1.5. The original definition of slice spheres. *In [HHR16, Definition 4.1] these spectra were called* **slice cells** *and the set of them was defined to be*

$$\{\widehat{S}(m,H), \Sigma^{-1}\widehat{S}(m,H) \mid m \in \mathbf{Z}, H \subseteq G\};$$

it included the single desuspensions of the slice spheres of Definition 11.1.3. We learned later that the desuspensions were not needed and that the resulting slice tower of §11.1E has better properties without them. The details were first published in [Ull13].

The terminology in the following is meant to resemble that of Definition 8.4.14.

Definition 11.1.6. *A G-slice sphere is* **moving** *or* **induced up from** H *if it is of the form*

$$G \underset{H}{\ltimes} \widehat{S},$$

where $\widehat{S}$ is an H-slice sphere and $H \subset G$ is a proper subgroup; otherwise it is **stationary**. *It is* **free** *if H is the trivial group. A slice sphere is* **bound** *(called* **isotropic** *in [HHR16, Definition 1.12]) if it is not free.*

Since

$$[G \underset{H}{\ltimes} S, X]^G \cong [S, i_H^G X]^H \quad \text{and}$$
$$[X, G \underset{H}{\ltimes} S]^G \cong [i_H^G X, S]^H,$$

induction on $|G|$ usually reduces claims about stationary slice spheres, namely ones of the form $\Sigma^\infty S^{m\rho_G}$ for $m > 0$ and $S^{m\rho_G}$ for $m \leqslant 0$.

Definition 11.1.7. *The* **dimension** *of a slice sphere is defined by*

$$\dim \widehat{S}(m, H) = m|H|.$$

In other words the dimension of a slice sphere is that of its underlying spheres.

Remark 11.1.8. *The ordinary suspension or desuspension of a slice sphere need not be a slice sphere when the subgroup H is nontrivial.*

The following is immediate from the definition.

Proposition 11.1.9. Restrictions of slice spheres. *Let $H \subset G$ be a subgroup. If $\widehat{S}$ is a G-slice sphere of dimension n, then $i_H^G \widehat{S}$ is a wedge of H-slice spheres of dimension n. If $\widehat{S}$ is an H-slice sphere of dimension n then $G \underset{H}{\ltimes} \widehat{S}$ is a G-slice sphere of dimension n.*

The slice spheres behave well under the norm.

Proposition 11.1.10. Norms of slice spheres. *Let $H \subset G$ be a subgroup. If $\widehat{W}$ is a wedge of H-slice spheres, then $N_H^G \widehat{W}$ is a wedge of G-slice spheres.*

Proof The wedges of H-slice spheres are exactly the indexed wedges (as in Definition 2.9.6) of spectra of the form $S^{m\rho_K}$ for $K \subset H$, and $m \in \mathbf{Z}$. Since regular representations induce up to regular representations, Proposition 9.7.8 and the indexed distributive law (Proposition 2.9.20) show that the norm of such an indexed wedge is an indexed wedge of $S^{m\rho_K}$ with $K \subset G$ and $m \in \mathbf{Z}$. The claim follows. □

11.1C Slice connected and slice coconnected spectra

Underlying the theory of the Postnikov tower is the notion of "connectivity" and the class of $(n-1)$-connected spectra. In this section we describe the slice analogues of these ideas. We will see in §11.1D that slice connectivity as in Definition 11.1.11 below coincides with the geometric connectivity of Definition 9.11.7. This means that slice connectivity plays nicely with smash products, as stated in Corollary 11.1.28 below.

Definition 11.1.11. *A G-spectrum Y is* **slice n-coconnected** *(called* **slice n-null** *in [HHR16]), written*

$$Y < n \quad \text{or} \quad Y \leqslant n-1$$

if for every slice sphere $\widehat{S}$ with $\dim \widehat{S} \geqslant n$ the space $\mathcal{S}p^G(\widehat{S}, Y)$ is contractible. A G-spectrum X is **slice n-connected** *(***slice n-positive** *in [HHR16]), written*

$$X > n \quad \text{or} \quad X \geqslant n+1$$

if it is in the localizing subcategory (Definition 6.3.12) generated by the set

$$\left\{\widehat{S}(m,K)\colon m|K| > n\right\}, \tag{11.1.12}$$

which we denote by $\mathcal{S}p^G_{>n}$ or $\mathcal{S}p^G_{\geqslant n+1}$.

Similarly, the full subcategory of $\mathcal{S}p^G$ consisting of X with $X < n$ will be denoted by $\mathcal{S}p^G_{<n}$ or $\mathcal{S}p^G_{\leqslant n-1}$. We will use the terms **slice connected** *and* **slice coconnected** *instead of "slice 0-connected" and "slice 0-coconnected."*

The **nth slice layer category** *$\mathcal{S}p^G_{=n}$ (whose objects are* **n-slices** *as in Definition 11.1.42 below) is the intersection*

$$\mathcal{S}p^G_{\geqslant n} \cap \mathcal{S}p^G_{\leqslant n},$$

the category of spectra which are both slice $(n-1)$-connected and $(n+1)$-coconnected.

Remark 11.1.13. A generator for $\mathcal{S}p^G_{>n}$. *As noted in Remark 6.3.15, the generating set of (11.1.12) could be replaced by the singleton consisting of the wedge of all the spectra in the set. In the case at hand we can get by with just a finite wedge of slice spheres. The reader can verify that*

$$S^{\rho_G} \wedge \widehat{S}(m,K) = \widehat{S}(m+|G/K|, K).$$

This means that $\mathcal{S}p^G_{>n}$ is generated by

$$\bigvee_{\substack{K \subseteq G \\ n < m|K| \leqslant n+|G|}} \widehat{S}(m,K).$$

Remark 11.1.14. Notation in other papers. *The category $\mathcal{S}p^G_{\geqslant n}$ as defined above is denoted by $\overline{\tau}^G_n$ in [Ull13], and he denotes by τ^G_n the localizing subcategory generated by*

$$\left\{G \underset{K}{\ltimes} S^{m\rho_K}\colon m|K| \geqslant n\right\} \cup \left\{G \underset{K}{\ltimes} S^{m\rho_K - 1}\colon m|K| - 1 \geqslant n\right\}.$$

This latter subcategory is denoted by $\mathcal{S}p^G_{\geqslant n}$ in [HHR16] and by $\tau^G_{\geqslant n}$ in [Hil12]. Ullman shows [Ull13, Proposition 3.1] that $\Sigma\tau^G_n = \overline{\tau}^G_{n+1}$. We will denote the subcategory generated by the first set above (Ullman's $\overline{\tau}^G_n$) by τ^G_n here; see Definition 11.1.22 and Theorem 11.1.27 below.

Lemma 11.1.15. Contractibility of $\mathcal{S}p_G(\widehat{S}, X)$. *For a G-spectrum X, if for all slice spheres $\widehat{S}$ with $\dim \widehat{S} \geqslant n$, $[\widehat{S}, X]^G = 0$ (in particular if $\mathcal{S}p^G(\widehat{S}, X)$ is contractible), then the G-space $\mathcal{S}p_G(\widehat{S}, X)$ is equivariantly contractible.*

Proof We will prove this by induction on $|G|$, so we need to do it first for trivial G. The statement is that the space $\mathcal{S}p(\Sigma^\infty S^k, X)$ is contractible for $k \geqslant n$ when $[\Sigma^\infty S^\ell, X] = 0$ for $\ell \geqslant n$. Now $\pi_i \mathcal{S}p(\Sigma^\infty S^k, X) = \pi_{i+k} X$, and by hypothesis this group vanishes for $i + k \geqslant n$, i.e., for $i \geqslant n - k$. Since $k \geqslant n$ we have $\pi_i = 0$ for $i \geqslant 0$, so the space $\mathcal{S}p(\Sigma^\infty S^k, X)$ is contractible as desired.

For nontrivial G we may assume by the induction hypothesis that the G-space

$$\mathcal{S}p_G(\widehat{S}, X)$$

is equivariantly contractible for all moving slice spheres $\widehat{S}$ with $\dim \widehat{S} \geqslant n$, and that for all slice spheres $\widehat{S}$ with $\dim \widehat{S} \geqslant n$, and all proper $H \subset G$, the space

$$\mathcal{S}p_G(\widehat{S}, X)^H$$

is contractible. We therefore also know that the G-space

$$\mathcal{S}p_G(T \wedge \widehat{S}, X)$$

is contractible for all slice spheres $\widehat{S}$ with $\dim \widehat{S} \geqslant n$ and all pointed G-CW complexes T which are built entirely from G-cells of the form $G/H \rtimes D^m$ with $H \subset G$ a proper subgroup, and $m \geqslant 0$. Equivalently,

$$\mathcal{S}p_G(T \wedge \widehat{S}, \Sigma X)$$

is contractible for all slice spheres $\widehat{S}$ with $\dim \widehat{S} \geqslant n$ and all G-CW complexes T which are built entirely from moving cells of nonnegative dimension. This condition on a T is equivalent to requiring that $T^G = *$ and that for all proper $H \subset G$, the space T^H be connected.

We must show that the groups $[S^t \wedge S^{m\rho_G}, X]^G = 0$ for $t \geqslant 0$ and $m|G| \geqslant n$. They are zero by assumption when $t = 0$. Let T be quotient G-CW complex

$$T = S^{t\rho_G}/S^t,$$

and consider the exact sequence (see Proposition 9.4.3(i))

$$[S^{t\rho_G} \wedge S^{m\rho_G}, X]^G \to [S^t \wedge S^{m\rho_G}, X]^G \to [T \wedge S^{m\rho_G}, \Sigma X]^G.$$

The leftmost group is zero since $S^{t\rho_G} \wedge S^{m\rho_G}$ is a slice sphere of dimension $(t + m)|G| \geqslant n$. The rightmost group is zero by the induction hypothesis as T is easily checked to be a have the fixed point properties described above. It follows from exactness that the middle group is zero. □

Remark 11.1.16. *The fiber of a map of slice n-connected spectra is not assumed to be slice n-connected, and need not be. For example, the fiber of*

$* \to S^{\rho_G}$ is $S^{\overline{\rho}_G}$ which is not slice $(|G|-1)$-connected, even though both $*$ and S^{ρ_G} are.

Proposition 11.1.17. The slice connectivity of G-cells. *For each $n \geqslant 0$ and each subgroup $H \subseteq G$, The spectrum $G \underset{H}{\ltimes} \Sigma^n S^{-0}$ is in $\mathcal{S}p^G_{\geqslant n}$.*

Proof Since $G \ltimes \Sigma^n S^{-0} = \widehat{S}(n, G)$ is a generator, the statement is true for trivial G, and it suffices to prove that $G \underset{H}{\ltimes} \Sigma^n S^{-0}$ is in $\mathcal{S}p^G_{\geqslant n}$ for each nontrivial subgroup H. We do this by induction on $|G|$. The inductive hypothesis gives it to us for each proper subgroup, so it suffices to show it for $\Sigma^n S^{-0}$.

For this we use the cofiber sequence

$$S(n\overline{\rho}_G)_+ \to S^0 \to S^{n\overline{\rho}_G}$$

in $\mathcal{T}^G$. (Compare with the discussion in Example 8.5.17.) Smashing with $\Sigma^n S^{-0}$ gives

$$S(n\overline{\rho}_G) \ltimes \Sigma^n S^{-0} \to \Sigma^n S^{-0} \to S^{n\rho_G} \wedge S^{-0}.$$

Now the spectrum on the left is made entirely of moving G-cells (Definition 8.4.14) and is therefore in $\mathcal{S}p^G_{\geqslant n}$ by induction, while the one on the right is a generator of $\mathcal{S}p^G_{\geqslant |G|n}$. It follows that $\Sigma^n S^{-0}$ is $\mathcal{S}p^G_{\geqslant n}$ as claimed. □

For $n = 0$ and $n = 1$, the notions of slice n-coconnected and slice n-connected coincide with the usual notions of connectivity and coconnectivity.

Proposition 11.1.18. *For a G-spectrum X the following hold*

(i) $X \geqslant 0 \iff X$ *is (-1)-connected, i.e.* $\underline{\pi}_k X = 0$ *for* $k < 0$*;*
(ii) $X < 0 \iff X$ *is 0-coconnected, i.e.* $\underline{\pi}_k X = 0$ *for* $k \geqslant 0$*;*
(iii) $X \geqslant 1 \iff X$ *is 0-connected, i.e.* $\underline{\pi}_k X = 0$ *for* $k < -1$*;*
(iv) $X < 1 \iff X$ *is 1-coconnected, i.e.* $\underline{\pi}_k X = 0$ *for* $k \geqslant -1$*;*

Proof The first two statements are equivalent, so we only need to prove the first one. The only generators of $\mathcal{S}p^G_{\geqslant 0}$ which are not 0-connected are the spectra $\widehat{S}(0, H) = G \underset{H}{\ltimes} S^{-0}$. This means that the category contains all (-1)-connected G-CW spectra and hence all (-1)-connected G-spectra.

The last two statements are also equivalent, so we need only prove the third. We will do so by showing that $G \underset{H}{\ltimes} \Sigma S^{-0}$ is in $\mathcal{S}p^G_{\geqslant 1}$ for each H. We do this by induction on $|G|$.

Since $G \ltimes \Sigma S^{-0} = \widehat{S}(1, G)$ is a generator, the statement is true for trivial G, and it suffices to prove that $G \underset{H}{\ltimes} \Sigma S^{-0}$ is in $\mathcal{S}p^G_{\geqslant 1}$ for each nontrivial subgroup H. The inductive hypothesis gives it to us for each proper subgroup, so it suffices to show it for ΣS^{-0}. This is the case $n = 1$ of Proposition 11.1.17. □

Remark 11.1.19. Slice connectivity and ordinary connectivity. *It is not the case that if $Y > 0$ then $\underline{\pi}_0 Y = 0$. In Proposition 11.1.37 we will see that the fiber F of $S^0 \to H\underline{\mathbf{Z}}$ has the property that $F > 0$. On the other hand $\underline{\pi}_0 F$ is the augmentation ideal of the Burnside ring. Proposition 11.3.3 below gives a characterization of slice connected spectra.*

The classes of slice n-coconnected and slice n-connected spectra are preserved under change of group.

Proposition 11.1.20. The effect of restriction and induction on slice connectivity. *Suppose $H \subset G$, that X is a G-spectrum and Y is an H-spectrum. The following implications hold*

$$\begin{aligned} X > n &\implies i_H^G X > n \\ X < n &\implies i_H^G X < n \\ Y > n &\implies G \underset{H}{\ltimes} Y > n \\ Y < n &\implies G \underset{H}{\ltimes} Y < n. \end{aligned}$$

Proof The second and third implications are straightforward consequences of Proposition 11.1.9. The fourth implication follows from the stable analog of the Wirthmüller isomorphism (8.0.16) and Proposition 11.1.9, and the first implication is an immediate consequence of the fourth. □

11.1D Slice connectivity and geometric connectivity

The following definition was introduced by the first author and Carolyn Yarnall in [HY18]. Recall Definition 9.11.7 of the geometric fixed point spectrum $\Phi^H X$ for a subgroup $H \subseteq G$ and the corresponding notion of geometric connectivity.

For a rational number x, we will denote the largest integer not exceeding x by $\lfloor x \rfloor$, the **floor of** x, and the smallest integer not exceeded by x by $\lceil x \rceil$, the **ceiling of** x. Note that

$$\lfloor x + y \rfloor \geqslant \lfloor x \rfloor + \lfloor y \rfloor, \quad \lceil x + y \rceil \leqslant \lceil x \rceil + \lceil y \rceil \quad \text{and} \quad \lfloor -x \rfloor = -\lceil x \rceil. \tag{11.1.21}$$

Definition 11.1.22. Some localizing subcategories of $\mathcal{S}p^G$. *Let τ_n^G be the full subcategory of $\mathcal{S}p^G$ whose objects are G-spectra X satisfying*

$$\pi_k \Phi^H X = 0 \qquad \text{for } k < n/|H|.$$

Proposition 11.1.23. Properties of τ_n^G.

(i) The subcategory τ_n^G is a localizing subcategory (Definition 6.3.12) of $\mathcal{S}p^G$.
(ii) The spectrum $\Sigma^\infty S^{\rho_G}$ is in $\tau_{|G|}^G$, and $S^{-\rho_G}$ is in $\tau_{-|G|}^G$.
(iii) If X is in τ_m^G and Y is in τ_n^G, then $X \wedge Y$ is in τ_{m+n}^G.

(iv) For each integer n there is an equivalence of categories $\tau_n^G \to \tau_{n+|G|}^G$ given by $X \mapsto X \wedge S^{\rho_G}$ with inverse given by $Y \mapsto Y \wedge S^{-\rho_G}$.

Proof The first statement follows immediately from the definitions and (ii) is a consequence of Theorem 9.11.8 (iii).

For (iii), observe that if X is in τ_m^G, the first nontrivial homotopy group of $\Phi^H X$ occurs in dimension at least $\lceil m/|H| \rceil$ for each $H \subseteq G$. Hence that of $(\phi^H X) \wedge (\phi^H Y)$ occurs in dimension at least $\lceil m/|H| \rceil + \lceil n/|H| \rceil$. Since

$$\lceil m/|H| \rceil + \lceil n/|H| \rceil \geqslant \lceil (m+n)/|H| \rceil$$

and

$$\Phi^H(X \wedge Y) \simeq (\Phi^H X) \wedge (\Phi^H Y)$$

by Theorem 9.11.8 (iv), $\Phi^H(X \wedge Y)$ has the required connectivity.

The last statement follows easily from (ii) and (iii). □

Proposition 11.1.24. Geometric connectivity of $\mathcal{S}p_{\geqslant n}^G$. *The localizing subcategory $\mathcal{S}p_{\geqslant n}^G$ as in Definition 11.1.11 is contained in the category τ_n^G of Definition 11.1.22.*

Proof We know that X is in $\mathcal{S}p_{\geqslant n}^G$ iff $S^{\rho_G} \wedge X$ is in $\mathcal{S}p_{\geqslant n+|G|}^G$; see Remark 11.1.13. Similarly, X is in τ_n^G iff $S^{\rho_G} \wedge X$ is in $\tau_{n+|G|}^G$ by Proposition 11.1.23 (iv). Hence it suffices to treat the case $n \geqslant 0$.

Since $\widehat{S}(m, K)$ for $m \geqslant 0$ is a suspension spectrum, the connectivity of its geometric fixed point set Φ^H coincides with that of the ordinary fixed points of the corresponding space. These are easily seen to be what is needed to place $\widehat{S}(m, K)$ in $\tau_{m|K|}$. □

We want to show the converse of Proposition 11.1.24, i.e., that $\tau_n^G \subseteq \mathcal{S}p_{\geqslant n}^G$, so that geometric connectivity coincides with slice connectivity. We will argue by induction on $|G|$, the statement being immediate for trivial G.

To proceed further we need two lemmas.

Lemma 11.1.25. Using the inductive hypothesis. *If $\tau_n^H = \mathcal{S}p_{\geqslant n}^H$ for all proper subgroups $H \subset G$, Y is in τ_n^G and $\Phi^G Y$ is weakly contractible, then Y is in $\mathcal{S}p_{\geqslant n}^G$.*

Proof If $\phi^G Y$ is weakly contractible, then Y is equivalent to $E\mathcal{P} \ltimes Y$ by Proposition 9.11.11(iv). Thus Y is built out of moving H-cells as in Definition 8.4.14. By hypothesis these are induced up from cells in $\mathcal{S}p_{\geqslant n}^H$, so they are in $\mathcal{S}p_{\geqslant n}^G$ by Proposition 11.1.20. The conclusion follows. □

Lemma 11.1.26. The inductive step. *If Y is in τ_n^G and $\Phi^H Y$ is weakly contractible for all proper subgroups $H \subseteq G$ then Y is in $\mathcal{S}p_{\geqslant n}^G$.*

Proof The hypotheses implies that $\pi_k \Phi^G Y = 0$ for $k < n/|G|$ and hence for

$k < \lceil n/|G|\rceil$. This means that Y is in the smaller subcategory $\tau^G_{m|G|}$ where $m = \lceil n/|G|\rceil$.

We also know that $Y \simeq \widetilde{E}\mathcal{P} \wedge Y$ by Proposition 9.11.11 (iii). Hence Proposition 9.11.12 implies that for all proper subgroups $H \subseteq G$ and all integers ℓ, $\mathcal{S}p^G(G \underset{H}{\ltimes} S^\ell, Y)$ is contractible and the inclusion map $S^\ell \to \widehat{S}(\ell, G)$ induces an isomorphism $[\widehat{S}(\ell, G), Y]^G \to \pi^G_\ell Y = \pi_\ell \Phi^G Y$.

This tells us that for such Y, the Postnikov filtration coincides, after rescaling by a factor of $|G|$, with both the slice filtration and the geometric connectivity filtration. Therefore Y is in $\mathcal{S}p^G_{\geqslant m|G|}$ and hence in the larger subcategory $\mathcal{S}p^G_{\geqslant n}$. □

Theorem 11.1.27. Geometric connectivity characterization of $\mathcal{S}p^G_{\geqslant n}$. *The localizing subcategories $\mathcal{S}p^G_{\geqslant n}$ (Definition 11.1.11) and τ^G_n (Definition 11.1.22) are equal.*

Proof We know that $\mathcal{S}p^G_{\geqslant n} \subseteq \tau^G_n$ by Proposition 11.1.24. We will prove the converse by induction on $|G|$. To start the induction, note that for trivial G each category is that of $(n-1)$-connected spectra.

For the inductive step, let X be in τ^G_n and consider the isotropy separation sequence of §9.11A,

$$E\mathcal{P} \ltimes X \to X \to \widetilde{E}\mathcal{P} \wedge X,$$

where $E\mathcal{P}$ is the space of Definition 8.6.15. Since τ^G_n is closed under homotopy colimits, both the left and right spectra are in it, and their restrictions are in $\tau^H_n = \mathcal{S}p^H_{\geqslant n}$ for each proper subgroup H. We claim the spectra on the left and right satisfy the hypotheses of Lemma 11.1.25 and Lemma 11.1.26 respectively. It follows that X is in $\mathcal{S}p^G_{\geqslant n}$.

The claim about $Y = E\mathcal{P} \ltimes X$ is that $\Phi^G Y$ is weakly contractible. For this we have

$$\begin{aligned}
\Phi^G(E\mathcal{P} \ltimes X) &\simeq \Phi^G\Sigma^\infty E\mathcal{P} \ltimes \Phi^G X && \text{by Theorem 9.11.8(iv)}\\
&\simeq \Sigma^\infty E\mathcal{P}^G_+ \wedge \Phi^G X && \text{by Theorem 9.11.8(iii)}\\
&\simeq * \wedge \Phi^G X \simeq * && \text{by (9.11.3).}
\end{aligned}$$

The claim about $Y = \widetilde{E}\mathcal{P} \wedge X$ is that $\Phi^H Y$ is weakly contractible for all proper subgroups H. For this we have

$$\begin{aligned}
\Phi^H(\widetilde{E}\mathcal{P} \wedge X) &\simeq \Phi^H\Sigma^\infty \widetilde{E}\mathcal{P} \wedge \Phi^H X && \text{by Theorem 9.11.8(iv)}\\
&\simeq \Sigma^\infty \widetilde{E}\mathcal{P}^H \wedge \Phi^H X && \text{by Theorem 9.11.8(iii)}\\
&\simeq * \wedge \Phi^H X \simeq * && \text{by (9.11.3).}
\end{aligned}$$
□

With the above characterization of $\mathcal{S}p^G_{\geqslant n}$ in hand, the following is a consequence of Proposition 11.1.23(iii).

Corollary 11.1.28. Slice connectivity of smash products. *If X is in $\mathcal{S}p^G_{\geqslant m}$ and Y is in $\mathcal{S}p^G_{\geqslant n}$, then $X \wedge Y$ is in $\mathcal{S}p^G_{\geqslant m+n}$.*

Another advantage of this characterization of $\mathcal{S}p^G_{\geqslant n}$ is that it enables us to say which of these subcategories contain other representation spheres.

Proposition 11.1.29. Slice connectivity of representation spheres and Yoneda spectra. *Let V be a representation of G of degree d.*

(i) $|V^H| \geqslant \lfloor d/|H| \rfloor$ *for all subgroups* $H \subseteq G$ *iff* $\Sigma^\infty S^V$ *is in* τ^G_d.
(ii) $|V^H| \leqslant \lfloor d/|H| \rfloor$ *for all subgroups* $H \subseteq G$ *iff* S^{-V} *is in* τ^G_{-d}.

Proof For (i), if the conditions on $|V^H|$ are met, we have $\pi_k \Phi^H \Sigma^\infty S^V = 0$ for $k < |V^H|$ because

$$\Phi^H \Sigma^\infty S^V \simeq \Sigma^\infty S^{V^H} \qquad \text{by Theorem 9.11.8(iii).}$$

Since $|V^H| \geqslant \lfloor d/|H| \rfloor$, this implies that $\pi_k \Phi^H \Sigma^\infty S^V = 0$ for $k < d/|H|$, so $\Sigma^\infty S^V$ is in τ^G_d. We leave the converse to the reader.

For (ii), if the conditions on $|V^H|$ are met, Theorem 9.11.8(iii) gives $\Phi^H S^{-V} \simeq S^{-V^H}$, so $\pi_k \Phi^H S^{-V} = 0$ for $k < -|V^H|$. We also know that

$$|V^H| \leqslant \lfloor d/|H| \rfloor \qquad \text{implies} \qquad -|V^H| \geqslant -\lfloor d/|H| \rfloor = \lceil -d/|H| \rceil ,$$

so $\pi_k \Phi^H S^{-V} = 0$ for $k < \lceil -d/|H| \rceil$ and hence for $k < -d/|H|$. Again we leave the converse to the reader. ∎

Remark 11.1.30. The smallest τ^G_n containing $\Sigma^\infty S^V$ and S^{-V}. *The suspension spectrum $\Sigma^\infty S^V$ as in Proposition 11.1.29 is not in τ^G_{d+1} since $\pi^u_d \Sigma^\infty S^V = \mathbf{Z}$. Similarly S^{-V} is not in τ^G_{1-d}. If the conditions on $|V^H|$ are not met, then the largest d' with $\Sigma^\infty S^V$ in $\tau^G_{d'}$ is some number less that d. A similar statement holds for S^{-V}.*

Remark 11.1.31. The spectra $G \underset{K}{\ltimes} \Sigma^\infty S^V$ and $G \underset{K}{\ltimes} S^{-V}$ for a representation V of a proper subgroup $K \subset G$. *Similar statements to those of Proposition 11.1.29 about these spectra, with conditions on $|V^H|$ for $H \subseteq K$, can be proved in a similar fashion. For either of them, the geometric fixed point set $\Phi^{H'}(-)$ for $K \subset H' \subseteq G$ is contractible. We leave the details to the reader.*

Corollary 11.1.32. Smashing with representation spheres. *Suppose there is a representation V of degree d and an integer n such that*

$$\left\lceil \frac{n}{|H|} \right\rceil + |V^H| = \left\lceil \frac{n+d}{|H|} \right\rceil \qquad \textit{for all } H \subseteq G.$$

Then $S^V \wedge (-) : \tau^G_n \to \tau^G_{n+d}$ is an equivalence of categories whose inverse is $S^{-V} \wedge (-)$.

Proof The defining condition for $X \in \tau_n^G$, namely $\pi_k \Phi^H X = 0$ for $k < n/|H|$, is equivalent to $\pi_k \Phi^H X = 0$ for $k < \lceil n/|H| \rceil$, and similarly for $S^V \wedge X \in \tau_{n+d}^G$. □

The situation when the conditions of Corollary 11.1.32 are met for two adjacent values of n is the subject of Proposition 11.1.48 below.

Corollary 11.1.33. Relations between slice connective covers and between slice sections. *Suppose n and V satisfy the hypothesis of Corollary 11.1.32. Then the natural maps*

$$S^V \wedge P_{n+1} X \to P_{n+d+1} \left(S^V \wedge X \right)$$
$$S^V \wedge P^n X \to P^{n+d} \left(S^V \wedge X \right)$$

are weak equivalences.

In particular the natural maps

$$S^{m\rho_G} \wedge P_{k+1} X \to P_{k+m|G|+1} \left(S^{m\rho_G} \wedge X \right)$$
$$S^{m\rho_G} \wedge P^k X \to P^{k+m|G|} \left(S^{m\rho_G} \wedge X \right)$$

are weak equivalences for all m and k.

Example 11.1.34. Some equivalences among the subcategories τ_n^G.

(i) *Let G be any finite group and $V = \rho_G$. Then the conditions of Corollary 11.1.32 hold for any n. Hence $S^{\rho_G} \wedge (-)$ induces an equivalence between τ_n^G and $\tau_{n+|G|}^G$ for all n. This is a restatement of Proposition 11.1.23(iv).*

(ii) *Let G be any finite group and $V = \overline{\rho}_G$, the reduced regular representation. The conditions of Corollary 11.1.32 hold for any n congruent to 1 mod $|G|$. Hence $S^{\overline{\rho}_G} \wedge (-)$ induces an equivalence between τ_1^G and $\tau_{|G|}^G$.*

(iii) *Let $G = C_2$. Then the two previous examples show that each τ_n^G is equivalent to τ_0^G.*

(iv) *Let $G = C_4$. Then $V = \sigma$ leads to an equivalence between τ_2^G and τ_3^G, while $V = \overline{\rho}_G$ (the reduced regular representation) leads to one between τ_1^G and τ_4^G. Hence each τ_n^G is equivalent to either τ_0^G or τ_2^G.*

(v) *Let $G = C_8$. Let σ be the sign representation and let λ and λ' be rotations of order 8 and 4 respectively. Then the representations σ, $\sigma + \lambda$, $\sigma + \lambda + \lambda'$ and $\overline{\rho} = \sigma + 2\lambda + \lambda'$ lead respectively to equivalences $\tau_4^G \to \tau_5^G$, $\tau_3^G \to \tau_6^G$, $\tau_2^G \to \tau_7^G$ and $\tau_1^G \to \tau_8^G$. Thus there are four equivalence classes corresponding the four even values of n mod 8.*

(vi) *Let $G = C_p$ for p an odd prime, and let $V = \lambda$, a 2-dimensional rotation matrix of order p. Then the conditions of Corollary 11.1.32 hold provided n is not congruent to 0 or -1 mod p. Hence we get equivalences*

$$\tau_1^G \to \tau_3^G \to \cdots \to \tau_p^G \qquad \text{and} \qquad \tau_2^G \to \tau_4^G \to \cdots \to \tau_{p-1}^G.$$

Combining these with the first example shows that each τ_n^G is equivalent to τ_1^G or τ_2^G.

(vii) *Let $G = C_{p^2}$ for p an odd prime, and let λ and λ' denote rotations of orders p^2 and p respectively. Then $V = \lambda$ leads to equivalences $\tau_n^G \to \tau_{n+2}^G$ for n not congruent to 0 or -1 mod p. Similarly $V = (p-1)\lambda + \lambda'$ leads to equivalences $\tau_n^G \to \tau_{n+2p}^G$ for $1 \leqslant n \leqslant p^2 - 2p$. Thus there are four equivalence classes, those of $\tau_{1+a_0+a_1 p}^G$ for $0 \leqslant a_0, a_1 \leqslant 1$.*

11.1E The slice tower

Let $P^n X$ be the Bousfield localization, or Dror Farjoun nullification (Theorem 6.3.17) of X with respect to τ_{n+1}^G (Definition 11.1.22), and $P_{n+1}X$ the homotopy fiber (as in Definition 5.8.43) of $X \to P^n X$. Equivalently (by Theorem 11.1.27) it is localization with respect to the subcategory $\mathcal{S}p_{>n}^G$ of Definition 11.1.11. Thus, by definition, there is (up to weak equivalence) a functorial fiber sequence

$$P_{n+1}X \to X \to P^n X.$$

Definition 11.1.35. *The spectra $P^n X$ and $P_{n+1}X$ are respectively the n***th slice section** *of and* **slice n-connected cover** *of X.*

The functor $P^n X$ can be constructed (up to weak equivalence) as the colimit of a sequence of functors

$$W_0 X \to W_1 X \to \cdots .$$

The $W_i X$ are defined inductively starting with $W_0 X = X$, and taking $W_k X$ to be the cofiber of

$$\bigvee_{L_k} \widehat{S} \to W_{k-1}X,$$

in which the indexing set L_k is the union of the sets $[\widehat{S}, \operatorname*{colim}_k W_{k-1}X]^G$ over slice spheres $\widehat{S}$ of dimensions $> n$. Equivalently, W_k is the pushout in the diagram (compare with Quillen's diagram (4.2.11))

$$\begin{tikzcd} \bigvee_{L_k} \widehat{S} \arrow[r] \arrow[d] & W_{k-1} \arrow[d] \\ \bigvee_{L_k} C\widehat{S} \arrow[r] & W_k, \arrow[ul, phantom, "\lrcorner"] \end{tikzcd} \tag{11.1.36}$$

where $C\widehat{S}$ denotes the cone on $\widehat{S}$. Then we have

$$[\widehat{S}, \operatorname*{colim}_k W_k X]^G = 0$$

for such $\widehat{S}$, by Lemma 11.1.15 the G-space $\mathcal{S}p_G(\widehat{S}, \operatorname*{colim}_k W_k X)$ is weakly contractible.

Proposition 11.1.37. A filtration for slice n-connected spectra. *A spectrum X is slice n-connected if and only if it admits (up to weak equivalence) a filtration*

$$X_0 \subset X_1 \subset \cdots$$

whose associated graded spectrum $\bigvee X_k/X_{k-1}$ is a wedge of slice spheres of dimension greater than n. For any spectrum X, $P_{n+1}X$ is slice n-connected.

Proof This follows easily from the construction of $P^n X$ described above. □

The map $P_{n+1}X \to X$ is characterized up to a contractible space of choices by the properties

i) for all X, $P_{n+1}X \in \tau_{n+1}^G$;
ii) for all $A \in \tau_{n+1}^G$ and all X, the map $\mathcal{S}p_G(A, P_{n+1}X) \to \mathcal{S}p_G(A, X)$ is a weak equivalence of G-spaces.

In other words, $P_{n+1}X \to X$ is the "universal map" from an object of τ_{n+1}^G to X. Similarly $X \to P^n X$ is the universal map from X to a slice $(n+1)$-coconnected G-spectrum Z. More specifically

iii) the spectrum $P^n X$ is slice $(n+1)$-coconnected;
iv) for any slice $(n+1)$-coconnected Z, the map

$$\mathcal{S}p_G(P^n X, Z) \to \mathcal{S}p_G(X, Z)$$

is a weak equivalence.

These conditions lead to a useful recognition principle.

Lemma 11.1.38. Recognition of the nth slice section. *Suppose X is a G-spectrum and that*

$$\widetilde{P}_{n+1} \to X \to \widetilde{P}^n$$

is a fiber sequence with the property that $\widetilde{P}^n \leqslant n$ and $\widetilde{P}_{n+1} > n$. Then the canonical maps $\widetilde{P}_{n+1} \to P_{n+1}X$ and $P^n X \to \widetilde{P}^n$ are weak equivalences.

Proof We show that the map $X \to \widetilde{P}^n$ satisfies the universal property of $P^n X$. Suppose that $Z \leqslant n$, and consider the fiber sequence of G-spaces

$$\mathcal{S}p_G(\widetilde{P}^n, Z) \to \mathcal{S}p_G(X, Z) \to \mathcal{S}p_G(\widetilde{P}_{n+1}, Z)$$

The rightmost space is contractible since $\widetilde{P}_{n+1} > n$, so the left map is a weak equivalence. □

The following consequence of Lemma 11.1.38 is used in the proof of the Reduction Reduction Theorem 12.4.8.

Corollary 11.1.39. Cofibrations of slice connected covers. *Suppose that $X \to Y \to Z$ is a cofiber sequence, and that the mapping cone of $P^n X \to P^n Y$ is slice $(n+1)$-coconnected. Then both*

$$P^n X \to P^n Y \to P^n Z$$

and

$$P_{n+1} X \to P_{n+1} Y \to P_{n+1} Z$$

are cofiber sequences.

Remark 11.1.40. The functors P^n and P_{n+1} do not preserve cofiber sequences in general. *The hypothesis about the coconnectivity of the cofiber of $P^n X \to P^n Y$ in Corollary 11.1.39 is essential. In the nonequivariant case consider the cofiber sequence*

$$\Sigma^\infty S^n \to * \to \Sigma^\infty S^{n+1}. \tag{11.1.41}$$

Applying the functor P^n to the first map gives

$$\Sigma^n H\mathbf{Z} \to *,$$

whose cofiber $\Sigma^{n+1} H\mathbf{Z}$ is not $(n+1)$-coconnected. Applying the functor P^n to (11.1.41) gives

$$\Sigma^n H\mathbf{Z} \to * \to *,$$

which is **not** *a cofiber sequence. The functors P_{n+1} and P_n^n (see Definition 11.1.43 below) also fail to give cofiber sequences.*

Proof of Corollary 11.1.39 Consider the diagram

$$\begin{tikzcd}
P_{n+1}X \arrow[r] \arrow[d] & P_{n+1}Y \arrow[r] \arrow[d] & \widetilde{P}_{n+1}Z \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
P^n X \arrow[r] & P^n Y \arrow[r] & \widetilde{P}^n Z
\end{tikzcd}$$

in which the rows and columns are cofiber sequences. By construction, $\widetilde{P}_{n+1} Z$ is slice n-connected since τ_{n+1}^G is closed under cofibers. If $\widetilde{P}^n Z \leqslant n$ then the right column satisfies the condition of Lemma 11.1.38, and the result follows. □

Since $\tau_{n+1}^G \subset \tau_n^G$, there is a natural transformation

$$P^n X \to P^{n-1} X.$$

Definition 11.1.42. *The* **slice tower** *of X is the tower $\{P^n X : n \in \mathbf{Z}\}$.*

When considering more than one group, we will write

$$P^n X = P^n_G X \qquad \text{and} \qquad P_n X = P^G_n X.$$

The risk of ambiguity here is minimal since the image of τ^G_n under the restriction functor i^G_H is τ^H_n. On the other hand, the image of τ^H_n under the right adjoint $G \underset{H}{\ltimes} (-)$ is not τ^G_n, but the subcategory of it generated by slice spheres induced up from H.

Let $P^n_n X$ be the fiber of the map

$$P^n X \to P^{n-1} X.$$

Definition 11.1.43. *The n-**slice** of a spectrum X is $P^n_n X$. A spectrum X is an n-**slice** if $X = P^n_n X$.*

The following is a consequence of Corollary 11.1.39.

Corollary 11.1.44. Cofibrations of slices. *Suppose the hypothesis of Corollary 11.1.39 holds for both $n = m$ and $n = m + 1$. Then the cofiber of $P^m_m X \to P^m_m Y$ is equivalent to $P^m_m Z$.*

The spectrum $P_n X$ is analogous to the $(n-1)$-connected cover of X, and for $n = 0$ they coincide. The following is a straightforward consequence of Proposition 11.1.18.

Proposition 11.1.45. The (-1)-connected slice cover. *For any spectrum X, $P_0 X$ is the (-1)-connected cover of X. The 0-slice of X is given by*

$$P^0_0 X = H\underline{\pi}_0 X.$$

The formation of slice sections and therefore of the slices themselves behave well with respect to change of group.

Proposition 11.1.46. Slice sections and change of group. *The functor P^n commutes with both restriction to a subgroup and induction. More precisely, given $H \subset G$ there are natural weak equivalences*

$$i^G_H(P^n_G X) \to P^n_H(i^G_H X)$$

and

$$G \underset{H}{\ltimes} (P^n_H X) \to P^n_G(G \underset{H}{\ltimes} X).$$

Proof This is follows from Lemma 11.1.38 and Proposition 11.1.20. $\square$

Remark 11.1.47. *When G is the trivial group the slice spheres are just ordinary spheres and the slice tower becomes the Postnikov tower. It therefore follows from Proposition 11.1.46 that the tower of non-equivariant spectra underlying the slice tower is the Postnikov tower.*

Proposition 11.1.48. Relations among slices. *Let V be a representation of degree d of a finite group G. Suppose that m is an integer such that the conditions of Corollary 11.1.32 are met for both $n = m$ and $n = m + 1$. Then the smash product of S^V with any m-slice (Definition 11.1.43) is an $(m + d)$-slice, and that of S^{-V} with any $(m + d)$-slice is an m-slice.*

Proof The hypotheses imply that

$$S^V \wedge P^{m-1}X \simeq P^{m+d-1}(S^V \wedge X) \text{ and } S^V \wedge P^m X \simeq P^{m+d}(S^V \wedge X),$$

so

$$S^V \wedge P^m_m X \simeq P^{m+d}_{m+d}(S^V \wedge X). \qquad \square$$

The following special cases follow from parts (i), (vi) and (vii) of Example 11.1.34.

Corollary 11.1.49. Some specific slice relations.

(i) For any integer n and any finite group G, X is an n-slice iff $S^{\rho_G} \wedge X$ is an $(n + |G|)$-slice. Thus any slice is equivalent to the smash product of some power of S^{ρ_g} with a k-slice for $0 \leqslant k < |G|$.

(ii) Let $G = C_p$ or C_{p^2} for a prime $p \geqslant 5$. Then for n not congruent to 0, -1 or -2 mod p (equivalently for $\binom{n+2}{3}$ not divisible by p), X is an n-slice iff $S^\lambda \wedge X$ is an $(n + 2)$-slice. For $G = C_p$, each k-slice for $0 < k < p$ can be obtained from a from a 1-slice or a 2-slice by smashing with a power of S^λ. For $G = C_{p^2}$, each $k_0 + k_1 p$-slice for $0 < k_0 < p$ and $0 \leqslant k_1 < p$ can be obtained from a from a $(1 + k_1 p)$-slice or a $(2 + k_1 p)$-slice by smashing with a power of S^λ.

(iii) Let C_{p^2} for a prime $p \geqslant 5$ and let V be as in Example 11.1.34(vii). Then X is an n-slice iff $S^V \wedge X$ is an $(n + 2p)$-slice when $n \equiv k$ mod p^2 for $1 \leqslant k \leqslant p^2 - 2p - 1$. In particular each $(k_0 + k_1 p)$-slice for $1 \leqslant k_0 \leqslant 2$ and $0 \leqslant k_1 < p$ can be obtained from a k_0-slice or a $(k_0 + p)$-slice by smashing with a power of S^V, and each $k_1 p$-slice for $0 < k_1 < p$ can be so obtained from a p-slice or a $2p$-slice.

11.1F Multiplicative properties of the slice tower

The theme of this section is that the functor P^n for $n \geqslant 0$ plays nicely with multiplicative structures on connective spectra. One important result is Corollary 11.1.54 asserting that the slice sections of a (-1)-connected homotopy commutative or associative algebra have similar properties. Three more precise results along these lines, which will be proved in §11.4, are stated for convenience here as Proposition 11.1.55, Proposition 11.1.56 and Proposition 11.1.57.

The following definition should be compared to Definition 6.2.1.

Definition 11.1.50. *A map $X \to Y$ is a* **slice P^n-equivalence** *if $P^nX \to P^nY$ is a weak equivalence. Equivalently, $X \to Y$ is a slice P^n-equivalence if for every $Z < n$, the map*

$$\mathcal{S}p_G(Y, Z) \to \mathcal{S}p_G(X, Z)$$

is a weak equivalence.

Lemma 11.1.51. The fiber of a slice P^n-equivalence. *If the homotopy fiber F of $f : X \to Y$ is in τ^G_{n+1}, then f is a slice P^n-equivalence.*

Proof This follows immediately from the fiber sequence

$$\mathcal{S}p_G(Y, Z) \to \mathcal{S}p_G(X, Z) \to \mathcal{S}p_G(F, Z). \qquad \square$$

Remark 11.1.52. *The converse of the above result is not true. For instance, $* \to S^0$ is a P^{-1}-equivalence, but the fiber S^{-1} is not in τ^G_0.*

Lemma 11.1.53. The smash product of a slice P^n-equivalence with a slice connected spectrum.

(i) If $X \to Y$ is a slice P^n-equivalence and $Z \geqslant 0$, then $X \wedge Z \to Y \wedge Z$ is a slice P^n-equivalence;

(ii) For $X_1, \ldots, X_k \in \mathcal{S}p^G_{\geqslant 0}$, the map

$$X_1 \wedge \cdots \wedge X_k \to P^nX_1 \wedge \cdots \wedge P^nX_k$$

is a slice P^n-equivalence.

Proof Since $P_{n+1}X$ and $P_{n+1}Y$ are both slice n-connected, the vertical maps in the square below are slice P^n-equivalences by Lemma 11.1.51 and Proposition 11.1.23(iv).

$$\begin{tikzcd} X \wedge Z \arrow[r] \arrow[d] & Y \wedge Z \arrow[d] \\ P^nX \wedge Z \arrow[r] & P^nY \wedge Z . \end{tikzcd}$$

The bottom row is a weak equivalence by assumption. It follows that the top row is a slice P^n-equivalence.

The second assertion is proved by induction on k, the case $k = 1$ being trivial. For the induction step consider

$$\begin{tikzcd} X_1 \wedge \cdots \wedge X_{k-1} \wedge X_k \arrow[r] & P^nX_1 \wedge \cdots \wedge P^nX_{k-1} \wedge X_k \arrow[d] \\ & P^nX_1 \wedge \cdots \wedge P^nX_{k-1} \wedge P^nX_k. \end{tikzcd}$$

The first map is a slice P^n-equivalence by the induction hypothesis and part 1. The second map is a slice P^n-equivalence by part 1. $\square$

Corollary 11.1.54. Multiplicative structures preserved by P^n. *Let R be a (-1)-connected G-spectrum. If R is a homotopy commutative or homotopy associative algebra, then so is P^nR for all n.*

The following additional results are proved in §11.4. The first two are Proposition 11.4.4 and Proposition 11.4.10, while the third, which is a more precise variant of Corollary 11.1.54, is easily deduced from Theorem 11.4.13.

Proposition 11.1.55. Slice connectivity is preserved by the norm. *Suppose that $n \geqslant 0$ is an integer. If A is a slice $(n-1)$-connected H-spectrum then $N_H^G A$ is a slice $(n-1)$-connected G-spectrum.*

Proposition 11.1.56. Slice connectivity is preserved by symmetric powers. *Suppose that $n \geqslant 0$ is an integer. If A is a slice $(n-1)$-connected G-spectrum then for every $m > 0$, the symmetric smash power $\mathrm{Sym}^m A$ is slice $(n-1)$-connected.*

Proposition 11.1.57. Equivariant commutativity is preserved by P^n. *Suppose that $n \geqslant 0$ is an integer. If R is a (-1)-connected equivariant commutative ring, then the slice section P^nR can be given the structure of an equivariant commutative ring in such a way that $R \to P^nR$ is a commutative ring homomorphism. Moreover this commutative ring structure is unique.*

This result insures that the slice spectral sequence, to be studied in the next section, is one of algebras when applied to a commutative ring spectrum such as $MU_{\mathbf{R}}$ and its norms, the subject of the next chapter. This means that differentials are derivations as expected. This will enable us to prove Theorem 13.3.23 and hence the Periodicity Theorem of §1.1C.

11.2 The slice spectral sequence

The **slice spectral sequence** is the homotopy spectral sequence of the slice tower of Definition 11.1.42. The main point of this section is to establish strong convergence of the slice spectral sequence, and to show that for any X, the E_2-term is distributed in the gray region of Figure 11.1, as explained in Corollary 11.2.12. We begin with some results relating the slice sections to Postnikov sections.

11.2A Connectivity and the slice filtration

Our convergence result for the slice spectral sequence depends on knowing how slice spheres are constructed from G-cells.

Definition 11.2.1. Decomposition of a G-spectrum. *A space or spectrum X* **decomposes** *into the elements of a collection of spaces or spectra $\{T_\alpha\}$ if X is weakly equivalent to a spectrum $\tilde{X}$ admitting an increasing filtration*

$$X_0 \subset X_1 \subset \cdots$$

with the property that X_n/X_{n-1} (with $X_{-1} = *$) is weakly equivalent to a wedge of T_α.

Remark 11.2.2. *The suspension spectrum of a G-CW complex decomposes into the collection of spectra $\{\Sigma^\infty G/H \ltimes S^m \mid H \subseteq G, m \geqslant 0\}$. More generally, a $(n-1)$-connected G-spectrum X decomposes into the collection of spectra*

$$\{\Sigma^\infty G/H \ltimes S^m \mid H \subseteq G, m \geqslant n\}.$$

Remark 11.2.3. *To say that X decomposes into the elements of a collection of compact objects $\{T_\alpha\}$ means that X is in the localizing subcategory (Definition 6.3.12) generated by the T_α.*

Lemma 11.2.4. The cellular structure of slice spheres. *For $m \geqslant 0$, $\widehat{S}(m, K)$ decomposes into the spectra $\Sigma^\infty G/H \ltimes S^k$ with $m \leqslant k \leqslant m|K|$ and $H \subseteq K$. For $m < 0$ it has a similar decomposition with $m|K| \leqslant k \leqslant m$.*

Proof The cell structure of $S^{\overline{\rho}_G}$ described in Example 8.5.17 (the one associated with the barycentric subdivision) has G-cells ranging in dimension from 0 to $|G|-1$, and suspends to a cell decomposition of S^{ρ_G} with G-cells whose dimensions ranges from 1 to $|G|$. The case $\widehat{S} = \Sigma^\infty S^{m\rho_G}$ with $m \geqslant 0$ is handled by smashing these together and passing to suspension spectra, giving G-cells whose dimensions range from m to $m|G|$. For $m < 0$, Spanier-Whitehead duality gives (see §8.0C) an equivariant cell decomposition of $S^{-|m|\rho_G}$ into cells whose dimensions range from $-|m||G|$ to $-|m|$. Finally, the case in which $\widehat{S}$ is induced from a subgroup $K \subset G$ is proved by left inducing its K-equivariant cell decomposition. □

Corollary 11.2.5. Bredon cofibrant decomposition of spectra in τ_n^G. *Let $X \in \tau_n^G$. If $n \geqslant 0$, then X can be decomposed into the spectra $G/H \ltimes S^m \wedge S^{-0}$ with $m \geqslant \lceil n/|G| \rceil$. If $n < 0$ then X can be decomposed into $G/H \ltimes S^m \wedge S^{-|n|}$ with $m \geqslant 0$.*

Proof The class of G-spectra X which can be decomposed into $\Sigma^\infty G/H \ltimes S^m$ with $m \geqslant \lceil n/|G| \rceil$ is closed under weak equivalences, homotopy colimits, and extensions. By Lemma 11.2.4 it contains the slice spheres $\widehat{S}$ with $\dim \widehat{S} \geqslant n$. It therefore contains all $X \in \tau_n^G$. A similar argument handles the case $n < 0$. □

Proposition 11.2.6. The relation between slice connectivity and ordinary connectivity.

(i) *If $n \geqslant 0$, then $(G/H) \ltimes \Sigma^\infty S^n$ is in τ_n^G.*

(ii) If $n < 0$, then $(G/H) \ltimes S^{-|n|}$ is in $\tau^G_{-|n||G|}$.
(iii) If Y is in τ^G_n for $n \geqslant 0$, then $\underline{\pi}_k Y = 0$ for $k < \lceil n/|G| \rceil$.
(iv) If Y is in τ^G_n for $n < 0$, then $\underline{\pi}_k Y = 0$ for $k < n$.
(v) If X is an $(n-1)$-connected G-spectrum with $n \geqslant 0$ then X is in τ^G_n.

Proof (i) We will prove the claim by induction on $|G|$, the case of the trivial group being obvious. Using Proposition 11.1.20 we may assume by induction that $\Sigma^\infty(G/H) \ltimes S^n \geqslant n$ when $n \geqslant 0$ and $H \subset G$ is a proper subgroup. This implies that if T is an equivariant CW spectrum built from G-cells of the form $\Sigma^\infty(G/H) \ltimes S^n$ with and $H \subset G$ a proper subgroup, then $T \geqslant n$. The homotopy fiber of the natural inclusion

$$\Sigma^\infty S^n \to \Sigma^\infty S^{n\rho_G}$$

can be identified with the suspension spectrum of $S(n\rho_G - n) \ltimes S^n$, and so is such a T. Since $\Sigma^\infty S^{n\rho_G} \geqslant n|G| \geqslant n$ the fiber sequence

$$T \to \Sigma^\infty S^n \to \Sigma^\infty S^{n\rho_G}$$

exhibits $\Sigma^\infty S^n$ as an extension of two slice $(n-1)$-connected spectra, making it slice $(n-1)$-connected.

(ii) For $n > 0$ we have

$$(G/H) \ltimes S^{-n} \simeq (G/H) \ltimes S^{n(\overline{\rho}_G)} \wedge S^{-n\rho_G}.$$

Since $n > 0$, the spectrum $\Sigma^\infty S^{n(\overline{\rho}_G)}$ is a suspension spectrum of a finite G-CW complex, so

$$(G/H) \ltimes S^{-n} \geqslant -n|G|.$$

The third and fourth assertions are immediate from Corollary 11.2.5.

(v) The class of $(n-1)$-connected spectra is exactly the class of spectra which decompose into terms of the form $G/H \ltimes S^m$ with $m \geqslant n$. By (i) these are in τ^G_n. □

11.2B The spectral sequence

The slice spectral sequence is the spectral sequence associated to the tower of fibrations $\{P^n X\}$ of Definition 11.1.42, and it takes the form

$$E_2^{s,t} X := \pi^G_{t-s} P^t_t X \implies \pi^G_{t-s} X. \tag{11.2.7}$$

It has variants in which the functor π^G_* is replaced by π^H_* for a subgroup H, including the trivial subgroup for which we use the notaion π^u, where "u" stands for "underlying." We can also apply the Mackey functor valued functor $\underline{\pi}$, which is discussed in §9.4B. The integer t (the second superscript) can be replaced by an element $V \in RO(G)$ in the orthogonal representation ring of G. However **the first superscript s (the filtration degree) and the differential index r below are always ordinary integers**.

We have chosen our indexing so that the display of the spectral sequence is in accord with the classical Adams spectral sequence: the $E_r^{s,t}$-term is placed in the plane in position $(t-s,s)$. The situation is depicted in Figure 11.1. The differential d_r maps $E_r^{s,t}$ to $E_r^{s+r,t+r-1}$, or in terms the display in the plane, the group in position $(t-s,s)$ to the group in position $(t-s-1,s+r)$.

The following is an easy consequnce of Proposition 11.1.23 (iii).

Proposition 11.2.8. The external pairing induced by the smash product. *For G-spectra X and Y there is a spectral squence pairing*

$$E_r^{s,t}(X) \otimes E_r^{s',t'}(Y) \to E_r^{s+s',t+t'}(X \wedge Y)$$

representing the pairing $\underline{\pi}_* X \wedge \underline{\pi}_* Y \to \underline{\pi}_*(X \wedge Y)$.

Remark 11.2.9. An E_1-term for the slice spectral sequence. *In many cases of interest, $P_t^t X$ is contractible for t odd, and for even t it has the form $W_t \wedge H\underline{\mathbf{Z}}$ where W_t is a wedge of slice spheres $\widehat{S}(m,K)$ with $m|K| = t$. This means that $\underline{\pi}_* P_t^t X = \underline{H}_* W_t$. For $t \geqslant 0$, W_t is the suspension spectrum of a finite G-CW complex. Its cellullar chain complex could be regarded as the graded group $E_1^{*,t}$. For $t < 0$ W_t is the smash product of $H\underline{\mathbf{Z}}$ with the dual to such spectrum, and a similar remark applies.*

The following is an immediate consequence of Proposition 11.2.6.

Theorem 11.2.10. The homotopy groups of $P^n X$. *Let X be a G-spectrum. The map $X \to P^n X$ induces an isomorphism in Mackey functor homotopy groups $\underline{\pi}_k$*

$$\text{for } \begin{cases} k < \lceil (n+1)/|G| \rceil & \text{if } n \geqslant 0 \\ k < n+1 & \text{if } n < 0. \end{cases}$$

We also have

$$\underline{\pi}_k P^n X = 0 \text{ for} \begin{cases} k \geqslant n+1 & \text{if } n \geqslant 0 \\ k \geqslant \lceil (n+1)/|G| \rceil & \text{if } n < 0. \end{cases}$$

Thus for any X, $\operatorname*{colim}_n P^n X$ is contractible, the map $X \to \lim_n P^n X$ is a weak equivalence, and for each k, the map

$$\{\underline{\pi}_k X\} \to \{\underline{\pi}_k P^n X\}$$

from the constant tower to the slice tower of Mackey functors is a pro-isomorphism.

Proof The fiber of the map $X \to P^n X$ is $P_{n+1} X$, so there is an exact sequence

$$\underline{\pi}_k P_{n+1} X \to \underline{\pi}_k X \to \underline{\pi}_k P^n X \to \underline{\pi}_{k-1} P_{n+1} X.$$

Since $P_{n+1}X$ is in τ^G_{n+1}, the vanishing statements of Proposition 11.2.6(iii)–(iv) give the desired isomorphisms in $\underline{\pi}_k$.

The vanishing of $\underline{\pi}_k P^n X$ for the stated values of k follows from that fact any map to $P^n X$ from an object in τ^G_{n+1} is null homotopic, and the latter contains the G-cells indicated in Proposition 11.2.6(i)–(ii).

The remaining statements follow easily from the first two. □

Corollary 11.2.11. The homotopy groups of n-slices. *If Y is an n-slice, then $\underline{\pi}_k Y = 0$ unless*

$$\begin{cases} \lceil n/|G|\rceil \leqslant k \leqslant n & \text{for } n \geqslant 0 \\ n \leqslant k < \lceil (n+1)/|G|\rceil & \text{for } n < 0. \end{cases}$$

Corollary 11.2.12. Vanishing regions for the slice E_2-term. *In the slice spectral sequence for a G-spectrum X,*

$$E_2^{s,t} = \pi_{t-s} P^t_t X = 0 \text{ for } \begin{cases} t \geqslant 0 \text{ and } t-s < \left\lceil \dfrac{t}{|G|} \right\rceil \\ \qquad \text{(first quadrant above line of slope } |G|-1\text{)} \\ t < 0 \text{ and } s > 0 \\ \qquad \text{(entire second quadrant)} \\ t < 0 \text{ and } t-s > \left\lceil \dfrac{t+1}{|G|} \right\rceil \\ \qquad \text{(third quadrant below line of slope } |G|-1\text{)} \\ t \geqslant 0 \text{ and } s < 0 \\ \qquad \text{(entire fourth quadrant).} \end{cases}$$

Theorem 11.2.10 gives the strong convergence of the slice spectral sequence, while Corollary 11.2.12 shows that the E_2-term vanishes outside of a restricted range of dimensions. The situation is depicted in Figure 11.1 for a group of order 4. The homotopy groups of individual slices lie along lines of slope -1, and the groups contributing to $\underline{\pi}_* P^n X$ lie to the left of a line of slope -1 intersecting the $(t-s)$-axis at $(t-s) = n$. All of the groups outside the gray regions are zero.

4/4/20. We should remove the circled numbers from Figure 11.1.

Remark 11.2.13. The peculiarity of the slice spectral sequence. *The vanishing lines depicted in Figure 11.1 hold for* **any** *G-spectrum X. In particular, replacing X by $\Sigma^k X$ does* **not** *move the entire chart k units to the right as one might expect based on experience with the Adams spectral sequence.*

The slice filtration itself does not play nicely with ordinary suspension. It is not true that $P^t_t \Sigma^k X$ is the same as $\Sigma^k P^t_t X$, although it is the case that

$$P^t_t \Sigma^{k\rho_G} X \cong \Sigma^{k\rho_G} P^t_t X.$$

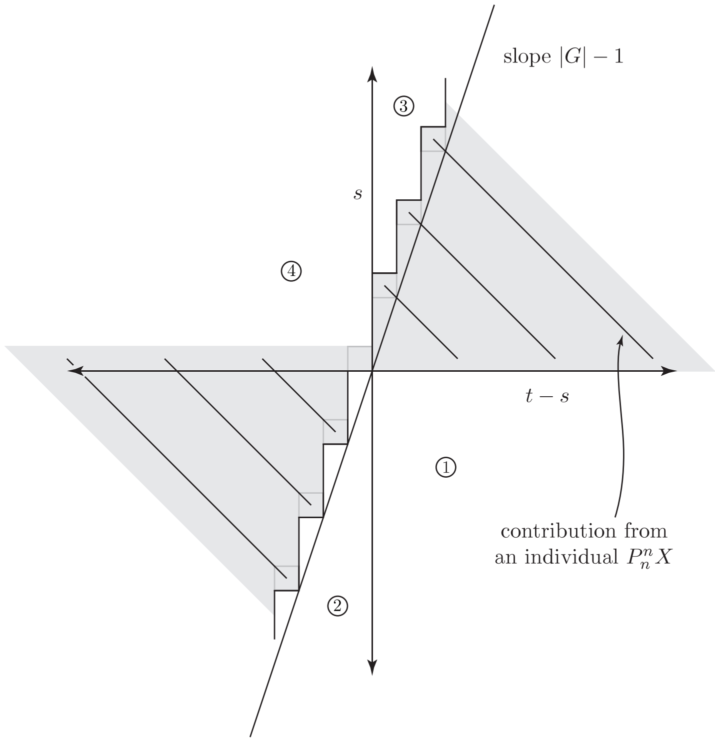

Figure 11.1 The integer graded slice spectral sequence for $G = C_4$.

The integer Eilenberg-Mac Lane spectrum $H\mathbf{Z}$ of Theorem 9.1.47 is a 0-slice, but for a virtual representation V, its Vth suspension is in general not a $|V|$-slice. The determination of the slice filtration for $\Sigma^V H\mathbf{Z}$ is a topic of ongoing research. For some results in this direction, see [Hil12], [HHR17b], [Yar17], [HY18] and [GY18].

Remark 11.2.14. The difficulty of using the slice spectral sequence. *In general it is very difficult to determine the slices $P_t^t X$ for a G-spectrum X. Since the restriction of the slice filtration to the trivial group is the same as the ordinary Postnikov filtration, identifying the slices entails identifying the underlying homotopy groups of X.*

Fortunately this information **is** *available in the cases we need to consider, namely various equivariant relatives of the complex cobordism spectrum MU. Their construction and properties are the subject of the first two sections of the next chapter. In § 12.4 we develop some handy tools for identifying their slices with complete precision.* **The mainspring of this calculation**

is the Reduction Theorem 12.4.8, *which identifies the 0-slice as the integer Eilenberg-Mac Lane spectrum in certain cases.*

We end this section with an application. The next result says that if a tower looks like the slice tower, then it is the slice tower.

Proposition 11.2.15. Slice tower recognition. *Suppose that $X \to \{\widetilde{P}^n\}$ is a map from X to a tower of fibrations with the properties*

(i) the map $X \to \lim_n \widetilde{P}^n$ is a weak equivalence;

(ii) the spectrum $\operatorname{colim}_n \widetilde{P}^n$ is contractible;

(iii) for all n, the fiber of the map $\widetilde{P}^n \to \widetilde{P}^{n-1}$ is an n-slice.

Then $\left\{\widetilde{P}^n\right\}$ is the slice tower of X.

Proof We first show that $\widetilde{P}^n$ is slice $(n+1)$-coconnected. We will use the criteria of Lemma 11.1.15. Suppose that $\widehat{S}$ is a slice sphere with $\dim \widehat{S} > n$. By condition (iii), the maps

$$[\widehat{S}, \widetilde{P}^n]^G \to [\widehat{S}, \widetilde{P}^{n-1}]^G \to [\widehat{S}, \widetilde{P}^{n-2}]^G \to \cdots$$

are all monomorphisms. Since $\widehat{S}$ is finite, the map

$$\operatorname*{colim}_{k \leqslant n} [\widehat{S}, \widetilde{P}^k]^G \to [\widehat{S}, \operatorname*{colim}_{k \leqslant n} \widetilde{P}^k]^G$$

is an isomorphism. It then follows from assumption (ii) that $[\widehat{S}, \widetilde{P}^n]^G = 0$. This shows that $\widetilde{P}^n$ is slice $(n+1)$-coconnected. Now let $\widetilde{P}_{n+1}$ be the homotopy fiber of the map $X \to \widetilde{P}^n$. By Lemma 11.1.38, the result will follow if we can show $\widetilde{P}_{n+1} > n$. By assumption (iii), for any $N > n+1$, the spectrum

$$\widetilde{P}_{n+1} \cup C\widetilde{P}_N$$

admits a finite filtration whose layers are m-slices, with $m \geqslant n+1$. It follows that

$$\widetilde{P}_{n+1} \cup C\widetilde{P}_N > n.$$

In view of the cofiber sequence

$$\widetilde{P}_N \to \widetilde{P}_{n+1} \to \widetilde{P}_{n+1} \cup C\widetilde{P}_N,$$

to show that $\widetilde{P}_{n+1} > n$ it suffices to show that $\widetilde{P}_N > n$ for **some** $N > n$.

Let Z be any slice $(n+1)$-coconnected spectrum. We need to show that the G-space

$$\mathcal{S}p_G(\widetilde{P}_N, Z)$$

is contractible. We do this by studying the Mackey functor homotopy groups of the spectra involved, and appealing to an argument using the usual equivariant notion of connectivity. By 11.2.10, there is an integer m with the property that for $k > m$,

$$\underline{\pi}_k Z = 0.$$

By Corollary 11.2.11 and assumption (iii), for $N \gg 0$ and any $N' > N$,

$$\underline{\pi}_k \widetilde{P}_N \cup C\widetilde{P}_{N'} = 0, \quad k \leqslant m,$$

so

$$\underline{\pi}_k \widetilde{P}_{N'} \to \underline{\pi}_k \widetilde{P}_N$$

is an isomorphism for $k \leqslant m$. Since $\operatorname{holim}_{N'} \widetilde{P}_{N'}$ is contractible this implies that for $N \gg 0$

$$\underline{\pi}_k \widetilde{P}_N = 0, \quad k \leqslant m.$$

Thus for $N \gg 0$, $\widetilde{P}_N$ is m-connected in the usual sense and so

$$\mathcal{S}p_G(\widetilde{P}_N, Z)$$

is contractible. □

11.2C The $RO(G)$-graded slice spectral sequence

Applying $RO(G)$-graded homotopy groups to the slice tower leads to an $RO(G)$-graded slice spectral sequence

$$E_2^{s,V} = \pi^G_{V-s} P^{|V|}_{|V|} X \implies \pi^G_{V-s} X.$$

The grading convention is chosen so that it restricts to the one of § 11.2B when V is a trivial virtual representation. The rth differential is a map

$$d_r : E_2^{s,V} \to E_2^{s+r,V+r-1}.$$

Remark 11.2.16. Gradings in the slice spectral sequence. *Note that while V is an element of $RO(G)$, the indices r (the index of the differential), s (the filtration degree) and $|V|$ (the slice degree) are ordinary integers. Thus the spectral sequence is not bigraded over $RO(G)$, but rather it is graded over $\mathbf{Z} \times RO(G)$. Differentials preserve the image of $V - s$ in $RO(G)/\mathbf{Z}$.*

The $RO(G)$-graded slice spectral sequence is thus a sum of spectral sequences bigraded over $\mathbf{Z}$, one for each element of $RO(G)/\mathbf{Z}$. If one wants to depict this spectral sequence in the usual way with a 2-dimensional chart, one would need a **different chart for each element of** *$RO(G)/\mathbf{Z}$. Of course this is rarely done in practice.*

The quotient $RO(G)/\mathbf{Z}$ is isomorphic to the subring generated by virtual representations of degree 0. For any actual representation V, the virtual representation $V - |V|$ (where $|V|$ denotes a vector space having the same dimension as V but with trivial G-action) has degree 0. For $G = C_2$, this subring is generated by the reduced sign representation $\sigma - 1$

We will call the spectral sequence corresponding to the coset $V + \mathbf{Z} \in RO(G)/\mathbf{Z}$ the **slice spectral sequence for $\pi^G_{V+*}X$.** This spectral sequence can be displayed on the (x, y)-plane, and we will do so following the Adams convention, with the term $E_2^{s,V+t}$ displayed at a position with x-coordinate $(|V| + t - s)$ and y-coordinate s.

Corollary 11.2.12 implies the following.

Corollary 11.2.17. Vanishing regions for the $RO(G)$-graded slice E_2-term. *In the slice spectral sequence for a G-spectrum X,*

$$E_2^{s,V} = 0 \text{ for } \begin{cases} |V| \geqslant 0 \text{ and } |V| - s < \left\lceil \dfrac{|V|}{|G|} \right\rceil \\ \qquad \text{(first quadrant above line of slope } |G| - 1\text{)} \\ |V| < 0 \text{ and } s > 0 \\ \qquad \text{(entire second quadrant)} \\ |V| < 0 \text{ and } |V| - s > \left\lceil \dfrac{|V| + 1}{|G|} \right\rceil \\ \qquad \text{(third quadrant below line of slope } |G| - 1\text{)} \\ |V| \geqslant 0 \text{ and } s < 0 \\ \qquad \text{(entire fourth quadrant),} \end{cases}$$

where the quadrants and sloped lines are in a hypothetical chart for each fixed value of $V - |V|$ as in Remark 11.2.16.

11.3 Spherical slices

In this section we define **spectra with spherical slices** in Definition 11.3.14. Our main result (Theorem 11.3.17) asserts that a map $X \to Y$ of G-spectra with spherical slices is a weak equivalence if and only if the underlying map of non-equivariant spectra is. We need it for the proof of the Reduction Theorem 12.4.8. We will also describe methods for determining the slices of spectra, and introduce a convenient class of equivariant spectra.

Our first results make use of the isotropy separation sequence (§ 9.11A) obtained by smashing with the cofiber sequence of pointed G-spaces

$$E\mathcal{P}_+ \to S^0 \to \widetilde{E}\mathcal{P}.$$

Lemma 11.3.1. Slice connectivity of $E\mathcal{P} \ltimes X$. *Fix an integer d. If X is* *a G-spectrum with the property that $i_H^G X > d$ for all proper $H \subset G$, then $E\mathcal{P} \ltimes X > d$.*

Proof Suppose that $Z \leqslant d$. Then

$$\mathcal{S}p_G(E\mathcal{P} \ltimes X, Z) \cong \mathcal{T}_G(E\mathcal{P}_+, \mathcal{S}p_G(X, Z)).$$

By the assumption on X, the G-space $\mathcal{S}p_G(X, Z)$ has contractible H fixed points for all proper $H \subset G$. The space $E\mathcal{P}_+$ is an equivariant CW complex built from G-cells of the form $(G/H) \ltimes S^n$ with $H \subset G$ a proper subgroup. It follows that if W is a pointed G-space whose H-fixed points are contractible for all proper $H \subset G$, then $\mathcal{T}_G(E\mathcal{P}_+, W)$ is contractible. The result follows. □

Lemma 11.3.2. Slice connectivity of $\widetilde{E}\mathcal{P}$ and its suspension. *The suspension spectrum of $\widetilde{E}\mathcal{P}$ is in τ_0^G but not in τ_1^G, while that of its single suspension is in $\tau_{|G|}^G$ but not in $\tau_{1+|G|}^G$.*

Proof The suspension spectrum of $\widetilde{E}\mathcal{P}$ is in τ_0^G, since it is (-1)-connected (Proposition 11.1.18). To see that it is not in τ_1^G, note that

$$\Phi^G(\Sigma^\infty \widetilde{E}\mathcal{P}) \simeq \Sigma^\infty (\widetilde{E}\mathcal{P})^G \simeq \Sigma^\infty S^0,$$

so $\pi_0 \Phi^G(\Sigma^\infty \widetilde{E}\mathcal{P}) \neq 0$.

Similarly, $\Phi^G(\Sigma^\infty \Sigma \widetilde{E}\mathcal{P}) \simeq \Sigma^\infty S^1$. The map $\widetilde{E}\mathcal{P} \wedge S^1 \to \widetilde{E}\mathcal{P} \wedge S^{\rho_G}$ is a weak equivalence (Theorem 9.11.8 and Proposition 9.11.10). Thus Example 11.1.34(i) shows that $\Sigma^\infty \Sigma \widetilde{E}\mathcal{P}$ is in $\tau_{|G|}^G$ but not in $\tau_{1+|G|}^G$. □

The following result is due to Yan Zou.

Proposition 11.3.3. Ordinary connectivity and slice connectivity. *Let k be an integer that is not divisible by any prime factor of $|G|$. Suppose that X is a G-spectrum in τ_k^G and that $\pi_k^u X = 0$, i.e., the non-equivariant spectrum $i_0^G X$ (for i_e^G as in Definition 2.1.29(v)) underlying X is k-connected. Then X is in τ_{k+1}^G, i.e., X is slice k-connected. In particular this holds for $k = 1$.*

Proof For any $H \subseteq G$ we have

$$\pi_i \Phi^H X = 0 \qquad \text{for } i < \frac{k}{|H|}.$$

Our hypothesis on k implies that for nontrivial H, $k/|H|$ is not an integer. This means that for an integer i, the condition $i < k/|H|$ is equivalent to $i \leqslant k/|H|$, which is equivalent to $i < (k+1)/|H|$.

For H trivial, $\pi_* \Phi^H X = \pi_*^u X$, so $\pi_* \Phi^H X = 0$ for $i < k+1$ by hypothesis. Hence X meets all the conditions to be in τ_{k+1}^G. □

Proposition 11.3.4. A property of Eilenberg-Mac Lane 1-slices. *Suppose that $\Sigma H\underline{M}$ is a 1-slice for a Mackey functor $\underline{M}$. Then for each pair of subgroups $K \subseteq H \subseteq G$, the restriction map $\mathrm{Res}_K^H : \underline{M}(G/H) \to \underline{M}(G/K)$ is one to one.*

Proof Suppose that $f : S \to S'$ is a surjective map of finite G-sets, and let C be the cofiber of the map $S_+ \to S'_+$. Hence C is finite wedge of circles that are permuted by G. Then $X = \Sigma^\infty C$ is in τ_0^G with $\pi_0^u X = 0$, so ΣX is in τ_1^G with $\pi_1^u \Sigma X = 0$. Hence ΣX is in τ_2^G by Proposition 11.3.3.

This means $[\Sigma X, \Sigma H\underline{M}] = [X, H\underline{M}]$ is trivial, so the map

$$[\Sigma^\infty S'_+, H\underline{M}] \to [\Sigma^\infty S_+, H\underline{M}]$$

is one to one. Under the identification of (9.4.8), this means that the map $\underline{M}(S') \to \underline{M}(S)$ is one to one. When the map $S \to S'$ is $G/K \to G/H$ for $K \subseteq H \subseteq G$, this is the restriction map Res_K^H for the Mackey functor $\underline{M}$. □

Thus we have a necessary condition on $\underline{M}$ for $\Sigma H\underline{M}$ to be a 1-slice. The Proposition below shows that this is also a sufficient condition.

Theorem 11.3.5. Characterization of 0-slices and 1-slices.

(i) A spectrum X is a 0-slice if and only if it is of the form $X = H\underline{M}$ for a Mackey functor $\underline{M}$.
(ii) A spectrum X is a 1-slice if and only if it is of the form $\Sigma H\underline{M}$ with $\underline{M}$ a Mackey functor all of whose restriction maps are monomorphisms.

Remark 11.3.6. The original definition of the slice filtration. *Under the definition of the slice filtration used in [HHR16], (see Remark 11.1.5) the above was a statement about (-1)-slices and 0-slices. The spectrum $\Sigma^\infty S^{\overline{\rho}} = \Sigma^{-1}\widehat{S}(1,G)$ was a $(|G|-1)$-slice sphere. Now $\Sigma^{-1}\widehat{S}(1,G)$ is in τ_0^G but not in τ_1^G. In particular, for $G = C_2$, this is true for $\Sigma^\infty S^\sigma$, where σ is the sign representation.*

Remark 11.3.7. The G-sets $G \times S$ and $G \times S'$. *The condition on $\underline{M}$ in Theorem 11.3.5(ii) is that if $S \to S'$ is a surjective map of finite G-sets then $\underline{M}(S') \to \underline{M}(S)$ is a monomorphism. Let G act on $G \times S$ and $G \times S'$ through its left action on G. Then $G \times S \to G \times S'$ has a section, so $\underline{M}(G \times S') \to \underline{M}(G \times S)$ is always a monomorphism. Using this one easily checks that this condition is also equivalent to requiring that for every finite G-set S', the map $\underline{M}(S') \to \underline{M}(G \times S')$, induced by the action mapping $G \times S' \to S'$, is a monomorphism.*

Proof of Theorem 11.3.5. The first assertion is immediate from Proposition 11.1.45, which, combined with Proposition 11.2.6(i), also shows that a 0-slice is an Eilenberg-Mac Lane spectrum.

For the second assertion, Proposition 11.2.6(iii) tells us that for Y in τ_1^G (in particular for Y a 1-slice), $\underline{\pi}_k Y = 0$ for $k \leqslant 0$. The slice coconnectivity condition for a 1-slice is that it admits no essential maps from any spectrum in τ_2^G. By Proposition 11.2.6(iii) this means $\underline{\pi}_n Y = 0$ for $n \geqslant 2$, so $Y = \Sigma H\underline{M}$ for some Mackey functor $\underline{M}$. Proposition 11.3.4 tells us that this $\underline{M}$ must have monomorphic restriction maps.

It remains to show that $\Sigma H\underline{M}$ is a 1-slice for any such $\underline{M}$. Consider the cofiber sequence

$$P_2\Sigma H\underline{M} \to \Sigma H\underline{M} \to P^1\Sigma H\underline{M}.$$

Since $P_2\Sigma H\underline{M} \geqslant 1$ it is 0-connected, and so $P_2\Sigma H\underline{M}$ is an Eilenberg-Mac Lane spectrum. For convenience, write

$$\begin{aligned} \underline{M}' &= \underline{\pi}_1 P_2\Sigma H\underline{M} \\ \underline{M}'' &= \underline{\pi}_1 P^1\Sigma H\underline{M} \end{aligned}$$

so that there is a short exact sequence

$$0 \to \underline{M}' \to \underline{M} \to \underline{M}'' \to 0.$$

Suppose that S is any finite G-set and consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{M}'(S) \arrow[r] \arrow[d] & \underline{M}(S) \arrow[r] \arrow[d] & \underline{M}''(S) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{M}'(G \times S) \arrow[r] & \underline{M}(G \times S) \arrow[r] & \underline{M}''(G \times S) \arrow[r] & 0 \end{tikzcd}$$

in which the rows are exact, and the vertical maps are induced by the action mapping, as in Remark 11.3.7. The bottom right arrow is an isomorphism since $i_0^G\Sigma H\underline{M} \to i_0^G P^1\Sigma H\underline{M}$ is an equivalence. Thus $\underline{M}'(G \times S) = 0$. The middle vertical arrow is one to one by Remark 11.3.7, so $\underline{M}'(S) = 0$ and therefore $\underline{M}' = 0$. □

Corollary 11.3.8. If $X = \Sigma H\underline{M}$ is a 1-slice and $\pi_1^u X = 0$ then X is contractible.

Corollary 11.3.9. The 1-slice of $\Sigma H\underline{M}$ for arbitrary $\underline{M}$. *Given a Mackey functor $\underline{M}$, let $\underline{M}'$ be given by*

$$\underline{M}'(G/H) = \ker Res_e^H : \underline{M}(G/H) \to \underline{M}(G/e),$$

and let $\underline{M}'' = \underline{M}/\underline{M}'$. Then $P_1^1\Sigma H\underline{M} \simeq P^1\Sigma H\underline{M} \simeq \Sigma H\underline{M}''$.

Proof The Mackey functor $\underline{M}''$ satisfies the condition of Theorem 11.3.5(ii), so $\Sigma H\underline{M}''$ is a 1-slice. We also have $\underline{M}'(G/e) = 0$. We can use Proposition 11.3.3 to show that $\Sigma H\underline{M}'$ is in τ_2^G. It follows that $P_1^1\Sigma H\underline{M} \simeq \Sigma H\underline{M}''$ as claimed. □

Corollary 11.3.10. The 0-slice of S^{-0} is $H\underline{\pi}_0 S^{-0}$, and the 1-slice of ΣS^{-0} is $\Sigma H\underline{\mathbf{Z}}$.

Proof The first assertion follows easily from Theorem 11.3.5 (i). For the second assertion note that the $S^1 \to \Sigma H\underline{\pi}_0 S^{-0}$ is a P^1-equivalence, so the 1-slice of ΣS^{-0} is $P^1\Sigma H\underline{\pi}_0 S^{-0}$. This is $\Sigma H\underline{\mathbf{Z}}$ by Corollary 11.3.9. □

Remark 11.3.11. The tom Dieck theorem. *It follows from a theorem of tom Dieck [tD79] (which is also discussed in [LMSM86, Chapter VII.11] and in [Sch14, §6]) that* $\underline{\pi}_0 S^{-0} \cong \underline{A}$, *the Burnside Mackey functor of Definition 8.2.7. For our present purposes, all we need to know about* $\underline{\pi}_0 S^{-0}$ *is that its value on* G/e, *which is* $\pi_0^u S^{-0}$, *is* $\mathbf{Z}$, *and that the generator of this group, which is represented by the identity map on* S^{-0}, *is in the image of every restriction map. It follows that* $\underline{\mathbf{Z}}$ *is the image of the surjective map from* $\underline{\pi}_0 S^{-0}$ *obtained as in Corollary 11.3.9.*

Corollary 11.3.12. The bottom slices for slice spheres and their suspensions. *For* $K \subset G$, *the* $m|K|$*-slice of* $\widehat{S}(m,K)$ *is*

$$H\underline{\pi}_0 S^{-0} \wedge \widehat{S}(m,K)$$

and the $(m|K|+1)$*-slice of* $\Sigma\widehat{S}(m,K)$ *is*

$$H\underline{\mathbf{Z}} \wedge \Sigma\widehat{S}(m,K).$$

In particular, for $K = \{e\}$, *the* m*-slice of* $\widehat{S}(m,e)$ *is* $H\underline{\mathbf{Z}} \wedge \Sigma^m G_+$.

Proof Using the fact that $G \underset{K}{\ltimes} (-)$ commutes with the formation of the slice tower (Proposition 11.1.46) it suffices to consider the case $K = G$. The result then follows from Proposition 11.1.23(iv) and Corollary 11.3.10.

For K trivial, $\widehat{S}(m,K) = G \ltimes S^m = \Sigma\widehat{S}(m-1,K)$, so the m-slice of $\widehat{S}(m,K)$ is the $((m-1)+1)$-slice of $\widehat{S}(m-1,K)$. □

While the bottom slice of $\widehat{S}(m,K)$ is not $H\underline{\mathbf{Z}} \wedge \widehat{S}(m,K)$, the latter is nevertheless a slice.

Proposition 11.3.13. Smash products of slice spheres with $H\underline{\mathbf{Z}}$. *For any integer* m *and subgroup* $K \subseteq G$, *the spectrum* $H\underline{\mathbf{Z}} \wedge \widehat{S}(m,K)$ *is an* $m|K|$*-slice.*

Proof We can argue by induction on $|G|$ using Proposition 11.1.23 (iv), so it suffices to treat the case $K - G$. We know that $H\underline{\mathbf{Z}}$ is a 0-slice by Theorem 11.3.5. The result then follows from Corollary 11.1.49(i). □

Definition 11.3.14. Spherical slices and pure spectra. *A* d*-slice is* **spherical** *(***cellular** *in [HHR16, Definition 4.56]) if it is of the form* $H\underline{\mathbf{Z}} \wedge \widehat{W}$, *where* $\widehat{W}$ *is a wedge of slice spheres of dimension* d. *(Such spectra are slices by Proposition 11.3.13.) A spherical slice is* **bound** *(isotropic in [HHR16, Definition 1.12]) if* $\widehat{W}$ *can be written as a wedge of slice spheres, none of which is free (i.e., of the form* $G \ltimes S^n$*). A* G*-spectrum* X **has spherical slices** *if* $P_n^n X$ *is spherical for all* n, *and is* **pure** *if in addition its slices are all bound.*

Spherical slices are not to be confused with the slice spheres of §11.1B. The former involve smash products of the latter with $H\underline{\mathbf{Z}}$.

This terminology differs from that of [HHR16], where "pure" meant that all slices had summands of the form

$$H\underline{\mathbf{Z}} \wedge \widehat{S}(m, K) \quad \text{rather than} \quad H\underline{\mathbf{Z}} \wedge \Sigma^{-1}\widehat{S}(m, K).$$

A pivotal result of this book is that the real cobordism spectrum $MU_{\mathbf{R}}$ (the subject of Chapter 12) and certain related spectra are all pure. This is Slice Theorem 12.4.1, which is proved in Chapter 13.

Remark 11.3.15. Slices of pure spectra. *The n-slice of a pure spectrum is contractible if n is prime to $|G|$ since the only slice spheres in such dimensions are free. In particular if G is a 2-group, the slices of a pure spectrum are concentrated in even dimesions.*

Lemma 11.3.16. Maps between spherical d-slices. *Suppose that $f : X \to Y$ is a map of spherical d-slices and $\pi_d^u f$ is an isomorphism. Then f is a weak equivalence of G-spectra.*

Proof The proof is by induction on $|G|$. If G is the trivial group, the result is obvious since X and Y are Eilenberg-Mac Lane spectra. Now suppose we know the result for all proper $H \subset G$, and consider the map of isotropy separation sequences

$$\begin{tikzcd} E\mathcal{P} \ltimes X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \widetilde{E}\mathcal{P} \wedge X \arrow[d] \\ E\mathcal{P} \ltimes Y \arrow[r] & Y \arrow[r] & \widetilde{E}\mathcal{P} \wedge Y. \end{tikzcd}$$

By the induction hypothesis, the left vertical map is a weak equivalence. If d is not congruent to 0 modulo $|G|$ then the rightmost terms are contractible, since every slice sphere of dimension d is moving. Smashing with $S^{m\rho_G}$ for suitable m, we may therefore assume $d = 0$. We assume that $X = H\underline{M}_0$ and $Y = H\underline{M}_1$ where $\underline{M}_0$ and $\underline{M}_1$ are permutation Mackey functors. The result then follows from part (iv) of Lemma 8.2.12. □

Theorem 11.3.17. Maps between spectra with spherical slices. *Suppose that X and Y have spherical slices. If $f : X \to Y$ has the property that $\pi_*^u f$ is an isomorphism, then f is a weak equivalence of G-spectra.*

Proof It suffices to show that for each d the induced map of slices

$$P_d^d X \to P_d^d Y \tag{11.3.18}$$

is a weak equivalence. Since the map of ordinary spectra underlying the slice tower is the Postnikov tower, the map satisfies the conditions of Lemma 11.3.16, and the result follows. □

The following will be used to describe spectra related to MU in the next chapter.

Definition 11.3.19. *Suppose that X is a G-spectrum with the property that $\pi_d^u X$ is a free abelian group. A* **refinement of** $\pi_d^u X$ *is an equivariant map $c_d : \widehat{W}_d \to X$ in which $\widehat{W}_d$ is a wedge of slice spheres of dimension d, with the property that the map $\pi_d^u \widehat{W}_d \to \pi_d^u X$ is an isomorphism.*

Suppose further that $\pi_d^u X$ is a free abelian group for all d. Then a **refinement of** $\pi_*^u X$ *is an equivariant map $c : \widehat{W} \to X$ in which $\widehat{W}$ is a wedge of slice spheres of varying dimensions, such that for each d the restriction of c to the d-dimensional summands of $\widehat{W}$ is a refinement of $\pi_d^u X$.*

11.4 The slice tower, symmetric powers and the norm

The main goal of this section is to prove Theorem 11.4.13, which says that if R is an equivariant commutative ring in τ_0^G (see Definition 11.1.22), and $n \geqslant 0$ is an integer, then there is a variant ,

$$\left\{P^n_{\mathrm{alg}} R : n \geqslant 0\right\} \qquad \text{to be defined in §11.4B,}$$

of the slice tower in which each section is also an equivariant commutative ring in τ_0^G. The proof uses the results of §10.9B to show that cofibrant commutative rings are equifibrantly flat.

The results here are needed to prove Proposition 11.1.55, Proposition 11.1.56 and Proposition 11.1.57. We need them to show that differentials in slice spectral sequence play nicely with products, i.e., they are derivations.

The reader may wish to look again through §11.1 for the basic definitions concerning the slice tower. Our presentation there was homotopy theoretic, and the slice sections P^n and related constructions were given up to weak equivalence.

Here we will use some explicit constructions, and some care needs to be taken to ensure that the derived functors we are ultimately interested in can be computed on the objects that arise. Using the fact that indexed smash products (Theorem 10.4.7) and indexed symmetric powers (Theorem 10.5.10) of cofibrant spectra are cofibrant, one can check that this is indeed the case.

Definition 11.4.1. The cofibrant slice tower. *The n*th **cofibrant slice section** $P_c^n X$ *is the spectrum obtained by replacing the slice spheres in (11.1.36) by their cofibrant replacements as in (11.1.2).*

This functor is homotopical, the map $X \to P_c^n X$ is a positive equifibrant cofibration, and its codomain is cofibrant when X is.

Our task will be to show that something functorially weakly equivalent to P_c^n takes commutative rings in τ_0^G to commutative rings in τ_0^G.

We begin with the interaction of the slice filtration with the formation of indexed smash products. As in Chapter 10 we fix a finite G-set T and work with the homotopy theory of equivariant T-diagrams of orthogonal spectra. We define slice spheres and the slice filtration in the evident manner, so that the slice filtration on equivariant T-diagrams corresponds to the product of slice filtrations on G_t-spectra under the equivalence

$$\mathcal{S}p^{\mathcal{B}_T G} \cong \prod_t \mathcal{S}p^{G_t},$$

where G_t is the stabilizer of t as in (10.1.1).

11.4A Slice connectivity of indexed products

The proposition below follows easily from Proposition 11.1.20.

Proposition 11.4.2. Indexed wedges preserve slice connectivity. *Suppose that T is a non-empty G-set, X is a cofibrant equivariant T-diagram, and n is an integer. If each X_t is slice $(n-1)$-connected, then the indexed wedge*

$$\bigvee_{t \in T} X_t$$

is slice $(n-1)$-connected.

The next two results make use of the implication

$$X \geqslant 0 \quad \text{and} \quad Y \geqslant k \implies X \wedge Y \geqslant k \tag{11.4.3}$$

of Proposition 11.1.23(iii).

Proposition 11.4.4. Indexed smash products preserve slice connectivity. *Suppose that T is a non-empty G-set, X is a cofibrant equivariant T-diagram, and $n \geqslant 0$ is an integer. If each X_t is slice $(n-1)$-connected, then the indexed smash product*

$$\bigwedge_{t \in T} X_t$$

is slice $(n-1)$-connected.

Proof By induction on $|G|$ we may suppose that $i_H^G X^{\wedge T}$ is slice $(n-1)$-connected for any proper subgroup $H \subset G$. This implies that $K \wedge X^{\wedge T} \geqslant n$ if K is any G-CW complex built entirely from moving G-cells. Since the formation of indexed smash products commutes with filtered colimits, it suffices by Proposition 11.1.37 to consider a cofibration $A \to B$ of equivariant T-diagrams in which B/A is a wedge of slice spheres of dimension greater than n, and show that

$$A^{\wedge T} \geqslant n \implies B^{\wedge T} \geqslant n. \tag{11.4.5}$$

Using the filtration of §2.9C for the identity pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\lrcorner"] & B, \end{tikzcd}$$

gives a filtration of $B^{\wedge T}$ whose stages fit into cofibration sequences

$$\mathrm{fil}_{m-1}B^{\wedge T} \to \mathrm{fil}_m B^{\wedge T} \to \bigvee A^{\wedge T_0} \wedge (B/A)^{\wedge T_1} \tag{11.4.6}$$

in which the indexing G-set for the coproduct on the right is the set of all set theoretic decomposition $T = T_0 \coprod T_1$ with $|T_1| = m$. The implication (11.4.3) and Proposition 11.4.2 above reduce the claim to showing that if $T_1 \neq \varnothing$, then $(B/A)^{\wedge T_1}$ (regarded as an equivariant spectrum for the stabilizer of T_1) is slice $(n-1)$-connected. In other words, it suffices to prove the proposition when X is a wedge of slice spheres of dimension greater than or equal to n.

Making use of the distributive law, (11.4.3) and Proposition 11.4.2, one reduces to the case in which $T = G/H$ is a single orbit, and X corresponds to $S^{k\rho_H}$ with $k|H| \geqslant n$. Then we have

$$X^{\wedge T} \cong S^{k\rho_G}$$

has dimension $k|G| \geqslant k|H| \geqslant n$. □

Remark 11.4.7. A simplification brought about by the use of Definition 11.1.3 instead of the original definition of slice spheres. *The above result is [HHR16, Proposition B.169]. In the proof there it was also necessary (see Remark 11.1.5) to consider the case $X = S^{k\rho_H - 1}$ with $k|H| - 1 \geqslant n$, referred to there as "the second case." The argument for it was more complicated than that for "the first case," $X = S^{k\rho}$, presented above. It also contains a typo: the definition of the representation W should be $(\mathrm{Ind}_H^G 1) - 1$, not $\mathrm{Ind}_H^G - 1$.*

Proposition 11.4.8. The slice connectivity of a cofibrant indexed smash product. *For X and T be as in Proposition 11.4.4,*

$$\Sigma^{-1}(\Sigma X)^{\wedge T} \geqslant n. \tag{11.4.9}$$

Proof Rewrite the spectrum in (11.4.9) as

$$\left(\Sigma^{-1}(S^1)^{\wedge T}\right) \wedge (X^{\wedge T}).$$

The factor $\Sigma^{-1}(S^1)^{\wedge T}$ is weakly equivalent to the sphere S^V with $V = \mathbf{R}^T - 1$. This gives

$$\Sigma^{-1}(S^1)^{\wedge T} \geqslant 0$$

and the relation (11.4.9) then follows from Proposition 11.4.4 and (11.4.3). □

We next turn to indexed symmetric powers. As in §10.5 we consider a finite G-set T, a G-stable subgroup $\Lambda \subset \Sigma_T$, and the indexed symmetric power

$$\mathrm{Sym}^T_\Lambda X = X^{\wedge T}/\Lambda.$$

Proposition 11.4.10. Indexed symmetric powers and slice connectivity. *Let $n \geqslant 0$ be an integer, T a nonempty finite G-set, and X a cofibrant equivariant T-diagram. If X is slice $(n-1)$-connected then both the indexed symmetric power $\mathrm{Sym}^T_\Lambda X$ and $\Sigma^{-1}\mathrm{Sym}^T_\Lambda(\Sigma X)$ are slice $(n-1)$-connected.*

Proof Using the equivalences

$$E_G\Lambda \underset{\Lambda}{\ltimes} X^{\wedge T} \cong \mathrm{Sym}^T_\Lambda X$$
$$\Sigma^{-1}E_G\Lambda \underset{\Lambda}{\ltimes} (\Sigma X)^{\wedge T} \cong \Sigma^{-1}\mathrm{Sym}^T_\Lambda(\Sigma X)$$

of Lemma 10.5.18 and working through an equivariant cell decomposition of $E_G\Lambda$ reduces the claim to showing that

$$S \underset{\Lambda}{\ltimes} X^{\wedge T} \quad \text{and} \quad \Sigma^{-1} S \underset{\Lambda}{\ltimes} (\Sigma X)^{\wedge T} \tag{11.4.11}$$

are slice $(n-1)$-connected when S is a finite Λ-free $\Lambda \rtimes G$-set. But the first spectrum in (11.4.11) is an indexed wedge of indexed smash products of X (see the proof of Lemma 10.5.16), hence slice $(n-1)$-connected by Proposition 11.4.4 and Proposition 11.4.2. The second spectrum is an indexed wedge of desuspensions of indexed smash products of ΣX, hence slice $(n-1)$-connected by Proposition 11.4.8 and Proposition 11.4.2. □

11.4B The cofibrant slice tower for a commutative ring

We can now investigate the slice sections of commutative rings. Let

$$P^n_{\mathrm{alg}} : \mathbf{Comm}^G \to \mathbf{Comm}^G,$$

the nth algebraic slice section, be the multiplicative analogue of P^n_c, constructed for an equivariant commutative ring R as the colimit of a sequence of functors

$$P^{n,0}_{\mathrm{alg}} R \to P^{n,1}_{\mathrm{alg}} R \to \cdots .$$

The $P^{n,k}_{\mathrm{alg}} R$ are defined inductively on k starting with $P^{n,0}_{\mathrm{alg}} R = R$, and in which $P^{n,k}_{\mathrm{alg}} R$ is defined by the pushout square (compare with (11.1.36), Definition 11.4.1, and Quillen's diagram (4.2.11))

$$\begin{tikzcd} \mathrm{Sym}\left(\bigvee_{L_{n,k}} \Sigma^t \widehat{S}_c\right) \arrow[r] \arrow[d] & P_{\mathrm{alg}}^{n,k-1} R \arrow[d] \\ \mathrm{Sym}\left(\bigvee_{L_{n,k}} C\,\Sigma^t \widehat{S}_c\right) \arrow[r] \arrow[ur, phantom, "\lrcorner"] & P_{\mathrm{alg}}^{n,k} R \end{tikzcd} \tag{11.4.12}$$

in which the indexing set $L_{n,k}$ is the set of maps $\Sigma^t \widehat{S}_c \to P_{\mathrm{alg}}^{n,k-1} R$ with $\widehat{S}_c > n$ a cofibrant slice sphere and $t \geqslant 0$. The functor P_{alg}^n is homotopical and for any R, the map $R \to P_{\mathrm{alg}}^n R$ is a cofibration of equivariant commutative rings. The arrow $R \to P_{\mathrm{alg}}^n R$ is characterized up to weak equivalence by the following universal property: if S is an equivariant commutative ring whose underlying spectrum is slice $(n+1)$-coconnected then the map

$$\mathrm{Ho}\,\mathbf{Comm}^G(P_{\mathrm{alg}}^n R, S) \to \mathrm{Ho}\,\mathbf{Comm}^G(R, S)$$

is an isomorphism.

Let U be the forgetful functor

$$U : \mathbf{Comm}^G \to \mathcal{S}p^G.$$

By the small object argument, the spectrum $UP_{\mathrm{alg}}^n R$ is slice $(n+1)$-coconnected, so there is a natural transformation

$$P_c^n UR \to UP_{\mathrm{alg}}^n R$$

of functors to $\mathcal{S}p^G$.

Theorem 11.4.13. The multiplicative slice tower of a commutative ring spectrum. *If R is a slice (-1)-connected cofibrant equivariant commutative ring, then for all $n \in \mathbf{Z}$, the map*

$$P_c^n UR \to UP_{alg}^n R$$

is a weak equivalence. Thus R has a slice tower in which each section is itself an equivariant commutative ring.

Proof When n is negative, $P_c^n UR$ is contractible, and $P_{\mathrm{alg}}^n R$ is a commutative ring whose unit is null homotopic, hence also contractible. We may therefore assume n is non-negative.

It suffices to show that each of the maps

$$R_1 := UP_{\mathrm{alg}}^{n,k-1} R \to UP_{\mathrm{alg}}^{n,k} R =: R_2$$

is a slice P^n-equivalence. We do this by working through the pushout ring filtration of Definition 2.9.47, whose successive terms are related by the homotopy coCartesian square

$$\begin{tikzcd} R_1 \wedge \partial_A \mathrm{Sym}^m B \arrow[r] \arrow[d] & R_1 \wedge \mathrm{Sym}^m B \arrow[d] \\ \mathrm{fil}_{m-1}^{R_1} R_2 \arrow[r] & \mathrm{fil}_m^{R_1} R_2, \end{tikzcd}$$

in which $A \to B$ is the map

$$\bigvee_{L_{n,k}} \Sigma^t \widehat{S}_c \to \bigvee_{L_{n,k}} C\Sigma^t \widehat{S}_c. \tag{11.4.14}$$

By induction we may assume that the maps

$$UR \to UP_{\mathrm{alg}}^{n,k-1} R \to \mathrm{fil}_{m-1} P_{\mathrm{alg}}^{n,k} R$$

are P^n equivalences, so the three spectra are all in τ_0^G. The homotopy fiber of $\mathrm{fil}_{m-1} P_{\mathrm{alg}}^{n,k} R \to \mathrm{fil}_m P_{\mathrm{alg}}^{n,k} R$ is

$$UP_{\mathrm{alg}}^{n,k-1} R \wedge \Sigma^{-1} \mathrm{Sym}^m (B/A).$$

Now B/A is the suspension of the left term in (11.4.14) which is slice n-connected. It follows (Proposition 11.4.10) that $\Sigma^{-1}\mathrm{Sym}^m(B/A)$ is also slice n-connected hence so is $UP_{\mathrm{alg}}^{n,k-1} R \wedge \Sigma^{-1} \mathrm{Sym}^m (B/A)$ since

$$UP_{\mathrm{alg}}^{n,k-1} R \geqslant 0.$$

The fact that $\mathrm{fil}_{m-1} P_{\mathrm{alg}}^{n,k} R \to \mathrm{fil}_m P_{\mathrm{alg}}^{n,k} R$ is a slice P^n-equivalence is now a consequence of Lemma 11.1.51. $\square$

12

The construction and properties of $MU_{\mathbf{R}}$

In this chapter we give a construction of the real bordism spectrum $MU_{\mathbf{R}}$ as a commutative algebra in $\mathcal{S}p^{C_2}$. This construction owes a great deal to the Stefan Schwede's construction of MU in [Sch07, Chapter 2]. We are indebted to him for some very helpful correspondence concerning these matters.

We assume that the reader is familiar with the ordinary spectrum MU. If not, please consult [Rav86, Section 4.1], [Sto68b] or [Mil60]. In particular we will make use of the facts that

$$H_*(MU;\mathbf{Z}) = \mathbf{Z}[b_1, b_2, \dots], \quad \text{with } b_i \in H_{2i} \tag{12.0.1}$$

and

$$\pi_*(MU) = \mathbf{Z}[x_1, x_2, \dots], \quad \text{with } x_i \in \pi_{2i}. \tag{12.0.2}$$

The homology generators b_i can be defined in terms of complex projective spaces, but there is no simple way to define the homotopy generators x_i. We will define specific generators for equivariant homotopy below in Corollary 12.3.9.

Our goal is to construct a C_2-equivariant commutative ring $MU_{\mathbf{R}}$ admitting the canonical homotopy presentation (see §7.4F)

$$MU_{\mathbf{R}} \cong \operatorname{hocolim} S^{-n\mathbf{C}} \wedge MU(n), \tag{12.0.3}$$

where $\mathbf{C}$ denotes the complex numbers with conjugation regarded as a real orthogonal representation of C_2, and $MU(n)$ is the Thom complex of the universal bundle over $BU(n)$, the classifying space of the group $U(n)$ of $n \times n$ unitary matrices. The group C_2 acts on everything by complex conjugation, so we could also write this expression as

$$MU_{\mathbf{R}} \cong \operatorname{hocolim} S^{-n\rho_2} \wedge MU(n). \tag{12.0.4}$$

The map

$$S^{-\rho_2} \wedge MU(1) \to MU_{\mathbf{R}}$$

defines a real orientation. These properties form the basis for everything we will prove about $MU_{\mathbf{R}}$.

The most natural construction of $MU_{\mathbf{R}}$ realizes this structure in the category $\mathcal{S}p_{\mathbf{R}}$ of **real spectra**, which is related to the category of C_2-equivariant orthogonal spectra by a multiplicative Quillen equivalence

$$r_! : \mathcal{S}p_{\mathbf{R}} \underset{\perp}{\rightleftarrows} \mathcal{S}p^{C_2} : r^*.$$

We will construct a commutative algebra $\mathcal{MU}_{\mathbf{R}} \in \mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}}$ (not to be confused with $MU_{\mathbf{R}}$), whose underlying real spectrum has a canonical homotopy presentation of the form

$$\begin{tikzcd} & \operatorname*{hocolim}_n S^{-n\mathbf{C}} \wedge MU(n) \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\ \mathcal{MU}_{\mathbf{R}} & & (\operatorname*{hocolim}_n S^{-n\mathbf{C}} \wedge MU(n))_f, \end{tikzcd} \tag{12.0.5}$$

as in Definition 7.4.63. In this case $S^{-n\rho_2} \wedge MU(n)$ is already cofibrant, so there is no need for a cofibrant replacement. The map on the right is fibrant replacement.

Applying $r_!$ to (12.0.5) and making the identification $r_! S^{-\mathbf{C}} = S^{-\rho_2}$ leads to the diagram

$$\begin{tikzcd} & \operatorname*{hocolim}_n S^{-n\rho_2} \wedge MU(n) \arrow[dl] \arrow[dr, "\simeq"] & \\ r_!\mathcal{MU}_{\mathbf{R}} & & (\operatorname*{hocolim}_n S^{-n\rho_2} \wedge MU(n))_f, \end{tikzcd} \tag{12.0.6}$$

We define $MU_{\mathbf{R}}$ to be the spectrum $r_!\mathcal{MU}'_{\mathbf{R}}$, where $\mathcal{MU}'_{\mathbf{R}} \to \mathcal{MU}_{\mathbf{R}}$ is a cofibrant commutative algebra approximation. The functor $r_!$ is strictly monoidal, so $MU_{\mathbf{R}}$ is a commutative ring in $\mathcal{S}p^{C_2}$. The map on the right in (12.0.6) is a weak equivalence since $r_!$ is a left Quillen functor. The problem is to show that the one on the left is.

This involves two steps. The first is to show that the forgetful functor

$$\mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}} \to \mathcal{S}p_{\mathbf{R}}$$

creates a model structure on $\mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}}$. This involves analyzing the symmetric powers of cofibrant real spectra. The second is to show that the functor $r_!$ is homotopical on a subcategory of $\mathcal{S}p_{\mathbf{R}}$ containing the real spectra underlying cofibrant real commutative rings. As in our analysis of norms of commutative rings, this involves a generalized notion of flatness. The role of the model category structure on $\mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}}$ is to identify the cofibrant real commutative algebras. But the only real work in establishing the model structure is showing that what one thinks is a cofibrant approximation is actually a weak equivalence, and that is what is needed to show that every real commutative algebra is weakly equivalent to a cofibrant one.

12.1 Real and complex spectra

In this section we describe the basics of **real and complex spectra**. The additive results follow from the results of [MM02], but the important multiplicative properties require a separate analysis.

For finite dimensional complex Hermitian vector spaces A and B, let $U(A, B)$ be the Stiefel manifold of unitary embeddings $A \hookrightarrow B$. There is a natural Hermitian inner product on the complexification $V_{\mathbf{C}}$ of a real orthogonal vector space V, so there is a natural embedding

$$O(V, W) \to U(V_{\mathbf{C}}, W_{\mathbf{C}}).$$

The group C_2 acts on $U(V_{\mathbf{C}}, W_{\mathbf{C}})$ by complex conjugation, and the fixed point space is $O(V, W)$. The complex numbers $\mathbf{C}$ with conjugation is isomorphic to ρ_2 as a 2-dimensional orthogonal representation of C_2. It follows that there is a natural embedding

$$U(V_{\mathbf{C}}, W_{\mathbf{C}}) \to O(V_{\rho_2}, W_{\rho_2}). \tag{12.1.1}$$

where V_{ρ_2} denotes $V \otimes \rho_2$.

The following two definitions should be compared with Definition 8.9.24 and Definition 9.0.2.

Definition 12.1.2. *The* **complex Mandell-May category** $\mathscr{J}_{\mathbf{C}}$ *is* $\mathscr{J}^{\mathbf{U}}$ *as in Definition 7.2.4(iv). It is topological category whose objects are finite dimensional Hermitian vector spaces, and whose morphism space* $\mathscr{J}_{\mathbf{C}}(V, W)$ *is the Thom complex*

$$\mathscr{J}_{\mathbf{C}}(V, W) = Thom(U(V, W); W - V).$$

Here $U(V, W)$ denotes the space (a complex Steifel manifold) of unitary embeddings of V into W. As in the orthogonal case, each such embedding $i : V \to W$ has a unitary complement which we denote by $W - i(V)$. This defines a complex vector bundle over the complex Stiefel manifold $U(V, W)$, and the morphism object is its Thom space.

The **real Mandell-May category** $\mathscr{J}_{\mathbf{R}}$ *is the C_2-equivariant topological category (meaning a category enriched over $\mathcal{T}^{C_2}$) whose objects are finite dimensional orthogonal real vector spaces V, and with*

$$\mathscr{J}_{\mathbf{R}}(V, W) := \mathscr{J}_{\mathbf{C}}(V_{\mathbf{C}}, W_{\mathbf{C}}),$$

where $V_{\mathbf{C}}$ denotes $V \otimes \mathbf{C}$, on which C_2 acts by complex conjugation.

It follows that $\mathscr{J}_{\mathbf{R}}(\mathbf{m},\mathbf{n})$ is a certain subspace of $\mathscr{J}_{C_2}(m\rho_2, n\rho_2)$ related to the embedding of (12.1.1).

The following is a consequence of the unitary case of Proposition 7.2.24.

Proposition 12.1.3. The categories $\mathscr{J}_{\mathbf{C}}$ and $\mathscr{J}_{\mathbf{R}}$ are both $\mathscr{J}^{\mathbf{O}}$-algebras *as in Definition 7.2.19.*

Definition 12.1.4. *The category $\mathcal{S}p_{\mathbf{C}}$ of* **complex spectra** *is the topological category of enriched functors*

$$\mathscr{J}_{\mathbf{C}} \to \mathcal{T}.$$

The category $\mathcal{S}p_{\mathbf{R}}$ of **real spectra** *is the topological category of C_2-enriched functors*

$$\mathscr{J}_{\mathbf{R}} \to \mathcal{T}^{C_2}.$$

We will write

$$V \mapsto X_{V_{\mathbf{C}}}$$

for a typical real spectrum X, and let $S^{-V_{\mathbf{C}}} \in \mathcal{S}p_{\mathbf{R}}$ be the functor corepresented by $V \in \mathscr{J}_{\mathbf{R}}$. From the Yoneda lemma there is a natural isomorphism

$$\mathcal{S}p_{\mathbf{R}}(S^{-V_{\mathbf{C}}}, X) = X_{V_{\mathbf{C}}}.$$

As with smashable spectra in general, every real spectrum X has a tautological presentation as in Proposition 7.2.57,

$$\bigvee_{V,W\in\mathscr{J}_{\mathbf{R}}} S^{-W_{\mathbf{C}}} \wedge \mathscr{J}_{\mathbf{R}}(V,W) \wedge X_{W_{\mathbf{C}}} \rightrightarrows \bigvee_{V\in\mathscr{J}_{\mathbf{R}}} S^{-V_{\mathbf{C}}} \wedge X_{V_{\mathbf{C}}} \to X. \quad (12.1.5)$$

A similar apparatus exists for complex spectra.

Remark 12.1.6. Skeletal subcategories of $\mathscr{J}_{\mathbf{R}}$ and $\mathscr{J}_{\mathbf{C}}$. *The category $\mathscr{J}_{\mathbf{R}}$ is equivalent to its full subcategory with objects $\mathbf{R}^n$, and similarly $\mathscr{J}_{\mathbf{C}}$ is equivalent to its full subcategory with objects $\mathbf{C}^n$. Thus a real spectrum X is specified by the spaces $X_{V_{\mathbf{C}}}$ with $V = \mathbf{R}^n$ together with the structure maps between them, and an object $Y \in \mathcal{S}p_{\mathbf{C}}$ is specified by its spaces $Y_{\mathbf{C}^n}$, together with the structure maps between them.*

The group C_2 acts on $\mathcal{S}p_{\mathbf{C}}$ through its action on $\mathscr{J}_{\mathbf{C}}$. We write this as $X \mapsto \overline{X}$, where

$$\overline{X}_V = X_{\overline{V}}$$

A fixed point for this action is a complex spectrum X equipped with an isomorphism $X \to \overline{X}$ having the property that

$$X \to \overline{X} \to \overline{\overline{X}} = X$$

is the identity map. Restricting to the spaces $X_{\mathbf{C}^n}$ and using the standard basis to identify $\mathbf{C}^n$ with $\overline{\mathbf{C}}^n$ one sees that a fixed point for this C_2-action consists of a sequence C_2-spaces $X_{\mathbf{C}^n}$, together with an associative family of C_2-equivariant maps

$$\mathscr{J}_{\mathbf{C}}(\mathbf{C}^n, \mathbf{C}^m) \wedge_{U(n)} X_{\mathbf{C}^n} \to X_{\mathbf{C}^m},$$

where C_2 is acting by conjugation. But this is the same thing as giving a real spectrum indexed on the spaces $\mathbf{R}^n$. This shows that the category of fixed points for the C_2-action on $\mathcal{S}p_{\mathbf{C}}$ is $\mathcal{S}p_{\mathbf{R}}$.

12.1A Smash products and indexed smash products

The direct sum operation makes $\mathscr{J}_{\mathbf{C}}$ into a $\mathcal{T}$-enriched symmetric monoidal category and $\mathscr{J}_{\mathbf{R}}$ an $\mathcal{T}^{C_2}$-enriched symmetric monoidal category. Using this one can define the smash product $X \wedge Y$ giving both $\mathcal{S}p_{\mathbf{R}}$ and $\mathcal{S}p_{\mathbf{C}}$ the structure of symmetric monoidal categories. The smash product in $\mathcal{S}p_{\mathbf{R}}$ is specified by the formula

$$S^{-V_{\mathbf{C}}} \wedge S^{-W_{\mathbf{C}}} \cong S^{-(V \oplus W)_{\mathbf{C}}}$$

and the fact that it commutes with colimits in each variable. A similar characterization holds for $\mathcal{S}p_{\mathbf{C}}$.

There are indexed monoidal products in this context. Let T be a finite set with a C_2-action. The actions of C_2 on T and on $\mathcal{S}p_{\mathbf{C}}$ combine to give an action on the product category $\mathcal{S}p_{\mathbf{C}}^T$. The category of $\mathcal{S}p_{\mathbf{R}}^T$ of **real T-diagrams** is the category of fixed points for this action. The category of real T-diagrams for $T = \{\mathrm{pt}\}$ is equivalent to $\mathcal{S}p_{\mathbf{R}}$. When $T = C_2$, the category of real T-diagrams is equivalent to $\mathcal{S}p_{\mathbf{C}}$. For general $T = n_1 + n_2 C_2$, one has an equivalence

$$\mathcal{S}p_{\mathbf{R}}^T \cong \mathcal{S}p_{\mathbf{R}}^{n_1} \times \mathcal{S}p_{\mathbf{C}}^{n_2}.$$

There are indexed wedges and indexed smash products from $\mathcal{S}p_{\mathbf{R}}^T$ to $\mathcal{S}p_{\mathbf{R}}$.

12.1B Homotopy theory of real and complex spectra

We now turn to the homotopy theory of real and complex spectra. We describe the case of $\mathcal{S}p_{\mathbf{R}}$ and leave the analogous case of $\mathcal{S}p_{\mathbf{C}}$ to the reader.

Suppose that X is a real spectrum. For $H \subset C_2$ and $k \in \mathbf{Z}$ set

$$\pi_k^H(X) = \operatorname*{colim}_V \pi_{k+V_{\mathbf{C}}}^H X_{V_{\mathbf{C}}}.$$

The colimit is taken over the poset of finite dimensional orthogonal vector spaces over $\mathbf{R}$, ordered (in agreement with Definition 8.9.9) by dimension. A **stable weak equivalence** in $\mathcal{S}p_{\mathbf{R}}$ is a map $X \to Y$ inducing an isomorphism $\pi_k^H X \to \pi_k^H Y$ for all $H \subset C_2$ and $k \in \mathbf{Z}$. For fixed k, the groups π_k^H form a Mackey functor which we denote $\underline{\pi}_k$.

Equipped with the stable weak equivalences, the category $\mathcal{S}p_{\mathbf{R}}$ becomes a homotopical category. We refine it to a model category by defining a map to be a **fibration** if for each **non-zero** V, the map $X_{V_{\mathbf{C}}} \to Y_{V_{\mathbf{C}}}$ is a fibration in $\mathcal{T}^{C_2}$. The cofibrations are the maps having the left lifting property against the trivial fibrations. This is the **positive stable model structure** on $\mathcal{S}p_{\mathbf{R}}$.

The positive stable model structure is cofibrantly generated. The generating cofibrations can be taken to be the maps of the form

$$S^{-V_{\mathbf{C}}} \wedge \left(S_+^{n-1} \to D_+^n\right)$$

and

$$C_2 \ltimes S^{-V_{\mathbf{C}}} \wedge \left(S_+^{n-1} \to D_+^n\right)$$

with $V > 0$. The generating trivial cofibrations are the analogous maps

$$S^{-V_{\mathbf{C}}} \wedge \left(I_+^n \to I^{n+1}\right)$$

and

$$C_2 \ltimes S^{-V_{\mathbf{C}}} \wedge \left(I_+^n \to I^{n+1}\right)$$

together with the corner maps formed by smashing

$$S^{-V_{\mathbf{C}} \oplus W_{\mathbf{C}}} \wedge S^{W_{\mathbf{C}}} \to \tilde{S}^{V_{\mathbf{C}}, W_{\mathbf{C}}} \tag{12.1.7}$$

with the maps $S_+^{n-1} \to D_+^n$ and $C_2 \ltimes (S_+^{n-1} \to D_+^n)$. We assume $V > 0$, while W need not be. The map (12.1.7) is extracted from the factorization

$$S^{-V_{\mathbf{C}} \oplus W_{\mathbf{C}}} \wedge S^{W_{\mathbf{C}}} \to \tilde{S}^{V_{\mathbf{C}}, W_{\mathbf{C}}} \to S^{-V_{\mathbf{C}}}$$

formed by applying the small object construction with the generating cofibrations. As in the case of the equifibrant positive stable model structure on $\mathcal{S}p^G$, the map $\tilde{S}^{V_{\mathbf{C}}, W_{\mathbf{C}}} \to S^{-V_{\mathbf{C}}}$ is a homotopy equivalence. The verification of the model category axioms is a special case of Theorem 7.4.52.

12.1C The relation between real spectra and C_2-spectra

Let

$$r : \mathscr{J}_{\mathbf{R}} \to \mathscr{J}_{C_2}$$

be the functor sending V to

$$V_{\rho_2} = V \otimes \rho_2.$$

Then the precomposition functor

$$r^* : \mathcal{S}p^{C_2} \to \mathcal{S}p_{\mathbf{R}}$$

has both a left and right adjoint which we denote $r_!$ and r_* respectively. The left adjoint sends $S^{-V_{\mathbf{C}}}$ to $S^{-V_{\rho_2}}$, and is described in general by applying the functor termwise to the tautological presentation.

Since the functor r^* is symmetric monoidal as in Definition 2.6.20, the left adjoint $r_!$ is oplax symmetric monoidal by Proposition 2.6.21. This means that for real spectra X and Y, there is a natural map

$$\mu_{X,Y} : r_!(X \wedge Y) \to r_! X \wedge r_! Y$$

in $\mathcal{S}p^{C_2}$ with properties spelled out in Definition 2.6.19.

Proposition 12.1.8. A Quillen equivalence between real spectra and C_2-spectra. *The functors*

$$\begin{tikzcd} r_! : \mathcal{S}p_{\mathbf{R}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{S}p^{C_2} : r^* \arrow[l, shift left] \end{tikzcd}$$

form a Quillen equivalence.

Remark 12.1.9. *A similar argument leads to a Quillen equivalence*

$$\begin{tikzcd} \mathcal{S}p_{\mathbf{C}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{S}p. \arrow[l, shift left] \end{tikzcd}$$

Proof Since $r_!$ is a left adjoint and

$$r_!(S^{-V_{\mathbf{C}}} \wedge A) = S^{-V_{\rho_2}} \wedge A$$

for a pointed C_2-space A, it is immediate that $r_!$ sends the generating (trivial) cofibrations to (trivial) cofibrations, and hence is a left Quillen functor. Using the tautological presentations of the two categories (see Proposition 3.2.33), one can easily check that a map $X \to Y$ in $\mathcal{S}p^{C_2}$ is a weak equivalence if and only if $r^*X \to r^*Y$ is. This means that to show that $r_!$ and r^* form a Quillen equivalence it suffices to show that the unit map

$$X \to r^* r_! X \tag{12.1.10}$$

is a weak equivalence for every cofibrant $X \in \mathcal{S}p_{\mathbf{R}}$. Since r^* is also a left adjoint, it preserves colimits, and therefore so does $r^* r_!$. Since both functors also commute with smashing with a C_2-space, we are reduced to checking that for each $0 \neq V \in \mathscr{J}_{\mathbf{R}}$, the map

$$S^{-V_{\mathbf{C}}} \to r^* S^{-V_{\rho_2}} \tag{12.1.11}$$

is a weak equivalence.

For $W \in \mathscr{J}_{\mathbf{R}}$, the $W_{\mathbf{C}}$th space of $S^{-V_{\mathbf{C}}}$ is

$$\mathscr{J}_{\mathbf{R}}(V, W) = Thom(U(V_{\mathbf{C}}, W_{\mathbf{C}}); W_C - V_{\mathbf{C}})$$

and the Wth space of $r^* S^{-V_{\rho_2}}$ is

$$\mathscr{J}_{C_2}(V_{\rho_2}, W_{\rho_2}) = Thom(O(V_{\rho_2}, W_{\rho_2}); W_{\rho_2} - V_{\rho_2}).$$

The unit of the adjunction is derived from the inclusion

$$U(V_{\mathbf{C}}, W_{\mathbf{C}}) \to O(V_{\rho_2}, W_{\rho_2}) \qquad \text{of (12.1.1).}$$

We must therefore show that for each integer k, the map

$$\operatorname*{colim}_{W \in \mathscr{J}_{\mathbf{R}}} \underline{\pi}_{k+W_{\mathbf{C}}} \mathscr{J}_{\mathbf{R}}(V,W) \to \operatorname*{colim}_{W \in \mathscr{J}_{\mathbf{R}}} \underline{\pi}_{k+W_{\mathbf{C}}} \mathscr{J}_{C_2}(V_{\rho_2}, W_{\rho_2}) \tag{12.1.12}$$

is an isomorphism.

We may suppose that $|W| \geqslant |V|$ since otherwise both spaces above are points. For a fixed W choose an orthogonal embedding $V \subset W$, write $W = V \oplus U$, and consider the diagram

$$\begin{tikzcd}
\mathscr{J}_{\mathbf{R}}(0,U) = S^{U_{\mathbf{C}}} \arrow[r] \arrow[d, "\cong"'] & \mathscr{J}_{\mathbf{R}}(V,W) \arrow[d] \\
\mathscr{J}_{C_2}(0,U) = S^{U_{\rho_2}} \arrow[r] & \mathscr{J}_{C_2}(V_{\rho_2}, W_{\rho_2}) .
\end{tikzcd}$$

The left vertical map is an equivariant homeomorphism. A straightforward argument using the connectivity of Stiefel manifolds shows that for $|W| \gg 0$ the horizontal maps are isomorphisms in both $\pi^u_{k+W_{\mathbf{C}}}$ and $\pi^{C_2}_{k+W_{\mathbf{C}}}$. It follows that the right vertical map is as well, and hence so is (12.1.12). □

For later reference, we record one fact that emerged in the proof of Proposition 12.1.8.

Lemma 12.1.13. The precomposition functor r^* detects weak equivalences. *A morphism in $\mathcal{S}p^{C_2}$ is a weak equivalence if and only if its image under r^* is one.*

12.1D Multiplicative properties of real spectra

The multiplicative homotopy theory of real spectra is similar to that of $\mathcal{S}p^G$ as described in Chapter 10. There does not seem to be a simple way to deduce the results directly from the case of $\mathcal{S}p^{C_2}$, but the proofs are very similar.

Proposition 12.1.14. Indexed corner maps of real spectra. *If T is a finite C_2-set and $X \to Y$ is a cofibration of cofibrant real T-diagrams, then both the indexed corner map $\partial_X Y^{\wedge T} \to Y^{\wedge T}$ and the absolute map $X^{\wedge T} \to Y^{\wedge T}$ are cofibrations between cofibrant objects. They are weak equivalences if $X \to Y$ is.*

Proof This is an analogue of Proposition 10.3.8 and Proposition 10.4.5, and is proved in the same way, using the arrow category and the target exponent filtration of §2.9C. □

For the symmetric powers, we fix a finite C_2-set T and a C_2-stable subgroup $\Lambda \subset \Sigma_T$. The following three results are analogs of Lemma 10.5.18, Theorem 10.5.10 and Theorem 10.7.2, making use of Proposition 12.1.14, Lemma 12.1.15 and Theorem 12.1.16 respectively.

Lemma 12.1.15. *If $X \in \mathcal{S}p_{\mathbf{R}}$ is cofibrant and Z is any real spectrum equipped with an action of $\Lambda \rtimes C_2$ extending the C_2-action, then the map*

$$(E_{C_2}\Lambda) \underset{\Lambda}{\ltimes} (X^{\wedge T} \wedge Z) \to (X^{\wedge T} \wedge Z)/\Lambda.$$

is a weak equivalence.

Theorem 12.1.16. Cofibrations and indexed symmetric powers. *Given a cofibration of cofibrant real spectra $A \to B$ and a finite C_2-set T, in the diagram*

$$\begin{tikzcd} E_{C_2}\Lambda \underset{\Lambda}{\ltimes} \partial_A B^{\wedge T} \arrow[r] \arrow[d] & E_{C_2}\Lambda \underset{\Lambda}{\ltimes} B^{\wedge T} \arrow[d] \\ \partial_A \mathrm{Sym}^T B \arrow[r] & \mathrm{Sym}^T B \end{tikzcd}$$

the upper row is a cofibration between cofibrant objects, the vertical maps are weak equivalences and remain so after smashing with any object, and the bottom row is an h-cofibration of flat spectra. The horizontal maps are weak equivalences if $A \to B$ is.

Theorem 12.1.17. A model structure on commutative algebras in $\mathcal{S}p_{\mathbf{R}}$. *The forgetful functor*

$$U : \mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}} \to \mathcal{S}p_{\mathbf{R}}$$

creates a model structure on $\mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}}$, in which a map of commutative algebras is a fibration or weak equivalence if and only if the underlying map of real spectra is.

12.1E C_2-flat real spectra

Our next task is to show that the left derived functor of $r_!$ of Proposition 12.1.8 can be computed on a subcategory of real spectra containing those which underlie real commutative rings.

The following should be compared with Definition 10.9.6, in which the functor in place of $r_!$ is $(p^*-)^{\wedge K/L}$. Theorem 12.1.22 below is the analogue of Theorem 10.9.9.

Definition 12.1.18. C_2-flatness. *A real spectrum $X \in \mathcal{S}p_{\mathbf{R}}$ is* **C_2-flat** *if it satisfies the following property: for every cofibrant approximation $\tilde{X} \to X$ in $\mathcal{S}p_{\mathbf{R}}$ and every weak equivalence $\widetilde{Z} \to Z$ in $\mathcal{S}p^{C_2}$, the map*

$$r_!\tilde{X} \wedge \widetilde{Z} \to r_!X \wedge Z \tag{12.1.19}$$

is a weak equivalence.

Remark 12.1.20. *Since $r_!$ is a left Quillen functor and cofibrant objects of $\mathcal{S}p^{C_2}$ are flat as in Definition 5.1.20, cofibrant objects of $\mathcal{S}p_{\mathbf{R}}$ are C_2-flat.*

Remark 12.1.21. *If (12.1.19) is a weak equivalence for one cofibrant approximation of X, it is a weak equivalence for any cofibrant approximation of X.*

Our main result is

Theorem 12.1.22. C_2-flatness of cofibrant commutative algebras. *If $R \in \mathcal{S}p_{\mathbf{R}}$ is a cofibrant commutative algebra, then it is C_2-flat.*

The proof of Theorem 12.1.22 follows the argument for the proof of Theorem 10.9.9.

Lemma 12.1.23. C_2-flatness of symmetric powers. *If $A \in \mathcal{S}p_{\mathbf{R}}$ is cofibrant, and $n \geqslant 1$, then $\mathrm{Sym}^n A$ is C_2-flat.*

Proof By Lemma 12.1.15, the map

$$(E_{C_2}\Sigma_n) \underset{\Sigma_n}{\ltimes} A^{\wedge n} \to \mathrm{Sym}^n A$$

is a cofibrant approximation. Since $r_!$ is a continuous left adjoint, we may identify

$$r_!\big((E_{C_2}\Sigma_n) \underset{\Sigma_n}{\ltimes} A^{\wedge n}\big) \wedge \widetilde{Z} \to r_!\big(\mathrm{Sym}^n A\big) \wedge Z \tag{12.1.24}$$

with

$$(E_{C_2}\Sigma_n) \underset{\Sigma_n}{\ltimes} (r_! A)^{\wedge n} \wedge \widetilde{Z} \to \mathrm{Sym}^n(r_! A) \wedge Z. \tag{12.1.25}$$

Since $r_!$ is a left Quillen functor, $r_!(A)$ is cofibrant, and Lemma 10.5.18 implies that (12.1.25), hence (12.1.24) is a weak equivalence. □

We also require an analogue of Lemma 10.9.25, though the statement and proof are much simpler in this case, since $r_!$ is a left adjoint.

Lemma 12.1.26. C_2-flatness in cofiber sequences. *If $S \to T$ is an h-cofibration in $\mathcal{S}p_{\mathbf{R}}$, and two of S, T, T/S are C_2-flat, then so is the third.*

Proof We may choose a map $\tilde{S} \to \tilde{T}$ of cofibrant approximations which is a cofibration, hence an h-cofibration. Our assumption is that two of the vertical maps in

$$\begin{tikzcd}
r_!\tilde{S} \wedge \widetilde{Z} \arrow[r] \arrow[d] & r_!\tilde{T} \wedge \widetilde{Z} \arrow[r] \arrow[d] & r_!(\tilde{T}/\tilde{S}) \wedge \widetilde{Z} \arrow[d] \\
r_! S \wedge Z \arrow[r] & r_! T \wedge Z \arrow[r] & r_!(T/S) \wedge Z
\end{tikzcd}$$

are weak equivalences. This implies that the third is, since the two left horizontal maps are h-cofibrations hence flat. □

Lemma 12.1.27. C_2-flatness of pushouts. *Consider a pushout square in $\mathcal{S}p_{\mathbf{R}}$,*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{12.1.28}$$

in which $S \to T$ is an h-cofibration. If T, T/S and X are C_2-flat, then so is Y.

Proof Since T and T/S are C_2-flat, so is S by Lemma 12.1.26. We may choose cofibrant approximations of everything fitting into a pushout diagram

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] & \tilde{T} \arrow[d] \\ \tilde{X} \arrow[r] & \tilde{Y} \end{tikzcd}$$

in which the top row is an h-cofibration. Now consider

$$\begin{tikzcd} r_!\tilde{X} \wedge \tilde{Z} \arrow[d] & r_!\tilde{S} \wedge \tilde{Z} \arrow[l] \arrow[r] \arrow[d] & r_!\tilde{T} \wedge \tilde{Z} \arrow[d] \\ r_!X \wedge Z & r_!S \wedge Z \arrow[l] \arrow[r] & r_!T \wedge Z \end{tikzcd}$$

The left horizontal maps are h-cofibrations, hence flat, and the vertical maps are weak equivalences by assumption. It follows that the map of pushouts is a weak equivalence. □

Proof of Theorem 12.1.22 It suffices to show that if $A \to B$ is a generating cofibration in $\mathcal{S}p_{\mathbf{R}}$ then

$$\begin{tikzcd} \mathrm{Sym}\, A \arrow[r] \arrow[d] & \mathrm{Sym}\, B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is a pushout square of commutative algebras in $\mathcal{S}p_{\mathbf{R}}$, and if X is C_2-flat, then Y is C_2-flat. We induct over the filtration described in §2.9C. Since $\mathrm{fil}_0 Y = X$, the induction starts. For the inductive step, consider the pushout square

$$\begin{tikzcd} X \wedge \partial_A \mathrm{Sym}^m B \arrow[r] \arrow[d] & X \wedge \mathrm{Sym}^m B \arrow[d] \\ \mathrm{fil}_{m-1} Y \arrow[r] & \mathrm{fil}_m Y\,, \end{tikzcd} \tag{12.1.29}$$

and assume that $\mathrm{fil}_{m-1} Y$ is C_2-flat. Both $\mathrm{Sym}^m B$ and

$$\mathrm{Sym}^m B/\partial_A \mathrm{Sym}^m B = \mathrm{Sym}^m (B/A)$$

are C_2-flat by Lemma 12.1.23. Since smash products of C_2-flat spectra are C_2-flat, both $X \wedge \mathrm{Sym}^m B$ and $X \wedge \mathrm{Sym}^m(B/A)$ are C_2-flat. The top row of (12.1.29) is an h-cofibration, so Lemma 12.1.27 implies that $\mathrm{fil}_m Y$ is C_2-flat. This completes the inductive step, and the proof. □

Though we don't quite need the following result, having come this far we record it for future reference.

Proposition 12.1.30. A Quillen equivalence between real and C_2-equivariant commutative rings. *The functors $r_!$ and r^* restrict to a Quillen equivalence*

$$r_! : \mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Comm}\,\mathcal{S}p^{C_2} : r^*$$

Proof It is immediate from the definition of the model structures on $\mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}}$ and $\mathbf{Comm}\,\mathcal{S}p^{C_2}$, and the fact that

$$r_! : \mathcal{S}p_{\mathbf{R}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}p^{C_2} : r^*$$

is a Quillen pair, that

$$r^* : \mathbf{Comm}\,\mathcal{S}p^{C_2} \to \mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}}$$

preserves the classes of fibrations and trivial fibrations. It remains to show that if $A \in \mathbf{Comm}\,\mathcal{S}p_{\mathbf{R}}$ is cofibrant, then the composition

$$A \to r^* r_! A \to r^*(r_! A_f)$$

is a weak equivalence, where $r_! A \to r_! A_f$ is a fibrant replacement. Since r^* reflects weak equivalences (Lemma 12.1.13) this is equivalent to showing that

$$A \to r^* r_! A$$

is a weak equivalence. Let $A' \to A$ be a cofibrant approximation in $\mathcal{S}p_{\mathbf{R}}$, and consider the following diagram in $\mathcal{S}p_{\mathbf{R}}$

$$\begin{tikzcd} A' \arrow[r, "\sim"] \arrow[d, "\sim"'] & r^* r_! A' \arrow[d, "\sim"] \\ A \arrow[r] & r^* r_! A\,. \end{tikzcd} \tag{12.1.31}$$

By Theorem 12.1.22 the map $r_! A' \to r_! A$ is a weak equivalence. The rightmost arrow in (12.1.31) is therefore a weak equivalence. The top arrow is a weak equivalence by Proposition 12.1.8, and the left arrow is a weak equivalence by assumption. This implies that the bottom arrow is a weak equivalence. □

12.2 The real bordism spectrum

We begin with a real spectrum that will lead us to the C_2-spectrum $MU_{\mathbf{R}}$.

Definition 12.2.1. The real spectrum for complex cobordism $\mathcal{MU}_{\mathbf{R}}$ *is given by*

$$(\mathcal{MU}_{\mathbf{R}})_V = MU(V_{\mathbf{C}}) = Thom(BU(V_{\mathbf{C}}), V_{\mathbf{C}}) \qquad \text{for } V \in \mathscr{J}_{\mathbf{R}},$$

the Thom complex of the bundle

$$EU(V_{\mathbf{C}}) \underset{U(V_{\mathbf{C}})}{\times} V_{\mathbf{C}}$$

over $BU(V_{\mathbf{C}})$ *(the classifying space of Proposition 3.4.15(iii)), equipped with the* C_2*-action of complex conjugation.*

This means that the functor $\mathscr{J}_{\mathbf{R}} \to \mathcal{T}_{C_2}$

$$V \mapsto MU(V_{\mathbf{C}}) \tag{12.2.2}$$

that defines $\mathcal{MU}_{\mathbf{R}}$ is lax symmetric monoidal as in Definition 2.6.19. It follows by an argument similar to that of Proposition 9.1.39 that $\mathcal{MU}_{\mathbf{R}}$ is a commutative ring.

Definition 12.2.3. *The* **real bordism spectrum** *is the* C_2*-spectrum*

$$MU_{\mathbf{R}} = r_! \mathcal{MU}'_{\mathbf{R}},$$

where $\mathcal{MU}'_{\mathbf{R}} \to \mathcal{MU}_{\mathbf{R}}$ *is a cofibrant approximation to* $\mathcal{MU}_{\mathbf{R}}$ *in* **Comm** $\mathcal{S}p_{\mathbf{R}}$ *with the model structure of Theorem 12.1.17.*

Thus $MU_{\mathbf{R}}$ is algebraically cofibrant in **Comm** $\mathcal{S}p^{C_2}$ as in Definition 10.7.1.

To get at the homotopy type of $MU_{\mathbf{R}}$, we examine the canonical homotopy presentation of $\mathcal{MU}_{\mathbf{R}}$ as in Definition 7.4.63. This gives a weak equivalence

$$\underset{n}{\operatorname{hocolim}}\, S^{-\mathbf{C}^n} \wedge MU(n) \xrightarrow{\sim} \mathcal{MU}'_{\mathbf{R}} \tag{12.2.4}$$

in which $MU(n) = MU(\mathbf{C}^n)$. Applying $r_!$ and using Theorem 12.1.22 gives

$$\underset{n}{\operatorname{hocolim}}\, S^{-n\rho_2} \wedge MU(n) \xrightarrow{\sim} MU_{\mathbf{R}}.$$

In this presentation the universal real orientation of $MU_{\mathbf{R}}$ (Example 12.2.11) is given by restricting to the term $n = 1$

$$S^{-\rho_2} \wedge MU(1) \to MU_{\mathbf{R}}.$$

The next result summarizes some further consequences of the presentation (12.2.4).

Proposition 12.2.5. Properties of $MU_{\mathbf{R}}$.

(i) The non-equivariant spectrum underlying $MU_{\mathbf{R}}$ *is stably equivalent to the usual complex cobordism spectrum* MU.

(ii) The equivariant cohomology theory represented by $MU_{\mathbf{R}}$ coincides with the one studied in [Lan68], [Fuj76],[Ara79] and [HK01].

(iii) There is an equivalence

$$\Phi^{C_2} MU_{\mathbf{R}} \simeq MO.$$

(iv) The Schubert cell decomposition of complex Grassmannians described by Milnor and Stasheff in [MS74, §6] leads to a Bredon cofibrant approximation of $MU_{\mathbf{R}}$ by a C_2-CW spectrum with one 0-cell (S^{-0}) and the remaining cells of the form $e^{m\rho_2} \wedge S^{-0}$, with $m > 0$.

12.2A The spectrum $MU^{((G))}$

Assume now that $G = C_{2^n}$, and for convenience we **localize all spectra at the prime** 2. Write $g = 2^n$ and let $\gamma \in G$ be a fixed generator. We now introduce our equivariant variation on the complex cobordism spectrum by defining

$$MU^{((G))} = N_{C_2}^G MU_{\mathbf{R}}, \tag{12.2.6}$$

where $MU_{\mathbf{R}}$ is the C_2-equivariant **real bordism spectrum** of Definition 12.2.3. It is a algebraically cofibrant as in Definition 10.7.1. The norm is taken along the unique inclusion $C_2 \subset G$. Since the norm is symmetric monoidal, and its left derived functor may be computed on the spectra underlying cofibrant commutative rings (Theorem 10.9.9), the spectrum $MU^{((G))}$ is an equivariant commutative ring spectrum. For $H \subset G$ the unit of the restriction-norm adjunction (Corollary 10.7.5) gives a canonical commutative algebra map

$$MU^{((H))} \to i_H^G MU^{((G))}. \tag{12.2.7}$$

By analogy with the shorthand i_e^G for restriction along the inclusion of the trivial group, we will employ the shorthand notation

$$i_2^G = i_{C_2}^G$$

for the restriction map $\mathcal{S}p^G \to \mathcal{S}p^{C_2}$ induced by the unique inclusion $C_2 \subset G$. Restricting, one has a C_2-equivariant smash product decomposition

$$i_2^G MU^{((G))} = \bigwedge_{j=0}^{g/2-1} \gamma^j MU_{\mathbf{R}}. \tag{12.2.8}$$

12.2B Real bordism, real orientations and formal groups

We begin by reviewing work of [Ara79] and [HK01] on real bordism.

Consider the complex projective spaces $\mathbf{C}P^n$ and $\mathbf{C}P^\infty$ as pointed C_2-spaces under the action of complex conjugation, with $\mathbf{C}P^0$ as the base point. The fixed point spaces are the real projective spaces $\mathbf{R}P^n$ and $\mathbf{R}P^\infty$. There are homeomorphisms

$$\mathbf{C}P^n/\mathbf{C}P^{n-1} \cong S^{n\rho_2}, \tag{12.2.9}$$

and in particular an identification $\mathbf{C}P^1 \cong S^{\rho_2}$.

Definition 12.2.10 ([Ara79])**.** *Let E be a C_2-equivariant homotopy commutative ring spectrum. A* **real orientation** *of E is a class $\overline{x} \in \tilde{E}^{\rho_2}_{C_2}(\mathbf{C}P^\infty)$ whose restriction to*

$$\tilde{E}^{\rho_2}_{C_2}(\mathbf{C}P^1) = \tilde{E}^{\rho_2}_{C_2}(S^{\rho_2}) \cong E^0_{C_2}(pt)$$

is a unit. A **real oriented spectrum** *is a C_2-equivariant ring spectrum E equipped with a real orientation.*

If $(E, \overline{x})$ is a real oriented spectrum and $f : E \to E'$ is an equivariant multiplicative map, then

$$f_*(\overline{x}) \in (E')^{\rho_2}(\mathbf{C}P^\infty)$$

is a real orientation of E'. We will often not distinguish in notation between $\overline{x}$ and $f_*\overline{x}$.

Example 12.2.11. The real orientations for $MU_{\mathbf{R}}$ and its norms. *The zero section $\mathbf{C}P^\infty \to MU(1)$ is an equivariant equivalence, and defines a real orientation*

$$\overline{x} \in MU^{\rho_2}_{\mathbf{R}}(\mathbf{C}P^\infty),$$

making $MU_{\mathbf{R}}$ into a real oriented spectrum. From the map

$$MU_{\mathbf{R}} \to i^G_2 MU^{((G))}$$

provided by (12.2.7), the spectrum $i^G_2 MU^{((G))}$ gets a real orientation which we'll also denote

$$\overline{x} \in (MU^{((G))})^{\rho_2}(\mathbf{C}P^\infty).$$

Example 12.2.12. Real orientations on smash products. *If $(H, \overline{x}_H)$ and $(E, \overline{x}_E)$ are two real oriented spectra then $H \wedge E$ has two real orientations given by*

$$\overline{x}_H = \overline{x}_H \otimes 1 \text{ and } \overline{x}_E = 1 \otimes \overline{x}_E.$$

The following result of Araki follows easily from the homeomorphisms (12.2.9).

Theorem 12.2.13 ([Ara79])**. The real oriented cohomology of $\mathbf{C}P^\infty$ and $\mathbf{C}P^\infty \times \mathbf{C}P^\infty$.** *Let E be a real oriented cohomology theory. There are isomorphisms*

$$E^\star(\mathbf{C}P^\infty) \cong E^\star[[\overline{x}]]$$
$$\text{and} \qquad E^\star(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) \cong E^\star[[\overline{x} \otimes 1, 1 \otimes \overline{x}]].$$

Because of 12.2.13, the map $\mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ classifying the tensor product of the two tautological line bundles defines a formal group law over $\pi_\star^G E$. Using this, much of the theory relating formal groups, complex cobordism, and complex oriented cohomology theories works for C_2-equivariant spectra, with $MU_{\mathbf{R}}$ playing the role of MU. For information beyond the discussion below, see [Ara79, HK01].

Remark 12.2.14. *A real orientation $\overline{x}$ corresponds to a* **coordinate** *on the corresponding formal group. Because of this we will use the terms interchangeably, preferring "coordinate" when the discussion predominantly concerns the formal group, and "real orientation" when it concerns spectra.*

The standard formulae from the theory of formal groups give elements in the $RO(C_2)$-graded homotopy groups $\pi_\star^{C_2} E$ of real oriented E. For example, there is a map from the Lazard ring to $\pi_\star^{C_2} E$ classifying the formal group law. Using Quillen's theorem to identify the Lazard ring with the complex cobordism ring, this map can be written as

$$\pi_* MU \to \pi_\star^{C_2} E. \tag{12.2.15}$$

It sends $\pi_{2n} MU$ to $\pi_{n\rho_2}^{C_2} E$. When $E = MU_{\mathbf{R}}$ this splits the forgetful map

$$\pi_{n\rho_2}^{C_2} MU_{\mathbf{R}} \to \pi_{2n}^u MU_{\mathbf{R}} = \pi_{2n} MU, \tag{12.2.16}$$

which is therefore surjective. A similar discussion applies to iterated smash products of $MU_{\mathbf{R}}$ giving

Proposition 12.2.17. The relation between underlying and equivariant homotopy of smash powers of $MU_{\mathbf{R}}$. *For every $m > 0$, the above construction gives a ring homomorphism*

$$\bigoplus_j \pi_{2j}^u \bigwedge^m MU_{\mathbf{R}} \to \bigoplus_j \pi_{j\rho_2}^{C_2} \bigwedge^m MU_{\mathbf{R}} \tag{12.2.18}$$

splitting the forgetful map

$$\bigoplus_j \pi_{j\rho_2}^{C_2} \bigwedge^m MU_{\mathbf{R}} \to \bigoplus_j \pi_{2j}^u \bigwedge^m MU_{\mathbf{R}}. \tag{12.2.19}$$

In particular, (12.2.19) is a split surjection.

It is a result of Hu-Kriz[HK01] that (12.2.19) is in fact an isomorphism. This result, and a generalization to $MU^{((G))}$ can be recovered from the slice spectral sequence.

The class

$$\overline{x}_H \in H_{C_2}^{\rho_2}(\mathbf{C}P^\infty; \underline{\mathbf{Z}}_{(2)})$$

corresponding to $1 \in H_{C_2}^0(\mathrm{pt}, \underline{\mathbf{Z}}_{(2)})$ under the isomorphism

$$H_{C_2}^{\rho_2}(\mathbf{C}P^\infty; \underline{\mathbf{Z}}_{(2)}) \cong H_{C_2}^{\rho_2}(\mathbf{C}P^1; \underline{\mathbf{Z}}_{(2)}) \cong H_{C_2}^0(\mathrm{pt}; \underline{\mathbf{Z}}_{(2)})$$

defines a real orientation of $H\underline{\mathbf{Z}}_{(2)}$. As in Example 12.2.12, the classes $\overline{x}$ and $\overline{x}_H$ give two orientations of $E = H\underline{\mathbf{Z}}_{(2)} \wedge MU_{\mathbf{R}}$. By 12.2.13 these are related by a power series

$$\overline{x}_H = \log_F(\overline{x}) = \overline{x} + \sum_{i>0} \overline{m}_i \overline{x}^{i+1},$$

with

$$\overline{m}_i \in \pi^{C_2}_{i\rho_2} H\underline{\mathbf{Z}}_{(2)} \wedge MU_{\mathbf{R}}. \tag{12.2.20}$$

This power series is the **logarithm** of F. Similarly, the invariant differential on F gives classes $(n+1)\overline{m}_n \in \pi^{C_2}_{n\rho_2} MU_{\mathbf{R}}$. The coefficients of the formal sum give

$$\overline{a}_{i,j} \in \pi^{C_2}_{(i+j-1)\rho_2} MU_{\mathbf{R}}. \tag{12.2.21}$$

Remark 12.2.22. Action of C_2 on the homology of $S^{n\rho_2}$. *Since the generator of C_2 acts by $(-1)^n$ on*

$$H_{2n} i_0^G S^{n\rho_2} = \pi^u_{2n} H\underline{\mathbf{Z}} \wedge S^{n\rho_2},$$

it acts also acts by $(-1)^n$ on the non-equivariant class m_n underlying $\overline{m}_n$ and by $(-1)^n$ on $\pi^u_{2n} \bigwedge^m MU_{\mathbf{R}} = \pi_{2n} \bigwedge^m MU$.

If $(E, \overline{x}_E)$ is a real oriented spectrum then $E \wedge MU_{\mathbf{R}}$ has two orientations

$$\overline{x}_E = \overline{x}_E \otimes 1$$
$$\overline{x}_R = 1 \otimes \overline{x}.$$

These two orientations are related by a power series

$$\overline{x}_R = \sum \overline{b}_i x_E^{i+1} \tag{12.2.23}$$

defining classes

$$\overline{b}_i = \overline{b}_i^E \in \pi^{C_2}_{i\rho_2} E \wedge MU_{\mathbf{R}}.$$

The power series (12.2.23) is an isomorphism over $\pi^{C_2}_\star E \wedge MU_{\mathbf{R}}$

$$F_E \to F_R$$

of the formal group law for $(E, \overline{x}_E)$ with the formal group law for $(MU_{\mathbf{R}}, \overline{x})$.

Theorem 12.2.24 ([Ara79])**. The real oriented homology of $MU_{\mathbf{R}}$.** *The map*

$$E_\star[\overline{b}_1, \overline{b}_2, \dots] \to \pi^{C_2}_\star E \wedge MU_{\mathbf{R}}$$

is an isomorphism.

Araki's theorem has an evident geometric counterpart. For each j choose a map

$$S^{j\rho_2} \wedge S^{-0} \to E \wedge MU_{\mathbf{R}} \tag{12.2.25}$$

representing $\bar{b}_j$. As in §10.10, let

$$S^{-0}[\bar{b}_j] \simeq \bigvee_{k \geqslant 0} S^{k \cdot j\rho_2} \qquad \text{as in Definition 10.10.2}$$

$$\text{and} \qquad E[\bar{b}_j] = E \wedge S^{-0}[\bar{b}_j] \qquad \text{as in (10.10.12).}$$

be the free associative E-algebra on $S^{j\rho_2}$ and

$$S^{-0}[\bar{b}_j] \to E \wedge MU_{\mathbf{R}}$$

the homotopy associative algebra map extending that of (12.2.25). Using the multiplication map, smash these together to form a map of spectra

$$E[\bar{b}_1, \bar{b}_2, \dots] \to E \wedge MU^{((G))}, \tag{12.2.26}$$

where

$$E[\bar{b}_1, \bar{b}_2, \dots] := \operatorname*{hocolim}_k E[\bar{b}_1, \bar{b}_2, \dots, \bar{b}_k] \tag{12.2.27}$$

The map on $RO(C_2)$-graded homotopy groups induced by (12.2.26) is the isomorphism of Araki's theorem. This proves

Corollary 12.2.28. The weak homotopy type of $E \wedge MU_{\mathbf{R}}$. *If E is a real oriented spectrum then there is a weak equivalence*

$$E \wedge MU_{\mathbf{R}} \cong E[\bar{b}_1, \bar{b}_2, \dots].$$

Remark 12.2.29. *If E is strictly associative then (12.2.26) is a map of associative algebras, and the above identifies $E \wedge MU_{\mathbf{R}}$ as a twisted monoid ring over E.*

As in (12.2.27), write

$$S^{-0}[\bar{b}_1, \bar{b}_2, \dots] = \operatorname*{hocolim}_k S^{-0}[\bar{b}_1] \wedge S^{-0}[\bar{b}_2] \wedge \cdots \wedge S^{-0}[\bar{b}_k],$$

and

$$S^{-0}[G \cdot \bar{b}_1, G \cdot \bar{b}_2, \dots] = N_{C_2}^G S^{-0}[\bar{b}_1, \bar{b}_2, \dots].$$

Using Proposition 11.1.10 one can easily check that $S^{-0}[G \cdot \bar{b}_1, G \cdot \bar{b}_2, \dots]$ is a wedge of bound slice spheres. In particulat, let

$$MU^{((G))}[G \cdot \bar{b}_1, G \cdot \bar{b}_2, \dots] = MU^{((G))} \wedge S^{-0}[G \cdot \bar{b}_1, G \cdot \bar{b}_2, \dots].$$

Corollary 12.2.30. The restriction of $MU^{((G))}$ to a subgroup. *For $H \subset G$ of index 2, there is an equivalence of H-equivariant associative algebras*

$$i_H^G MU^{((G))} \cong MU^{((H))}[H \cdot \bar{b}_1, H \cdot \bar{b}_2, \dots].$$

Proof Apply $N_{C_2}^H$ to the decomposition of Corollary 12.2.28 with $E = MU_{\mathbf{R}}$. □

12.2C The unoriented cobordism ring

Passing to geometric fixed points from

$$\bar{x} : \mathbf{C}P^\infty \to \Sigma^{\rho_2} MU_{\mathbf{R}}$$

gives the canonical inclusion

$$a : \mathbf{R}P^\infty = MO(1) \to \Sigma MO,$$

defining the MO Euler class of the tautological line bundle. There are isomorphisms

$$MO^*(\mathbf{R}P^\infty) \cong MO^*[\![a]\!]$$
$$MO^*(\mathbf{R}P^\infty \times \mathbf{R}P^\infty) \cong MO^*[\![a \otimes 1, 1 \otimes a]\!]$$

and the multiplication map $\mathbf{R}P^\infty \times \mathbf{R}P^\infty \to \mathbf{R}P^\infty$ gives a formal group law over MO_*. By Quillen [Qui69a], it is the universal formal group law F over a ring of characteristic 2 for which $F(a, a) = 0$.

As described by Quillen [Qui71, age 53], the formal group can be used to give convenient generators for the unoriented cobordism ring. Let

$$e \in H^1(\mathbf{R}P^\infty; \mathbf{Z}/2)$$

be the $H\mathbf{Z}/2$ Euler class of the tautological line bundle. Over $\pi_* H\mathbf{Z}/2 \wedge MO$ there is a power series relating e and the image of the class a

$$e = \ell(a) = a + \sum \alpha_n a^{n+1}.$$

Lemma 12.2.31. A power series over $\pi_* MO$. *The composite series*

$$\left(a + \sum_{j>0} \alpha_{2^j-1} a^{2^j}\right)^{-1} \circ \ell(a) = a + \sum_{j>0} x_j a^{j+1} \tag{12.2.32}$$

has coefficients in $\pi_* MO$. The classes x_j with $j + 1 = 2^k$ are zero. The remaining x_j are polynomial generators for the unoriented cobordism ring

$$\pi_* MO = \mathbf{Z}/2[x_j, j \neq 2^k - 1]. \tag{12.2.33}$$

The unoriented cobordism ring was originally determined by Thom in [Tho54, Théorème IV.12]. Our notation for the generators is that of Stong [Sto68a, page 40].

Proof The assertion that $x_j = 0$ for $j + 1 = 2^k$ is straightforward. Since the sequence

$$\pi_* MO \to \pi_* H\mathbf{Z}/2 \wedge MO \rightrightarrows \pi_* H\mathbf{Z}/2 \wedge H\mathbf{Z}/2 \wedge MO \tag{12.2.34}$$

is a split equalizer, to show that the remaining x_j are in $\pi_* MO$ it suffices to show that they are equalized by the parallel maps in (12.2.34). This works out to showing that the series (12.2.32) is invariant under substitutions of the form

$$e \mapsto e + \sum e_m e^{2^m}, \tag{12.2.35}$$

The series (12.2.32) is characterized as the unique isomorphism of the formal group law for unoriented cobordism with the additive group, having the additional property that the coefficients of a^{2^k} are zero. This condition is stable under the substitutions (12.2.35). The last assertion follows from Quillen's characterization of $\pi_* MO$. $\square$

Remark 12.2.36. A class in $MO^1(\mathbf{R}P^\infty)$. *Recall the real orientation $\overline{x}$ of $i_2^G MU^{((G))}$ of Example 12.2.11. Applying the $RO(G)$-graded cohomology norm (§9.7C) to $\overline{x}$, and then passing to geometric fixed points, gives a class*

$$\Phi^G N(\overline{x}) \in MO^1(\mathbf{R}P^\infty).$$

One can easily check that $\Phi^G N(\overline{x})$ coincides with the MO Euler class a defined at the beginning of this section. Similarly one has

$$\Phi^G N(\overline{x}_H) = e.$$

Applying $\Phi^G N$ to $\log_{\overline{F}}$ and using the fact that it is a ring homomorphism (Proposition 9.11.52) gives

$$e = a + \sum \Phi^G N(\overline{m}_k) a^{k+1}.$$

It follows that

$$\Phi^G N(\overline{m}_k) = \alpha_k.$$

12.2D Refinement of homotopy groups

We begin by stating a simple consequence of Proposition 12.2.17.

Proposition 12.2.37. Refining the C_2-equivariant homotopy of smash powers of $MU_{\mathbf{R}}$. *For every $m > 1$, every element of*

$$\pi_{2k}\left(\bigwedge^m MU\right)$$

can be refined (see Definition 11.3.19) to an equivariant map

$$S^{k\rho_2} \to \bigwedge^m MU_{\mathbf{R}}.$$

This result expresses an important property of the C_2-spectra given by iterated smash products of $MU_{\mathbf{R}}$. Our goal in this section is to formulate a generalization to the case $G = C_{2^n}$.

Remark 12.2.38. The underlying bottom homotopy group of a slice sphere. *Let $\sigma_G(\mathbf{Z})$ be the sign representation of G on $\mathbf{Z}$, also known as $\mathbf{Z}_-$. We use the former notation here because we will consider it for more than one group. There is an G-module isomorphism*

$$\pi^u_{|G|} S^{\rho_G} \cong \sigma_G(\mathbf{Z}),$$

and more generally

$$\pi^u_{n|H|}\big(G \underset{H}{\ltimes} \wedge S^{n\rho_H}\big) \cong \mathrm{Ind}_H^G \sigma_H(\mathbf{Z})^{\otimes n}.$$

This implies that when k is even, a necessary condition for $\pi^u_k X$ to admit a refinement is that it be isomorphic as a G-module to a sum

$$\bigoplus_{H\subset G} M_{H,k}$$

where $M_{H,k}$ is zero unless $|H|$ divides k and is a sum of copies of $\mathrm{Ind}_H^G\big(\sigma_H(\mathbf{Z})^{\otimes \ell}\big)$ when $k = \ell|H|$. Adding the further condition that for every $H \subset G$, with $k = \ell|H|$, every element in $\pi^u_k X$ transforming in $\sigma_H(\mathbf{Z})^{\otimes\ell}$ refines to an element of $\pi^H_{\ell\rho_H} X$ makes it sufficient. A similar analysis describes the case in which k is odd.

Remark 12.2.39. *Using Remark 12.2.38 one can check that a refinement of $\pi^u_k X$ consists of bound slice spheres if and only if $\pi^u_k X$ does not contain a free G-module as a summand.*

The splitting (12.2.18) used to prove Proposition 12.2.37 is multiplicative. This too has an important analogue.

Definition 12.2.40. *Suppose that R is an equivariant associative algebra. A* **multiplicative refinement of homotopy** *is an associative algebra map $\widehat{W} \to R$ which, when regarded as a map of G-spectra is a refinement of homotopy.*

Proposition 12.2.41. The refinement of $\pi^u_* MU^{((G))}$. *For every $m \geqslant 1$ there exists a multiplicative refinement of homotopy*

$$\widehat{W} \to \bigwedge^m MU^{((G))},$$

with $\widehat{W}$ a wedge of bound slice spheres.

Two ingredients form the proof of Proposition 12.2.41. The first, Lemma 12.2.42 below, is a description of $\pi^u_* MU^{((G))}$ as a G-module. The computation is of interest in its own right, and is used elsewhere in this book. It will be proved in §12.3. The second is the classical description of $\pi^u_*(\bigwedge^m MU^{((G))})$, $m > 1$, as a $\pi^u_* MU^{((G))}$-module.

Lemma 12.2.42. $\pi_*^u MU^{((G))}$ as a polynomial algebra. *There is a sequence of elements $r_i \in \pi_{2i}^u MU^{((G))}$ with the property that*

$$\pi_*^u MU^{((G))} = \mathbf{Z}_{(2)}[G \cdot r_1, G \cdot r_2, \dots], \tag{12.2.43}$$

in which $G \cdot r_i$ stands for the sequence

$$(r_i, \dots \gamma^{\frac{g}{2}-1} r_i)$$

of length $g/2$.

We refer to the condition (12.2.43) by saying that the elements $r_i \in \pi_{2i}^u MU^{((G))}$ form a set of **G-algebra generators for $\pi_*^u MU^{((G))}$.**

Remark 12.2.44. *Lemma 12.2.42 completely describes $\pi_*^u MU^{((G))}$* **as a representation of** *G. To spell it out, recall from Remark 12.2.22 that the action of the generator of C_2 on $\pi_{2i}^u MU^{((G))}$ is by $(-1)^i$. The elements $r_i \in \pi_{2i}^u MU^{((G))}$ therefore satisfy $\gamma^{\frac{g}{2}} r_i = (-1)^i r_i$ and transform in the representation induced from the sign representation of C_2 if i is odd and in the representation induced from the trivial representation of C_2 if i is even. Lemma 12.2.42 implies that the map from the symmetric algebras on the sum of these representations to $\pi_*^u MU^{((G))}$ is an isomorphism.*

Note that **we are not defining the** *$r_i \in \pi_{2i}^u MU^{((G))}$* **explicitly** *in Lemma 12.2.42. We will give a precise definition of related elements $\overline{r}_i \in \pi_{i\pi_{C_2}}^{C_2} i_2^G MU^{((G))}$ in Definition 12.3.6 below.*

Remark 12.2.45. The refinement of $\pi_*^u MU^{((G))}$ is bound. *The fact that the action of C_2 on $\pi_{2i}^u MU^{((G))}$ is either a sum of sign or trivial representations means that it cannot contain a summand which is free. The same is therefore true of the G-action. By Remark 12.2.39 this implies that only bound slice spheres may occur in a refinement of $\pi_{2i}^u MU^{((G))}$.*

Over $\pi_*^u MU^{((G))} \wedge MU^{((G))}$, there are two formal group laws, F_L and F_R coming from the canonical orientations of the left and right factors. There is also a canonical isomorphism between them, which can be written as

$$x_R = \sum b_j x_L^{j+1}.$$

Write

$$G \cdot b_i$$

for the sequence

$$b_i, \gamma b_i, \dots, \gamma^{g/2-1} b_i.$$

The following result is a standard computation in complex cobordism.

Lemma 12.2.46. The ring $\pi_*^u MU^{((G))} \wedge MU^{((G))}$ *is given by*

$$\pi_*^u MU^{((G))} \wedge MU^{((G))} = \pi_*^u MU^{((G))}[G \cdot b_1, G \cdot b_2, \dots].$$

For $m > 1$,

$$\pi_*^u \bigwedge^m MU^{((G))} = \pi_*^u \left(MU^{((G))} \wedge \bigwedge^{m-1} MU^{((G))} \right)$$

is the polynomial ring

$$\pi_*^u MU^{((G))}[G \cdot b_i^{(j)} \colon i > 0, 0 < j < m],$$

The element $b_i^{(j)}$ is the class b_i arising from the jth factor of $MU^{((G))}$ in $\bigwedge^{m-1} MU^{((G))}$.

Proof The second assertion follows from the first and the Künneth formula. If not for the fact that G acts on both factors of $i_0^G MU^{((G))}$, the first assertion would also follow immediately from the Künneth formula and the usual description of $MU_* MU$. The quickest way to deduce it from the apparatus we have describe so far is to let $G \subset \hat{G}$ be an embedding of index 2 into a cyclic group, write

$$MU^{((G))} \wedge MU^{((G))} \cong i_G^{\hat{G}} MU^{((\hat{G}))}$$

and use Corollary 12.2.30. □

Remark 12.2.47. *As with Lemma 12.2.42, the lemma above actually determines* **the structure of $\pi_*^u MU^{((G))} \wedge MU^{((G))}$ as a G-equivariant $\pi_*^u MU^{((G))}$-algebra**. *See Remark 12.2.44.*

Proof of Proposition 12.2.41, assuming Lemma 12.2.42. This is a straightforward application of the method of twisted monoid rings of §10.10. To keep the notation simple we begin with the case $m = 1$. Choose a sequence $r_i \in \pi_{2i}^u MU^{((G))}$ with the property described in Lemma 12.2.42. Since $MU^{((G))}$ is a commutative algebra, the method of twisted monoid rings can be used to construct an associative algebra map

$$S^{-0}[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots] \to MU^{((G))}, \tag{12.2.48}$$

Using Proposition 11.1.10 one can easily check that $S^{-0}[G \cdot \bar{r}_1, G \cdot \bar{r}_2, \dots]$ is a wedge of bound G-slice spheres. Using Lemma 12.2.42 one then easily checks that (12.2.48) is multiplicative refinement of homotopy. The case $m \geqslant 1$ is similar, using in addition Lemma 12.2.46 and the collection $\{r_i, b_i(j)\}$. □

12.3 Algebra generators for $\pi_*^u MU^{((G))}$

In this section we will describe convenient algebra generators for $\pi_*^u MU^{((G))}$. Our two main results are Proposition 12.3.5, which gives a criterion for a sequence of elements r_i to "generate" $\pi_*^u MU^{((G))}$ as a G-algebra as in Lemma 12.2.42, and Corollary 12.3.9, which specifies a particular sequence of r_i. Proposition 12.3.5 implies Lemma 12.2.42.

We remind the reader that the notation $H_*^u X$ refers to the homology groups $H_*(i_0^G X)$ of the non-equivariant spectrum underlying X.

12.3A A criterion for a generating set

Let

$$m_i \in H_{2i}MU = \pi_{2i}^u H\underline{\mathbf{Z}} \wedge MU_{\mathbf{R}}$$

be the coefficient of the universal logarithm. Using the identification (12.2.8)

$$i_2^G MU^{((G))} = \bigwedge_{j=0}^{g/2-1} \gamma^j MU_{\mathbf{R}}$$

and the Künneth formula, one has

$$H_*^u MU^{((G))} = \mathbf{Z}_{(2)}[\gamma^j m_i],$$

where

$$\begin{aligned} i &= 1, 2, \ldots, \\ j &= 0, \ldots, g/2 - 1. \end{aligned}$$

By the definition of the $\gamma^j m_i$ and Remark 12.2.22, the action of G on $H_*^u MU^{((G))}$ is given by

$$\gamma \cdot \gamma^j m_i = \begin{cases} \gamma^{j+1} m_i & j < g/2 - 1 \\ (-1)^i m_i & j = g/2 - 1\,. \end{cases} \tag{12.3.1}$$

Let

$$\begin{aligned} I &= \ker \pi_*^u MU^{((G))} \to \mathbf{Z}_{(2)} \\ I_H &= \ker H_*^u MU^{((G))} \to \mathbf{Z}_{(2)} \end{aligned}$$

denote the augmentation ideals, and

$$\begin{aligned} Q_* &= I/I^2 \\ QH_* &= I_H/I_H^2 \end{aligned}$$

the modules of indecomposable, with Q_{2m} and QH_{2m} indicating the homogeneous parts of degree $2m$ (the odd degree parts are zero). The module QH_* is the free abelian group with basis $\{\gamma^j m_i\}$, and from Milnor [Mil60], one knows that the Hurewicz homomorphism gives an isomorphism

$$Q_{2i} \to QH_{2i}$$

if $2i$ is not of the form $2(2^n - 1)$, and an exact sequence

$$Q_{2(2^n-1)} \rightarrowtail QH_{2(2^n-1)} \twoheadrightarrow \mathbf{Z}/2 \qquad (12.3.2)$$

in which the rightmost map is the one sending each $\gamma^j m_i$ to 1.

Formula (12.3.1) implies that the G-module QH_{2i} is the module induced from the sign representation of C_2 if i is odd and from the trivial representation if i is even.

Lemma 12.3.3. The $\mathbf{Z}_{(2)}[G]$-module structure of the indecomposable homology. *For a given $i > 0$, let*

$$r = \sum_j w_{i,j}\gamma^j m_i \in QH_{2i}.$$

The unique G-module map

$$\begin{aligned} \mathbf{Z}_{(2)}[G] &\to QH_{2i} \\ 1 &\mapsto r \end{aligned}$$

factors through a map

$$\mathbf{Z}_{(2)}[G]/(\gamma^{g/2} - (-1)^i) \to QH_{2i}$$

which is an isomorphism if and only if $\sum w_{i,j} \equiv 1 \mod 2$.

Proof The factorization is clear, since $\gamma^{g/2}$ acts with eigenvalue $(-1)^i$ on QH_{2i}. Use the unique map $\mathbf{Z}_{(2)}[G] \to QH_{2i}$ sending 1 to m_i to identify QH_{2i} with $A = \mathbf{Z}_{(2)}[G]/(\gamma^{g/2} - (-1)^i)$. The main assertion is then that an element $r = \sum w_j\gamma^j \in A$ is a unit if and only if $\sum w_j \equiv 1 \mod 2$. Since A is a finitely generated free module over the Noetherian local ring $\mathbf{Z}_{(2)}$, Nakayama's lemma implies that the map $A \to A$ given by multiplication by r is an isomorphism if and only if it is after reduction modulo 2. So r is a unit if and only if it is after reduction modulo 2. But $A/(2) = \mathbf{Z}/2[\gamma]/(\gamma^{g/2} - 1)$ is a local ring with nilpotent maximal ideal $(\gamma - 1)$. The residue map

$$A/(2) \to A/(2, \gamma - 1) = \mathbf{Z}/2$$

sends $\sum w_{i,j}\gamma^j m_i$ to $\sum w_{i,j}$. The result follows. □

Lemma 12.3.4. The structure in dimension $2(2^n - 1)$. *The G-module $Q_{2(2^n-1)}$ is isomorphic to the module induced from the sign representation of C_2. For $y \in QH_{2(2^n-1)}$, the unique G-map*

$$\begin{aligned} \mathbf{Z}_{(2)}[G] &\to QH_{2(2^n-1)} \\ 1 &\mapsto y \end{aligned}$$

factors through a map

$$A = \mathbf{Z}_{(2)}[G]/(\gamma^{g/2} + 1) \to Q_{2(2^n-1)}$$

which is an isomorphism if and only if $y = (1-\gamma)r$ where $r \in QH_{2(2^n-1)}$ satisfies the condition $\sum w_j = 1 \mod 2$ of Lemma 12.3.3.

Proof Identify $QH_{2(2^n-1)}$ with A by the map sending 1 to m_{2^n-1}. In this case A is isomorphic to $\mathbf{Z}_{(2)}[\zeta]$, with ζ a primitive gth root of unity, and in particular is an integral domain. Under this identification, the rightmost map in (12.3.2) is the quotient of A by the principal ideal $(\zeta - 1)$. Since A is an integral domain, this ideal is a rank 1 free module generated by any element of the form $(1-\gamma)r$ with $r \in A$ a unit. The result follows. □

This discussion proves

Proposition 12.3.5. Recognizing polynomial generators. *Let*

$$\{r_1, r_2, \dots\} \subset \pi_*^u MU^{((G))}$$

be any sequence of elements whose images

$$s_i \in QH_{2i}$$

have the property that

$$s_i = \begin{cases} (1-\gamma)\sum_j w_{i,j}\gamma^j m_i & \text{for } i = 2^n - 1 \\ \sum_j w_{i,j}\gamma^j m_i & \text{otherwise.} \end{cases}$$

where $\sum_j w_{i,j}$ is odd for each $i > 0$. Then the sequence

$$\{r_1, \dots \gamma^{\frac{g}{2}-1} r_1, r_2, \dots, \gamma^{\frac{g}{2}-1} r_2, \dots\}$$

generates the ideal I, and the map

$$\mathbf{Z}_{(2)}[r_1, \dots \gamma^{\frac{g}{2}-1} r_1, r_2, \dots, \gamma^{\frac{g}{2}-1} r_2, \dots] \to \pi_*^u MU^{((G))}$$

is an isomorphism.

12.3B Specific generators

We now use the action of G on $i_0^G MU^{((G))}$ to define specific elements $\bar{r}_i \in \pi_{i\rho_2}^{C_2} MU^{((G))}$ refining a sequence satisfying the condition of Proposition 12.3.5.

Write

$$\overline{F}(\overline{x}, \overline{y})$$

for the formal group law over $\pi_\star^{C_2} MU^{((G))}$, and

$$\log_{\overline{F}}(\overline{x}) = \overline{x} + \sum_{i>0} \overline{m}_i \overline{x}^{i+1}$$

for its logarithm. This defines elements

$$\overline{m}_i \in \pi_{i\rho_2}^{C_2} H\underline{\mathbf{Z}}_{(2)} \wedge MU^{((G))} \cong \underline{H}_{i\rho_2}(MU^{((G))}; \underline{\mathbf{Z}}_{(2)})(G/C_2).$$

It is known that for $G = C_2$, $i\overline{m}_i$ is the image of $[\mathbf{C}P^i]$ (the element of $\pi_{2i}MU$ corresponding to the cobordism class of the complex manifold $\mathbf{C}P^i$) under the composite of the 2-local Hurewicz map and the map of (12.2.15).

Definition 12.3.6. Specific generators $\bar{r}_i \in \pi^{C_2}_{i\rho_2} MU^{((G))}$. *The elements*

$$\bar{r}_i = \bar{r}_i^G \in \pi^{C_2}_{i\rho_2} MU^{((G))}$$

are the coefficients of the unique strict isomorphism of $\overline{F}$ with the 2-typification of $\overline{F}^{\gamma}$. The Hurewicz images

$$\bar{r}_i \in \pi^{C_2}_{i\rho_2} H\underline{\mathbf{Z}}_{(2)} \wedge MU^{((G))}$$

are given by the power series identity

$$\sum \bar{r}_i \bar{x}^{i+1} = \Big(\bar{x} + \sum \gamma(\overline{m}_{2^n-1})\bar{x}^{2^{\ell}}\Big)^{-1} \circ \log_{\overline{F}}(\bar{x}). \tag{12.3.7}$$

Modulo decomposables this becomes

$$\bar{r}_i \equiv \begin{cases} (1-\gamma)\overline{m}_i & i = 2^n - 1 \\ \overline{m}_i & \text{otherwise.} \end{cases} \tag{12.3.8}$$

This shows that the elements $\bar{r}_i$ satisfy the condition of Proposition 12.3.5, hence

Corollary 12.3.9. Our specific polynomial generators. *The classes*

$$r_i = i_0^G \bar{r}_i$$

form a set of G-algebra generators for $\pi^u_* MU^{((G))}$.

The $\bar{r}_i$ of Definition 12.3.6 are the specific generators we will use. In §13.3 we will need to consider the classes $\bar{r}_i$ for a group G and for a subgroup $H \subset G$. We will then use the notation

$$\bar{r}_i^H \text{ and } \bar{r}_i^G$$

to distinguish them.

The following result establishes an important property of these specific $\bar{r}_i$. In the statement below, the symbol N is the norm map $N_{C_2}^G$ on the $RO(G)$-graded homotopy groups of commutative rings.

Proposition 12.3.10. The image of Φ^G on norms of our generators. *For all $i > 0$*

$$\Phi^G N(\bar{r}_i) = x_i \in \pi_i MO,$$

where the x_i are the classes defined in Lemma 12.2.31. In particular, the set

$$\{\Phi^G N(\bar{r}_i) \mid i \neq 2^n - 1\}$$

is a set of polynomial algebra generators of $\pi_* MO$, and for all ℓ

$$\Phi^G N(\bar{r}_{2^n-1}) = h_{2^n-1} = 0.$$

Proof From Remark 12.2.36 we know that

$$\begin{aligned}\Phi^G N\overline{x} &= a\\ \Phi^G N\overline{x}_H &= e\\ \Phi^G N\overline{m}_n &= \alpha_n.\end{aligned}$$

Corollary 10.7.7 implies that

$$\Phi^G N\gamma\overline{m}_n = \Phi^G N\overline{m}_n,$$

so we also know that

$$\Phi^G N\gamma\overline{m}_n = \alpha_n.$$

Since the Hurewicz homomorphism

$$\begin{tikzcd}
\pi_*\Phi^G MU^{((G))} \arrow[r] \arrow[d, "\cong"'] & \pi_*\Phi^G(H\underline{\mathbf{Z}}_{(2)} \wedge MU^{((G))}) \arrow[d, "\cong"] \\
\pi_* MO \arrow[r] & \pi_* H\mathbf{Z}/2[b] \wedge MO
\end{tikzcd}$$

is a monomorphism, we can calculate $\Phi^G N\overline{r}_i$ using (12.3.7). Applying $\Phi^G N$ to (12.3.7), and using the fact that it is a ring homomorphism gives

$$\begin{aligned}a+&\sum(\Phi^G N\overline{r}_i)a^{i+1}\\ &= \left(a+\sum(\Phi^G N\gamma\overline{m}_{2^n-1})a^{2^\ell}\right)^{-1}\circ\left(a+\sum(\Phi^G N\overline{m}_i)a^{i+1}\right)\\ &= \left(a+\sum\alpha_{2^n-1}a^{2^\ell}\right)^{-1}\circ\left(a+\sum\alpha_i a^{i+1}\right).\end{aligned}$$

But this is the identity defining the classes x_i. □

In addition to

$$h_i = \Phi^G N(\overline{r}_i) \in \pi_i\Phi^G MU^{((G))} = \pi_i MO$$

there are some important classes f_i attached to these specific $\overline{r}_i$.

Definition 12.3.11. The elements f_i in $\pi_*^G MU^{((G))}$. *Set*

$$f_i = a_{\overline{\rho}_G}^i N\overline{r}_i \in \pi_i^G MU^{((G))},$$

where $\overline{\rho}_G$ is the reduced regular representation.

The relationship between these classes is displayed in the following commutative diagram.

$$\begin{tikzcd}
& S^i \arrow[dl, "a_{\overline{\rho}_G}^i"'] \arrow[d, "f_i"] \arrow[dr, "x_i"] & \\
S^{i\rho_G} \arrow[r, "N\overline{r}_i"] & MU^{((G))} \arrow[r] & \widetilde{E}\mathcal{P} \wedge MU^{((G))}.
\end{tikzcd} \tag{12.3.12}$$

12.4 The slice structure of $MU^{((G))}$

The results of this section are critical to the calculations that follow. Here we identify the slices of the spectrum $MU^{((G))}$ of (12.2.6) for a finite cyclic 2-group $G \cong C_{2^n}$.

Using the method of twisted monoid rings of §10.10C, one can show the Slice Theorem 12.4.1 and the Reduction Theorem 12.4.8 to be equivalent. In §12.4 we formally state the Reduction Theorem, and assuming it, prove the Slice Theorem. In §12.4B we establish a converse, for associative algebras R which are pure and which admit a multiplicative refinement of homotopy by a polynomial algebra. Both assertions are used in the proof of the Reduction Theorem itself in §12.4E.

12.4A The slice theorem

We now state the Slice Theorem, which will enable us to identify the slices of the spectrum $MU^{((G))}$ of (12.2.6).

Slice Theorem 12.4.1. *The spectrum $MU^{((G))}$ is pure as in Definition 11.3.14.*

For the proof of the slice theorem, let

$$A_G = S^{-0}[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \dots] \to MU^{((G))}$$

be the multiplicative refinement of homotopy constructed in Proposition 12.2.41 using the method of twisted monoid rings of §10.10, and the specific generators of Definition 12.3.6. Let T be the left G-set defined by

$$T = \coprod_{i>0} G/C_2. \tag{12.4.2}$$

This is a disjoint union of copies of G/C_2, one for each positive integer i. Hence each element $t \in T$ has an integer $i(t) > 0$ associated with it.

As described in §10.10, the spectrum A_G is the indexed wedge

$$A_G = S^{-0} \wedge \bigvee_{f \in \mathbf{N}_0^T} S^{\rho_f}, \tag{12.4.3}$$

where the wedge is over all finitely supported $\mathbf{N}_0$-valued functions f on T. The group G acts on the set of such functions, so each function f has a stabilizer group $G_f \subseteq G$. It always contains the subgroup $C_2 \subseteq G$ since the latter acts trivially on T. In (12.4.3), ρ_f is the unique multiple of the regular representation of G_f having dimension

$$\dim \rho_f = 2 \sum_{t \in T} i(t)\, f(t). \tag{12.4.4}$$

The spectrum A_G is a wedge of bound (as in Definition 11.1.6) slice spheres of various dimensions.

Example 12.4.5. The case $G = C_4$. *Let $\gamma \in G$ be a generator. The stabilizer group G_f of a function f is all of G iff $f(\gamma t) = f(t)$ for all $t \in T$ as in (12.4.2). This means the sum in (12.4.4) is even, so the dimension of ρ_f is divisible by 4, as we would expect since ρ_f is a multiple of ρ_G.*

Similarly $G_f = C_2$ iff there is a $t \in T$ for which $f(\gamma t) \neq f(t)$. In that case the function γf defined by $\gamma f(t) = f(\gamma t)$ is distinct from f. Both ρ_f and $\rho_{\gamma f}$ are multiples of ρ_{C_2}, The corresponding summand of A_G is

$$S^{-0} \wedge (S^{\rho_f} \vee S^{\rho_{\gamma f}}) \cong S^{-0} \wedge G \underset{C_2}{\ltimes} S^{\rho_f} \cong \widehat{S}((\dim \rho_f)/2, C_2)$$

as in Definition 11.1.1.

As in Example 10.10.9(ii), let

$$M_d \subset A_G$$

be the monomial ideal consisting of the indexed wedge of the $S^{-0} \wedge S^{\rho_f}$ in A_G with $\dim \rho_f \geqslant d$. Then $M_{2d-1} = M_{2d}$, and the M_{2d} fit into a sequence

$$\cdots \hookrightarrow M_{2d+2} \hookrightarrow M_{2d} \hookrightarrow M_{2d-2} \hookrightarrow \cdots .$$

The quotient

$$M_{2d}/M_{2d+2}$$

is the indexed wedge

$$\widehat{W}_{2d} = S^{-0} \wedge \bigvee_{\dim \rho_f = 2d} S^{\rho_f}$$

on which A_G is acting through the multiplicative map $A_G \to S^{-0}$ (Example 10.10.9(ii) and Example 2.9.58). This G-spectrum is a wedge of bound slice spheres of dimension $2d$.

Replace $MU^{((G))}$ with a cofibrant right A_G-module (see Proposition 10.8.1 for the model structure for A_G-modules), and form

$$K_{2d} = MU^{((G))}{}_A\, G \wedge M_{2d}.$$

The K_{2d} fit into a sequence

$$\cdots \hookrightarrow K_{2d+2} \hookrightarrow K_{2d} \hookrightarrow \cdots .$$

Lemma 12.4.6. Some modules over the associative algebra A_G. *The sequences*

$$K_{2d+2} \to K_{2d} \to K_{2d}/K_{2d+2}$$
$$K_{2d}/K_{2d+2} \to MU^{((G))}/K_{2d+2} \to MU^{((G))}/K_{2d}$$

are weakly equivalent to cofibration sequences. There is an equivalence

$$K_{2d}/K_{2d+2} \cong R(\infty) \wedge \widehat{W}_{2d}$$

in which

$$R_G(\infty) = MU^{((G))} \underset{A_G}{\wedge} S^{-0}. \tag{12.4.7}$$

Proof Since the map $K_{2d+2} \to K_{2d}$ is the inclusion of a wedge summand it is a cofibration of spectra, the first assertion follows from Proposition 9.4.3 (iii) and Corollary 10.8.4. The second assertion follows from the associativity of the smash product

$$MU^{((G))} \underset{A_G}{\wedge} (M_{2d}/M_{2d+1}) \cong (MU^{((G))} \underset{A_G}{\wedge} S^{-0}) \wedge \widehat{W}_{2d} \cong R_G(\infty) \wedge \widehat{W}_{2d}.$$

This completes the proof. □

The Thom map

$$MU^{((G))} \to H\underline{\mathbf{Z}}_{(2)}$$

factors uniquely through an $MU^{((G))}$-module map

$$R_G(\infty) \to H\underline{\mathbf{Z}}_{(2)}.$$

The following important result will be proved in §12.4E.

Reduction Theorem 12.4.8. *The map*

$$R_G(\infty) \to H\underline{\mathbf{Z}}_{(2)}$$

(for $R_G(\infty)$ as in (12.4.7)) is a weak equivalence.

The case $G = C_2$ of this result is proved by Hu-Kriz as [HK01, Proposition 4.9]. Its analogue in motivic homotopy theory appears in unpublished work of the second author and Morel.

To deduce the Slice Theorem 12.4.1 from the Reduction Theorem 12.4.8 we need two simple lemmas.

Lemma 12.4.9. Connectivity. *The spectrum K_{2d+2} is slice $(2d+1)$-connected.*

Proof The class of left A_G-modules M for which $M \underset{A_G}{\wedge} M_{2d+2} > 2d+1$ is closed under homotopy colimits and extensions. It contains every module of the form $\Sigma^k G/H \ltimes A_G$, with $k \geqslant 0$. Since A_G is (-1)-connected this means it contains every (-1)-connected cofibrant A_G-module. In particular it contains the cofibrant replacement of $MU^{((G))}$. □

Lemma 12.4.10. Coconnectivity. *If the Reduction Theorem 12.4.8 holds, then*

$$MU^{((G))}/K_{2d+2} \leqslant 2d.$$

Proof This is easily proved by induction on d, using the fact that

$$R_G(\infty) \wedge \widehat{W}_{2d} \to MU^{((G))}/K_{2d+2} \to MU^{((G))}/K_{2d}.$$

is weakly equivalent to a cofibration sequence (Lemma 12.4.6). □

Proof of the Slice Theorem 12.4.1 assuming the Reduction Theorem 12.4.8. It follows from the fibration sequence

$$K_{2d+2} \to MU^{((G))} \to MU^{((G))}/K_{2d+2},$$

Lemma 12.4.9 and Lemma 12.4.10 above, and Lemma 11.1.38 that

$$P^{2d+1}MU^{((G))} \cong P^{2d}MU^{((G))} \cong MU^{((G))}/K_{2d+2}.$$

Thus the odd slices of $MU^{((G))}$ are contractible and the $2d$-slice is weakly equivalent to

$$R_G(\infty) \wedge \widehat{W}_{2d} \cong H\underline{\mathbf{Z}}_{(2)} \wedge \widehat{W}_{2d}.$$

This completes the proof. □

12.4B A converse to the slice theorem

The arguments of the previous subsection can be reversed. Suppose that R is a (-1)-connected associative algebra which we know in advance to be pure, and that $A \to R$ is a multiplicative refinement of homotopy, with

$$A = S^{-0}[G \cdot \bar{x}_1, \dots]$$

a twisted monoid ring having the property that $|\bar{x}_i| > 0$ for all i. Note that this implies that $\pi_0^u R = \mathbf{Z}$ and that $P_0^0 R = H\underline{\mathbf{Z}}$. Let $M_{d+1} \subset A$ be the monomial ideal consisting of the slice spheres in A of dimension $> d$, write

$$\widetilde{P}_{d+1}R = M_{d+1} \underset{A}{\wedge} R$$

and

$$\widetilde{P}^d R = R/\widetilde{P}_{d+1}R \cong (A/M_{d+1}) \underset{A}{\wedge} R.$$

Then the $\widetilde{P}^d R$ form a tower. Since $M_{d+1} > d$ and $R \geqslant 0$ (Proposition 11.1.45), the spectrum $\widetilde{P}_{d+1}R$ is slice d-connected. There is therefore a map

$$\widetilde{P}^d R \to P^d R, \tag{12.4.11}$$

compatible with variation in d.

Proposition 12.4.12. The slice tower of a pure associative algebra. *The map of (12.4.11) is a weak equivalence. The tower $\{\tilde{P}^d R\}$ is the slice tower for R.*

By analogy with the slice tower, write $\widetilde{P}^d_{d'}R$ for the homotopy fiber of the map

$$\widetilde{P}^d R \to \widetilde{P}^{d'-1}R,$$

when $d' \leqslant d$.

We start with a lemma concerning the case $d = 0$.

Lemma 12.4.13. Leveraging the 0-slice. *Let $n \geqslant 0$. If the map*

$$\widetilde{P}^0 R \to P^0 R$$

becomes an equivalence after applying P^n, then for every $d \geqslant 0$ the map

$$\widetilde{P}^d_d R \to P^d_d R$$

becomes an equivalence after applying P^{d+n}.

Proof Write $\widehat{W}_d = M_d/M_{d+1}$. Then there are equivalences

$$\widetilde{P}^d_d R \cong \widehat{W}_d \underset{A}{\wedge} R \cong \widehat{W}_d \wedge (S^{-0} \underset{A}{\wedge} R) \cong \widehat{W}_d \wedge \widetilde{P}^0_0 R.$$

Since $A \to R$ is a refinement of homotopy and R is pure, the analogous map

$$\widehat{W}_d \wedge P^0_0 R \to P^d_d R$$

is also a weak equivalence. Now consider the following diagram

$$\begin{tikzcd}
\widehat{W}_d \wedge P^n(\widetilde{P}^0_0 R) \arrow[r, "\sim"] \arrow[d] & \widehat{W}_d \wedge P^n(P^0_0 R) \arrow[d] \\
P^{d+n}\widehat{W}_d \wedge \widetilde{P}^0_0 R) \arrow[r] \arrow[d, "\sim"'] & P^{d+n}\widehat{W}_d \wedge P^0_0 R) \arrow[d, "\sim"] \\
P^{d+n}(\widetilde{P}^d_d R) \arrow[r] & P^{d+n}(P^d_d R)
\end{tikzcd}$$

The top map is an equivalence by assumption. The bottom vertical maps are the result of applying P^{d+n} to the weak equivalences just described. Since $\widehat{W}_d$ is a wedge of slice spheres of dimension d, Corollary 11.1.33 implies that the upper vertical maps are weak equivalences. It follows that the bottom horizontal map is a weak equivalence as well. □

Proof of Proposition 12.4.12. We will show by induction on k that for all d, the map

$$P^{d+k}(\tilde{P}^d R) \to P^{d+k}(P^d R)$$

is a weak equivalence. By the strong convergence of the slice tower (11.2.10) this will give the result. The induction starts with $k = 0$ since $\widetilde{P}_{d+1} R > d$ and so $R \to \tilde{P}^d R$ is a P^d-equivalence. For the induction step, suppose we know the result for some $k \geqslant 0$, and consider

$$\begin{tikzcd}
P^{d+k}\widetilde{P}^d_d R \arrow[r] \arrow[d, "\sim"'] & P^{d+k}(\widetilde{P}^d R) \arrow[r] \arrow[d, "\sim"'] & P^{d+k}(\widetilde{P}^{d-1} R) \arrow[d] \\
P^{d+k}(P^d_d R) \arrow[r] & P^{d+k}(P^d R) \arrow[r] & P^{d+k}(P^{d-1} R)
\end{tikzcd}$$

The bottom row is a cofibration sequence since it can be identified with

$$P_d^d R \to P^d R \to P^{d-1} R.$$

The middle vertical map is a weak equivalence by the induction hypothesis, and the left vertical map is a weak equivalence by the induction hypothesis and Lemma 12.4.13. It follows that the cofiber of the upper left map is weakly equivalent to $P^{d+k}(P^{d-1}R)$ and hence is $(d+k+1)$-slice null (in fact d slice null). The top row is therefore a cofibration sequence by Corollary 11.1.39, and so the rightmost vertical map is a weak equivalence. This completes the inductive step and the proof. □

12.4C The inductive approach to the Reduction Theorem

We will prove the Reduction Theorem 12.4.8 by induction on $|G|$. The case in which G is the trivial group follows from Quillen's results. We may therefore assume that we are working with a non-trivial group G and that the Reduction Theorem is known for all proper subgroups of G. In this section we will collect some consequences of this induction hypothesis. The proof of the inductive step is in §12.4E.

This next result holds for general G.

Lemma 12.4.14. Smashing with bound slice spheres preserves purity. *Suppose that X is a pure spectrum (Definition 11.3.14) and $\widehat{W}$ is a wedge of bound slice spheres as in Definition 11.1.6. Then $\widehat{W} \wedge X$ is pure.*

Proof Using Proposition 11.1.46 one reduces to the case in which $\widehat{W} = S^{m\rho_G}$. In that case the claim follows from Corollary 11.1.33. □

Proposition 12.4.15. Purity over the index two subgroup. *Let $G' \subset G$ be the index 2 subgroup. If the Slice Theorem holds for G' then the spectrum $i_{G'}^G MU^{((G))}$ is pure.*

Proof This is an easy consequence of Corollary 12.2.30, which gives an associative algebra equivalence

$$i_{G'}^G MU^{((G))} \cong MU^{((G'))}[G' \cdot \bar{b}_1, G' \cdot \bar{b}_2, \ldots].$$

This shows that $i_{G'}^G MU^{((G))}$ is a wedge of smash products of bound slice spheres with $MU^{((G'))}$, and hence (by Lemma 12.4.14) a pure spectrum since $MU^{((G'))}$ is. □

Proposition 12.4.16. The restriction of $R_G(\infty)$ to the index 2 subgroup. *Suppose $G' \subset G$ has index 2. If the Slice Theorem holds for G' then the map*

$$i_{G'}^G R_G(\infty) \to i_{G'}^G H\underline{\mathbf{Z}}_{(2)}$$

is an equivalence.

Proof By Proposition 12.4.15 we know that $i^G_{G'}MU^{((G))}$ is pure. The claim then follows from Proposition 12.4.12. $\square$

12.4D Some auxiliary spectra

Our proof of the Reduction Theorem will require certain auxiliary spectra. For an integer $\ell > 0$ we define

$$R_G(\ell) = MU^{((G))}/(G\cdot \bar{r}_1, \ldots, G\cdot \bar{r}_\ell) = MU^{((G))} \underset{A_{G,\ell}}{\wedge} A'_{G,\ell} \qquad (12.4.17)$$

where

$$\begin{aligned} A_{G,\ell} &= S^{-0}[G\cdot \bar{r}_1, G\cdot \bar{r}_2, \ldots, G\cdot \bar{r}_\ell] \\ A'_{G,\ell} &= S^{-0}[G\cdot \bar{r}_{\ell+1}, G\cdot \bar{r}_{\ell+2}, \ldots]. \end{aligned}$$

The spectrum $R_G(\ell)$ is a right $A'_{G,\ell}$-module. As in the case of $MU^{((G))}$ described in §12.4A, the filtration of A'_G by the "dimension" monomial ideals leads to a filtration of $R_G(\ell)$ whose associated graded spectrum is

$$R_G(\infty) \wedge A'_G.$$

Thus the reduction theorem also implies that $R_G(\ell)$ is a pure spectrum.

Remark 12.4.18. $R_G(\ell)$ as a quotient of $MU^{((G))}$. *Alternatively,*

$$R_G(\ell) = MU^{((G))} \underset{A''_{G,\ell}}{\wedge} S^{-0},$$

where

$$A''_{G,\ell} = S^{-0}[G\cdot \bar{r}_1, G\cdot \bar{r}_2, \ldots, G\cdot \bar{r}_\ell].$$

By an argument similar to that of Proposition 12.4.15 we have the following.

Proposition 12.4.19. Purity of $R_G(\ell)$ over the index two subgroup. *Let $G' \subset G$ be the index 2 subgroup. If the Slice Theorem holds for G' then the spectrum $i^G_{G'}R_G(\ell)$ is pure. In particular, its 0-slice is $i^G_{G'}H\underline{\mathbf{Z}}_{(2)}$. More generally, for even m*

$$P^m_m i^G_{G'} R_G(\ell) \cong i^G_{G'}(H\underline{\mathbf{Z}}_{(2)} \wedge \widehat{W}'_{G,m})$$

where $\widehat{W}'_{G,m} \subset A'_G$ is the summand consisting of the wedge of slice spheres of dimension m. For odd m the slice above is contractible.

When m is odd Proposition 12.4.19 implies that $T \wedge P^m_m R_G(k)$ is contractible for any G-CW complex T built entirely from moving G-cells as in Definition 8.4.14. In particular, the equivariant homotopy groups of $E\mathcal{P} \ltimes R_G(k)$ may be investigated by smashing the slice tower of $R_G(k)$ with $E\mathcal{P}_+$, and we will do so in §12.4E, where we will exploit some very elementary aspects of the situation.

12.4E The proof of the Reduction Theorem

As mentioned at the beginning of the chapter, our proof of the Reduction Theorem is by induction on $|G|$, the case of the trivial group being a result of Quillen. We may therefore assume that G is non-trivial, and that the result is known for all proper subgroups $H \subset G$. By Proposition 12.4.16 this implies that the map

$$R_G(\infty) \to H\underline{\mathbf{Z}}_{(2)}$$

becomes a weak equivalence after applying i_H^G.

For the induction step we smash the map in question with the isotropy separation sequence (9.11.4)

$$\begin{tikzcd} E\mathcal{P} \ltimes R_G(\infty) \arrow[r] \arrow[d, "f"'] & R_G(\infty) \arrow[r] \arrow[d, "g"] & \widetilde{E}\mathcal{P} \wedge R_G(\infty) \arrow[d, "h"] \\ E\mathcal{P} \ltimes H\underline{\mathbf{Z}}_{(2)} \arrow[r] & H\underline{\mathbf{Z}}_{(2)} \arrow[r] & \widetilde{E}\mathcal{P} \wedge H\underline{\mathbf{Z}}_{(2)} . \end{tikzcd}$$

By the induction hypothesis, the map f is an equivalence. It therefore suffices to show that the map h is, and that, as discussed in Proposition 9.11.10, is equivalent to showing that

$$\pi_*^G h : \pi_* \Phi^G R_G(\infty) \to \pi_* \Phi^G H\underline{\mathbf{Z}}_{(2)} \tag{12.4.20}$$

is an isomorphism.

We first show that the two groups are abstractly isomorphic.

Proposition 12.4.21. Geometric fixed points of $H\underline{\mathbf{Z}}_{(2)}$. *The ring $\pi_*\Phi^G H\underline{\mathbf{Z}}_{(2)}$ is given by*

$$\pi_* \Phi^G H\underline{\mathbf{Z}}_{(2)} = \mathbf{Z}/2[b],$$

with

$$b = u_{2\sigma} a_\sigma^{-2} \in \pi_2 \Phi^G H\underline{\mathbf{Z}}_{(2)} \subset a_\sigma^{-1} \pi_\star^G H\underline{\mathbf{Z}}_{(2)}.$$

The groups $\pi_*\Phi^G R_G(\infty)$ are given by

$$\pi_n \Phi^G R_G(\infty) = \begin{cases} \mathbf{Z}/2 & n \geqslant 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Proof The first assertion is a restatement of Theorem 9.11.20. For the second we will make use of the monoidal geometric fixed point functor Φ_M^G. The main technical issue is to take care that at key points in the argument we are working with spectra X for which $\Phi^G X$ and $\Phi_M^G X$ are weakly equivalent.

Recall the definition

$$R_G(\infty) = MU_c^{((G))} \underset{A_G}{\wedge} S^{-0},$$

where for emphasis we have written $MU_c^{((G))}$ as a reminder that $MU^{((G))}$ has been replaced by a cofibrant A_G-module (see §10.10). Proposition 10.8.12 implies that $R_G(\infty)$ is cofibrant, so there is an isomorphism

$$\pi_*\Phi^G R_G(\infty) \cong \pi_*\Phi^G_M R_G(\infty)$$

by Proposition 9.11.49. For the monoidal geometric fixed point functor, Proposition 10.8.12 gives an isomorphism

$$\Phi^G_M(R_G(\infty)) = \Phi^G_M(MU_c^{((G))} \underset{A_G}{\wedge} S^{-0}) \cong \Phi^G_M MU_c^{((G))} \underset{\Phi^G_M A_G}{\wedge} S^{-0}.$$

We next claim that there are associative algebra **isomorphisms**

$$\Phi^G_M A_G \cong S^{-0}[\Phi^G N\bar{r}_1, \Phi^G N\bar{r}_2, \dots] \cong S^{-0}[\Phi^{C_2}\bar{r}_1, \Phi^{C_2}\bar{r}_2, \dots].$$

For the first, decompose A_G into an indexed wedge, and use Proposition 9.11.40. For the second use the fact that the monoidal geometric fixed point functor distributes over wedges, and for V and W representations of C_2, can be computed in terms of the isomorphisms

$$\Phi^G_M(N^G_{C_2}(S^{-W} \wedge S^V)) \cong \Phi^G_M(S^{-\operatorname{Ind}^G_{C_2} W} \wedge S^{\operatorname{Ind}^G_{C_2} V}) \cong \Phi^{C_2}_M(S^{-W} \wedge S^V).$$

By Proposition 10.8.6, $\Phi^G_M MU_c^{((G))}$ is a cofibrant $\Phi^G_M A_G$-module, and so

$$\Phi^G_M MU_c^{((G))} \underset{\Phi^G_M A_G}{\wedge} S^{-0} \cong \Phi^G_M MU^{((G))}/(\Phi^G_M N\bar{r}_1, \Phi^G_M N\bar{r}_2, \dots).$$

Since $MU_c^{((G))}$ is a cofibrant A_G-module, and the polynomial algebra A_G has the property that $S^{-1} \wedge A_G$ is cofibrant, the spectrum underlying $MU_c^{((G))}$ is cofibrant by Corollary 10.8.11. This means that

$$\Phi^G_M MU_c^{((G))}$$

and

$$\Phi^G MU_c^{((G))} \sim \Phi^G MU^{((G))} \sim MO$$

are related by a functorial zigzag of weak equivalences (Proposition 9.11.49). Putting all of this together, we arrive at the equivalence

$$\Phi^G R_G(\infty) \sim MO/(\Phi^{C_2}\bar{r}_1, \Phi^{C_2}\bar{r}_2, \dots).$$

By Proposition 12.3.10

$$\Phi^G \bar{r}_i = \begin{cases} h_i & i \neq 2^n - 1 \\ 0 & i = 2^n - 1. \end{cases}$$

From this is an easy matter to compute $\pi_* MO/(\Phi^G\bar{r}_1, \Phi^G\bar{r}_2, \dots)$ using the cofibration sequences described at the end of § 10.10D. The outcome is as asserted. □

Before going further we record a simple consequence of the above discussion which will be used in §13.3A.

Proposition 12.4.22. The effect of the map $MU^{((G))} \to H\underline{\mathbf{Z}}_{(2)}$ on geometric fixed points. *The map*

$$\pi_* \Phi^G MU^{((G))} = \pi_* MO \to \pi_* \Phi^G H\underline{\mathbf{Z}}_{(2)}$$

is zero for $* > 0$.

A simple multiplicative property reduces the problem of showing that (12.4.20) is an isomorphism to showing that it is surjective in dimensions which are a power of 2.

Lemma 12.4.23. A criterion for surjectivity. *If for every $k \geqslant 1$, the class $b^{2^{k-1}}$ is in the image of*

$$\pi_{2^k} \Phi^G MU^{((G))}/(G \cdot \bar{r}_{2^n-1}) \to \pi_{2^k} \Phi^G H\underline{\mathbf{Z}}_{(2)}, \tag{12.4.24}$$

then (12.4.20) is surjective, hence an isomorphism.

Proof By writing

$$R_G(\infty) = MU^{((G))}/(G \cdot \bar{r}_1) \underset{MU^{((G))}}{\wedge} MU^{((G))}/(G \cdot \bar{r}_2) \underset{MU^{((G))}}{\wedge} \cdots$$

we see that if for every $k \geqslant 1$, $b^{2^{k-1}}$ is in the image of (12.4.24), then all products of the $b^{2^{k-1}}$ are in the image of

$$\pi_* \Phi^G R_G(\infty) \to \pi_* \Phi^G H\underline{\mathbf{Z}}_{(2)}.$$

Hence every power of b is in the image of this map. □

In view of Lemma 12.4.23, the Reduction Theorem follows from

Proposition 12.4.25. Meeting the surjectivity criterion. *For every $n > 0$, the class $b^{2^{n-1}}$ is in the image of*

$$\pi_{2^n} \Phi^G (MU^{((G))}/(G \cdot \bar{r}_{2^n-1})) \to \pi_{2^n} \Phi^G (H\underline{\mathbf{Z}}_{(2)}).$$

To simplify some of the notation, write

$$c_n = 2^n - 1 \qquad \text{and} \qquad M_{(n)} = MU^{((G))}/(G \cdot \bar{r}_{c_n}). \tag{12.4.26}$$

Since $S^{c_n \rho_G}$ is obtained from S^{c_n} by attaching moving G-cells (Definition 8.4.14), the restriction map

$$\pi^G_{c_n \rho_G + 1} \widetilde{E}\mathcal{P} \wedge M_{(n)} \to \pi^G_{c_n + 1} \widetilde{E}\mathcal{P} \wedge M_{(n)}$$

is an isomorphism (Proposition 9.11.12). The element of interest in this group (the one hitting $b^{2^{n-1}}$) arises from the class

$$N\bar{r}_{c_n} \in \pi^G_{c_n \rho_G} MU^{((G))}$$

and from the fact that it is zero for two reasons in $\pi^G_{c_n\rho_G}\widetilde{E}\mathcal{P}\wedge M_{(n)}$: it has been coned off in the formation of $M_{(n)}$, and it is zero in $\pi^G_{c_n\rho_G}\widetilde{E}\mathcal{P}\wedge MU^{((G))} = \pi_{c_n}MO$ by Proposition 12.3.10. We make this more precise and prove Proposition 12.4.25 by chasing the class $N\bar{r}_{c_n}$ around the sequences of equivariant homotopy groups arising from the diagram

$$\begin{tikzcd}
E\mathcal{P}\ltimes MU^{((G))} \arrow[r] \arrow[d] & MU^{((G))} \arrow[r] \arrow[d] & \widetilde{E}\mathcal{P}\wedge MU^{((G))} \arrow[d] \\
E\mathcal{P}\ltimes M_{(n)} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] & \widetilde{E}\mathcal{P}\wedge M_{(n)} \arrow[d] \\
E\mathcal{P}\ltimes H\underline{\mathbf{Z}}_{(2)} \arrow[r] & H\underline{\mathbf{Z}}_{(2)} \arrow[r] & \widetilde{E}\mathcal{P}\wedge H\underline{\mathbf{Z}}_{(2)}\,.
\end{tikzcd} \tag{12.4.27}$$

We start with the top row. By Proposition 12.3.10 the image of $N\bar{r}_{c_n}$ in

$$\pi^G_{c_n\rho_G}\widetilde{E}\mathcal{P}\wedge MU^{((G))} \cong \pi^G_{c_n}\widetilde{E}\mathcal{P}\wedge MU^{((G))} \cong \pi_{c_n}MO$$

is zero. There is therefore a class

$$y_n \in \pi^G_{c_n\rho_G}E\mathcal{P}\ltimes MU^{((G))} \tag{12.4.28}$$

mapping to $N\bar{r}_{c_n}$. The key computation, from which everything follows is

Proposition 12.4.29. Nontriviality of the image of y_n. *The image of any choice of y_n as in (12.4.28) under*

$$\pi^G_{c_n\rho_G}E\mathcal{P}\ltimes MU^{((G))} \to \pi^G_{c_n\rho_G}E\mathcal{P}\ltimes H\underline{\mathbf{Z}}_{(2)},$$

is non-zero.

Proof of Proposition 12.4.25 assuming Proposition 12.4.29. We continue chasing around the diagram (12.4.27). By construction the image of y_n in $\pi^G_{c_n\rho_G}E\mathcal{P}\ltimes M_{(n)}$ maps to zero in $\pi^G_{c_n\rho_G}M_{(n)}$. It therefore comes from a class

$$\tilde{y}_n \in \pi^G_{c_n\rho_G+1}\widetilde{E}\mathcal{P}\wedge M_{(n)}.$$

The image of $\tilde{y}_n$ in $\pi^G_{c_n\rho_G+1}\widetilde{E}\mathcal{P}\wedge H\underline{\mathbf{Z}}_{(2)}$ is non-zero since it has a non-zero image in

$$\pi^G_{c_n\rho_G}E\mathcal{P}\ltimes H\underline{\mathbf{Z}}_{(2)}$$

by Proposition 12.4.29. Now consider the commutative square below, in which the horizontal maps are the isomorphisms (Proposition 9.11.12) given by restriction along the fixed point inclusion $S^{2^n} \subset S^{c_n\rho_G+1}$:

$$
\begin{tikzcd}
\pi^G_{c_n\rho_G+1}\widetilde{E}\mathcal{P} \wedge M_{(n)} \arrow[r, "\cong"] \arrow[d] & \pi^G_{2^n}\widetilde{E}\mathcal{P} \wedge M_{(n)} \arrow[d] \\
\pi^G_{c_n\rho_G+1}\widetilde{E}\mathcal{P} \wedge H\underline{\mathbf{Z}}_{(2)} \arrow[r, "\cong"'] & \pi^G_{2^n}\widetilde{E}\mathcal{P} \wedge H\underline{\mathbf{Z}}_{(2)} .
\end{tikzcd}
$$

The group on the bottom right is cyclic of order 2, generated by $b^{2^{n-1}}$. We've just shown that the image of $\tilde{y}_n$ under the left vertical map is non-zero. It follows that the right vertical map is non-zero and hence that $b^{2^{n-1}}$ is in its image. □

The remainder of this section is devoted to the proof of Proposition 12.4.29. The advantage of this statement is that it entirely involves G-spectra which have been smashed with $E\mathcal{P}_+$, and which (as discussed in §12.4D) therefore fall under the jurisdiction of the induction hypothesis. In particular, the map

$$E\mathcal{P} \ltimes MU^{((G))} \to E\mathcal{P} \ltimes H\underline{\mathbf{Z}}_{(2)} \tag{12.4.30}$$

can be studied by smashing the slice tower of $MU^{((G))}$ with $E\mathcal{P}_+$.

We can cut down some the size of things by making use of the spectra introduced in §12.4D. Factor (12.4.30) as

$$E\mathcal{P} \ltimes MU^{((G))} \to E\mathcal{P} \ltimes R_G(2^n-2) \to E\mathcal{P} \ltimes H\underline{\mathbf{Z}}_{(2)},$$

and replace y_n with its image

$$y'_n \in \pi^G_{c_n\rho_G} E\mathcal{P} \ltimes R_G(2^n-2).$$

Lemma 12.4.31. On $\pi_{c_n\rho_G}$ of the smash product of $E\mathcal{P}_+$ with some slices of $R_G(2^n-2)$.

(i) For $0 < m < c_n g$,

$$\pi_{c_n\rho_G} E\mathcal{P} \ltimes P^m_m R_G(2^n-2) = 0.$$

(ii) There is an exact sequence

$$
\begin{tikzcd}
\pi^G_{c_n\rho_G} E\mathcal{P} \ltimes P_{c_n g} R_G(2^n-2) \arrow[r] & \pi^G_{c_n\rho_G} E\mathcal{P} \ltimes R_G(2^n-2) \arrow[d] \\
 & \pi^G_{c_n\rho_G} E\mathcal{P} \ltimes H\underline{\mathbf{Z}}_{(2)} = \mathbf{Z}/2 .
\end{tikzcd}
$$

Proof (i) Because of the induction hypothesis, we know that the spectrum

$$E\mathcal{P} \ltimes P^m_m R_G(2^n-2)$$

is contractible when m is odd, and that when m is even it is equivalent to

$$E\mathcal{P} \ltimes H\mathbf{Z} \wedge \widehat{W}_m,$$

where $\widehat{W}_m \subset S^{-0}[G\cdot\overline{r}_{c_n},\dots]$ is the summand consisting of the wedge of slice spheres of dimension m. Since $1 < m < c_n g$ all of these slice spheres are moving as in Definition 11.1.6. This implies that the map

$$E\mathcal{P} \ltimes H\mathbf{Z} \wedge \widehat{W}_m \to H\mathbf{Z} \wedge \widehat{W}_m$$

is an equivalence, since

$$E\mathcal{P}_+ \to S^{-0}$$

is an equivalence after restricting to any proper subgroup of G. But

$$\pi^G_{c_n\rho_G} H\mathbf{Z} \wedge \widehat{W}_m = \pi^G_0 H\mathbf{Z} \wedge S^{-c_n\rho_G} \wedge \widehat{W}_m = 0$$

since

$$H\mathbf{Z} \wedge S^{-c_n\rho_G} \wedge \widehat{W}_m$$

is an $(m-c_n g)$-slice and $m-c_n g < 0$. This proves the first assertion. It implies that the map

$$\pi^G_{c_n\rho_G} E\mathcal{P} \ltimes P_{c_n g} R_G(2^n-2) \to \pi^G_{c_n\rho_G} E\mathcal{P} \ltimes P_1 R_G(2^n-2)$$

is surjective.

(ii) By Proposition 12.4.19,

$$P^0_0 i^G_{G'} R_G(2^n-2) = i^G_{G'} H\underline{\mathbf{Z}}_{(2)},$$

so

$$E\mathcal{P} \ltimes P^0_0 R_G(2^n-2) = E\mathcal{P} \ltimes H\underline{\mathbf{Z}}_{(2)},$$

and the second assertion follows from the exact sequence of the fibration

$$E\mathcal{P} \ltimes \big(R_G(2^n-2) \to R_G(2^n-2) \to P^0_0 R_G(2^n-2)\big)\,. \qquad \square$$

The exact sequence in Lemma 12.4.31 converts the problem of showing that y_n has non-zero image in $\pi^G_{c_n\rho_G} E\mathcal{P} \ltimes H\underline{\mathbf{Z}}_{(2)}$ to showing that it is not in the image of

$$\pi^G_{c_n\rho_G} E\mathcal{P} \ltimes P_{c_n g} R_G(2^n-2).$$

We now isolate a property of this image that is not shared by y_n. Recall that γ is a fixed generator of G.

Proposition 12.4.32. Divisibility by $1-\gamma$. *The image of*

$$\pi^G_{c_n\rho_G} E\mathcal{P} \ltimes P_{c_n g} R_G(2^n-2) \to \pi^G_{c_n\rho_G} R_G(2^n-2) \xrightarrow{i^G_0} \pi^u_{c_n g} R_G(2^n-2)$$

is contained in the image of $(1-\gamma)$.

The class y_n does not have the property described in Proposition 12.4.32. Its image in $\pi^u_{c_n g} R_G(2^n-2)$ is $i_0^G N\bar{r}_{c_n}$, which generates a sign representation of G occurring as a summand of $\pi^u_{c_n g} R_G(2^n-2)$. Thus once Proposition 12.4.32 is proved, the proof of the Reduction Theorem is complete.

The proof of Proposition 12.4.32 makes use of the Mackey functor

$$\underline{\pi}_{c_n \rho_G}(X)$$

and the transfer map

$$\underline{\pi}_{c_n \rho_G}(X)(G/G') \to \underline{\pi}_{c_n \rho_G}(X)(G/G),$$

in which $G' \subseteq G$ is the subgroup of index two. By Definition 9.4.14, this map is given by the map of equivariant homotopy groups

$$\pi^G_{c_n \rho_G}(X \rtimes G/G') \to \pi^G_{c_n \rho_G}(X)$$

induced by the unique surjective map $G/G' \to \mathrm{pt}$.

There are two steps in the proof of Proposition 12.4.32. First it is shown in Corollary 12.4.35 that the image of

$$\pi^G_{c_n \rho_G} E\mathcal{P} \ltimes P_{c_n g} R_G(2^n-2) \to \pi^G_{c_n \rho_G} R_G(2^n-2)$$

is contained in the image of the transfer map just described. Then in Lemma 12.4.36 we will show that the image of the transfer map in $\pi^u_{c_n g} R_G(2^n-2)$ is in the image of $(1-\gamma)$.

Lemma 12.4.33. The transfer map and $E\mathcal{P}_+$. *Let $M \geqslant 0$ be a G-spectrum, and regard C_2 as a finite G-set using the unique surjective map $G \to G/G' \cong C_2$. The image of*

$$\pi_0^G E\mathcal{P} \ltimes M \to \pi_0^G M$$

is the image of the transfer map

$$\pi_0^G M \rtimes G/G' \to \pi_0^G M.$$

Proof As mentioned in Example 9.11.6, the space $E\mathcal{P}_+$ can be taken to be the space S^∞_+ on which γ acts through the antipodal action. The standard cell decomposition in this model has 0-skeleton G/G'_+. Since M is (-1)-connected (Proposition 11.1.18) this implies that $\pi_0^G G/G' \ltimes M \to \pi_0^G E\mathcal{P} \ltimes M$ is surjective, and the claim follows. □

Corollary 12.4.34. The image of the transfer map in a connective cover of $R_G(2^n-2)$. *The image of*

$$\pi^G_{c_n \rho_G} E\mathcal{P} \ltimes P_{c_n g} R_G(2^n-2) \to \pi^G_{c_n \rho_G} P_{c_n g} R_G(2^n-2)$$

is contained in the image of the transfer map.

Proof This follows from Lemma 12.4.33 above, after the identification

$$\pi^G_{c_n\rho_G} P_{c_n g} R_G(2^n-2) \cong \pi^G_0 S^{-c_n\rho_G} \wedge P_{c_n g} R_G(2^n-2)$$

and the observation that

$$S^{-c_n\rho_G} \wedge P_{c_n g} R_G(2^n-2) \cong P_0(S^{-c_n\rho_G} \wedge R_G(2^n-2))$$

is $\geqslant 0$. □

Corollary 12.4.35. The image of the transfer map in $R_G(2^n-2)$ itself. *The image of*

$$\pi^G_{c_n\rho_G} E\mathcal{P} \ltimes P_{c_n g} R_G(2^n-2) \to \pi^G_{c_n\rho_G} R_G(2^n-2)$$

is contained in the image of the transfer map.

Proof Immediate from Corollary 12.4.34 and the naturality of the transfer. □

The remaining step is the special case $X = P_{c_n g} R_G(2^n-2)$, $V = c_n\rho_G$ of the next result.

Lemma 12.4.36. The image of a fold map. *Let X be a G-spectrum, V a virtual representation of G of virtual dimension d, and regard C_2 as a finite G-set through the unique surjective map $G \to G/G'$. Write $\epsilon \in \{\pm 1\}$ for the degree of*

$$\gamma : i^G_0 S^V \to i^G_0 S^V.$$

The image of

$$\pi^G_V(X \rtimes C_2) \to \pi^G_V X \to \pi^u_d X$$

is contained in the image of

$$(1+\epsilon\gamma) : \pi^u_d X \to \pi^u_d X.$$

Proof Consider the diagram

$$\begin{tikzcd} \pi^G_V(X \rtimes C_2) \arrow[r] \arrow[d] & \pi^G_V X \arrow[d] \\ \pi^u_d(X \rtimes C_2) \arrow[r] & \pi^u_d X\,. \end{tikzcd}$$

The non-equivariant identification

$$S^{-0} \rtimes C_2 \cong S^{-0} \vee S^{-0}$$

gives an isomorphism of groups of non-equivariant stable maps

$$[S^V, X \rtimes C_2] \cong [S^V, X] \oplus [S^V, X],$$

and so an isomorphism of the group in the lower left hand corner with

$$\pi_d^u X \oplus \pi_d^u X$$

under which the generator $\gamma \in G$ acts as

$$(a, b) \mapsto (\epsilon\gamma b, \epsilon\gamma a).$$

The map along the bottom is $(a, b) \mapsto a + b$. Now the image of the left vertical map is contained in the set of elements invariant under γ which, in turn, is contained in the set of elements of the form

$$(a, \epsilon\gamma a).$$

The result follows. □

Proof of Proposition 12.4.32. As described after its statement, Proposition 12.4.32 is a consequence of Corollary 12.4.35 and Lemma 12.4.36. □

13

The proofs of the Gap, Periodicity and Detection Theorems

This chapter is the payoff, the reason for developing all the machinery of the previous eleven chapters. We will prove (in reverse order) the three theorems listed in §1.1C, which we restate here for convenience.

Key properties of the C_8 fixed point spectrum Ξ.

(i) **Detection Theorem.** *It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each θ_j is nontrivial.* **This means that if θ_j exists, we will see its image in $\pi_*(\Xi)$.**

(ii) **Periodicity Theorem.** *It is 256-periodic, meaning that $\pi_k(\Xi)$ depends only on the reduction of k modulo 256. As in the case of Bott periodicity, we have an equivalence $\Omega^{256}\Xi \simeq \Xi$.*

(iii) **Gap Theorem.** $\pi_k(\Xi) = 0$ *for* $-4 < k < 0$.

We will identify the spectrum Ξ in Definition 13.3.27. We will state and prove the Gap Theorem 13.2.6 (for a class of spectra that includes Ξ) in §13.2, the Periodicity Theorem as Corollary 13.3.30 in §13.3, and the Detection Theorem 13.4.2 in §13.4.

We begin by describing the slice spectral sequence (which was developed in Chapter 11) for the C_2-spectrum $MU_{\mathbf{R}}$, the subject of Chapter 12. This computation is relatively simple. The E_2-term is a ring generated by four types of elements listed in §13.1A. Three of them, a_σ, $e_{i\sigma}$ for $i > 0$ and $u_{2\sigma}$ (all introduced in Definition 9.9.7)) are part of the equivariant homotopy of the integral Eilenberg-Mac Lane spectrum. The fourth is the family of polynomial generators

$$\overline{r}_i \in \pi^{C_2}_{i\rho_2} MU_{\mathbf{R}} \qquad \text{for } i > 0$$

for the equivariant homotopy of $MU_{\mathbf{R}}$ given in Definition 12.3.6. Of these, all but $u_{2\sigma}$ are permanent cycles. Its powers support a family of differentials listed in Theorem 13.1.1.

Typically theorems in stable homotopy theory about differentials in spectral sequences have proofs involving some geometry (such as an extended power construction) beyond that needed to construct the spectral sequence in the first place. In this case that additional geometry has to do with the geometric fixed point spectra of §9.11.

We know by Proposition 12.2.5(iii) that the geometric fixed point spectrum of $MU_{\mathbf{R}}$ is the unoriented cobordism spectrum MO. Hence it follows from Theorem 9.11.20 that $\pi_* MO$ is the integer graded portion of

$$a_\sigma^{-1}\pi_\star^{C_2}(MU_{\mathbf{R}}).$$

This means that formally inverting the permanent cycle a_σ in the slice spectral sequence for $MU_{\mathbf{R}}$ will give a spectral sequence whose $\mathbf{Z}$-graded portion converges to $\pi_* MO$, which is a graded polynomial algebra over $\mathbf{Z}/2$ on generators x_i of dimension i for $i > 0$ **not of the form** $2^n - 1$**.** We show that Thom's x_i correspond to the elements

$$f_i = a_\sigma^i \overline{r}_i \in E_2^{i,2i}$$

of Definition 12.3.11. For $i = 2^n - 1$, this element has to die because it vanishes in $\pi_* MO$. This forces the pattern of differentials described in Theorem 13.1.1. In Slice Differentials Theorem 13.3.9 we will leverage these differentials to get similar ones in the slice spectral sequence for $\pi_\star^G MU^{((G))}$.

Returning the slice spectral sequence for $\pi_\star^{C_2} MU_{\mathbf{R}}$, in §13.1C we explain how inverting the element $\overline{r}_{2^n-1}$ (by passing from $MU_{\mathbf{R}}$ to a telescope derived from it) leads to some interesting permanent cycles, which in turn lead to periodicities. The simplest case is the 8-dimensional periodicity of Example 13.1.8.

13.1 A warmup: the slice spectral sequence for $MU_{\mathbf{R}}$

We will study the $RO(G)$-graded spectral sequence of Mackey functors for the group $G = C_2$ converging to $\underline{\pi}_\star MU_{\mathbf{R}}$. We denote its E_2-term by $\underline{E}_2^{*,\star}$. The underline is meant to remind us that it is a Mackey functor as discussed in §8.2. The first superscript is an integer while the second lies in $RO(G)$. We will use the Adams grading convention, which means that $\underline{E}_\infty^{s,V}$ is a subquotient of $\underline{\pi}_{V-s}$. The rth differential (for r an ordinary integer $\geqslant 2$) has the form

$$d_r : \underline{E}_r^{s,V} \to \underline{E}_r^{s+r,V+r-1}$$

It raises the filtration degree (the first superscript) by r and lowers the topological degree (in $RO(G)$) by one.

Another account of this spectral sequence, stated in terms of $BP_{\mathbf{R}}$ rather than $MU_{\mathbf{R}}$, can be found in [LSWX19, §3]. The authors also study the Johnson-Wilson analogs $E\mathbf{R}(n)$, and they have results about the C_2-equivariant Adams spectral sequence for these spectra.

By the Slice Theorem 12.4.1, each oddly indexed slice of $MU_{\mathbf{R}}$ is contractible, while the $2i$th slice is the wedge of a certain number $p(i)$ copies of $S^{i\rho_2} \wedge H\mathbf{Z}$. The value of

$$\underline{\pi}_\star S^{i\rho_2} \wedge H\mathbf{Z} = \underline{H}_* S^{i\rho_2}$$

is given in Theorem 9.9.19.

The number $p(i)$ is the partition function defined by the generating function

$$\begin{aligned}\sum_{d\geqslant 0} p(i)t^i &= \prod_{\ell>0} \frac{1}{1-t^\ell} \\ &= 1+t+2t^2+3t^3+5t^4+7t^5+11t^6+15t^7+22t^8\cdots.\end{aligned}$$

It is also the number of monomials of degree i in the variables $\overline{r}_\ell$, where the degree of $\overline{r}_\ell$ is ℓ. Equivalently it is the rank of the free abelian group $\pi_{2i}MU$ as described in (12.0.2).

13.1A The $\underline{E}_2$-term.

The slice spectral sequence for $MU_{\mathbf{R}}$ has the following elements in its $\underline{E}_2$-term.

- The image of the element a_σ of Definition 9.9.7(i) in $\underline{E}_2^{1,1-\sigma}(C_2/C_2)$. It has order 2 and its restriction in $\underline{E}_2^{1,1-\sigma}(C_2/e)$ is trivial.
- The images in $\underline{E}_2^{0,i\sigma-i}$ of the elements $e_{i\sigma}$ of Definition 9.9.7(ii) for $i>0$.
- The image of the element $u_{2\sigma}$ of Definition 9.9.7(iii) in $\underline{E}_2^{0,2-2\sigma}(C_2/C_2)$. We also have $u_\sigma \in \underline{E}_2^{0,1-\sigma}(C_2/e)$, which is invertible by Lemma 9.9.8. The restriction of $u_{2\sigma}$ is u_σ^2. That of $\overline{r}_i$ is $u_\sigma^i r_i$ and r_i is congruent of x_i as in (12.0.2) modulo decomposables in $\pi_*^u MU_{\mathbf{R}} \cong \pi_* MU$.
- The images of the generators

  $$\overline{r}_i \in \underline{\pi}_{i\rho_2} MU_{\mathbf{R}}(C_2/C_2);$$

  of Definition 12.3.6 in $\underline{E}_2^{0,i\rho_2}(C_2/C_2)$.

Each of the generators listed above is a permanent cycle except $u_{2\sigma}$. Its powers support the differentials of Theorem 13.1.1 below.

13.1B The differentials

The following will be generalized to larger cyclic 2-groups below in the Slice Differentials Theorem 13.3.9.

Theorem 13.1.1. Slice differentials. *With notation as above, the following differentials and no others (save multiplicative consequences of these) occur in the slice spectral sequence for $\underline{\pi}_\star MU_{\mathbf{R}}$:*

$$d_{2^{n+1}-1}(u_{2\sigma}^{2^{n-1}}) = a_\sigma^{2^{n+1}-1}\overline{r}_{2^n-1} \qquad \textit{for } n>0.$$

Proof The key idea here is that by Theorem 9.11.20, inverting a_σ and taking the $\mathbf{Z}$-graded portion must yield a spectral sequence converging to

$$\pi_* \Phi^G MU_{\mathbf{R}} = \pi_* MO.$$

The $RO(G)$-graded E_2-term is

$$E_2^{*,\star} = \mathbf{Z}/2[a_\sigma^{\pm 1}, u_{2\sigma}][\overline{r}_i : i > 0].$$

The elements $e_{i\sigma}$ for $i > 0$ are not present here because each of them is killed by a_σ by Lemma 9.9.10(v).

Its $\mathbf{Z}$-graded subring is

$$E_2^{*,*} = \mathbf{Z}/2[b][f_i : i > 0], \tag{13.1.2}$$

where $b = u_{2\sigma} a_\sigma^{-2} \in E_2^{-2,4}$ as in Theorem 9.11.20 and $f_i = a_\sigma^i \overline{r}_i \in E_2^{i,2i}$ as in Definition 12.3.11. In terms of these generators, the stated slice differentials would be

$$d_{2^{n+1}-1}(b^{2^{n-1}}) = f_{2^n-1} \qquad \text{for } n > 0, \tag{13.1.3}$$

leaving

$$E_\infty = \mathbf{Z}/2[f_i : i > 0, i \neq 2^n - 1].$$

On the other hand, we know from Proposition 12.3.10 and (12.3.12) that f_i maps to $x_i \in \pi_i MO$ and that $x_{2^n-1} = 0$. It follows that **for each** $n > 0$**,** f_{2^n-1} **must be killed by a differential in the slice spectral sequence**. (This will be generalized in Remark 13.3.10 below.) Since b is the only generator in (13.1.2) that is not a permanent cycle, the pattern of differentials of (13.1.3) is the only way to kill all the generators f_{2^n-1}. □

It is not easy to describe the resulting value $\underline{E}_\infty(G/G)$, which is the associated bigraded (over $\mathbf{Z} \times RO(C_2)$) object for $\pi_\star^{C_2} MU_{\mathbf{R}}$.

Example 13.1.4. Toda brackets in $\pi_\star^{C_2} MU_{\mathbf{R}}$**.** *We refer the reader to [Rav86, A1.4 and Remark 7.4.9] for an introduction to Massey products, and to [Koc82], [Koc78] and [Koc80] for the relation between Massey products and Toda brackets.*

The first two slice differentials are

$$d_3(u_{2\sigma}) = a_\sigma^3 \overline{r}_1 \qquad \textit{and} \qquad d_7(u_{2\sigma}^2) = a_\sigma^7 \overline{r}_3. \tag{13.1.5}$$

It follows that

$$d_7(\overline{r}_1 u_{2\sigma}^2) = a_\sigma^7 \overline{r}_1 \overline{r}_3 = a_\sigma^4 \overline{r}_3 d_3(u_{2\sigma}).$$

This means that the intended target of the d_7 on $\overline{r}_1 u_{2\sigma}^2$ has been killed by an earlier differential (a d_3), so it survives even though $u_{2\sigma}^2$ does not. (See

Corollary 13.3.14 below for a generalization to larger cyclic 2-groups.) It is the Massey product

$$\langle \overline{r}_1, a_\sigma^7, \overline{r}_3 \rangle,$$

which is defined in $\underline{E}_8$*, and it represents a corresponding Toda bracket in* $\pi^{C_2}_{5-3\sigma}$*. This element is killed by* a_σ^3 *since* $d_3(u_{2\sigma}^3) = a_\sigma^3 \overline{r}_1 u_{2\sigma}$*.*

One can construct more elements of this sort by computing the differential on some power of $u_{2\sigma}$ *and dividing the target by the appropriate power of* a_σ*. If we do it for* $u_{2\sigma}^m$*, the power of* a_σ *will depend on the 2-adic valaution of* m*, and the order of the Massey product will depend on* $\alpha(m)$*, the number of digits in the binary expansion of* m*.*

We illustrate with the case $m = 14$*, for which* $\alpha(m) = 3$*. We have*

$$d_7(u_{2\sigma}^{14}) = a_\sigma^7 \overline{r}_3 u_{2\sigma}^{12}$$

and $\overline{r}_3 u_{2\sigma}^{12}$ *represents the Massey product*

$$\langle \overline{r}_3, a_\alpha^{15}, \langle \overline{r}_7, a_\sigma^{31}, \overline{r}_{15} \rangle \rangle = \langle \langle \overline{r}_3, a_\alpha^{15}, \overline{r}_7 \rangle, a_\sigma^{31}, \overline{r}_{15} \rangle,$$

which is defined in $\underline{E}_{32}$ *and represents a Toda bracket in* $\pi^{C_2}_{27-21\sigma}$*.*

13.1C Inversion and periodicity

Since each $\overline{r}_i$ is a permanent cycle, it represents a map $S^{i\rho} \to MU_{\mathbf{R}}$. We can use the multiplication m on $MU_{\mathbf{R}}$ to form a self map, namely

$$\Sigma^{i\rho} MU_{\mathbf{R}} \xrightarrow{\overline{r}_i \wedge MU_{\mathbf{R}}} MU_{\mathbf{R}} \wedge MU_{\mathbf{R}} \xrightarrow{\quad m \quad} MU_{\mathbf{R}}.$$

This map, which we denote abusively by $\overline{r}_i$, can be iterated to form a diagram

$$MU_{\mathbf{R}} \xrightarrow{\quad \overline{r}_i \quad} \Sigma^{-i\rho} MU_{\mathbf{R}} \xrightarrow{\quad \overline{r}_i \quad} \Sigma^{-2i\rho} MU_{\mathbf{R}} \xrightarrow{\quad \overline{r}_i \quad} \cdots . \tag{13.1.6}$$

We denote its homotopy colimit or telescope by

$$\overline{r}_i^{-1} MU_{\mathbf{R}} = \operatorname*{hocolim}_k \Sigma^{-ki\rho} MU_{\mathbf{R}}. \tag{13.1.7}$$

This is an orthogonal C_2-spectrum with

$$\underline{\pi}_\star \overline{r}_i^{-1} MU_{\mathbf{R}} \cong \overline{r}_i^{-1} \underline{\pi}_\star MU_{\mathbf{R}}.$$

This construction is most interesting when $i = 2^n - 1$ for some $n > 0$.

Example 13.1.8. Inverting $\overline{r}_1$ leads to 8-dimensional periodicity. *Suppose we invert* $\overline{r}_1 \in \pi_\rho^G MU_{\mathbf{R}}$ *as above. Then the first slice differential (see (13.1.5)),*

$$d_3(u_{2\sigma}) = a_\sigma^3 \overline{r}_1,$$

can be rewritten as

$$d_3(\overline{r}_1^{-1} u_{2\sigma}) = a_\sigma^3,$$

so $a_\sigma^3 = 0$ in $\underline{\pi}_\star \overline{r}_1^{-1} MU_{\mathbf{R}}$. This makes the target of the next slice differential, $a_\sigma^7 \overline{r}_3$, trivial, so $u_{2\sigma}^2$ **is a permanent cycle**.

The unit inclusion

$$S^{-0} \to \overline{r}_1^{-1} MU_{\mathbf{R}}$$

gives a map

$$H\underline{\mathbf{Z}} = P_0^0 S^{-0} \to P_0^0 \overline{r}_1^{-1} MU_{\mathbf{R}}$$

and hence defines an elements

$$u_{2\sigma} \in \pi_{2-2\sigma} P_0^0 R = E_2^{0,2-2\sigma}$$

in the E_2-term of the $RO(G)$-graded slice spectral sequence for $\pi_\star^G \overline{r}_1^{-1} MU_{\mathbf{R}}$. Since $u_{2\sigma}^2$ is a permanent cycle, it gives us a map

$$S^8 \wedge \overline{r}_1^{-1} MU_{\mathbf{R}} \xrightarrow{u_{2\sigma}^2} S^{4+4\sigma} \wedge \overline{r}_1^{-1} MU_{\mathbf{R}} \xrightarrow{\overline{r}_1^4} \overline{r}_1^{-1} MU_{\mathbf{R}}.$$

which underlain by a weak equivalence.

We claim that this map induces a weak equivalence of underlying spectra. *This follows from the fact that*

$$u_{2\sigma}^2 \in \underline{E}_\infty^{0,4-4\sigma}(C_2/C_2)$$

restricts to a unit in

$$u_{2\sigma}^2 \in \underline{E}_\infty^{0,4-4\sigma}(C_2/e)$$

by Lemma 9.9.8. This means it induces a weak equivalence on the homotopy fixed point spectrum by Theorem 9.11.22,

$$(S^8 \wedge \overline{r}_1^{-1} MU_{\mathbf{R}})^{hC_2} \to (\overline{r}_1^{-1} MU_{\mathbf{R}})^{hC_2}.$$

This makes the homotopy fixed spectrum $(\overline{r}_1^{-1} MU_{\mathbf{R}})^{hC_2}$ 8-periodic.

The Homotopy Fixed Point Theorem 13.3.28 below shows that the map

$$\eta : (\overline{r}_1^{-1} MU_{\mathbf{R}})^{C_2} \to (\overline{r}_1^{-1} MU_{\mathbf{R}})^{hC_2}$$

of (5.8.2) is a weak equivalence, so the ordinary fixed spectrum $(\overline{r}_1^{-1} MU_{\mathbf{R}})^{C_2}$ is also 8-periodic.

A similar argument can be made if we invert $\overline{r}_{2^n-1}$. In that case $u_{2\sigma}^{2^n}$ is a permanent cycle and we have

$$u^{2^n(2^n-1)} \overline{r}_{2^n-1}^{2^{n+1}} \in \pi_{2^{n+2}(2^n-1)}^{C_2} (\overline{r}_{2^n-1}^{-1} MU_{\mathbf{R}})$$

This leads to the following periodicity theorem.

Theorem 13.1.9. Periodicities for $MU_{\mathbf{R}}$. *For each integer $n > 0$, the fixed point spectrum*

$$\left(\overline{r}_{2^n-1}^{-1} MU_{\mathbf{R}}\right)^{C_2}$$

is $2^{n+2}(2^n-1)$-periodic.

13.2 The Gap Theorem

In this section we will prove §1.1C(iii).

Proposition 13.2.1. A gap in cohomology. *Let G be any nontrivial finite group and $m \geqslant 0$ an integer. Except for the case $G = C_3$, $k = 3$, and $m = 1$, the groups*

$$H_G^k(S^{m\rho_G}; \underline{\mathbf{Z}})$$

are trivial for $0 < k < 4$. In the exceptional case one has

$$H_G^k(S^{\rho_{C_3}}; \underline{\mathbf{Z}}) = \begin{cases} 0 & \text{for } k = 1 \text{ and } k = 2 \\ \mathbf{Z} & \text{for } k = 3. \end{cases}$$

Proof Since

$$H_G^k(S^{m\rho_G}; \underline{\mathbf{Z}}) \cong H_G^{k-m}(S^{m(\overline{\rho}_G)}; \underline{\mathbf{Z}}),$$

connectivity and Example 8.5.4 show that $H_G^k(S^{m\rho_G}; \underline{\mathbf{Z}}) = 0$ for $k \leqslant m + 1$. This takes care of the cases in which $m + 1 \geqslant 3$, leaving only $m = 1$, and in that case only the group

$$H_G^2(S^{\overline{\rho}_G}; \underline{\mathbf{Z}})$$

which is isomorphic to

$$H^2(S^{\overline{\rho}_G}/G; \mathbf{Z}).$$

Since the orbit space $S^{\overline{\rho}_G}/G$ is simply connected, the universal coefficient theorem gives an inclusion

$$H^2(S^{\overline{\rho}_G}/G; \mathbf{Z}) \to H^2(S^{\overline{\rho}_G}/G; \mathbf{Q}).$$

It therefore suffices to show that

$$H^2(S^{\overline{\rho}_G}/G; \mathbf{Q}) = 0.$$

But since G is finite, this group is just the G-invariant part of

$$H^2(S^{\overline{\rho}_G}; \mathbf{Q})$$

which is zero since G does not have order 3. When G does have order 3 the group is $\mathbf{Q}$. The claim follows since the homology groups are finitely generated. □

Given the Slice Theorem, the Gap Theorem is a consequence of the following special case of Proposition 13.2.1.

Proposition 13.2.2. The cohomology gap for regular representation spheres. *Suppose that $G = C_{2^n}$ is a non-trivial group, and $m \geqslant 0$. Then*

$$H_G^k(S^{m\rho_G}; \underline{\mathbf{Z}}_{(2)}) = 0 \qquad \textit{for } 0 < k < 4.$$

Lemma 13.2.3. The gap for slice spheres. *Let $G = C_{2^n}$ for some $n > 0$. If $\widehat{S}$ is an bound (Definition 11.1.6) slice sphere (Definition 11.1.3) of even dimension, then the groups $\pi_k^G H\underline{\mathbf{Z}}_{(2)} \wedge \widehat{S}$ are zero for $-4 < k < 0$.*

Proof Suppose that

$$\widehat{S} = G \underset{H}{\ltimes} S^{m\rho_H}$$

with $H \subset G$ non-trivial. By the Wirthmüller isomorphism of (8.0.16),

$$\pi_k^G H\underline{\mathbf{Z}}_{(2)} \wedge \widehat{S} \cong \pi_k^H H\underline{\mathbf{Z}}_{(2)} \wedge S^{m\rho_H},$$

so the assertion is reduced to the case $\widehat{S} = S^{m\rho_G}$ with G non-trivial. If $m \geqslant 0$ then $\pi_k^G H\underline{\mathbf{Z}}_{(2)} \wedge \widehat{S} = 0$ for $k < 0$. For the case $m < 0$ write $m' = -m > 0$, and

$$\pi_k^G H\underline{\mathbf{Z}}_{(2)} \wedge \widehat{S} = H_G^k(S^{m'\rho_G}; \underline{\mathbf{Z}}_{(2)}).$$

The result then follows from Proposition 13.2.2. □

Theorem 13.2.4. The gap for pure bound spectra. *If X is pure and bound, then*

$$\pi_k^G X = 0 \qquad -4 < k < 0.$$

Proof This is immediate from Lemma 13.2.3 and the slice spectral sequence for X. □

Corollary 13.2.5. *If Y can be written as a directed homotopy colimit of bound pure spectra, then*

$$\pi_k^G X = 0 \qquad -4 < k < 0.$$

Gap Theorem 13.2.6. *Let $G = C_{2^n}$ with $n > 0$ and let $D \in \pi_{\ell\rho_G} MU^{((G))}$ for $\ell > 0$ be any class. Then for $-4 < k < 0$*

$$\pi_k^G D^{-1} MU^{((G))} = 0.$$

Proof The spectrum $D^{-1}MU^{((G))}$ is the homotopy colimit

$$\operatorname*{hocolim}_j \Sigma^{-j\,\ell\rho_G} MU^{((G))}.$$

By the Slice Theorem, $MU^{((G))}$ is pure and bound. But then the spectrum

$$\Sigma^{-j\,\ell\rho_G} MU^{((G))}$$

is also pure and bound, since for any X

$$P_m^m \Sigma^{\rho_G} X \cong \Sigma^{\rho_G} P_{m-g}^{m-g} X$$

by Corollary 11.1.33. The result then follows from Corollary 13.2.5. □

13.3 The Periodicity Theorem

Let G be the finite cyclic 2-group of order g. In this section we will describe a general method for producing periodicity results for spectra obtained from $MU^{((G))}$ by inverting suitable elements of $\pi_\star^G MU^{((G))}$. We prove Theorem 13.3.23, which has § 1.1C(ii) as a special case. The proof relies on a small amount of computation in the e $RO(G)$-graded slice spectral sequence for $\pi_\star^G MU^{((G))}$.

13.3A The $RO(G)$-graded slice spectral sequence for $MU^{((G))}$

Let $\sigma = \sigma_G$ be the real sign representation of G, and let

$$u = u_{2\sigma} \in \pi^G_{2-2\sigma} H\underline{\mathbf{Z}}_{(2)}$$

be the element defined in Definition 9.9.7(iii). Since

$$P_0^0 MU^{((G))} = H\underline{\mathbf{Z}},$$

the powers u^m define elements

$$u^m \in E_2^{0,2m-2m\sigma} = \pi^G_{2m-2m\sigma} P_0^0 MU^{((G))} \tag{13.3.1}$$

in the E_2-term of the $RO(G)$-graded slice spectral sequence

$$E_2^{s,V} = \pi^G_{V-s} P_{|V|}^{|V|} MU^{((G))} \implies \pi_{V-s} MU^{((G))},$$

with $V \in -2m\sigma + \mathbf{Z}$. Our periodicity theorems depend on the fate of these elements. To study them it is convenient to consider odd negative multiples of σ as well as even ones, and to investigate the slice spectral sequences for $\pi_{*+k(\sigma-1)}$ for $k \leqslant 0$. See Remark 11.2.16.

It turns out to be enough to investigate the groups $E_2^{s,V}$ with

$$V = |V| - k(\sigma - 1) \qquad \text{for} \qquad s \geqslant (2^n - 1)(|V| - s).$$

The situation is depicted in Figures 13.1–13.3 for the group $G = C_8$. Note that each chart has a vanishing line of slope 7, as required by Corollary 11.2.17.

We already know the groups in this range. Note that

$$\begin{aligned} E_2^{s,V} &= \pi^G_{V-s} P_{|V|}^{|V|} MU^{((G))} P_{|V|}^{|V|} MU^{((G))} = \pi^G_{|V|-s+k-k\sigma} P_{|V|}^{|V|} MU^{((G))} \\ &= \pi^G_{|V|-s+k} S^{k\sigma} \wedge P_{|V|}^{|V|} MU^{((G))}. \end{aligned}$$

Since $S^{k\sigma} \wedge P_{|V|}^{|V|} MU^{((G))} \geqslant |V|$, Proposition 11.2.6(iii) tells us that this group vanishes if

$$|V| - s + k < \lfloor |V|/g \rfloor,$$

and hence if

$$s > (g-1)\big(V - s + k\sigma\big) + k.$$

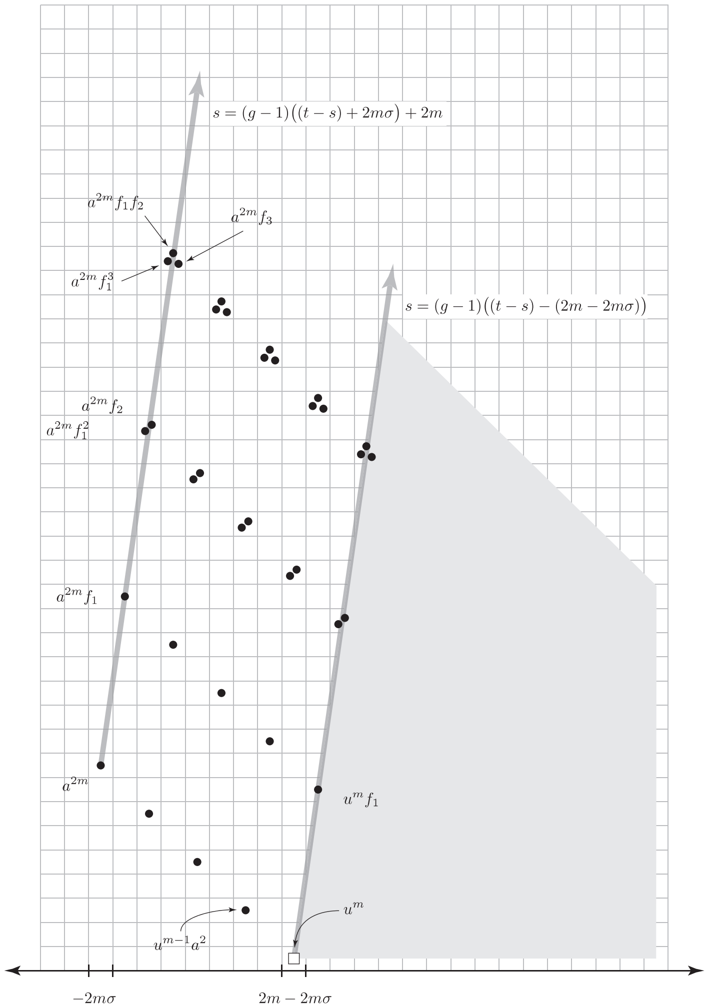

Figure 13.1 The slice spectral sequence for $\pi^G_{-2m\sigma+*}MU^{((G))}$.

This gives the vanishing line depicted in Figure 13.1 and Figure 13.3 for $k = 2m$ and in Figure 13.2 for $k = 2m+1$.

Now $P^{|V|}_{|V|}MU^{((G))}$ is contractible unless $|V|$ is even, in which case it a wedge of G-spectra of the form $H\underline{\mathbf{Z}} \wedge \widehat{S}$ where $\widehat{S}$ is a slice sphere of dimension $|V|$. Since the restriction of the sign representation σ to any proper subgroup $H \subset G$ is trivial, when $\widehat{S} = G \underset{H}{\ltimes} \wedge S^{\ell'\rho_H}$ is a moving slice sphere (Definition 11.1.6), there are isomorphisms

$$S^{k\sigma} \wedge \widehat{S} \wedge H\underline{\mathbf{Z}} \cong S^{k\sigma} \wedge G \underset{H}{\ltimes} \wedge \big(S^{\ell'\rho_H} \wedge H\underline{\mathbf{Z}}\big)$$

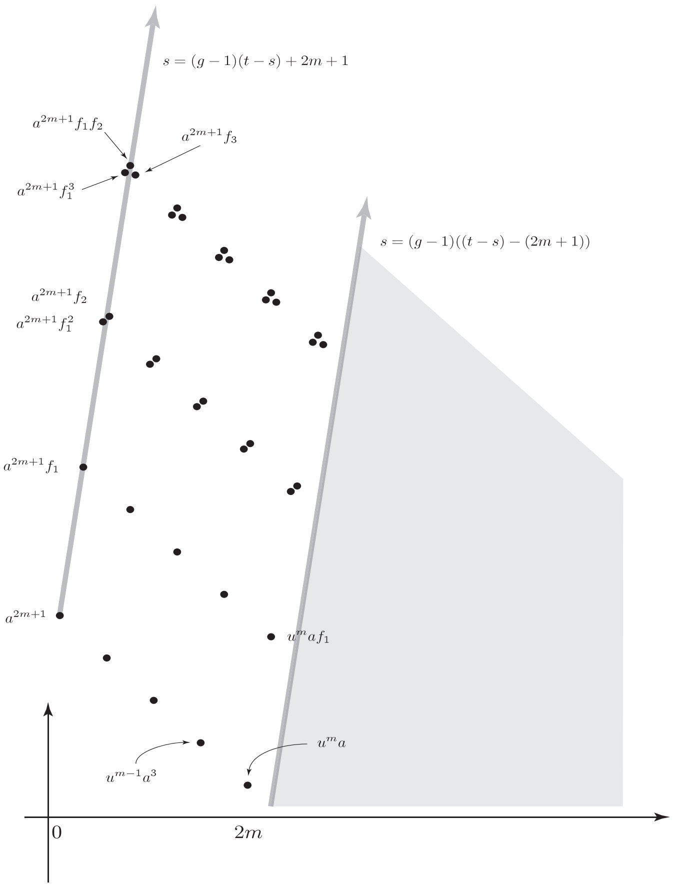

Figure 13.2 The slice spectral sequence for $\pi^G_{-(2m+1)\sigma+*} MU^{((G))}$.

$$\cong G \underset{H}{\ltimes} \wedge \big(S^{k+\ell'\rho_H} \wedge H\underline{\mathbf{Z}}\big),$$

so $\pi^G_{|V|-s+k} S^{k\sigma} \wedge H\underline{\mathbf{Z}} \wedge \widehat{S}$ is isomorphic to

$$\pi^H_{|V|-s} H\underline{\mathbf{Z}} \wedge S^{\ell'\rho_H}.$$

Proposition 11.2.6(iii) tells us that this group vanishes if

$$|V| - s < \ell' = |V|/h \qquad (h = |H|),$$

so certainly when

$$|V| - s \leqslant |V|/g,$$

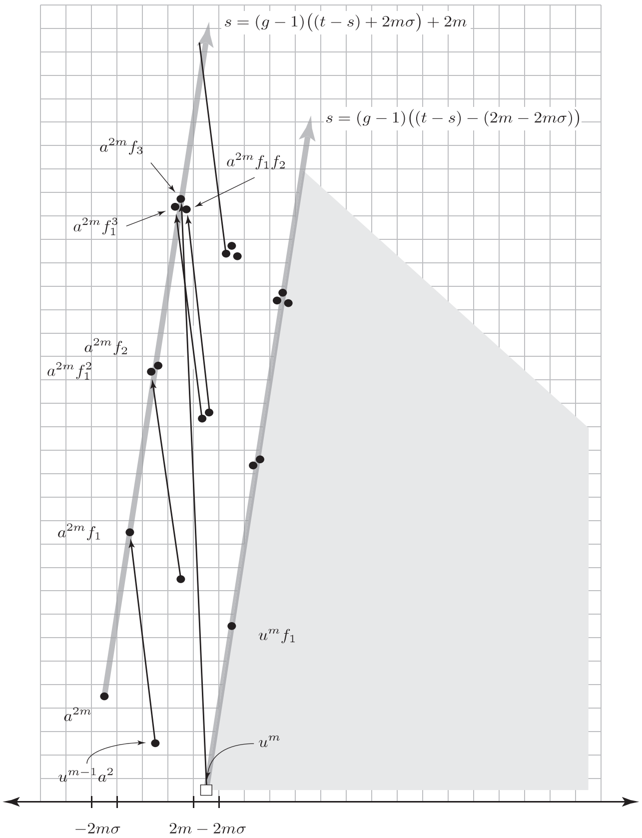

Figure 13.3 Differentials on u^m.

or, equivalently when

$$s \geqslant (g-1)\big((V-s) - (k - k\sigma)\big).$$

Thus **in this range only the stationary slice spheres contribute**; see Definition 11.1.6.

The only stationary slice spheres are those of the form $S^{\ell\rho_G}$. We are therefore studying the groups

$$\pi_j^G H\underline{\mathbf{Z}} \wedge S^{k\sigma + \ell\rho_G}$$

with $j \leqslant \ell + k$ and $k, \ell \geqslant 0$.

Lemma 13.3.2. The homology of $S^{k\sigma+\ell\rho_G}$ in low dimensions. *For $k, \ell \geqslant$*

0 and $j \leqslant \ell + k$ the group

$$\pi_j^G H\underline{\mathbf{Z}} \wedge S^{k\sigma+\ell\rho_G} = \underline{H}_j S^{k\sigma+\ell\rho_G}(G/G)$$

is given by

$$\pi_j^G H\underline{\mathbf{Z}} \wedge S^{k\sigma+\ell\rho_G} = \begin{cases} \mathbf{Z}/2\{a_\sigma^{k+2\ell-j} u_{2\sigma}^{(j-\ell)/2} a_{\rho'}^\ell\} & \\ & \text{for } \ell \leqslant j < k+\ell \text{ and } j-\ell \text{ even} \\ \mathbf{Z}_{(2)}\{u_{2\sigma}^{k/2}\} & \text{for } j = k,\ k \text{ even and } \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here $\rho' = \overline{\rho}_G - \sigma$ and $a_{\rho'}$ is as in Definition 9.9.7(i).

Proof The sphere we are studying is

$$S^{k\sigma+\ell\rho} = S^{\ell+(k+\ell)\sigma+\ell\rho'}.$$

It is a sum of 2-dimensional rotations described in Proposition 9.9.1. Hence Corollary 9.9.5 tells us that

$$\underline{H}_j S^{k\sigma+\ell\rho} \cong \underline{H}_j S^{\ell+(k+\ell)\sigma} \cong \underline{H}_{j-\ell} S^{(k+\ell)\sigma} \qquad \text{for } j < k+2\ell,$$

and the groups are as stated. The factor $a_{\rho'}^\ell$ is there because we are mapping $S^{\ell+(k+\ell)\sigma}$ into $S^{k\sigma+\ell\rho}$. This covers the case $j \leqslant k+\ell$ except when $\ell = 0$, which covered by Example 9.9.21 □

To complete the description of the E_2-term of the $RO(G)$-graded slice spectral sequence in this range we need to identify the summand of stationary slices (Definition 11.1.6) of $MU^{((G))}$. From the associative algebra equivalence

$$\bigvee_{k\in\mathbf{Z}} P_k^k MU^{((G))} \sim H\mathbf{Z} \wedge S^{-0}[G.\overline{r}_1, \dots]$$

this is equivalent to identifying the summand of stationary slice spheres in the twisted monoid ring

$$S^{-0}[G \cdot \overline{r}_1, \dots].$$

Since the smash product of an induced spectrum with any spectrum is induced, we can do this by identifying the summand of stationary slice spheres in each

$$S^{-0}[G \cdot \overline{r}_i]$$

and smashing them together.

Take the generating inclusion

$$\overline{r}_i : S^{i\rho_{C_2}} \to S^{-0}[\overline{r}_i],$$

apply $N_{C_2}^G$ to obtain

$$N\overline{r}_i : S^{i\rho_G} \to S^{-0}[G \cdot \overline{r}_i],$$

and extend it to an associative algebra map

$$S^{-0}[N\overline{r}_i] \to S^{-0}[G \cdot \overline{r}_i]. \tag{13.3.3}$$

Lemma 13.3.4. A twisted monoid subring.

The map of (13.3.3) is the inclusion of the summand of stationary slice spheres.

Proof The distributive law expresses $S^{-0}[G\cdot\overline{r}_i] = N_{C_2}^G S^{-0}[\overline{r}_i]$ as an indexed wedge (see §2.9B)

$$S^{-0}[G \cdot \overline{r}_i] \cong \bigvee_{f:G/C_2 \to \mathbf{N}_0} S^{V_f},$$

and $V_f = \bigoplus_{i=1}^{g/2} \gamma^i f(\gamma^i)\rho_{C_2}$. We now decompose the right hand side into an ordinary wedge over the G-orbits. Since an indexed wedge over a G-orbit is induced from the stabilizer of any element of the orbit, the summand of stationary slice spheres consists of those f which are constant. If $f : G/C_2$ is the constant function with value n, then $V_f = n\rho_G$, so the summand of stationary slice spheres is

$$\bigvee_{\underline{n}} S^{n\rho_G}.$$

The result follows easily from this. $\square$

Smashing these together gives

Corollary 13.3.5. A bigger twisted monoid subring. *The associative algebra map*

$$S^{-0}[N\overline{r}_1, \dots] \to S^{-0}[G \cdot \overline{r}_1, \dots]$$

is the inclusion of the summand of stationary slice spheres.

To put this all together, consider the $\mathbf{Z} \times RO(G)$-graded ring

$$\mathbf{Z}_{(2)}[a, f_i, u]/(2a, 2f_i)$$

with

$$\begin{aligned} |a| &= (1, 1-\sigma) \\ |f_i| &= (i(g-1), ig) \\ |u| &= (0, 2-2\sigma). \end{aligned}$$

Define a map

$$\mathbf{Z}_{(2)}[a, f_i, u]/(2a, 2f_i) \to \bigoplus_{\substack{s,k \geqslant 0 \\ t \in * - k\sigma}} \underline{E}_2^{s,t}(G/G) \tag{13.3.6}$$

by

$$\left.\begin{array}{llll} f_i \mapsto a_{\bar\rho}^i N_2^g \bar r_i & \in \underline{E}_2^{i(g-1),ig}(G/G) & = \pi_i^G P_{ig}^{ig} MU^{((G))} \\ a \mapsto a_\sigma & \in \underline{E}_2^{1,1-\sigma}(G/G) & = \pi_{-\sigma} P_0^0 MU^{((G))} \\ a \mapsto a_\sigma & \in \underline{E}_2^{0,2-2\sigma}(G/G) & = \pi_{2-2\sigma} P_0^0 MU^{((G))} \end{array}\right\} \tag{13.3.7}$$

where the image of u is as in (13.3.1). The combination of Lemma 13.3.2 and Lemma 13.3.4 gives

Proposition 13.3.8. The E_2-term near the vanishing line. *The map*

$$\mathbf{Z}_{(2)}[a, f_i, u]/(2a, 2f_i) \to \bigoplus_{\substack{s,k\geqslant 0\\ t\in *-k\sigma}} E_2^{s,t}$$

is an isomorphism in the range

$$s \geqslant (g-1)((t-s)-(k-k\sigma)).$$

We now turn to the differentials. By construction, the f_i are the representatives at the E_2-term of the slice spectral sequence of the elements defined in Definition 12.3.11 (and also called f_i). They are therefore permanent cycles. Similarly, the element a is the representative of a_σ and also a permanent cycle. This leaves the powers of u. The case $G = C_2$ of the following result appears in unpublished work of Araki and in Hu-Kriz [HK01].

Slice Differentials Theorem 13.3.9. *In the slice spectral sequence for $\pi_\star^G MU^{((G))}$ (where G is a finite cyclic 2-group of order g), the differentials $d_i(u^{2^{n-1}})$ are zero for $i < r = 1 + (2^n - 1)g$, and*

$$d_r(u^{2^{n-1}}) = a^{2^n} f_{2^n-1}.$$

Remark 13.3.10. Elements on the vanishing line which must die. *It follows from Proposition 13.3.8 that what lies on the "vanishing line"*

$$s = (g-1)\big((t-s)+k\sigma\big)+k$$

is the algebra

$$\mathbf{Z}_{(2)}[a, f_i]/(2a, 2f_i).$$

In Proposition 12.3.10 it was shown that the kernel of the map

$$\mathbf{Z}_{(2)}[a_\sigma, f_i]/(2a, 2f_i) \to \pi_\star^G MU^{((G))} \to \pi_\star^G \Phi^G MU^{((G))} = \pi_* MO[a_\sigma^{\pm 1}]$$

is the ideal $(2, f_1, f_3, f_7, \ldots)$. The only possible non-trivial differentials into the vanishing line must therefore land in this ideal.

For the following, the reader may find it helpful to consult Figure 13.3.

Proof of the Slice Differentials Theorem 13.3.9. We establish the differential by induction on k. Assume the result for $k' < k$. Then what is left in the range

$$s \geqslant (g-1)(t-s-k)$$

after the differentials assumed by induction is the sum of two modules over $\mathbf{Z}_{(2)}[f_i]/(2f_i)$. One is generated by a^{2^n} and is free over the quotient ring

$$\mathbf{Z}/2[f_i]/(f_1, f_3, \ldots, f_{2^{n-1}-1}).$$

The other is generated by $u^{2^{n-1}}$. Since the differential must take its value in the ideal $(2, a, f_1, f_3, \ldots)$, the next (and only) possible differential on $u^{2^{n-1}}$ is the one asserted in the theorem. So all we need do is show that the classes $u^{2^{n-1}}$ do not survive the spectral sequence. For this it suffices to do so after inverting a. Consider the map

$$a_\sigma^{-1}\pi_\star^G MU^{((G))} \to a_\sigma^{-1}\pi_\star^G H\underline{\mathbf{Z}}_{(2)}.$$

We know the $\mathbf{Z}$-graded homotopy groups of both sides, since they can be identified with the homotopy groups of the geometric fixed point spectrum. If $u^{2^{n-1}}$ is a permanent cycle, then the class $a^{-2^n}u^{2^{n-1}}$ is as well, and represents a class with non-zero image in $\pi_*^G \Phi^G H\underline{\mathbf{Z}}_{(2)}$. This contradicts Proposition 12.4.22. $\square$

Remark 13.3.11. The effect of inverting a_σ. *After inverting a_σ, the differentials described in the Slice Differentials Theorem 13.3.9 describe completely the $RO(G)$-graded slice spectral sequence. The spectral sequence starts from*

$$\mathbf{Z}/2[f_i, a^{\pm 1}, u].$$

The class $u^{2^{n-1}}$ hits a unit multiple of f_{2^n-1}, and so the E_∞-term is

$$\mathbf{Z}/2[f_i, i \neq 2^n-1][a^{\pm 1}] = MO_*[a^{\pm 1}]$$

which we know to be the correct answer since $\Phi^G MU^{((G))} = MO$. This also shows that the class $u^{2^{n-1}}$ is a permanent cycle modulo $(\bar{r}_{2^n-1})$. This fact corresponds to the main computation in the proof of the Reduction Theorem 12.4.8 (which, of course we used in the above proof). The logic can be reversed, and for the group $G = C_2$ the results are established in the reverse order in [HK01].

Definition 13.3.12. The element $\bar{\mathfrak{d}}_n = \bar{\mathfrak{d}}_n^G \in \pi^G_{(2^n-1)\rho_G} MU^{((G))}$ for $n > 0$. *Applying the norm functor $N_{C_2}^G$ to the map*

$$\bar{r}_{2^n-1}^G : S^{(2^n-1)\rho_2} \wedge S^{-0} \to i_{C_2}^G MU^{((G))}$$

of Definition 12.3.6 gives a map from $S^{(2^n-1)\rho_G} \wedge S^{-0}$ to

$$N_{C_2}^G\left(i_{C_2}^G MU^{((G))}\right) \cong N_{C_2}^G\left(MU_{\mathbf{R}}^{\wedge(g/2)}\right) \cong \left(MU^{((G))}\right)^{\wedge(g/2)}.$$

$\overline{\mathfrak{d}}_n^G$ *is its composite with the multiplication map*

$$\left(MU^{((G))}\right)^{\wedge(g/2)} \to MU^{((G))}$$

given by the ring structure of $MU^{((G))}$*.*

For $H \subseteq G$*, we have*

$$\overline{r}_i^H \in \pi_{i\rho_2}^{C_2} MU^{((H))} \subset \pi_{i\rho_2}^{C_2} MU^{((G))},$$

and we can define

$$\overline{\mathfrak{d}}_n^H = N_{C_2}^H(\overline{r}_{2^n-1}^H) \in \pi_{(2^n-1)\rho_H}^H MU^{((G))}$$

in the same way.

The element $\overline{\mathfrak{d}}_n = \overline{\mathfrak{d}}_n^G$ is the map of (9.7.9) with

$$H = C_2, \qquad X = S^{(2^n-1)\rho_2} \wedge S^{-0} \qquad \text{and} \qquad R = MU^{((G))}.$$

Note that with this notation, the element f_{2^n-1} of (13.3.7) becomes

$$f_{2^n-1} = a_{\bar\rho}^{2^n-1}\overline{\mathfrak{d}}_n.$$

In the proof of Corollary 13.3.14 below, we will make use of the identity

$$f_{2^{n+1}-1}\overline{\mathfrak{d}}_n = a_{\bar\rho}^{2^{n+1}-1}\overline{\mathfrak{d}}_{n+1}\overline{\mathfrak{d}}_n = f_{2^n-1}a_{\bar\rho}^{2^n}\overline{\mathfrak{d}}_{n+1}. \tag{13.3.13}$$

The map

$$\overline{\mathfrak{d}}_n : S^{(2^n-1)\rho_G} \wedge S^{-0} \to MU^{((G))}$$

is represented at the E_2-term of the $RO(G)$-graded slice spectral sequence by a map

$$S^{(2^n-1)\rho_G} \wedge S^{-0} \to P_{(2^n-1)g}^{(2^n-1)g} MU^{((G))}$$

which we will also call $\overline{\mathfrak{d}}_n$. Multiplying, this defines elements $\overline{\mathfrak{d}}_n u^{2^n}$ in the E_2-term of the $RO(G)$-graded slice spectral sequence.

Corollary 13.3.14. Some permanent cycles. *In the* $RO(G)$*-graded slice spectral sequence for* $MU^{((G))}$*, the class* $\overline{\mathfrak{d}}_n u^{2^n}$ *is a permanent cycle.*

Proof Write

$$r = 1 + (2^{n+1} - 1)g.$$

The Slice Differentials Theorem 13.3.9 implies that differentials

$$d_i(\overline{\mathfrak{d}}_n u^{2^n}) = \overline{\mathfrak{d}}_n d_i(u^{2^n})$$

are zero for $i < r$, and

$$d_r(\overline{\mathfrak{d}}_n u^{2^n}) = \overline{\mathfrak{d}}_n a^{2^{n+1}} f_{2^{n+1}-1} = a^{2^{n+1}} f_{2^n-1} a_{\bar\rho}^{2^n}\overline{\mathfrak{d}}_{n+1},$$

the second equality coming from (13.3.13). But from the earlier differential

$$d_{r'}u^{2^{n-1}} = a^{2^n} f_{2^n-1}$$

where $r' = 1 + (2^n - 1)g < r$, we also have

$$d_{r'}(u^{2^{n-1}} a^{2^n} a_{\bar{\rho}}^{2^n} \bar{\mathfrak{d}}_{n+1}) = a^{2^{n+1}} f_{2^n-1} a_{\bar{\rho}}^{2^n} \bar{\mathfrak{d}}_{n+1}$$

so that in fact $d_r(\bar{\mathfrak{d}}_n u^{2^n}) = 0$. The target of the remaining differentials work out to be in a region of the spectral sequence which is already zero at the E_2-term. So once we check this, the proof is complete.

To check the claim about the vanishing region first note that with our conventions, differential d_{i+1} of the $RO(G)$-graded slice spectral sequence maps a sub-quotient of

$$\pi_m^G P_n^n X$$

to a sub-quotient of

$$\pi_{m-1}^G P_{n+i}^{n+i} X.$$

The class in question starts out at the E_2-term as

$$\bar{\mathfrak{d}}_n u^{2^n} \in \pi^G_{2^n(2-2\sigma)+(2^n-1)\rho_G} P^{(2^n-1)g}_{(2^n-1)g} MU^{((G))}$$

so we are interested in the groups

$$\pi^G_{2^n(2-2\sigma)+(2^n-1)\rho_G-1} P^{(2^n-1)g+i}_{(2^n-1)g+i} MU^{((G))}$$

or, equivalently

$$\pi^G_{2^{n+1}-1}\big(S^{2^{n+1}\sigma} \wedge S^{-(2^n-1)\rho_G} \wedge P^{(2^n-1)g+i}_{(2^n-1)g+i} MU^{((G))}\big)$$

with $i + 1 > r = 1 + (2^{n+1} - 1)g$. To simplify the notation, write

$$X_i = S^{-(2^n-1)\rho_G} \wedge P^{(2^n-1)g+i}_{(2^n-1)g+i} MU^{((G))},$$

so that the group we are interested in is

$$\pi^G_{2^{n+1}-1}\big(S^{2^{n+1}\sigma} \wedge X_i\big). \tag{13.3.15}$$

Now $X_i \geqslant i$, so Proposition 11.2.6 implies that

$$\pi_j^G X_i = 0 \qquad \text{for } j < \lfloor i/g \rfloor.$$

Since $S^{2^{n+1}\sigma}$ is (-1)-connected this means that if $i \geqslant 2^{n+1}g$ the group of (13.3.15) is trivial. The remaining values of i are strictly between $(2^{n+1} - 1)g$ and $(2^{n+1})g$, and hence not divisible by g. But since $MU^{((G))}$ is pure, when i is not divisible by g the spectrum $P^{(2^n-1)g+i}_{(2^n-1)g+i} MU^{((G))}$ is induced from a proper subgroup of G, hence so is X_i. This means

$$S^{2^{n+1}\sigma} \wedge X_i \cong S^{2^{n+1}} \wedge X_i,$$

so

$$\pi^G_{2^{n+1}-1}\big(S^{2^{n+1}\sigma} \wedge X_i\big) = \pi^G_{2^{n+1}-1}\big(S^{2^{n+1}} \wedge X_i\big) = 0$$

since $X_i \geqslant 0$. □

13.3B Periodicity theorems

We now turn to our main periodicity theorem. As will be apparent to the reader, the technique can be used to get a much more general result. We have chosen to focus on a case which contains what is needed for the proof of §1.1C(ii), and yet can be stated for a general cyclic 2-group $G = C_{2^n}$.

Our motivating example is the spectrum $K_{\mathbf{R}}$ of "real" K-theory first studied by Atiyah in [Ati66]. Multiplication by the real Bott class $\bar{r}_1 \in \pi_{\rho_2} K_{\mathbf{R}}$ is an isomorphism, giving $K_{\mathbf{R}}$ an S^{ρ_2}-periodicity. On the other hand, the representation $4\rho_2$ admits a Spin structure, and the construction of the KO-orientation of Spin bundles leads to a "Thom" class $u \in \pi_8^{C_2} K_{\mathbf{R}} \wedge S^{4\rho_2}$. This class is represented at the E_2-term of the slice spectral sequence by $u_{4\rho_2}$. Multiplication by $\bar{r}_1^4 u$ is then an equivariant map $S^8 \wedge K_{\mathbf{R}} \to K_{\mathbf{R}}$ whose underlying map of non-equivariant spectra is an equivalence. It therefore gives an equivalence $S^8 \wedge K_{\mathbf{R}}^{hC_2} \cong K_{\mathbf{R}}^{hC_2}$. Since the map $KO \to K_{\mathbf{R}}^{hC_2}$ is an equivalence, this gives the 8-fold periodicity of KO.

In our situation we begin with an equivariant commutative ring spectrum R, a representation V of G, and an element $D \in \pi_V^G R$. We manually create a spectrum with S^V-periodicity by working with the homotopy colimit, $D^{-1}R$, of the sequence

$$R \xrightarrow{D} S^{-V} \wedge R \xrightarrow{S^{-V} \wedge D} S^{-2V} \wedge R \xrightarrow{S^{-2V} \wedge D} \cdots \tag{13.3.16}$$

as in (5.8.18).

The unit inclusion

$$S^{-0} \to D^{-1}R$$

gives a map

$$H\underline{\mathbf{Z}} = P_0^0 S^{-0} \to P_0^0 D^{-1}R$$

and hence defines, for every oriented representation V of G, elements

$$u_V \in \pi_{|V|-V} P_0^0 R = E_2^{0,|V|-V}$$

in the E_2-term of the $RO(G)$-graded slice spectral sequence for $\pi_\star^G D^{-1}R$. We will show, under certain hypotheses on D, that there is an integer $k > 0$ with the property that u_{kV} is a permanent cycle. Let $u \in \pi_\star^G D^{-1}R$ be any element representing u_{kV}. Then the equivariant map

$$S^{k|V|} \wedge D^{-1}R \xrightarrow{u} S^{kV} \wedge D^{-1}R \xrightarrow{D^k} D^{-1}R$$

induces an equivalence of underlying, non-equivariant spectra, and hence an equivalence of homotopy fixed point spectra

$$\left(S^{k|V|} \wedge D^{-1}R\right)^{hG} \to \left(D^{-1}R\right)^{hG}$$

by Theorem 9.11.22. This establishes a periodicity theorem for the homotopy fixed point spectrum $(D^{-1}R)^{hG}$.

The exposition is cleanest when one exploits multiplicative properties of the spectrum $D^{-1}R$. There are some easy general things to say at first. The spectrum $D^{-1}R$ is certainly an R-module, and inherits a homotopy commutative multiplication (over R) from R. The technique of [EKMM97, §VIII.4] can be used to show that the non-equivariant spectrum underlying $D^{-1}R$ has a unique commutative algebra structure for which the map $i_H^G R \to i_H^G D^{-1}R$ is a map of commutative rings.

With an additional assumption on D, one can go further. Let $H \subset G$ be a subgroup, and suppose that there is an $m > 0$ for which the norm $N_H^G\left(i_H^G D\right)$ divides D^m. We will abbreviate $N_H^G\left(i_H^G D\right)$ by $N_H^G D$ and write $D^m = D_H \cdot N_H^G D$. Then there is a commutative diagram

$$\begin{tikzcd}
N_H^G R \arrow[r, "N_H^G(D)"] \arrow[d] & N_H^G\left(S^{-V} \wedge R\right) \arrow[r, "N_H^G(D)"] \arrow[d] & N_H^G\left(S^{-2V} \wedge R\right) \arrow[r] \arrow[d] & \cdots \\
R \arrow[r, "N_H^G(D)"] \arrow[d, "1"] & S^{-V'} \wedge R \arrow[r, "N_H^G(D)"] \arrow[d, "D_H"] & S^{-2V'} \wedge R \arrow[r] \arrow[d, "D_H{}^2"] & \cdots \\
R \arrow[r, "D^m"] & S^{-mV} \wedge R \arrow[r, "D^m"] & S^{-2mV} \wedge R \arrow[r] & \cdots
\end{tikzcd}$$

in which $V' = \mathrm{Ind}_H^G V$. Passing to the homotopy colimit gives a map

$$N_H^G i_H^G\left(D^{-1}R\right) \to D^{-1}R$$

extending the iterated multiplication. This allows one to form norms of elements in $\pi_\star^H D^{-1}R$ as if $D^{-1}R$ were an **equivariant** commutative ring, that is a commutative ring (in the category of orthogonal G-spectra) in which **indexed** products, as well as ordinary ones, are defined.

A necessary condition for $D^{-1}R$ to actually be an equivariant commutative ring, is that for **every** $H \subset G$, the norm $N_H^G i_H^G D$ divides a power of D. In fact the condition is also sufficient. The proof of the following result is described in [HH14, §4], and more details can be found in [HH16].

Proposition 13.3.17. A criterion for the telescope of an equivariant commutative ring to be an equivariant commutative ring. *Let R be an equivariant commutative ring and $D \in \pi_\star^G R$. If D has the property that for every $H \subset G$, the element $N_H^G i_H^G D$ divides a power of D, then the spectrum $D^{-1}R$ has a unique equivariant commutative algebra structure for which the map $R \to D^{-1}R$ is a map of commutative rings.*

We will not make use of Proposition 13.3.17, as the *ad hoc* formation of norms from the non-trivial subgroups of G is sufficient for our purpose.

Suppose that $V \in RO(H)$ and $u \in \pi_V^H D^{-1}R$ is represented at the E_2-term of the $RO(H)$-graded slice spectral sequence by the image of $u' \in \pi_V^H H\underline{\mathbf{Z}}$ under the map $\pi_V^H H\underline{\mathbf{Z}} \to \pi_V^H P_0^0 D^{-1}R$ induced by the unit. We then have an H-equivariant commutative diagram

$$\begin{tikzcd} & S^V \arrow[dl, "u"'] \arrow[d] \arrow[drr, "u'"] & & \\ D^{-1}R & P_0 D^{-1}R \arrow[l] \arrow[r] & P_0^0 D^{-1}R & H\underline{\mathbf{Z}}. \arrow[l] \end{tikzcd} \tag{13.3.18}$$

The maps in the bottom row are maps of homotopy commutative ring spectra. Since the formation of slice sections commutes with filtered colimits, if $N_H^G D$ divides a power of D then the spectra along the bottom row also come equipped with maps $\nu : N_H^G(-) \to (-)$ extending the iterated multiplication, and compatible with the maps between them. This means we may apply the norm to the diagram of (13.3.18) and use the maps ν on the bottom row to produce

$$\begin{tikzcd} & S^{\mathrm{Ind}_H^G V} \arrow[dl, "N_H^G u"'] \arrow[d] \arrow[drr, "N_H^G u'"] & & \\ D^{-1}R & P_0 D^{-1}R \arrow[l] \arrow[r] & P_0^0 D^{-1}R & H\underline{\mathbf{Z}}, \arrow[l] \end{tikzcd} \tag{13.3.19}$$

showing that $N_H^G u'$ is a permanent cycle representing the class

$$N_H^G u \in \pi^G_{\mathrm{Ind}_H^G V} D^{-1}R.$$

We will take R to be the spectrum $MU^{((G))}$. In order to specify the element D we need to consider all of the spectra $MU^{((H))}$ for $H \subset G$, and we will need to distinguish some of the important elements of the homotopy groups we have specified. We use (12.2.7) to map

$$\pi_\star^H MU^{((H))} \to \pi_\star^H MU^{((G))},$$

and make all of our computations in $\pi_\star^H MU^{((G))}$. In addition to $g = |G|$ we will write $h = |H|$ for $H \subset G$ and N_h^g for N_H^G. We will sometimes write $\rho_G = \rho_g$, and $\sigma_G = \sigma_g$, $\rho_H = \rho_h$, and $\sigma_H = \sigma_h$. Let

$$\bar{\mathfrak{d}}_n^H = N_{C_2}^H(\bar{r}_{2^n-1}^H) \in \pi^H_{(2^n-1)\rho_H} MU^{((G))}.$$

be as in Definition 13.3.12.

Theorem 13.3.20. An $RO(G)$-graded permanent cycle. *Let*

$$D \in \pi^G_{\ell\rho_G} MU^{((G))}$$

be a class such that for every nontrivial $H \subset G$, the image of D in $\pi_\star^H MU^{((G))}$ is divisible by $\overline{\mathfrak{d}}_{g/h}^H$ and the element $N_H^G i_H^G D$ divides a power of D. Then the class $u_{2\rho_G}^{2^{g/2}}$ is a permanent cycle in the $RO(G)$-graded slice spectral sequence for $\pi_\star^G D^{-1} MU^{((G))}$.

Proof By Corollary 13.3.14, for each nontrivial subgroup $H \subset G$, the class $\overline{\mathfrak{d}}_{g/h}^H u_{2\sigma_H}^{2^{g/h}}$ is a permanent cycle in the $RO(H)$-graded slice spectral sequence for $\pi_\star^H MU^{((G))}$. Since $i_H^G D$ is divisible by $\overline{\mathfrak{d}}_{g/h}^H$, the class $u_{2\sigma_H}^{2^{g/h}}$ is then a permanent cycle in the $RO(G)$-graded slice spectral sequence for $\pi_\star^H D^{-1} MU^{((G))}$. From this inventory of permanent cycles, and the *ad hoc* norm of (13.3.19), we will show that $u_{2\rho_G}^{2^{g/2}}$ is also a permanent cycle.

To begin, note that if $H \subset G$ has index 2, then $\mathrm{Ind}_H^G 1 = 1 + \sigma_G$. It follows from Corollary 9.9.13 that

$$u_{2\rho_G} = u_{2\sigma_G}^{g/2} N_H^G u_{2\rho_H}.$$

For $G = C_2$, C_4 and C_8, this gives

$$\begin{aligned} u_{2\rho_2} &= u_{2\sigma_2}, \\ u_{2\rho_4} &= u_{2\sigma_4}^2 N_2^4(u_{2\rho_2}) = u_{2\sigma_4}^2 N_2^4(u_{2\sigma_2}) \\ \text{and} \qquad u_{2\rho_8} &= u_{2\sigma_8}^4 N_4^8(u_{2\rho_4}) = u_{2\sigma_8}^4 N_4^8(u_{2\sigma_4}^2 N_2^4(u_{2\sigma_2})) \\ &= u_{2\sigma_8}^4 N_4^8(u_{2\sigma_4}^2) N_2^8(u_{2\sigma_2}). \end{aligned}$$

For a cyclic 2-group G, a similar calculation gives

$$u_{2\rho_g}^\ell = \prod_{e \neq H \subset G} N_h^g \left(u_{2\sigma_H}^{\ell h/2} \right).$$

When $\ell = 2^{g/2}$ we have $\ell h/2 = 2^{g/2} h/2 \geqslant 2^{g/h}$ for every $h \neq 1$ dividing g, so every term in the product is a permanent cycle (the inequality is an equality only when $h = 2$). This completes the proof. □

Corollary 13.3.21. An integer graded permanent cycle. *In the situation of Theorem 13.3.20, let $\Delta^G = u_{2\rho_G} \overline{\mathfrak{d}}_1^G$. Then the class*

$$(\Delta^G)^{2^{g/2}} = u_{2\rho_G}^{2^{g/2}} (\overline{\mathfrak{d}}_1^G)^{2 \cdot 2^{g/2}} \tag{13.3.22}$$

is a permanent cycle. Any class in $\pi_{2 \cdot g \cdot 2^{g/2}}^G D^{-1} MU^{((G))}$ represented by (13.3.22) restricts to a unit in $\pi_*^u D^{-1} MU^{((G))}$.

Proof The fact that (13.3.22) is a permanent cycle is immediate from Theorem 13.3.20. Since the slice tower refines the Postnikov tower, the restriction of an element in the $RO(G)$-graded group $\pi_\star^G D^{-1} MU^{((G))}$ to $\pi_*^u D^{-1} MU^{((G))}$ is determined entirely by any representative at the E_2-term of the slice spectral sequence. Since $u_{2\rho_G}$ restricts to 1, the restriction of any representative of (13.3.22) is equal to the restriction of $(\overline{\mathfrak{d}}_1^G)^{2 \cdot 2^{g/2}}$, which is a unit since $\overline{\mathfrak{d}}_1^G$ divides D. □

This gives

Theorem 13.3.23. The periodicity theorem for homotopy fixed point spectra. *With the notation of Theorem 13.3.20, if M is any equivariant $D^{-1}MU^{((G))}$-module, then multiplication by $(\Delta^G)^{2^{g/2}}$ is a weak equivalence*

$$\Sigma^{2\cdot g\cdot 2^{g/2}} i_H^G M \to i_H^G M$$

and hence an isomorphism

$$(\Delta^G)^{2^{g/2}} : \pi_* M^{hG} \to \pi_{*+2\cdot g\cdot 2^{g/2}} M^{hG}.$$

For example, in the case of $G = C_2$ the groups $\pi_*(D^{-1}MU^{((G))})^{hG}$ are periodic with period $2\cdot 2\cdot 2 = 8$, and for $G = C_4$ there is a periodicity of $2\cdot 4\cdot 2^2 = 32$. For $G = C_8$ we have a period of $2\cdot 8\cdot 2^4 = 256$. For the next case, $G = C_{16}$, the period is $2^{13} = 8192$.

Remark 13.3.24. *Suppose that $D \in \pi_\star^G R$ is of the form*

$$D = N_{C_2}^G x.$$

Then for $C_2 \subset H \subset G$ one has

$$N_H^G i_H^G D = D^{g/h}.$$

Indeed,

$$N_H^G i_H^G D = N_H^G i_H^G N_{C_2}^G x = N_H^G (N_{C_2}^H)^{g/h} = N_{C_2}^G x^{g/h} = D^{g/h}.$$

Since each $\bar{\mathfrak{d}}_n^H$ has this form, any class D which is a product of $N_H^G \bar{\mathfrak{d}}_n^H$ has the property required for Theorems 13.3.20 and 13.3.23.

Corollary 13.3.25. The Periodicity Theorem of §1.1C for homotopy fixed point spectra. *Let $G = C_8$, and*

$$D = \left(N_2^8 \bar{\mathfrak{d}}_4^{C_2}\right)\left(N_4^8 \bar{\mathfrak{d}}_2^{C_4}\right)\left(\bar{\mathfrak{d}}_1^{C_8}\right) \in \pi_{19\rho_G}^G MU^{((G))}.$$

Then multiplication by $(\Delta^G)^{16}$ gives an isomorphism

$$\pi_*\left(D^{-1}MU^{((G))}\right)^{hG} \to \pi_{*+256}\left(D^{-1}MU^{((G))}\right)^{hG}.$$

For $G = C_{2^n}$, the dimension of D is

$$\sum_{k=0}^{n-1}\left(2^{2^n} - 1\right)\rho_G.$$

The Periodicity Theorem of §1.1C(ii) itself is stated in terms of **ordinary fixed points** rather than homotopy fixed points. We will see in the next subsection that for $D^{-1}MU^{((G))}$, the two are equivalent.

Remark 13.3.26. Our choice of D. *For a periodicity theorem, one gets a sufficient inventory of powers of $u_{2\sigma_H}$ as permanent cycles as long as for each H, some $\overline{\mathfrak{d}}_j^H$ is inverted. This is also enough to prove the Homotopy Fixed Point Theorem 13.3.28. Our particular choice of $\overline{\mathfrak{d}}_{g/h}^H$ is dictated by the requirements of the Detection Theorem, specifically Lemma 13.4.6.*

With this in mind, we can at last make the definition promised in the beginning of this chapter and in the beginning of the book.

Definition 13.3.27. The spectra $\Xi_{\mathbf{O}}$ and Ξ *are respectively the telescope $D^{-1}MU^{((G))}$ as in (13.3.16) and its fixed point spectrum $\left(D^{-1}MU^{((G))}\right)^G$, where $G = C_8$ and D is as in Corollary 13.3.25.*

13.3C The homotopy fixed point theorem

We now consider a more general situation similar to the one in §13.3B.

Homotopy Fixed Point Theorem 13.3.28. *Let*

$$D \in \pi^G_{\ell\rho_G} MU^{((G))}$$

have the property that for all non-trivial $H \subset G$ the restriction of D to $\pi^H_* MU^{((G))}$ is divisible by $\overline{\mathfrak{d}}_n^H$ for some n, which may depend on H.

Then for any module M over $D^{-1}MU^{((G))}$ is cofree as in Definition 9.11.23, and

$$\pi^G_* M \to \pi_* M^{hG}$$

is an isomorphism.

Proof We will show that $D^{-1}MU^{((G))}$ satisfies Lemma 9.11.27(i). The result will then follow from Corollary 9.11.28. Suppose that $H \subset G$ is non-trivial. Then

$$\Phi^H(D^{-1}MU^{((G))}) \approx \Phi^H(D)^{-1}\Phi^H(MU^{((G))}).$$

But D is divisible by $\overline{\mathfrak{d}}_n^H$, and so $\Phi^H(D)$ is divisible by

$$\Phi^H(\overline{\mathfrak{d}}_n^H) = \Phi^H(N_{C_2}^H(\overline{r}^H_{2^n-1}))y = \Phi^{C_2}(\overline{r}^H_{2^n-1})$$

which is zero by Proposition 12.3.10. This completes the proof. □

Corollary 13.3.29. *In the situation of Corollary 13.3.25, the map "multiplication by Δ^G" gives an isomorphism*

$$\pi^G_*(D^{-1}MU^{((G))}) \to \pi^G_{*+256}(D^{-1}MU^{((G))}).$$

Proof In the diagram

$$\begin{tikzcd}
\pi_*^G(D^{-1}MU^{((G))}) \arrow[r] \arrow[d] & \pi_{*+256}^G(D^{-1}MU^{((G))}) \arrow[d] \\
\pi_*(D^{-1}MU^{((G))})^{hG} \arrow[r] & \pi_{*+256}^G(D^{-1}MU^{((G))})^{hG}
\end{tikzcd}$$

the vertical maps are isomorphisms by the Homotopy Fixed Point Theorem 13.3.28, and the bottom horizontal map is an isomorphism by Corollary 13.3.25. □

Corollary 13.3.30. The Periodicity Theorem of § 1.1C(ii) for ordinary fixed point spectra. *For $G = C_8$, and D as in Corollary 13.3.25, multiplication by $(\Delta^G)^{16}$ gives an isomorphism*

$$\pi_*(D^{-1}MU^{((G))})^G \to \pi_{*+256}(D^{-1}MU^{((G))})^G.$$

13.4 The Detection Theorem

The account given here differs substantially from that of [HHR16, §11]. In particular, it includes an explanation of why C_8 is the smallest cyclic 2-group G such that $(N_{C_2}^G MU_{\mathbf{R}})^G$ detects the Kervaire invariant elements θ_j; see Remark 13.4.17.

13.4A θ_j in the Adams-Novikov spectral sequence

Browder's theorem says that θ_j is detected in the classical Adams spectral sequence by

$$h_j^2 \in \mathrm{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbf{Z}/2, \mathbf{Z}/2),$$

where $\mathcal{A}$ denotes the mod 2 Steenrod algebra. This element is known to be the only one in its bidegree.

It is more convenient for us to work with the Adams-Novikov spectral sequence, which maps to the Adams spectral sequence. It has a family of elements in filtration 2, namely

$$\beta_{i/j} \in \mathrm{Ext}_{BP_*(BP)}^{2,6i-2j}(BP_*, BP_*)$$

for certain values of of i and j. The subscript i/j here is **not** to be interpreted as a fraction. When $j = 1$, it is customary to omit it from the notation and denote the element by β_i. The definition of these elements can be found in [Rav86, Chapter 5].

Here are the first few of these in the relevant bidegrees.

bidegree of θ_2 :$\beta_{2/2}$

bidegree of θ_3 :$\beta_{4/4}$ and β_3
bidegree of θ_4 :$\beta_{8/8}$ and $\beta_{6/2}$
bidegree of θ_5 :$\beta_{16/16}$, $\beta_{12/4}$ and β_{11}

and so on.

The sequence of integers appearing as subscripts of the last element in the list for oddly indexed θ_js, namely

$$3 = \frac{1+2^3}{3}, \qquad 11 = \frac{1+2^5}{3}, \qquad 43 = \frac{1+2^7}{3}, \qquad \cdots$$

converges 2-adically to $1/3$. The analogous sequence for an odd prime p converges p-adically to $1/(p+1)$.

In the bidegree of θ_j, only $\beta_{2^{j-1}/2^{j-1}}$ has a nontrivial image (namely h_j^2) in the Adams spectral sequence. There is an additional element in this bidegree, namely $\alpha_1\alpha_{2^j-1}$. According to [Shi81], [Rav86, Corollary 5.4.5], a basis for

$$\mathrm{Ext}^{2,2^{j+1}}_{BP_*(BP)}(BP_*, BP_*)$$

for $j > 1$ is given by

$$\begin{gathered} \{\alpha_1\alpha_{2^j-1}\} \cup \left\{\beta_{c(j,k)/2^{j-1-2k}} : 0 \leqslant k < j/2\right\}, \\ \text{where} \quad c(j,k) = \frac{2^{j-1-2k}(1+2^{2k+1})}{3}. \end{gathered} \tag{13.4.1}$$

For $j = 1$, the Ext group is spanned by α_1^2. For $j > 1$, $\alpha_1\alpha_{2^j-1}$ supports a nontrivial d_3, but we do not need this fact. None of these elements is divisible by 2. The element $\beta_{c(j,k)/2^{j-1-2k}}$ is represented by the chromatic fraction (see [Rav86, Chapter 5])

$$\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}};$$

no correction terms are needed in the numerator.

We need to show that any element mapping to h_j^2 in the classical Adams spectral sequence has nontrivial image the Adams-Novikov spectral sequence for the spectrum Ξ of §1.1C.

Detection Theorem 13.4.2. *Let*

$$u \in \mathrm{Ext}^{2,2^{j+1}}_{BP_*(BP)}(BP_*, BP_*)$$

be any element whose image in $\mathrm{Ext}^{2,2^{j+1}}_{\mathcal{A}}(\mathbf{Z}/2, \mathbf{Z}/2)$ *is* h_j^2 *with* $j \geqslant 6$. *Then the image of* u *in* $H^2(C_8; \pi_*\Xi_{\mathbf{O}})$ *is nonzero.*

We will prove this by showing the same is true after we map the latter to a simpler object involving another algebraic tool, **the theory of formal** A**-modules**, where A is the ring of integers in a suitable field. We will show that the image $\beta_{2^{j-1}/2^{j-1}}$ is nontrivial (see Lemma 13.4.10), while those of the other elements of (13.4.1) are trivial. The latter will be demonsatted following the proof of Lemma 13.4.15 at the end of §13.4C.

13.4B Formal A-modules

Recall that a formal group law over a ring R is a power series

$$F(x, y) = x + y + \sum_{i,j>0} a_{i,j} x^i y^j \in R[[x, y]]$$

with certain properties.

For positive integers m one has power series $[m](x) \in R[[x]]$ defined recursively by $[1](x) = x$ and

$$[m](x) = F(x, [m-1](x)).$$

These satisfy

$$[m+n](x) = F([m](x), [n](x)) \text{ and } [m]([n](x)) = [mn](x).$$

With these properties we can define $[m](x)$ uniquely for all integers m, and we get a homomorphism

$$\tau : \mathbf{Z} \to \mathrm{End}(F) \qquad (13.4.3)$$

where $\mathrm{End}(F)$ is the endomorphism ring of F.

Equivalently, the power series $f_m(x) = [m](x)$ is characterized by

$$f'_m(0) = m \qquad \text{and} \qquad f_m(F(x, y)) = F(f_m(x), f_m(y)).$$

If the ground ring R is an algebra over the p-local integers $\mathbf{Z}_{(p)}$ or the p-adic integers $\mathbf{Z}_p$, then we can make sense of $[m](x)$ for m in $\mathbf{Z}_{(p)}$ or $\mathbf{Z}_p$.

Now suppose R is an algebra over a larger ring A, such as the ring of integers in a number field or a finite extension of the p-adic numbers.

Definition 13.4.4. Formal A-modules. *A formal group law F over an A-algebra R is a* **formal A-module** *if the homomorphism τ of (13.4.3) extends to A in such a way that*

$$[a](x) \equiv ax \bmod (x^2) \textit{ for } a \in A.$$

Equivalently for each $a \in A$ there is a power series $f_a(X) \in A[[x]]$ with

$$f'_a(0) = a \qquad \textit{and} \qquad f_a(F(x, y)) = F(f_a(x), f_a(y)).$$

The theory of formal A-modules is well developed. Jonathan Lubin and John Tate (1925–2019) used it to do local class field theory in [LT65]. A good reference for this is Michiel Hazewinkel's book [Haz78, Chapter 21].

The example of interest to us is $A = \mathbf{Z}_2[\zeta_8]$, where ζ_8 is a primitive 8th root of unity. We will generalize this to $A = \mathbf{Z}_2[\zeta_g]$, where $g = 2^m$ and ζ_g is a primitive 2^mth root of unity for some $n > 0$. This will enable us to consider the norms $N_{C_2}^G MU_{\mathbf{R}}$ for all cyclic 2-groups $G = C_{2^m}$. We will eventually see that the proof of a would be detection theorem breaks down for $m < 3$, and that the cases $m > 3$ lead to longer periodicities. Hence the optimal group G for our purposes is C_8, as stated at the end of §1.1C.

The maximal ideal of $A = \mathbf{Z}_2[\zeta_g]$ is generated by $\pi = \zeta_g - 1$, and $\pi^{g/2}$ is a unit multiple of 2 in A. In [Haz78, 24.5.2 and 25.3.16] it is shown that there is a formal A-module F over $R_* = A[w^{\pm 1}]$ (with $|w| = 2$) with logarithm

$$\log_F(x) = \sum_{n \geqslant 0} \frac{w^{2^n-1}x^{2^n}}{\pi^n}. \tag{13.4.5}$$

where

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y).$$

What does this formal A-module (for the case $G = C_8$) have to do with our C_8-spectrum $\Xi_{\mathbf{O}} = D^{-1}MU^{((C_8))}$? Recall (Definition 13.3.12) that

$$\overline{\mathfrak{d}}_n^H = N_2^h \overline{r}_{2^n-1}^H \in \pi_{(2^n-1)\rho_H}^H MU^{((H))} \qquad \text{for } h = |H|,$$

where H is a nontrivial subgroup of G, and $\overline{r}_{2^n-1}^H$ is as in Definition 12.3.6. This can be mapped into

$$\pi_{(2^n-1)\rho_H}^H MU^{((G))}$$

using (12.2.7). Equivalently, we can map

$$\overline{r}_{2^n-1}^H \in \pi_{(2^n-1)\rho_{C_2}}^H MU^{((H))}$$

itself into $\pi_{(2^n-1)\rho_{C_2}} MU^{((G))}$ in the same way and apply N_2^h there. Then we have

$$\begin{aligned} D &= \left(N_2^8 \overline{\mathfrak{d}}_4^{C_2}\right)\left(N_4^8 \overline{\mathfrak{d}}_2^{C_4}\right)\left(\overline{\mathfrak{d}}_1^{C_8}\right) \\ &= N_2^8\left(\overline{r}_{15}^{C_2}\right) N_4^8\left(N_2^4 \overline{r}_3^{C_4}\right) N_2^8\left(\overline{r}_1^{C_8}\right) \\ &= N_2^8\left(\overline{r}_{15}^{C_2}\overline{r}_3^{C_4}\overline{r}_1^{C_8}\right) \in \pi_{19\rho_{C_8}}^{C_8} MU^{((C_8))}. \end{aligned}$$

For $G = C_{2^m}$, the element to be inverted is

$$D = \prod_{0 \leqslant \ell < m}\left(N_{2^{\ell+1}}^{2^m} \overline{\mathfrak{d}}_{2^{m-\ell}-1}^{C_{2^{\ell+1}}}\right) = N_2^{2^m}\left(\prod_{0 \leqslant \ell < m} \overline{r}_{2^{2^{m-\ell-1}}-1}^{C_{2^{\ell+1}}}\right) \in \pi_{c(m)\rho_G}^G MU^{((G))}$$

where $c(m)$ is defined by induction on m by

$$c(m) = \begin{cases} 1 & \text{for } m = 1 \\ 2^{g/2} - 1 + c(m-1) & \text{for } m > 1. \end{cases}$$

We saw in Theorem 13.3.20 that inverting a product of this sort is needed to get a spectrum such as $\Xi_{\mathbf{O}}$ with a periodic fixed point set such as Ξ, but we did not explain the choice of subscripts of $\bar{\mathfrak{d}}$; see Remark 13.3.26. For the purposes of periodicity, any choice would do. The subscripts indicated above the smallest ones that satisfy the second part of the following, as will be explained in the last paragraph of its proof.

Lemma 13.4.6. The homomorphism classifying the formal A-module. *The classifying homomorphism $\lambda : \pi_* MU \to R_*$ for the formal group law F of (13.4.5) factors through $\pi_* MU^{\wedge(g/2)}$ in such a way that*

(i) the homomorphism $\lambda^{(g/2)} : \pi_* MU^{\wedge(g/2)} \to R_*$ is equivariant, where G acts on $\pi_* MU^{\wedge(g/2)}$ as before, it acts trivially on A, and $\gamma(w) = \zeta_g w$ for a generator γ of G, and

(ii) the element $i_0^G D \in \pi_* MU^{\wedge(g/2)}$, for D as in Corollary 13.3.25, maps to a unit in R_*.

We will prove this in §13.4D. For $G = C_8$, this D is the element that we invert to get $i_0^* \Xi_{\mathbf{O}}$ in Definition 13.3.27.

13.4C The proof of the Detection Theorem

As before, let G be the cyclic group C_{2^m}, making $g = |G| = 2^m$. It follows from Lemma 13.4.6 that we have a map

$$H^*(G; \pi_*(i_0^* D)^{-1} MU^{(g/2)}) \to H^*(G; R_*).$$

The source here is the E_2-term of the homotopy fixed point spectral sequence for M, and the target is easy to calculate. We will use it to prove Detection Theorem 13.4.2 by showing that the image of $i_0^G D$ in $H^{2,2^{j+1}}(C_8; R_*)$ is nonzero.

We will calculate with BP-theory. Recall that

$$BP_*(BP) = BP_*[t_1, t_2, \dots] \quad \text{where } |t_i| = 2(2^i - 1).$$

We will abbreviate $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, M)$ (for a $BP_*(BP)$-comodule M) by $\mathrm{Ext}^{s,t}(M)$.

We recall the description of it given in [Rav04, A2.1], starting in the paragraph preceding [Rav04, Lemma A2.1.26]. The pair $(BP_*, BP_*(BP))$ represents the functor that assigns to each $\mathbf{Z}_{(p)}$-algebra R the groupoid of p-typical formal group laws and strict isomorphisms between them. The formal group law over R_* of (13.4.5) is 2-typical and is thus induced by a homomorphism $\lambda : BP_* \to R_*$.

[Rav04, Lemma A2.1.26] says that if F is a p-typical formal group law over some ring R and $\phi : F \to G$ is an isomorphism, then G is p-typical if

$$\phi^{-1}(x) = \sum_{i \geqslant 0}{}^F \phi_i x^{p^i} \tag{13.4.7}$$

for $\phi_i \in R$ with ϕ_0 a unit. The isomorphism is strict when $\phi_0 = 1$. In that case ϕ corresponds to a map $BP_*(BP) \to R$ sending t_i to ϕ_i.

Suppose F and G are both the formal group law of (13.4.5) and the isomorphism is the series $[z^{-1}](x)$ for some gth root of unity z. Then (13.4.7) reads

$$[z](x) = \sum_{i\geqslant 0}{}^F \phi_i(z)x^{2^i} \in R_*[[x]]. \tag{13.4.8}$$

Here we write $\phi_i(z) \in R_*$ to emphasize its dependence on the choice of the gth root of unity z. Let

$$\overline{\phi}_i = w^{1-2^i}\phi_i.$$

Taking the logarithm of both sides of (13.4.8) gives the following.

$$\begin{aligned}
z\log_F(x) &= \sum_{i\geqslant 0} \log_F(w^{2^i-1}\overline{\phi}_i x^{2^i}) \\
z\sum_{n\geqslant 0} \frac{w^{2^n-1}x^{2^n}}{\pi^n} &= \sum_{i\geqslant 0}\sum_{j\geqslant 0} \frac{w^{2^j-1}(w^{2^i-1}\overline{\phi}_i x^{2^i})^{2^j}}{\pi^j} \\
&= \sum_{i,j\geqslant 0} \frac{w^{2^j-1}(w^{2^i-1}\overline{\phi}_i)^{2^j} x^{2^{i+j}}}{\pi^j} \\
&= \sum_{n\geqslant 0} w^{2^n-1}x^{2^n} \sum_{0\leqslant j\leqslant n} \frac{\overline{\phi}_j^{2^{n-j}}}{\pi^{n-j}}
\end{aligned}$$

Equating coefficients of x^{2^n} for each $n \geqslant 0$ we have

$$\begin{aligned}
z &= \sum_{0\leqslant j\leqslant n} \pi^j \overline{\phi}_j^{2^{n-j}} = \pi^n\overline{\phi}_n + \sum_{0\leqslant j\leqslant n-1} \pi^j\overline{\phi}_j^{2^{n-j}} \\
\overline{\phi}_n(z) &= \pi^{-n}\left(z - \sum_{0\leqslant j\leqslant n-1} \pi^j\overline{\phi}_j(z)^{2^{n-j}}\right).
\end{aligned}$$

Thus we have $\overline{\phi}_0(z) = z$ and

$$\overline{\phi}_1(z) = \pi^{-1}(z - \overline{\phi}_0^2) = \frac{z - z^2}{\pi} = \frac{z(1-z)}{\pi}. \tag{13.4.9}$$

This is a unit whenever z is a primitive 2^mth root of unity, that is and odd power of $\zeta_g = \pi$. Each $\overline{\phi}_n \in (A\otimes\mathbf{Q})[z]$ is a polynomial in Z over $A\otimes\mathbf{Q}$ which is **numerical** in the sense of taking values in A for all $z \in A$.

The Hopf algebroid associated with $H^*(G; R_*)$ has the form $(R_*, R_*(G))$, where $R_*(G)$ denotes the ring of R_*-valued functions on G. Its left unit sends R_* to the set of constant functions, and the right unit is determined by the group action on R_* via the formula

$$\eta_R(r)(\gamma) = \gamma(r) \qquad \text{for } r \in R_* \text{ and } \gamma \in G.$$

This map is A-linear and G has a generator γ for which $\eta_R(w)(\gamma^k) = \zeta_g^k w$.

We identify the coproduct

$$\Delta : R_*(G) \to R_*(G) \otimes_{R_*} R_*(G)$$

by composing it with the isomorphism

$$R_*(G) \otimes_{R_*} R_*(G) \to R_*(G \times G)$$

given by

$$(\phi' \otimes \phi'')(\gamma_1, \gamma_2) = \phi'(\gamma_1)\gamma_1(\phi''(\gamma_2)),$$

where the factor $\gamma_1(\phi''(\gamma_2))$ refers to the action of G on R_*. The resulting composite

$$\delta : R_*(G) \to R_*(G \times G)$$

is defined by $(\delta\phi)(\gamma_1, \gamma_2) = \phi(\gamma_1\gamma_2)$.

Lemma 13.4.10. An element in $\mathrm{Ext}^{2,2^{j+1}}(BP_*)$ **detected by** $H^*C_{2^m}$ **for** $m \geqslant 3$**.** *Let*

$$b_{1,j-1} = \frac{1}{2} \sum_{0<i<2^j} \binom{2^j}{i} \left[t_1^i | t_1^{2^j-i} \right] \in \mathrm{Ext}^{2,2^{j+1}}(BP_*)$$

Its image in $H^{2,2^{j+1}}(G; R_*)$ *is nontrivial for* $j \geqslant 2$*, where* $G = C_{2^m}$*.*

This element is known to be cohomologous to $\beta_{2^{j-1}/2^{j-1}}$ and to have order 2; see [Rav86, Theorem 5.4.6(a)].

Proof Let $\gamma \in G$ be the generator with $\gamma(w) = \zeta_g w$, where ζ_g is a primitvie 2^mth root of unity. Then $H^*(G; R_*)$ is the cohomology of the cochain complex of $R_*[G]$-modules

$$R_* \xrightarrow{\gamma-1} R_* \xrightarrow{\mathrm{Trace}} R_* \xrightarrow{\gamma-1} \cdots \qquad (13.4.11)$$

where Trace is multiplication by $1 + \gamma + \cdots + \gamma^{g-1}$. Note that

$$(1-\gamma)w^\ell = \begin{cases} \pi^{2^i} w^\ell & \text{for } \ell \equiv 2^i \bmod 2^{i+1} \text{ with } 0 \leqslant i < m \\ 0 & \text{for } \ell \equiv 0 \bmod 2^m \end{cases}$$

and

$$\mathrm{Trace}(w^\ell) = \begin{cases} 2^m w^\ell & \text{for } 2^m | \ell \\ 0 & \text{otherwise.} \end{cases} \qquad (13.4.12)$$

It follows that the cohomology groups $H^s(G; R_*)$ for $s > 0$ are periodic in

s with period 2. We have

$$\left.\begin{aligned} H^0(G;R_{2\ell}) &= \ker(\zeta_g^\ell-1) \\ &= \begin{cases} A & \text{for } \ell\equiv 0 \bmod 2^m \\ 0 & \text{otherwise} \end{cases} \\ H^1(G;R_{2\ell}) &= \ker(1+\zeta_g^\ell+\cdots+\zeta_g^{(g-1)\ell})/\operatorname{im}(\zeta_g^\ell-1) \\ &= \begin{cases} A/(\zeta_g^\ell-1)\cong w^\ell A/(\pi^{2^i})\cong(\mathbf{Z}/2)^{2^i} \\ \quad \text{for } \ell\equiv 2^i \bmod 2^{i+1} \\ \quad \text{where } 0\leqslant i<m \\ 0 \quad \text{for } \ell\equiv 0 \bmod 2^m \end{cases} \\ H^2(G;R_{2\ell}) &= \ker(\zeta_g^\ell-1)/\operatorname{im}(1+\zeta_g^\ell+\cdots+\zeta_g^{(g-1)\ell}) \\ &= \begin{cases} w^\ell A/(2^m) & \text{for } \ell\equiv 0 \bmod 2^m \\ 0 & \text{otherwise} \end{cases} \end{aligned}\right\} \tag{13.4.13}$$

Note that the A-modules occurring above are

$$A/(\pi^{2^i})\cong\begin{cases} \mathbf{Z}/2[\pi]/(\pi^{2^i}) & \text{for } 0\leqslant i\leqslant m-2, \\ A/(2)\cong\mathbf{Z}/2[\pi]/(\pi^{2^{m-1}}), & \text{for } i=m-1 \\ \mathbf{Z}/2^m[\pi]/(1+(1+\pi)^{2^{m-1}}) & \text{for } i=m. \end{cases}$$

We also have a map

$$\operatorname{Ext}^{*,*}_{BP_*(BP)}(BP_*,BP_*/2)\to H^*(G;R_*/(2))$$

Reducing the complex of (13.4.11) mod 2 makes the trace map trivial by (13.4.12), so for $t\geqslant 0$ we have

$$\left.\begin{aligned} H^{2t}(G;R_{2\ell}/2) &= \begin{cases} \pi^{2^{m-1}-2^i}A/(\pi^{2^i}) \\ \qquad \text{for } \ell\equiv 2^i \bmod 2^{i+1}, \\ \qquad\quad \text{where } 0\leqslant i\leqslant m-2 \\ A/(2) \quad \text{for } \ell\equiv 0 \bmod 2^{m-1} \end{cases} \\ H^{2t+1}(G;R_{2\ell}/2) &= \begin{cases} A/(\pi^{2^i}) & \text{for } \ell\equiv 2^i \bmod 2^{i+1}, \\ & \quad \text{where } 0\leqslant i\leqslant m-2 \\ A/(2) & \text{for } \ell\equiv 0 \bmod 2^{m-1} \end{cases} \end{aligned}\right\} \tag{13.4.14}$$

For $j\geqslant 0$, the image of the class

$$[t_1^{2^j}]\in\operatorname{Ext}^{1,2^{j+1}}_{BP_*(BP)}BP_*/2$$

in $H^1(G;R_{2^{j+1}}/(2))$ is a unit in $A/(2)=\mathbf{Z}/2[\pi]/(\pi^{2^{m-1}})$ since by (13.4.9) the function $\overline{\phi}_1$, and hence any power of it, is not divisible by π. For $j\geqslant m$, consider the following diagram.

$$\begin{tikzcd}
\mathrm{Ext}^{1,2^{j+1}}(BP_*) \arrow[r] \arrow[d, "\lambda"] & \mathrm{Ext}^{1,2^{j+1}}(BP_*/(2)) \arrow[r, "\delta"] \arrow[d, "\lambda"] & \mathrm{Ext}^{2,2^{j+1}}(BP_*) \arrow[d, "\lambda"] \\
H^1(G;R_{2^j}) \arrow[r] \arrow[d, equal] & H^1(G;R_{2^j}/(2)) \arrow[r, "\delta'"] \arrow[d, equal] & H^2(G;R_{2^j}) \arrow[d, equal] \\
0 \arrow[r] & A/(2) \arrow[r] & A/(2^m)
\end{tikzcd}$$

Here δ and δ' are the evident connecting homomorphisms, $\lambda : BP_* \to R_*$ is the classifying map for our formal A-module and the rows are exact. The values of cohomolgy groups indicated in the bottom row follow from (13.4.13) and (13.4.14). The connecting homomorphism δ sends $[t_1^{2^j}]$ to $b_{1,j-1}$, and $\lambda([t_1^{2^j}])$ is a unit, so $\lambda(b_{1,j-1})$ has the desired property. □

To finish the proof of the Detection Theorem we need to show that $\alpha_1\alpha_{2^j-1}$ and the other βs in the same bidegree map to zero. We will do this for $j \geqslant 6$. The appropriate Ext group was described in (13.4.1). Note that

$$\beta_{c(j,0)/2^{j-1}} = \beta_{2^{j-1}/2^{j-1}},$$

so we need to show that the elements $\beta_{c(j,k)/2^{j-1-2^k}}$ with $k > 0$ map to zero.

Lemma 13.4.15. The images of v_1 and v_2 in R_*. *Let $\lambda : BP_* \to R_*$ be the classifying map for the formal A-module of (13.4.5) for the group $G = C_{2^m}$. Then*

$$\lambda(v_1) = \frac{2w}{\pi},$$

$$\textit{and} \qquad \lambda(v_2) = \frac{(2\pi-4)w^3}{\pi^3} = \begin{cases} w^3 & \textit{for } m = 1 \\ (-1+2\sqrt{-1})w^3 & \textit{for } m = 2 \\ \dfrac{2w^3}{\pi^2}\left(1-\dfrac{2}{\pi}\right) & \textit{for } m > 2 \end{cases}$$

Proof The logarithm of the formal group law over BP_* is

$$x + \sum_{n>0} \ell_n x^{2^n}$$

where the relation between the ℓ_ns and Hazewinkel's v_ns is given recursively by

$$2\ell_n = \sum_{0 \leqslant i < n} \ell_i v_{n-i}^{2^i}.$$

Hence under the classifying map $\ell_n \mapsto w^{2^n-1}/\pi^n$ we find that

$$\begin{aligned} v_1 &\mapsto \frac{2w}{\pi} \\ v_2 + \frac{v_1^3}{2} &\mapsto \frac{2w^3}{\pi^2} \end{aligned}$$

$$v_2 \mapsto \frac{2w^3}{\pi^2} - \frac{4w^3}{\pi^3} = \frac{2w^3}{\pi^2}\left(1 - \frac{2}{\pi}\right)$$

When $m = 1$, $\pi = -2$, so v_2 maps to w^3. When $m = 2$, $\pi = \sqrt{-1} - 1$, so v_2 maps to the indicated element. □

We can define a valuation $||\cdot||$ on R_* by setting $||\pi|| = 2^{1-m}$ (so $||2|| = 1$) and $||w|| = 0$. We can define one on $BP_*(BP)$ by defining

$$||v_i|| = ||\lambda(v_i)|| \qquad \text{and} \qquad ||t_i|| = ||\overline{\phi}_i|| \qquad \text{for } i > 0. \tag{13.4.16}$$

The valuation extends in an obvious way to the cobar complex and to the chromatic modules M^n, such as

$$M^2 = v_2^{-1}BP_*/(2^\infty, v_1^\infty).$$

From there we can extend it to the chromatic cobar complex defined in [Rav86, 5.1.10]. Thus we get valuations on the groups

$$\mathrm{Ext}^0_{BP_*(BP)}\left(M^2\right) \longrightarrow \mathrm{Ext}^2_{BP_*(BP)}\left(BP_*\right) \longrightarrow H^2(C_8; R_*)$$

The left group contains the chromatic fractions $\beta_{i/j}$. The homomorphisms cannot lower (but may raise) this valuation. We will show that for $n = 3$ (meaning the group is C_8), the valuation of the relevant chromatic fractions is $\geqslant 3$. This valuation is a lower bound on the one in $H^*(C_8; R_*)$, where every group has exponent at most 8. Hence a valuation $\geqslant 3$ means the β-element has trivial image.

Hence for $j \geqslant 6$ we have

$$\begin{aligned} ||\alpha_1\alpha_{2^j-1}|| &\geqslant ||\frac{v_1}{2}|| + ||\frac{v_1^{2^j-1}}{2}|| \\ &= \frac{3}{4} - 1 + \frac{3(2^j-1)}{4} - 1 \geqslant 46, \end{aligned}$$

and for $1 \leqslant k \leqslant (j-1)/2$,

$$\begin{aligned} ||\beta_{c(j,k)/2^{j-1-2k}}|| &\geqslant \left|\left|\frac{v_2^{c(j,k)}}{2v_1^{2^{j-1-2k}}}\right|\right| \\ &= \frac{c(j,k)}{2} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\ &= \frac{2^j + 2^{j-1-2k}}{6} - \frac{3\cdot 2^{j-1-2k}}{4} - 1 \\ &\qquad \text{by (13.4.1)} \\ &= \frac{(4^k-7)2^{j-3-2k}}{3} - 1 \geqslant 5. \end{aligned}$$

This means $\beta_{c(j,k)/2^{j-1-2k}}$ maps to an element that is divisible by 8 and therefore zero. We leave the analogous computation for $m > 3$, which leads to a similar conclusion, as an exercise for the reader.

Remark 13.4.17. This argument does not work for the groups C_2 and C_4. *In those cases the image of v_2 in R_* is a unit by Lemma 13.4.15, and* **the computation above does show that we can detect θ_j.**

This completes the proof of the Detection Theorem assuming Lemma 13.4.6.

4/4/20. Stopped listing notations here today.

13.4D The proof of Lemma 13.4.6

We will specialize here to the case $G = C_8$, the argument in the general case being similar. To prove Lemma 13.4.6(i), consider the following diagram for an arbitrary ring K.

$$\begin{tikzcd}
 & MU_*(MU) \arrow[d, equal] & \\
\pi_*MU \arrow[ur, "\eta_L"] \arrow[dr, "\lambda_1"'] & \pi_*MU^{\wedge 2} \arrow[d, dotted, "\lambda^{(2)}"'] & \pi_*MU \arrow[ul, "\eta_R"'] \arrow[dl, "\lambda_2"] \\
 & K &
\end{tikzcd}$$

The maps λ_1 and λ_2 classify two formal group laws F_1 and F_2 over K. The Hopf algebroid $MU_*(MU)$ represents strict isomorphisms between formal group laws. Hence the existence (and choice) of $\lambda^{(2)}$ is equivalent to that of a strict isomorphism between F_1 and F_2.

Similarly consider the diagram

$$\begin{tikzcd}
 & & \pi_*MU^{\wedge 4} \arrow[dd, dotted, "\lambda^{(4)}"] & & \\
\pi_*MU \arrow[urr, "\eta_1"] \arrow[drr, "\lambda_1"'] & \pi_*MU \arrow[ur, "\eta_2"'] \arrow[dr, "\lambda_2"] & & \pi_*MU \arrow[ul, "\eta_3"] \arrow[dl, "\lambda_3"'] & \pi_*MU \arrow[ull, "\eta_4"'] \arrow[dll, "\lambda_4"] \\
 & & K & &
\end{tikzcd}$$

where the homomorphisms η_j are unit maps corresponding to the four smash product factors of $MU^{(4)}$. The existence of $\lambda^{(4)}$ is equivalent to that of strict isomorphisms between the formal group laws

$$F_1 \to F_2 \to F_3 \to F_4$$

induced by the four maps λ_j.

Now suppose that $K = R_*$ and each λ_j classifies the formal A-module given by (13.4.5). Then we have the required isomorphisms, so $\lambda^{(4)}$ exists. The inclusions η_j are related by the action of C_8 on $\pi_* MU^{\wedge 4}$ via

$$\gamma\eta_j = \eta_{j+1} \qquad \text{for } 1 \leqslant j \leqslant 3$$

and $\gamma\eta_4$ differs from η_1 by the $(-1)^i$ in dimension $2i$. The λ_j can be chosen to satisfy a similar relation to the C_8-action on R_*. It follows that $\lambda^{(4)}$ is equivariant with respect to the C_8-actions on its source and target. This proves Lemma 13.4.6(i).

For Lemma 13.4.6(ii), recall from Corollary 13.3.25 that

$$D = N_2^8(\overline{r}_{15}^{C_2}\overline{r}_3^{C_4}\overline{r}_1^{C_8}).$$

The norm sends products to products, and $N_2^8(x)$ is a product of conjugates of x under the action of C_8. Hence its image in R_* is a unit multiple of that of a power of x, so it suffices to show that each of the three elements $\overline{r}_{15}^{C_2}$, $\overline{r}_3^{C_4}$ and $\overline{r}_1^{C_8}$ maps to a unit in R_*.

The generators $\overline{r}_i^H$ are defined by (12.3.7), which we rewrite as

$$\overline{x} + \sum_{i>0} \overline{m}_i \overline{x}^{i+1} = \left(\overline{x} + \sum_{k>0} \gamma_H(\overline{m}_{2^n-1})\overline{x}^{2^n}\right) \circ \left(\overline{x} + \sum_{i>0} \overline{r}_i^H \overline{x}^{i+1}\right)$$

where $\gamma_H = \gamma^{8/h}$ denotes a generator of $H \subset G$ and γ is a generator of C_8. Note here that the $\overline{m}_i$ are independent of the choice of subgroup H. For our purposes we can replace this by the corresponding equation in underlying homotopy groups, namely

$$x + \sum_{i>0} m_i x^{i+1} = \left(x + \sum_{k>0} \gamma^{8/h}(m_{2^n-1})x^{2^n}\right) \circ \left(x + \sum_{i>0} r_i^H x^{i+1}\right).$$

We have such an equation for each subgroup $H \subseteq C_8$.

Applying the homomorphism $\lambda^{(4)} : \pi_* MU^{\wedge 4} \to R_*$, we get

$$\begin{aligned} x + \sum_{k>0} \frac{w^{2^n-1}}{\pi^k} x^{2^n} \\ = \left(x + \sum_{j>0} \frac{\zeta^{8/h} w^{2^j-1}}{\pi^j} x^{2^j}\right) \circ \left(x + \sum_{i>0} \lambda^{(4)}(r_i^H) x^{i+1}\right). \end{aligned} \tag{13.4.18}$$

For brevity, let $s_{H,i} = \lambda^{(4)}(r_i^H)$ and

$$s_H(x) = x + \sum_{i>0} s_{H,i} x^{i+1}, \tag{13.4.19}$$

so (13.4.18) reads

$$
\begin{aligned}
x+\sum_{k>0}\frac{w^{2^n-1}}{\pi^k}x^{2^n} &= \left(x+\sum_{j>0}\frac{\zeta^{8/h}w^{2^j-1}}{\pi^j}x^{2^j}\right)\circ s_H(x)\\
&= s_H(x)+\sum_{j>0}\frac{\zeta^{8/h}w^{2^j-1}}{\pi^j}s_H(x)^{2^j}.
\end{aligned}
\tag{13.4.20}
$$

Recall that 2 is a unit multiple of π^4. For each H, we can solve (13.4.20) directly for $s_{H,2^\ell-1}$ for various ℓ. Doing so gives

$$
\left.
\begin{aligned}
s_{C_2,1} &= \left(-\pi^3-4\pi^2-6\pi-4\right)w \;=\; \pi^3\cdot\mathrm{unit}\cdot w\\
s_{C_2,3} &= \left(-4\pi^3-5\pi^2+14\pi+26\right)w^3 \;=\; \pi^2\cdot\mathrm{unit}\cdot w^3\\
s_{C_2,7} &= \left(-6182\pi^3-21426\pi^2-22171\pi-1052\right)w^7\\
&= \pi\cdot\mathrm{unit}\cdot w^7\\
s_{C_2,15} &= \big(306347134\pi^3-3700320563\pi^2\\
&\qquad -15158766469\pi-16204677587\big)\,w^{15}\\
&= \mathrm{unit}\cdot w^{15}\\
s_{C_4,1} &= (-\pi-2)\,w \;=\; \pi\cdot\mathrm{unit}\cdot w\\
s_{C_4,3} &= \left(8\pi^3+26\pi^2+25\pi-1\right)w^3 \;=\; \mathrm{unit}\cdot w^3\\
s_{C_8,1} &= -w,
\end{aligned}
\right\}
\tag{13.4.21}
$$

where each unit is in A.

Hence the images under $\lambda^{(4)}$ of $r_{15}^{C_2}$, $r_3^{C_4}$ and $r_1^{C_8}$ are units as required. Thus we have shown that each factor of i_0^*D and hence i_0^*D itself maps to a unit in R_*, thereby proving the lemma. □

On the other hand, we see from (13.4.21) that the images of $r_1^{C_2}$, $r_3^{C_2}$, $r_7^{C_2}$, and $r_1^{C_4}$ under $\lambda^{(4)}$ are not units in R_*. For this reason, **smaller subscripts of $\overline{\mathfrak{d}}$ in the definition of D would not lead to a detection theorem**.

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Table of notations

3/28/20. These need to be alphabetized. The set of entries grouped together as "cofibrant generating sets" should be left intact under this sorting. The same goes for "categories of spectra."

Symbol	Location	Meaning
1_X	2.1.1	Identity morphism on an object X.
$1_{\mathcal{C}}$	2.1.12	Identity functor on a category $\mathcal{C}$.
α_A and $\alpha_{A,X,Y}$	2.6.6	Left addition functor and induced map of morphism sets.
ω_A and $\omega_{A,X,Y}$	2.6.6	Right addition functor and induced map of morphism sets.
$\mathcal{A}b$	2.2.30(i)	Category of abelian groups.
$\mathcal{C}(X,Y)$	2.1.1	Set of morphisms $X \to Y$ in a category $\mathcal{C}$.
$c_{X,Y,X}$	2.1.1	Composition pairing in a category $\mathcal{C}$.
$\underline{\mathcal{C}}(-,-)$ or $(-)^-$	2.6.33	Internal Hom functor..
$\mathcal{C}^{op}$	2.1.3	Opposite category of $\mathcal{C}$.
$\mathcal{C}^J$	2.1.15	Category of functors $J \to \mathcal{C}$.
$\mathcal{C}_1$	2.6.55	Arrow category of $\mathcal{C}$.
$\Delta : \mathcal{C} \to \mathcal{C}^J$	2.3C	Diagonal functor.
$\mathcal{C}^G = \mathcal{C}^{\mathcal{B}G}$	2.1.15	Category of G-objects in $\mathcal{C}$.
$[\mathcal{C},\mathcal{D}]$	2.1.15 and 3.2.18	Category of functors $\mathcal{C} \to \mathcal{D}$.
$\mathcal{V}ect_k$	2.2.5 a	Category of k-vector spaces.
$\mathcal{T}op$ and $\mathcal{T}$	2.1.48	Categories of topological spaces.
$\mathcal{T}op^G$ and $\mathcal{T}^G$	3.1.59	Categories of G-spaces.
$\mathcal{T}op_G$ and $\mathcal{T}_G$	3.1.59	Categories of G-spaces and all continuous maps.

Continued on next page

Symbol	Location	Meaning
$\mu : T^2 \Rightarrow T$	2.2.40	Natural transformation for monad.
$\mu : F(-) \otimes F(-) \Rightarrow F(- \oplus -)$	2.6.19	Natural transformation for lax monoidal functor.
$\iota : F(\mathbf{0}) \to \mathbf{1}$	2.6.19	Structure map for lax monoidal functor.
ι_2	4.6.2	Inclusion map for solid cylinder object in model category.
$e_G(V, W)$	8.9.23	Thom space embedding.
μ_H^G	(2.2.26) and (8.3.20)	Relative action map for $H \subseteq G$.
ψ_H^G	(2.2.26) and (8.3.20)	Relative coaction map for $H \subseteq G$.
μ_R and μ_L	2.2.32 and 3.1.69	Right and left actions of an endomorphism object on a morphism object.
$\mathbf{\Delta}$	3.4	Category of finite ordered sets.
$\mathcal{S}et_{\mathbf{\Delta}}$ and $\mathcal{S}et^{\mathbf{\Delta}}$	3.4.1	Category of simplicial or cosimplicial sets.
$\mathcal{C}_{\mathbf{\Delta}}$ and $\mathcal{C}^{\mathbf{\Delta}}$	3.4.1	Categories of simplicial and cosimplicial objects in $\mathcal{C}$.
$\mathcal{C}h_R$	4.2	Category of chain complexes of R-modules.
$\mathcal{CAT}$	2.1.14 and 2.7.2(ii)	Category of categories.
$\mathcal{C}at$	2.1.14 and 2.7.2(ii)	Category of small categories.
$\mathcal{CAT}_{ad}$	2.7.2(iv)	The 2-category of adjunctions.
$\mathcal{M}od$	2.7.2(v)	The 2-category of model categories.
$\mathcal{M}on\mathcal{CAT}$	2.7.2(vi)	The 2-category of monoidal categories.
$\mathcal{VCAT}$	3.1.20	The 2-category of $\mathcal{V}$-categories.
$\mathcal{VC}at$	3.1.20	The 2-category of small $\mathcal{V}$-categories.
$\mathcal{C} \otimes \mathcal{C}'$	3.1.28	The product of two $\mathcal{V}$-categories.
$\mathfrak{M}_G$	8.2.3	Category of Mackey functors on G.
$\mathscr{B}_G^+$ and $\mathscr{B}_G$	8.2.4	Lindner and Burnside categories.
よ^A	2.2.10	Yoneda functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathcal{S}et$.
よ	2.2.12	Yoneda embedding $\mathcal{C}^{op} \to [\mathcal{C}, \mathcal{S}et]$.

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Symbol	Location	Meaning
$\mathscr{P}(S)$, $\mathscr{P}_0(S)$ and $\mathscr{P}_1(S)$	2.3.53	Categories of subsets of a finite set S.
	Indexing categories for spectra	
$\mathscr{J}_K^{\mathbf{N}}$	7.2.4	Presymmetric spectra in 7.2.32 and 7.2.33.
$\mathscr{J}_K^{\mathbf{\Sigma}}$	7.2.4	Symmetric spectra in 7.2.33.
$\mathscr{J}_K^{\mathbf{O}}$	7.2.4	Orthogonal spectra in 7.2.33.
$\mathscr{J}_K^{\mathbf{U}}$ and $\mathscr{J}_{\mathbf{C}}$	7.2.4	Unitary and complex spectra in 7.2.33 and 12.1.4.
$\mathscr{J}_{\mathbf{R}}$	12.1.2	Real spectra in 12.1.4.
$\mathscr{J}_G$ and $\mathscr{J}_G^+$	8.9.24	Mandell-May categories for G-spectra.
	Categories of spectra.	
$\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, T)$	7.1.13	Hovey spectra.
$\mathcal{S}p^{\mathbf{N}}(\mathcal{M}, K)$	7.2.33	Presymmetric spectra $[\mathscr{J}_K^{\mathbf{N}}, \mathcal{M}]$.
$\mathcal{S}p^{\mathbf{\Sigma}}(\mathcal{M}, K)$	7.2.33	Symmetric spectra $[\mathscr{J}_K^{\mathbf{\Sigma}}, \mathcal{M}]$.
$\mathcal{S}p^{\mathbf{O}}(\mathcal{M}, K)$	7.2.33	Orthogonal spectra $[\mathscr{J}_K^{\mathbf{O}}, \mathcal{M}]$.
$\mathcal{S}p^{\mathbf{U}}(\mathcal{M}, K)$	7.2.33	Unitary spectra $[\mathscr{J}_K^{\mathbf{U}}, \mathcal{M}]$.
$\mathcal{S}p^{\mathbf{F}}(\mathcal{M}, K)$	7.2.33	Extraorthogonal spectra $[\mathscr{J}_K^{\mathbf{F}}, \mathcal{M}]$.
$\mathcal{S}p^G$	7.2.33	Extraorthogonal spectra $[\mathscr{J}_K^{\mathbf{F}}, \mathcal{M}]$.
$\mathcal{S}p^G$	9.0.2	Orthogonal G-spectra $[\mathscr{J}_G, \mathcal{T}]$.
$\mathcal{S}p^G_{naive}$	9.3.2	Naive G-spectra .
$\mathcal{S}p^{\mathcal{B}_{G/H}G}$	9.3.18	Spectra with groupoid action.
$\mathcal{S}p^{\mathcal{B}_T G}$	9.3.24	Spectra over a G-set.
$\mathcal{S}p^G_{\mathrm{fl}}$	(9.6.7)	Subcategory of flat objects in $\mathcal{S}p^G$.
$\mathbf{Comm}^G$	9.7.1	$\mathbf{Comm}\,\mathcal{S}p^G$.
$\mathbf{Alg}^G$	9.7.1	$\mathbf{Assoc}\,\mathcal{S}p^G$.
$\mathbf{Alg}^G$	9.7.1	$\mathbf{Assoc}\,\mathcal{S}p^G$.
$\mathcal{S}p^{\mathcal{B}_T G}$	10.1	T-diagrams of orthogonal spectra.
$\mathcal{S}p$	12.1.4	Complex spectra.
	12.1.4	Real spectra.
$\mathbf{Comm}\,\mathcal{C}$	2.6.58	Commutative monoids in $\mathcal{C}$.
$T(-)$	2.6.66	Free associative algebra functor.
$\mathrm{Sym}\,(-)$	2.6.63	Free commutative algebra functor.
Φ	2.7.10	Generic pseudofunctor.
$\mathcal{B}_T G$ and $\mathcal{B}G$	2.9.1	Groupoid associated with a G-set T.
N	2.3.63	Sequential colimit category.
N^{op}	2.3.63	Sequential limit category

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Symbol	Location	Meaning
$\mathrm{Ob}\,\mathcal{C}$	2.1.1	Collection of objects in a category $\mathcal{C}$.
$\mathrm{Arr}\,\mathcal{C}$	2.1.1	Collection of arrows in a category $\mathcal{C}$.
$\mathrm{Dom}\,f$	2.1.1	Domain or source or a morphism f.
$\mathrm{Cod}\,f$	2.1.1	Codomain or target or a morphism f.
$\dashv$	2.2D	Adjunction symbol, the turnstile.
$\epsilon : F \Rightarrow 1_{\mathcal{C}}$	2.2.8	Augmentation for endofunctor.
$\eta : 1_{\mathcal{C}} \Rightarrow F$	2.2.8	Coaugmentation endofunctor.
$\eta : 1_{\mathcal{C}} \Rightarrow GF$	2.2.20	Unit of the adjunction $F \dashv G$.
$\epsilon : FG \Rightarrow 1_{\mathcal{D}}$	2.2.20	Counit of the adjunction $F \dashv G$.
θ_X	2.2.1	Morphism $F(X) \to G(X)$ associated with a natural transformation θ.
$\theta_{(Y,Z)}$	2.2.6 and 3.1.40	$\theta_{(Y,Z)} : \mathcal{S}et(Y,Z) \times F(Y) \to F(Z)$ composition at Y in $\mathcal{S}et$.
$\widehat{\theta}_{(Y,Z)}$	3.1.40	Adjoint to $\theta_{(Y,Z)}$ in (3.1.42).
$\kappa_{(W,X)}$	2.2.6 and 3.1.40	$\kappa_{(W,X)} : G(X) \times \mathcal{S}et(W,X) \to G(W)$ precomposition at X in $\mathcal{S}et$.
$\widehat{\kappa}_{(W,X)}$	3.1.40	Adjoint to $\kappa_{(W,X)}$ in (3.1.43).
$\epsilon^F_{A,B}$	(3.1.38)	Enriched composition map.
$\eta^F_{A,B}$	(3.1.38)	Enriched cocomposition map.
$\int^J H(j,j)$ and $\int_J H(j,j)$	2.4.5	End and coend of a functor $H : J^{op} \times J \to \mathcal{C}$.
$\int \Phi$	2.8.8	Grothendieck construction for a pseudofunctor Φ.
$(Lan_K F, \eta)$	2.5 and 3.2.35	Left Kan extension of a functor $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathcal{D}$.
$(Ran_K F, \epsilon)$	2.5 and 3.2.35	Right Kan extension of a functor $F : \mathcal{C} \to \mathcal{E}$ along $K : \mathcal{C} \to \mathcal{D}$.
$\partial_G X$	2.3.57	Boundary of a functor $G : \mathscr{P}(S) \to \mathcal{D}$ for a cocomplete category $\mathcal{D}$.
$f_A = \square_{\alpha \in A} f_\alpha$	2.9.29	Indexed corner map.
$\square$	2.6.1 and 2.6.12	Generic binary operation or pushout corner map.
$- \square -$	8.2.14	Box product of Mackey functors.
$a_{-,-,-}$	2.6.1 and 2.6.42	Associator isomorphism.
λ_{-}	2.6.1	Left unitor isomorphism.
		Continued on next page

Symbol	Location	Meaning
ρ_{-}	2.6.1	Right unitor isomorphism.
$\tau_{-,-}$	2.6.1	Twist isomorphism.
$\diamond$	2.6.12	Pullback corner map.
μ	2.6.19	Natural transformation for lax monoidal functor.
ϕ	2.2D	Adjunction isomorphism.
ϕ_ℓ and ϕ_r	2.6.26	two variable adjunction isomorphisms.
Hom_ℓ and Hom_r	2.6.26	two variable adjunction functors.
$\otimes^A$	2.9	Iterated monoidal product functor.
fil_*	2.9.34	Target exponent filtration.
$\mathcal{C}_\diamond(i,p)$	2.3.14	Lifting test map.
d_i	3.4	Face map.
s_i	3.4	Degeneracy map.
$X_\bullet$ and $X^\bullet$	3.4.1	Simplicial or cosimplicial set.
$cs_*(-)$	3.4.1	Constant simplicial object.
$cc_*(-)$	3.4.1	Constant cosimplicial object.
$\Delta^\bullet$	3.4.2	Cosimplicial standard simplex.
Δ^n_i	3.4.2	ith face of the standard n-simplex.
Λ^n_i	3.4.2	ith horn of the standard n-simplex.
$\lvert - \rvert$	3.4.3	Geometric realization.
$\lvert - \rvert$	2.1.58	Degree of object.
$\mathcal{S}ing(-)$	3.4.7	Singular functor.
$N(-)$	3.4.12	Nerve of a small category.
$B-$	3.4.12	Classifying space.
M_{-} and M'_{-}	3.5.1	(Reduced) mapping sylinder.
$E\mathscr{F}$	8.6.15	Universal $\mathscr{F}$-space.
$\mathscr{F}$	8.6.10	Family of subgroups.
$\mathcal{P}$	8.6.11	Family of proper subgroups.
Q	4.1.20	Functorial cofibrant replacement.
R	4.1.20	Functorial fibrant replacement.
$Cyl(-)$	4.3.7	Cylinder object in a model category.
$Path(-)$	4.3.7	Path object in a model category.
$\overset{\ell}{\simeq}$ $\overset{r}{\simeq}$ and $\simeq$	4.3.6	Left, right and 2 sided homotopy.
L_{-}	4.3.15	Localization of category.
$\mathrm{Ho}\,-$	4.3.16	Quillen homotopy category.
$\mathbf{n}$	7.2.4	$\{1,2,\ldots,n\}$.
$[n]$	3.4A	$\{0,1,2,\ldots,n\}$.

Continued on next page

Symbol	Location	Meaning
$\mathbf{N}_0$	7.2.4	Set of natural numbers.
$\mathbf{L}F$	4.4.7	Total left derived functor of F
$\mathbf{R}F$	4.4.7	Total right derived functor of F
LF	4.4.5	Left derived functor of F
RF	4.4.5	Right derived functor of F
$Sol(-)$	4.6.2	Solid cylinder object
$Surf(-)$	4.6.2	Surface object
$\pi_1^\ell(A, B;\ f_0, f_1)$ and $\pi_1^r(A, B;\ f_0, f_1)$	(4.6.4)	Higher homotopy classes of left and right homotopies between f_0 and f_1.
$\pi_1(A, B)$	4.6.16	Quillen's fundamental group.
$\pi^\ell(A, B)$, $\pi^r(A, B)$ and $\pi(A, B)$	4.3.11	Sets of homotopy classes of maps $A \to B$.
$\Sigma(-)$	4.6.17	Suspension object.
$\Omega(-)$	4.6.17	Loop object.
Cofibrant generating sets for various model categories.		
$\mathcal{I}$	2.6.15 and (5.2.10)	$\{i_n : S^{n-1} \to D^n : n \geqslant 0\}$ in $\mathcal{T}op$.
$\mathcal{J}$	(5.2.11)	$\{j_n : (\{0\} \to [0,1]) \times I^n : n \geqslant 0\}$.
$\mathcal{I}_+$ and $\mathcal{J}_+$	(5.2.13)	Pointed analogs of $\mathcal{I}$ and $\mathcal{J}$.
$\mathcal{I}_{\mathscr{K}}$ and $\mathcal{J}_{\mathscr{K}}$	5.6.38	Induced model structure on $[\mathscr{J}, \mathcal{M}]$.
$\mathcal{I}_T$ and $\mathcal{J}_T^{proj}$	7.1.33	Hovey projective model structure.
$\mathcal{J}_T = \mathcal{J}_T^{\mathrm{proj}} \cup (\mathcal{I} \square \mathcal{S})$	(7.3.28)	Hovey stable model structure.
$\mathcal{I}_G^e$ and $\mathcal{J}_G^e$	8.6.2	Naive or underlying model structure.
$\mathcal{I}_G^{\mathcal{A}\ell\ell}$ and $\mathcal{J}_G^{\mathcal{A}\ell\ell}$, or $\mathcal{I}_G$ and $\mathcal{J}_G$	8.6.2	Genuine or Bredon model structure.
$\mathcal{I}_G^{\mathscr{F}}$ and $\mathcal{J}_G^{\mathscr{F}}$	8.6.13	Model structure in $\mathcal{T}^G$ for a subgroup family $\mathscr{F}$.
$\mathcal{I}_L^{\mathbf{F}}$, $\mathcal{I}_L^{\mathbf{F},+}$, $\mathcal{J}_L^{\mathbf{F}}$, $\mathcal{J}_L^{\mathbf{F},+}$, $\mathcal{K}_L^{\mathbf{F}}$ and $\mathcal{K}_L^{\mathbf{F},+}$	(7.4.40)	Maps for smashable spectra.
$\mathcal{I}^G$, $\mathcal{I}^{G,+}$, $\mathcal{J}^G$, $\mathcal{J}^{G,+}$, $\mathcal{K}^G$ and $\mathcal{K}^{G,+}$	(9.2.9)	Generating maps for G-spectra.
$\tilde{\mathcal{I}}^G$, $\tilde{\mathcal{I}}^{G,+}$, $\tilde{\mathcal{J}}^G$, $\tilde{\mathcal{J}}^{G,+}$, $\tilde{\mathcal{K}}^G$ and $\tilde{\mathcal{K}}^{G,+}$	(9.2.12)	Equifibrant maps for G-spectra.
$\between_-$	4.6.22	Pinch map.
$\ker f$	4.1.27	Kernel of f in a model category.
$\operatorname{coker} f$	4.1.27	Cokernel of f in a model category.
CX	4.1.28	Reduced cone object or space.
PX	4.1.28	Reduced path object or space.

Continued on next page

Symbol	Location	Meaning
F_f	4.1.28	Homotopy fiber of f.
C_f	4.1.28	Homotopy cofiber of f.
$i \boxslash p$	2.3.10	Lifting properties of i and p.
$\mathcal{X}$-inj $= \mathcal{X}^{\boxslash}$	2.3.10	Morphisms with right lifting property.
$\mathcal{X}$-proj $= {}^{\boxslash}\mathcal{X}$	2.3.10	Morphisms with left lifting property.
$\mathrm{cofib}(\mathcal{X}) = {}^{\boxslash}(\mathcal{X}^{\boxslash})$	2.3.10	$\mathcal{X}$-cofibrations.
$\mathrm{fib}(\mathcal{X}) = ({}^{\boxslash}\mathcal{X})^{\boxslash}$	2.3.10	$\mathcal{X}$-fibrations.
$\mathcal{C}_{\diamond}(i,p)$	2.3.14	Pullback corner map of 2.3.9.
$\diamond(i,p)$	2.3.14	Pullback set of 2.3.5.
$(\mathcal{L},\mathcal{R})$	2.3.19	Weak factorization system.
$\mathcal{L}$	2.3.19	Left class of factorization system.
$\mathcal{R}$	2.3.19	Right class of factorization system.
$\mathcal{R}eg(\mathcal{I})$	4.8.13	Regular class generated by $\mathcal{I}$.
$\mathcal{S}at(\mathcal{I})$	4.8.13	Saturated class generated by $\mathcal{I}$.
$\flat$	5.1.22	Precofibration as in 5.1.15.
$\mathscr{W}$	4.1.1	Class of weak equivalence.
$\mathscr{C}$	4.1.1	Class of cofibrations.
$\mathscr{F}$	4.1.1	Class of fibrations.
A^{disc}	2.1.7	Discrete category for a set A.
$\lvert J\rvert$	2.1.7	Discrete category for category J .
$\mathbf{2}$	2.1.6	Walking arrow or interval category.
α^*	2.2.54	Precomposition functor induced by α.
$\alpha_!$	2.2.54	Left adjoint of α^*.
α_*	2.2.54	Right adjoint of α^*.
$\ltimes$	2.1.49	Left half smash product.
$\rtimes$	2.1.49	Right half smash product.
$\ltimes_H$	5.5.33	Left half smash product over H.
${}_H\rtimes$	5.5.33	Right half smash product over H.
π_α	5.6.5	Homotopy invariant.
End_A	2.2.32	Endomorphism category of A.
$\mathrm{Ev}: X \times \mathcal{S}et(X,Y) \to Y$	2.1.16(v)	Evaluation map.
$\mathrm{Ev}_A : [\mathcal{C},\mathcal{E}] \to \mathcal{E}$	2.2.35	Evaluation functor.
$F^A : \mathcal{E} \to [\mathcal{C},\mathcal{E}]$	2.2.35	Coevaluation functor.
Σ^{-1}	5.7.3	Desuspension functor.
Ω^{-1}	5.7.3	Deloopingfunctor.
$\Theta^k = \Omega^k\Omega^{-k}$	5.7.3	Endofunctor.

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Symbol	Location	Meaning
Θ^∞	5.7.3	$\operatorname{hocolim}_k \Theta^k$.
holim and hocolim	5.8.1	Homotopy limit and colimit.
$B(-,-,-)$	5.8.11	Two sided bar construction.
$L_E(-)$	6.1A and 6.1B	Bousfield localization with respect to homology theory E.
$\mathscr{E}$	6.2.1	Class of morphisms.
$L_{\mathscr{E}}$	6.2.5	Left Bousfield localization with respect to morphism class $\mathscr{E}$.
$P^n(-)$	6.2.13	nth Postnikov section.
Υ	6.2.15	Homotopy idempotent functor.
$\mathcal{M}_{enla}$, $\mathcal{M}_{conf}$ and $\mathcal{M}_{enco}$	6.2C	Altered model structures on $\mathcal{M}$.
$\Lambda(\mathcal{S})$	6.3.8	Full set of $\mathcal{S}$-horns.
τ and $\tau^\perp$	6.3.12	Localizing subcategory and complement.
P^τ and P_τ	6.3.17	Dror Farjoun localization functor and fiber.
ϵ_n^X	7.1.1	Structure map of spectrum.
η_n^X	7.1.6	Costructure map of spectrum.
T	7.1.1	Hovey's generalization of suspension.
Ω_T	7.1.1	Hovey's generalization of looping.
Ev_m	7.1.20	mth evaluation functor.
T^{-m}	7.1.20	mth tensored Yoneda functor.
R_m	7.1.24	Right adjoint of Ev_m.
$T^{-m}S$	7.1.30	mth Yoneda spectrum.
$\pi_{\alpha,n}(-)$	(7.2.21)	$\pi_0\mathcal{M}(A_\alpha \wedge L^{\wedge n}, -)$.
$R\mathbf{F}$	7.2.26	Indexing group of $\mathscr{J}^{\mathbf{O}}$-algebra.
S^V	7.2.29	Structured sphere $\mathscr{J}_L^{\mathbf{F}}(0,V)$.
$\Omega^V(-)$	7.2.29	Structured loop functor $\mathcal{M}(S^W, -)$.
$\pi_V(-)$	7.2.30	Structured homotopy group of space.
$\pi_V(-)$	7.3.14	Stable homotopy group of spectrum.
$i_{\mathbf{N}}^{\mathbf{\Sigma}}$, $i_{\mathbf{\Sigma}}^{\mathbf{O}}$, $i_{\mathbf{O}}^{\mathbf{U}}$ and $i_{\mathbf{O}}^{\mathbf{F}}$	7.2.34	Inclusion functors.
$\overline{\epsilon}_{V,W}^X$ and $\widetilde{\epsilon}_{V,W}^X$	7.2.38	Structure maps.
$\xi_{V,W}$	(7.2.63)	Stabilizing map.
s_V	(7.2.68)	Stabilizing map.
s_m^M	7.3.1	Stabilizing map for Hovey spectra.
$\sigma^k : \Sigma^k\Sigma^{-k} \Rightarrow 1_{\mathcal{S}p}$	5.7.3(i)	Natural transformation.
$\theta^k : 1_{\mathcal{S}p} \Rightarrow \Theta^k$	5.7.3(i)	Natural transformation.

Continued on next page

Symbol	Location	Meaning
$\widehat{\theta} : 1_{\mathcal{M}} \Rightarrow \Theta_T^k$	7.3.50	Natural transformation.
$\widehat{\sigma} : T^k T^{-k} \Rightarrow 1_{\mathcal{M}}$	7.3.50	Natural transformation.
$\overline{T}$ and $\overline{T}^{-1}$	7.3.52	Reduced Hovey suspension.
$\overline{\Omega}_T$ and $\overline{\Omega}_T^{-1}$	7.3.52	Reduced Hovey looping.
$\overline{\sigma}^k$ and $\widehat{\sigma}^k$	7.3.52(ii)	Natural transformations.
θ^k and $\widehat{\theta}^k$	7.3.52(ii)	Natural transformations.
$\widehat{\mathcal{S}}$ and $\mathcal{S}$	7.4.8	Sets of stabilizing maps.
C_{p^n}	8.0.1	Cyclic group of order p^n.
Res_K^H	8.2.3 and 9.4.14	Mackey functor restriction map.
Tr_K^H	8.2.3 and 9.4.14	Mackey functor transfer map.
$\underline{M}$	8.2.3	Mackey functor.
$\underline{M}$	8.2.8	Fixed point Mackey functor.
$\widehat{\underline{M}}$	8.2.8	Fixed quotient Mackey functor.
$\underline{\mathbf{Z}}$	8.2.6	Constant Mackey functor.
$\underline{A}$	8.2.7	Burnside Mackey functor.
$\underline{A}_S$	8.2.7	Free Mackey functor.
$\underline{I}$	8.2.7	Augmentation ideal Mackey functor.
$\uparrow_H^G \underline{M}$	8.2.9	Induced G-Mackey functor.
$\downarrow_H^G \underline{M}$	8.2.9	Restricted H-Mackey functor.
$\underline{M}_S$	8.2.9	Precomposite Mackey functor.
$\underline{\mathbf{Z}}\{S\}$	8.2.11	Permutation Mackey functor.
X_G or X/G	8.3.8	Orbit space.
X_{hG}	8.3.8	Homtopy orbit space.
X^G	8.3.8	Fixed point space.
X^{hG}	8.3.8	Homotopy fixed point space.
i_H^G	2.2.25	Forgetful or restriction functor.
N_H^G	8.3.23	Norm functor.
S^V	8.3.26	Representation sphere of V.
$S(V)$	8.3.26	Unit sphere of V.
$D(V)$	8.3.26	Unit disk of V.
ρ_G and $\overline{\rho}_G$	8.3.27	Regular representations.
$\mathrm{Ind}_H^G V$	2.5.8(iv)	Induced representation.
$C_*(X; \underline{M})$	8.5.14	Bredon chain comp-lex.
$H_*(X; \underline{M})$	8.5.14	Bredon homology.
$\mathscr{O}_G$	8.6.22	Orbit category of G.
$\mathscr{O}_{\mathscr{F}}$	8.6.22	Orbit category of a family $\mathscr{F}$.

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Symbol	Location	Meaning
$\Phi : \mathcal{T}^G \to \mathscr{O}_G\mathcal{T}$	8.6.27	Fixed point functor.
$\mathcal{F}_G$	8.1.1	Category of finite G-sets.
$\Psi : \mathscr{O}_G\mathcal{T} \to \mathcal{T}^G$	8.8.1	Elmendorf's coalescence functor.
$K(\underline{M}, n)$	8.8.4	Eilenberg-Mac Lane space for $\underline{M}$.
$\Sigma^V(\text{-})$	8.9.3	Twisted suspension.
$\Omega^V(\text{-})$	8.9.3	Twisted loop space.
$\pi_*^u(-)$	8.9.6	Underlying homotopy of G-space .
$S^V = \bigwedge_{t\in T} S^{V_t}$	8.9.10(i)	Sphere for representation of G-set.
$\lvert(T, V)\rvert$	8.9.10(iv)	Degree of (T, V).
V_t	8.9.10	Image of t under representation V.
$O(V, W)$	8.9.15	Space of orthogonal embeddings.
$\mathscr{I}_G$ and $\mathscr{I}_G^+$	8.9.19	Stiefel categories.
$f(V)^\perp$	8.9.22	Orthogonal complement.
$\pi_V^G(-)$	9.1.1	Equivariant stable homotopy group.
$BO^{\oplus K}$	(9.1.42)	Variant of BO.
$H\underline{M}$	9.1.47	Eilenberg-Mac Lane spectrum for $\underline{M}$.
$\mathcal{S}_{naive}$	(9.3.3)	Naive stabilizing maps.
$\mathcal{S}_{genuine}$	(9.3.13)	Genuine stabilizing maps.
Θ^∞_{naive}	(9.3.4)	Naive fibrant replacement.
$\Theta^\infty_{genuine}$	(9.3.14)	Genuine fibrant replacement.
$\pi^G_{V,naive}X$	(9.3.5)	Naive stable homotopy group.
$p_*^\wedge$ and $p_*^\vee$	5.5.34	Functors induced by covering.
$\underline{\pi}_V$	(9.4.9)	$RO(G)$-graded homotopy group.
$\tilde{u}_H^G(Y, X)$	9.4.10	Equivariant homeomorphism.
$\underline{\pi}_{H,V}$	9.4.17	$\underline{RO}(G)$-graded homotopy group .
$\underline{r}_K^{H,V}$	9.4.19	Group action restriction map.
$\underline{t}_K^{H,V}$	9.4.19	Group action transfer map.
$H_\cap$ and $H_\cup$	9.4.20	Intersection and union of subgroups.
$\mathcal{SW}^G$	9.5.1	Adams category.
$\pi^{\mathrm{st}}\mathcal{S}p^G$	9.5.4	Homotopical approximation to $\mathcal{S}p^G$.
$R_H^0(X)$	(9.7.9)	$[X, i_H^G R]^H$ for R in $\mathbf{Comm}\,\mathcal{S}p^G$.
a_V	9.9.7(i)	Element in $\underline{\pi}_{-V}H\underline{\mathbf{Z}}(G/G)$.
e_V	9.9.7(ii)	Element in $\underline{\pi}_{V-\lvert V\rvert}H\underline{\mathbf{Z}}(G/G_V)$.
u_V	9.9.7(iii)	Element in $\underline{\pi}_{\lvert V\rvert-V}H\underline{\mathbf{Z}}(G/G)$.
FX and F^GX	9.10.2	Schwede fixed point spectrum.
R^GX	9.10.2	Fibrant fixed point spectrum.
$\tilde{E}\mathcal{P}$	9.11.2	Isotropy separation space.
$\Phi^G(-)$	9.11.7	Geometric fixed point functor.

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Symbol	Location	Meaning
$\Phi^G_M(-)$ and $\tilde{\Phi}^G_M(-)$	9.11.38, 9.11.49	Monoidal geometric fixed points.
$\mathrm{Sym}^n(-)$	10.5.1	nth symmetric power of spectrum.
$\mathrm{Sym}^T_\Lambda X$	10.5.6	Indexed symmetric power.
$\mathrm{Sym}_\Lambda f_T$	10.5.7	Indexed symmetric corner map.
$- \wedge A \wedge -$	(9.7.2)	Smash product over A.
$S^{-0}[S^V] = S^{-0}[\overline{x}]$	10.10.2	Twisted monoid ring on S^V.
$S^{-0}[G \cdot S^V]$	(10.10.5)	Normed up twisted monoid ring.
$\overset{\mathbf{L}}{\wedge}$	10.4.7	Derived smash product.
$\widehat{S}(m, H)$	11.1.1	Slice sphere.
$\widehat{S}_c(m, H)$	(11.1.2)	Cofibrant slice sphere.
$\mathcal{S}p^G_{>n}$ and $\mathcal{S}p^G_{\leqslant n}$	11.1.11	Slice categories.
$X > n$ and $X < n$	11.1.11	Slice connectivity and coconnectivity.
τ^G_n	11.1.22	Geometric connectivity category.
$\lfloor - \rfloor$ and $\lceil - \rceil$	11.1D	Floor and ceiling of a number.
P^n- and $P_{n+1}-$	11.1.35	Slice section and slice cover.
$\widetilde{P}^n$ and $\widetilde{P}_{n+1}$	11.1.38	Putative slice section and cover.
P^n_n-	11.1.43	n-slice.
$E_2^{s,t}X$	(11.2.7)	Slice E_2-term.
$\widehat{W}$	11.3.19	Wedge of dimensional slice spheres.
P^n_c-	11.4.1	nth cofibrant slice section.
P^n_{alg}	(11.4.12)	nth algebraic slice section.
$V_{\mathbf{C}}$	12.1	Complexification of V.
V_{ρ_2}	12.1	$V \otimes \rho_2$.
$\mathcal{MU}_{\mathbf{R}}$	12.2.1	Real spectrum for MU.
$MU_{\mathbf{R}}$	12.2.3	Real bordism spectrum.
$MU^{((G))}$	(12.2.6)	Normed real bordism spectrum.
$\overline{m}_i \in \underline{H}^{C_2}_{i\rho_2} MU_{\mathbf{R}}$	(12.2.20)	Coefficient of C_2 logarithm.
$m_i \in H_{2i}MU$	12.3A	Coefficient of ordinary logarithm.
$\overline{a}_{i,j} \in \pi^{C_2}_\star MU_{\mathbf{R}}$	(12.2.21)	Coefficient of C_2 formal group law.
$\overline{b}_i \in \pi^{C_2}_{i\rho_2} E \wedge MU_{\mathbf{R}}$	(12.2.23)	Generator of $E_* MU_{\mathbf{R}}$.
MU	(12.0.1)	Complex cobordism spectrum.
MO	12.2C	Unoriented cobordism spectrum.
r_i	12.2.42	Generator of $\pi^u_{2i} MU^{((G))}$.
$\bar{r}_i$	12.3.6	Generator of $\pi^{C_2}_{i\rho_2} MU^{((G))}$.
$f_i \in \pi^G_i MU^{((G))}$	12.3.11	$a^i_{\overline{\rho}_G} N\overline{r}_i$.
M_d	10.10.9(ii)	Monoidal ideal.
A_G	12.4A	Associative algebra refining $MU^{((G))}$.

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Symbol	Location	Meaning
$R_G(\infty)$	(12.4.7)	$MU^{((G))} \underset{A_G}{\wedge} S^{-0}$.
$R_G(k)$	(12.4.17)	Spectrum related to $MU^{((G))}$.
$b \in \pi_2 \Phi^G H\underline{\mathbf{Z}}_{(2)}$	12.4.21	$u_{2\sigma} a_\sigma^{-2}$.
y_n	(12.4.28)	Preimage of $N\overline{r}_{c_n}$.
c_n	(12.4.26)	$2^n - 1$.
$M_{(n)}$	(12.4.26)	$MU^{((G))}/(G \cdot \overline{r}_{c_n})$.
$w_{i,j}$	12.3.3	Coefficient in QH_{2i}.
$\overline{\mathfrak{d}}_n = \overline{\mathfrak{d}}_n^G$	13.3.12	Element in $\pi_\star^G MU^{((G))}$.
A	13.4.4	Ring of integers in field.
$\|\| - \|\|$	(13.4.16)	Detection Theorem valuation.
$s_{H,i} \in R_*$	(13.4.19)	$\lambda^{(4)}(r_i^H)$.

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