Quillen's work on formal group laws and complex cobordism theory

Conference honoring the legacy of Daniel Quillen
October 6-8, 2012

Doug Ravenel
University of Rochester

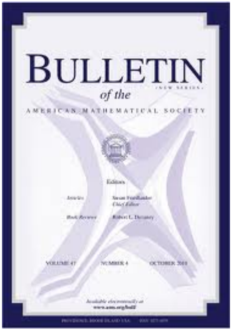
The six author paper

THE mod-p LOWER CENTRAL SERIES AND THE ADAMS SPECTRAL SEQUENCE†

A. K. BOUSFIELD, E. B. CURTIS, D. M. KAN, D. G. QUILLEN, D. L. RECTOR and J. W. SCHLESINGER

(*Received* 18 *March* 1966)

Quillen's cryptic and insightful masterpiece: Six pages that rocked our world



ON THE FORMAL GROUP LAWS OF UNORIENTED AND COMPLEX COBORDISM THEORY

BY DANIEL QUILLEN[1]

Communicated by Frank Peterson, May 16, 1969

Six pages that rocked our world (continued)

ON THE FORMAL GROUP LAWS OF UNORIENTED AND COMPLEX COBORDISM THEORY

BY DANIEL QUILLEN[1]

Communicated by Frank Peterson, May 16, 1969

Six pages that rocked our world (continued)

ON THE FORMAL GROUP LAWS OF UNORIENTED AND COMPLEX COBORDISM THEORY

BY DANIEL QUILLEN[1]

Communicated by Frank Peterson, May 16, 1969

Table of contents

Six pages that rocked our world (continued)

Quillen begins by defining formal group laws

Six pages that rocked our world (continued)

Quillen begins by defining formal group laws just as we define them today.

Six pages that rocked our world (continued)

Quillen begins by defining formal group laws just as we define them today.

Definition

A formal group law over a ring R is a power series $F(X, Y) \in R[[X, Y]]$ with

$$\begin{aligned} F(X,0) = F(0,X) &= X \\ F(Y,X) &= F(X,Y) \\ F(X,F(Y,Z)) &= F(F(X,Y),Z). \end{aligned}$$

Six pages that rocked our world (continued)

Quillen begins by defining formal group laws just as we define them today.

Definition

A formal group law over a ring R is a power series $F(X, Y) \in R[[X, Y]]$ with

$$\begin{aligned} F(X,0) = F(0,X) &= X \\ F(Y,X) &= F(X,Y) \\ F(X,F(Y,Z)) &= F(F(X,Y),Z). \end{aligned}$$

Examples:

- $X + Y$, the additive formal group law.

Six pages that rocked our world (continued)

Quillen begins by defining formal group laws just as we define them today.

Definition

A formal group law over a ring R is a power series $F(X, Y) \in R[[X, Y]]$ with

$$\begin{aligned} F(X,0) = F(0,X) &= X \\ F(Y,X) &= F(X,Y) \\ F(X,F(Y,Z)) &= F(F(X,Y),Z). \end{aligned}$$

Examples:

- $X + Y$, the additive formal group law.
- $X + Y + XY$, the multiplicative formal group law.

Six pages that rocked our world (continued)

Quillen begins by defining formal group laws just as we define them today.

Definition

A formal group law over a ring R is a power series $F(X, Y) \in R[[X, Y]]$ with

$$\begin{aligned} F(X,0) = F(0,X) &= X \\ F(Y,X) &= F(X,Y) \\ F(X,F(Y,Z)) &= F(F(X,Y),Z). \end{aligned}$$

Examples:

- $X + Y$, the additive formal group law.
- $X + Y + XY$, the multiplicative formal group law.
- Euler's addition formula for a certain elliptic integral,

$$\frac{X\sqrt{1-Y^4} + Y\sqrt{1-X^4}}{1-X^2Y^2} \in \mathbf{Z}[1/2][[X,Y]].$$

Six pages that rocked our world (continued)

He then defines the formal group law of complex cobordism in terms of the first Chern class of the tensor product of two complex line bundles,

Six pages that rocked our world (continued)

He then defines the formal group law of complex cobordism in terms of the first Chern class of the tensor product of two complex line bundles, just as we define it today.

Definition

Let L_1 and L_2 be complex line bundles over a space X with Conner-Floyd Chern classes

$$c_1(L_1),\ c_1(L_2) \in MU^2(X) = \Omega^2(X)$$

Six pages that rocked our world (continued)

He then defines the formal group law of complex cobordism in terms of the first Chern class of the tensor product of two complex line bundles, just as we define it today.

Definition

Let L_1 and L_2 be complex line bundles over a space X with Conner-Floyd Chern classes

$$c_1(L_1),\ c_1(L_2) \in MU^2(X) = \Omega^2(X)$$

Then the formal group law over the complex cobordism ring is

$$F^{\Omega}(c_1(L_1), c_1(L_2)) = c_1(L_1 \otimes L_2).$$

Six pages that rocked our world (continued)

Then he states his first theorem.

Six pages that rocked our world (continued)

Then he states his first theorem.

THEOREM 1. *Let E be a complex vector bundle of dimension n, let $f\colon PE' \to X$ be the associated projective bundle of lines in the dual E' of E, and let $\mathcal{O}(1)$ be the canonical quotient line bundle on PE'. Then the Gysin homomorphism $f_*\colon \Omega^q(PE') \to \Omega^{q-2n+2}(X)$ is given by the formula*

$$f_*(u(\xi)) = \operatorname{res} \frac{u(Z)\omega(Z)}{\prod_{j=1}^{n} F^{\Omega}(Z, I\lambda_j)} . \tag{4}$$

Here $u(Z) \in \Omega(X)[Z]$, $\xi = c_1^{\Omega}(\mathcal{O}(1))$, ω and I are the invariant differential form and inverse respectively for the group law F^{Ω}, and the λ_j are the dummy variables of which $c_q^{\Omega}(E)$ is the qth-elementary symmetric function.

The hardest part of this theorem is to define the residue; we specialize to dimension one an unpublished definition of Cartier, which has also been used in a related form by Tate [7].

Six pages that rocked our world (continued)

Then he states his first theorem.

THEOREM 1. *Let E be a complex vector bundle of dimension n, let $f: PE' \to X$ be the associated projective bundle of lines in the dual E' of E, and let $\mathcal{O}(1)$ be the canonical quotient line bundle on PE'. Then the Gysin homomorphism $f_*: \Omega^q(PE') \to \Omega^{q-2n+2}(X)$ is given by the formula*

$$f_*(u(\xi)) = \operatorname{res} \frac{u(Z)\omega(Z)}{\prod_{j=1}^{n} F^{\Omega}(Z, I\lambda_j)}. \tag{4}$$

Here $u(Z) \in \Omega(X)[Z]$, $\xi = c_1^{\Omega}(\mathcal{O}(1))$, ω and I are the invariant differential form and inverse respectively for the group law F^{Ω}, and the λ_j are the dummy variables of which $c_q^{\Omega}(E)$ is the qth-elementary symmetric function.

The hardest part of this theorem is to define the residue; we specialize to dimension one an unpublished definition of Cartier, which has also been used in a related form by Tate [7].

This mysterious statement leads to a new determination of the logarithm of the formal group law.

Six pages that rocked our world (continued)

Then he states his first theorem.

THEOREM 1. *Let E be a complex vector bundle of dimension n, let $f\colon PE' \to X$ be the associated projective bundle of lines in the dual E' of E, and let $\mathcal{O}(1)$ be the canonical quotient line bundle on PE'. Then the Gysin homomorphism $f_*\colon \Omega^q(PE') \to \Omega^{q-2n+2}(X)$ is given by the formula*

$$f_*(u(\xi)) = \operatorname{res} \frac{u(Z)\omega(Z)}{\prod_{j=1}^{n} F^{\Omega}(Z, I\lambda_j)} . \tag{4}$$

Here $u(Z) \in \Omega(X)[Z]$, $\xi = c_1^{\Omega}(\mathcal{O}(1))$, ω and I are the invariant differential form and inverse respectively for the group law F^{Ω}, and the λ_j are the dummy variables of which $c_q^{\Omega}(E)$ is the qth-elementary symmetric function.

The hardest part of this theorem is to define the residue; we specialize to dimension one an unpublished definition of Cartier, which has also been used in a related form by Tate [7].

This mysterious statement leads to a new determination of the logarithm of the formal group law.

Applying the theorem to the map $f\colon CP^n \to pt$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n, the cobordism class of CP^n in $\Omega^{-2n}(pt)$. From (2) we obtain the

COROLLARY (MYSHENKO [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$l(X) = \sum_{n \geq 0} P_n \frac{X^{n+1}}{n+1} . \tag{5}$$

Six pages that rocked our world (continued)

Applying the theorem to the map $f\colon CP^n \to pt$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n, the cobordism class of CP^n in $\Omega^{-2n}(pt)$. From (2) we obtain the

COROLLARY (MYSHENKO [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$l(X) = \sum_{n \geq 0} P_n \frac{X^{n+1}}{n+1}. \tag{5}$$

Six pages that rocked our world (continued)

Applying the theorem to the map $f\colon CP^n \to pt$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n, the cobordism class of CP^n in $\Omega^{-2n}(pt)$. From (2) we obtain the

COROLLARY (MYSHENKO [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$l(X) = \sum_{n \geq 0} P_n \frac{X^{n+1}}{n+1}. \tag{5}$$

The logarithm $\ell(X)$ of a formal group law is a power series defining an isomorphism (after tensoring with the rationals) with the additive formal group law,

Six pages that rocked our world (continued)

Applying the theorem to the map $f: CP^n \rightarrow pt$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n, the cobordism class of CP^n in $\Omega^{-2n}(pt)$. From (2) we obtain the

COROLLARY (MYSHENKO [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$l(X) = \sum_{n \geq 0} P_n \frac{X^{n+1}}{n+1}. \tag{5}$$

The logarithm $\ell(X)$ of a formal group law is a power series defining an isomorphism (after tensoring with the rationals) with the additive formal group law, so we have

$$\ell(F(X, Y)) = \ell(X) + \ell(Y).$$

Six pages that rocked our world (continued)

Applying the theorem to the map $f\colon CP^n \to pt$, we find that the coefficient of $X^n dX$ in $\omega(X)$ is P_n, the cobordism class of CP^n in $\Omega^{-2n}(pt)$. From (2) we obtain the

COROLLARY (MYSHENKO [6]). *The logarithm of the formal group law of complex cobordism theory is*

$$l(X) = \sum_{n \geq 0} P_n \frac{X^{n+1}}{n+1}. \tag{5}$$

The logarithm $\ell(X)$ of a formal group law is a power series defining an isomorphism (after tensoring with the rationals) with the additive formal group law, so we have

$$\ell(F(X, Y)) = \ell(X) + \ell(Y).$$

The above Corollary identifies the logarithm for the formal group law associated with complex cobordism theory.

Six pages that rocked our world (continued)

Then he shows that the formal groups law for complex cobordism is universal.

Six pages that rocked our world (continued)

Then he shows that the formal groups law for complex cobordism is universal.

THEOREM 2. *The group law F^{Ω} over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \to R$ carrying F^{Ω} to F.*

Six pages that rocked our world (continued)

Then he shows that the formal groups law for complex cobordism is universal.

THEOREM 2. *The group law F^{Ω} over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \to R$ carrying F^{Ω} to F.*

His proof uses two previously known facts:

Six pages that rocked our world (continued)

Then he shows that the formal groups law for complex cobordism is universal.

THEOREM 2. *The group law F^{Ω} over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \to R$ carrying F^{Ω} to F.*

His proof uses two previously known facts:

- 

  Michel Lazard had determined the ring L over which the universal formal group law F^L is defined.

Six pages that rocked our world (continued)

Then he shows that the formal groups law for complex cobordism is universal.

THEOREM 2. *The group law F^{Ω} over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \to R$ carrying F^{Ω} to F.*

His proof uses two previously known facts:

-  Michel Lazard had determined the ring L over which the universal formal group law F^L is defined. The previous corollary implies that the map $L \to \Omega^{ev}(pt)$ carrying F^L to F^{Ω} is a rational isomorphism.

Six pages that rocked our world (continued)

Then he shows that the formal groups law for complex cobordism is universal.

THEOREM 2. *The group law F^{Ω} over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \to R$ carrying F^{Ω} to F.*

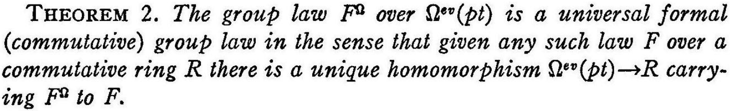
His proof uses two previously known facts:

- 

  Michel Lazard had determined the ring L over which the universal formal group law F^L is defined. The previous corollary implies that the map $L \to \Omega^{ev}(pt)$ carrying F^L to F^{Ω} is a rational isomorphism. The target was known to be torsion free,

Six pages that rocked our world (continued)

Then he shows that the formal groups law for complex cobordism is universal.



THEOREM 2. *The group law F^Ω over $\Omega^{ev}(pt)$ is a universal formal (commutative) group law in the sense that given any such law F over a commutative ring R there is a unique homomorphism $\Omega^{ev}(pt) \to R$ carrying F^Ω to F.*

His proof uses two previously known facts:

- 

  Michel Lazard had determined the ring L over which the universal formal group law F^L is defined. The previous corollary implies that the map $L \to \Omega^{ev}(pt)$ carrying F^L to F^Ω is a rational isomorphism. The target was known to be torsion free, so it suffices to show the map is onto.

Six pages that rocked our world (continued)

John Milnor

Sergei Novikov

- Milnor and Novikov had determined the structure of the ring MU_*.

Six pages that rocked our world (continued)

John Milnor

Sergei Novikov

- Milnor and Novikov had determined the structure of the ring MU_*. It is torsion free and generated by as a ring by the cobordism classes of the Milnor hypersurfaces,

$$H^{m,n} \subset \mathbf{C}P^m \times \mathbf{C}P^n.$$

Six pages that rocked our world (continued)

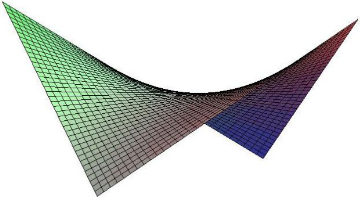
John Milnor

Sergei Novikov

- Milnor and Novikov had determined the structure of the ring MU_*. It is torsion free and generated by as a ring by the cobordism classes of the Milnor hypersurfaces,

$$H^{m,n} \subset \mathbf{C}P^m \times \mathbf{C}P^n.$$

$H^{m,n}$ is the zero locus of a bilinear function on $\mathbf{C}P^m \times \mathbf{C}P^n$.



Six pages that rocked our world (continued)

These imply that it suffices to show that the cobordism classes of the $H^{m,n}$ can be defined in terms of the formal group law.

Six pages that rocked our world (continued)

These imply that it suffices to show that the cobordism classes of the $H^{m,n}$ can be defined in terms of the formal group law.

Denote the latter as usual by

$$F(X,Y) = \sum_{i,j\geq 0} a_{i,j} X^i Y^j \qquad \text{where } a_{i,j} \in MU_{2(i+j-1)}$$

Six pages that rocked our world (continued)

These imply that it suffices to show that the cobordism classes of the $H^{m,n}$ can be defined in terms of the formal group law.

Denote the latter as usual by

$$\begin{aligned} F(X,Y) &= \sum_{i,j\geq 0} a_{i,j}X^iY^j \qquad \text{where } a_{i,j}\in MU_{2(i+j-1)} \\ \text{with} \qquad P(X) &= \sum_{n\geq 0} P_nX^n \\ &= \ell'(X) \qquad \text{where } \ell(X) \text{ is the logarithm} \end{aligned}$$

Six pages that rocked our world (continued)

These imply that it suffices to show that the cobordism classes of the $H^{m,n}$ can be defined in terms of the formal group law.

Denote the latter as usual by

$$\begin{aligned} F(X,Y) &= \sum_{i,j\geq 0} a_{i,j}X^iY^j \qquad \text{where } a_{i,j}\in MU_{2(i+j-1)} \\ \text{with} \qquad P(X) &= \sum_{n\geq 0} P_nX^n \\ &= \ell'(X) \qquad \text{where } \ell(X) \text{ is the logarithm} \\ \text{and} \qquad H(X,Y) &= \sum_{m,n\geq 0} [H^{m,n}]X^mY^n. \end{aligned}$$

Six pages that rocked our world (continued)

These imply that it suffices to show that the cobordism classes of the $H^{m,n}$ can be defined in terms of the formal group law.

Denote the latter as usual by

$$\begin{aligned} F(X,Y) &= \sum_{i,j\geq 0} a_{i,j}X^iY^j \qquad \text{where } a_{i,j}\in MU_{2(i+j-1)} \\ \text{with} \qquad P(X) &= \sum_{n\geq 0} P_nX^n \\ &= \ell'(X) \qquad \text{where } \ell(X) \text{ is the logarithm} \\ \text{and} \qquad H(X,Y) &= \sum_{m,n\geq 0} [H^{m,n}]X^mY^n. \end{aligned}$$

These are related by the formula

$$H(X,Y) = P(X)P(Y)F(X,Y),$$

Six pages that rocked our world (continued)

These imply that it suffices to show that the cobordism classes of the $H^{m,n}$ can be defined in terms of the formal group law.

Denote the latter as usual by

$$\begin{aligned} F(X,Y) &= \sum_{i,j\geq 0} a_{i,j}X^iY^j \qquad \text{where } a_{i,j}\in MU_{2(i+j-1)} \\ \text{with} \qquad P(X) &= \sum_{n\geq 0} P_nX^n \\ &= \ell'(X) \qquad \text{where } \ell(X) \text{ is the logarithm} \\ \text{and} \qquad H(X,Y) &= \sum_{m,n\geq 0}[H^{m,n}]X^mY^n. \end{aligned}$$

These are related by the formula

$$H(X,Y) = P(X)P(Y)F(X,Y),$$

so the cobordism class each Milnor hypersurface is defined in terms of the formal group law.

Six pages that rocked our world (continued)

These imply that it suffices to show that the cobordism classes of the $H^{m,n}$ can be defined in terms of the formal group law.

Denote the latter as usual by

$$\begin{aligned} F(X,Y) &= \sum_{i,j\geq 0} a_{i,j}X^iY^j \qquad \text{where } a_{i,j}\in MU_{2(i+j-1)} \\ \text{with} \qquad P(X) &= \sum_{n\geq 0} P_nX^n \\ &= \ell'(X) \qquad \text{where } \ell(X) \text{ is the logarithm} \\ \text{and} \qquad H(X,Y) &= \sum_{m,n\geq 0} [H^{m,n}]X^mY^n. \end{aligned}$$

These are related by the formula

$$H(X,Y) = P(X)P(Y)F(X,Y),$$

so the cobordism class each Milnor hypersurface is defined in terms of the formal group law. This is Quillen's proof of Theorem 2.

Six pages that rocked our world (continued)

He proved a similar result about unoriented cobordism.

Six pages that rocked our world (continued)

He proved a similar result about unoriented cobordism. Here there is a formal group law defined in terms of Stiefel-Whitney classes instead of Chern classes.

Six pages that rocked our world (continued)

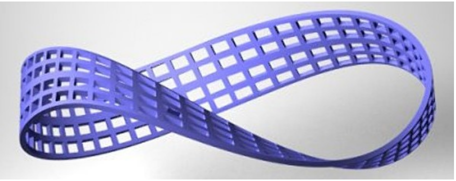
He proved a similar result about unoriented cobordism. Here there is a formal group law defined in terms of Stiefel-Whitney classes instead of Chern classes.

As in the complex case, the cobordism ring is generated by real analogs of the Milnor hypersurfaces.

Six pages that rocked our world (continued)

He proved a similar result about unoriented cobordism. Here there is a formal group law defined in terms of Stiefel-Whitney classes instead of Chern classes.

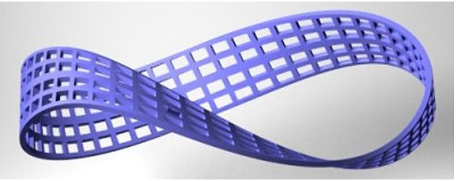
As in the complex case, the cobordism ring is generated by real analogs of the Milnor hypersurfaces. Unlike the complex case, the tensor product square of any real line bundle is trivial.



Six pages that rocked our world (continued)

He proved a similar result about unoriented cobordism. Here there is a formal group law defined in terms of Stiefel-Whitney classes instead of Chern classes.

As in the complex case, the cobordism ring is generated by real analogs of the Milnor hypersurfaces. Unlike the complex case, the tensor product square of any real line bundle is trivial.



This forces the formal group law to have characteristic 2 and satisfy the relation

$$F(X,X) = 0.$$

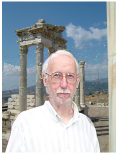
ON THE FORMAL GROUP LAWS OF UNORIENTED AND COMPLEX COBORDISM THEORY

BY DANIEL QUILLEN[1]

Communicated by Frank Peterson, May 16, 1969

Table of contents

The Brown-Peterson theorem

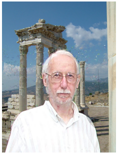


A SPECTRUM WHOSE Z_p COHOMOLOGY IS THE ALGEBRA OF REDUCED p^{th} POWERS

EDGAR H. BROWN, JR. and FRANKLIN P. PETERSON†

(*Received* 3 *September* 1965)

The Brown-Peterson theorem



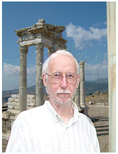
A SPECTRUM WHOSE Z_p COHOMOLOGY IS THE ALGEBRA OF REDUCED p^{th} POWERS

EDGAR H. BROWN, JR. and FRANKLIN P. PETERSON†

(*Received* 3 *September* 1965)

In 1966 Brown and Peterson showed that after localization at a prime p, Ω (or MU) splits into a wedge of smaller spectra now known as BP and denoted by Quillen as ΩT.

The Brown-Peterson theorem



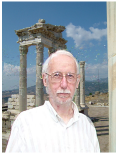
A SPECTRUM WHOSE Z_p COHOMOLOGY IS THE ALGEBRA OF REDUCED p^{th} POWERS

EDGAR H. BROWN, JR. and FRANKLIN P. PETERSON†

(*Received* 3 *September* 1965)

In 1966 Brown and Peterson showed that after localization at a prime p, Ω (or MU) splits into a wedge of smaller spectra now known as BP and denoted by Quillen as ΩT. This splitting is suggested by a corresponding decomposition of H^*MU as a module over the mod p Steenrod algebra.

The Brown-Peterson theorem



A SPECTRUM WHOSE Z_p COHOMOLOGY IS THE ALGEBRA OF REDUCED p^{th} POWERS

EDGAR H. BROWN, JR. and FRANKLIN P. PETERSON†

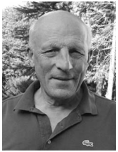
(*Received* 3 *September* 1965)

In 1966 Brown and Peterson showed that after localization at a prime p, Ω (or MU) splits into a wedge of smaller spectra now known as BP and denoted by Quillen as ΩT. This splitting is suggested by a corresponding decomposition of H^*MU as a module over the mod p Steenrod algebra. Their theorem was a diamond in the rough.

p-typicality

Quillen made it sparkle!

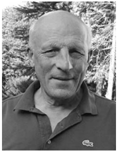
Quillen made it sparkle!



By using some algebra developed by Pierre Cartier, he gave a much cleaner form of the splitting,

p-typicality

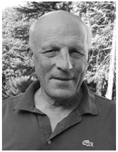
Quillen made it sparkle!



By using some algebra developed by Pierre Cartier, he gave a much cleaner form of the splitting, thereby showing that *BP* is a ring spectrum.

p-typicality

Quillen made it sparkle!

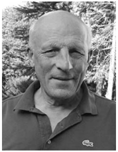


By using some algebra developed by Pierre Cartier, he gave a much cleaner form of the splitting, thereby showing that BP is a ring spectrum.

A formal group law F over a ring R defines a group structure on the set of curves over R, meaning power series with trivial constant term.

p-typicality

Quillen made it sparkle!



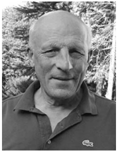
By using some algebra developed by Pierre Cartier, he gave a much cleaner form of the splitting, thereby showing that BP is a ring spectrum.

A formal group law F over a ring R defines a group structure on the set of curves over R, meaning power series with trivial constant term. Given a curve $f(X)$ and a positive integer n, let

$$(F_n f)(X) = \sum_{i=1}^{n}{}^{F} f(\zeta_i X^{1/n}),$$

p-typicality

Quillen made it sparkle!



By using some algebra developed by Pierre Cartier, he gave a much cleaner form of the splitting, thereby showing that BP is a ring spectrum.

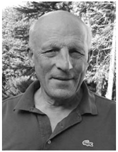
A formal group law F over a ring R defines a group structure on the set of curves over R, meaning power series with trivial constant term. Given a curve $f(X)$ and a positive integer n, let

$$(F_n f)(X) = \sum_{i=1}^{n}{}^{F} f(\zeta_i X^{1/n}),$$

where the ζ_i are the nth roots of unity,

p-typicality

Quillen made it sparkle!



By using some algebra developed by Pierre Cartier, he gave a much cleaner form of the splitting, thereby showing that BP is a ring spectrum.

A formal group law F over a ring R defines a group structure on the set of curves over R, meaning power series with trivial constant term. Given a curve $f(X)$ and a positive integer n, let

$$(F_n f)(X) = \sum_{i=1}^{n}{}^F f(\zeta_i X^{1/n}),$$

where the ζ_i are the nth roots of unity, and the addition on the right is defined by the formal group law F.

p-typicality (continued)

$$(F_n f)(X) = \sum_{i=1}^{n}{}^{F} f(\zeta_i X^{1/n}).$$

p-typicality (continued)

$$(F_n f)(X) = \sum_{i=1}^{n}{}^F f(\zeta_i X^{1/n}).$$

Note that if we replace the formal sum by an ordinary one and

$$f(X) = \sum_{j>0} f_j X^j,$$

p-typicality (continued)

$$(F_n f)(X) = \sum_{i=1}^{n}{}^F f(\zeta_i X^{1/n}).$$

Note that if we replace the formal sum by an ordinary one and

$$f(X) = \sum_{j>0} f_j X^j, \qquad \text{then} \qquad (F_n f)(X) = n \sum_{j>0} f_{nj} X^j.$$

p-typicality (continued)

$$(F_n f)(X) = \sum_{i=1}^{n}{}^{F} f(\zeta_i X^{1/n}).$$

Note that if we replace the formal sum by an ordinary one and

$$f(X) = \sum_{j>0} f_j X^j, \qquad \text{then} \qquad (F_n f)(X) = n \sum_{j>0} f_{nj} X^j.$$

The curve f is said to be p-typical if $F_q f = 0$ for each prime $q \neq p$.

p-typicality (continued)

$$(F_n f)(X) = \sum_{i=1}^{n}{}^F f(\zeta_i X^{1/n}).$$

Note that if we replace the formal sum by an ordinary one and

$$f(X) = \sum_{j>0} f_j X^j, \qquad \text{then} \qquad (F_n f)(X) = n \sum_{j>0} f_{nj} X^j.$$

The curve f is said to be p-typical if $F_q f = 0$ for each prime $q \neq p$. In the case of ordinary summation this means that f has the form

$$f(X) = \sum_{k \geq 0} f_{(k)} X^{p^k}.$$

p-typicality (continued)

$$(F_n f)(X) = \sum_{i=1}^{n}{}^F f(\zeta_i X^{1/n}).$$

Note that if we replace the formal sum by an ordinary one and

$$f(X) = \sum_{j>0} f_j X^j, \qquad \text{then} \qquad (F_n f)(X) = n \sum_{j>0} f_{nj} X^j.$$

The curve f is said to be p-typical if $F_q f = 0$ for each prime $q \neq p$. In the case of ordinary summation this means that f has the form

$$f(X) = \sum_{k \geq 0} f_{(k)} X^{p^k}.$$

The formal group law itself is said to be p-typical if the curve X is p-typical with respect to it.

p-typicality (continued)

$$(F_n f)(X) = \sum_{i=1}^{n}{}^F f(\zeta_i X^{1/n}).$$

Note that if we replace the formal sum by an ordinary one and

$$f(X) = \sum_{j>0} f_j X^j, \qquad \text{then} \qquad (F_n f)(X) = n \sum_{j>0} f_{nj} X^j.$$

The curve f is said to be p-typical if $F_q f = 0$ for each prime $q \neq p$. In the case of ordinary summation this means that f has the form

$$f(X) = \sum_{k \geq 0} f_{(k)} X^{p^k}.$$

The formal group law itself is said to be p-typical if the curve X is p-typical with respect to it. Over a torsion free ring, this is equivalent to the logarithm having the form above.

p-typicality (continued)

Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra,

p-typicality (continued)

Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one.

p-typicality (continued)

Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one.

Quillen used this to define an idempotent map $\hat{\xi}$ on $\Omega_{(p)} = MU_{(p)}$ whose telescope is $\Omega T = BP$.

p-typicality (continued)

Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one.

Quillen used this to define an idempotent map $\hat{\xi}$ on $\Omega_{(p)} = MU_{(p)}$ whose telescope is $\Omega T = BP$. This construction is much more convenient than that of Brown and Peterson.

p-typicality (continued)

Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one.

Quillen used this to define an idempotent map $\hat{\xi}$ on $\Omega_{(p)} = MU_{(p)}$ whose telescope is $\Omega T = BP$. This construction is much more convenient than that of Brown and Peterson.

This process changes the logarithm from

$$\sum_{n \geq 0} \frac{[\mathbf{C}P^n] X^{n+1}}{n+1}$$

p-typicality (continued)

Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one.

Quillen used this to define an idempotent map $\hat{\xi}$ on $\Omega_{(p)} = MU_{(p)}$ whose telescope is $\Omega T = BP$. This construction is much more convenient than that of Brown and Peterson.

This process changes the logarithm from

$$\sum_{n\geq 0} \frac{[\mathbf{C}P^n]X^{n+1}}{n+1} \quad \text{to} \quad \sum_{k\geq 0} \frac{[\mathbf{C}P^{p^k-1}]X^{p^k}}{p^k}.$$

p-typicality (continued)

Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one.

Quillen used this to define an idempotent map $\hat{\xi}$ on $\Omega_{(p)} = MU_{(p)}$ whose telescope is $\Omega T = BP$. This construction is much more convenient than that of Brown and Peterson.

This process changes the logarithm from

$$\sum_{n\geq 0} \frac{[\mathbf{C}P^n]X^{n+1}}{n+1} \quad \text{to} \quad \sum_{k\geq 0} \frac{[\mathbf{C}P^{p^k-1}]X^{p^k}}{p^k}.$$

This new logarithm is much simpler than the old one.

p-typicality (continued)

Cartier showed that when R is a $\mathbf{Z}_{(p)}$-algebra, there is a canonical coordinate change that converts any formal group law into a p-typical one.

Quillen used this to define an idempotent map $\hat{\xi}$ on $\Omega_{(p)} = MU_{(p)}$ whose telescope is $\Omega T = BP$. This construction is much more convenient than that of Brown and Peterson.

This process changes the logarithm from

$$\sum_{n\geq 0} \frac{[\mathbf{C}P^n]X^{n+1}}{n+1} \quad \text{to} \quad \sum_{k\geq 0} \frac{[\mathbf{C}P^{p^k-1}]X^{p^k}}{p^k}.$$

This new logarithm is much simpler than the old one.

An analogous construction converts the unoriented cobordism spectrum MO to the mod 2 Eilenberg-Mac Lane spectrum $H\mathbf{Z}/2$.

Operations in BP-theory

In order to get the most use out of a cohomology theory E^* represented by a spectrum E,

Operations in BP-theory

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

Operations in BP-theory

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

In favorable cases one can set up an E-based Adams spectral sequence and wonder about its E_2-term.

Operations in BP-theory

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

In favorable cases one can set up an E-based Adams spectral sequence and wonder about its E_2-term. This is usually some Ext group defined in terms of the algebra of operations A^E.

Operations in BP-theory

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

In favorable cases one can set up an E-based Adams spectral sequence and wonder about its E_2-term. This is usually some Ext group defined in terms of the algebra of operations A^E.

Finding it explicitly is a daunting task.

Operations in BP-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = H\mathbf{Z}/2$, the algebra A^E is the mod 2 Steenrod algebra, which has proven to be a fertile source of theorems in algebraic topology.

Operations in BP-theory (continued)

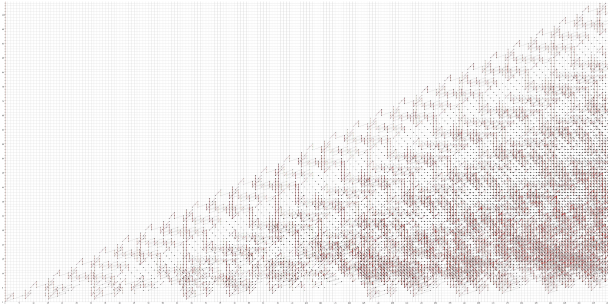
In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = H\mathbf{Z}/2$, the algebra A^E is the mod 2 Steenrod algebra, which has proven to be a fertile source of theorems in algebraic topology. The corresponding Ext group has been extensively studied.

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = H\mathbf{Z}/2$, the algebra A^E is the mod 2 Steenrod algebra, which has proven to be a fertile source of theorems in algebraic topology. The corresponding Ext group has been extensively studied.



210 dimensions worth of Adams Ext groups, as computed by Christian Nassau in 1999.

Operations in BP-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = MU$, A^E was determined in 1967 by Novikov.

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = MU$, A^E was determined in 1967 by Novikov. He found a small but extremely rich portion of its Ext group,

Operations in *BP*-theory (continued)

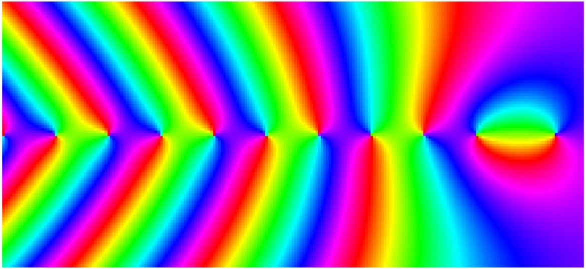
In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = MU$, A^E was determined in 1967 by Novikov. He found a small but extremely rich portion of its Ext group, rich enough to include the denominator of the value of the Riemann zeta function at each negative odd integer!

Operations in BP-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = MU$, A^E was determined in 1967 by Novikov. He found a small but extremely rich portion of its Ext group, rich enough to include the denominator of the value of the Riemann zeta function at each negative odd integer!



The color coded argument of $\zeta(x+iy)$ for $-20 \leq x \leq 2$ and $-5 \leq y \leq 5$.

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = BP$, A^E was determined by Quillen.

Operations in BP-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = BP$, A^E was determined by Quillen. The details are too technical for this talk.

Operations in BP-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = BP$, A^E was determined by Quillen. The details are too technical for this talk. He gave a precise description in less than two pages,

Operations in BP-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = BP$, A^E was determined by Quillen. The details are too technical for this talk. He gave a precise description in less than two pages, with little indication of proof.

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = BP$, A^E was determined by Quillen. The details are too technical for this talk. He gave a precise description in less than two pages, with little indication of proof.

  The resulting Ext group is the same as Novikov's localized at the prime p.

Operations in BP-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = BP$, A^E was determined by Quillen. The details are too technical for this talk. He gave a precise description in less than two pages, with little indication of proof.

  The resulting Ext group is the same as Novikov's localized at the prime p. The underlying formulas are easier to use than Novikov's,

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = BP$, A^E was determined by Quillen. The details are too technical for this talk. He gave a precise description in less than two pages, with little indication of proof.

  The resulting Ext group is the same as Novikov's localized at the prime p. The underlying formulas are easier to use than Novikov's, once one knows how to use them.

Operations in *BP*-theory (continued)

In order to get the most use out of a cohomology theory E^* represented by a spectrum E, one needs to understand the graded algebra A^E of maps from E to itself.

- For $E = BP$, A^E was determined by Quillen. The details are too technical for this talk. He gave a precise description in less than two pages, with little indication of proof.

  The resulting Ext group is the same as Novikov's localized at the prime p. The underlying formulas are easier to use than Novikov's, once one knows how to use them.

  It took the rest of us about 5 years to figure that out.

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement.

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Part II. QUILLEN'S WORK ON FORMAL GROUPS AND COMPLEX COBORDISM

J.F. Adams
1930-1989

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Part II. QUILLEN'S WORK ON FORMAL GROUPS AND COMPLEX COBORDISM

J.F. Adams
1930-1989

Adams explained all the proofs with great care.

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Part II. QUILLEN'S WORK ON FORMAL GROUPS AND COMPLEX COBORDISM

J.F. Adams
1930-1989

Adams explained all the proofs with great care. His book became the definitive reference for Quillen's results.

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Part II. QUILLEN'S WORK ON FORMAL GROUPS AND COMPLEX COBORDISM

J.F. Adams
1930-1989

Adams explained all the proofs with great care. His book became the definitive reference for Quillen's results.

He also introduced a very helpful but counterintuitive point of view.

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Part II. QUILLEN'S WORK ON FORMAL GROUPS AND COMPLEX COBORDISM

J.F. Adams 1930-1989

Adams explained all the proofs with great care. His book became the definitive reference for Quillen's results.

He also introduced a very helpful but counterintuitive point of view. Instead of studying the algebra A^{BP},

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Part II. QUILLEN'S WORK ON FORMAL GROUPS AND COMPLEX COBORDISM

J.F. Adams
1930-1989

Adams explained all the proofs with great care. His book became the definitive reference for Quillen's results.

He also introduced a very helpful but counterintuitive point of view. Instead of studying the algebra A^{BP}, one should compute in terms of its suitably defined linear dual.

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Part II. QUILLEN'S WORK ON FORMAL GROUPS

AND COMPLEX COBORDISM

J.F. Adams
1930-1989

Adams explained all the proofs with great care. His book became the definitive reference for Quillen's results.

He also introduced a very helpful but counterintuitive point of view. Instead of studying the algebra A^{BP}, one should compute in terms of its suitably defined linear dual.

> *Quillen's formal variables t_i are crying out to be located in $BP_*(BP)$.*

Complex cobordism theory after Quillen

Quillen never wrote a detailed account of his 6 page 1969 Bulletin announcement. Frank Adams did it for him.

Part II. QUILLEN'S WORK ON FORMAL GROUPS

AND COMPLEX COBORDISM

J.F. Adams
1930-1989

Adams explained all the proofs with great care. His book became the definitive reference for Quillen's results.

He also introduced a very helpful but counterintuitive point of view. Instead of studying the algebra A^{BP}, one should compute in terms of its suitably defined linear dual.

> *Quillen's formal variables t_i are crying out to be located in $BP_*(BP)$.*

This proved to be a huge technical simplification.

Complex cobordism theory after Quillen (continued)

Quillen's work was a bridge connecting algebraic topology with algebraic geometry and number theory.

Complex cobordism theory after Quillen (continued)

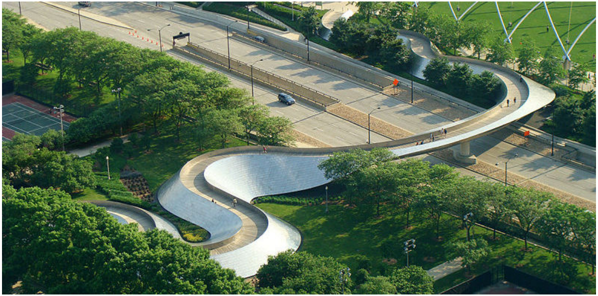
Quillen's work was a bridge connecting algebraic topology with algebraic geometry and number theory.

Homotopy theorists have been expanding that bridge ever since.

Complex cobordism theory after Quillen (continued)

Quillen's work was a bridge connecting algebraic topology with algebraic geometry and number theory.

Homotopy theorists have been expanding that bridge ever since.



The BP Bridge in Chicago, where
Adams lectured on Quillen's work in 1971.

Expanding the bridge: the work of Jack Morava

Expanding the bridge: the work of Jack Morava

In the early 1970s Morava applied deeper results (due mostly to Lazard) from the theory of formal group laws to algebraic topology.

Expanding the bridge: the work of Jack Morava

In the early 1970s Morava applied deeper results (due mostly to Lazard) from the theory of formal group laws to algebraic topology. These included

- A classification of formal group laws over the algebraic closure of the field F_p.

Expanding the bridge: the work of Jack Morava

In the early 1970s Morava applied deeper results (due mostly to Lazard) from the theory of formal group laws to algebraic topology. These included

- A classification of formal group laws over the algebraic closure of the field F_p. There is a complete isomorphism invariant called the height,

Expanding the bridge: the work of Jack Morava

In the early 1970s Morava applied deeper results (due mostly to Lazard) from the theory of formal group laws to algebraic topology. These included

- A classification of formal group laws over the algebraic closure of the field F_p. There is a complete isomorphism invariant called the height, which can be any positive integer or infinity.

Expanding the bridge: the work of Jack Morava

In the early 1970s Morava applied deeper results (due mostly to Lazard) from the theory of formal group laws to algebraic topology. These included

- A classification of formal group laws over the algebraic closure of the field F_p. There is a complete isomorphism invariant called the height, which can be any positive integer or infinity.
- The automorphism group of a height n formal group law is a certain p-adic Lie group.

Expanding the bridge: the work of Jack Morava

In the early 1970s Morava applied deeper results (due mostly to Lazard) from the theory of formal group laws to algebraic topology. These included

- A classification of formal group laws over the algebraic closure of the field F_p. There is a complete isomorphism invariant called the height, which can be any positive integer or infinity.
- The automorphism group of a height n formal group law is a certain p-adic Lie group. It is now known as the Morava stabilizer group S_n.

Expanding the bridge: the work of Jack Morava

In the early 1970s Morava applied deeper results (due mostly to Lazard) from the theory of formal group laws to algebraic topology. These included

- A classification of formal group laws over the algebraic closure of the field F_p. There is a complete isomorphism invariant called the height, which can be any positive integer or infinity.
- The automorphism group of a height n formal group law is a certain p-adic Lie group. It is now known as the Morava stabilizer group S_n.
- He defined a cohomology theory associated with height n formal group laws now known as Morava K-theory $K(n)$.

The work of Jack Morava (continued)

He also studied the affine variety *Spec*(MU_*) and defined an action on it by a group of power series substitutions.

The work of Jack Morava (continued)

He also studied the affine variety *Spec*(MU_*) and defined an action on it by a group of power series substitutions. It is now known as the moduli stack of formal group laws.

The work of Jack Morava (continued)

He also studied the affine variety $Spec(MU_*)$ and defined an action on it by a group of power series substitutions. It is now known as the moduli stack of formal group laws.

After passage to characteristic p, the orbits under this action are isomorphism classes of formal group laws as classified by Lazard.

The work of Jack Morava (continued)

He also studied the affine variety $Spec(MU_*)$ and defined an action on it by a group of power series substitutions. It is now known as the moduli stack of formal group laws.

After passage to characteristic p, the orbits under this action are isomorphism classes of formal group laws as classified by Lazard.

The Zariski closures of these orbits form a nested sequence of affine subspaces of the affine variety.

The work of Jack Morava (continued)

He also studied the affine variety $Spec(MU_*)$ and defined an action on it by a group of power series substitutions. It is now known as the moduli stack of formal group laws.

After passage to characteristic p, the orbits under this action are isomorphism classes of formal group laws as classified by Lazard.

The Zariski closures of these orbits form a nested sequence of affine subspaces of the affine variety.

The isotropy group of the height n orbit is the height n automorphism group S_n,

The work of Jack Morava (continued)

He also studied the affine variety $Spec(MU_*)$ and defined an action on it by a group of power series substitutions. It is now known as the moduli stack of formal group laws.

After passage to characteristic p, the orbits under this action are isomorphism classes of formal group laws as classified by Lazard.

The Zariski closures of these orbits form a nested sequence of affine subspaces of the affine variety.

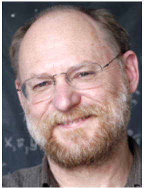
The isotropy group of the height n orbit is the height n automorphism group S_n, hence the name stabilizer group.

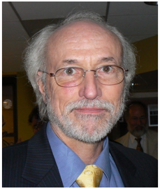
Chromatic homotopy theory

Morava's insights led to the formulation of the chromatic point of view in stable homotopy theory.

Chromatic homotopy theory

Morava's insights led to the formulation of the chromatic point of view in stable homotopy theory. In the late 1970s Haynes Miller, Steve Wilson and I showed that Morava's stratification leads to a nice filtration of the Adams-Novikov E_2-term.

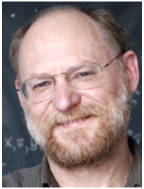


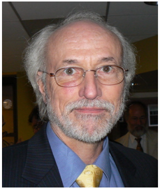


Chromatic homotopy theory

Morava's insights led to the formulation of the chromatic point of view in stable homotopy theory. In the late 1970s Haynes Miller, Steve Wilson and I showed that Morava's stratification leads to a nice filtration of the Adams-Novikov E_2-term.

In the early 80s we learned that the stable homotopy category itself possesses a filtration similar to the one found by Morava in $Spec(MU_*)$.

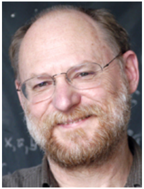


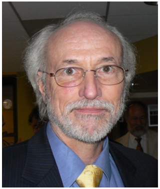


Chromatic homotopy theory

Morava's insights led to the formulation of the chromatic point of view in stable homotopy theory. In the late 1970s Haynes Miller, Steve Wilson and I showed that Morava's stratification leads to a nice filtration of the Adams-Novikov E_2-term.

In the early 80s we learned that the stable homotopy category itself possesses a filtration similar to the one found by Morava in $Spec(MU_*)$. A key technical tool in defining it is Bousfield localization.

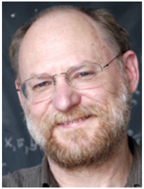


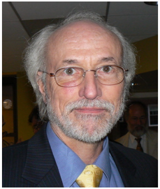


Pete Bousfield

Chromatic homotopy theory

Morava's insights led to the formulation of the chromatic point of view in stable homotopy theory. In the late 1970s Haynes Miller, Steve Wilson and I showed that Morava's stratification leads to a nice filtration of the Adams-Novikov E_2-term.





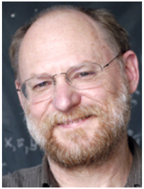
In the early 80s we learned that the stable homotopy category itself possesses a filtration similar to the one found by Morava in $Spec(MU_*)$. A key technical tool in defining it is Bousfield localization.

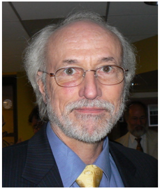
Pete Bousfield

As in the algebraic case, for each positive integer n,

Chromatic homotopy theory

Morava's insights led to the formulation of the chromatic point of view in stable homotopy theory. In the late 1970s Haynes Miller, Steve Wilson and I showed that Morava's stratification leads to a nice filtration of the Adams-Novikov E_2-term.





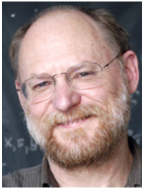
In the early 80s we learned that the stable homotopy category itself possesses a filtration similar to the one found by Morava in $Spec(MU_*)$. A key technical tool in defining it is Bousfield localization.

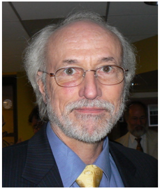
Pete Bousfield

As in the algebraic case, for each positive integer n, there is a layer of the stable homotopy category (localized at a prime p)

Chromatic homotopy theory

Morava's insights led to the formulation of the chromatic point of view in stable homotopy theory. In the late 1970s Haynes Miller, Steve Wilson and I showed that Morava's stratification leads to a nice filtration of the Adams-Novikov E_2-term.





In the early 80s we learned that the stable homotopy category itself possesses a filtration similar to the one found by Morava in $Spec(MU_*)$. A key technical tool in defining it is Bousfield localization.

Pete Bousfield

As in the algebraic case, for each positive integer n, there is a layer of the stable homotopy category (localized at a prime p) related to height n formal group laws.

Chromatic homotopy theory (continued)

Roughly speaking, the structure of the nth layer is controlled by the cohomology of the nth Morava stabilizer group S_n.

Chromatic homotopy theory (continued)

Roughly speaking, the structure of the nth layer is controlled by the cohomology of the nth Morava stabilizer group S_n.

Homotopy groups of objects in the nth layer tend to repeat themselves every $2p^k(p^n - 1)$ dimensions for various k.

Chromatic homotopy theory (continued)

Roughly speaking, the structure of the nth layer is controlled by the cohomology of the nth Morava stabilizer group S_n.

Homotopy groups of objects in the nth layer tend to repeat themselves every $2p^k(p^n - 1)$ dimensions for various k. This is known as v_n-periodicity.

Chromatic homotopy theory (continued)

Roughly speaking, the structure of the nth layer is controlled by the cohomology of the nth Morava stabilizer group S_n.

Homotopy groups of objects in the nth layer tend to repeat themselves every $2p^k(p^n - 1)$ dimensions for various k. This is known as v_n-periodicity.

The term chromatic refers to this separation into varying frequencies.

Chromatic homotopy theory (continued)

Roughly speaking, the structure of the nth layer is controlled by the cohomology of the nth Morava stabilizer group S_n.

Homotopy groups of objects in the nth layer tend to repeat themselves every $2p^k(p^n - 1)$ dimensions for various k. This is known as v_n-periodicity.

The term chromatic refers to this separation into varying frequencies.

The first known example of this phenomenon was the Bott Periodicity Theorem of 1956,

Chromatic homotopy theory (continued)

Roughly speaking, the structure of the *n*th layer is controlled by the cohomology of the *n*th Morava stabilizer group S_n.

Homotopy groups of objects in the *n*th layer tend to repeat themselves every $2p^k(p^n - 1)$ dimensions for various k. This is known as v_n-periodicity.

The term chromatic refers to this separation into varying frequencies.

The first known example of this phenomenon was the Bott Periodicity Theorem of 1956, describing the homotopy of the stable unitary and orthogonal groups.

Chromatic homotopy theory (continued)

Roughly speaking, the structure of the nth layer is controlled by the cohomology of the nth Morava stabilizer group S_n.

Homotopy groups of objects in the nth layer tend to repeat themselves every $2p^k(p^n - 1)$ dimensions for various k. This is known as v_n-periodicity.

The term chromatic refers to this separation into varying frequencies.

The first known example of this phenomenon was the Bott Periodicity Theorem of 1956, describing the homotopy of the stable unitary and orthogonal groups. It is an example of v_1-periodicity.

Raoul Bott
1923-2005

Chromatic homotopy theory (continued)

The motivating problem behind this work was understanding the stable homotopy groups of spheres.

Chromatic homotopy theory (continued)

The motivating problem behind this work was understanding the stable homotopy groups of spheres. Research on them in the 1950s and 60s indicated a very disorganized picture, a zoo of erratic phenomena.

Chromatic homotopy theory (continued)

The motivating problem behind this work was understanding the stable homotopy groups of spheres. Research on them in the 1950s and 60s indicated a very disorganized picture, a zoo of erratic phenomena.

It was seen then to contain one systematic pattern related to Bott periodicity.

Chromatic homotopy theory (continued)

The motivating problem behind this work was understanding the stable homotopy groups of spheres. Research on them in the 1950s and 60s indicated a very disorganized picture, a zoo of erratic phenomena.

It was seen then to contain one systematic pattern related to Bott periodicity. The known homotopy groups of the stable orthogonal group mapped to the unknown stable homotopy groups of spheres by the Hopf-Whitehead J-homomorphism.

Chromatic homotopy theory (continued)

The motivating problem behind this work was understanding the stable homotopy groups of spheres. Research on them in the 1950s and 60s indicated a very disorganized picture, a zoo of erratic phenomena.

It was seen then to contain one systematic pattern related to Bott periodicity. The known homotopy groups of the stable orthogonal group mapped to the unknown stable homotopy groups of spheres by the Hopf-Whitehead J-homomorphism.

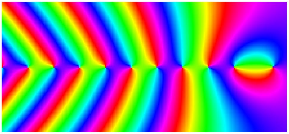
Its image was determined by Adams.

Chromatic homotopy theory (continued)

The motivating problem behind this work was understanding the stable homotopy groups of spheres. Research on them in the 1950s and 60s indicated a very disorganized picture, a zoo of erratic phenomena.

It was seen then to contain one systematic pattern related to Bott periodicity. The known homotopy groups of the stable orthogonal group mapped to the unknown stable homotopy groups of spheres by the Hopf-Whitehead J-homomorphism.

Its image was determined by Adams. It contained the rich arithmetic structure detected by the Novikov calculation referred to earlier.



Chromatic homotopy theory (continued)

In the early 1970s some more systematic patterns were found independently by Larry Smith and Hirosi Toda.

Chromatic homotopy theory (continued)

In the early 1970s some more systematic patterns were found independently by Larry Smith and Hirosi Toda.

The aim of chromatic theory was find a unified framework for such patterns.

Chromatic homotopy theory (continued)

In the early 1970s some more systematic patterns were found independently by Larry Smith and Hirosi Toda.

The aim of chromatic theory was find a unified framework for such patterns. It was very successful.

Chromatic homotopy theory (continued)

In the early 1970s some more systematic patterns were found independently by Larry Smith and Hirosi Toda.

The aim of chromatic theory was find a unified framework for such patterns. It was very successful. A milestone result in it is the Nilpotence Theorem of Ethan Devinatz, Mike Hopkins and Jeff Smith, proved in 1985.

Chromatic homotopy theory (continued)

In the early 1970s some more systematic patterns were found independently by Larry Smith and Hirosi Toda.

The aim of chromatic theory was find a unified framework for such patterns. It was very successful. A milestone result in it is the Nilpotence Theorem of Ethan Devinatz, Mike Hopkins and Jeff Smith, proved in 1985.

Of this result Adams said

Chromatic homotopy theory (continued)

In the early 1970s some more systematic patterns were found independently by Larry Smith and Hirosi Toda.

The aim of chromatic theory was find a unified framework for such patterns. It was very successful. A milestone result in it is the Nilpotence Theorem of Ethan Devinatz, Mike Hopkins and Jeff Smith, proved in 1985.

Of this result Adams said

> *At one time it seemed that homotopy theory was utterly without system;*

Chromatic homotopy theory (continued)

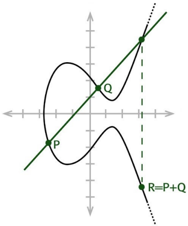
In the early 1970s some more systematic patterns were found independently by Larry Smith and Hirosi Toda.

The aim of chromatic theory was find a unified framework for such patterns. It was very successful. A milestone result in it is the Nilpotence Theorem of Ethan Devinatz, Mike Hopkins and Jeff Smith, proved in 1985.

Of this result Adams said

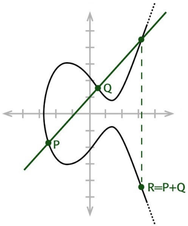
> *At one time it seemed that homotopy theory was utterly without system; now it is almost proved that systematic effects predominate.*

Elliptic cohomology and topological modular forms



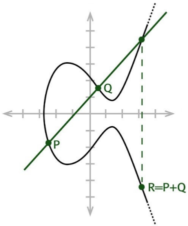
For over a century elliptic curves have stood at the center of mathematics.

Elliptic cohomology and topological modular forms



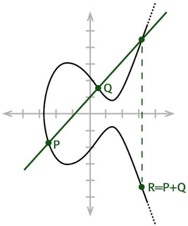
For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it.

Elliptic cohomology and topological modular forms



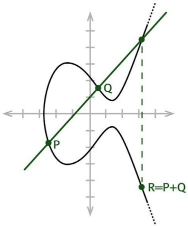
For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it. This means there is a homomorphism to R from MU_*.

Elliptic cohomology and topological modular forms



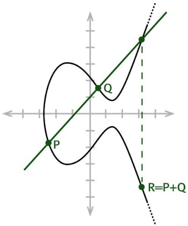
For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it. This means there is a homomorphism to R from MU_*. It is known that its mod p reduction (for any prime p) of this formal group law

Elliptic cohomology and topological modular forms



For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it. This means there is a homomorphism to R from MU_*. It is known that its mod p reduction (for any prime p) of this formal group law has height 1 or 2.

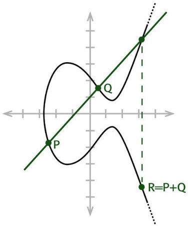
Elliptic cohomology and topological modular forms



For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it. This means there is a homomorphism to R from MU_*. It is known that its mod p reduction (for any prime p) of this formal group law has height 1 or 2.

This led to the definition of the elliptic genus by Serge Ochanine in 1984

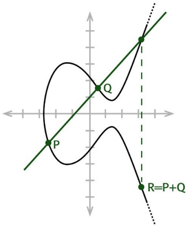
Elliptic cohomology and topological modular forms



For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it. This means there is a homomorphism to R from MU_*. It is known that its mod p reduction (for any prime p) of this formal group law has height 1 or 2.

This led to the definition of the elliptic genus by Serge Ochanine in 1984 and the definition of elliptic cohomology by Peter Landweber, Bob Stong and me a few years later.

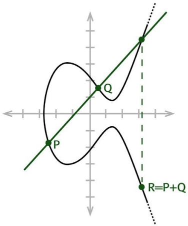
Elliptic cohomology and topological modular forms



For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it. This means there is a homomorphism to R from MU_*. It is known that its mod p reduction (for any prime p) of this formal group law has height 1 or 2.

This led to the definition of the elliptic genus by Serge Ochanine in 1984 and the definition of elliptic cohomology by Peter Landweber, Bob Stong and me a few years later. Attempts to interpret the former analytically have been made by Ed Witten, Stephan Stolz and Peter Teichner.

Elliptic cohomology and topological modular forms



For over a century elliptic curves have stood at the center of mathematics. Every elliptic curve over a ring R has a formal group law attached to it. This means there is a homomorphism to R from MU_*. It is known that its mod p reduction (for any prime p) of this formal group law has height 1 or 2.

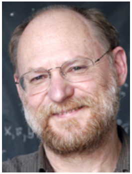
This led to the definition of the elliptic genus by Serge Ochanine in 1984 and the definition of elliptic cohomology by Peter Landweber, Bob Stong and me a few years later. Attempts to interpret the former analytically have been made by Ed Witten, Stephan Stolz and Peter Teichner.

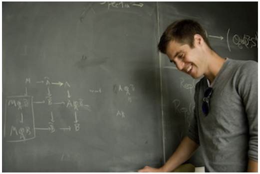
Elliptic cohomology and topological modular forms (continued)

A deeper study of the connection between ellitpic curves and algebraic topology led to the theory of topological modular forms in the past decade.

Elliptic cohomology and topological modular forms (continued)

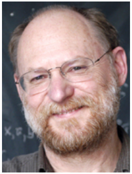
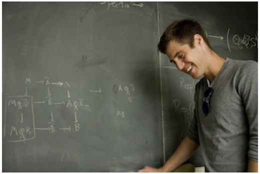
A deeper study of the connection between ellitpic curves and algebraic topology led to the theory of topological modular forms in the past decade. The main players here are Mike Hopkins, Haynes Miller, Paul Goerss and Jacob Lurie.



Elliptic cohomology and topological modular forms (continued)

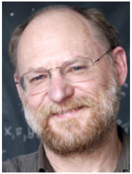
A deeper study of the connection between ellitpic curves and algebraic topology led to the theory of topological modular forms in the past decade. The main players here are Mike Hopkins, Haynes Miller, Paul Goerss and Jacob Lurie.

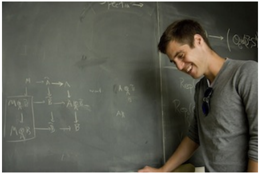
   

Algebraic geometers study objects like elliptic curves by looking at moduli spaces for them.

Elliptic cohomology and topological modular forms (continued)

A deeper study of the connection between ellitpic curves and algebraic topology led to the theory of topological modular forms in the past decade. The main players here are Mike Hopkins, Haynes Miller, Paul Goerss and Jacob Lurie.

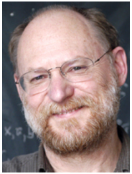



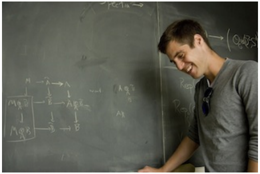


Algebraic geometers study objects like elliptic curves by looking at moduli spaces for them. Roughly speaking, the moduli space (or stack) $\mathcal{M}_{E\ell\ell}$ for elliptic curves is a topological space with an elliptic curve attached to each point.

Elliptic cohomology and topological modular forms (continued)

A deeper study of the connection between ellitpic curves and algebraic topology led to the theory of topological modular forms in the past decade. The main players here are Mike Hopkins, Haynes Miller, Paul Goerss and Jacob Lurie.



Algebraic geometers study objects like elliptic curves by looking at moduli spaces for them. Roughly speaking, the moduli space (or stack) $\mathcal{M}_{E\ell\ell}$ for elliptic curves is a topological space with an elliptic curve attached to each point. The theory of elliptic curves is in a certain sense controlled by the geometry of this space.

Elliptic cohomology and topological modular forms (continued)

To each open subset in the moduli stack $\mathcal{M}_{E\ell\ell}$ one can associate a certain commutative ring of functions related to the corresponding collection of elliptic curves.

Elliptic cohomology and topological modular forms (continued)

To each open subset in the moduli stack $\mathcal{M}_{E\ell\ell}$ one can associate a certain commutative ring of functions related to the corresponding collection of elliptic curves. This collection is called a sheaf of rings $\mathcal{O}_{E\ell\ell}$ over $\mathcal{M}_{E\ell\ell}$.

Elliptic cohomology and topological modular forms (continued)

To each open subset in the moduli stack $\mathcal{M}_{E\ell\ell}$ one can associate a certain commutative ring of functions related to the corresponding collection of elliptic curves. This collection is called a sheaf of rings $\mathcal{O}_{E\ell\ell}$ over $\mathcal{M}_{E\ell\ell}$.

Such a sheaf has a ring of global sections $\Gamma(\mathcal{O}_{E\ell\ell})$, which encodes a lot of useful information.

Elliptic cohomology and topological modular forms (continued)

To each open subset in the moduli stack $\mathcal{M}_{E\ell\ell}$ one can associate a certain commutative ring of functions related to the corresponding collection of elliptic curves. This collection is called a sheaf of rings $\mathcal{O}_{E\ell\ell}$ over $\mathcal{M}_{E\ell\ell}$.

Such a sheaf has a ring of global sections $\Gamma(\mathcal{O}_{E\ell\ell})$, which encodes a lot of useful information. Elements of this ring are closely related to modular forms, which are complex analytic functions with certain arithmetic properties that have fascinated number theorists for over a century.

Elliptic cohomology and topological modular forms (continued)

To each open subset in the moduli stack $\mathcal{M}_{E\ell\ell}$ one can associate a certain commutative ring of functions related to the corresponding collection of elliptic curves. This collection is called a sheaf of rings $\mathcal{O}_{E\ell\ell}$ over $\mathcal{M}_{E\ell\ell}$.

Such a sheaf has a ring of global sections $\Gamma(\mathcal{O}_{E\ell\ell})$, which encodes a lot of useful information. Elements of this ring are closely related to modular forms, which are complex analytic functions with certain arithmetic properties that have fascinated number theorists for over a century. They were a key ingredient in Wiles' proof of Fermat's Last Theorem.

Elliptic cohomology and topological modular forms (continued)

Hopkins, Lurie *et al.* have found a way to enrich this theory by replacing every ring R in sight with a commutative ring spectrum E with suitable formal properties.

Elliptic cohomology and topological modular forms (continued)

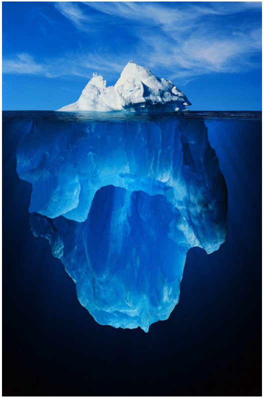
Hopkins, Lurie *et al.* have found a way to enrich this theory by replacing every ring R in sight with a commutative ring spectrum E with suitable formal properties. We can think of E as an iceberg whose tip is R.

Elliptic cohomology and topological modular forms (continued)

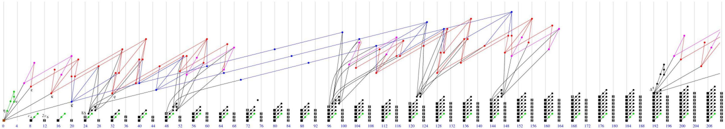
Hopkins, Lurie *et al.* have found a way to enrich this theory by replacing every ring R in sight with a commutative ring spectrum E with suitable formal properties. We can think of E as an iceberg whose tip is R. The one associated with $\Gamma(\mathcal{O}_{E\ell\ell})$ is known as ***tmf***, the ring spectrum of topological modular forms.

Elliptic cohomology and topological modular forms (continued)

Hopkins, Lurie *et al.* have found a way to enrich this theory by replacing every ring R in sight with a commutative ring spectrum E with suitable formal properties. We can think of E as an iceberg whose tip is R. The one associated with $\Gamma(\mathcal{O}_{E\ell\ell})$ is known as ***tmf***, the ring spectrum of topological modular forms. This ring spectrum is an iceberg whose tip is the classical theory of modular forms.



Elliptic cohomology and topological modular forms (continued)



The 2-primary homotopy of *tmf* illustrated by Andre Henriques.

Conclusion

Quillen's work on formal group laws and complex cobordism opened a new era in algebraic topology.

Conclusion

Quillen's work on formal group laws and complex cobordism opened a new era in algebraic topology. It led to a chain of discoveries that is unabated to this day.

Conclusion

Quillen's work on formal group laws and complex cobordism opened a new era in algebraic topology. It led to a chain of discoveries that is unabated to this day.

Thank you!