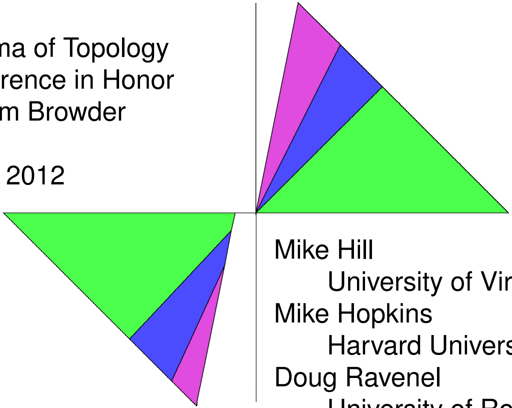
Browder's work on Arf-Kervaire invariant problem

Panorama of Topology
A Conference in Honor of William Browder

May 10, 2012



Mike Hill
University of Virginia
Mike Hopkins
Harvard University
Doug Ravenel
University of Rochester

Browder's theorem and its impact

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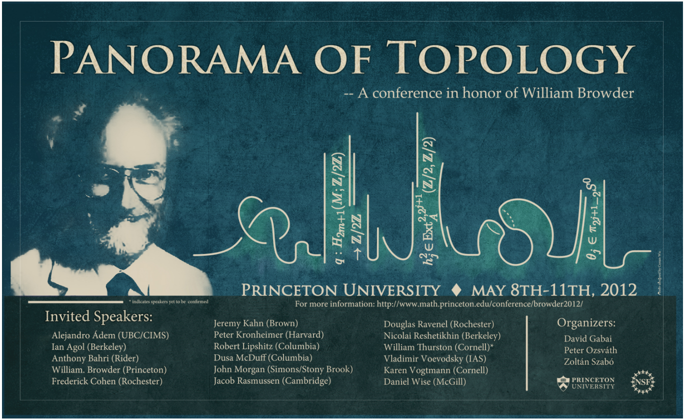

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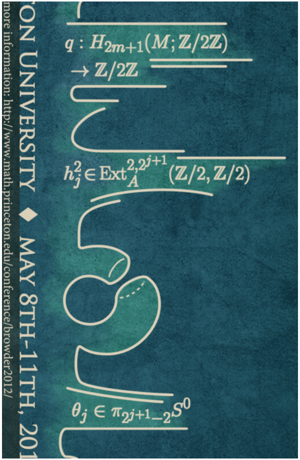

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Mark Mahowald's sailboat

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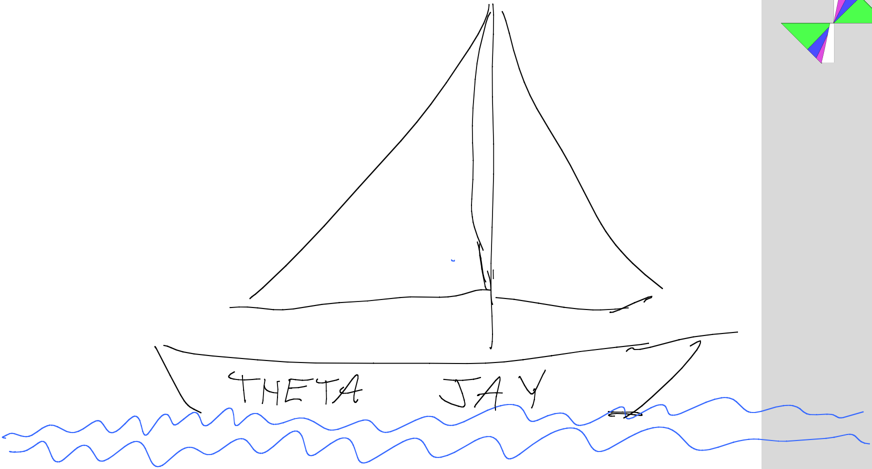

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London, Ontario 1981

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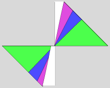

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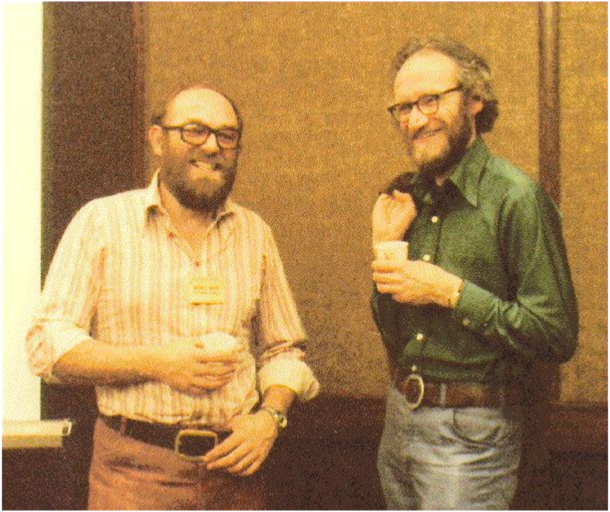

Vic Snaith and Bill Browder in 1981
Photo by Clarence Wilkerson

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to 2009

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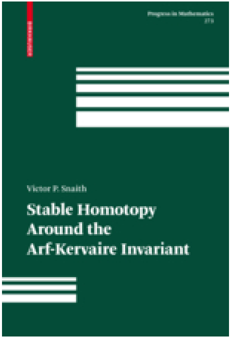

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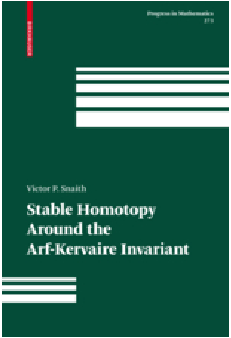

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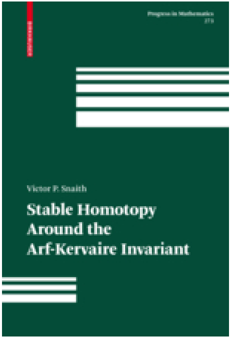

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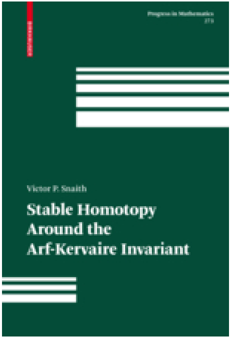

"For a brief period overnight we were convinced that we had the method to make all the sought after framed manifolds - a feeling which must have been shared by many topologists working on this problem. All in all, the temporary high of believing that one had the construction was sufficient to maintain in me at least an enthusiastic spectator's interest in the problem."

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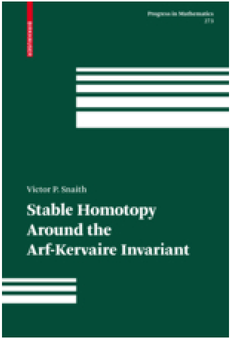

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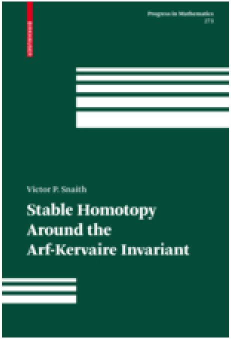

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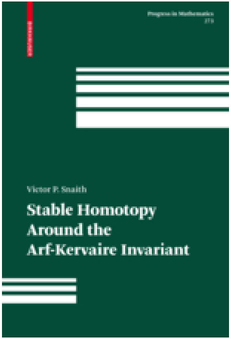

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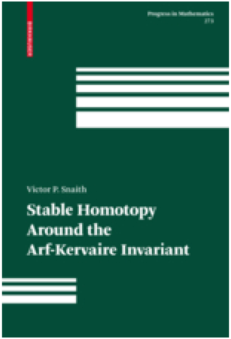

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Pontryagin's early work on homotopy groups of spheres

Back to the 1930s

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- Pick a regular value $y \in S^k$. Its inverse image will be a smooth n-manifold M in S^{n+k}.
- By studying such manifolds, Pontryagin was able to deduce things about maps between spheres.

Pontryagin's early work on homotopy groups of spheres (continued)

$$\begin{tikzcd}
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There is a way to reverse this procedure. A framed manifold $M^n \subset S^{n+k}$ determines a map $f : S^{n+k} \to S^k$.

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Suppose there is homotopy $h : S^{n+k} \times [0,1] \to S^k$ between two such maps $f_1, f_2 : S^{n+k} \to S^k$.

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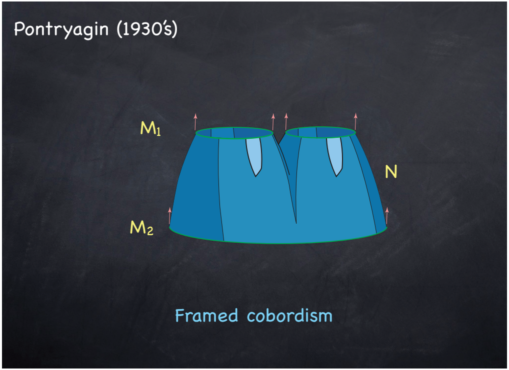

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The determination of the stable homotopy groups π^S_n is an ongoing problem in algebraic topology.

The Kervaire-Milnor classification of exotic spheres

Into the 60s again

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  John Milnor's *On manifolds homeomorphic to the 7-sphere*, 1956.

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Mike Hill
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Doug Ravenel

The Kervaire-Milnor classification of exotic spheres

Into the 60s again

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The Kervaire-Milnor classification of exotic spheres (continued)

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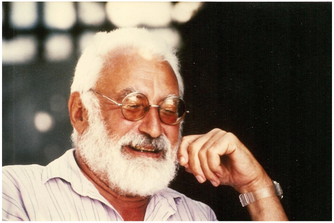
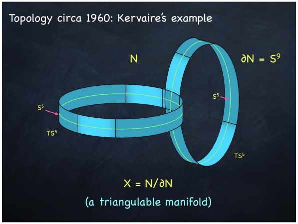
- 

  Michel Kervaire 1927-2007

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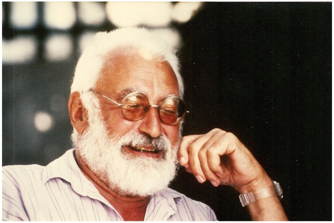
- 

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  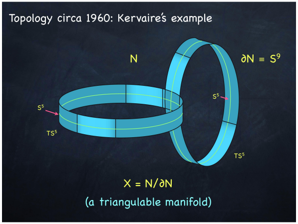

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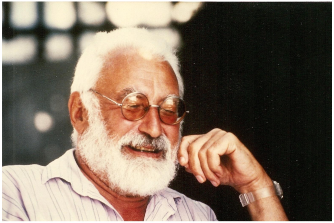
- 

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  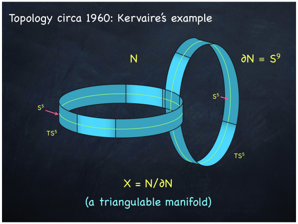

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For example, for $n = 1, 2, 3, \cdots, 18$, it will be shown that the order of the group Θ_n is respectively:

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$[\Theta_n]$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16.

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Exotic spheres as framed manifolds

Following Kervaire-Milnor, let Θ_n denote the group of diffeomorphism classes of exotic n-spheres Σ^n.

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Two framings of an exotic sphere $\Sigma^n \subset S^{n+k}$ differ by a map to the special orthogonal group $SO(k)$, and this map does not depend on the differentiable structure on Σ^n.

Exotic spheres as framed manifolds (continued)

Varying the framing on the standard sphere S^n leads to a homomorphism

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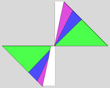

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They denote the kernel of p by bP_{n+1}, the group of exotic n-spheres bounding parallelizable $(n+1)$-manifolds.

Exotic spheres as framed manifolds (continued)

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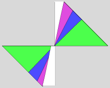

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We now know the value of bP_{4m+2} in every case except $m = 31$.

Exotic spheres as framed manifolds (continued)

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$$0 \longrightarrow \Theta_{4m+2} \xrightarrow{p} \pi^S_{4m+2}/\mathrm{Im}\, J \longrightarrow \mathbf{Z}/2 \longrightarrow bP_{4m+2} \longrightarrow 0$$

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To say more about this we need to define the Kervaire invariant of a framed manifold.

The Arf invariant of a quadratic form in characteristic 2

Back to the 1940s

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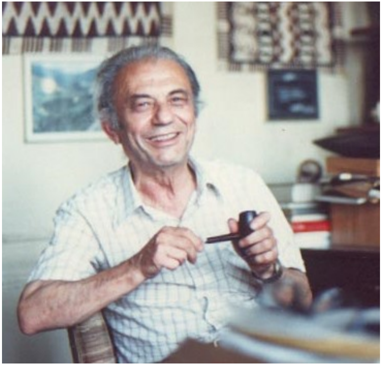

Cahit Arf 1910-1997

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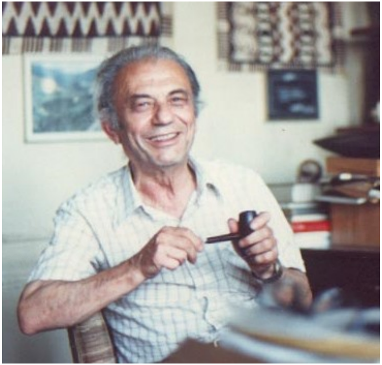

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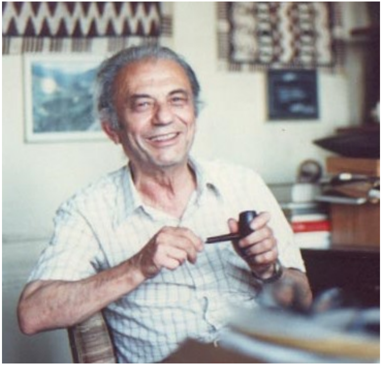

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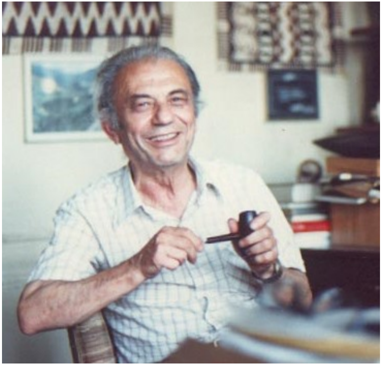

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$$\lambda(a_i, a_{i'}) = 0 \qquad \lambda(b_j, b_{j'}) = 0 \qquad \text{and} \qquad \lambda(a_i, b_j) = \delta_{i,j}.$$

The Arf invariant of a quadratic form in characteristic 2 (continued)

Mike Hill
Mike Hopkins
Doug Ravenel

In other words, $\overline{H}$ has a basis for which the bilinear form's matrix has the symplectic form

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

The Arf invariant of a quadratic form in characteristic 2 (continued)

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In 1941 Arf proved that this invariant (along with the number n) determines the isomorphism type of q.

Money talks: Arf's definition republished in 2009

Cahit Arf 1910-1997

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The Kervaire invariant of a framed $(4m+2)$-manifold

Into the 60s
a third time

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$$q : H_1(T^2; \mathbf{Z}/2) \to \mathbf{Z}/2$$

as follows.

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Into the 60s
a third time

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The Kervaire invariant of a framed $(4m+2)$-manifold (continued)

For $M = T^2 \subset S^3$ and $x \in H_1(T^2; \mathbf{Z}/2)$, $q(x)$ is the number of full twists in a cylinder V neighboring a curve representing x.

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Kervaire-Milnor Theorem (1963)

$bP_{4m+2} = 0$ iff there is a smooth framed $(4m+2)$-manifold M with $\Phi(M)$ nontrivial.

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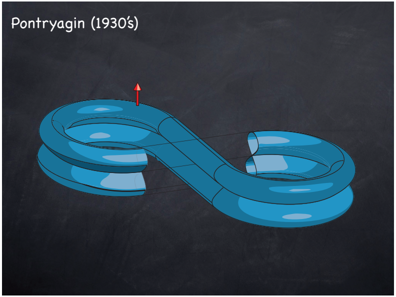

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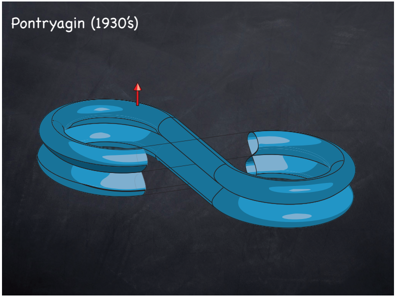

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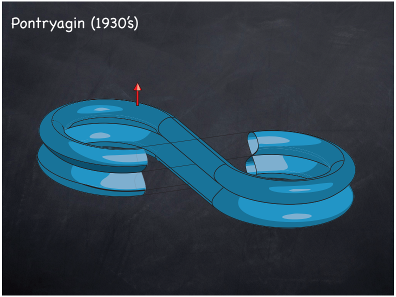

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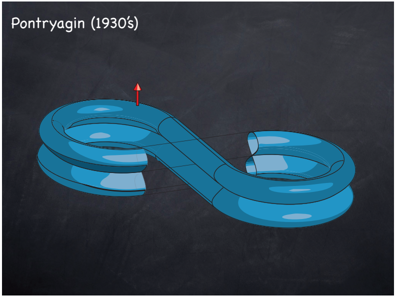

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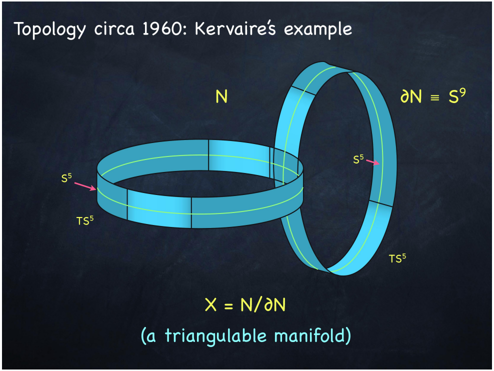

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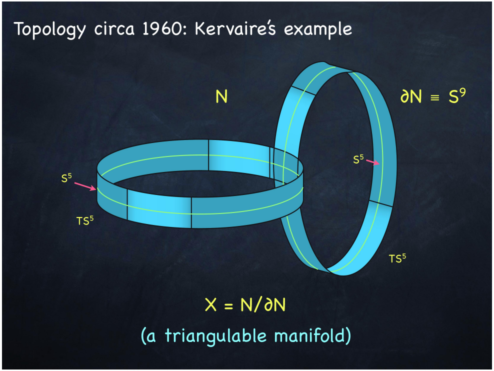
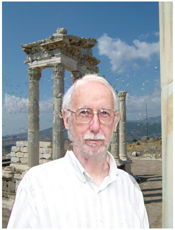
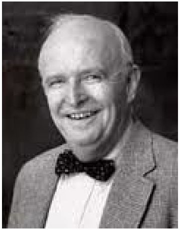


This construction generalizes to higher m, but Kervaire's proof that the boundary is exotic does not.

Some theorems about $\phi(M)^{4m+2}$ (continued)



Ed Brown



Frank Peterson
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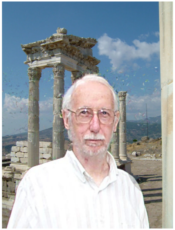

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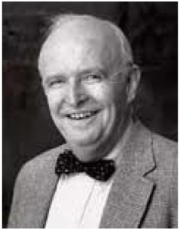

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Brown-Peterson (1966) showed that it vanishes for all positive even m. This means $bP_{8\ell+2} = \mathbf{Z}/2$ for $\ell > 0$.

Browder's theorem

Browder's Theorem (1969)

The Kervaire invariant of a smooth framed $(4m+2)$*-manifold* M *can be nontrivial only if* $m = 2^{j-1} - 1$ *for some* $j > 0$*. This happens iff the element* h_j^2 *is a permanent cycle in the Adams spectral sequence.*

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$$K_n = K(\mathbf{Z}/2, n).$$

A sketch of Browder's proof

Now consider the diagram

$$\begin{tikzcd} & \Sigma K_{2m+1} \arrow[dl, dotted, "\widehat{i}"'] \arrow[d, "i"] \arrow[dr, "*"] & \\ F_{2m+2} \arrow[r] & K_{2m+2} \arrow[r, "Sq^{2m+2}"] & K_{4m+4} \end{tikzcd}$$

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The space F_{2m+2} has two nontrivial homotopy groups,

$$\pi_n F_{2m+2} = \begin{cases} \mathbf{Z}/2 & \text{for } n = 2m+2 \\ \mathbf{Z}/2 & \text{for } n = 4m+3 \\ 0 & \text{otherwise.} \end{cases}$$

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The map $\widehat{i}$ is an equivalence thru dimension $4m+3$ and

$$\pi_{4m+2+k}\Sigma^k K_{2m+1} = \mathbf{Z}/2 \qquad \text{for } k > 0.$$

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Hence $v_n(\xi)$ for each $n > 0$ is a certain polynomial in the Stiefel-Whitney classes.

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$$v(\xi) = \left(Sq^{-1}w(\xi)\right)^{-1}.$$

For a $(4m+2)$-manifold M we define $v_i(M) \in H^i M$ to be the ith Wu class of its normal bundle. It is known that for $x \in H^{4m+2-i}M$,

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$$\begin{tikzcd}
 & M \arrow[dl, dotted, "\widehat{\nu}"'] \arrow[d, "\nu"] \arrow[dr, "*"] & \\
B\langle v_{2m+2}\rangle \arrow[r, "\pi"] & BO \arrow[r, "v_{2m+2}"] & K_{2m+2}
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The Browder spectrum

$$
\begin{tikzcd}
 & & M \arrow[dl, dotted, "\widehat{\nu}"'] \arrow[d, "\nu"] \arrow[dr, "*"] & \\
K_{2m+1} \arrow[r] & B\langle v_{2m+2}\rangle \arrow[r, "\pi"] & BO \arrow[r, "v_{2m+2}"] & K_{2m+2}
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We now consider the Thom spectra associated the universal bundle over BO and its pullbacks.

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We now consider the Thom spectra associated the universal bundle over BO and its pullbacks. The diagram becomes

$$\begin{tikzcd}
 & & T(\nu_M) \arrow[dl, dotted, "T\widehat{\nu}"'] \arrow[d, "T\nu"] \\
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\end{tikzcd}$$

where $T(\nu_M)$ is the Thom spectrum for the normal bundle of M, K_{2m+1} here denotes the suspension spectrum of the space K_{2m+1} and Br_{2m+2}, the mth Browder spectrum, is the Thom spectrum associated with $B\langle v_{2m+2}\rangle$.

The Browder spectrum (continued)

$$\begin{tikzcd}
 & & T(\nu_M) \arrow[dl, dotted, "T\widehat{\nu}"'] \arrow[d, "T\nu"] \\
\Sigma^\infty K_{2m+1} \arrow[r] & \mathrm{Br}_{2m+2} \arrow[r, "\overline{p}"] & MO
\end{tikzcd}$$

The Browder spectrum (continued)

$$\begin{tikzcd} & & T(\nu_M) \arrow[dl, dotted, "T\widehat{\nu}"'] \arrow[d, "T\nu"] \\ \Sigma^\infty K_{2m+1} \arrow[r] & \mathrm{Br}_{2m+2} \arrow[r, "\overline{p}"] & MO \end{tikzcd}$$

The Spanier-Whitehead dual of $T(\nu_M)$ is $\Sigma^{-4m-2}M_+$, so we have a map

$$D\mathrm{Br}_{2m+2} \xrightarrow{\eta} \Sigma^{-4m-2}M_+.$$

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Both of these spectra have no cells in positive dimensions and Sq^{2m+2} maps trivially to H^0.

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$$
\begin{tikzcd}
D\mathrm{Br}_{2m+2} \arrow[r, "\eta"] \arrow[d, equal] & \Sigma^{-4m-2}M_+ \arrow[d, "x"] \\
X \arrow[r, "g"] & \Sigma^{-4m-2}K_{2m+1} \arrow[r, equal] & K
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The Browder spectrum (continued)

Let $q = 2m + 1$, so our diagram reads

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Consider the following diagram with exact rows in black:

$$\begin{tikzcd} \color{red}0 & \color{red}\iota_q \arrow[l, mapsto, red] \arrow[dddd, mapsto, red, bend right] & \color{red}\alpha \arrow[l, mapsto, red] \arrow[dddd, mapsto, red, bend right] \\ H^{-q}X & H^{-q}K \arrow[l, "g^*"'] \arrow[d, "Sq^{q+1}"] & H^{-q}(K,X) \arrow[l] \arrow[d, "Sq^{q+1}"] & H^{-1-q}X \arrow[l] \arrow[d, "0"] \\ & H^1K & H^1(K,X) \arrow[l] & H^0X \arrow[l] & H^0K \arrow[l, "0"'] \\ & & & \color{red}0 & \color{red}Sq^q\iota_q \arrow[l, mapsto, red] \\ & \color{red}0 & \color{red}Sq^{q+1}\alpha \arrow[l, mapsto, red] & \color{red}\psi(x) \arrow[l, mapsto, red] \end{tikzcd}$$

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& H^1 K & H^1(K,X) \arrow[l] & H^0 X \arrow[l] & H^0 K \arrow[l, "0"'] \\
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 & H^1 K & H^1(K,X) \arrow[l] & H^0 X \arrow[l] & H^0 K \arrow[l, "0"'] \\
 & & & 0 & Sq^q \iota_q \arrow[l, mapsto] \\
 & 0 & Sq^{q+1}\alpha \arrow[l, mapsto] & \psi(x) \arrow[l, mapsto] &
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& H^1 K & H^1(K, X) \arrow[l] & H^0 X \arrow[l] & H^0 K \arrow[l, "0"'] \\
& & & \color{red}{0} & \color{red}{Sq^q \iota_q} \arrow[l, mapsto, red] \\
& \color{red}{0} & \color{red}{Sq^{q+1}\alpha} \arrow[l, mapsto, red] & \color{red}{\psi(x)} \arrow[l, mapsto, red] &
\end{tikzcd}$$

The diagram chase is shown in red. The element $\psi(x)$ is independent of the choice of α. Browder shows that the operation ψ is quadratic.

The Browder spectrum (continued)

If the manifold M has a framing F we get

The Browder spectrum (continued)

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If the manifold M has a framing F we get

$$\begin{tikzcd}
 & S^0 \arrow[d] & T(\nu_M) \arrow[l] \arrow[dl, dotted, "T\widehat{\nu}"'] \arrow[d, "T\nu"] \\
\Sigma^\infty K_{2m+1} \arrow[r] & \mathrm{Br}_{2m+2} \arrow[r, "\overline{p}"] & MO
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\end{tikzcd}$$

This means we can replace $X = D\mathrm{Br}_{2m+2}$ by S^0, so the next diagram becomes

$$\begin{tikzcd}
S^0 \arrow[r, "p_F"] \arrow[d, equal] & \Sigma^{-4m-2} M_+ \arrow[d, "x"] \\
S^0 \arrow[r] & \Sigma^{-4m-2} K_{2m+1}
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\end{tikzcd}$$

This is Browder's interpretation of the quadratic operation ψ described earlier.

The homotopy type of Br_{2m+2}

A framed $(4m+2)$-manifold M with nontrivial Kervaire invariant represents, via Pontryagin's isomorphism, a nontrivial map

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Browder shows that the composite map to the Browder spectrum

$$S^{4m+2} \xrightarrow{\theta} S^0 \longrightarrow \mathrm{Br}_{2m+2}$$

must also be nontrivial.

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He analyzes the homotopy type of Br_{2m+2} and gets a diagram

$$\begin{tikzcd}
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & \mathrm{Br}_{2m+2}{}^{(1)} \arrow[l] \arrow[d, "h"] & \mathrm{Br}_{2m+2}{}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
MO & K_{2m+1} \wedge MO & K_{4m+2} &
\end{tikzcd}$$

The homotopy type of Br_{2m+2} (continued)

$$
\begin{tikzcd}
S^0 \arrow[d] & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & {\mathrm{Br}_{2m+2}}^{(1)} \arrow[l] \arrow[d, "h"] & {\mathrm{Br}_{2m+2}}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
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S^0 \arrow[d] & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
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MO & K_{2m+1} \wedge MO & K_{4m+2} &
\end{tikzcd}
$$

Here each horizontal map is the inclusion of the fiber of the following vertical map.

The homotopy type of Br_{2m+2} (continued)

$$
\begin{tikzcd}
S^0 \arrow[d] & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & {\mathrm{Br}_{2m+2}}^{(1)} \arrow[l] \arrow[d, "h"] & {\mathrm{Br}_{2m+2}}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c} (4m+2)\text{-} \\ \text{connected} \\ \text{fiber} \end{array}\right) \arrow[l] \\
MO & K_{2m+1} \wedge MO & K_{4m+2} &
\end{tikzcd}
$$

Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that MO is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra.

The homotopy type of Br_{2m+2} (continued)

$$
\begin{tikzcd}
S^0 & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & \mathrm{Br}_{2m+2}{}^{(1)} \arrow[l] \arrow[d, "h"] & \mathrm{Br}_{2m+2}{}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
MO & K_{2m+1} \wedge MO & K_{4m+2} &
\arrow[from=1-1, to=2-1]
\end{tikzcd}
$$

Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that *MO* is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra. This means that Br_{2m+2} is a 3-stage Postnikov system in the relevant range of dimensions.

The homotopy type of Br_{2m+2} (continued)

$$
\begin{tikzcd}
S^0 \arrow[d] & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & \mathrm{Br}_{2m+2}{}^{(1)} \arrow[l] \arrow[d, "h"] & \mathrm{Br}_{2m+2}{}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
MO & K_{2m+1} \wedge MO & K_{4m+2} &
\end{tikzcd}
$$

Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that MO is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra. This means that Br_{2m+2} is a 3-stage Postnikov system in the relevant range of dimensions.

It follows that θ must be detected by an element on the 2-line of the Adams spectral sequence.

The homotopy type of Br_{2m+2} (continued)

$$
\begin{tikzcd}
S^0 \arrow[d] & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & \mathrm{Br}_{2m+2}{}^{(1)} \arrow[l] \arrow[d, "h"] & \mathrm{Br}_{2m+2}{}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
MO & K_{2m+1} \wedge MO & K_{4m+2} &
\end{tikzcd}
$$

Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that MO is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra. This means that Br_{2m+2} is a 3-stage Postnikov system in the relevant range of dimensions.

It follows that θ must be detected by an element on the 2-line of the Adams spectral sequence. An explicit description of the map k rules out all elements other than h_j^2,

The homotopy type of Br_{2m+2} (continued)

$$
\begin{tikzcd}
S^0 \arrow[d] & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & \mathrm{Br}_{2m+2}{}^{(1)} \arrow[l] \arrow[d, "h"] & \mathrm{Br}_{2m+2}{}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
MO & K_{2m+1} \wedge MO & K_{4m+2} &
\end{tikzcd}
$$

Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that MO is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra. This means that Br_{2m+2} is a 3-stage Postnikov system in the relevant range of dimensions.

It follows that θ must be detected by an element on the 2-line of the Adams spectral sequence. An explicit description of the map k rules out all elements other than h_j^2, which is shown to detect the Kervaire invariant in dimension $2^{j+1} - 2$.

The homotopy type of Br_{2m+2} (continued)

$$\begin{tikzcd}
S^0 \arrow[d] & & S^{4m+2} \arrow[ll, "\theta"'] \arrow[d, dotted] & \\
\mathrm{Br}_{2m+2} \arrow[d, "\overline{p}"] & \mathrm{Br}_{2m+2}{}^{(1)} \arrow[l] \arrow[d, "h"] & \mathrm{Br}_{2m+2}{}^{(2)} \arrow[l] \arrow[d, "k"] & \left(\begin{array}{c}(4m+2)\text{-}\\ \text{connected}\\ \text{fiber}\end{array}\right) \arrow[l] \\
MO & K_{2m+1} \wedge MO & K_{4m+2} &
\end{tikzcd}$$

Here each horizontal map is the inclusion of the fiber of the following vertical map. We know that MO is a wedge of suspensions of mod 2 Eilenberg-Mac Lane spectra. This means that Br_{2m+2} is a 3-stage Postnikov system in the relevant range of dimensions.

It follows that θ must be detected by an element on the 2-line of the Adams spectral sequence. An explicit description of the map k rules out all elements other than h_j^2, which is shown to detect the Kervaire invariant in dimension $2^{j+1} - 2$.

This completes the proof of the theorem.