Model categories and spectra

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Mike Hopkins Harvard University
Doug Ravenel University of Rochester

Special Session on Algebraic and Geometric Topology
Fudan University, Shanghai, China
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Introduction

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Mike Hill
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These two definitions are known to be equivalent.

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Iintroduction (continued)

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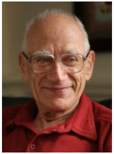

Dan Kan
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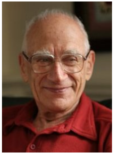

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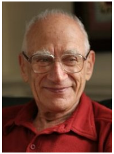

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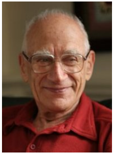

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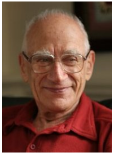

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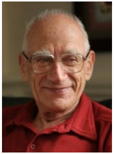


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We will see that the passage from strict equivalence to stable equivalence is a form of Bousfield localization.

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Quillen model categories

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Quillen model categories (continued)

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Quillen model categories (continued)

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$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y, \end{tikzcd}$$

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Quillen model categories (continued)

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$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\text{cofibration } i"'] & X \arrow[d, "p \text{ trivial fibration}"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y, \end{tikzcd}$$

a morphism h (called a lifting) exists for i and p as indicated.

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cofibration / trivial cofibration (i) — trivial fibration / fibration (p)

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Quillen model categories (continued)

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cofibration $= \alpha(f)$, $\beta(f)$ $=$ trivial fibration

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Quillen model categories (continued)

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This last axiom is the hardest one to verify in practice.

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Two classical examples

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for each $n \geqslant 0$, where I^n is the unit n-cube.

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Two classical examples

Let $\mathcal{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups. Fibrations are Serre fibrations, that is is maps $p : X \to Y$ with the right lifting property

$$\begin{tikzcd} I^n \arrow[r, "f"] \arrow[d, "j_n"'] & X \arrow[d, "p"] \\ I^{n+1} \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y, \end{tikzcd}$$

for each $n \geqslant 0$, where I^n is the unit n-cube.

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Similar definitions can be made for $\mathcal{T}$, the category of pointed topological spaces and basepoint preserving maps.

Mike Hill
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Some definitions

Recall that we denote the initial and terminal objects of $\mathcal{M}$ by $\varnothing$ and $*$. When they are the same, we say that $\mathcal{M}$ is pointed.

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These maps to and from X are called cofibrant and fibrant approximations. The objects QX and RX are called cofibrant and fibrant replacements of X.

Cofibrant generation

In $\mathcal{T}op$, let

$$\mathcal{I} = \{i_n : S^{n-1} \to D^n, n \geqslant 0\} \text{ and } \mathcal{J} = \{j_n : I^n \to I^{n+1}, n \geqslant 0\}.$$

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It is known that every (trivial) cofibration in $\mathcal{T}op$ can be derived from the ones in $(\mathcal{J})$ $\mathcal{I}$ by iterating certain elementary constructions.

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In $\mathcal{T}$, the category of pointed topological spaces, one can define similar sets $\mathcal{I}_+$ and $\mathcal{J}_+$, by adding disjoint basepoints to the above.

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Cofibrant generation (continued)

Definition

A cofibrantly generated model category $\mathcal{M}$

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Cofibrant generation (continued)

Definition

A cofibrantly generated model category $\mathcal{M}$ is one with morphism sets $\mathcal{I}$ and $\mathcal{J}$ having similar properties to the ones in $\mathcal{T}op$.

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In practice, specifying the generating sets $\mathcal{I}$ and $\mathcal{J}$, and defining weak equivalences

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Cofibrant generation (continued)

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In practice, specifying the generating sets $\mathcal{I}$ and $\mathcal{J}$, and defining weak equivalences is the most convenient way to describe a model category.

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Bousfield localization

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Around 1975 Pete Bousfield had a brilliant idea.

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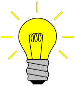
Around 1975 Pete Bousfield had a brilliant idea. 

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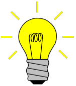

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Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

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The hardest part of this is showing that the new classes of weak equivalences and fibrations, along with the original class of cofibrations, satisfy the second Factorization Axiom MC5. The proof involves some delicate set theory.

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An elementary examples of Bousfield localization

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Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

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Choose an integer $n > 0$.

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An elementary examples of Bousfield localization

Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leqslant n$.

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Let $\mathcal{T}op$ be the category of topological spaces with its usual model structure.

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The strict model structure on the category of spectra

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Recall that a spectrum X is a sequence of pointed spaces $\{X_0, X_1, X_2, \ldots\}$

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Recall that a spectrum X is a sequence of pointed spaces $\{X_0, X_1, X_2, \ldots\}$ with structure maps $\epsilon_n^X : \Sigma X_n \to X_{n+1}$.

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Definition

The m**th Yoneda spectrum** S^{-m} *is given by*

$$(S^{-m})_n = \begin{cases} * & \text{for } n < m \\ S^{n-m} & \text{otherwise,} \end{cases}$$

with the evident structure maps.

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In particular, S^{-0} is the sphere spectrum.

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The strict model structure on the category of spectra (continued)

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The strict model structure on the category of spectra (continued)

Definition

The strict or projective model structure on $\mathcal{S}p$.

The strict model structure on the category of spectra (continued)

Definition

The strict or projective model structure on $\mathcal{S}p$. *A map of spectra $f : X \to Y$ is a weak equivalence or a fibration*

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The strict model structure on the category of spectra (continued)

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The strict or projective model structure on $\mathcal{S}p$. *A map of spectra $f : X \to Y$ is a weak equivalence or a fibration if f_n is one for each $n \geqslant 0$.*

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The strict model structure on the category of spectra (continued)

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The strict model structure on the category of spectra (continued)

Definition

The strict or projective model structure on $\mathcal{S}p$. *A map of spectra $f : X \to Y$ is a weak equivalence or a fibration if f_n is one for each $n \geqslant 0$. A map is a cofibration if it has the left lifting property with respect to all trivial fibrations.*

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Each space $(\Lambda X)_n$ is an infinite loop space,

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is a weak equivalence for all n, so ΛX is an Ω-spectrum.

The stable model structure (continued)

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Again, $(\Lambda X)_n$ is the homotopy colimit

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We can use it to define the stable homotopy groups of X by

$$\pi_k X := \pi_{n+k}(\Lambda X)_n,$$

which is independent of n. We say a map $f : X \to Y$ is a stable equivalence if $\pi_* f$ is an isomorphism.

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Thus we have expanded the class of weak equivalences, so we can use Bousfield localization to construct the stable model structure. It turns out that the fibrant objects are precisely the Ω-spectra and that our functor Λ is fibrant replacement!

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Cofibrant generation

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We will now describe cofibrant generating sets for the stable model structure on the category of spectra $\mathcal{S}p$.

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In order to do so, we need another construction, the pushout product or corner map.

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Cofibrant generation (continued)

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Cofibrant generation (continued)

Suppose $f : A \to B$ and $g : C \to D$ are maps of pointed spaces.

Cofibrant generation (continued)

Suppose $f : A \to B$ and $g : C \to D$ are maps of pointed spaces. Consider the diagram

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A \wedge C \arrow[r, "f \wedge C"] \arrow[d, "A \wedge g"'] & B \wedge C \arrow[dr, bend left, "B \wedge g"] & \\
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Here P is the pushout of the two maps from $A \wedge C$.

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Here P is the pushout of the two maps from $A \wedge C$. Since the outer diagram commutes, there is a unique map from it to $B \wedge D$

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Model categories and spectra

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This construction also makes sense if $f : A \to B$ is a map of spectra,

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This construction also makes sense if $f : A \to B$ is a map of spectra, with $g : C \to D$ still being a map of spaces.

Cofibrant generation (continued)

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Cofibrant generation (continued)

Recall that we need to enlarge the generating set of trivial cofibrations,

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Recall that we need to enlarge the generating set of trivial cofibrations,

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We will do so by defining a set $\mathcal{S}$ of stable equivalences of spectra and adjoining the set $\mathcal{S} \square \mathcal{I}_+$ to $\mathcal{J}^{strict}$.

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Recall the Yoneda spectrum S^{-k} given by

$$(S^{-k})_n = \begin{cases} * & \text{for } n < k \\ S^{n-k} & \text{otherwise.} \end{cases}$$

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It follows that $S^{-k} \wedge S^k$ given by

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Cofibrant generation (continued)

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$$\mathcal{J}^{strict} = \bigcup_{m \geqslant 0} S^{-m} \wedge \mathcal{J}_+.$$

We will do so by defining a set $\mathcal{S}$ of stable equivalences of spectra and adjoining the set $\mathcal{S} \square \mathcal{I}_+$ to $\mathcal{J}^{strict}$.

Recall the Yoneda spectrum S^{-k} given by

$$(S^{-k})_n = \begin{cases} * & \text{for } n < k \\ S^{n-k} & \text{otherwise.} \end{cases}$$

It follows that $S^{-k} \wedge S^k$ given by

$$(S^{-k} \wedge S^k)_n = \begin{cases} * & \text{for } n < k \\ S^n & \text{otherwise.} \end{cases}$$

This is the same as the sphere spectrum S^{-0} for $n \geqslant k$.

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Cofibrant generation (continued)

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Cofibrant generation (continued)

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where $s_k : S^{-k} \wedge S^k \to S^{-0}$ is the stable equivalence defined above.

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