The Chromatic Conjectures

Doug Ravenel
University of Rochester

Chromatic Homotopy Theory: Journey to the Frontier
University of Colorado
May 17, 2018


BOULDER HIGH SCHOOL
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Minnie and Jake

Chromatic homotopy theory in 1977

Annals of Mathematics, **106** (1977), 469–516

Periodic phenomena in the Adams-Novikov spectral sequence

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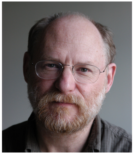

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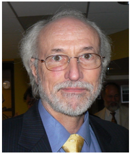

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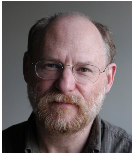

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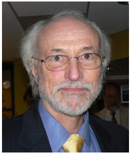

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What would happen if we replace $I_n = (p, \ldots v_{n-1})$ by a smaller invariant regular ideal with n generators, and look for a self map inducing multiplication by some power of v_n?

The chromatic resolution

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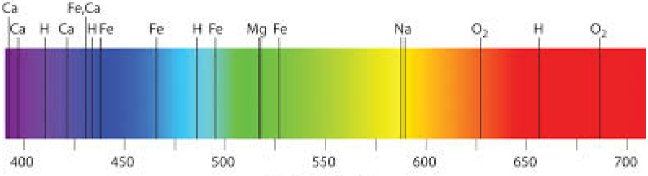

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$$\begin{tikzcd} 0 \arrow[r] & N^n \arrow[r, "v_n^{-1}"] \arrow[d, equal] & M^n \arrow[r] \arrow[dd, equal] & N^{n+1} \arrow[r] \arrow[d, equal] & 0 \\ & BP_*/(p^\infty, \dots v_{n-1}^\infty) & & BP_*/(p^\infty, \dots v_n^\infty) & \\ & & v_n^{-1}BP_*/(p^\infty, \dots v_{n-1}^\infty) & & \end{tikzcd}$$

The chromatic resolution (continued)

$$\begin{tikzcd}
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IS THERE A SIMILAR CONSTRUCTION IN THE STABLE HOMOTOPY CATEGORY?

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The chromatic resolution (continued)

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IS THERE A SIMILAR CONSTRUCTION, AND THE BEAUTIFUL ALGEBRA THAT GOES ALONG WITH IT, IN THE STABLE HOMOTOPY CATEGORY?

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OR IS IT JUST AN ARTIFACT OF COMPLEX COBORDISM THEORY?

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OR IS IT JUST AN ARTIFACT OF COMPLEX COBORDISM THEORY?

This question occupied me for several years.

The chromatic resolution (continued)

LOCALIZATION WITH RESPECT TO CERTAIN PERIODIC HOMOLOGY THEORIES

By Douglas C. Ravenel*

This paper represents an attempt, only partially successful, to get at what appear to be some deep and hitherto unexamined properties of the stable homotopy category. This work was motivated by the discovery of the pervasive manifestation of various types of periodicity in the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres. In section 3 of [34] and section 8 of [41], we introduced the chromatic spectral sequence, which converges to the above E_2-term. Unlike most spectral sequences, its input is in some sense more interesting than its output, as the former displays many appealing patterns which are somewhat hidden in the latter (see section 8 of [41] for a more detailed discussion). It is not so much a computational aid (although it has been used [34] for computing the Novikov 2-line) as a conceptual tool for understanding certain qualitative aspects of the Novikov E_2-term.

Since the Novikov E_2-term is a reasonably good approximation to sta-

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This was easy enough for $n = 0$. We knew then how to invert a prime p homotopically. The resulting N^1 is the Moore spectrum for the group $\mathbf{Q}/\mathbf{Z}_{(p)}$. But how would we invert v_1 to do the next step?

Bousfield localization (continued)

As luck would have it,

Bousfield localization (continued)

As luck would have it, Bousfield localization had just been invented!

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THE LOCALIZATION OF SPECTRA WITH RESPECT TO HOMOLOGY

A. K. BOUSFIELD†

(*Received* 3 *January* 1979)

Bousfield localization (continued)

Suppose we have a generalized homology theory represented by a spectrum E.

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$$f^* : [B, Z] \to [A, Z]$$

is also an isomorphism.

Bousfield localization (continued)

Theorem (Bousfield localization of spectra 1979)

For a given E there is a coaugmented functor L_E such that for any spectrum X, $L_E X$ is E-local and the map $X \to L_E X$ is an E_*-equivalence.

Bousfield localization (continued)

Theorem (Bousfield localization of spectra 1979)

For a given E there is a coaugmented functor L_E such that for any spectrum X, L_EX is E-local and the map $X \to L_EX$ is an E_*-equivalence.

It turns out that when E and X are both connective, then L_EX can be described in arithmetic terms.

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For a given E there is a coaugmented functor L_E such that for any spectrum X, $L_E X$ is E-local and the map $X \to L_E X$ is an E_*-equivalence.

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It turns out that when E and X are both connective, then $L_E X$ can be described in arithmetic terms. It is either obtained from X by inverting some set of primes, or it is the p-adic completion for a single prime p.

Things can be much more interesting when either E or X (or both) fail to be connective.

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It is closely related to the fancier Morava spectrum E_n, but the latter had not been invented yet. It turns out that both lead to the same localization functor.

Bousfield equivalence

Recall that a spectrum Z is E-local if, whenever $f : A \to B$ is an E_*-equivalence, that is a map inducing an isomorphism $E_*A \to E_*B$, then the induced map

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It is easy to check that wedges and smash products of Bousfield classes are well defined, that is we can define

$$\langle E \rangle \vee \langle F \rangle := \langle E \vee F \rangle \text{ and } \langle E \rangle \wedge \langle F \rangle := \langle E \wedge F \rangle.$$

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The collection of classes with complements forms a Boolean algebra **BA**.

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The chromatic tower

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For a fixed prime p, let $L_n = L_{E(n)}$. Then for any spectrum X we get a diagram

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Harmonic and dissonant spectra

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In 2014 Tobias Barthel proved a p-local connective spectrum X is chromatically convergent when $BP_* X$ has finite projective dimension as a BP_*-module. Such spectra were previously known to be harmonic.

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 & & \| & & \| & & \| & & \\
 & & BP_*/(p^\infty, \ldots v_{n-1}^\infty) & & \| & & BP_*/(p^\infty, \ldots v_n^\infty) & & \\
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Some conjectures (continued)

Periodicity Theorem (Hopkins-Smith 1998)

Let X be a p-local finite spectrum of chromatic type n, meaning that $K(n-1)_*X = 0$, but $K(n)_*X \neq 0$. Then there is a map $v : \Sigma^d X \to X$ (a v_n self-map) with $K(n)_*(v)$ an isomorphism and $H_*(v; \mathbf{Z}/p) = 0$. If $n = 0$ then $d = 0$, and when $n > 0$, d is a multiple of $2p^n - 2$.

Given a second such map $w : \Sigma^e X \to X$, there are positive integers i and j such that $id = je$ and $v^i = w^j$. In other words, v is assymptotically unique.

It follows that the cofiber of v (or of any of its iterates) is a p-local finite spectrum of chromatic type $n+1$. This means that there are finite complexes of all chromatic types. Finite complexes of arbitrary chromatic type were first constructed by Steve Mitchell in 1985.

HENCE THERE ARE LOTS OF PERIODIC FAMILIES IN $\pi_* S$.

Some conjectures (continued)

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Theorem (The class invariance conjecture)

The Bousfield class of a p-local finite spectrum X is determined by its chromatic type, i.e., the smallest n for which $K(n)_*X \neq 0$.

Some conjectures (continued)

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Theorem (The class invariance conjecture)

The Bousfield class of a p-local finite spectrum X is determined by its chromatic type, i.e., the smallest n for which $K(n)_*X \neq 0$. In particular if H_*X is not all torsion, then $\langle X \rangle = \langle S_{(p)} \rangle$.

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Note that it is independent of the choice of v. Since v is a $K(n)$-equivalence and therefore an $E(n)$-equivalence, we have maps

$$X \longrightarrow \widehat{X} \xrightarrow{\ \lambda\ } L_n X.$$

The telescope conjecture (continued)

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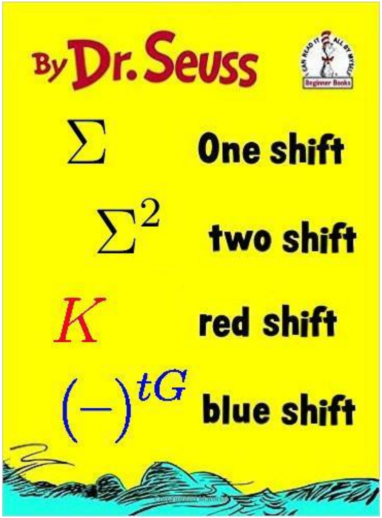
Telescope Conjecture

For any p-local spectrum X of chromatic type n, the map $\lambda : \widehat{X} \to L_n X$ *is an equivalence.*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Σ One shift
Σ^2 two shift
K red shift
$(-)^{tG}$ blue shift

THANK YOU!